haha

Tuesday 19th September, 2017

1 Introduction

2 bounds for the radius of confidence balls

These results are from Cai and Low 2004. I adapted it from “All of Nonparametric Statistics”.

Let $\mathbf{Z}^n = (Z_1, \ldots, Z_n)$ where $Z_i = \theta_i + \sigma_n \epsilon_n$, $i = 1, \ldots, n$, $\epsilon_1, \ldots, \epsilon_n$ are independent $N(0,1)$ random variables, $\theta^n = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ is a vector of unknown parameters and σ_n is assumed known.

Theorem 1 (Cai and Low 2004)**.** *Fix $0 < \alpha < 1/2$. Let $\mathcal{B}_n = \{\theta : \|\hat{\theta} - \theta\| \leq s_n\}$ be such that*

$$\inf_{\theta \in \mathbb{R}^n} \mathrm{P}_\theta(\theta \in \mathcal{B}_n) \geq 1 - \alpha.$$

Then, for every $0 < \epsilon < 1/2 - \alpha$,

$$\inf_{\theta \in \mathbb{R}^n} \mathrm{E}_\theta(s_n) \geq \frac{1}{2} \sigma_n (1 - 2\alpha - \epsilon) n^{1/4} (\log(1 + \epsilon^2))^{1/4}.$$

Proof. Let

$$a = \frac{\sigma_n}{n^{1/4}} (\log(1 + \epsilon^2))^{1/4}$$

and define

$$\Omega = \{\theta = (\theta_1, \ldots, \theta_n) : |\theta_i| = a,\ i = 1, \ldots, n\}.$$

Note that Ω contains 2^n elements. Let f_θ denote the density of a multivariate normal with mean θ and covariance $\sigma_n^2 I$ where I is the identity matrix. Define the mixture

$$q(y) = \frac{1}{2^n} \sum_{\theta \in \Omega} f_\theta(y).$$

Let f_0 denote the density of a multivariate normal with mean $(0, \ldots, 0)$ and covariance $\sigma_n^2 I$.

It can be proved that $\int |f_0(x) - q(x)| dx \leq \epsilon$.

Define two events, $A = \{(0, \ldots, 0) \in \mathcal{B}_n\}$ and $B = \{\Omega \cap \mathcal{B}_n \neq \emptyset\}$. Every $\theta \in \Omega$ has norm

$$\|\theta\| = \sqrt{na^2} \stackrel{def}{=} c_n.$$

Hence, $A \cap B \subset \{2s_n \geq c_n\}$. For all $\theta \in \Omega$, we have

$$\mathrm{P}_\theta(\Omega \cap \mathcal{B}_n \neq \emptyset) \geq \mathrm{P}_\theta(\theta \in \mathcal{B}_n) \geq 1 - \alpha.$$

Hence, $Q(B) \geq 1 - \alpha$ and thus $\mathrm{P}_0(B) \geq 1 - \alpha - \epsilon$. Then

$$\mathrm{P}_0(2s_n \geq c_n) \geq \mathrm{P}_0(A \cap B) \geq \mathrm{P}_0(A) + \mathrm{P}_0(B) - 1 \geq 1 - 2\alpha - \epsilon$$

□

Theorem 2. *Fix* $0 < \alpha < 1/2$. *Let* $\mathcal{B}_n = \{\theta : \|\hat{\theta} - \theta\| \leq s_n\}$ *be such that*

$$\inf_{\theta \in \mathbb{R}^n} \mathrm{P}_\theta(\theta \in \mathcal{B}_n) \geq 1 - \alpha.$$

Then, for every $0 < \epsilon < 1/2 - \alpha$,

$$\sup_{\theta \in \mathbb{R}^n} \mathrm{E}_\theta(s_n) \geq \epsilon \sigma_n z_{\alpha+2\epsilon} \sqrt{n} \sqrt{\frac{\epsilon}{1 - \alpha - \epsilon}}.$$

Acknowledgements

References