Notes on the theory of Bayesian statistics

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Abstract

This document provides notes on the theory of Bayesian statistics.

1 Consistency

Nonparametric convergence rate. The work of [6] is seminal. A simlar result was obtained by [8]. The work of [6] focused on iid case. [9] generalized the results to non iid case. This line of research relies on the assumption that there exists a sequence of uniformly consistent test.

2 Results in [9]

For each $n \in \mathbb{N}$ and $\theta \in \Theta$, let $P_\theta^{(n)}$ admit densities $p_\theta^{(n)}$ relative to a σ-finite measure $\mu^{(n)}$. Assume that $(x, \theta) \mapsto p_\theta^{(n)}(x)$ is jointly measurable relative to $\mathscr{A} \otimes \mathscr{B}$, where $\mathscr{B}$ is a σ-field on Θ. The posterior distribution is given by

$$\Pi_n(B|X^{(n)}) = \frac{\int_B p_\theta^{(n)}(X^{(n)})\, \mathrm{d}\Theta_n(\theta)}{\int_\Theta p_\theta^{(n)}(X^{(n)})\, \mathrm{d}\Theta_n(\theta)}, \quad B \in \mathscr{B}.$$

Here $X^{(n)}$ is generated according to $P_{\theta_0}^{(n)}$ for some given $\theta_0 \in \Theta$.

Assumption 1. *For each n, let d_n and e_n be semimetrics on Θ with the property that there exist universal constants $\xi > 0$ and $K > 0$ such that for every $\epsilon > 0$ and for each $\theta_1 \in \Theta$ with $d_n(\theta_1, \theta_0) > \epsilon$, there exists a test ϕ_n such that*

$$P_{\theta_0}^{(n)} \phi_n \le e^{-Kn\epsilon^2}, \qquad \sup_{\theta \in \Theta : e_n(\theta, \theta_1) < \epsilon\xi} P_\theta^{(n)}(1 - \phi_n) \le e^{-Kn\epsilon^2}.$$

Lemma 1. *Suppose Assumption 1 holds. Suppose that for some nonincreasing function $\epsilon \mapsto N(\epsilon)$ and some $\epsilon_n \ge 0$,*

$$N\left(\frac{\epsilon\xi}{2}, \{\theta \in \Theta : d_n(\theta, \theta_0) < \epsilon\}, e_n\right) \le N(\epsilon) \quad \text{for all } \epsilon > \epsilon_n.$$

Then for every $\epsilon > \epsilon_n$, there exist tests ϕ_n, $n \geq 1$, (depending on ϵ) such that $P_{\theta_0}^{(n)} \phi_n \leq N(\epsilon) e^{-Kn\epsilon^2}/(1 - e^{-Kn\epsilon^2})$ and $P_\theta^{(n)}(1 - \phi_n) \leq e^{-Kn\epsilon^2 j^2}$ for all $\theta \in \Theta$ such that $d_n(\theta, \theta_0) > j\epsilon$ and for every $j \in \mathbb{N}$.

Proof. For a given $j \in \mathbb{N}$, let $\Theta_j = \{\theta \in \Theta : j\epsilon < d_n(\theta, \theta_0) \leq (j+1)\epsilon\}$ and choose a set $\Theta'_j \subset \Theta_j$ such that $\{B_{e_n}(\theta_{(j,i)}, j\epsilon\xi), i = 1, \ldots, |\Theta'_j|\}$ is a minimal $j\epsilon\xi$-covering. Since $\Theta_j \subset \{\theta \in \Theta : d_n(\theta, \theta_0) \leq 2j\epsilon\}$, we have

$$|\Theta'_j| \leq N\left(j\epsilon\xi, \{\theta \in \Theta : d_n(\theta, \theta_0) < 2j\epsilon\}, e_n\right) \leq N(2j\epsilon).$$

By assumption, for every point $\theta_{(j,i)} \in \Theta'_j$, there exists a test $\phi_n^{(j,i)}$ with the following properties

$$P_{\theta_0}^{(n)} \phi_n^{(j,i)} \leq e^{-Knj^2\epsilon^2}, \qquad \sup_{\theta \in B_{e_n}(\theta_{(j,i)}, j\epsilon\xi)} P_\theta^{(n)}(1 - \phi_n^{(j,i)}) \leq e^{-Knj^2\epsilon^2}.$$

Let

$$\phi_n = \sup_{\{(j,i): i \in 1, \ldots, |\Theta'_j|, j \in \mathbb{N}\}} \phi_n^{(j,i)}.$$

Then

$$P_{\theta_0}^{(n)} \phi_n \leq \sum_{j=1}^{\infty} \sum_{i=1}^{|\Theta'_j|} P_{\theta_0}^{(n)} \phi_n^{(j,i)} \leq \sum_{j=1}^{\infty} |\Theta'_j| e^{-Knj^2\epsilon^2} \leq N(\epsilon) \sum_{j=1}^{\infty} e^{-Knj\epsilon^2} = N(\epsilon) \frac{e^{-Kn\epsilon^2}}{1 - e^{-Kn\epsilon^2}}.$$

On the other hand, for any $\theta \in \Theta$ such that $d_n(\theta, \theta_0) > j\epsilon$, there exists (j', i') such that $j' \geq j$, $\theta \in B_{e_n}(\theta_{(j,i)}, j\epsilon\xi)$.

$$P_\theta^{(n)}(1 - \phi_n) \leq P_\theta^{(n)}(1 - \phi_n^{(j',i')}) \leq \sup_{\theta \in B_{e_n}(\theta_{(j,i)}, j\epsilon\xi)} P_\theta^{(n)}(1 - \phi_n^{(j',i')}) \leq e^{-Knj'^2\epsilon^2} \leq e^{-Knj^2\epsilon^2}.$$

This completes the proof.

□

Corollary 1. *If the conclution of Lemma 1 holds, then for $d_n(\theta, \theta_0) > \epsilon$,*

$$P_\theta^{(n)}(1 - \phi_n) \leq \exp\left\{-\frac{K}{4} n d_n^2(\theta, \theta_0)\right\}.$$

Proof. Lemma 1 asserts that $P_\theta^{(n)}(1 - \phi_n) \leq e^{-Kn\epsilon^2 j^2}$ for all $\theta \in \Theta$ such that $d_n(\theta, \theta_0) > j\epsilon$ and for every $j \in \mathbb{N}$. Then $P_\theta^{(n)}(1 - \phi_n) \leq e^{-Kn\epsilon^2 j^2}$ for $\theta \in \Theta$ such that $j\epsilon < d_n(\theta, \theta_0) \leq (j+1)\epsilon$ for every $j \in \mathbb{N}$. Thus if $d_n(\theta, \theta_0) > \epsilon$,

$$\begin{aligned} P_\theta^{(n)}(1 - \phi_n) &\leq e^{-Kn\epsilon^2 j^2} \\ &= e^{-Kn\epsilon^2 (j+1)^2 \frac{j^2}{(j+1)^2}} \\ &\leq e^{-\frac{K}{4} n d_n^2(\theta, \theta_0)} \end{aligned}$$

□

For a given $k > 1$, let

$$B_n(\theta_0, \epsilon; k) = \left\{\theta \in \Theta : K\left(p_{\theta_0}^{(n)}, p_\theta^{(n)}\right) \le n\epsilon^2, V_{k,0}\left(p_{\theta_0}^{(n)}, p_\theta^{(n)}\right) \le n^{k/2}\epsilon^k\right\},$$

where $V_{k,0}(f,g) = \int f|\log(f/g) - K(f,g)|^k \,\mathrm{d}\mu$.

Lemma 2. *For $k \ge 2$, every $\epsilon > 0$ and every probability measure $\bar{\Pi}_n$ supported on the set $B_n(\theta_0, \epsilon; k)$, we have, for every $C > 0$,*

$$P_{\theta_0}^{(n)}\left(\int \frac{p_\theta^{(n)}}{p_{\theta_0}^{(n)}} \,\mathrm{d}\bar{\Pi}_n(\theta) \le e^{-(1+C)n\epsilon^2}\right) \le \frac{1}{C^k(n\epsilon^2)^{k/2}}$$

Proof. By Jenson's inequality applied to the logarithm, with $\ell_{n,\theta} = \log\left(p_\theta^{(n)}/p_{\theta_0}^{(n)}\right)$, we have $\log\int(p_\theta^{(n)}/p_{\theta_0}^{(n)})\,\mathrm{d}\bar{\Pi}_n(\theta) \ge \int \ell_{n,\theta}\,\mathrm{d}\bar{\Pi}_n(\theta)$. Thus,

$$\begin{aligned}
&P_{\theta_0}^{(n)}\left(\int \frac{p_\theta^{(n)}}{p_{\theta_0}^{(n)}} \,\mathrm{d}\bar{\Pi}_n(\theta) \le e^{-(1+C)n\epsilon^2}\right)\\
\le& P_{\theta_0}^{(n)}\left(\int \ell_{n,\theta}\,\mathrm{d}\bar{\Pi}_n(\theta) \le -(1+C)n\epsilon^2\right)\\
=& P_{\theta_0}^{(n)}\left(\int \left(\ell_{n,\theta} - P_{\theta_0}^{(n)}\ell_{n,\theta}\right)\mathrm{d}\bar{\Pi}_n(\theta) \le -(1+C)n\epsilon^2 - \int P_{\theta_0}^{(n)}\ell_{n,\theta}\,\mathrm{d}\bar{\Pi}_n(\theta)\right)\\
=& P_{\theta_0}^{(n)}\left(\int \left(\ell_{n,\theta} - P_{\theta_0}^{(n)}\ell_{n,\theta}\right)\mathrm{d}\bar{\Pi}_n(\theta) \le -(1+C)n\epsilon^2 + \int K\left(p_{\theta_0}^{(n)}, p_\theta^{(n)}\right)\mathrm{d}\bar{\Pi}_n(\theta)\right)\\
\le& P_{\theta_0}^{(n)}\left(\int \left(\ell_{n,\theta} - P_{\theta_0}^{(n)}\ell_{n,\theta}\right)\mathrm{d}\bar{\Pi}_n(\theta) \le -Cn\epsilon^2\right)\\
\le& \frac{P_{\theta_0}^{(n)}\int|\ell_{n,\theta} - P_{\theta_0}^{(n)}\ell_{n,\theta}|^k\,\mathrm{d}\bar{\Pi}_n(\theta)}{(Cn\epsilon^2)^k}\\
=& \frac{\int V_{k,0}(f,g)\,\mathrm{d}\bar{\Pi}_n(\theta)}{(Cn\epsilon^2)^k}\\
\le& \frac{1}{C^k(n\epsilon^2)^{k/2}}.
\end{aligned}$$

□

Theorem 1. *Suppose Assumption 1 holds. Let $\epsilon_n > 0$, $\epsilon_n \to 0$, $(n\epsilon_n^2)^{-1} = O(1)$, $k > 1$, and $\Theta_n \subset \Theta$ be such that,*

$$\sup_{\epsilon > \epsilon_n} \log N\left(\frac{1}{2}\epsilon\xi, \{\theta \in \Theta_n : d_n(\theta, \theta_0) < \epsilon\}, e_n\right) \le n\epsilon_n^2, \tag{1}$$

for some $C > 0$,

$$\frac{e^{(1+C)n\epsilon_n^2}}{\Pi_n(B_n(\theta_0, \epsilon_n; k))}\int_{\{\theta\in\Theta_n : d_n(\theta,\theta_0)\ge M_n\epsilon_n\}} \exp\left\{-\frac{K}{4}nd_n^2(\theta,\theta_0)\right\}\mathrm{d}\Pi_n(\theta) \to 0. \tag{2}$$

Then for every $M_n \to \infty$, we have that

$$P_{\theta_0}^{(n)} \Pi_n \left(\theta \in \Theta_n : d_n(\theta, \theta_0) \geq M_n \epsilon_n | X^{(n)} \right) \to 0.$$

Proof. From Corollary 1, applied with $N(\epsilon) = \exp\left(n\epsilon_n^2\right)$ and $\epsilon = M_n \epsilon_n$ (W.L.O.G $M_n \geq 1$) in its assertion, there exist tests ϕ_n that satisfy

$$P_{\theta_0}^{(n)} \phi_n \leq e^{n\epsilon_n^2} \frac{e^{-KnM_n^2\epsilon_n^2}}{1 - e^{-KnM_n^2\epsilon_n^2}},$$

$$P_{\theta}^{(n)}(1 - \phi_n) \leq \exp\left\{ -\frac{K}{4} n d_n^2(\theta, \theta_0) \right\} \text{ for all } \theta \in \Theta_n \text{ s.t. } d_n(\theta, \theta_0) > M_n \epsilon_n.$$

The first assertion implies that if M_n is sufficiently large to ensure that $KM_n^2 - 1 > KM_n^2/2$, then as $n \to \infty$, we have

$$P_{\theta_0}^{(n)} \left[\Pi_n \left(\theta \in \Theta_n : d_n(\theta, \theta_0) \geq M_n \epsilon_n | X^{(n)} \right) \phi_n \right] \leq P_{\theta_0}^{(n)} \phi_n \lesssim \exp\left\{ -KM_n^2 n\epsilon_n^2/2 \right\}.$$

Setting $\Theta_n^\dagger = \{\theta \in \Theta_n : d_n(\theta, \theta_0) \geq M_n \epsilon_n\}$, we obtain, by Fubini's theorem,

$$\begin{aligned}
& P_{\theta_0}^{(n)} \left[\int_{\{\theta \in \Theta_n : d_n(\theta,\theta_0) \geq M_n \epsilon_n\}} \frac{p_\theta^{(n)}}{p_{\theta_0}^{(n)}} \, \mathrm{d}\Pi_n(\theta) \, (1 - \phi_n) \right] \\
&= \int_{\mathcal{X}^{(n)}} \int_{\{\theta \in \Theta_n : d_n(\theta,\theta_0) \geq M_n \epsilon_n\}} \frac{p_\theta^{(n)}(X^{(n)})}{p_{\theta_0}^{(n)}(X^{(n)})} \, \mathrm{d}\Pi_n(\theta) \left(1 - \phi_n(X^{(n)})\right) p_{\theta_0}^{(n)}(X^{(n)}) \, \mathrm{d}\mu^{(n)} \\
&= \int_{\{\theta \in \Theta_n : d_n(\theta,\theta_0) \geq M_n \epsilon_n\}} P_\theta^{(n)}(1 - \phi_n) \, \mathrm{d}\Pi_n(\theta) \\
&\leq \int_{\{\theta \in \Theta_n : d_n(\theta,\theta_0) \geq M_n \epsilon_n\}} \exp\left\{ -\frac{K}{4} n d_n^2(\theta, \theta_0) \right\} \mathrm{d}\Pi_n(\theta)
\end{aligned}$$

Fix some $C > 0$. By Lemma 2, we have, on an event A_n with probability at least $1 - C^{-k}(n\epsilon_n^2)^{-k/2}$,

$$\int_\Theta \frac{p_\theta^{(n)}}{p_{\theta_0}^{(n)}} \, \mathrm{d}\Pi_n(\theta) \geq \int_{B_n(\theta_0, \epsilon_n; k)} \frac{p_\theta^{(n)}}{p_{\theta_0}^{(n)}} \, \mathrm{d}\Pi_n(\theta) \geq e^{-(1+C)n\epsilon_n^2} \Pi_n\left(B_n\left(\theta_0, \epsilon_n; k\right)\right).$$

Thus,

$$
\begin{aligned}
&P_{\theta_0}^{(n)}\left[\Pi_n\left(\theta\in\Theta_n : d_n(\theta,\theta_0)>\epsilon_n M_n|X^{(n)}\right)(1-\phi_n)\mathbf{1}_{A_n}\right]\\
=&P_{\theta_0}^{(n)}\left[\frac{\int_{\{\theta\in\Theta_n:d_n(\theta,\theta_0)\geq M_n\epsilon_n\}}\frac{p_\theta^{(n)}}{p_{\theta_0}^{(n)}}\,\mathrm{d}\Pi_n(\theta)}{\int_\Theta\frac{p_\theta^{(n)}}{p_{\theta_0}^{(n)}}\,\mathrm{d}\Pi_n(\theta)}(1-\phi_n)\mathbf{1}_{A_n}\right]\\
\leq&\frac{e^{(1+C)n\epsilon_n^2}}{\Pi_n\left(B_n\left(\theta_0,\epsilon_n;k\right)\right)}P_{\theta_0}^{(n)}\left[\int_{\{\theta\in\Theta_n:d_n(\theta,\theta_0)\geq M_n\epsilon_n\}}\frac{p_\theta^{(n)}}{p_{\theta_0}^{(n)}}\,\mathrm{d}\Pi_n(\theta)(1-\phi_n)\mathbf{1}_{A_n}\right]\\
\leq&\frac{e^{(1+C)n\epsilon_n^2}}{\Pi_n\left(B_n\left(\theta_0,\epsilon_n;k\right)\right)}P_{\theta_0}^{(n)}\left[\int_{\{\theta\in\Theta_n:d_n(\theta,\theta_0)\geq M_n\epsilon_n\}}\frac{p_\theta^{(n)}}{p_{\theta_0}^{(n)}}\,\mathrm{d}\Pi_n(\theta)(1-\phi_n)\right]\\
\leq&\frac{e^{(1+C)n\epsilon_n^2}}{\Pi_n\left(B_n\left(\theta_0,\epsilon_n;k\right)\right)}\int_{\{\theta\in\Theta_n:d_n(\theta,\theta_0)\geq M_n\epsilon_n\}}\exp\left\{-\frac{K}{4}nd_n^2(\theta,\theta_0)\right\}\mathrm{d}\Pi_n(\theta).
\end{aligned}
$$

This completes the proof.

□

3 Fractional posteriors

Consistency:

[11], [2], [Alquier and Ridgway] (variational fractional posterior, also contains an example of mixture model).

nonregular models: [5], [4], [3].

Martingale methods: [10], [12].

Fractional posterior with power t is defined as

$$\Pi^{(t)}(\theta\in B|\mathbf{X}^{(n)})=\frac{\int_B\left[p_n(\mathbf{X}^{(n)}|\theta)\right]^t\Pi_n(\mathrm{d}\theta)}{\int_\Theta\left[p_n(\mathbf{X}^{(n)}|\theta)\right]^t\Pi_n(\mathrm{d}\theta)}.$$

For two parameters θ_1 and θ_2, the α order Rényi divergence ($0<\alpha<1$) of P_{θ_1} from P_{θ_2} (suppose $P_{\theta_1}\ll P_{\theta_2}$) is defined to be

$$D_\alpha(\theta_1\|\theta_2)=-\frac{1}{1-\alpha}\log\rho_\alpha(\theta_1,\theta_2),$$

where $\rho_\alpha(\theta_1,\theta_2)=\int_{\mathcal{X}}p(X|\theta_1)^\alpha p(X|\theta_2)^{1-\alpha}\mathrm{d}\mu$ is the so-called Hellinger integral.

The Kullback-Leibler between P and Q is

$$D_1(P\|Q)=\begin{cases}\int_{\mathcal{X}}\log\frac{\mathrm{d}P}{\mathrm{d}Q}\mathrm{d}P, & \text{if } P\ll Q,\\ +\infty & \text{otherwise.}\end{cases}$$

It is known that $\lim_{\alpha\uparrow 1}D_\alpha(\theta_1\|\theta_2)=D_1(\theta_1\|\theta_2)$.

3.1 Results in [7], Section 8.6

Lemma 3. *Suppose v are nonnegative measurable function from Θ to $\mathbb{R}$ such that*

$$0 < \int_\Theta v(\theta)\Pi(\mathrm{d}\theta) < \infty.$$

Let

$$\mathcal{W} = \left\{ w : w \textit{ is a nonnegative measurable function from } \Theta \textit{ to } \mathbb{R} \textit{ s.t. } \int_\Theta w(\theta)\Pi(\mathrm{d}\theta) = 1 \right\}.$$

Then

$$\log \int_\Theta v(\theta)\Pi(\mathrm{d}\theta) = \sup_{w\in\mathcal{W}} \int_\Theta \left[-\log w(\theta) + \log v(\theta)\right] w(\theta)\Pi(\mathrm{d}\theta).$$

The equality holds when

$$w(\theta) = \frac{v(\theta)}{\int_\Theta v(\theta)\,\Pi(\mathrm{d}\theta)}.$$

Remark 1. This lemma seems well known. One can find it in [**?**], Lemma 1.1.3.

Proof. The conclusion follows from

$$D_1\left(w\Pi(\mathrm{d}\theta) \middle\| \frac{v\Pi(\mathrm{d}\theta)}{\int_\Theta v\,\Pi(\mathrm{d}\theta)} \right) \geq 0.$$

□

Theorem 2. *For any numbers $\epsilon > 0$, $\beta \in (0,1)$, $\gamma \geq 0$, for $\alpha = (\gamma+\beta)/(\gamma+1)$,*

$$\frac{1-\beta}{\gamma+1} P_{\theta_0}^{(n)} \int_\Theta D_\beta(\theta\|\theta_0)\Pi^{(\alpha)}(\mathrm{d}\theta|\mathbf{X}^{(n)}) \leq -\log \int_\Theta e^{-\alpha D_1(\theta_0\|\theta)}\Pi(\mathrm{d}\theta).$$

Proof. Apply Lemma 3 twice, first with $v(\theta) = (p_\theta/p_{\theta_0})^\delta(X)$, and second with $v(\theta) = (p_\theta/p_{\theta_0})^\beta(X)/\rho_\beta(\theta,\theta_0)$. We have

$$\int_\Theta \left(-\log w(\theta) + \delta \log \frac{p_\theta}{p_{\theta_0}}(X) \right) w(\theta)\Pi(\mathrm{d}\theta) \leq \log \int_\Theta \left(\frac{p_\theta}{p_{\theta_0}}\right)^\delta (X)\Pi(\mathrm{d}\theta),$$

$$\int_\Theta \left(-\log w(\theta) + \beta \log \frac{p_\theta}{p_{\theta_0}}(X) - \log \rho_\beta(\theta,\theta_0) \right) w(\theta)\Pi(\mathrm{d}\theta) \leq \log \int_\Theta \left(\frac{p_\theta}{p_{\theta_0}}\right)^\beta (X)\frac{1}{\rho_\beta(\theta,\theta_0)}\Pi(\mathrm{d}\theta).$$

Adding the second inequality to γ times the first inequality yields

$$\begin{aligned} &\int_\Theta \left(-(\gamma+1)\log w(\theta) + (\gamma\delta+\beta)\log \frac{p_\theta}{p_{\theta_0}}(X) - \log \rho_\beta(\theta,\theta_0) \right) w(\theta)\Pi(\mathrm{d}\theta) \\ \leq &\log \int_\Theta \left(\frac{p_\theta}{p_{\theta_0}}\right)^\beta (X)\frac{1}{\rho_\beta(\theta,\theta_0)}\Pi(\mathrm{d}\theta) + \gamma \log \int_\Theta \left(\frac{p_\theta}{p_{\theta_0}}\right)^\delta (X)\Pi(\mathrm{d}\theta). \end{aligned}$$

Note that the right hand side does not depend on $w(\theta)$, and

$$\begin{aligned}
&P_{\theta_0} \log \int_\Theta \left(\frac{p_\theta}{p_{\theta_0}}\right)^\delta (X)\Pi(\mathrm{d}\theta) \\
\leq &\log P_{\theta_0} \int_\Theta \left(\frac{p_\theta}{p_{\theta_0}}\right)^\delta (X)\Pi(\mathrm{d}\theta) = \log \int_\Theta P_{\theta_0} \left(\frac{p_\theta}{p_{\theta_0}}\right)^\delta (X)\Pi(\mathrm{d}\theta) \\
= &\log \int_\Theta P_{\theta_0} \left(\frac{p_\theta}{p_{\theta_0}}\right)^\delta (X)\Pi(\mathrm{d}\theta) = \log \int_\Theta \rho_\delta(\theta,\theta_0)\Pi(\mathrm{d}\theta) \\
\leq &\log \int_\Theta \Pi(\mathrm{d}\theta) \leq 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&P_{\theta_0} \log \int_\Theta \left(\frac{p_\theta}{p_{\theta_0}}\right)^\beta (X)\frac{1}{\rho_\beta(\theta,\theta_0)}\Pi(\mathrm{d}\theta) \\
\leq &\log P_{\theta_0} \int_\Theta \left(\frac{p_\theta}{p_{\theta_0}}\right)^\beta (X)\frac{1}{\rho_\beta(\theta,\theta_0)}\Pi(\mathrm{d}\theta) = \log \int_\Theta P_{\theta_0} \left(\frac{p_\theta}{p_{\theta_0}}\right)^\beta (X)\frac{1}{\rho_\beta(\theta,\theta_0)}\Pi(\mathrm{d}\theta) \\
= &\log \int_\Theta P_{\theta_0} \left(\frac{p_\theta}{p_{\theta_0}}\right)^\beta (X)\frac{1}{\rho_\beta(\theta,\theta_0)}\Pi(\mathrm{d}\theta) = \log \int_\Theta \Pi(\mathrm{d}\theta) \leq 0.
\end{aligned}$$

It follows that for every choice of $w(\theta)$ ($w(\theta)$ may depende on X), we have

$$P_{\theta_0} \int_\Theta \left(-(\gamma+1)\log w(\theta) + (\gamma\delta+\beta)\log\frac{p_\theta}{p_{\theta_0}}(X) - \log\rho_\beta(\theta,\theta_0)\right) w(\theta)\Pi(\mathrm{d}\theta) \leq 0.$$

Note that

$$\int_\Theta -\log\rho_\beta(\theta,\theta_0)w(\theta)\Pi(\mathrm{d}\theta) \geq 0.$$

Then for any $w(\theta)$,

$$P_{\theta_0} \int_\Theta -\log\rho_\beta(\theta,\theta_0)w(\theta)\Pi(\mathrm{d}\theta) \leq P_{\theta_0} \int_\Theta \left((\gamma+1)\log w(\theta) - (\gamma\delta+\beta)\log\frac{p_\theta}{p_{\theta_0}}(X)\right) w(\theta)\Pi(\mathrm{d}\theta).$$

Divided both sides by $\gamma+1$, we have

$$\frac{1}{\gamma+1}P_{\theta_0} \int_\Theta -\log\rho_\beta(\theta,\theta_0)w(\theta)\Pi(\mathrm{d}\theta) \leq P_{\theta_0} \int_\Theta \left(\log w(\theta) - \log\left(\frac{p_\theta}{p_{\theta_0}}(X)\right)^\alpha\right) w(\theta)\Pi(\mathrm{d}\theta),$$

where $\alpha = \frac{\gamma\delta+\beta}{\gamma+1}$. From Lemma 3, the right hand side is minimized (for $w \in \mathcal{W}$ and w can depends on X) when

$$w(\theta) = \frac{p_\theta^\alpha(X)}{\int_\Theta p_\theta^\alpha(X)\Pi(\mathrm{d}\theta)}.$$

And the minimum value of the right hand side is $-P_{\theta_0}\log\int_\Theta \left(\frac{p_\theta}{p_{\theta_0}}(X)\right)^\alpha \Pi(\mathrm{d}\theta)$. In this case,

$w(\theta)\Pi(\mathrm{d}\theta) = \Pi^{\alpha}(\mathrm{d}\theta|\mathbf{X}^{(n)})$. Thus,

$$\begin{aligned}
&\frac{1}{\gamma+1}P_{\theta_0}\int_{\Theta} -\log \rho_{\beta}(\theta,\theta_0)\Pi^{\alpha}(\mathrm{d}\theta|\mathbf{X}^{(n)}) \\
\leq& - P_{\theta_0}\log\int_{\Theta}\left(\frac{p_\theta}{p_{\theta_0}}(X)\right)^{\alpha}\Pi(\mathrm{d}\theta) \\
=& P_{\theta_0}\inf_{w\in\mathcal{W}}\int_{\Theta}\left(\log w(\theta) - \log\left(\frac{p_\theta}{p_{\theta_0}}(X)\right)^{\alpha}\right)w(\theta)\Pi(\mathrm{d}\theta) \\
\leq& \inf_{w\in\mathcal{W}} P_{\theta_0}\int_{\Theta}\left(\log w(\theta) - \log\left(\frac{p_\theta}{p_{\theta_0}}(X)\right)^{\alpha}\right)w(\theta)\Pi(\mathrm{d}\theta) \\
=& \inf_{w\in\mathcal{W}}\int_{\Theta}\left(\log w(\theta) - \alpha P_{\theta_0}\log\frac{p_\theta}{p_{\theta_0}}(X)\right)w(\theta)\Pi(\mathrm{d}\theta) \\
=& \inf_{w\in\mathcal{W}}\int_{\Theta}\left(\log w(\theta) - \log e^{-\alpha D_1(\theta_0\|\theta)}\right)w(\theta)\Pi(\mathrm{d}\theta) \\
=& -\log\int_{\Theta} e^{-\alpha D_1(\theta_0\|\theta)}\Pi(\mathrm{d}\theta).
\end{aligned}$$

This completes the proof.

□

4 Hausdorff entropy

[?], [?], [?], [?],

Their results can be generalized to non iid case.

Lemma 4. *For $x > 0$, $y \geq 1$, $\log(1+x) \leq yx^{1/y}$.*

Proposition 1. *Consistency for $t = 1, 2$.*

Proof. Let

$$\left\{\theta : \|\theta-\theta_0\| \geq M_n/\sqrt{n}\right\} = \cup_{i=1}^{\infty} A_i.$$

Let $\theta_i \in A_i$. Let $\pi_i(\theta) = \mathbf{1}_{A_i}(\theta)\pi(\theta)/\Pi(A_i)$ be the restriction of $\pi(\cdot)$ on A_i. Then

$$\begin{aligned}
&\int_{\left\{\theta:\|\theta-\theta_0\|\geq M_n/\sqrt{n}\right\}} e^{-tR_n(\theta_0\|\theta)}\pi(\theta)\,\mathrm{d}\theta \\
\leq& \sum_{i=1}^{\infty}\Pi(A_i)\int_{A_i} e^{-tR_n(\theta_0\|\theta)}\pi_i(\theta)\,\mathrm{d}\theta \\
=& \sum_{i=1}^{\infty}\Pi(A_i)e^{-tR_n(\theta_0\|\theta_i)}\left(1+\int_{A_i}(e^{-tR_n(\theta_i\|\theta)}-1)\pi_i(\theta)\,\mathrm{d}\theta\right) \\
\leq& \sum_{i=1}^{\infty}\Pi(A_i)e^{-tR_n(\theta_0\|\theta_i)} + \sum_{i=1}^{\infty}\Pi(A_i)e^{-tR_n(\theta_0\|\theta_i)}\int_{A_i}|e^{-tR_n(\theta_i\|\theta)}-1|\pi_i(\theta)\,\mathrm{d}\theta.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\log\left(1+\int_{\{\theta:\|\theta-\theta_0\|\geq M_n/\sqrt{n}\}} e^{-tR_n(\theta_0\|\theta)}\pi(\theta)\,\mathrm{d}\theta\right)\\
\leq&\sum_{i=1}^{\infty}\log\left(1+\Pi(A_i)e^{-tR_n(\theta_0\|\theta_i)}\right)+\sum_{i=1}^{\infty}\log\left(1+\Pi(A_i)e^{-tR_n(\theta_0\|\theta_i)}\int_{A_i}|e^{-tR_n(\theta_i\|\theta)}-1|\pi_i(\theta)\,\mathrm{d}\theta\right)\\
\leq&2t\sum_{i=1}^{\infty}\Pi(A_i)^{\frac{1}{2t}}e^{-\frac{1}{2}R_n(\theta_0\|\theta_i)}+\frac{4}{3}t\sum_{i=1}^{\infty}\Pi(A_i)^{\frac{3}{4t}}e^{-\frac{3}{4}R_n(\theta_0\|\theta_i)}\left(\int_{A_i}|e^{-tR_n(\theta_i\|\theta)}-1|\pi_i(\theta)\,\mathrm{d}\theta\right)^{\frac{3}{4t}}.
\end{aligned}
$$

We have

$$
P_{\theta_0}^n e^{-\frac{1}{2}R_n(\theta_0\|\theta_i)}=e^{-\frac{n}{2}D_{1/2}(\theta_0\|\theta_i)}.
$$

$$
\begin{aligned}
&P_{\theta_0}^n e^{-\frac{3}{4}R_n(\theta_0\|\theta_i)}\left(\int_{A_i}|e^{-tR_n(\theta_i\|\theta)}-1|\pi_i(\theta)\,\mathrm{d}\theta\right)^{\frac{3}{4t}}\\
=&P_{\theta_i}^n e^{-\frac{1}{4}R_n(\theta_i\|\theta_0)}\left(\int_{A_i}|e^{-tR_n(\theta_i\|\theta)}-1|\pi_i(\theta)\,\mathrm{d}\theta\right)^{\frac{3}{4t}}\\
\leq&\sqrt{P_{\theta_i}^n e^{-\frac{1}{2}R_n(\theta_i\|\theta_0)}}\sqrt{P_{\theta_i}^n\left(\int_{A_i}|e^{-tR_n(\theta_i\|\theta)}-1|\pi_i(\theta)\,\mathrm{d}\theta\right)^{\frac{3}{2t}}}
\end{aligned}
$$

□

5 ρ-estimator

Lucien Birgé and Yannick Baraud's work.

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