Astronomy Exercise 2

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1. Kepler's laws

a) State the three laws of Kepler

1. Law of ellipses: Bodies move around a body with way greater mass in an ellipitcal path and the greater body being at one of the focal points of the ellipses.

2. Law of equal areas: The body moving around the other doesn't move with a constant velocity around in the ellipitcal path as it covers always the same area that is drawn on the ellipses in the same time, meaning if the body is further away from the focal point it moves slower as it would when it is close to it.

3. Law of harmonic time: The planet's orbital period is proportional to the cube of the length of the semi-major axis of the ellipses.

b) The comet Tschurjumow-Gerassimenko has a perihelion of 1.21 AU and an orbital period of 6.43 yr. What is the aphelion of the comet's orbit? Give the answer in both AU and in km.

With Kepler's third law we can determine the lenght of the semi-major axis since the orbital period is given:

$$\begin{aligned}
\frac{a^3}{T^2} &= \frac{G(M+m)}{4\pi^2} \approx \frac{GM}{4\pi^2} \\
a^3 &= \frac{GMT^2}{4\pi^2} \\
a &= \sqrt[3]{\frac{GM}{4\pi^2} \cdot T^2} \\
&= \sqrt[3]{\frac{T^2}{yr^2}} \cdot \sqrt[3]{\frac{GM}{4\pi^2} \cdot yr^2}
\end{aligned}$$

The term $\sqrt[3]{\frac{GM}{4\pi^2} \cdot yr^2}$ is the definition of the astronomical unit AU, which is why we receive:

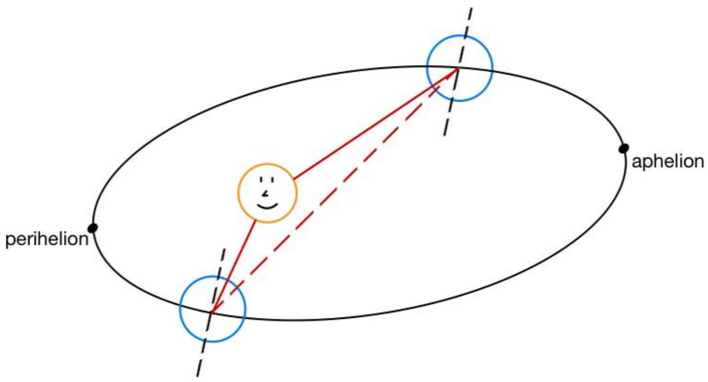
$$\begin{aligned}
a &= \sqrt[3]{\frac{T^2}{yr^2}} AU \\
&= \sqrt[3]{\frac{6.43^2 \cdot yr^2}{yr^2}} AU \\
&= \sqrt[3]{6.43^2} AU \\
&\approx 3.4579 AU
\end{aligned}$$

The aphelion we receive by subtracting the perihelion from the full major axis:

$$\begin{aligned}
d_{aphelion} &= 2a - d_{perihelion} \\
&= 2a - d_{perihelion} \\
&= 2 \cdot 3.4579AU - 1.21AU \\
&= 5.7058AU \\
&= 5.7058 \cdot 1,496e+8km \\
&\approx 8,535755e+12km
\end{aligned}$$

c) This and next year's spring equinox happend on March 20, while this year's fall equinox occured on September 22. If you count the days, there are roughly 186 days between the spring and fall equinox, but only 179 days between fall and the next spring equinox. Using Kepler's laws, explain why the nothern winter seems to take less time than the northern summer. During which season is the Earth closer to the Sun?

An equinox occurs when the Sun's declanation is $0°$. The Earth's axis is tilted in perspective to the Sun. Also the orbital eccentricity of the Earth is about 0.017, which means that the Sun is at one of the foci and there is aphelion and perihelion which the Earth reaches. The two equinox points draw a line throught the ellipses of Earth's orbit deviding the ellipses in a not mirror symmetrical but symmetrical manner. That means that the two lengths which we receive from the circumference are equal. The perihelion and aphelion are each on one of these lengths. Due to Kepler's second law these to lengths will be run with two different velocities, one being faster and one slower. Due to the slight ellipitcal shape of the orbit this difference will be noticed in only a few days as stated in the problem setting.



During the northern winter the Earth is closer to the Sun, which means that the Earth has a higher velocity on it's circular path and which is why this season takes less time.

2. Time systems

a) Define sideral time, true solar time and mean solar time.

textbf1. Sideral time: The sideral time is the measure of time based on the Earth's rotation relative to distant stars. A sideral day is approximately 23 hours, 56 minutes and 4 seconds.

textbf2. True solar time: The true solar time is based on the position of the Sun in the sky. It varies throughout the year due to the Earth's elliptical orbit and axis tilt.

textbf3. Mean solar time: The mean solar time, is the solar time that would be measured by observation if the Sun traveled at a uniform apparent speed throughout the year rather than, as it actually does as it's speed is varying slightly.

b) Calculate the difference between the civil time and mean solar time for the following cities: Berlin $\lambda = 13.40°$ E; Barcelona $\lambda = 2.16°$ E; Warsaw $\lambda = 21.012°$ E. Note all these cities follow the same time zone (CET) defined by meridian with longitude $\lambda = 15°$ E.

We can calculate the mean solar time with:

$$t_{diff} = 4\frac{min}{\circ} \cdot (\lambda - 15°)$$

If we insert the longitudes we receive the time difference from the time of the CET zone in minutes:

$$\begin{aligned}
t_{diff} &= 4\frac{min}{\circ} \cdot (13.40^\circ - 15^\circ) \\
&= 4\frac{min}{\circ} \cdot -1.60^\circ \\
&= 4min \cdot -1.60 \\
&= -6.40min
\end{aligned}$$

Meaning that the mean solar time of Berlin is: CET $-6.40min$.

$$\begin{aligned}
t_{diff} &= 4\frac{min}{\circ} \cdot (2.16^\circ - 15^\circ) \\
&= 4\frac{min}{\circ} \cdot -12.84^\circ \\
&= 4min \cdot -12.84 \\
&= -51.36min
\end{aligned}$$

Meaning that the mean solar time of Barcelona is: CET $-51.36min$.

$$\begin{aligned}
t_{diff} &= 4\frac{min}{\circ} \cdot (21.012^\circ - 15^\circ) \\
&= 4\frac{min}{\circ} \cdot +6.012^\circ \\
&= 4min \cdot +6.012 \\
&= 24.048min
\end{aligned}$$

Meaning that the mean solar time of Barcelona is: CET $+24.048min$.

3. The Virial Theorem

a) In space there are many gravitationally bound systems. If a system is roughly in equilibrium, the Virial Theorem states that the kinetic energy is equal to minus one half the potential energy ($\langle T\rangle = -\frac{1}{2}\langle V\rangle$). Consider a light particle in a circular orbit around a heavier one. Prove the Virial Theorem from this system's equation of motion.

When we consider a light particle in a circular orbit around a havier one. Then the light has the mass m and the heavier one has the mass M. The light orbits around the object at the radius R. Then the potential energy is:

$$V = -\frac{GmM}{R}$$

To figure out the kinetic energy we remeber that the gravitational force is:

$$F_{grav} = -\frac{GmM}{R^2}$$

while the centrifugal force is:

$$F_{cent} = \frac{mv^2}{R}$$

In a circular orbit these counteract each other perfectly, so we must have:

$$\frac{mv^2}{R} = \frac{GmM}{R^2}$$

Thus the kinetic energy of the light particle is:

$$T = \frac{mv^2}{2} = \frac{GmM}{2R}$$

while the kinetic energy of the heavier one is negligible, putting the previous equations in perspective we receive:

$$\begin{aligned}
V &= -\frac{GmM}{R} \\
-V &= \frac{GmM}{R} \\
T &= \frac{GmM}{2R} \\
2T &= \frac{GmM}{R} \\
2T &= -V \\
T &= -\frac{1}{2}V
\end{aligned}$$

4. Luminosity

a) What do you understand by irradiance? Calculate the irradiance received from the Sun above the absorbing atmospheres of planet Mercury, Earth, and Uranus. Given, luminosity of the Sun $L_\odot = 3.839 \times 10^{26} W$.

Irradiance is the power of the electromagnetic energy received by a surface. Given from the lecture we have the following equation for power intake of a planet:

$$L_{in} = (1 - A)\frac{L_\odot \pi R_P^2}{4\pi d_P^2}$$

With that equation we receive the power intake of the whole atmosphere of the planet. If we devide it now by the area of its atmosphere we get the irradiance or power intake per area unit:

$$\begin{aligned}
E_e &= \frac{L_{in}}{A_P} \\
&= (1 - A)\frac{L_\odot \pi R_P^2}{4\pi d_P^2} \cdot \frac{1}{\pi R_P^2} \\
&= (1 - A)\frac{L_\odot}{4\pi d_P^2}
\end{aligned}$$

We take a look at the albedos and distance Sun to planet:

Planet	Albedo	Distance to Sun
Mercury	0.06	$0.307499AU$
Earth	0.31	$1AU$
Uranus	0.66	$19.201AU$

Mercury:

$$\begin{aligned}
E_e &= (1 - 0.06)\frac{3.839 \times 10^{26} W}{4\pi 0.307499^2 AU^2} \\
&\approx (1 - 0.06)\frac{3.839 \times 10^{26} W}{4\pi 0.307499^2 \cdot (1.496 \times 10^{11})^2 m^2} \\
&\approx 13571 \frac{W}{m^2}
\end{aligned}$$

Earth:

$$\begin{aligned}
E_e &= (1 - 0.31)\frac{3.839 \times 10^{26} W}{4\pi 1^2 AU^2} \\
&\approx (1 - 0.31)\frac{3.839 \times 10^{26} W}{4\pi 1^2 \cdot (1.496 \times 10^{11})^2 m^2} \\
&\approx 941.19037 \frac{W}{m^2}
\end{aligned}$$

Uranus:

$$\begin{aligned}
E_e &= (1-0.66)\frac{3.839\times 10^{26}W}{4\pi 19.201^2 AU^2}\\
&\approx (1-0.66)\frac{3.839\times 10^{26}W}{4\pi 19.201^2\cdot(1.496\times 10^{11})^2 m^2}\\
&\approx 1.259\frac{W}{m^2}
\end{aligned}$$

b) The irradiance received by the Sun above the Earth's atmosphere per unit area, is also known as the "solar irradiance". Find the distance from where a 60-Watt lightbulb has its irradiance equal to the solar irradiance.

The Eath has a solar irradiance of about $941.19037\frac{W}{m^2}$. With that in mind we receive the following statement:

$$941.19037\frac{W}{m^2} = \frac{60W}{d_P^2}$$

With d_P being the distance from lightbulb to Earth.

$$\begin{aligned}
d_P^2 &= \frac{60W}{941.19037\frac{W}{m^2}}\\
&= \sqrt{60W\cdot\frac{1}{941.19037\frac{W}{m^2}}}\\
&= \sqrt{60W\cdot\frac{1}{941.19037\frac{W}{m^2}}}\\
&= \sqrt{60W\cdot\frac{1}{941.19037}\cdot\frac{1}{\frac{W}{m^2}}}\\
&= \sqrt{60W\cdot\frac{1}{941.19037}\cdot\frac{m^2}{W}}\\
&= \sqrt{60\cdot\frac{1}{941.19037}\cdot m^2}\\
&= \sqrt{\frac{60}{941.19037}}m\\
&\approx 0.25248m
\end{aligned}$$

The lightbulb must be around 0.25248 metre above the atmosphere of the Earth to cause the same irradiance as the Sun to Earth.

c) The energy emitted per second by a star is $L = 4\pi R^2\sigma T_{eff}^4 = S\sigma T_{eff}^4$, where S is the surface area of the star. A person also emits radiation. Under normal conditions, the temperature of the human body is around $37°C$ or $T\approx 310K$. An area of a person's body is on the order of $S\approx 1.7m^2$. What is the energy emitted per second by a person and what is the characteristic wavelength of this emission?

The constant σ is here the Stefan-Boltzmann constant, which is about $5.67\times 10^{-8}\frac{W}{m^2K^4}$. We can then insert the values given in the problem setting:

$$\begin{aligned}
L &= S\sigma T_{eff}^4\\
&= 1.7m^2\cdot 5.67\times 10^{-8}\frac{W}{m^2K^4}\cdot 310^4K^4\\
&= 1.7\cdot 5.67\times 10^{-8}W\cdot 310^4\\
&= 1.7\cdot 5.67\times 10^{-8}W\cdot 9.23521\times 10^9\\
&= 1.7\cdot 5.67W\cdot 9.23521\cdot 10\\
&= 890.2W
\end{aligned}$$

Since realistically a person can't be perfectly still, we assume that the person is walking at walking speed. Meanin that the person would have a velocity of $1.42\frac{m}{s}$. Then we get the characteristic wavelength with the following equation:

$$\lambda = \frac{h}{p}$$

where h is the Planck constant of about $6.626 \times 10^{-34} \frac{m^2 kg}{s}$. And p being the momentum of the person:

$$p = m \cdot v$$

An average person ways about $80kg$. Then we receive:

$$\begin{aligned}
\lambda &= \frac{h}{m \cdot v} \\
&= 6.626 \times 10^{-34} \frac{m^2 kg}{s} \cdot \frac{1}{80kg \cdot 1.42 \frac{m}{s}} \\
&= 6.626 \times 10^{-34} m \cdot \frac{1}{80 \cdot 1.42} \\
&= 5.83275 \times 10^{-36} m
\end{aligned}$$