Astrophysics Exercise 1

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1. Elliptical orbits

a) Consider a planet on an elliptical orbit around the Sun, with semimajor axis a and numerical excentricity ϵ. Derive a formula for the maximum and the minimum distance of the planet to the Sun, given a and ϵ.

In the lecture given, is the formula for the excentricity calculated with the distance of the centre of an ellipse to the focal point f the following:

$$\epsilon = \frac{f}{a}$$

with that we can derive a formula to calculate f:

$$f = \epsilon \cdot a$$

Additional with that distance we can derive formulas of the maximum d_{max}:

$$\begin{aligned} d_{max} &= a + f \\ &= a + a \cdot \epsilon \\ &= a \cdot (\epsilon + 1) \end{aligned}$$

and the minimum distance d_{min}:

$$\begin{aligned} d_{min} &= a - f \\ &= a - a \cdot \epsilon \\ &= a \cdot (\epsilon - 1) \end{aligned}$$

b) Kepler's second law states that the orbital speed on an elliptical orbit varies in a systematic way. Derive a formula similar to a), but now for the maximum and the minimum orbital speed.

With the vis-viva equation we can determine the velocity of a body on an elliptical sphere:

$$\begin{aligned} v^2 &= G(M+m)\left(\frac{2}{r} - \frac{1}{a}\right) \\ v &= \sqrt{G(M+m)\left(\frac{2}{r} - \frac{1}{a}\right)} \end{aligned}$$

If we now insert the maximum and the minimum distances we receive the formula for the maximum and the minimum orbital speed respectively:

$$v_{max} = \sqrt{G(M+m)\left(\frac{2}{d_{max}} - \frac{1}{a}\right)}$$

and:

$$v_{min} = \sqrt{G(M+m)\left(\frac{2}{d_{min}} - \frac{1}{a}\right)}$$

c) Calculate the ratio between the maximum and the minimum distances to the Sun for the Earth and for the planet Mercury. Calculate also the ratio between the maximum and the minimum orbital velocities for both planets.

We use the following formula to calculate the ratio of the maximum and minimum distances ratio:

$$\begin{aligned}\frac{d_{max}}{d_{min}} &= \frac{a \cdot (\epsilon + 1)}{a \cdot (\epsilon - 1)} \\ &= \frac{\epsilon + 1}{\epsilon - 1}\end{aligned}$$

Earth:

$$\begin{aligned}\frac{d_{max}}{d_{min}} &= \frac{\epsilon + 1}{\epsilon - 1} \\ &= \frac{0.017 + 1}{0.017 - 1} \\ &= \frac{0.017 + 1}{0.017 - 1} \\ &= \frac{1.017}{-0.983} \\ &\approx -1.0346\end{aligned}$$

Mercury:

$$\begin{aligned}\frac{d_{max}}{d_{min}} &= \frac{\epsilon + 1}{\epsilon - 1} \\ &= \frac{0.206 + 1}{0.206 - 1} \\ &= \frac{0.206 + 1}{0.206 - 1} \\ &= \frac{1.206}{-0.794} \\ &\approx -1.5189\end{aligned}$$

For the ratio of the maximum and minimum orbital velocity we can use:

$$\begin{aligned}\frac{v_{max}}{v_{min}} &= \frac{\sqrt{G(M+m)\left(\frac{2}{d_{max}} - \frac{1}{a}\right)}}{\sqrt{G(M+m)\left(\frac{2}{d_{min}} - \frac{1}{a}\right)}} \\ &= \frac{\sqrt{\left(\frac{2}{d_{max}} - \frac{1}{a}\right)}}{\sqrt{\left(\frac{2}{d_{min}} - \frac{1}{a}\right)}}\end{aligned}$$

Earth:

$$\begin{aligned}\frac{v_{max}}{v_{min}} &= \frac{\sqrt{\left(\frac{2}{d_{max}} - \frac{1}{a}\right)}}{\sqrt{\left(\frac{2}{d_{min}} - \frac{1}{a}\right)}} \\ &= \frac{\sqrt{\left(\frac{2}{1.017} - \frac{1}{1}\right)}}{\sqrt{\left(\frac{2}{0.983} - \frac{1}{1}\right)}} \\ &= \frac{\sqrt{\left(\frac{2}{1.017} - 1\right)}}{\sqrt{\left(\frac{2}{0.983} - 1\right)}}\end{aligned}$$

Mercury:

$$\frac{v_{max}}{v_{min}} = \frac{\sqrt{\left(\frac{2}{d_{max}} - \frac{1}{a}\right)}}{\sqrt{\left(\frac{2}{d_{min}} - \frac{1}{a}\right)}} = \frac{\sqrt{\left(\frac{2}{1.206} - \frac{1}{1}\right)}}{\sqrt{\left(\frac{2}{0.794} - \frac{1}{1}\right)}} = \frac{\sqrt{\left(\frac{2}{1.206} - 1\right)}}{\sqrt{\left(\frac{2}{0.794} - 1\right)}}$$

2. Artificial satellites on circular orbits

a) The Hubble Space Telescope needs 95 minutes for one complete revolution arount the Earth. Why is it's orbit not stable over several decades?

The Hubble Space Telescope orbits only about 550 km above earth. At this low altitude there is still a tenuous atmosphere. Meaning that there are still thin layers of gas paricles, which drag a little on the spacecraft. Over time that adds up and over several decades the orbit won't be stable. Besides that the Earth's mass is not equally distributed, which has an impact on the satellites orbit at that low altitude.

b) Re-derive Equation (4.15) in the lecture notes.

c) Suppose that a spacecraft moves on a circular orbit around the Earth at a height of $H = 1000$ km above the surface. Now the spacecraft fires its rockets in tangential direction, accelerating its speed to 1.2x its equilibrium orbital velocity. Calculate the new equilibrium orbit that will result from this action (always assuming circular motion). At what height H' above Earth will the spacecraft end up?

d) What would happen if the spacecraft in c) was instead accelerated to 1.5x its equilibrium orbital velocity?