High-dimensional two-sample test under spiked covariance

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Abstract

This paper considers testing the means of two p-variate normal samples in high dimensional setting. The covariance matrices are assumed to be spiked, which often arises in practice. We propose a new test procedure through projection on the orthogonal complement of principal space. The asymptotic normality of the new test statistic is proved and the power function of the test is given. Theoretical and simulation results show that the new test outperforms existing methods substantially when the covariance matrices are spiked. Even when the covariance matrices are not spiked, the new test is acceptable.

Keywords: high dimension, mean test, orthogonal complement of principal space, spiked covariance

1. Introduction

Suppose that $X_{k,1}, \ldots, X_{k,n_k}$ are independent identically distributed (i.i.d.) as $N_p(\mu_k, \Sigma)$, where μ_k and Σ are unknown, $k = 1, 2$. We consider the hypothesis testing problem:

$$H_0 : \mu_1 = \mu_2 \quad \text{vs.} \quad H_1 : \mu_1 \neq \mu_2. \tag{1}$$

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In this paper, high dimensional setting is adopted, i.e., the dimension p varies as n increase, where $n = n_1 + n_2$ is the total sample size. Testing hypotheses (1) is important in many applications, including biology, finance and economics. Quite often, these data have strong correlations between variables. When strong correlations exist, covariance matrices are often spiked in the sense that a few eigenvalues are distinctively larger than the others. This paper is devoted to testing hypotheses (1) in high dimensional setting with spiked covariance.

If $\Sigma_1 = \Sigma_2 = \Sigma$ is unknown, a classical test for hypotheses (1) is Hotelling's T^2 test. Hotelling's test statistic is $(\bar{X}_1 - \bar{X}_2)^T S^{-1}(\bar{X}_1 - \bar{X}_2)$, where S is the pooled sample covariance matrix. However, Hotelling's test is not defined when $p \geq n-1$. Moreover, Bai and Saranadasa (1996) showed that even if $p < n-1$, Hotelling's test suffers from low power when p is comparable to n. Perhaps, the main reason for low power of Hotelling's test is due to that S is a poor estimator of Σ when p is large compared with n. See Chen and Qin (2010) and the references therein. In high dimensional setting, many test statistics in the literatures are based on an estimator of $(\mu_1 - \mu_2)^T A(\mu_1 - \mu_2)$ for a given positive definite matrix A. For example, Bai and Saranadasa (1996) proposed a test based on

$$T_{BS} = \|\bar{X}_1 - \bar{X}_2\|^2 - (\frac{1}{n_1} + \frac{1}{n_2})\mathrm{tr}S,$$

which is an unbiased estimator of $\|\mu_1 - \mu_2\|^2$. Chen and Qin (2010) modified T_{BS} by removing terms $\sum_{i=1}^{n_k} X_{ki}^T X_{ki}$, $k = 1, 2$ and proposed a test based on

$$\begin{aligned} T_{CQ} &= \frac{\sum_{i\neq j}^{n_1} X_{1i}^T X_{1j}}{n_1(n_1-1)} + \frac{\sum_{i\neq j}^{n_2} X_{2i}^T X_{2j}}{n_2(n_2-1)} - 2\frac{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} X_{1i}^T X_{2j}}{n_1 n_2} \\ &= \|\bar{X}_1 - \bar{X}_2\|^2 - \frac{1}{n_1}\mathrm{tr}S_1 - \frac{1}{n_2}\mathrm{tr}S_2, \end{aligned}$$

where S_1 and S_2 are sample covariance matrices. Statistic T_{CQ} is also an unbiased estimator of $\|\mu_1 - \mu_2\|^2$. Choosing $A = [\mathrm{diag}(\Sigma)]^{-1}$, Srivastava and Du (2008) proposed a test based on

$$T_S = (\bar{X}_1 - \bar{X}_2)^T [\mathrm{diag}(S)]^{-1} (\bar{X}_1 - \bar{X}_2),$$

where $\mathrm{diag}(A)$ is a diagonal matrix with the same diagonal elements as A's.

As Ma et al. (2015) pointed out, however, these test procedures may not be valid if strong correlations exist, i.e., Σ is far away from diagonal matrix. For example, the assumption

$$\mathrm{tr}(\Sigma^4) = o\big(\mathrm{tr}^2(\Sigma^2)\big) \tag{2}$$

adopted by Chen and Qin (2010) can be violated when $\Sigma = (1-c)I_p + c\mathbf{1}_p\mathbf{1}_p^T$ where $-1/(p-1) < c < 1$, I_p is the p dimensional identity matrix and $\mathbf{1}_p$ is the p dimensional vector with elements 1. To characterize strong correlations, Ma et al. (2015) considered a factor model and proposed a parameter bootstrap procedure to adjust Chen and Qin (2010)'s critical value.

Strong correlations between variables do exist in practice. In gene expression analysis, genes are correlated due to genetic regulatory networks (see Thulin (2014)). Chen et al. (2011) pointed out that in terms of pathway analysis in proteomic studies, test level can not be guaranteed if correlations are incorrectly assumed to be absent. As Ma et al. (2015) argued, there're strong correlations between different stock returns since they are all affected by the market index.

Incorrectly assuming the absence of correlation between variables will result in level inflation and low power for a test procedure. A class of test procedures is proposed through random projection (see Lopes et al. (2011), Thulin (2014) and Srivastava et al. (2015)). The idea is to project data on some random lower-dimensional subspaces. It has been shown that these procedures perform well under strong correlations.

In many situations, the correlations are determined by a small number of factors. As a consequence, Σ is spiked. See, for example, Cai et al. (2013) and Shen et al. (2013). The random projection methods imply that test procedures are improved when data are projected on certain subspaces. We will see that the ideal subspace is the orthogonal complement of the principal space. Fortunately, the principal space can be estimated consistently even in high dimensional setting by the theory of principal component analysis (PCA). With the assumption of spiked covariance model, we propose a new test procedure through projection on the (estimated) ideal subspace. The asymptotic distribution of the test statistic is derived and hence asymptotic power is given. We will see that the test is more powerful than T_{CQ}. Moreover, even there's no strong correlation showing up, we prove that the new test performs equally well as T_{CQ} does. The idea is also generalized to the unequal variance setting and similar results still hold.

The rest of the paper is organized as follows. In Section 2, the model and some assumptions are given. In Section 3, we propose a test procedure under $\Sigma_1 = \Sigma_2$. Section 4 exploits properties of the test. In Section 5, we generalize our test procedure to the situation of $\Sigma_1 \neq \Sigma_2$. In Section 6, simulations are carried out and a real data example is given. Section 7 contains some discussion. All the technical details are in appendix.

2. Model and assumptions

Let $X_{k,1}, \ldots, X_{k,n_k}$ be i.i.d. observations with common distribution $N(\mu_k, \Sigma)$, $k = 1, 2$. Throughout the paper, we assume $p \to \infty$ as $n \to \infty$ and $n_1/n_2 \to \xi \in (0, +\infty)$, that is, we consider high dimensional and balanced data.

In the rest of this section, we introduce some notations that will be used. Let $\tau = (n_1 + n_2)/(n_1 n_2)$, S be the pooled sample covariance:

$$S = \frac{1}{n-2} \sum_{k=1}^{2} \sum_{i=1}^{n_k} (X_{k,i} - \bar{X}_k)(X_{k,i} - \bar{X}_k)^T = \frac{(n_1 - 1)S_1 + (n_2 - 1)S_2}{n-2},$$

where $S_k = (n_k - 1)^{-1} \sum_{i=1}^{n_k} (X_{k,i} - \bar{X}_k)(X_{k,i} - \bar{X}_k)^T$ is the sample covariance of the sample k, $k = 1, 2$. Denote by $\text{Wishart}_p(m, \Psi)$ the p dimensional Wishart distribution with parameter Ψ and m degrees of freedom.

For random variable ξ and η, we write $\xi \sim \eta$ to denote they have the same distribution. Let $\mathcal{L}(\xi)$ be the distribution of ξ and $\mathcal{L}(\xi|\eta)$ be the conditional distribution of ξ given η. We denote by "$\xrightarrow{a.s.}$", "$\xrightarrow{P}$" and "$\xrightarrow{\mathcal{L}}$" the almost surely convergence, convergence in probability and weak convergence.

For nonrandom positive sequence $\{a_n\}$ and $\{b_n\}$, $a_n \asymp b_n$ represents $a_n = O(b_n)$ and $b_n = O(a_n)$ as $n \to \infty$.

We denote by $\|\cdot\|$ and $\|\cdot\|_F$ the operator and Frobenius norm of matrix, separately. For $p \geq q$, define

$$\mathbb{O}_{p\times q} = \{O | \, O \text{ is } p \times q \text{ column orthonormal matrix } \}.$$

In Chen and Qin (2010), the asymptotic normality of T_{CQ} is derived under condition (2). To determine the critical value of T_{CQ}, they used the limiting normal distribution to approximate the finite sample distribution. The resulting test can asymptotically preserve the test level if and only if the asymptotic normality is correct. While Chen and Qin (2010) only proved (2) is sufficient for the asymptotic normality of T_{CQ}, one may ask: is (2) also necessary for asymptotic normality of T_{CQ}?

To answer this question, we first note that T_{CQ} is a quadratic form of standard normal random vector. To see this, let $Z_{k,i} = \Sigma^{-1/2} X_{k,i}$ so that $Z_{k,i} \sim N_p(0, I_p)$. Write all $Z_{k,i}$ in a long vector form $Z = (Z_{1,1}^T, \ldots, Z_{1,n_1}^T, Z_{2,1}^T, \ldots, Z_{2,n_2}^T)^T$. Then T_{CQ} is a quadratic form of Z and $T_{CQ} = Z^T \big(B_n \otimes \Sigma\big) Z$, where $\otimes$ is the Kronecker product,

$$B_n = \begin{pmatrix} \frac{1}{n_1(n_1-1)}(n_1\gamma_1\gamma_1^T - I_{n_1}) & -\frac{1}{\sqrt{n_1 n_2}}\gamma_1\gamma_2^T \\ -\frac{1}{\sqrt{n_1 n_2}}\gamma_2\gamma_1^T & \frac{1}{n_2(n_2-1)}(n_2\gamma_2\gamma_2^T - I_{n_2}) \end{pmatrix},$$

γ_1 is an n_1 dimensional vector with all elements equal to $1/\sqrt{n_1}$ and γ_2 is an n_2 dimensional vector with all elements equal to $1/\sqrt{n_2}$.

Using characteristic function method, one can prove the following propostion which gives a necessary and sufficient condition for the asymptotic normality of the quadratic form of a standard normal random vector.

Proposition 1. *Suppose Y_n is a k_n dimensional standard normal random vector and A_n is a $k_n \times k_n$ symmetric matrix. Then as $n \to \infty$, a necessary and sufficient condition for*

$$\frac{Y_n^T A_n Y_n - \mathrm{E} Y_n^T A_n Y_n}{[\mathrm{Var}(Y_n^T A_n Y_n)]^{1/2}} \xrightarrow{\mathcal{L}} N(0,1) \tag{3}$$

is that

$$\frac{\lambda_{\max}(A_n^2)}{\mathrm{tr}(A_n^2)} \to 0. \tag{4}$$

To apply Proposition 1 for T_{CQ}, we need to calculate the eigenvalues of $B_n \otimes \Sigma$ which in turn relies on the eigenvalues of B_n. It can be seen that the eigenvalues of B_n are $-1/n_1(n_1-1)$, $-1/n_2(n_2-1)$, $(n_1+n_2)/n_1n_2$ and 0 with multiplicities n_1-1, n_2-1, 1 and 1 respectively. Thus,

$$\operatorname{tr}\left(B_n \otimes \Sigma\right)^2 = \operatorname{tr}(B_n^2)\operatorname{tr}\Sigma^2 = (\frac{1}{n_1(n_1-1)} + \frac{1}{n_1(n_1-1)} + \frac{2}{n_1n_2})\operatorname{tr}\Sigma^2,$$

and

$$\lambda_{\max}\Big(\left(B_n \otimes \Sigma\right)^2\Big) = \lambda_{\max}(B_n^2)\lambda_{\max}(\Sigma^2) = \Big(\frac{1}{n_1} + \frac{1}{n_2}\Big)^2 \lambda_{\max}(\Sigma^2).$$

It can be seen that

$$\frac{\lambda_{\max}\Big(\left(B_n \otimes \Sigma\right)^2\Big)}{\operatorname{tr}\left(B_n \otimes \Sigma\right)^2} \to 0$$

is in turn equivalent to $\lambda_{\max}(\Sigma^2)/\operatorname{tr}\Sigma^2 \to 0$. From the inequality

$$\frac{\lambda_1(\Sigma)^4}{(\sum_{i=1}^p \lambda_i(\Sigma)^2)^2} \le \frac{\sum_{i=1}^p \lambda_i(\Sigma)^4}{(\sum_{i=1}^p \lambda_i(\Sigma)^2)^2} \le \frac{\lambda_1(\Sigma)^2 \sum_{i=1}^p \lambda_i(\Sigma)^2}{(\sum_{i=1}^p \lambda_i(\Sigma)^2)^2} = \frac{\lambda_1(\Sigma)^2}{\sum_{i=1}^p \lambda_i(\Sigma)^2}$$

we can see that $\lambda_{\max}^2(\Sigma)/\operatorname{tr}(\Sigma^2) \to 0$ is equivalent to (2). By Lemma 1, (2) is a necessary and sufficient condition for

$$\frac{T_{CQ} - \mathrm{E}T_{CQ}}{\left[\operatorname{Var}(T_{CQ})\right]^{1/2}} \xrightarrow{L} N(0,1).$$

While condition (2) is necessary for the validity of Chen and Qin (2010)'s test, it requires the eigenvalues of Σ to be concentrated, which can be violated in many applications. For example, in a class of applications, the correlations between variables are mainly driven by several common factors so that Σ has a few eigenvalues which are much larger than the others. See, for example, Jung and Marron (2009), Cai et al. (2013) and Fan and Wang (2017). It is meaningful to consider testing problem for data of such type. To characterize such correlations between variables, we consider the following spiked covariance structure:

Assumption 1. *The covariance matrix* Σ *has structure* $\Sigma = V\Lambda V^T + \sigma^2 I_p$, *where* $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$, $\lambda_1 \ge \cdots \ge \lambda_r > 0$, *$r$ is a known number, V is a*

$p \times r$ *orthonormal matrix and* $\sigma^2 > 0$. *As* n, p *tend to infinity,* r, σ^2 *are fixed and* Λ *satisfies*

$$\kappa p^\beta \geq \lambda_1 \geq \cdots \geq \lambda_r \geq \kappa^{-1} p^\beta.$$

where $\kappa > 1$ *and* $\beta \geq 1/2$ *are absolute constants.*

The covariance structure in Assumption 1 is commonly adopted in PCA study. See, for example, Cai et al. (2013). Spiked covariance also has connection to factor model. In fact, the factor model in Ma et al. (2015) corresponds to $\beta = 1$ in Assumption 1. Under the covariance structure in Assumption 1, condition (2) is equivalent to $\beta < 1/2$. So we consider the case of $\beta \geq 1/2$.

Under Assumption 1, VV^T is the orthogonal projection matrix on the column space of V. Let $\tilde{V}$ be a $p \times (p-r)$ full column rank orthonormal matrix orthogonal to columns of V. Although such $\tilde{V}$ is not unique, the projection matrix $\tilde{V}\tilde{V}^T$ is unique and $\tilde{V}\tilde{V}^T = I - VV^T$.

Theorem 1. *Under assumption 1, suppose* $\lambda_i / p^\beta \to l_i$ *with* $0 < l_i < +\infty$, $i = 1, \ldots, r$.

$$T_{CQ}$$

Proof. Since $(n_k - 1)S_k \sim \mathrm{Wishart}_p(n_k - 1, \Sigma)$, $k = 1, 2$, we have

$$\mathrm{E}\Big(\frac{1}{n_1}\operatorname{tr} S_1 + \frac{1}{n_2}\operatorname{tr} S_2\Big) = \tau \operatorname{tr}\Sigma,$$

and

$$\begin{aligned}&\mathrm{Var}\Big(\frac{1}{n_1}\operatorname{tr} S_1 + \frac{1}{n_2}\operatorname{tr} S_2\Big) = \Big(\frac{2}{n_1^2(n_1-1)} + \frac{2}{n_2^2(n_2-1)}\Big)\operatorname{tr}\Sigma^2 \\ =&O(\frac{1}{n^3}(p^{2\beta} + p)) = O\big(\frac{1}{n}(\tau \operatorname{tr}\Sigma)^2\big).\end{aligned}$$

It follows that

$$\frac{1}{n_1}\operatorname{tr} S_1 + \frac{1}{n_2}\operatorname{tr} S_2 = \tau \operatorname{tr}\Sigma\Big(1 + O_P(\frac{1}{\sqrt{n}})\Big).$$

Note that we have $\|\bar{X}_1 - \bar{X}_2\|^2 \sim \tau(\sum_{i=1}^r \lambda_i(\Sigma) Z_i + \sigma^2 W)$, where $Z_i \overset{i.i.d.}{\sim} \chi_1^2$ and $W \sim \chi_{p-r}^2$ is independent of Z_i's. Then

$$\frac{1}{\tau p^\beta}\|\bar{X}_1 - \bar{X}_2\|^2 \sim \sum_{i=1}^r \frac{\lambda_i + \sigma^2}{p^\beta} Z_i + \frac{\sigma^2}{p^\beta} W.$$

It's easy to see that

$$\sum_{i=1}^{r} \frac{\lambda_i + \sigma^2}{p^\beta} Z_i \xrightarrow{\mathcal{L}} \sum_{i=1}^{r} l_i Z_i.$$

By central limit theorem, we have

$$\frac{1}{\sqrt{2(p-r)}} \big(W - (p-r)\big) \xrightarrow{\mathcal{L}} N(0,1).$$

If $\beta = 1/2$, by Slutsky's theorem we have

$$\frac{1}{\tau p^\beta} \|\bar{X}_1 - \bar{X}_2\|^2 - \frac{\sigma^2(p-r)}{p^\beta} \xrightarrow{\mathcal{L}} \sum_{i=1}^{r} l_i Z_i + \sqrt{2}\sigma^2 \epsilon.$$

where $\epsilon \sim N(0,1)$

$\square$

3. Methodology

In this section, we describe our new test procedure for hypotheses (1). For simplicity, we now work on equal covariance setting. Unequal covariance setting will be considered latter.

3.1. Motivation

The Proposition **??** implies that under spiked covariance, Chen and Qin (2010)'s method can not guarantee the test level. To this end, Ma et al. (2015) proposed a test procedure which is based on T_{CQ} and has the correct asymptotic test level. In their paper, Σ's first few eigenvalues are assumed to be of order p.

Note that T_{BS}, T_{CQ} and Ma et al. (2015)'s method are all based on $\tau\|\bar{X}_1 - \bar{X}_2\|^2$, which can be written as the sum of two parts

$$\tau\|V^T(\bar{X}_1 - \bar{X}_2)\|^2 + \tau\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2. \tag{5}$$

Under the null hypotheses, we have

$$\operatorname{Var}\left(\tau\|V^T(\bar{X}_1 - \bar{X}_2)\|^2\right) = \sum_{i=1}^{r} 2(\lambda_i + \sigma^2)^2, \quad \operatorname{Var}\left(\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2\right) = 2\sigma^4(p-r).$$

The ratio of the two variance is

$$\frac{\sum_{i=1}^{r} 2(\lambda_i + \sigma^2)^2}{2\sigma^4(p-r)} \asymp p^{2\beta-1},$$

which tends to ∞ as $p \to \infty$ for $\beta > 1/2$. On the other hand, $\tau\|V^T(\bar{X}_1 - \bar{X}_2)\|^2$ only involves the signal from r dimension. Thus, compared with the second term of (5), the first term has larger variance and tends to contain much weaker signal. This motivates us to drop the first part of (5) and only use the second part. After adjustment of expectation, we define the following statistic

$$T_1 = \|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2 - \frac{1}{n_1}\mathrm{tr}(\tilde{V}^T S_1 \tilde{V}) - \frac{1}{n_2}\mathrm{tr}(\tilde{V}^T S_2 \tilde{V}).$$

Proposition 2 shows that the asymptotic distribution of T_1 is normal.

Proposition 2. *Suppose that Assumptions* **??**-**??** *holds and* $\frac{n}{p}\|\mu_1 - \mu_2\|^2 = o(1)$. *We have*

$$\frac{T_1 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2}{\sigma^2\sqrt{2\tau^2 p}} \xrightarrow{\mathcal{L}} N(0,1).$$

In another point of view, T_1 is obtained by transforming $X_{k,i}$ to $\tilde{V}^T X_{k,i}$ ($i = 1, \ldots, n_k$, $k = 1, 2$) and then invoking the statistic of Chen and Qin (2010). A class of test procedures have been proposed through random projection to lower dimensional space, for example, Lopes et al. (2011), Thulin (2014) and Srivastava et al. (2015). It is known that random projection based methods offer higher power when the variables are dependent. However, these test procedures are randomized, which is undesirable in practice. This raise the question: is there an optimal projection which is nonrandomized?

It can be seen that

$$\tilde{V} = \underset{O \in \mathbb{O}_{p\times(p-r)}}{\arg\min} \mathrm{Var}\left(\|O^T(\bar{X}_1 - \bar{X}_2)\|^2\right).$$

Thus, transformation by $\tilde{V}$ is optimal in the sense of variance reduction. Based on $\tilde{V}^T X_{ki}$, the likelihood ratio test statistic for hypothesis (1) is $\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2$ which coincides with our proposal. In this view, T_1 can be regarded as a restricted likelihood ratio test.

3.2. New Test

We denote by $\hat{V}$ and $\hat{\hat{V}}$ the first r and last $p-r$ eigenvectors of S respectively. Similarly, we denote by $\hat{V}$ and $\hat{\hat{V}}_k$ the first r and last $p-r$ eigenvectors of S_k respectively, $k = 1, 2$. As estimators of their population counterparts, these simple statistics actually reach the optimal convergence rate (See Cai et al. (2013)).

Note that T_1 relies on the subspace $\tilde{V}\tilde{V}^T$ which is unknown and thus should be estimated. The first part of T_1, $\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2$, can be directly estimated by $\|\hat{\hat{V}}^T(\bar{X}_1 - \bar{X}_2)\|^2$. Note that $n_1^{-1}\mathrm{tr}(\tilde{V}^T S_1 \tilde{V})$, the second part of T_1, only involves sample one. We estimate it by $n_1^{-1}\mathrm{tr}(\hat{\hat{V}}_1^T S_1 \hat{\hat{V}}_1)$. Similarly, we estimate the third part of T_1 by $n_2^{-1}\mathrm{tr}(\hat{\hat{V}}_2^T S_2 \hat{\hat{V}}_2)$. Define

$$T_2 = \|\hat{\hat{V}}^T(\bar{X}_1 - \bar{X}_2)\|^2 - \frac{1}{n_1}\mathrm{tr}(\hat{\hat{V}}_1^T S_1 \hat{\hat{V}}_1) - \frac{1}{n_2}\mathrm{tr}(\hat{\hat{V}}_2^T S_2 \hat{\hat{V}}_2).$$

The asymptotic result of Proposition 2 involves σ^2. In order to formulate a test procedure by asymptotic distribution, σ^2 needs to be consistently estimated. Note that σ^2 can be written as $\sigma^2 = (p-r)^{-1}\sum_{i=r+1}^{p} \lambda_i(\Sigma)$. Thus it can be estimated by

$$\hat{\sigma}^2 = \frac{1}{p-r}\sum_{i=r+1}^{p} \lambda_i(S).$$

Now we propose our new test statistic as

$$Q = \frac{T_2}{\hat{\sigma}^2\sqrt{2\tau^2 p}}. \tag{6}$$

In next section, it will be proved that the asymptotic null distribution of Q is $N(0,1)$. Thus, the null hypothesis is rejected when Q is larger than the upper α quantile of $N(0,1)$.

Remark 1. When both samples are simultaneously transformed by shift and orthogonal transformation, the statistic T_2 is invariant. More precisely, T_2 is invariant under the following transformation:

$$X_{1,i} \mapsto OX_{1,i} + \mu \text{ and } X_{2,j} \mapsto OX_{2,j} + \mu,\ i = 1, \ldots, n_1,\ j = 1, \ldots, n_2,$$

where $\mu \in \mathbb{R}^p$ and $O \in \mathbb{O}_{p\times p}$.

Theoretical results will show that the asymptotic variance of T_2 is significantly smaller than T_{CQ}. Since the new test statistic estimates $\|\tilde{V}^T(\mu_1-\mu_2)\|^2$, the superiority of the new test will be established if

$$\frac{\|\tilde{V}^T(\mu_1-\mu_2)\|}{\|\mu_1-\mu_2\|}\approx 1. \tag{7}$$

Obviously, (7) is not always the case since there always exists some $\tilde{V}$ and $\mu_1-\mu_2$ such that $\|\tilde{V}^T(\mu_1-\mu_2)\|=0$. However, (7) is reasonable since $\tilde{V}\tilde{V}^T$ is nearly an identity matrix in the sense that $\|I_p-\tilde{V}\tilde{V}^T\|_F^2/\|I_p\|_F^2 = r/p \to 0$. In bayesian framework, if we assume that the elements of μ_k are independently generated from certain prior distribution, it can be established that $\|\tilde{V}(\mu_1-\mu_2)\|/\|\mu_1-\mu_2\| \xrightarrow{P} 1$. Such assumption for μ_k will be used in Theorem **??**.

4. Theoretical results

4.1. Asymptotic properties

In this section, we study the asymptotic behavior of the new test procedure.

We first give a result of the convergence rate of $\hat{\sigma}^2$. In particular, it can be seen that $\hat{\sigma}^2$ is a consistent estimator of σ^2. Our proof relies on the Weyl's inequality.

Proposition 3. *Under Assumptions* **??**-**??**, *we have that*

$$\hat{\sigma}^2=\sigma^2+O_P\Big(\frac{\max(n,p)}{np}\Big).$$

In the construction of T_2, we replace $\tilde{V}$ by $\hat{\tilde{V}}$. The asymptotic property of T_2 is closely related the asymptotic property of $\hat{\tilde{V}}\hat{\tilde{V}}^T$ as an estimator of $\tilde{V}\tilde{V}^T$. However, $\hat{\tilde{V}}\hat{\tilde{V}}^T$ can not always consistently estimate $\tilde{V}\tilde{V}^T$ in high dimensional setting. In fact, Cai et al. (2014)'s Theorem 5 implies that it is possible only when $p^{1-\beta}/n\to 0$, see Lemma 2 in appendix. The asymptotic normality of T_2 requires a stronger condition.

Assumption 2. *Assume* $p/n^2\to 0$.

When $\beta = 1/2$, Assumption 2 is equivalent to $p^{1-\beta}/n$, otherwise 2 is stronger than $p^{1-\beta}/n$. The following theorem establishes the asymptotic normality of T_2.

Theorem 2. *Under Assumptions* **??**-*2, if the local alternative holds, that is,*

$$\frac{n}{\sqrt{p}}\|\mu_1 - \mu_2\|^2 = O(1),$$

then

$$\frac{T_2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2}{\sigma^2\sqrt{2\tau^2 p}} \xrightarrow{\mathcal{L}} N(0,1).$$

Now the power function of the new test can be obtained immediately.

Corollary 1. *Suppose Assumptions* **??**-*2 holds. If the null hypothesis is rejected when Q is larger than $1-\alpha$ quantile of $N(0,1)$, then the asymptotic power function of the new test is*

$$\Phi\Big(- \Phi^{-1}(1-\alpha) + \frac{\|\tilde{V}(\mu_1 - \mu_2)\|^2}{\sigma^2\sqrt{2\tau^2 p}}\Big).$$

Note that the power of T_{CQ} is of the form

$$\Phi\Big(- \Phi^{-1}(1-\alpha) + \frac{\|\mu_1 - \mu_2\|^2}{\sqrt{2\tau^2 \mathrm{tr}\Sigma^2}}\Big).$$

The relative efficiency of our test with respect to Chen's test is

$$\sqrt{\frac{\mathrm{tr}\Sigma^2}{(p-r)\sigma^4}}\frac{\|\tilde{V}(\mu_1 - \mu_2)\|^2}{\|\mu_1 - \mu_2\|^2} \sim p^{\beta - 1/2}\frac{\|\tilde{V}(\mu_1 - \mu_2)\|^2}{\|\mu_1 - \mu_2\|^2},$$

which is large when $\beta > 1/2$ and $\|\tilde{V}(\mu_1 - \mu_2)\|/\|\mu_1 - \mu_2\|$ is close to 1.

It is natural to ask if Assumption 2 can be relaxed in Theorem 2. The next theorem shows that the asymptotics are different from Theorem 2 when $p/n^2 \to \infty$.

Theorem 3. *Suppose that Assumptions* **??**-**??** *hold, $\lambda_1 = \cdots = \lambda_r = \kappa p^\beta$, $\mu_1 = \mu_2$, $\beta > 1/2$ and $p/n^2 \to \infty$. We have*

$$\frac{\kappa n p^\beta + p\sigma^2}{\tau\kappa p^{\beta+1}\sigma^2}\big(\big\|\hat{\tilde{V}}^T\big(\bar{X}_1 - \bar{X}_2\big)\big\|^2 - \tau(p-r)\sigma^2\big) \xrightarrow{\mathcal{L}} \chi^2_r.$$

When Assumption 2 doesn't hold, permutation method can be used to determine the critical value. We will see from simulation results that the new test has good power behavior even if p is much large than n.

5. Numerical studies

5.1. Simulation results

In this section, we report the simulation performance of the proposed test and compare it with T_{CQ} and T_S.

In our simulation studies, samples are generated from the model described in Assumption 1, where $V \in \mathbb{O}_{p\times r}$ is randomly generated from Haar invariant distribution, $\lambda_{k,i}$ equals p^β plus a random error from $U(0,1)$ (Uniform distribution between 0 and 1) and $\sigma_k^2 = 1$, $k = 1, 2$. In unequal variance case, (V_1, Λ_1) and (V_2, Λ_2) are independently generated. We take nominal level $\alpha = 0.05$.

First, we simulate the level of the new test. We set factor number $r = 2$. Samples are repeatedly generated 1000 times to calculate empirical level. For comparison, we also give corresponding 'oracle' level which is calculated by 'statistics' $T_1/(\sigma^2\sqrt{2p\tau^2})$ in equal variance case and $T_1/\sqrt{\sigma_n^2}$ in unequal variance case. The results for equal variance case and unequal variance case are listed in Table 1 and 2, respectively. From the results, we can find that for small n and p, even oracle level is not satisfied. Level of the new test is a little inflated compared with oracle level. In all cases, the empirical level tends to be more close to 0.05 as n increases.

Next, we simulate the empirical power of the new test. The simulation results of Ma et al. (2015) have showed that the level of the Chen and Qin (2010)'s test can't be guaranteed when covariance is spiked. To be fair, critical values are all determined by permutation method. We set $\Sigma_1 = \Sigma_2$, under which permutation method can produce exact test procedures, see Lehmann and Romano (2005)'s Example 15.2.2. We permutate the sample 100 times to determine the critical value. We repeat the test procedure 500 times to obtain empirical power. We plot the empirical power versus signal-to-noise ratio SNR $= \|\mu_1 - \mu_2\|^2/(\sigma^2\sqrt{2\tau^2 p})$. The results are illustrated in figure 1 and 2, where 'NEW', 'CQ' and 'S' represent the new test, Chen and Qin (2010)'s test and Srivastava and Du (2008)'s test. From the results, we can find that when Σ is spiked, the new test outperforms T_{CQ} substantially; when Σ is not spiked, all

Table 1: Test level simulation. Equal variance case.

		β=0.5		β=1		β=2	
n	p	NEW	ORACLE	NEW	ORACLE	NEW	ORACLE
300	200	0.075	0.062	0.079	0.062	0.074	0.070
300	400	0.074	0.065	0.061	0.044	0.046	0.040
300	600	0.058	0.041	0.070	0.052	0.071	0.055
300	800	0.066	0.047	0.071	0.052	0.062	0.048
600	200	0.061	0.055	0.052	0.051	0.058	0.056
600	400	0.051	0.048	0.051	0.042	0.059	0.051
600	600	0.061	0.058	0.056	0.054	0.051	0.047
600	800	0.053	0.046	0.060	0.050	0.056	0.048

three tests have similar performance.

5.2. Real data analysis

In this section, we study the practical problem considered in Ma et al. (2015). The task is to test whether Monday stock returns are equal to those of other trading days on average. Define an observation be the log return of stocks in a day. Hence p is the total number of stocks. Let sample 1 and sample 2 be the observations on Monday and the other trading days, respectively. Then we would like to test $H_0 : \mu_1 = \mu_2$ v.s. $H_1 : \mu_1 \neq \mu_2$. We collected the data of $p = 710$ stocks of China from 01/04/2013 to 12/31/2014. There are total $n_1 = 95$ Monday and $n_2 = 388$ other trading days.

We assume $\Sigma_1 = \Sigma_2$. The first eigenvaule of S is 0.14, which is significantly larger than the others. In fact, the second eigenvalue is 0.02. Hence there's clearly a spiked eigenvalue. We set $r = 1$ and perform our new test. The p value is 0.149, which is obtained by 1000 permutations. Hence, the null hypothesis can not be rejected for $\alpha = 0.05$. We draw the same conclusion as Ma et al. (2015).

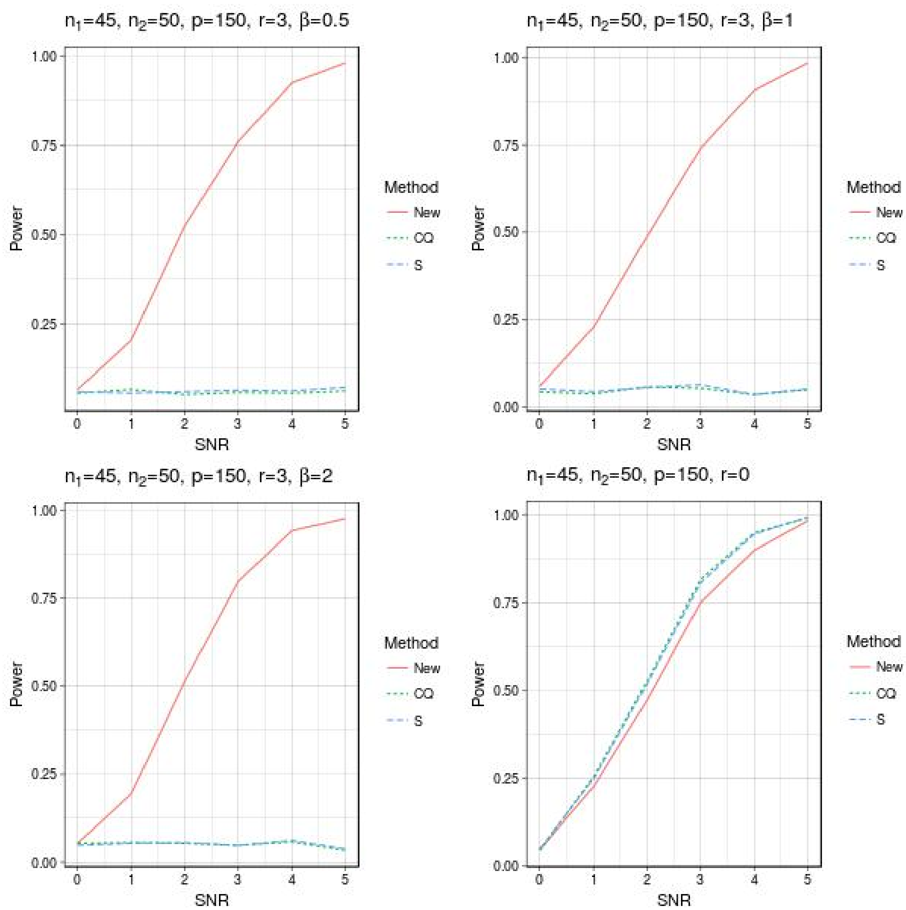

Figure 1: Empirical power simulation.

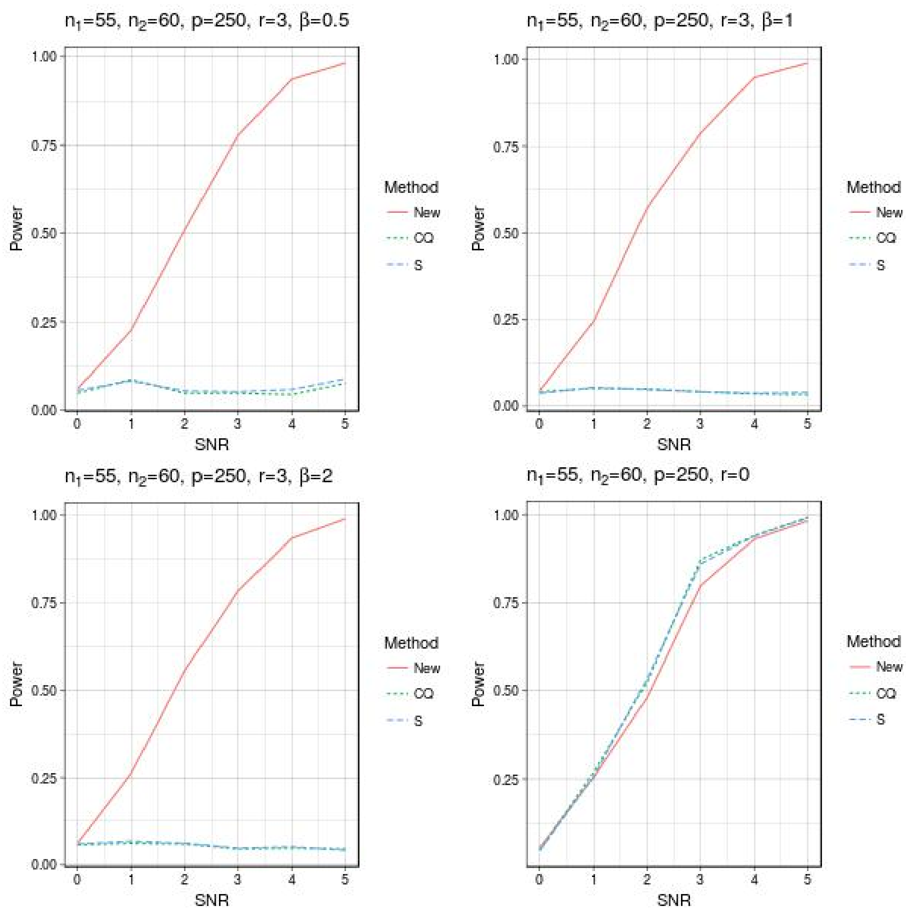

Figure 2: Empirical power simulation.

Table 2: Test level simulation. Unequal variance case.

		β=0.5		β=1		β=2	
n	p	NEW	ORACLE	NEW	ORACLE	NEW	ORACLE
300	200	0.066	0.058	0.057	0.054	0.054	0.050
300	400	0.063	0.047	0.063	0.047	0.069	0.052
300	600	0.070	0.058	0.091	0.059	0.086	0.053
300	800	0.069	0.040	0.097	0.055	0.083	0.056
600	200	0.054	0.056	0.049	0.049	0.052	0.051
600	400	0.060	0.054	0.067	0.059	0.060	0.055
600	600	0.041	0.034	0.069	0.062	0.055	0.049
600	800	0.077	0.063	0.066	0.058	0.071	0.058

6. Conclusion remark

If r is an unknown positive number, a consistent estimator of r is

$$\hat{r} = \text{argmax}_{l \leq R} \frac{\lambda_l(S)}{\lambda_{l+1}(S)}, \tag{8}$$

where R is a hyperparameter. See Ahn and Horenstein (2013) for detail. Thus, without loss of generality, we will assume that r is known throughout the paper.

This paper is concerned with the problem of testing the equality of means in the setting of high dimension and spiked covariance. We drop big variance terms from T_{CQ} and obtain a new test statistic. The asymptotic normality of the new statistic is proved and the asymptotic power is given.

In another paper, Zhao and Xu (2016) proved that their test statistic can be written in the form of projection. Their simulation results showed that their test performs well under strong correlations. Our work partially explains why their test performs well although the projections are slightly different.

Spiked covariance is an important correlation pattern and has been widely studied in terms of PCA. In PCA, authors focus on the principal subspace. However, in some circumstances, as our work have shown, the complement of principal subspace is more useful.

Our theoretical results rely on the assumption $\sqrt{p}/n \to 0$. In the situation of small sample or very large p, the critical value of the new test can be determined by permutation method. Our simulation shows that the new test still performs well. It remains a theoretical interest to study the asymptotic behavior of permutation based test in these situations.

Appendix

Lemma 1 (Weyl's inequality)**.** *Let H and P be two symmetric matrices and $M = H + P$. If $j + k - n \geq i \geq r + s - 1$, we have*

$$\lambda_j(H) + \lambda_k(P) \leq \lambda_i(M) \leq \lambda_r(H) + \lambda_s(P).$$

Corollary 2. *Let H and P be two symmetric matrices and $M = H + P$. If* $\text{rank}(P) < k$*, then*

$$\lambda_k(M) \leq \lambda_1(H).$$

Lemma 2 (Convergence rate of principal space estimation)**.** *Under the Assumption* **??**-**??***, we have*

$$E\|\hat{V}\hat{V}^T - VV^T\|_F^2 = O(\frac{p}{p^\beta n}).$$

Proof. Theorem 5 of Cai et al. (2013) asserts that sample principal subspace $\hat{V}\hat{V}^T$ is a minimax rate estimator of VV^T, namely, it reaches the minimax convergence rate

$$E\|\hat{V}\hat{V}^T - VV^T\|_F^2 \asymp r \wedge (p-r) \wedge \frac{r(p-r)}{(n_1+n_2-2)h(\lambda)} \tag{9}$$

as long as the right hand side tends to 0. Here $h(\lambda) = \frac{\lambda^2}{\lambda+1}$. In model of Assumption 1, r is fixed, $\lambda = cp^\beta$. It's obvious that the right hand side of (9) is of order $p^{1-\beta}/n$. We note that it is assumed $\beta \geq \frac{1}{2}$ in Assumption **??**, together with $\sqrt{p}/n \to 0$ we have $p^{1-\beta}/n \to 0$. Hence $\hat{V}\hat{V}^T$ reaches the convergence rate. $\square$

Lemma 3 (Bai-Yin's law)**.** *Suppose* $B_n = \frac{1}{q} Z Z^T$ *where* Z *is* $p \times q$ *random matrix composed of i.i.d. random variables with zero mean, unit variance and finite fourth moment. As* $q \to \infty$ *and* $\frac{p}{q} \to c \in [0, \infty)$*, the largest and smallest non-zero eigenvalues of* B_n *converge almost surely to* $(1+\sqrt{c})^2$ *and* $(1-\sqrt{c})^2$*, respectively.*

Remark 2. Lemma 3 is known as the Bai-Yin's law (Bai and Yin (1993)). As in Remark 1 of Bai and Yin (1993), the smallest non-zero eigenvalue is the $p - q + 1$ smallest eigenvalue of B for $c > 1$.

Corollary 3. *Suppose that* W_n *is a* $p \times p$ *matrix distributed as* $\mathrm{Wishart}_p(n, I_p)$*. Then as* $n \to \infty$*,*

$$\lambda_1(W_n) = O_P(\max(n, p)).$$

Proof. Since $[0, +\infty]$ is compact, for every subsequance $\{n_k\}$ of $\{n\}$, there is a further subsequance $\{n_{k_l}\}$ along which $p/n \to c \in [0, +\infty]$.

If $c \in [0, +\infty)$, by Lemma 3, we have that

$$\frac{\lambda_1(W_{n_{k_l}})}{n_{k_l}} \xrightarrow{P} (1+c)^2.$$

Hence the conclusion holds along this subsequance. If $c = +\infty$, suppose $W_n = Z_n Z_n^T$ where Z_n is a $p \times n$ matrix with all elements distributed as $N(0, 1)$. Then

$$\frac{\lambda_1(W_{n_{k_l}})}{p} = \frac{Z_{n_{k_l}}^T Z_{n_{k_l}}}{p} \xrightarrow{P} 1,$$

by Lemma 3, which proves the conclusion along the subsequance. Now the conclusion holds by a standard subsequance argument. □

Proof of Proposition 1. Let $\lambda_1(A_n) \geq \cdots \geq \lambda_{k_n}(A_n)$ be the eigenvalues of A_n, then

$$\frac{Y_n^T A_n Y_n - \mathrm{E} Y_n^T A_n Y_n}{[\mathrm{Var}(Y_n^T A_n Y_n)]^{1/2}} = \sum_{i=1}^{k_n} \frac{\lambda_i(A_n)}{[2\mathrm{tr}(A_n^2)]^{1/2}} (Z_{ni}^2 - 1), \tag{10}$$

where Z_{ni}'s ($i = 1, \ldots, k_n$) are independent standard normal random variables.

If 4 holds, then

$$\begin{aligned}
&\sum_{i=1}^{k_n} \mathrm{E}\Big[\frac{\lambda_i^2(A_n)}{2\mathrm{tr}(A_n^2)}(Z_{ni}^2-1)^2\Big\{\frac{\lambda_i^2(A_n)}{2\mathrm{tr}(A_n^2)}(Z_{ni}^2-1)^2 \geq \epsilon\Big\}\Big]\\
\leq&\sum_{i=1}^{k_n} \frac{\lambda_i^2(A_n)}{2\mathrm{tr}(A_n^2)}\mathrm{E}\Big[(Z_{n1}^2-1)^2\Big\{\frac{\lambda_{\max}(A_n^2)}{2\mathrm{tr}(A_n^2)}(Z_{n1}^2-1)^2 \geq \epsilon\Big\}\Big]\\
=&\frac{1}{2}\mathrm{E}\Big[(Z_{n1}^2-1)^2\Big\{\frac{\lambda_{\max}(A_n^2)}{2\mathrm{tr}(A_n^2)}(Z_{n1}^2-1)^2 \geq \epsilon\Big\}\Big] \to 0.
\end{aligned}$$

Hence 3 follows by Lindeberg's central limit theorem.

Conversely, if 3 holds, we will prove that there is a subsequence of $\{n\}$ along which 4 holds. Then 4 will hold by a standard contradiction argument.

Denote $c_{ni} = \lambda_i(A_n)/\big[2\mathrm{tr}(A_n^2)\big]^{1/2}$ $(i = 1, \ldots, k_n)$, we have $c_{ni} \in [-\sqrt{2}/2, \sqrt{2}/2]$. Since 3 holds, the characteristic function of $\sum_{i=1}^{k_n} c_{ni}(Z_{ni}^2 - 1)$ converges to $\exp(-t^2/2)$ for every t. For $t \in (-1, 1)$, we have

$$\begin{aligned}
&\log \mathrm{E} \exp\big(it\sum_{j=1}^{k_n} c_{nj}(Z_{nj}^2-1)\big) = -i(\sum_{j=1}^{k_n} c_{nj})t - \frac{1}{2}\sum_{j=1}^{k_n}\log(1-i2c_{nj}t)\\
=&-i(\sum_{j=1}^{k_n} c_{nj})t + \frac{1}{2}\sum_{j=1}^{k_n}\sum_{l=1}^{+\infty}\frac{1}{l}(i2c_{nj}t)^l = -i(\sum_{j=1}^{k_n} c_{nj})t + \frac{1}{2}\sum_{l=1}^{+\infty}\Big[\sum_{j=1}^{k_n}(c_{nj})^l\Big]\frac{1}{l}(i2t)^l\\
=&-\frac{1}{2}t^2 + \frac{1}{2}\sum_{l=3}^{+\infty}\Big[\sum_{j=1}^{k_n}(c_{nj})^l\Big]\frac{1}{l}(i2t)^l.
\end{aligned}$$

Denote $b_{nl} = \sum_{j=1}^{k_n}(c_{nj})^l$, $n = 1, 2, \cdots$ and $l = 3, 4, \cdots$. For $l \geq 3$, $\big|\sum_{j=1}^{k_n}(c_{nj})^l\big| \leq \big|\sum_{j=1}^{k_n}(c_{nj})^2\big| = 1/2$. By Helly's selection theorem, there's a subsequence of $\{n\}$ along which $\lim_{n\to\infty} b_{nl} = b_l$ exists for every l. Apply dominated convergence theorem to this subsequence we have $\log \mathrm{E}\exp\big(it\sum_{j=1}^{k_n} c_{nj}(Z_{nj}^2-1)\big) \to -\frac{1}{2}t^2 + \frac{1}{2}\sum_{l=3}^{+\infty} b_l \frac{1}{l}(i2t)^l$ for $t \in (-1/2, 1/2)$. By the property of power series, we have $b_l = 0$ for $l \geq 3$. Then 4 follows by noting that $b_{n4} \geq \max_j (c_{nj})^4$. $\square$

The rest of the Appendix is devoted to the proof of propositions and theorems in the paper.

Proof of Proposition 2. Let $Y_{k,i} = \tilde{V}^T(X_{k,i} - \mu_k)$, $i = 1, \ldots, n_k$, $k = 1, 2$. Then $Y_{k,i} \sim N(\tilde{V}^T\mu_k, \sigma^2 I_{p-r})$. Let $\bar{Y}_1$ and $\bar{Y}_2$ be the sample means of $\{Y_{1,i}\}_{i=1}^{n_1}$

and $\{Y_{2,i}\}_{i=1}^{n_2}$ respectively. Then

$$\begin{aligned}
&\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2 = \|\tilde{V}^T(\mu_1 - \mu_2) + (\bar{Y}_1 - \bar{Y}_2)\|^2 \\
=&\|\tilde{V}^T(\mu_1 - \mu_2)\|^2 + \|\bar{Y}_1 - \bar{Y}_2\|^2 + 2(\mu_1 - \mu_2)^T\tilde{V}(\bar{Y}_1 - \bar{Y}_2) \\
=&\|\tilde{V}^T(\mu_1 - \mu_2)\|^2 + \|\bar{Y}_1 - \bar{Y}_2\|^2 + o_P(\frac{\sqrt{p}}{n}).
\end{aligned} \tag{11}$$

The last equality holds since

$$\begin{aligned}
&2(\mu_1 - \mu_2)^T\tilde{V}(\bar{Y}_1 - \bar{Y}_2) \sim N(0, 4\sigma^2\tau\|\tilde{V}^T(\mu_1 - \mu_2)\|^2) \\
=&O_P(\sqrt{\tau}\|\tilde{V}^T(\mu_1 - \mu_2)\|) = o_P(\frac{\sqrt{p}}{n}).
\end{aligned}$$

For $k = 1, 2$, we have

$$\begin{aligned}
&\frac{1}{n_k}\mathrm{tr}(\tilde{V}^T S_k \tilde{V}) \sim \frac{\sigma^2}{n_k(n_k - 1)}\chi^2_{(p-r)(n_k-1)} \\
=&\sigma^2\frac{p - r}{n_k}(1 + O_P(\frac{1}{\sqrt{(p - r)(n_k - 1)}})),
\end{aligned}$$

where the last equality comes from central limit theorem. It follows that

$$\frac{1}{n_1}\mathrm{tr}(\tilde{V}^T S_1 \tilde{V}) + \frac{1}{n_2}\mathrm{tr}(\tilde{V}^T S_2 \tilde{V}) = \sigma^2\tau(p - r) + o_P(\frac{\sqrt{p}}{n}). \tag{12}$$

Equation (11) and (12) imply that

$$\frac{T_1 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2}{\sigma^2\sqrt{2\tau^2 p}} = \frac{\|\bar{Y}_1 - \bar{Y}_2\|^2 - \sigma^2\tau(p - r)}{\sigma^2\sqrt{2\tau^2 p}} + o_P(1). \tag{13}$$

Since $\|\tilde{V}^T(\bar{Z}_1 - \bar{Z}_2)\|^2 \sim \sigma^2\tau\chi^2_{p-r}$, the proposition follows by central limit theorem. □

Proof of Proposition 3. Note that $(n-2)S \sim \mathrm{Wishart}_p(n-2, \Sigma)$. Denote by $\Sigma = UEU^T$ the spectral decomposition of Σ, where $U = (V, \tilde{V})$ is an orthogonal matrix and $E = \mathrm{diag}(\lambda_1 + \sigma^2, \ldots, \lambda_r + \sigma^2, \sigma^2, \ldots, \sigma^2)$. Let Z be a $p \times (n - 2)$ random matrix with all elements i.i.d. distributed as $N(0, 1)$, then

$$S \sim \frac{1}{n - 2}UE^{1/2}ZZ^T E^{1/2}U^T.$$

Thus,

$$\hat{\sigma}^2 \sim \frac{1}{(p - r)(n - 2)}\sum_{i=r+1}^{p}\lambda_i(UE^{1/2}ZZ^T E^{1/2}U^T) = \frac{1}{(p - r)(n - 2)}\sum_{i=r+1}^{n-2}\lambda_i(Z^T EZ).$$

Denote $Z = (Z_{(1)}^T, Z_{(2)}^T)^T$, where $Z_{(1)}$ and $Z_{(2)}$ are the first r rows and last $p-r$ rows of Z. We have

$$Z^T EZ = Z_{(1)}^T(\Lambda + \sigma^2 I_r)Z_{(1)} + \sigma^2 Z_{(2)}^T Z_{(2)}.$$

The first term is of rank r. By Weyl's inequality, we have

$$\sigma^2\lambda_i(Z_{(2)}^T Z_{(2)}) \leq \lambda_i(Z^T EZ) \leq \sigma^2\lambda_{i-r}(Z_{(2)}^T Z_{(2)}), \quad i = r+1, \ldots, n-2.$$

Thus,

$$\sigma^2 \sum_{i=r+1}^{n-2} \lambda_i(Z_{(2)}^T Z_{(2)}) \leq \sum_{i=r+1}^{n-2} \lambda_i(Z^T EZ) \leq \sigma^2 \sum_{i=1}^{n-r-2} \lambda_i(Z_{(2)}^T Z_{(2)}).$$

It follows that

$$\begin{aligned}
&\Big|\frac{1}{(p-r)(n-2)} \sum_{i=r+1}^{n-2} \lambda_i(Z^T EZ) - \frac{1}{(p-r)(n-2)}\sigma^2 \sum_{i=1}^{n-2} \lambda_i(Z_{(2)}^T Z_{(2)})\Big| \\
\leq& r\sigma^2 \frac{1}{(p-r)(n-2)}\lambda_1(Z_{(2)}^T Z_{(2)}).
\end{aligned}$$

By Corollary 3, $\lambda_1(Z_{(2)}^T Z_{(2)}) = O_P(\max(n,p))$. Thus,

$$\begin{aligned}
&\frac{1}{(p-r)(n-2)} \sum_{i=r+1}^{n-2} \lambda_i(Z^T EZ) \\
=&\frac{1}{(p-r)(n-2)}\sigma^2 \sum_{i=1}^{n-2} \lambda_i(Z_{(2)}^T Z_{(2)}) + O_P(\frac{\max(n,p)}{np}) \\
=&\frac{1}{(p-r)(n-2)}\sigma^2 \mathrm{tr}(Z_{(2)}^T Z_{(2)}) + O_P(\frac{\max(n,p)}{np}) \\
=&\sigma^2 + O_P(\frac{1}{\sqrt{np}}) + O_P(\frac{\max(n,p)}{np}).
\end{aligned}$$

The last equality comes from central limit theorem. The theorem follows by noting that

$$O_P(\frac{1}{\sqrt{np}}) = O_P(\frac{\sqrt{np}}{np}) = O_P(\frac{\max(n,p)}{np}).$$

□

Proof of Theorem 2. Note that $\mathrm{tr}(\hat{\hat{V}}^T S_k \hat{\hat{V}}) = \sum_{i=r+1}^{p} \lambda_i(S_k)$, $k = 1, 2$. Similar to Proposition 3, we have $\mathrm{tr}(\hat{\hat{V}}^T S_k \hat{\hat{V}}) = (p-r)\sigma^2 + O_P(\max(n,p)/n)$,

$k = 1, 2$. Then

$$\frac{T_2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2}{\sigma^2\sqrt{2\tau^2 p}} = \frac{\|\hat{\tilde{V}}^T(\bar{X}_1 - \bar{X}_2)\|^2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 - \sigma^2\tau(p-r)}{\sigma^2\sqrt{2\tau^2 p}} + O_P(\frac{\max(n,p)}{n\sqrt{p}}).$$

By Assumption 2, $n^{-1}p^{-1/2}\max(n,p) = \max(p^{-1/2}, p^{1/2}/n) \to 0$. Note that

$$\begin{aligned}
&\frac{\|\hat{\tilde{V}}^T(\bar{X}_1 - \bar{X}_2)\|^2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 - \sigma^2\tau(p-r)}{\sigma^2\sqrt{2\tau^2 p}} \\
=&\frac{1}{\sigma^2\sqrt{2\tau^2 p}}\Big(\|\hat{\tilde{V}}^T\big((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)\big)\|^2 - \sigma^2\tau(p-r) + \\
&2(\mu_1 - \mu_2)^T\hat{\tilde{V}}\hat{\tilde{V}}^T\big((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)\big) + \|\hat{\tilde{V}}^T(\mu_1 - \mu_2)\|^2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2\Big).
\end{aligned}$$

Let

$$\begin{aligned}
P_1 &= \|\hat{\tilde{V}}^T\big((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)\big)\|^2 - \sigma^2\tau(p-r), \\
P_2 &= 2(\mu_1 - \mu_2)^T\hat{\tilde{V}}\hat{\tilde{V}}^T\big((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)\big), \\
P_3 &= \|\hat{\tilde{V}}^T(\mu_1 - \mu_2)\|^2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2.
\end{aligned}$$

To prove the theorem, it suffices to show that

$$\frac{P_1}{\sigma^2\sqrt{2\tau^2 p}} \xrightarrow{\mathcal{L}} N(0,1), \quad \frac{P_2}{\sigma^2\sqrt{2\tau^2 p}} \xrightarrow{P} 0 \quad \text{and} \quad \frac{P_3}{\sigma^2\sqrt{2\tau^2 p}} \xrightarrow{P} 0.$$

We first deal with P_2. To proves the convergence in probability, we only need to prove the convergence in L^2. Note that $\bar{X}_1$, $\bar{X}_2$, and S are mutually independent and $\hat{\tilde{V}}\hat{\tilde{V}}^T$ only depends on S. Thus

$$\begin{aligned}
&\mathrm{E}P_2^2 = \mathrm{E}[\mathrm{E}P_2^2|S] = 4\tau\mathrm{E}[(\mu_1 - \mu_2)^T\hat{\tilde{V}}\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}}\hat{\tilde{V}}^T(\mu_1 - \mu_2)] \\
\leq&4\tau\mathrm{E}[\lambda_1(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})(\mu_1 - \mu_2)^T\hat{\tilde{V}}\hat{\tilde{V}}^T(\mu_1 - \mu_2)] \leq 4\tau\|\mu_1 - \mu_2\|^2\mathrm{E}[\lambda_1(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})] \\
=&O(\frac{\sqrt{p}}{n^2})\mathrm{E}[\lambda_1(\hat{\tilde{V}}^T(V\Lambda V^T + \sigma^2 I_p)\hat{\tilde{V}})] \leq O(\frac{\sqrt{p}}{n^2})\big(\kappa p^\beta\mathrm{E}[\lambda_1(\hat{\tilde{V}}^T VV^T\hat{\tilde{V}})] + \sigma^2\big).
\end{aligned}$$

By the relationship

$$\lambda_1(\hat{\tilde{V}}^T VV^T\hat{\tilde{V}}) \leq \mathrm{tr}(\hat{\tilde{V}}^T VV^T\hat{\tilde{V}}) = \frac{1}{2}\|VV^T - \hat{V}\hat{V}^T\|_F^2$$

and Lemma 2, we have that

$$\mathrm{E}P_2^2 = O(\frac{\sqrt{p}}{n^2})\big(O(\frac{p}{n}) + \sigma^2\big) = o(\frac{p}{n^2}).$$

Next we deal with P_3. To prove the convergence in probability, we prove the convergence in L^1.

$$
\begin{aligned}
&\mathrm{E}|P_3| = \mathrm{E}\big|(\mu_1-\mu_2)^T(\hat{\tilde{V}}\hat{\tilde{V}}^T - \tilde{V}\tilde{V}^T)(\mu_1-\mu_2)\big| \le \|\mu_1-\mu_2\|^2\mathrm{E}\|\hat{\tilde{V}}\hat{\tilde{V}}^T - \tilde{V}\tilde{V}^T\| \\
=&\|\mu_1-\mu_2\|^2\mathrm{E}\|\hat{V}\hat{V}^T - VV^T\| \le \|\mu_1-\mu_2\|^2\sqrt{\mathrm{E}\|\hat{V}\hat{V}^T - VV^T\|^2} \\
\le&\|\mu_1-\mu_2\|^2\sqrt{\mathrm{E}\|\hat{V}\hat{V}^T - VV^T\|_F^2} = O(\frac{\sqrt{p}}{n})\sqrt{O(\frac{p}{p^\beta n})} = o(\frac{\sqrt{p}}{n}).
\end{aligned}
$$

Now we prove the asymptotic normality of P_1. To make clear the sense of convergence, we need a metric for weak convergence. For two distribution function F and G, the Levy metric ρ of F and G is defined as

$$
\rho(F,G) = \inf\{\epsilon : F(x-\epsilon) - \epsilon \le G(x) \le F(x+\epsilon) + \epsilon \quad \text{for all } x\}.
$$

It's well known that $\rho(F_n, F) \to 0$ if and only if $F_n \xrightarrow{\mathcal{L}} F$.

The conditional distribution of $\hat{\tilde{V}}^T\big((\bar{X}_1-\mu_1)-(\bar{X}_2-\mu_2)\big)$ given S is $N(0, \tau\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})$. It can be seen that

$$
\tau^{-1}\big\|\hat{\tilde{V}}^T\big((\bar{X}_1-\mu_1)-(\bar{X}_2-\mu_2)\big)\big\|^2 \sim \sum_{i=1}^{p-r}\lambda_i(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})\xi_i^2,
$$

where $\{\xi_i\}_{i=1}^{p-r}$ are i.i.d. standard normal random variables which are independent of $\hat{\tilde{V}}$. Note that

$$
\lambda_1(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}}) \le \frac{1}{2}\kappa p^\beta\|VV^T - \hat{V}\hat{V}^T\|_F^2 + \sigma^2.
$$

Hence $\lambda_i(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}}) = O_P(p/n+1)$, $i = 1,\ldots,r$. Moreover, by Weyl's inequality, $\lambda_i(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}}) = \sigma^2$, $i = r+1,\ldots,p-r$. Therefore

$$
\operatorname{tr}(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})^2 = \big(\frac{p}{n}+1\big)^2 O_P(1) + (p-2r)\sigma^4 = p\sigma^4(1+o_P(1)). \tag{14}
$$

It follows that

$$
\frac{\lambda_1^2(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})}{\operatorname{tr}(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})^2} = O_P\Big(\frac{(p/n+1)^2}{p}\Big) = o_P(1). \tag{15}
$$

Then for every subsequence of $\{n\}$, there's a further subsequence along which (15) holds almost surely. By Lemma 1, for every subsequence of $\{n\}$, there's a further subsequence along which we have

$$\rho\Big(\mathcal{L}\Big(\frac{\|\hat{\tilde{V}}^T\big((\bar{X}_1-\mu_1)-(\bar{X}_2-\mu_2)\big)\|^2-\tau\mathrm{tr}(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})}{\sqrt{2\tau^2\mathrm{tr}(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})^2}}\Big|S\Big),N(0,1)\Big)\xrightarrow{a.s.}0. \quad (16)$$

It means that (16) tends to 0 in probability. It can be seen that the weak convergence also holds unconditionally.

$$\frac{\|\hat{\tilde{V}}^T\big((\bar{X}_1-\mu_1)-(\bar{X}_2-\mu_2)\big)\|^2-\tau\mathrm{tr}(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})}{\sqrt{2\tau^2\mathrm{tr}(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})^2}}\xrightarrow{\mathcal{L}}N(0,1).$$

Similar to (14) we have

$$\mathrm{tr}(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})=(p-r)\sigma^2\big(1+O_P\big(\frac{1}{n}+\frac{1}{p}\big)\big). \quad (17)$$

By (14), (17) and Slutsky's theorem,

$$\frac{\|\hat{\tilde{V}}^T\big((\bar{X}_1-\mu_1)-(\bar{X}_2-\mu_2)\big)\|^2-\sigma^2\tau(p-r)}{\sigma^2\sqrt{2\tau^2p}}\xrightarrow{\mathcal{L}}N(0,1).$$

Now the desired asymptotic properties of P_1, P_2 and P_3 are established, the theorem follows. □

Proof of Theorem 3. Since we have assumed $p/n^2\to\infty$, throughout the proof, we assume $p>n$. From the proof of Theorem 2 we know that

$$\tau^{-1}\big\|\hat{\tilde{V}}^T(\bar{X}_1-\bar{X}_2)\big\|^2\sim\sum_{i=1}^{p-r}\lambda_i(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})\xi_i^2,$$

where $\{\xi_i\}_{i=1}^{p-r}$ are i.i.d. standard normal random variables which are independent of $\hat{\tilde{V}}$. Note that $\lambda_i(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})=\kappa p^\beta\lambda_i(\hat{\tilde{V}}^TVV^T\hat{\tilde{V}})+\sigma^2$, $i=1,\dots,p-r$. The rank of $\hat{\tilde{V}}^TVV^T\hat{\tilde{V}}$ is r. For $i=1,\dots,r$,

$$\begin{aligned}\lambda_i(\hat{\tilde{V}}^TVV^T\hat{\tilde{V}})&=\lambda_i(V^T\hat{\tilde{V}}\hat{\tilde{V}}^TV)=\lambda_i(V^T(I_p-\hat{V}\hat{V}^T)V)=\lambda_i(I_r-V^T\hat{V}\hat{V}^TV)\\&=\lambda_i(I_r-\hat{V}^TVV^T\hat{V})=\lambda_i(\hat{V}^T(I_p-VV^T)\hat{V})=\lambda_i(\hat{V}^T\tilde{V}\tilde{V}^T\hat{V})=\lambda_i(\tilde{V}^T\hat{V}\hat{V}^T\tilde{V}).\end{aligned}$$

It follows that

$$\sum_{i=1}^{p-r}\lambda_i(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})\xi_i^2=\kappa p^\beta\sum_{i=1}^{r}\lambda_i(\tilde{V}^T\hat{V}\hat{V}^T\tilde{V})\xi_i^2+\sigma^2\sum_{i=1}^{p-r}\xi_i^2. \quad (18)$$

For current purpose, we need the exact limit of $\tilde{V}^T\hat{V}\hat{V}^T\tilde{V}$ which, to the best of our knowledge, is not provided by existing PCA theory in the current context.

Denote by $\Sigma = UEU^T$ the spectral decomposition of Σ, where $U = (V, \tilde{V})$ and $E = \text{diag}(\lambda_1 + \sigma^2, \ldots, \lambda_r + \sigma^2, \sigma^2, \ldots, \sigma^2)$. Denote by $S = \hat{U}\hat{E}\hat{U}^T$ the spectral decomposition of S, where $\hat{U} = (\hat{V}, \hat{\tilde{V}})$ and $\hat{E} = \text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_p)$. To study the property of $\tilde{V}^T\hat{V}\hat{V}^T\tilde{V}$, we need to first consider $\hat{\lambda}_i$, the ith eigenvalue of S, $i = 1, \ldots, r$.

Let Z be a $p \times (n-2)$ random matrix with all elements i.i.d. distributed as $N(0, 1)$. Denote $Z = (Z_{(1)}^T, Z_{(2)}^T)^T$, where $Z_{(1)}$ and $Z_{(2)}$ are the first r rows and last $p - r$ rows of Z. We have

$$S \sim \frac{1}{n-2} U E^{1/2} Z Z^T E^{1/2} U^T. \qquad (19)$$

It can be seen that $\hat{\lambda}_i = \lambda_i(S) \sim (n-2)^{-1}\lambda_i(Z^T E Z)$, $i = 1, \ldots, r$. The equality $Z^T E Z = (\kappa p^\beta + \sigma^2) Z_{(1)}^T Z_{(1)} + \sigma^2 Z_{(2)}^T Z_{(2)}$ leads to

$$(\kappa p^\beta + \sigma^2)\lambda_i(Z_{(1)}^T Z_{(1)}) + \lambda_{\min}(Z_{(2)}^T Z_{(2)}) \le \lambda_i(Z^T E Z) \le (\kappa p^\beta + \sigma^2)\lambda_i(Z_{(1)}^T Z_{(1)}) + \lambda_{\max}(Z_{(2)}^T Z_{(2)}),$$

$i = 1, \ldots, r$. By Bai-Yin's law, we have

$$\lambda_{\max}(Z_{(2)}^T Z_{(2)}) = p(1 + o_P(1)), \quad \lambda_{\min}(Z_{(2)}^T Z_{(2)}) = p(1 + o_P(1)) \qquad (20)$$

By law of large numbers, $\lambda_i(Z_{(1)}^T Z_{(1)}) = \lambda_i(Z_{(1)} Z_{(1)}^T) = n(1 + o_P(1))$, $i = 1, \ldots, r$. Combining the above arguments leads to

$$\hat{\lambda}_i = \left(\kappa p^\beta + \frac{p}{n}\sigma^2\right)(1 + o_P(1)), \quad i = 1, \ldots, r. \qquad (21)$$

Next we deal with the eigenvectors of S, note that (19) implies $\sigma^{-2}\tilde{V}^T S \tilde{V} \sim (n-2)^{-1} Z_{(2)} Z_{(2)}^T$. Let $\hat{E}_1 = (\hat{\lambda}_1, \ldots, \hat{\lambda}_r)$, $\hat{E}_2 = (\hat{\lambda}_{r+1}, \ldots, \hat{\lambda}_p)$, then $S = \hat{V}\hat{E}_1\hat{V}^T + \hat{\tilde{V}}\hat{E}_2\hat{\tilde{V}}^T$. We have

$$\tilde{V}^T S \tilde{V} = \tilde{V}^T \hat{V}\hat{E}_1\hat{V}^T\tilde{V} + \tilde{V}^T\hat{\tilde{V}}\hat{E}_2\hat{\tilde{V}}^T\tilde{V} \sim \frac{\sigma^2}{n-2} Z_{(2)} Z_{(2)}^T.$$

The ranks of $\tilde{V}^T\hat{V}\hat{E}_1\hat{V}^T\tilde{V}$ and $\tilde{V}^T\hat{\tilde{V}}\hat{E}_2\hat{\tilde{V}}^T\tilde{V}$ are r and $n - 2 - r$ respectively. By Weyl's inequality, we have

$$\lambda_{n-2}(\tilde{V}^T S \tilde{V}) \le \lambda_r(\tilde{V}^T\hat{V}\hat{E}_1\hat{V}^T\tilde{V}) \le \lambda_1(\tilde{V}^T\hat{V}\hat{E}_1\hat{V}^T\tilde{V}) \le \lambda_1(\tilde{V}^T S \tilde{V}).$$

We apply (20) to get

$$\lambda_1(\tilde{V}^T\hat{V}\hat{E}_1\hat{V}^T\tilde{V}) = \frac{p}{n}\sigma^2(1+o_P(1)), \quad \lambda_r(\tilde{V}^T\hat{V}\hat{E}_1\hat{V}^T\tilde{V}) = \frac{p}{n}\sigma^2(1+o_P(1)).$$

This, together with inequality $\hat{\lambda}_1^{-1}\tilde{V}^T\hat{V}\hat{E}_1\hat{V}^T\tilde{V} \le \tilde{V}^T\hat{V}\hat{V}^T\tilde{V} \le \hat{\lambda}_r^{-1}\tilde{V}^T\hat{V}\hat{E}_1\hat{V}^T\tilde{V}$ and (21), implies that

$$\lambda_1(\tilde{V}^T\hat{V}\hat{V}^T\tilde{V}) = \frac{p\sigma^2}{\kappa np^\beta + p\sigma^2}(1+o_P(1)), \quad \lambda_r(\tilde{V}^T\hat{V}\hat{V}^T\tilde{V}) = \frac{p\sigma^2}{\kappa np^\beta + p\sigma^2}(1+o_P(1)).$$

Now come back to (18),

$$\begin{aligned}\sum_{i=1}^{p-r}\lambda_i(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})\xi_i^2 - (p-r)\sigma^2 &= \kappa p^\beta\sum_{i=1}^{r}\lambda_i(\tilde{V}^T\hat{V}\hat{V}^T\tilde{V})\xi_i^2 + \sigma^2\sum_{i=1}^{p-r}(\xi_i^2-1)\\ &= (1+o_P(1))\frac{\kappa p^{\beta+1}\sigma^2}{\kappa np^\beta + p\sigma^2}\sum_{i=1}^{r}\xi_i^2 + O_P(\sqrt{p}).\end{aligned}$$

Since we have assumed $p/n^2 \to \infty$ and $\beta > 1/2$, we have

$$\sqrt{p} = o\Big(\frac{\kappa p^{\beta+1}\sigma^2}{\kappa np^\beta + p\sigma^2}\Big).$$

Then

$$\frac{\kappa np^\beta + p\sigma^2}{\kappa p^{\beta+1}\sigma^2}\Big(\sum_{i=1}^{p-r}\lambda_i(\hat{\tilde{V}}^T\Sigma\hat{\tilde{V}})\xi_i^2 - (p-r)\sigma^2\Big) = \sum_{i=1}^{r}\xi_i^2 + o_P(1),$$

which proves the theorem.

□

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