Notes on Polish space

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1 Introduction

This document contains notes about Polish space which play an important role in probability and statistics. The materials are mainly from Cohn (2013), Chapter 8 and Dudley (2002), Chapter 13.

2 Polish space

Exercise 1 (Cohn (2013), Exercise 8.2.1)**.** *Let A be an uncountable analytic subset of the Polish space X. Then,*

(a) A has a subset that is homeomorphic to $\{0,1\}^{\mathbb{N}}$.

(b) A has the cardinality of the continuum.

Proof. From Cohn (2013), Corollary 8.2.8., there is a continuous function f from $\mathscr{N}$ onto A. By the axiom of choice, there is a set $S \subset \mathscr{N}$ such that the restriction of f on S is a bijection of S onto A. As a subspace of $\mathscr{N}$, S is an uncountable separable metrizable space. Let $S_0 \subset S$ be the set of all condensation points of the space S. From Cohn (2013), Lemma 8.2.12, S_0 is uncountable and each point of S_0 is a condensation point of S_0. Let $d_{\mathscr{N}}(\cdot,\cdot)$ be a metric on $\mathscr{N}$ which metrize the topology of $\mathscr{N}$. Let $d_X(\cdot,\cdot)$ be a metric on X which metrize the topology of X.

Now we construct a homeomorphic between a subset of X and $\{0,1\}^{\mathbb{N}}$. First, let x_0 and x_1 be two distinct points in S_0. Since the restriction of f on S_0 is injective, $f(x_0) \neq f(x_1)$. Hence there exists $0 < \epsilon_1 < 1$ such that $\overline{B(x_0,\epsilon_1)} \cap \overline{B(x_1,\epsilon_1)} = \emptyset$ and $f(\overline{B(x_0,\epsilon_1)}) \cap f(\overline{B(x_1,\epsilon_1)}) = \emptyset$. For $i = 0,1$, let $C(i) = B(x_i,\epsilon_1)$. Note that for $i = 0,1$, $C(i) \cap S_0$ is uncountable and each point of $C(i) \cap S_0$ is a condensation point of $C(i) \cap S_0$. Then there exist x_{i0}, $x_{i1} \in C(i) \cap S_0$ $(i = 0,1)$ and $0 < \epsilon_2 < 1/2$ such that for $j = 0,1$, $B(x_{ij},\epsilon_2) \subset B(x_i,\epsilon_1)$, $\overline{B(x_{i0},\epsilon_2)} \cap \overline{B(x_{i1},\epsilon_2)} = \emptyset$ and $f(\overline{B(x_{i0},\epsilon_2)}) \cap f(\overline{B(x_{i1},\epsilon_2)}) = \emptyset$. For $i,j \in \{0,1\}$, let $C(i,j) = B(x_{ij},\epsilon_2)$.

Inductively construct sets $C(n_1,n_2,\ldots,n_k)$, $n_i \in \{0,1\}$, $k \in \mathbb{N}$. Then for $\{n_k\}_{k=1}^{\infty} \in \mathscr{N}$, consider the set $\cap_{k=1}^{\infty}\overline{C(n_1,\ldots,n_k)}$. By the completeness of $\mathscr{N}$, $\cap_{k=1}^{\infty}\overline{C(n_1,\ldots,n_k)} \neq \emptyset$. Also, the diameter of $\overline{C(n_1,\dots,n_k)}$ tends to 0. Then there exists a unique point in $\cap_{k=1}^{\infty}\overline{C(n_1,\dots,n_k)}$. Let g be the function from $\mathscr{N}$ to X which maps $\{n_k\}_{k=1}^{\infty}$ to the unique point of $\cap_{k=1}^{\infty}\overline{C(n_1,\dots,n_k)}$.

By the construction of $C(n_1,\dots,n_k)$, g is continuous and injective. Then $f\circ g$ is continuous. To see that $f\circ g$ is injective, let $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ be two distinct points of $\{0,1\}^{\mathscr{N}}$. Let k_0 be the first k such that $n_k \neq m_k$. By the construction of $C(\cdot,\dots,\cdot)$, $f(\overline{C(n_1,\dots,n_{k_0})})\cap f(\overline{C(m_1,\dots,m_{k_0})})=\emptyset$. Since $g(\{n_k\}_{k=1}^{\infty})\subset\overline{C(n_1,\dots,n_{k_0})}$, $g(\{m_k\}_{k=1}^{\infty})\subset\overline{C(m_1,\dots,m_{k_0})}$. Then $f\circ g(\{n_k\}_{k=1}^{\infty})\neq f\circ g(\{m_k\}_{k=1}^{\infty})$.

Since $\{0,1\}^{\mathscr{N}}$ is compact, the inverse of $f\circ g$ is also continuous. This completes the proof of (a).

(a) implies that $\text{card}(A)\geq\mathfrak{c}$. On the other hand, Cohn (2013), Corollary 8.2.8. implies that $\text{card}(A)\leq\mathfrak{c}$. Thus, $\text{card}(A)=\mathfrak{c}$.

□

Exercise 2. *Let X be an uncoutable Polish space. Then the collection of analytic subsets of X and the collection of Borel subsets of X have the cardinality of the continuum.*

Proof. Exercise 1 implies that the cardinality of X is $\mathfrak{c}$. Since each single point of X is a Borel set, the cardinality of the collection of Borel subsets of X is at least $\mathfrak{c}$. We only need to prove that the cardinality of the collection of analytic subsets of X is at most $\mathfrak{c}$.

Cohn (2013), Proposition 8.2.9 implies that it suffices to upper bound the cardinality of the collection of closed subsets of the Polish space $\mathscr{N}\times X$. Let $\{U_i\}_{i=1}^{\infty}$ be a countable base of the topology of $\mathscr{N}\times X$. Then every closed subset of $\mathscr{N}\times X$ is the intersection of certain $U_i^{\complement}$, that is, $\cap_{i\in S}U_i^{\complement}$ where S is a subset of $\mathbb{N}$. Hence there is an injective map from the collection of closed subsets of $\mathscr{N}\times X$ to $2^{\mathbb{N}}$. Thus, the cardinality of the collection of closed subsets of $\mathscr{N}\times X$ is at most $\mathfrak{c}$.

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References

Cohn, D. L. (2013). *Measure Theory.* Birkhäuser, New York, 2nd edition.

Dudley, R. M. (2002). *Real Analysis and Probability.* Cambridge University Press.