

Notes on Polish space

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1 Introduction

This document contains notes about Polish space which play an important role in probability and statistics. The materials are mainly from Cohn (2013), Chapter 8 and Dudley (2002), Chapter 13.

2 Polish space

Exercise 1 (Cohn (2013), Exercise 8.1.3). *Let (X, \mathcal{A}) be a measurable space, let Y be a separable metrizable space, and let $f, g : X \rightarrow Y$ be measurable with respect to \mathcal{A} and $\mathcal{B}(Y)$. Then $\{x \in X : f(x) = g(x)\} \in \mathcal{A}$.*

Proof. For any $A, B \in \mathcal{B}(Y)$,

$$\{x : (f(x), g(x)) \in A \times B\} = f^{-1}(A) \cap f^{-1}(B) \in \mathcal{A}.$$

Hence the map $F : x \mapsto (f(x), g(x))$ is measurable with respect to \mathcal{A} and $\mathcal{B}(Y) \times \mathcal{B}(Y)$. Since Y is a separable metrizable space, $\mathcal{B}(Y) \times \mathcal{B}(Y) = \mathcal{B}(Y \times Y)$. Thus, the map F is measurable with respect to \mathcal{A} and $\mathcal{B}(Y \times Y)$. Let $\Delta = \{(y_1, y_2) \in Y \times Y : y_1 = y_2\}$. Then Δ is a closed subset of $Y \times Y$ and $\{x \in X : f(x) = g(x)\} = F^{-1}(\Delta)$. It follows that $\{x \in X : f(x) = g(x)\} \in \mathcal{A}$. \square

Exercise 2 (Cohn (2013), Exercise 8.2.1). *Let A be an uncountable analytic subset of the Polish space X . Then,*

(a) *A has a subset that is homeomorphic to $\{0, 1\}^{\mathbb{N}}$.*

(b) *A has the cardinality of the continuum.*

Proof. From Cohn (2013), Corollary 8.2.8., there is a continuous function f from \mathcal{N} onto A . By the axiom of choice, there is a set $S \subset \mathcal{N}$ such that the restriction of f on S is a bijection of S onto A . As a subspace of \mathcal{N} , S is an uncountable separable metrizable space. Let $S_0 \subset S$ be the set of all condensation points of the space S . From Cohn (2013), Lemma 8.2.12, S_0 is uncountable

and each point of S_0 is a condensation point of S_0 . Let $d_{\mathcal{N}}(\cdot, \cdot)$ be a metric on \mathcal{N} which metrize the topology of \mathcal{N} . Let $d_X(\cdot, \cdot)$ be a metric on X which metrize the topology of X .

Now we construct a homeomorphism between a subset of X and $\{0, 1\}^{\mathbb{N}}$. First, let x_0 and x_1 be two distinct points in S_0 . Since the restriction of f on S_0 is injective, $f(x_0) \neq f(x_1)$. Hence there exists $0 < \epsilon_1 < 1$ such that $\overline{B(x_0, \epsilon_1)} \cap \overline{B(x_1, \epsilon_1)} = \emptyset$ and $f(\overline{B(x_0, \epsilon_1)}) \cap f(\overline{B(x_1, \epsilon_1)}) = \emptyset$. For $i = 0, 1$, let $C(i) = B(x_i, \epsilon_1)$. Note that for $i = 0, 1$, $C(i) \cap S_0$ is uncountable and each point of $C(i) \cap S_0$ is a condensation point of $C(i) \cap S_0$. Then there exist $x_{i0}, x_{i1} \in C(i) \cap S_0$ ($i = 0, 1$) and $0 < \epsilon_2 < 1/2$ such that for $j = 0, 1$, $B(x_{ij}, \epsilon_2) \subset B(x_i, \epsilon_1)$, $\overline{B(x_{i0}, \epsilon_2)} \cap \overline{B(x_{i1}, \epsilon_2)} = \emptyset$ and $f(\overline{B(x_{i0}, \epsilon_2)}) \cap f(\overline{B(x_{i1}, \epsilon_2)}) = \emptyset$. For $i, j \in \{0, 1\}$, let $C(i, j) = B(x_{ij}, \epsilon_2)$.

Inductively construct sets $C(n_1, n_2, \dots, n_k)$, $n_i \in \{0, 1\}$, $k \in \mathbb{N}$. Then for $\{n_k\}_{k=1}^{\infty} \in \mathcal{N}$, consider the set $\cap_{k=1}^{\infty} \overline{C(n_1, \dots, n_k)}$. By the completeness of \mathcal{N} , $\cap_{k=1}^{\infty} \overline{C(n_1, \dots, n_k)} \neq \emptyset$. Also, the diameter of $\overline{C(n_1, \dots, n_k)}$ tends to 0. Then there exists a unique point in $\cap_{k=1}^{\infty} \overline{C(n_1, \dots, n_k)}$. Let g be the function from \mathcal{N} to X which maps $\{n_k\}_{k=1}^{\infty}$ to the unique point of $\cap_{k=1}^{\infty} \overline{C(n_1, \dots, n_k)}$.

By the construction of $C(n_1, \dots, n_k)$, g is continuous and injective. Then $f \circ g$ is continuous. To see that $f \circ g$ is injective, let $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ be two distinct points of $\{0, 1\}^{\mathcal{N}}$. Let k_0 be the first k such that $n_k \neq m_k$. By the construction of $C(\cdot, \dots, \cdot)$, $f(\overline{C(n_1, \dots, n_{k_0})}) \cap f(\overline{C(m_1, \dots, m_{k_0})}) = \emptyset$. Since $g(\{n_k\}_{k=1}^{\infty}) \in \overline{C(n_1, \dots, n_{k_0})}$, $g(\{m_k\}_{k=1}^{\infty}) \in \overline{C(m_1, \dots, m_{k_0})}$. Then $f \circ g(\{n_k\}_{k=1}^{\infty}) \neq f \circ g(\{m_k\}_{k=1}^{\infty})$.

Since $\{0, 1\}^{\mathcal{N}}$ is compact, the inverse of $f \circ g$ is also continuous. This completes the proof of (a).

(a) implies that $\text{card}(A) \geq \mathfrak{c}$. On the other hand, Cohn (2013), Corollary 8.2.8. implies that $\text{card}(A) \leq \mathfrak{c}$. Thus, $\text{card}(A) = \mathfrak{c}$. □

Exercise 3 (Cohn (2013), Exercise 8.2.2). *Let X be an uncountable Polish space. Then the collection of analytic subsets of X and the collection of Borel subsets of X have the cardinality of the continuum.*

Proof. Exercise 2 implies that the cardinality of X is \mathfrak{c} . Since each single point of X is a Borel set, the cardinality of the collection of Borel subsets of X is at least \mathfrak{c} . We only need to prove that the cardinality of the collection of analytic subsets of X is at most \mathfrak{c} .

Cohn (2013), Proposition 8.2.9 implies that it suffices to upper bound the cardinality of the collection of closed subsets of the Polish space $\mathcal{N} \times X$. Let $\{U_i\}_{i=1}^{\infty}$ be a countable base of the topology of $\mathcal{N} \times X$. Then every closed subset of $\mathcal{N} \times X$ is the intersection of certain U_i^c , that is, $\cap_{i \in S} U_i^c$ where S is a subset of \mathbb{N} . Hence there is an injective map from the collection of closed subsets of $\mathcal{N} \times X$ to $2^{\mathbb{N}}$. Thus, the cardinality of the collection of closed subsets of $\mathcal{N} \times X$ is at most \mathfrak{c} . □

Exercise 4 (Cohn (2013), Exercise 8.2.3).

(a) Let X be a nonempty zero-dimensional Polish space such that each nonempty open subset of X is not compact. Then X is homeomorphic to \mathcal{N} .

(b) the Space \mathcal{I} of irrational numbers in the interval $(0, 1)$ is homeomorphic to \mathcal{N} .

Proof. Let $d(\cdot, \cdot)$ be a complete metric for X . We begin by constructing a family $\{C(n_1, \dots, n_k)\}$ of subsets of X , indexed by the set of all finite sequences $\{(n_1, \dots, n_k)\}$ of positive integers, in such a way that

1. $C(n_1, \dots, n_k)$ is nonempty, open, closed and noncompact,
2. the diameter of $C(n_1, \dots, n_k)$ is at most $1/k$,
3. $\{C(n_1, \dots, n_{k-1}, n_k)\}_{n_k=1}^\infty$ are disjoint and $C(n_1, \dots, n_{k-1}) = \bigcup_{n_k=1}^\infty C(n_1, \dots, n_k)$,
4. $X = \bigcup_{n_1=1}^\infty C(n_1)$.

We do this by induction on k .

First, suppose that $k = 1$. Since X is assumed to be not compact, Cohn (2013), Lemma 8.2.11 gives a sequence $\{C(n_1)\}_{n_1=1}^\infty$ where terms are nonempty, open, closed and with diameter at most

1. By assumption, each $C(n_1)$ is not compact.

Now suppose that $k > 1$ and that $C(n_1, \dots, n_{k-1})$ has already been chosen. It is easy to use a modification of the construction of the $C(n_1)$'s, now applied to $C(n_1, \dots, n_{k-1})$ rather than to X , to produce sets $C(n_1, \dots, n_k)$, $n_k = 1, 2, \dots$ that satisfy conditions 1 to 4. With this, the induction step in our construction is complete.

We turn to the construction of a homeomorphism between \mathcal{N} and X . Let $\mathbf{n} = \{n_k\}$ be an element of \mathcal{N} . Then the sets $C(n_1)$, $C(n_1, n_2)$, \dots are decreasing nonempty closed sets whose diameters approach to 0. Since X is complete, there is a unique element in $\bigcap_{k=1}^\infty C(n_1, \dots, n_k)$. We can define a function $f : \mathcal{N} \rightarrow X$ by letting $f(\mathbf{n})$ be the unique member of $\bigcap_{k=1}^\infty C(n_1, \dots, n_k)$. Note that if \mathbf{m} and \mathbf{n} are elements of \mathcal{N} such that $m_i = n_i$ holds for $i = 1, \dots, k$, then $d(\mathbf{m}, \mathbf{n}) \leq 1/k$. It follows that f is continuous. Also, it is obvious that f is bijective. It remains to prove that the inverse of f is continuous. Suppose $f(\mathbf{n}^{(l)}) \rightarrow f(\mathbf{n})$. Fix $k > 0$. Then if l is large enough, $f(\mathbf{n}^{(l)}) \in C(n_1, \dots, n_k)$. By the construction of f , this implies that $n_i^{(l)} = n_i$ for $i = 1, \dots, k$. Thus, $\mathbf{n}^{(l)} \rightarrow \mathbf{n}$ as $l \rightarrow \infty$. This completes the proof of (a).

We turn to the proof of (b). The space \mathcal{I} is a G_δ set of $[0, 1]$, and hence is a Polish space. The family of intervals (a_i, b_i) where a_i and b_i is rational is a base that consists of sets that are both open and closed. It follows that \mathcal{I} is zero-dimensional. Each interval (a, b) is the union of $\{(a_i, b_i)\}_{i=1}^\infty$ where a_i, b_i are rational and $a_i \downarrow a$ and $b_i \uparrow b$. Hence each interval of \mathcal{I} is not compact. Then the conclusion follows from (a).

□

Exercise 5 (Cohn (2013), Exercise 8.2.3). *Each nonempty Polish space is the image of \mathcal{N} under a continuous open map.*

Proof. We mimic the proof of Cohn (2013), Proposition 8.2.7.

Let X be a nonempty Polish space, and let d be a complete metric for X . We begin by constructing a family $\{C(n_1, \dots, n_k)\}$ of subsets of X , indexed by the set of all finite sequences $\{n_1, \dots, n_k\}$ of positive integers, in such a way that

1. $C(n_1, \dots, n_k)$ is nonempty and open,
2. the diameter of $C(n_1, \dots, n_k)$ is at most $1/k$,
3. $\overline{C(n_1, \dots, n_{k-1}, n_k)} \subset C(n_1, \dots, n_{k-1})$ and $C(n_1, \dots, n_{k-1}) = \bigcup_{n_k=1}^{\infty} C(n_1, \dots, n_k)$,
4. $X = \bigcup_{n_1=1}^{\infty} C(n_1)$.

We do this by induction on k .

First, suppose that $k = 1$, and let $\{x_i\}_{i=1}^{\infty}$ be a sequence whose terms form a dense subset of X . The sequence $\{x_i\}_{i=1}^{\infty}$ may have duplicated elements. Let $\{C(n_1)\}_{n_1=1}^{\infty}$ be the collection of open balls which center at certain x_i and with rational radius not larger than $1/2$. Certainly each $C(n_1)$ is open and nonempty and has diameter at most 1. Furthermore, $X = \bigcup_{n_1=1}^{\infty} C(n_1)$.

Now suppose that $k > 1$ and that $C(n_1, \dots, n_{k-1})$ has already been chosen. Let $\{C(n_1, \dots, n_{k-1}, n_k)\}_{n_k=1}^{\infty}$ be the collection of open balls which center at certain x_i and with rational radius not larger than $1/(2k)$ and whose closure is contained in $C(n_1, \dots, n_{k-1})$. Certainly each $C(n_1, \dots, n_k)$ is open and nonempty and has diameter at most $1/k$. Now we prove that $C(n_1, \dots, n_{k-1}) = \bigcup_{n_k=1}^{\infty} C(n_1, \dots, n_k)$. Suppose $x \in C(n_1, \dots, n_{k-1})$. Since $C(n_1, \dots, n_{k-1})$ is open, there is a open ball $B(x, r) \subset C(n_1, \dots, n_{k-1})$ where r is rational and $r < 1/k$. Since $\{x_i\}_{i=1}^{\infty}$ is dense in X , there is an x_i such that $d(x, x_i) < r/3$. Then the ball $B(x_i, r/2)$ contains x . Also, the Closure of $B(x_i, r/2)$ has radius not larger than $1/(2k)$ and is contained in $C(n_1, \dots, n_{k-1})$. Thus, $B(x_i, r/2) = C(n_1, \dots, n_k)$ for some n_k . With this, the induction step in our construction is complete.

We turn to the construction of a continuous function that maps \mathcal{N} onto X . Let $\mathbf{n} = \{n_k\}$ be an element of \mathcal{N} . It follows from 3 that $\bigcap_{k=1}^{\infty} C(n_1, \dots, n_k) = \bigcap_{k=1}^{\infty} \overline{C(n_1, \dots, n_k)}$ which is intersection of a decreasing sequence of nonempty closed subsets of X whose diameters approach 0. Thus there is a unique element in the intersection of these sets, and we can define a function $f : \mathcal{N} \rightarrow X$ by letting $f(\mathbf{n})$ be the unique member of $\bigcap_k C(n_1, \dots, n_k)$. Note that if \mathbf{m} and \mathbf{n} are elements on \mathcal{N} such that $m_i = n_i$ holds for $i = 1, \dots, k$, then $d(f(\mathbf{m}), f(\mathbf{n})) \leq 1/k$. It follows that f is continuous. Also, 3 and 4 above imply that for each x in X there is an element $\mathbf{n} = \{n_k\}$ of \mathcal{N} such that $x \in \bigcap_k C(n_1, \dots, n_k)$ and hence such that $x = f(\mathbf{n})$. Thus f is surjective.

It remains to prove that f is an open map. Note that the sets of the form $\{n_1\} \times \dots \times \{n_k\} \times \mathbb{N} \times \dots$ is a base for the topology of \mathcal{N} . By the construction of f , for any n_1, \dots, n_k , $f(\{n_1\} \times \dots \times \{n_k\} \times \mathbb{N} \times \dots) = C(n_1, \dots, n_k)$ is an open set. This completes the proof.

□

Exercise 6 (Cohn (2013), Exercise 8.2.5). *Each Borel subset of a Polish space is the image under a continuous injective map of some Polish space.*

Proof. Let X be a Polish space. Let \mathcal{A} be the collection of Borel subsets of X which are the image under continuous injective maps of some Polish spaces. Then all open and closed subsets of X belong to \mathcal{A} since they are themselves Polish spaces.

Assume $A_1, \dots, A_n, \dots \in \mathcal{A}$ and A_1, \dots, A_n, \dots are disjoint. For each A_i , there is a Polish space X_i and a continuous injective map $f_i(\cdot)$ such that $f_i(X_i) = A_i$. Define $f : \cup_{i=1}^{\infty} X_i \mapsto \cup_{i=1}^{\infty} A_i$ by $f(x) = f_i(x)$ if $x \in X_i$. Here $\cup_{i=1}^{\infty} X_i$ is the disjoint union of X_i . Then $\cup_{i=1}^{\infty} X_i$ is Polish and f is injective and continuous. Then $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Assume $A_1, \dots, A_n, \dots \in \mathcal{A}$. For each A_i , there is a Polish space X_i and a continuous injective map $f_i(\cdot)$ such that $f_i(X_i) = A_i$. Define $f : \prod_{i=1}^{\infty} X_i \mapsto \prod_{i=1}^{\infty} A_i$ by $f(\{x_i\}_{i=1}^{\infty}) = \{f_i(x_i)\}_{i=1}^{\infty}$. Then f is injective and continuous onto $\prod_{i=1}^{\infty} A_i \subset \prod_{i=1}^{\infty} X_i$. Let $D = \{(x, x, \dots) : x \in X\}$. Define $g : D \mapsto X$ by $g(x, x, \dots) = x$. Then g is a homeomorphism between D and X . Consider $g \circ f$ defined on $f^{-1}(D)$. Then $g \circ f$ is injective and continuous from $f^{-1}(D)$ onto $\cap_{i=1}^{\infty} A_i$. Since $f^{-1}(D)$ is a closed subset of $\prod_{i=1}^{\infty} X_i$, it is Polish. Thus, $\cap_{i=1}^{\infty} A_i \in \mathcal{A}$.

From Cohn (2013), Lemma 8.2.4, \mathcal{A} contains all Borel subset of X . This completes the proof. □

Exercise 7 (Cohn (2013), Exercise 8.2.6). *If X is an uncountable Polish space, then there is an analytic subset of X that is not a Borel set.*

Proof. Let X be an uncountable Polish space. From Cohn (2013), Proposition 8.2.13, there is a continuous injective map $f : \mathcal{N} \rightarrow X$ such that $X - f(\mathcal{N})$ is countable. From Cohn (2013), Corollary 8.2.17, there is an analytic set $A \in \mathcal{N}$ that is not a Borel set. Then $f(A)$ is not a Borel set of X , or else $A = f^{-1}(f(A))$ would be a Borel set, a contradiction. On the other hand, $f(A)$ is analytic. This completes the proof. □

Exercise 8 (Cohn (2013), Exercise 8.3.1). *Let X and Y be Polish spaces, and let $f : X \rightarrow Y$ be a function whose graph is an analytic subset of $X \times Y$. Then f is Borel measurable.*

Remark 1. It follows from this conclusion and Cohn (2013), Proposition 8.1.8 that f is Borel measurable iff the graph of f is a Borel subset of $X \times Y$. Then the graph of f can not be an analytic set which is not a Borel set.

Proof. Let $G = \{(x, f(x)) : x \in X\}$ denote the graph of f . For any Borel subset B of Y , the sets $G \cap (X \times B)$ and $G \cap (X \times B^c)$ are analytic. Then the projection of these two sets on X , i.e. $f^{-1}(B)$ and $f^{-1}(B^c)$, are also analytic. From separation theorem, i.e. Cohn (2013), Theorem 8.3.1, B and B^c are Borel sets. Hence f is Borel measurable. □

Exercise 9 (Cohn (2013), Exercise 8.3.2). *Let X and Y be uncountable Polish spaces. Then the cardinality of the collection of Borel measurable functions from X to Y is that of the continuum.*

Proof. The cardinalities of X and Y are both \mathfrak{c} . For each $y \in Y$, the constant function $f(x) \equiv y$ is Borel measurable. Hence the cardinality of the collection of Borel measurable functions from X to Y is at least \mathfrak{c} .

On the other hand, since the graph of Borel measurable function f is a Borel subset of $X \times Y$, and the collection of Borel subsets of an uncountable Polish space has cardinality \mathfrak{c} , the cardinality of the collection of Borel measurable functions from X to Y is at most \mathfrak{c} . This completes the proof. \square

Exercise 10 (Cohn (2013), Exercise 8.3.2). *There is a Lebesgue measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that no real-valued Borel measurable function f_1 satisfies $f(x) \leq f_1(x)$ at each x in \mathbb{R} .*

Proof. Let K be the Cantor set. K is an uncountable Polish space. According to the preceding exercise, there is a bijection $x \mapsto g_x$ of K onto the set of real-valued Borel functions on K . Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} g_x(x) + 1 & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(x) = 0$ a.e., f is Lebesgue measurable. Suppose there is a real-valued Borel measurable function f_1 satisfying $f(x) \leq f_1(x)$ at each $x \in \mathbb{R}$. Then the restriction of f_1 on K is still Borel measurable. Hence there is an $x_1 \in K$ such that $f_1(x) = g_{x_1}(x)$ at each $x \in K$. Then $g_{x_1}(x) \geq g_x(x) + 1$ at each $x \in K$. But this is impossible when $x = x_1$. This completes the proof. \square

Exercise 11 (Cohn (2013), Exercise 8.3.3). *Let X be a Polish space, let μ be a Borel measure on X such that $\mu(X) = 1$, and let λ be Lebesgue measure on the Borel subsets of $[0, 1]$. Then there is a Borel measurable function $f : [0, 1] \rightarrow X$ such that $\mu = \lambda f^{-1}$.*

Proof. If X is countably infinite, let $\{x_i\}_{i=1}^\infty$ be an enumeration of the points of X . Then $\sum_{i=1}^\infty \mu(\{x_i\}) = \mu(\cup_{i=1}^\infty \{x_i\}) = 1$. We can construct f by letting $f(t) = x_i$ if $t \in [\sum_{j=1}^{i-1} \mu(\{x_j\}), \sum_{j=1}^i \mu(\{x_j\})]$, and $f(1) = x_1$. Then $\lambda f^{-1}(\{x_i\}) = \lambda([\sum_{j=1}^{i-1} \mu(\{x_j\}), \sum_{j=1}^i \mu(\{x_j\})]) = \mu(\{x_i\})$. Hence $\mu = \lambda f^{-1}$. If X is finite, the construction of f is similar.

Now suppose X is uncountable, from Cohn (2013), Theorem 8.3.6, there is a bijection g from \mathbb{R} onto X which is Borel isomorphism. Then the measure μg is a probability measure on the Borel sets of \mathbb{R} . From Cohn (2013), Proposition 10.1.15, there is a Borel measurable function h from $[0, 1]$ to \mathbb{R} such that $\mu g = \lambda h^{-1}$. Thus, for any $A \in \mathcal{B}(X)$,

$$\mu(A) = \mu(gg^{-1}A) = \mu g(g^{-1}A) = \lambda h^{-1}(g^{-1}A) = \lambda(h^{-1}g^{-1}A) = \lambda((g \circ h)^{-1}A) = \lambda(g \circ h)^{-1}(A).$$

The conclusion follows by letting $f = g \circ h$. \square

Exercise 12 (Cohn (2013), Exercise 8.4.1). *Let (X, \mathcal{A}) be a measurable space.*

- (a) *A function $f : X \rightarrow [-\infty, +\infty]$ is \mathcal{A}_* -measurable if and only if for each finite measure μ on (X, \mathcal{A}) there are \mathcal{A} -measurable functions $f_0, f_1 : X \rightarrow [-\infty, +\infty]$ that satisfy $f_0 \leq f \leq f_1$ everywhere on X and $f_0 = f_1$ μ -almost surely.*
- (b) *If $f : X \rightarrow [-\infty, +\infty]$ is \mathcal{A}_* -measurable and if the functions f_0 and f_1 in part (a) can be chosen independently of μ , then f is \mathcal{A} -measurable.*

Proof. It is understood that \mathcal{A}_* -measurable means $\mathcal{A}_*/\mathcal{B}([-\infty, +\infty])$ measurable.

Suppose $f : X \rightarrow [-\infty, +\infty]$ is \mathcal{A}_* -measurable. Then for each finite measure μ on (X, \mathcal{A}) , since $\mathcal{A} \subset \mathcal{A}_\mu$, f is \mathcal{A}_μ measurable. Then the existence of f_0, f_1 is implied by Cohn (2013), Proposition 2.2.5. Conversely, suppose for each finite measure μ , such f_0, f_1 exist. Then Cohn (2013), Proposition 2.2.5 implies that f is \mathcal{A}_μ measurable. Thus, for any $A \in \mathcal{B}([-\infty, +\infty])$, $f^{-1}(A) \in \cap_\mu \mathcal{A}_\mu = \mathcal{A}_*$. This completes the proof of (a).

We turn to (b). Consider the set $A = \{x : f_0(x) \neq f_1(x)\}$. By assumption, $\mu^*(A) = 0$ for any finite μ on \mathcal{A} . If A is not empty, let x be a point of A and δ_x be the point mass concentrated on x . Then $\delta_x^*(A) = 1$, a contradiction. It follows that $A = \emptyset$. Thus $f = f_0$, hence f is \mathcal{A} -measurable. \square

Exercise 13 (Cohn (2013), Exercise 8.4.2). *Let (X, \mathcal{A}) be a measurable space. Then*

- (a) $(\mathcal{A}_*)_* = \mathcal{A}_*$.
- (b) *If μ is a finite measure on (X, \mathcal{A}) , then $(\mathcal{A}_\mu)_* = \mathcal{A}_\mu$.*

Proof. First we proof (b). It is clear that $(\mathcal{A}_\mu)_* \supset \mathcal{A}_\mu$. On the other hand, $(\mathcal{A}_\mu)_* \subset (\mathcal{A}_\mu)_\mu = \mathcal{A}_\mu$. This completes the proof of (b).

We turn to (a). It is clear that $(\mathcal{A}_*)_* \supset \mathcal{A}_*$. On the other hand, let μ be any finite measure on \mathcal{A} . Then $(\mathcal{A}_*)_* \subset (\mathcal{A}_\mu)_* = \mathcal{A}_\mu$. Hence $(\mathcal{A}_*)_* \subset \cap_\mu \mathcal{A}_\mu = \mathcal{A}_*$. \square

Exercise 14 (Cohn (2013), Exercise 8.4.3). *There is a Lebesgue measurable subset of \mathbb{R} that is not universally measurable.*

Proof. Let C be the Cantor set on $[0, 1]$ (see Cohn (2013), example 1.4.6), let $f : [0, 1] \rightarrow [0, 1]$ be the Cantor function (see Cohn (2013), example 2.1.10). It is known that f is nondecreasing and continuous.

Let λ be the Lebesgue measure on \mathbb{R} . Let $\lambda_{[0,1]}$ be the Lebesgue measure on $[0, 1]$. Let μ be the measure on $[0, 1]$ with distribution function f . It is known that $\mu(C) = 1$. Define $\mu_1 = \lambda + \mu$. Then for $x \in [0, 1]$, $\mu_1([0, x]) = f(x) + x$. Define $g : [0, 1] \rightarrow [0, 2]$ by $g(x) = f(x) + x$. Then g is strict increasing and continuous. Thus, g is a Borel isomorphism from $([0, 1], \mathcal{B}([0, 1]), \mu_1)$ to $([0, 2], \mathcal{B}([0, 2]), \lambda)$. That is, $\lambda \circ g = \mu_1$. It can be easily seen that g is also an isomorphism from the completion space $([0, 1], \mathcal{B}([0, 1])_{\mu_1}, \mu_1)$ to the completion space $([0, 2], \mathcal{B}([0, 2])_\lambda, \lambda)$. It can be shown that $\lambda(g(C^c)) = 1$. Hence $\lambda(g(C)) = 2 - 1 = 1$. Then There is a Lebesgue **non**measurable

subset D of $g(C)$ (see Cohn (2013), exercise 1.4.6). Then $g^{-1}(D)$ is a $\mathcal{B}([0, 1])_{\mu_1}$ -nonmeasurable subset of C . Note also that $g^{-1}(D)$ is Lebesgue measurable since $g^{-1}(D)$ is a subset of C which has Lebesgue measure 0. Thus, $g^{-1}(D)$ is a Lebesgue measurable subset of \mathbb{R} which is not universally measurable.

□

References

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