# Notes on Polish space

Rui Wang

Saturday 18$^{\text{th}}$ May, 2019

## 1 Introduction

This document contains notes about Polish space which play an important role in probability and statistics. The materials are mainly from Cohn (2013), Chapter 8 and Dudley (2002), Chapter 13.

## 2 Polish space

**Exercise 1** (Cohn (2013), Exercise 8.1.3). *Let $(X, \mathscr{A})$ be a measurable space, let $Y$ be a separable metrizable space, and let $f, g : X \to Y$ be measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$. Then $\{x \in X : f(x) = g(x)\} \in \mathscr{A}$.*

*Proof.* For any $A, B \in \mathscr{B}(Y)$,

$$\{x : (f(x), g(x)) \in A \times B\} = f^{-1}(A) \cap f^{-1}(B) \in \mathscr{A}.$$

Hence the map $F : x \mapsto (f(x), f(x))$ is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y) \times \mathscr{B}(Y)$. Since $Y$ is a separable metrizable space, $\mathscr{B}(Y) \times \mathscr{B}(Y) = \mathscr{B}(Y \times Y)$. Thus, the map $F$ is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y \times Y)$. Let $\Delta = \{(y_1, y_2) \in Y \times Y : y_1 = y_2\}$. Then $\Delta$ is a closed subset of $Y \times Y$ and $\{x \in X : f(x) = g(x)\} = F^{-1}(\Delta)$. It follows that $\{x \in X : f(x) = g(x)\} \in \mathscr{A}$. $\square$

**Exercise 2** (Cohn (2013), Exercise 8.2.1). *Let $A$ be an uncountable analytic subset of the Polish space $X$. Then,*

*(a) $A$ has a subset that is homeomorphic to $\{0, 1\}^{\mathbb{N}}$.*

*(b) $A$ has the cardinality of the continuum.*

*Proof.* From Cohn (2013), Corollary 8.2.8., there is a continuous function $f$ from $\mathscr{N}$ onto $A$. By the axiom of choice, there is a set $S \subset \mathscr{N}$ such that the restriction of $f$ on $S$ is a bijection of $S$ onto $A$. As a subspace of $\mathscr{N}$, $S$ is an uncountable separable metrizable space. Let $S_0 \subset S$ be the set of all condensation points of the space $S$. From Cohn (2013), Lemma 8.2.12, $S_0$ is uncountable

1

and each point of $S_0$ is a condensation point of $S_0$. Let $d_{\mathcal{N}}(\cdot, \cdot)$ be a metric on $\mathcal{N}$ which metrize the topology of $\mathcal{N}$. Let $d_X(\cdot, \cdot)$ be a metric on $X$ which metrize the topology of $X$.

Now we construct a homeomorphic between a subset of $X$ and $\{0,1\}^{\mathbb{N}}$. First, let $x_0$ and $x_1$ be two distinct points in $S_0$. Since the restriction of $f$ on $S_0$ is injective, $f(x_0) \neq f(x_1)$. Hence there exists $0 < \epsilon_1 < 1$ such that $\overline{B(x_0, \epsilon_1)} \cap \overline{B(x_1, \epsilon_1)} = \emptyset$ and $f(\overline{B(x_0, \epsilon_1)}) \cap f(\overline{B(x_1, \epsilon_1)}) = \emptyset$. For $i = 0, 1$, let $C(i) = B(x_i, \epsilon_1)$. Note that for $i = 0, 1$, $C(i) \cap S_0$ is uncountable and each point of $C(i) \cap S_0$ is a condensation point of $C(i) \cap S_0$. Then there exist $x_{i0}, x_{i1} \in C(i) \cap S_0$ $(i = 0, 1)$ and $0 < \epsilon_2 < 1/2$ such that for $j = 0, 1$, $B(x_{ij}, \epsilon_2) \subset B(x_i, \epsilon_1)$, $\overline{B(x_{i0}, \epsilon_2)} \cap \overline{B(x_{i1}, \epsilon_2)} = \emptyset$ and $f(\overline{B(x_{i0}, \epsilon_2)}) \cap f(\overline{B(x_{i1}, \epsilon_2)}) = \emptyset$. For $i, j \in \{0, 1\}$, let $C(i, j) = B(x_{ij}, \epsilon_2)$.

Inductively construct sets $C(n_1, n_2, \ldots, n_k)$, $n_i \in \{0, 1\}$, $k \in \mathbb{N}$. Then for $\{n_k\}_{k=1}^{\infty} \in \mathcal{N}$, consider the set $\cap_{k=1}^{\infty} \overline{C(n_1, \ldots, n_k)}$. By the completeness of $\mathcal{N}$, $\cap_{k=1}^{\infty} \overline{C(n_1, \ldots, n_k)} \neq \emptyset$. Also, the diameter of $\overline{C(n_1, \ldots, n_k)}$ tends to 0. Then there exists a unique point in $\cap_{k=1}^{\infty} \overline{C(n_1, \ldots, n_k)}$. Let $g$ be the function from $\mathcal{N}$ to $X$ which maps $\{n_k\}_{k=1}^{\infty}$ to the unique point of $\cap_{k=1}^{\infty} \overline{C(n_1, \ldots, n_k)}$.

By the construction of $C(n_1, \ldots, n_k)$, $g$ is continuous and injective. Then $f \circ g$ is continuous. To see that $f \circ g$ is injective, let $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ be two distinct points of $\{0,1\}^{\mathcal{N}}$. Let $k_0$ be the first $k$ such that $n_k \neq m_k$. By the construction of $C(\cdot, \ldots, \cdot)$, $f(\overline{C(n_1, \ldots, n_{k_0})}) \cap f(\overline{C(m_1, \ldots, m_{k_0})}) = \emptyset$. Since $g(\{n_k\}_{k=1}^{\infty}) \subset \overline{C(n_1, \ldots, n_{k_0})}$, $g(\{m_k\}_{k=1}^{\infty}) \subset \overline{C(m_1, \ldots, m_{k_0})}$. Then $f \circ g(\{n_k\}_{k=1}^{\infty}) \neq f \circ g(\{m_k\}_{k=1}^{\infty})$.

Since $\{0,1\}^{\mathcal{N}}$ is compact, the inverse of $f \circ g$ is also continuous. This completes the proof of $(a)$.

(a) implies that $\operatorname{card}(A) \geq \mathfrak{c}$. On the other hand, Cohn (2013), Corollary 8.2.8. implies that $\operatorname{card}(A) \leq \mathfrak{c}$. Thus, $\operatorname{card}(A) = \mathfrak{c}$.

$\square$

**Exercise 3** (Cohn (2013), Exercise 8.2.2). *Let $X$ be an uncoutable Polish space. Then the collection of analytic subsets of $X$ and the collection of Borel subsets of $X$ have the cardinality of the continuum.*

*Proof.* Exercise 2 implies that the cardinality of $X$ is $\mathfrak{c}$. Since each single point of $X$ is a Borel set, the cardinality of the collection of Borel subsets of $X$ is at least $\mathfrak{c}$. We only need to prove that the cardinality of the collection of analytic subsets of $X$ is at most $\mathfrak{c}$.

Cohn (2013), Proposition 8.2.9 implies that it suffices to upper bound the cardinality of the collection of closed subsets of the Polish space $\mathcal{N} \times X$. Let $\{U_i\}_{i=1}^{\infty}$ be a countable base of the topology of $\mathcal{N} \times X$. Then every closed subset of $\mathcal{N} \times X$ is the intersection of certain $U_i^{\complement}$, that is, $\cap_{i \in S} U_i^{\complement}$ where $S$ is a subset of $\mathbb{N}$. Hence there is an injective map from the collection of closed subsets of $\mathcal{N} \times X$ to $2^{\mathbb{N}}$. Thus, the cardinality of the collection of closed subsets of $\mathcal{N} \times X$ is at most $\mathfrak{c}$.

$\square$

**Exercise 4** (Cohn (2013), Exercise 8.2.3)**.**

*(a) Let $X$ be a nonempty zero-dimensional Polish space such that each nonempty open subset of $X$ is not compact. Then $X$ is homeomorphic to $\mathcal{N}$.*

*(b) the Space $\mathscr{I}$ of irrational numbers in the interval $(0,1)$ is homeomorphic to $\mathcal{N}$.*

*Proof.* Let $d(\cdot, \cdot)$ be a complete metric for $X$. We begin by constructing a family $\{C(n_1, \ldots, n_k)\}$ of subsets of $X$, indexed by the set of all finite sequences $\{(n_1, \ldots, n_k)\}$ of positive integers, in such a way that

1. $C(n_1, \ldots, n_k)$ is nonempty, open, closed and noncompact,

2. the diameter of $C(n_1, \ldots, n_k)$ is at most $1/k$,

3. $\{C(n_1, \ldots, n_{k-1}, n_k)\}_{n_k=1}^{\infty}$ are disjoint and $C(n_1, \ldots, n_{k-1}) = \cup_{n_k=1}^{\infty} C(n_1, \ldots, n_k)$,

4. $X = \cup_{n_1=1}^{\infty} C(n_1)$.

We do this by induction on $k$.

First, suppose that $k = 1$. Since $X$ is assumed to be not compact, Cohn (2013), Lemma 8.2.11 gives a sequnce $\{C(n_1)\}_{n_1=1}^{\infty}$ where terms are nonempty, open, closed and with diameter at most 1. By assumption, each $C(n_1)$ is not compact.

Now suppose that $k > 1$ and that $C(n_1, \ldots, n_{k-1})$ has already been chosen. It is easy to use a modification of the construction of the $C(n_1)$'s, now applied to $C(n_1, \ldots, n_{k-1})$ rather than to $X$, to produce sets $C(n_1, \ldots, n_k)$, $n_k = 1, 2, \ldots$ that satisfy conditions 1 to 4. With this, the induction step in our construction is complete.

We turn to the construction of a homeomorphic between $\mathcal{N}$ and $X$. Let $\mathbf{n} = \{n_k\}$ be an element of $\mathcal{N}$. Then the sets $C(n_1)$, $C(n_1, n_2)$, $\ldots$ are decreasing nonempty closed sets whose diameters approach to 0. Since $X$ is complete, there is a unique element in $\cap_{k=1}^{\infty} C(n_1, \ldots, n_k)$. We can define a function $f : \mathcal{N} \to X$ by letting $f(\mathbf{n})$ be the unique member of $\cap_{k=1}^{\infty} C(n_1, \ldots, n_k)$. Note that if $\mathbf{m}$ and $\mathbf{n}$ are elements of $\mathcal{N}$ such that $m_i = n_i$ holds for $k = 1, \ldots, k$, then $d(\mathbf{m}, \mathbf{n}) \leq 1/k$. It follows that $f$ is continuous. Also, it is obvious that $f$ is bijective. It remain to prove that the inverse of $f$ is continuous. Suppose $f(\mathbf{n}^{(l)}) \to f(\mathbf{n})$. Fix $k > 0$. Then if $l$ is large enough, $f(\mathbf{n}^{(l)}) \in C(n_1, \ldots, n_k)$. By the construction of $f$, this implies that $n_i^{(l)} = n_i$ for $i = 1, \ldots, k$. Thus, $\mathbf{n}^{(l)} \to \mathbf{n}$ as $l \to \infty$. This completes the proof of $(a)$.

We turn to the proof of $(b)$. The space $\mathscr{I}$ is a $G_\delta$ set of $[0,1]$, and hence is a Polish space. The family of intervels $(a_i, b_i)$ where $a_i$ and $b_i$ is rational is a base that consists of sets that are both open and closed. It follows that $\mathscr{I}$ is zero-dimensional. Each interval $(a, b)$ is the union of $\{(a_i, b_i)\}_{i=1}^{\infty}$ where $a_i$, $b_i$ are rational and $a_i \downarrow a$ and $b_i \uparrow b$. Hence each interval of $\mathscr{I}$ is not compact. Then the conclusion follows from $(a)$.

$\square$

**Exercise 5** (Cohn (2013), Exercise 8.2.3)**.** *Each nonempty Polish space is the image of $\mathscr{N}$ under a continuous open map.*

*Proof.* We mimic the proof of Cohn (2013), Proposition 8.2.7.

Let $X$ be a nonempty Polish space, and let $d$ be a complete metric for $X$. We begin by constructing a family $\{C(n_1, \ldots, n_k)\}$ of subsets of $X$, indexed by the set of all finite sequences $\{n_1, \ldots, n_k\}$ of positive integers, in such a way that

1. $C(n_1, \ldots, n_k)$ is nonempty and open,

2. the diameter of $C(n_1, \ldots, n_k)$ is at most $1/k$,

3. $\overline{C(n_1, \ldots, n_{k-1}, n_k)} \subset C(n_1, \ldots, n_{k-1})$ and $C(n_1, \ldots, n_{k-1}) = \cup_{n_k=1}^{\infty} C(n_1, \ldots, n_k)$,

4. $X = \cup_{n_1=1}^{\infty} C(n_1)$.

We do this by induction on $k$.

First, suppose that $k = 1$, and let $\{x_i\}_{i=1}^{\infty}$ be a sequence whose terms form a dense subset of $X$. The sequence $\{X_i\}_{i=1}^{\infty}$ may have duplicated elements. Let $\{C(n_1)\}_{n_1=1}^{\infty}$ be the collection of open balls which center at certain $x_i$ and with rational radius not larger than $1/2$. Certainly each $C(n_1)$ is open and nonempty and has diameter at most 1. Furthermore, $X = \cup_{n_1} C(n_1)$.

Now suppose that $k > 1$ and that $C(n_1, \ldots, n_{k-1})$ has already been chosen. Let $\{C(n_1, \ldots, n_{k-1}, n_k)\}_{n_k=1}^{\infty}$ be the collection of open balls which center at centain $x_i$ and with rational radius not larger than $1/(2k)$ and whose closure is contained in $C(n_1, \ldots, n_{k-1})$. Certainly each $C(n_1, \ldots, n_k)$ is open and nonempty and has diameter at most $1/k$. Now we prove that $C(n_1, \ldots, n_{k-1}) = \cup_{n_k=1}^{\infty} C(n_1, \ldots, n_k)$. Suppose $x \in C(n_1, \ldots, n_{k-1})$. Since $C(n_1, \ldots, n_{k-1})$ is open, there is a open ball $B(x, r) \subset C(n_1, \ldots, n_{k-1})$ where $r$ is rational and $r < 1/k$. Since $\{x_i\}_{i=1}^{\infty}$ is dense in $X$, there is an $x_i$ such that $d(x, x_i) < r/3$. Then the ball $B(x_i, r/2)$ contains $x$. Also, the Closure of $B(x_i, r/2)$ has radius not larger than $1/(2k)$ and is contained in $C(n_1, \ldots, n_{k-1})$. Thus, $B(x_i, r/2) = C(n_1, \ldots, n_k)$ for some $n_k$. With this, the induction step in our construction is complete.

We turn to the construction of a continuous function that maps $\mathscr{N}$ onto $X$. Let $\mathbf{n} = \{n_k\}$ be an element of $\mathscr{N}$. It follows from 3 that $\cap_{k=1}^{\infty} C(n_1, \ldots, n_k) = \cap_{k=1}^{\infty} \overline{C(n_1, \ldots, n_k)}$ which is intersection of a decreasing sequence of nonempty closd subsets of $X$ whose diameters approach 0. Thus there is a unique element in the intersection of these sets, and we can define a function $f : \mathscr{N} \to X$ by letting $f(\mathbf{n})$ be the unique member of $\cap_k C(n_1, \ldots, n_k)$. Note that if $\mathbf{m}$ and $\mathbf{n}$ are elements on $\mathscr{N}$ such that $m_i = n_i$ holds for $i = 1, \ldots, k$, then $d(f(\mathbf{m}, \mathbf{n})) \leq 1/k$. It follows that $f$ is continuous. Also, 3 and 4 above imply that for each x in $X$ there is an element $\mathbf{n} = \{n_k\}$ of $\mathscr{N}$ such that $x \in \cap_k C(n_1, \ldots, n_k)$ and hence such that $x = f(\mathbf{n})$. Thus $f$ is surjective.

It remains to prove that $f$ is an open map. Note that the sets of the form $\{n_1\} \times \cdots \times \{n_k\} \times \mathbb{N} \times \ldots$ is a base for the topology of $\mathscr{N}$. By the construction of $f$, for any $n_1, .., n_k$, $f(\{n_1\} \times \cdots \times \{n_k\} \times \mathbb{N} \times \ldots) = C(n_1, \ldots, n_k)$ is an open set. This completes the proof.

□

**Exercise 6** (Cohn (2013), Exercise 8.2.5)**.** *Each Borel subet of a Polish space is the image under a continuous injective map of some Polish space.*

*Proof.* Let $X$ be a Polish space. Let $\mathcal{A}$ be the collection of Borel subsets of $X$ which are the image under continuous injective maps of some Polish spaces. Then all open and closed subsets of $X$ belong to $\mathcal{A}$ since they are themselves Polish spaces.

Assume $A_1, \ldots, A_n, \cdots \in \mathcal{A}$ and $A_1, \ldots, A_n, \ldots$ are disjoint. For each $A_i$, there is a Polish space $X_i$ and a continuous infective map $f_i(\cdot)$ such that $f_i(X_i) = A_i$. Define $f : \cup_{i=1}^\infty X_i \mapsto \cup_{i=1}^\infty A_i$ by $f(x) = f_i(x)$ if $x \in X_i$. Here $\cup_{i=1}^\infty X_i$ is the disjoint union of $X_i$. Then $\cup_{i=1}^\infty X_i$ is Polish and $f$ is injective and continuous. Then $\cup_{i=1}^\infty A_i \in \mathcal{A}$.

Assume $A_1, \ldots, A_n, \cdots \in \mathcal{A}$. For each $A_i$, there is a Polish space $X_i$ and a continuous infective map $f_i(\cdot)$ such that $f_i(X_i) = A_i$. Define $f : \prod_{i=1}^\infty X_i \mapsto \prod_{i=1}^\infty X$ by $f(\{x_i\}_{i=1}^\infty) = \{f_i(x_i)\}_{i=1}^\infty$. Then $f$ is injective and continuous onto $\prod_{i=1}^\infty A_i \subset \prod_{i=1}^\infty X$. Let $D = \{(x, x, \ldots) : x \in X\}$. Define $g : D \mapsto X$ by $g(x, x, \ldots) = x$. Then $g$ is a homeomorphism between $D$ and $X$. Consider $g \circ f$ defined on $f^{-1}(D)$. Then $g \circ f$ is injective and continuous from $f^{-1}(D)$ onto $\cap_{i=1}^\infty A_i$. Since $f^{-1}(D)$ is a closed subset of $\prod_{i=1}^\infty X_i$, it is Polish. Thus, $\cap_{i=1}^\infty A_i \in \mathcal{A}$.

From Cohn (2013), Lemma 8.2.4, $\mathcal{A}$ contains all Borel subset of $X$. This completes the proof. □

**Exercise 7** (Cohn (2013), Exercise 8.2.6)**.** *If $X$ is an uncountable Polish space, then there is an analytic subset of $X$ that is not a Borel set.*

*Proof.* Let $X$ be an uncountable Polish space. From Cohn (2013), Proposition 8.2.13, there is a continuous injective map $f : \mathcal{N} \to X$ such that $X - f(\mathcal{N})$ is countable. From Cohn (2013), Corollary 8.2.17, there is an analytic set $A \in \mathcal{N}$ that is not a Borel set. Then $f(A)$ is not a Borel set of $X$, or else $A = f^{-1}(f(A))$ would be a Borel set, a contradiction. On the other hand, $f(A)$ is analytic. This completes the proof.

□

# References

Cohn, D. L. (2013). *Measure Theory*. Birkhäuser, New York, 2nd edition.

Dudley, R. M. (2002). *Real Analysis and Probability*. Cambridge University Press.