Surface Integrals over n-Dimensional Spheres

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1 Introduction

This notes is adapted from Stan's library.

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, $\mathbf{r}$ the positive vector in $\mathbb{R}^n$ and $r = |\mathbf{r}|$ its norm

$$\mathbf{r} = (x_1, , \ldots, x_n), \quad r = |\mathbf{r}| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

We shall often use n-tuples of non-negative real exponents $\mathbf{p} = (p_1, \ldots, p_n)$, which, however, are not to be intended as elements of $\mathbb{R}^n$. The shorthand will be exploited in conventional expressions of the type

$$E(\mathbf{r}, \mathbf{p}) = \prod_{k=1}^{n} (x_k^2)^{p_k}.$$

An n-dimensional spherical surface $S(R)$ of radius R is defined by the condition

$$\sum_{k=1}^{n} \frac{x_k^2}{R^2} = 1.$$

An n-dimensional spherical volumes $V(R)$ of ratius R is defined by the condition

$$\sum_{k=1}^{n} \frac{x_k^2}{R^2} \leq 1.$$

We are interested in the evaluation of the following integrals over $S(R)$:

$$S_n(\mathbf{p}, R) = \int_{S(R)} E(\mathbf{r}, \mathbf{p}) \, \mathrm{d}\sigma, \tag{1}$$

where $\mathrm{d}\sigma$ is an $(n-1)$-dimensional surface element, and the following integrals over $V(R)$:

$$W_n(\mathbf{p}, R) = \int_{V(R)} E(\mathbf{r}, \mathbf{p})\, \mathrm{d}\tau,$$

where $\mathrm{d}\tau$ is the volume element.

2 Evaluation of the integrals

2.1 Volume integrals

This sub section is devoted to $W_n(\mathbf{p}, R)$. In Cartesian coordinates, the volumn element $\mathrm{d}\tau$ is given by

$$\mathrm{d}\tau = \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n.$$

By a coordinates-scaling transformation, we have

$$W_n(\mathbf{p}, R) = R^{2p+n} W_n(\mathbf{p}), \quad \text{where } p = \sum_{i=1}^{n} p_i \text{ and } W_n(\mathbf{p}) = \int_{V_n(1)} E(\mathbf{r}, \mathbf{p})\, \mathrm{d}\tau.$$

The integral $W_n(\mathbf{p})$ can be computed iteratively. Write

$$W_n(\mathbf{p}) = \int_{-1}^{1} \mathrm{d}x_n \int_{V_{n-1}(\sqrt{1-x_n^2})} E(\mathbf{r}, \mathbf{p})\, \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}.$$

Let $\mathbf{r}' = (x_1, \ldots, x_{n-1})$ and $\mathbf{p}' = (p_1, \ldots, p_{n-1})$. Then

$$\begin{aligned} W_n(\mathbf{p}) &= \int_{-1}^{1} x_n^{2p_n}\, \mathrm{d}x_n \int_{V_{n-1}(\sqrt{1-x_n^2})} E(\mathbf{r}', \mathbf{p}')\, \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} \\ &= \int_{-1}^{1} x_n^{2p_n} (\sqrt{1-x_n^2})^{2p'+n-1}\, \mathrm{d}x_n \int_{V_{n-1}(1)} E(\mathbf{r}', \mathbf{p}')\, \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}. \end{aligned}$$

Let $x_n = \cos(\theta)$, $\theta \in (0, \pi)$. Then

$$\begin{aligned} W_n(\mathbf{p}) &= \int_0^{\pi} \left(\cos^2(\theta)\right)^{p_n} \sin^{2p'+n}(\theta)\, \mathrm{d}\theta \int_{V_{n-1}(1)} E(\mathbf{r}', \mathbf{p}')\, \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} \\ &= \mathrm{Beta}(\frac{n+1}{2} + \sum_{k=1}^{n-1} p_k, \frac{1}{2} + p_n) W_{n-1}(\mathbf{p}'). \\ &= \frac{\Gamma(\frac{n+1}{2} + \sum_{k=1}^{n-1} p_k)\Gamma(\frac{1}{2} + p_n)}{\Gamma(\frac{n+2}{2} + \sum_{k=1}^{n} p_k)} W_{n-1}(\mathbf{p}'). \end{aligned}$$

By recurrence, we have

$$W_n(\mathbf{p}) = \frac{\prod_{k=1}^{n} \Gamma(p_k + 1/2)}{\Gamma(p + n + 2/2)}.$$

Then

$$W_n(\mathbf{p}, R) = R^{2p+n} \frac{\prod_{k=1}^{n} \Gamma(p_k + 1/2)}{\Gamma(p + (n+2)/2)}. \tag{2}$$

Here p_k is any real value greater than $-1/2$.

2.2 Integrals over spheres

In this sub section, we deal with $S_n(\mathbf{p}, R)$. The evaluation is considerably simplified by three facts:

- By (2), we have

$$W_n(\mathbf{p}, R) = R^{2p+n} \frac{2}{2p+n} \frac{\prod_{k=1}^{n} \Gamma(p_k + 1/2)}{\Gamma(p + n/2)}. \tag{3}$$

- Integral (1) has the nearly self-evident scaling property

$$S_n(\mathbf{p}, R) = R^{2p+n-1} S_n(\mathbf{p}, 1), \quad \text{where } p = \sum_{k=1}^{n} p_k,$$

  arising from the fact that $E(\mathbf{r}, \mathbf{p})$ scales with $2p$-th power of R and $\mathrm{d}\sigma$ scales with $(n-1)$-st power of R.

- For spheres, the volume integration can be carried out by summing the contributions of concentric shells defined by radii r and $r + \mathrm{d}r$, for r ranges from 0 to R. Hence

$$W_n(\mathbf{p}, R) = \int_0^R S_n(\mathbf{p}, r)\, \mathrm{d}r.$$

From these observation, it follows that

$$S_n(\mathbf{p}, R) = \frac{\partial}{\partial R} W_n(\mathbf{p}, R) = 2R^{2p+n-1} \frac{\prod_{k=1}^{n} \Gamma(p_k + 1/2)}{\Gamma(p + n/2)}.$$

Here p_k is any real value greater than $-1/2$.

3 Special cases

Setting all the p's equal to $v/2$, one obtains the following formula, valid for any $v > -1$:

$$\int_{S(R)} |x_1 x_2 \dots x_n|^v \, \mathrm{d}\sigma = 2R^{nv+n-1} \frac{\Gamma^n((v+1)/2)}{\Gamma(n(v+1)/2)}.$$

When only one of the p's equals $v/2$ and all the others are zero, we have, for any $v > -1$, that

$$\int_{S(R)} |x_1|^v \, \mathrm{d}\sigma = 2\pi^{(n-1)/2} R^{n+v-1} \frac{\Gamma((v+1)/2)}{\Gamma((v+n)/2)}.$$

The surface of the n-dimensional unit sphere is

$$S_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

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References