

# High-dimensional two-sample test under spiked covariance

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## Abstract

This paper considers testing the means of two  $p$ -variate normal samples in high dimensional setting. The covariance matrices are assumed to be spiked, which often arises in practice. We propose a new test procedure through projection on the orthogonal complement of principal space. The asymptotic normality of the new test statistic is proved and the power function of the test is given. Theoretical and simulation results show that the new test outperforms existing methods substantially when the covariance matrices are spiked.

*Keywords:* high dimension, mean test, orthogonal complement of principal space, spiked covariance

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## 1. Introduction

Suppose  $X_{k,1}, \dots, X_{k,n_k}$  are independent identically distributed (i.i.d.)  $p$ -dimensional normal random vectors with unknown mean vector  $\mu_k$  and covariance matrix  $\Sigma$ ,  $k = 1, 2$ . We consider the hypothesis testing problem

$$H_0 : \mu_1 = \mu_2 \quad \text{vs.} \quad H_1 : \mu_1 \neq \mu_2. \quad (1)$$

In this paper, the high dimensional setting is adopted, i.e., the dimension  $p$  varies as  $n$  increase, where  $n = n_1 + n_2$  is the total sample size. Testing hypotheses (1)

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is important in many fields, including biology, finance and economics.

A classical test for hypotheses (1) is Hotelling's  $T^2$  test. Hotelling's test statistic is  $(\bar{X}_1 - \bar{X}_2)^T S^{-1}(\bar{X}_1 - \bar{X}_2)$ , where  $\bar{X}_k = n_k^{-1} \sum_{i=1}^{n_k} X_{k,i}$  is the mean vector of sample  $k$ ,  $k = 1, 2$ , and  $S = (n - 2)^{-1} \sum_{k=1}^2 \sum_{i=1}^{n_k} (X_{k,i} - \bar{X}_k)(X_{k,i} - \bar{X}_k)^T$  is the pooled sample covariance matrix. However, Hotelling's test is not defined when  $p \geq n - 1$ . Moreover, Bai and Saranadasa (1996) showed that even if  $p < n - 1$ , Hotelling's test suffers from low power when  $p$  is comparable to  $n$ . Perhaps, the main reason for low power of Hotelling's test is due to that  $S$  is a poor estimator of  $\Sigma$  when  $p$  is large compared with  $n$ . See Chen and Qin (2010) and the references therein. In high dimensional setting, many test statistics are based on  $(\bar{X}_1 - \bar{X}_2)^T A(\bar{X}_1 - \bar{X}_2)$  for a positive definite matrix  $A$ . Bai and Saranadasa (1996) proposed a test based on

$$T_{BS} = \|\bar{X}_1 - \bar{X}_2\|^2 - \left(\frac{1}{n_1} + \frac{1}{n_2}\right) \text{tr} S,$$

which is an unbiased estimator of  $\|\mu_1 - \mu_2\|^2$ . Chen and Qin (2010) modified  $T_{BS}$  by removing terms  $\sum_{i=1}^{n_k} X_{ki}^T X_{ki}$ ,  $k = 1, 2$  and proposed a test based on

$$\begin{aligned} T_{CQ} &= \frac{\sum_{i \neq j}^{n_1} X_{1i}^T X_{1j}}{n_1(n_1 - 1)} + \frac{\sum_{i \neq j}^{n_2} X_{2i}^T X_{2j}}{n_2(n_2 - 1)} - 2 \frac{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} X_{1i}^T X_{2j}}{n_1 n_2} \\ &= \|\bar{X}_1 - \bar{X}_2\|^2 - \frac{1}{n_1} \text{tr} S_1 - \frac{1}{n_2} \text{tr} S_2, \end{aligned}$$

where  $S_k = (n_k - 1)^{-1} \sum_{i=1}^{n_k} (X_{k,i} - \bar{X}_k)(X_{k,i} - \bar{X}_k)^T$ ,  $k = 1, 2$ . Statistic  $T_{CQ}$  is also an unbiased estimator of  $\|\mu_1 - \mu_2\|^2$ . Srivastava and Du (2008) proposed a test based on

$$T_{SD} = (\bar{X}_1 - \bar{X}_2)^T [\text{diag}(S)]^{-1} (\bar{X}_1 - \bar{X}_2),$$

where  $\text{diag}(A)$  is a diagonal matrix with the same diagonal elements as  $A$ 's.

As Ma et al. (2015) pointed out, however, the asymptotic properties of these test procedures may not be valid if strong correlations exist. For example, the condition

$$\text{tr}(\Sigma^4) = o(\text{tr}^2(\Sigma^2)) \quad (2)$$

adopted by Chen and Qin (2010) is violated when  $\Sigma$  has uniform correlation structure, that is,  $\Sigma = (1 - \rho)I_p + \rho \mathbf{1}_p \mathbf{1}_p^T$  where  $0 < \rho < 1$ ,  $I_p$  is the  $p$  dimensional

identity matrix and  $\mathbf{1}_p$  is the  $p$  dimensional vector with elements 1. In this case,  $\Sigma$  has eigenvalues  $1+\rho(p-1)$  and  $1-\rho$  with multiplicities 1 and  $p-1$  respectively. Then (2) is violated since

$$\frac{\text{tr}(\Sigma^4)}{\text{tr}^2(\Sigma^2)} = \frac{(1+\rho(p-1))^4 + (1-\rho)^4(p-1)}{[(1+\rho(p-1))^2 + (1-\rho)^2(p-1)]^2} \rightarrow 1$$

as  $p \rightarrow \infty$ . Under uniform correlation structure, the leading eigenvalue of  $\Sigma$  is significantly larger than the rest of eigenvalues. This is a special case of the spiked covariance model

$$\Sigma = V\Lambda V^T + \sigma^2 I_p, \quad (3)$$

where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_r)$ ,  $\lambda_1 \geq \dots \geq \lambda_r > 0$ ,  $r$  is a known fixed number,  $V$  is a  $p \times r$  orthonormal matrix and  $\sigma^2 > 0$ . The spiked covariance model (3) is adopted by many theoretical studies, see Cai et al. (2013), Birnbaum et al. (2013), Passemier et al. (2017) and the references therein. The spiked covariance arises when variables are strongly correlated and the correlations are determined by a small number of factors.

Strong correlations between variables do exist in practice. In gene expression analysis, genes are correlated due to genetic regulatory networks (see Thulin (2014)). Chen et al. (2011) pointed out that in terms of pathway analysis in proteomic studies, test level can not be guaranteed if correlations are incorrectly assumed to be absent. As Ma et al. (2015) argued, there're strong correlations between different stock returns since they are all affected by the market index. In section 2, we derived the asymptotic distribution of  $T_{CQ}$  under spiked covariance model. We find that incorrectly assuming the absence of correlation between variables will result in level inflation. Hence we modify the critical value such that the resulting test procedure has correct asymptotic level under spiked covariance.

Recently, a class of test procedures are proposed through random projection. See Lopes et al. (2011), Thulin (2014) and Srivastava et al. (2015). The idea is to project data on some random lower-dimensional subspaces. It has been shown that these procedures perform well under strong correlations. The

random projection methods imply that test procedures are improved when data are projected on certain subspaces. We will see that the ideal subspace is the orthogonal complement of the principal space. Fortunately, the principal space can be estimated consistently even in high dimensional setting by the theory of principal component analysis (PCA). With the assumption of spiked covariance model, we propose a new test procedure through projection on the (estimated) ideal subspace. The asymptotic null distribution of the test statistic is derived and asymptotic power is also given. We will see that the test is more powerful than  $T_{CQ}$ .

The rest of the paper is organized as follows. In Section 2, we revisit Chen and Qin (2010)'s test. In Section 3, we propose a test procedure and exploit properties of the test. In Section 4, simulations are carried out and a real data example is given. Section 5 contains some discussion. All the technical details are in appendix.

## 2. Asymptotic properties of Chen and Qin (2010)'s test

Throughout the paper, we assume  $p \rightarrow \infty$  as  $n \rightarrow \infty$  and  $n_1/n_2 \rightarrow c \in (0, +\infty)$ , that is, we consider high dimensional and balanced data.

In Chen and Qin (2010), the asymptotic normality of  $T_{CQ}$  is derived under condition (2). Note that  $T_{CQ}$  is a quadratic form of a standard normal random vector. Let  $Z_{k,i} = \Sigma^{-1/2}X_{k,i}$ , hence  $Z_{k,i}$  has distribution  $N_p(0, I_p)$ . Write  $Z = (Z_{1,1}^T, \dots, Z_{1,n_1}^T, Z_{2,1}^T, \dots, Z_{2,n_2}^T)^T$ . Then  $T_{CQ} = Z^T(B_n \otimes \Sigma)Z$ , where  $\otimes$  is the Kronecker product and

$$B_n = \begin{pmatrix} \frac{1}{n_1(n_1-1)}(\mathbf{1}_{n_1}\mathbf{1}_{n_1}^T - I_{n_1}) & -\frac{1}{n_1n_2}\mathbf{1}_{n_1}\mathbf{1}_{n_2}^T \\ -\frac{1}{n_1n_2}\mathbf{1}_{n_2}\mathbf{1}_{n_1}^T & \frac{1}{n_2(n_2-1)}(\mathbf{1}_{n_2}\mathbf{1}_{n_2}^T - I_{n_2}) \end{pmatrix}.$$

Using characteristic function method, one can prove the following result which gives a necessary and sufficient condition for the asymptotic normality of the quadratic form of a standard normal random vector.

**Lemma 1.** *Suppose  $Y_n$  is a  $k_n$  dimensional standard normal random vector and  $A_n$  is a  $k_n \times k_n$  symmetric matrix. Then as  $n \rightarrow \infty$ , a necessary and*

sufficient condition for

$$\frac{Y_n^T A_n Y_n - \mathbb{E} Y_n^T A_n Y_n}{[\text{Var}(Y_n^T A_n Y_n)]^{1/2}} \xrightarrow{\mathcal{L}} N(0, 1) \quad (4)$$

is that

$$\frac{\lambda_{\max}(A_n^2)}{\text{tr}(A_n^2)} \rightarrow 0, \quad (5)$$

where “ $\xrightarrow{\mathcal{L}}$ ” means convergence of a sequence of random variables in law.

The eigenvalues of  $B_n \otimes \Sigma$  relies on the eigenvalues of  $B_n$  and  $\Sigma$ . The eigenvalues of  $B_n$  are  $-1/n_1(n_1 - 1)$ ,  $-1/n_2(n_2 - 1)$ ,  $(n_1 + n_2)/n_1 n_2$  and 0 with multiplicities  $n_1 - 1$ ,  $n_2 - 1$ , 1 and 1 respectively. Thus,

$$\text{tr}(B_n \otimes \Sigma)^2 = \text{tr}(B_n^2) \text{tr} \Sigma^2 = \left( \frac{1}{n_1(n_1 - 1)} + \frac{1}{n_2(n_2 - 1)} + \frac{2}{n_1 n_2} \right) \text{tr} \Sigma^2,$$

and

$$\lambda_{\max}((B_n \otimes \Sigma)^2) = \lambda_{\max}(B_n^2) \lambda_{\max}(\Sigma^2) = \left( \frac{1}{n_1} + \frac{1}{n_2} \right)^2 \lambda_{\max}(\Sigma^2).$$

Because  $n_1/n_2 \rightarrow c$ ,

$$\frac{\lambda_{\max}((B_n \otimes \Sigma)^2)}{\text{tr}(B_n \otimes \Sigma)^2} \rightarrow 0$$

is equivalent to  $\lambda_{\max}(\Sigma^2)/\text{tr} \Sigma^2 \rightarrow 0$ . From

$$\frac{\lambda_1(\Sigma)^4}{(\sum_{i=1}^p \lambda_i(\Sigma)^2)^2} \leq \frac{\sum_{i=1}^p \lambda_i(\Sigma)^4}{(\sum_{i=1}^p \lambda_i(\Sigma)^2)^2} \leq \frac{\lambda_1(\Sigma)^2 \sum_{i=1}^p \lambda_i(\Sigma)^2}{(\sum_{i=1}^p \lambda_i(\Sigma)^2)^2} = \frac{\lambda_1(\Sigma)^2}{\sum_{i=1}^p \lambda_i(\Sigma)^2},$$

we can see that  $\lambda_{\max}^2(\Sigma)/\text{tr}(\Sigma^2) \rightarrow 0$  is equivalent to (2). By Lemma 1, the condition (2) is a necessary and sufficient condition for

$$\frac{T_{CQ} - \mathbb{E} T_{CQ}}{[\text{Var}(T_{CQ})]^{1/2}} \xrightarrow{\mathcal{L}} N(0, 1).$$

Since the condition (2) is necessary for the asymptotic normality of  $T_{CQ}$ , Chen and Qin (2010)’s test procedure requires the eigenvalues of  $\Sigma$  to be concentrated around their average, which can be violated in many applications. In a class of applications, the correlations between variables are mainly driven by several common factors so that  $\Sigma$  has a few eigenvalues which are much larger than the

others. See, for example, Jung and Marron (2009), Cai et al. (2013) and Fan and Wang (2017). To characterize such correlations between variables, we consider the spiked covariance structure (3).

**Assumption 1.** *The covariance matrix  $\Sigma$  has structure  $\Sigma = V\Lambda V^T + \sigma^2 I_p$ , where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_r)$ ,  $\lambda_1 \geq \dots \geq \lambda_r > 0$ ,  $r$  is a known number,  $V$  is a  $p \times r$  orthonormal matrix and  $\sigma^2 > 0$ . As  $n, p$  tend to infinity,  $r, \sigma^2$  are fixed and  $\Lambda$  satisfies*

$$\kappa p^\beta \geq \lambda_1 \geq \dots \geq \lambda_r \geq \kappa^{-1} p^\beta.$$

where  $\kappa > 1$  and  $\beta \geq 1/2$  are constants.

The covariance structure in Assumption 1 is commonly adopted in PCA study. See Cai et al. (2013), Birnbaum et al. (2013), Passemier et al. (2017) and the references therein. The column space of  $V$ , the eigenspace of  $\Sigma$  corresponding to the leading eigenvalues, is called principal space. Since  $V$  is an orthonormal matrix,  $VV^T$  is the orthogonal projection matrix on the principal space. Let  $\tilde{V}$  be a  $p \times (p - r)$  full column rank orthonormal matrix orthogonal to columns of  $V$ . Although such  $\tilde{V}$  is not unique, the projection matrix  $\tilde{V}\tilde{V}^T = I - VV^T$  is unique and is the projection matrix on the orthogonal complement of principal space. Spiked covariance also has connection to factor model. In fact, the factor model in Ma et al. (2015) corresponds to  $\beta = 1$  in Assumption 1.

For positive sequence  $\{a_n\}$  and  $\{b_n\}$ , we write  $a_n \asymp b_n$  to denote  $a_n = O(b_n)$  and  $b_n = O(a_n)$  as  $n \rightarrow \infty$ . Then under Assumption 1, we have

$$\frac{\text{tr}(\Sigma^4)}{\text{tr}^2(\Sigma^2)} = \frac{\sum_{i=1}^r (\lambda_i + \sigma^2)^4 + (p - r)\sigma^8}{\left(\sum_{i=1}^r (\lambda_i + \sigma^2)^2 + (p - r)\sigma^4\right)^2} \asymp \frac{p^{4\beta} + p}{(p^{2\beta} + p)^2},$$

which tends to 0 if and only if  $p < 1/2$ . Our previous arguments assert that the asymptotic distribution of  $T_{CQ}$  won't be normal when  $\beta \geq 1/2$ . The following two theorems give the asymptotic distribution of  $T_{CQ}$  when  $\beta \geq 1/2$ .

**Theorem 1.** *Under Assumption 1, suppose  $\beta = 1/2$  and  $\lambda_i/p^\beta \rightarrow \omega_i \in (0, +\infty)$ ,  $i = 1, \dots, r$ . Let  $Z_0, Z_1, \dots, Z_r$  be i.i.d.  $N(0, 1)$  random variables, then the following results hold:*

(a) If  $\mu_1 = \mu_2$ , then

$$\frac{1}{\tau p^\beta} T_{CQ} \xrightarrow{w} \sqrt{2}\sigma^2 Z_0 + \sum_{i=1}^r \omega_i Z_i^2 - \sum_{i=1}^r \omega_i,$$

where  $\tau = 1/n_1 + 1/n_2$  and “ $\xrightarrow{w}$ ” denotes weak convergence.

(b) If  $(\tau p^\beta)^{-1/2} (V^T(\mu_1 - \mu_2))_i \rightarrow \zeta_i \in (-\infty, +\infty)$ ,  $i = 1, \dots, r$ , and  $(\tau p^\beta)^{-1} \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 \rightarrow \zeta^* \in [0, +\infty)$ , then

$$\frac{1}{\tau p^\beta} T_{CQ} \xrightarrow{w} \sqrt{2}\sigma^2 Z_0 + \sum_{i=1}^r (\sqrt{\omega_i} Z_i + \zeta_i)^2 + \zeta^* - \sum_{i=1}^r \omega_i.$$

**Theorem 2.** Under Assumption 1, suppose  $\beta > 1/2$  and  $\lambda_i/p^\beta \rightarrow \omega_i \in (0, +\infty)$ ,  $i = 1, \dots, r$ . Let  $Z_1, \dots, Z_r$  be i.i.d.  $N(0, 1)$  random variables, then the following results hold:

(a) If  $\mu_1 = \mu_2$ , then

$$\frac{1}{\tau p^\beta} T_{CQ} \xrightarrow{w} \sum_{i=1}^r \omega_i Z_i^2 - \sum_{i=1}^r \omega_i.$$

(b) If  $(\tau p^\beta)^{-1/2} (V^T(\mu_1 - \mu_2))_i \rightarrow \zeta_i \in (-\infty, +\infty)$ ,  $i = 1, \dots, r$ , and  $(\tau p^\beta)^{-1} \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 \rightarrow \zeta^* \in [0, +\infty)$ , then

$$\frac{1}{\tau p^\beta} T_{CQ} \xrightarrow{w} \sum_{i=1}^r (\sqrt{\omega_i} Z_i + \zeta_i)^2 + \zeta^* - \sum_{i=1}^r \omega_i.$$

It is possible to adjust the critical value of  $T_{CQ}$  such that the resulting test has correct level  $\alpha$  asymptotically. Let  $F(x; \lambda_1, \dots, \lambda_r, \sigma^2)$  be the cumulative distribution function of  $\sqrt{2p}\sigma^2 Z_0 + \sum_{i=1}^r \lambda_i Z_i^2 - \sum_{i=1}^r \lambda_i$ . By Theorem 1 and Theorem 2, if the critical value of  $\tau^{-1} T_{CQ}$  is defined as the upper  $\alpha$  quantile of  $F(x; \lambda_1, \dots, \lambda_r, \sigma^2)$ , the resulting test has level  $\alpha$  asymptotically. However, the distribution  $F(x; \lambda_1, \dots, \lambda_r, \sigma^2)$  involves some nuisance parameters. In order to consistently estimate  $F(x; \lambda_1, \dots, \lambda_r, \sigma^2)$ , we need to give ratio consistent estimators of  $\lambda_i$ ,  $i = 1, \dots, r$ , and  $\sigma^2$ . The following theorem shows that  $\lambda_i(S)$  can consistently estimate  $\lambda_i$ ,  $i = 1, \dots, r$ .

**Proposition 1.** Under Assumption 1, suppose  $n = O(p)$  and  $p = o(np^\beta)$ , then

$$\lambda_i(S) = \lambda_i(1 + o_P(1)), \quad i = 1, \dots, r.$$

In section 3, we will give an estimator  $\hat{\sigma}^2$  of  $\sigma^2$ . Proposition 3 asserts that  $\hat{\sigma}^2$  is consistent. Thus, if we reject the null hypothesis when  $\tau^{-1}T_{CQ}$  is larger than the upper  $\alpha$  quantile of  $F(x; \hat{\lambda}_1, \dots, \hat{\lambda}_r, \hat{\sigma}^2)$ , then under the conditions of Proposition 1, the resulting test has correct level asymptotically.

By Theorem 1 and 2, the adjusted  $T_{CQ}$  has trivial asymptotic power if  $\zeta_1 = \dots = \zeta_r = \zeta^* = 0$ , which is equivalent to  $\sum_{i=1}^r \zeta_i^2 + \zeta^* = 0$ , or

$$\frac{1}{\tau p^\beta} \|\mu_1 - \mu_2\|^2 \rightarrow 0.$$

Conversely, to make  $T_{CQ}$  have non-trivial power,  $\|\mu_1 - \mu_2\|^2$  is at least of order  $\tau p^\beta$ .

### 3. Methodology

#### 3.1. Motivation

Note that the main part of  $T_{CQ}$  is  $\tau \|\bar{X}_1 - \bar{X}_2\|^2$ , which is also the main part of  $T_{BS}$  and Ma et al. (2015)'s method. It can be written as the sum of two terms

$$\tau \|V^T(\bar{X}_1 - \bar{X}_2)\|^2 + \tau \|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2. \quad (6)$$

Under the null hypothesis, we have

$$\text{Var}(\tau \|V^T(\bar{X}_1 - \bar{X}_2)\|^2) = \sum_{i=1}^r 2(\lambda_i + \sigma^2)^2, \quad \text{Var}(\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2) = 2\sigma^4(p-r).$$

The ratio of the two variance is

$$\frac{\sum_{i=1}^r 2(\lambda_i + \sigma^2)^2}{2\sigma^4(p-r)} \asymp p^{2\beta-1},$$

which tends to  $\infty$  as  $p \rightarrow \infty$  for  $\beta > 1/2$ . While  $\tau \|V^T(\bar{X}_1 - \bar{X}_2)\|^2$  has relative large variance, it only involves the signal from  $r$  dimension. This motivates us to drop the first part of (6) and only use the second part. After adjustment of expectation, we define the following statistic

$$T_1 = \|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2 - \frac{1}{n_1} \text{tr}(\tilde{V}^T S_1 \tilde{V}) - \frac{1}{n_2} \text{tr}(\tilde{V}^T S_2 \tilde{V}).$$

Proposition 2 shows that  $T_1$  has normal limit distribution.



**Proposition 2.** *Under Assumption 1, suppose  $\frac{n}{p}\|\mu_1 - \mu_2\|^2 = o(1)$ , we have*

$$\frac{T_1 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2}{\sigma^2\sqrt{2\tau^2p}} \xrightarrow{\mathcal{L}} N(0, 1).$$

In another point of view,  $T_1$  is obtained by transforming  $X_{k,i}$  to  $\tilde{V}^T X_{k,i}$  ( $i = 1, \dots, n_k, k = 1, 2$ ) and then invoking the statistic of Chen and Qin (2010). A class of test procedures have been proposed through random projection to lower dimensional space. See, for example, Lopes et al. (2011), Thulin (2014) and Srivastava et al. (2015). It is known that random projection based methods offer higher power when the variables are dependent. However, these test procedures are randomized, which is undesirable in practice. This raises the question: is there an optimal projection which is nonrandomized?

For  $p \geq q$ , define  $\mathbb{O}_{p \times q} = \{O \mid O \text{ is } p \times q \text{ column orthonormal matrix}\}$ . Under null hypothesis, we have that

$$\tilde{V} = \arg \min_{O \in \mathbb{O}_{p \times (p-r)}} \text{Var}(\|O^T(\bar{X}_1 - \bar{X}_2)\|^2).$$

Thus, transformation by  $\tilde{V}$  is optimal in the sense of variance reduction. Based on  $\tilde{V}^T X_{ki}$ , the likelihood ratio test statistic for hypothesis (1) is then  $\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2$  which coincides with our proposal. In this view,  $T_1$  can be regarded as a restricted likelihood ratio test.

### 3.2. New test

We denote by  $\hat{V}$  and  $\hat{\hat{V}}$  the first  $r$  and last  $p-r$  eigenvectors of  $S$  respectively. Similarly, we denote by  $\hat{V}_k$  and  $\hat{\hat{V}}_k$  the first  $r$  and last  $p-r$  eigenvectors of  $S_k$  respectively,  $k = 1, 2$ . As estimators of their population counterparts, these simple statistics actually reach the optimal convergence rate (See Cai et al. (2013)).

Note that  $T_1$  relies on the subspace  $\tilde{V}\tilde{V}^T$  which is unknown and thus should be estimated. The first part of  $T_1$ ,  $\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2$ , can be directly estimated by  $\|\hat{\hat{V}}^T(\bar{X}_1 - \bar{X}_2)\|^2$ . Note that  $n_1^{-1}\text{tr}(\tilde{V}^T S_1 \tilde{V})$ , the second part of  $T_1$ , only involves sample one. We estimate it by  $n_1^{-1}\text{tr}(\hat{\hat{V}}_1^T S_1 \hat{\hat{V}}_1)$ . Similarly, we estimate

the third part of  $T_1$  by  $n_2^{-1}\text{tr}(\hat{V}_2^T S_2 \hat{V}_2)$ . Define

$$T_2 = \|\hat{V}^T(\bar{X}_1 - \bar{X}_2)\|^2 - \frac{1}{n_1}\text{tr}(\hat{V}_1^T S_1 \hat{V}_1) - \frac{1}{n_2}\text{tr}(\hat{V}_2^T S_2 \hat{V}_2).$$

The asymptotic result of Proposition 2 involves  $\sigma^2$ . In order to formulate a test procedure by asymptotic distribution,  $\sigma^2$  needs to be consistently estimated. Note that  $\sigma^2$  can be written as  $\sigma^2 = (p-r)^{-1} \sum_{i=r+1}^p \lambda_i(\Sigma)$ . So it can be estimated by

$$\hat{\sigma}^2 = \frac{1}{p-r} \sum_{i=r+1}^p \lambda_i(S).$$

Now we propose our new test statistic as

$$Q = \frac{T_2}{\hat{\sigma}^2 \sqrt{2\tau^2 p}}. \quad (7)$$

In the next section, it will be proved that the asymptotic null distribution of  $Q$  is  $N(0, 1)$ . Thus, the null hypothesis is rejected when  $Q$  is larger than the upper  $\alpha$  quantile of  $N(0, 1)$ .

**Remark 1.** When both samples are simultaneously transformed by shift and orthogonal transformation, the statistic  $T_2$  is invariant. More precisely,  $T_2$  is invariant under the following transformation:

$$X_{1,i} \mapsto OX_{1,i} + \mu \text{ and } X_{2,j} \mapsto OX_{2,j} + \mu, \quad i = 1, \dots, n_1, \quad j = 1, \dots, n_2,$$

where  $\mu \in \mathbb{R}^p$  and  $O \in \mathbb{O}_{p \times p}$ .

Theoretical results will show that the asymptotic variance of  $T_2$  is significantly smaller than  $T_{CQ}$ . Since the new test statistic estimates  $\|\tilde{V}^T(\mu_1 - \mu_2)\|^2$ , the superiority of the new test will be established if

$$\frac{\|\tilde{V}^T(\mu_1 - \mu_2)\|}{\|\mu_1 - \mu_2\|} \approx 1. \quad (8)$$

Obviously, (8) is not always the case since there always exists some  $\tilde{V}$  and  $\mu_1 - \mu_2$  such that  $\|\tilde{V}^T(\mu_1 - \mu_2)\| = 0$ . However, (8) is reasonable since  $\tilde{V}\tilde{V}^T$  is nearly an identity matrix in the sense that  $\|I_p - \tilde{V}\tilde{V}^T\|_F^2 / \|I_p\|_F^2 = r/p \rightarrow 0$ .

In bayesian framework, if we assume that the elements of  $\mu_k$  are independently generated from certain prior distribution, it can be established that  $\|\tilde{V}(\mu_1 - \mu_2)\|/\|\mu_1 - \mu_2\| \xrightarrow{P} 1$ .

### 3.3. Theoretical results

In this section, we study the asymptotic behavior of the new test procedure.

We first give a result of the convergence rate of  $\hat{\sigma}^2$ . In particular, this result shows that  $\hat{\sigma}^2$  is a consistent estimator of  $\sigma^2$ . Our proof relies on the Weyl's inequality.

**Proposition 3.** *Under Assumptions 1, we have*

$$\hat{\sigma}^2 = \sigma^2 + O_P\left(\frac{\max(n, p)}{np}\right).$$

In the construction of  $T_2$ , we replace  $\tilde{V}$  by  $\hat{\tilde{V}}$ . The asymptotic property of  $T_2$  is closely related the asymptotic property of  $\hat{\tilde{V}}\hat{\tilde{V}}^T$  as an estimator of  $\tilde{V}\tilde{V}^T$ . However,  $\hat{\tilde{V}}\hat{\tilde{V}}^T$  can not always consistently estimate  $\tilde{V}\tilde{V}^T$  in high dimensional setting. In fact, Cai et al. (2014)'s Theorem 5 implies that it is possible only when  $p^{1-\beta}/n \rightarrow 0$ , see Lemma 3 in appendix. The asymptotic normality of  $T_2$  requires a stronger condition.

**Assumption 2.** *Assume  $p/n^2 \rightarrow 0$ .*

When  $\beta = 1/2$ , Assumption 2 is equivalent to  $p^{1-\beta}/n$ , otherwise 2 is stronger than  $p^{1-\beta}/n$ . The following theorem establishes the asymptotic normality of  $T_2$ .

**Theorem 3.** *Under Assumptions 1 and 2, suppose*

$$\frac{n}{\sqrt{p}}\|\mu_1 - \mu_2\|^2 = O(1),$$

*we have*

$$\frac{T_2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2}{\sigma^2 \sqrt{2\tau^2 p}} \xrightarrow{\mathcal{L}} N(0, 1).$$

Now the power function of the new test can be obtained immediately.

**Corollary 1.** *Under the conditions of Theorem 3, the asymptotic power function of the new test is*

$$\Phi\left(-\Phi^{-1}(1-\alpha) + \frac{\|\tilde{V}(\mu_1 - \mu_2)\|^2}{\sigma^2 \sqrt{2\tau^2 p}}\right).$$

Recall that  $T_{CQ}$  has trivial power if  $\|\mu_1 - \mu_2\|^2 = o(\tau p^\beta)$ . Note that the power function of the new test do not affected by  $\beta$ . As a result, when  $\beta > 1/2$ , the new test tends to be much more powerful.

The proof of Theorem 3 implies that when Assumption 2 is not satisfied, the asymptotic normality is invalid. In this case, permutation method can be used to determine the critical value. We will see from simulation results that the new test has good power behavior even if  $p$  is much large than  $n$ .

## 4. Numerical studies

### 4.1. Simulation results

In this section, we report the simulation performance of the proposed test and compare it with  $T_{CQ}$  and  $T_S$ .

In our simulation studies, samples are generated from the model described in Assumption 1, where  $V \in \mathbb{O}_{p \times r}$  is randomly generated from Haar invariant distribution,  $\lambda_{k,i}$  equals  $p^\beta$  plus a random error from  $U(0, 1)$  (Uniform distribution between 0 and 1) and  $\sigma_k^2 = 1$ ,  $k = 1, 2$ . In unequal variance case,  $(V_1, \Lambda_1)$  and  $(V_2, \Lambda_2)$  are independently generated. We take nominal level  $\alpha = 0.05$ .

First, we simulate the level of the new test. We set factor number  $r = 2$ . Samples are repeatedly generated 1000 times to calculate empirical level. For comparison, we also give corresponding ‘oracle’ level which is calculated by ‘statistics’  $T_1/(\sigma^2 \sqrt{2p\tau^2})$  in equal variance case and  $T_1/\sqrt{\sigma_n^2}$  in unequal variance case. The results for equal variance case and unequal variance case are listed in Table 1 and 2, respectively. From the results, we can find that for small  $n$  and  $p$ , even oracle level is not satisfied. Level of the new test is a little inflated compared with oracle level. In all cases, the empirical level tends to be more close to 0.05 as  $n$  increases.

Table 1: Test level simulation. Equal variance case.

$n$	$p$	$\beta=0.5$		$\beta=1$		$\beta=2$	
		NEW	ORACLE	NEW	ORACLE	NEW	ORACLE
300	200	0.075	0.062	0.079	0.062	0.074	0.070
300	400	0.074	0.065	0.061	0.044	0.046	0.040
300	600	0.058	0.041	0.070	0.052	0.071	0.055
300	800	0.066	0.047	0.071	0.052	0.062	0.048
600	200	0.061	0.055	0.052	0.051	0.058	0.056
600	400	0.051	0.048	0.051	0.042	0.059	0.051
600	600	0.061	0.058	0.056	0.054	0.051	0.047
600	800	0.053	0.046	0.060	0.050	0.056	0.048

Next, we simulate the empirical power of the new test. The simulation results of Ma et al. (2015) have showed that the level of the Chen and Qin (2010)’s test can’t be guaranteed when covariance is spiked. To be fair, critical values are all determined by permutation method. We set  $\Sigma_1 = \Sigma_2$ , under which permutation method can produce exact test procedures, see Lehmann and Romano (2005)’s Example 15.2.2. We permute the sample 100 times to determine the critical value. We repeat the test procedure 500 times to obtain empirical power. We plot the empirical power versus signal-to-noise ratio  $\text{SNR} = \|\mu_1 - \mu_2\|^2 / (\sigma^2 \sqrt{2\tau^2 p})$ . The results are illustrated in figure 1 and 2, where ‘NEW’, ‘CQ’ and ‘S’ represent the new test, Chen and Qin (2010)’s test and Srivastava and Du (2008)’s test. From the results, we can find that when  $\Sigma$  is spiked, the new test outperforms  $T_{CQ}$  substantially; when  $\Sigma$  is not spiked, all three tests have similar performance.

#### 4.2. Real data analysis

In this section, we study the practical problem considered in Ma et al. (2015). The task is to test whether Monday stock returns are equal to those of other

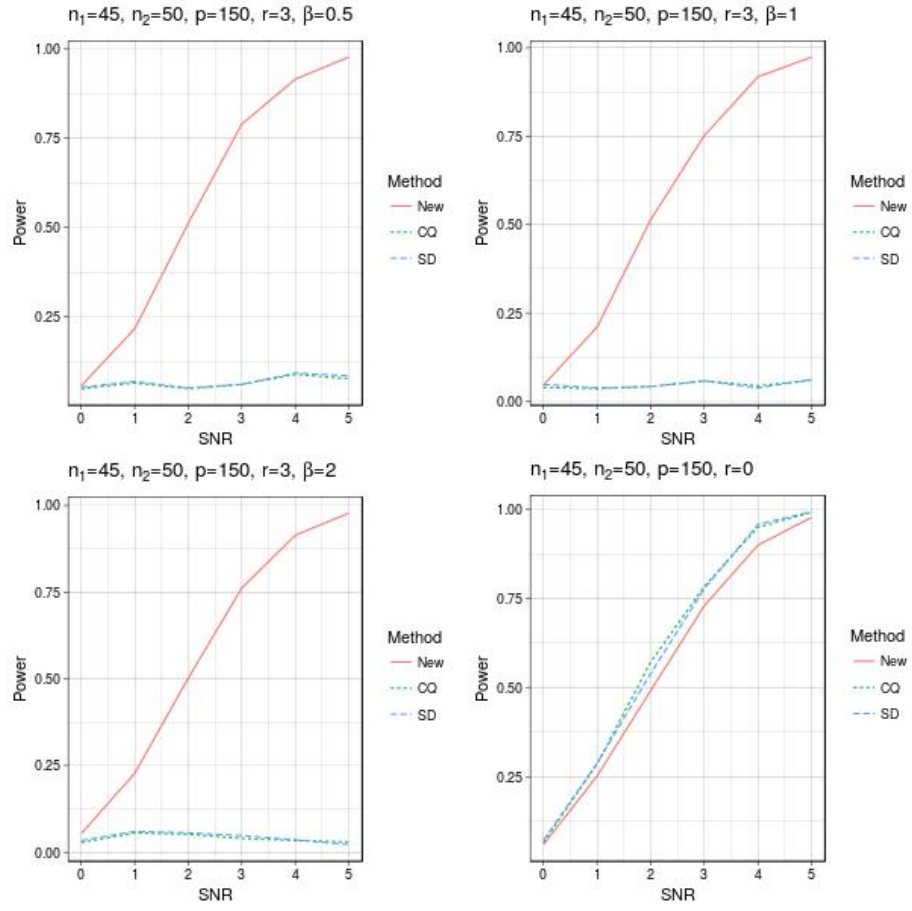


Figure 1: Empirical power simulation.

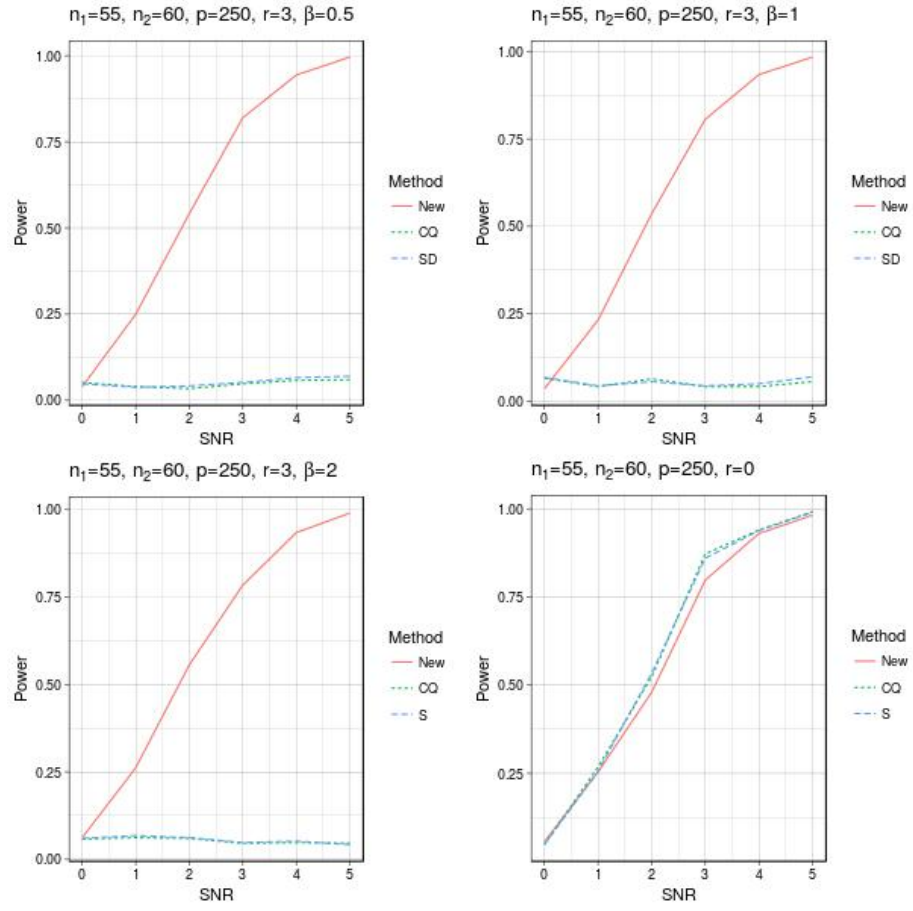


Figure 2: Empirical power simulation.

Table 2: Test level simulation. Unequal variance case.

$n$	$p$	$\beta=0.5$		$\beta=1$		$\beta=2$	
		NEW	ORACLE	NEW	ORACLE	NEW	ORACLE
300	200	0.066	0.058	0.057	0.054	0.054	0.050
300	400	0.063	0.047	0.063	0.047	0.069	0.052
300	600	0.070	0.058	0.091	0.059	0.086	0.053
300	800	0.069	0.040	0.097	0.055	0.083	0.056
600	200	0.054	0.056	0.049	0.049	0.052	0.051
600	400	0.060	0.054	0.067	0.059	0.060	0.055
600	600	0.041	0.034	0.069	0.062	0.055	0.049
600	800	0.077	0.063	0.066	0.058	0.071	0.058

trading days on average. Define an observation be the log return of stocks in a day. Hence  $p$  is the total number of stocks. Let sample 1 and sample 2 be the observations on Monday and the other trading days, respectively. Then we would like to test  $H_0 : \mu_1 = \mu_2$  v.s.  $H_1 : \mu_1 \neq \mu_2$ . We collected the data of  $p = 710$  stocks of China from 01/04/2013 to 12/31/2014. There are total  $n_1 = 95$  Monday and  $n_2 = 388$  other trading days.

We assume  $\Sigma_1 = \Sigma_2$ . The first eigenvalue of  $S$  is 0.14, which is significantly larger than the others. In fact, the second eigenvalue is 0.02. Hence there's clearly a spiked eigenvalue. We set  $r = 1$  and perform our new test. The  $p$  value is 0.149, which is obtained by 1000 permutations. Hence, the null hypothesis can not be rejected for  $\alpha = 0.05$ . We draw the same conclusion as Ma et al. (2015).

## 5. Conclusion remark

This paper is concerned with the problem of testing the equality of means in the setting of high dimension and spiked covariance. We drop big variance terms from  $T_{CQ}$  and obtain a new test statistic. The asymptotic normality of



the new statistic is proved and the asymptotic power is given.

In another paper, Zhao and Xu (2016) proved that their test statistic can be written in the form of projection. Their simulation results showed that their test performs well under strong correlations. Our work partially explains why their test performs well although the projections are slightly different.

Spiked covariance is an important correlation pattern and has been widely studied in terms of PCA. In PCA, authors focus on the principal subspace. However, in some circumstances, as our work have shown, the complement of principal subspace is more useful.

In our paper, we have assumed  $r$  is known. If  $r$  is an unknown positive number, a consistent estimator of  $r$  is

$$\hat{r} = \operatorname{argmax}_{l \leq R} \frac{\lambda_l(S)}{\lambda_{l+1}(S)}, \quad (9)$$

where  $R$  is a hyperparameter. See Ahn and Horenstein (2013) for detail.

Our theoretical results rely on the assumption  $\sqrt{p}/n \rightarrow 0$ . In the situation of small sample or very large  $p$ , the critical value of the new test can be determined by permutation method. Our simulation shows that the new test still performs well. It remains a theoretical interest to study the asymptotic behavior of permutation based test in these situations.

## Appendix

**Lemma 2** (Weyl's inequality). *Let  $H$  and  $P$  be two symmetric  $n \times n$  matrices and  $M = H + P$ . If  $r + s - 1 \leq i \leq j + k - n$ , we have*

$$\lambda_j(H) + \lambda_k(P) \leq \lambda_i(M) \leq \lambda_r(H) + \lambda_s(P).$$

*See, for example, Horn and Johnson (1985) Theorem 4.3.1.*

**Lemma 3** (Convergence rate of principal space estimation). *Under the Assumption 1, we have*

$$E\|\hat{V}\hat{V}^T - VV^T\|_F^2 = O\left(\frac{p}{p^\beta n}\right).$$

**Proof.** Theorem 5 of Cai et al. (2013) asserts that sample principal subspace  $\hat{V}\hat{V}^T$  is a minimax rate optimal estimator of  $VV^T$ , namely, it reaches the minimax convergence rate

$$E\|\hat{V}\hat{V}^T - VV^T\|_F^2 \asymp r \wedge (p - r) \wedge \frac{r(p - r)}{(n_1 + n_2 - 2)p^\beta} \quad (10)$$

as long as the right hand side tends to 0. It's obvious that the right hand side of (10) is of order  $p^{1-\beta}/n$ .  $\square$

**Lemma 4** (Bai-Yin's law). *Suppose  $B_n = \frac{1}{q}ZZ^T$  where  $Z$  is  $p \times q$  random matrix composed of i.i.d. random variables with zero mean, unit variance and finite fourth moment. As  $q \rightarrow \infty$  and  $\frac{p}{q} \rightarrow c \in [0, \infty)$ , the largest and smallest non-zero eigenvalues of  $B_n$  converge almost surely to  $(1 + \sqrt{c})^2$  and  $(1 - \sqrt{c})^2$ , respectively.*

**Remark 2.** Lemma 4 is known as the Bai-Yin's law (Bai and Yin (1993)). As in Remark 1 of Bai and Yin (1993), the smallest non-zero eigenvalue is the  $p - q + 1$  smallest eigenvalue of  $B$  for  $c > 1$ .

**Corollary 2.** *Suppose that  $W_n$  is a  $p \times p$  matrix distributed as  $\text{Wishart}_p(n, I_p)$  where  $\text{Wishart}_p(m, \Psi)$  is the  $p$  dimensional Wishart distribution with parameter  $\Psi$  and  $m$  degrees of freedom. Then as  $n, p \rightarrow \infty$ ,*

$$\lambda_1(W_n) = O_P(\max(n, p)).$$

**Proof.** Since  $[0, +\infty]$  is compact, for every subsequence  $\{n_k\}$  of  $\{n\}$ , there is a further subsequence  $\{n_{k_l}\}$  along which  $p/n \rightarrow c \in [0, +\infty]$ .

If  $c \in [0, +\infty)$ , by Lemma 4, we have that

$$\frac{\lambda_1(W_{n_{k_l}})}{n_{k_l}} \xrightarrow{P} (1 + c)^2.$$

Hence the conclusion holds along this subsequence. If  $c = +\infty$ , suppose  $W_n = Z_n Z_n^T$  where  $Z_n$  is a  $p \times n$  matrix with all elements independently distributed as  $N(0, 1)$ . Then

$$\frac{\lambda_1(W_{n_{k_l}})}{p} = \frac{\lambda_1(Z_{n_{k_l}}^T Z_{n_{k_l}})}{p} \xrightarrow{P} 1$$

by Lemma 4, which proves the conclusion along the subsequence. Now the conclusion holds by a standard subsequence argument.  $\square$

**Proof of Lemma 1.** Let  $\lambda_1(A_n) \geq \dots \geq \lambda_{k_n}(A_n)$  be the eigenvalues of  $A_n$ , then

$$\frac{Y_n^T A_n Y_n - \mathbb{E} Y_n^T A_n Y_n}{[\text{Var}(Y_n^T A_n Y_n)]^{1/2}} = \sum_{i=1}^{k_n} \frac{\lambda_i(A_n)}{[2\text{tr}(A_n^2)]^{1/2}} (Z_{ni}^2 - 1), \quad (11)$$

where  $Z_{ni}$ 's ( $i = 1, \dots, k_n$ ) are independent standard normal random variables.

If 5 holds, then

$$\begin{aligned} & \sum_{i=1}^{k_n} \mathbb{E} \left[ \frac{\lambda_i^2(A_n)}{2\text{tr}(A_n^2)} (Z_{ni}^2 - 1)^2 \left\{ \frac{\lambda_i^2(A_n)}{2\text{tr}(A_n^2)} (Z_{ni}^2 - 1)^2 \geq \epsilon \right\} \right] \\ & \leq \sum_{i=1}^{k_n} \frac{\lambda_i^2(A_n)}{2\text{tr}(A_n^2)} \mathbb{E} \left[ (Z_{n1}^2 - 1)^2 \left\{ \frac{\lambda_{\max}(A_n^2)}{2\text{tr}(A_n^2)} (Z_{n1}^2 - 1)^2 \geq \epsilon \right\} \right] \\ & = \frac{1}{2} \mathbb{E} \left[ (Z_{n1}^2 - 1)^2 \left\{ \frac{\lambda_{\max}(A_n^2)}{2\text{tr}(A_n^2)} (Z_{n1}^2 - 1)^2 \geq \epsilon \right\} \right] \rightarrow 0. \end{aligned}$$

Hence 4 follows by Lindeberg's central limit theorem.

Conversely, if 4 holds, we will prove that there is a subsequence of  $\{n\}$  along which 5 holds. Then 5 will hold by a standard contradiction argument.

Denote  $c_{ni} = \lambda_i(A_n) / [2\text{tr}(A_n^2)]^{1/2}$  ( $i = 1, \dots, k_n$ ), we have  $c_{ni} \in [-\sqrt{2}/2, \sqrt{2}/2]$ . Since 4 holds, the characteristic function of  $\sum_{i=1}^{k_n} c_{ni}(Z_{ni}^2 - 1)$  converges to  $\exp(-t^2/2)$  for every  $t$ . For  $t \in (-1, 1)$ , we have

$$\begin{aligned} & \log \mathbb{E} \exp \left( it \sum_{j=1}^{k_n} c_{nj} (Z_{nj}^2 - 1) \right) = -i \left( \sum_{j=1}^{k_n} c_{nj} \right) t - \frac{1}{2} \sum_{j=1}^{k_n} \log(1 - i2c_{nj}t) \\ & = -i \left( \sum_{j=1}^{k_n} c_{nj} \right) t + \frac{1}{2} \sum_{j=1}^{k_n} \sum_{l=1}^{+\infty} \frac{1}{l} (i2c_{nj}t)^l = -i \left( \sum_{j=1}^{k_n} c_{nj} \right) t + \frac{1}{2} \sum_{l=1}^{+\infty} \left[ \sum_{j=1}^{k_n} (c_{nj})^l \right] \frac{1}{l} (i2t)^l \\ & = -\frac{1}{2} t^2 + \frac{1}{2} \sum_{l=3}^{+\infty} \left[ \sum_{j=1}^{k_n} (c_{nj})^l \right] \frac{1}{l} (i2t)^l. \end{aligned}$$

Denote  $b_{nl} = \sum_{j=1}^{k_n} (c_{nj})^l$ ,  $n = 1, 2, \dots$  and  $l = 3, 4, \dots$ . For  $l \geq 3$ ,  $|\sum_{j=1}^{k_n} (c_{nj})^l| \leq |\sum_{j=1}^{k_n} (c_{nj})^2| = 1/2$ . By Helly's selection theorem, there's a subsequence of  $\{n\}$  along which  $\lim_{n \rightarrow \infty} b_{nl} = b_l$  exists for every  $l$ . Apply dominated convergence theorem to this subsequence we have  $\log \mathbb{E} \exp \left( it \sum_{j=1}^{k_n} c_{nj} (Z_{nj}^2 - 1) \right) \rightarrow$

$-\frac{1}{2}t^2 + \frac{1}{2}\sum_{l=3}^{+\infty} b_l \frac{1}{l}(i2t)^l$  for  $t \in (-1/2, 1/2)$ . By the property of power series, we have  $b_l = 0$  for  $l \geq 3$ . Then 5 follows by noting that  $b_{n4} \geq \max_j (c_{nj})^4$ .  $\square$

The rest of the Appendix is devoted to the proof of propositions and theorems in the paper.

**Proof of Theorem 1 and Theorem 2.** Note that in both 1 and 2, (a) is a corollary of (b). So we only need to proof (b).

For random variable  $\xi$  and  $\eta$ , we write  $\xi \sim \eta$  to denote they have the same distribution. Note that  $(n_k - 1)S_k \sim \text{Wishart}_p(n_k - 1, \Sigma)$ ,  $k = 1, 2$ , we have

$$\mathbb{E}\left(\frac{1}{n_1} \text{tr } S_1 + \frac{1}{n_2} \text{tr } S_2\right) = \tau \text{tr } \Sigma,$$

and

$$\begin{aligned} \text{Var}\left(\frac{1}{n_1} \text{tr } S_1 + \frac{1}{n_2} \text{tr } S_2\right) &= \left(\frac{2}{n_1^2(n_1 - 1)} + \frac{2}{n_2^2(n_2 - 1)}\right) \text{tr } \Sigma^2 \\ &= O\left(\frac{1}{n^3}(p^{2\beta} + p)\right) = O\left(\frac{p^{2\beta}}{n^3}\right). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{n_1} \text{tr } S_1 + \frac{1}{n_2} \text{tr } S_2 &= \tau \text{tr } \Sigma + O_P\left(\frac{1}{n\sqrt{n}}p^\beta\right) \\ &= \tau \sum_{i=1}^r (\lambda_i + \sigma^2) + \tau(p - r)\sigma^2 + O_P\left(\frac{1}{n\sqrt{n}}p^\beta\right) \\ &= \tau p^\beta \sum_{i=1}^r \omega_i + \tau(p - r)\sigma^2 + o_P\left(\frac{1}{n}p^\beta\right). \end{aligned}$$

Thus,

$$\frac{1}{\tau p^\beta} \left(\frac{1}{n_1} \text{tr } S_1 + \frac{1}{n_2} \text{tr } S_2\right) = \sum_{i=1}^r \omega_i + p^{1-\beta}\sigma^2 + o_P(1). \quad (12)$$

Next we deal with  $\|\bar{X}_1 - \bar{X}_2\|^2$ . Note that we have

$$\|\bar{X}_1 - \bar{X}_2\|^2 = \|V^T(\bar{X}_1 - \bar{X}_2)\|^2 + \|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2.$$

These two terms are independent. For the first term, note that  $V^T(\bar{X}_1 - \bar{X}_2) \sim N_r(V^T(\mu_1 - \mu_2), \tau(\Lambda + \sigma^2 I_r))$ , we have

$$\begin{aligned} \|V^T(\bar{X}_1 - \bar{X}_2)\|^2 &\sim \sum_{i=1}^r \left(\sqrt{\tau(\lambda_i + \sigma^2)}Z_i + (V^T(\mu_1 - \mu_2))_i\right)^2 \\ &= \tau p^\beta \sum_{i=1}^r \left(\sqrt{p^{-\beta}(\lambda_i + \sigma^2)}Z_i + \frac{1}{\sqrt{\tau p^\beta}}(V^T(\mu_1 - \mu_2))_i\right)^2. \end{aligned}$$

By the assumptions of the theorem, we have that

$$\frac{1}{\tau p^\beta} \|V^T(\bar{X}_1 - \bar{X}_2)\|^2 \xrightarrow{w} \sum_{i=1}^r (\sqrt{\omega_i} Z_i + \zeta_i)^2. \quad (13)$$

As for  $\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2$ , we have that

$$\begin{aligned} \|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2 &= \|\tilde{V}^T(\mu_1 - \mu_2) + \tilde{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 \\ &= \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 + \|\tilde{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 + 2(\mu_1 - \mu_2)^T \tilde{V} \tilde{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)). \end{aligned} \quad (14)$$

Since  $\tilde{V}^T(\bar{X}_1 - \bar{X}_2) \sim N_{p-r}(\tilde{V}^T(\mu_1 - \mu_2), \sigma^2 \tau I_{p-r})$ , by central limit theorem, we have

$$\frac{\|\tilde{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 - \sigma^2 \tau (p - r)}{\sigma^2 \tau \sqrt{2(p - r)}} \xrightarrow{\mathcal{L}} N(0, 1).$$

For the intersection term, we have

$$\begin{aligned} 2(\mu_1 - \mu_2)^T \tilde{V} \tilde{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)) &\sim N(0, 4\sigma^2 \tau \|\tilde{V}^T(\mu_1 - \mu_2)\|^2) \\ &= O_P(\sqrt{\tau} \|\tilde{V}^T(\mu_1 - \mu_2)\|) = o_P(\tau p^\beta). \end{aligned}$$

It follows that

$$\frac{1}{\tau p^\beta} (\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2 - \sigma^2 \tau (p - r) - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2) \xrightarrow{\mathcal{L}} \sqrt{2} \sigma^2 \delta_{\{\frac{1}{2}\}}(\beta) Z_0, \quad (15)$$

where  $\delta_{\frac{1}{2}}(\beta)$  equals 1 if  $\beta = 1/2$  and equals 0 otherwise.

Combining (12) (13) and (15) leads to

$$\begin{aligned} \frac{1}{\tau p^\beta} T_{CQ} &= \frac{1}{\tau p^\beta} (\|\bar{X}_1 - \bar{X}_2\|^2 - \frac{1}{n_1} \text{tr } S_1 - \frac{1}{n_2} \text{tr } S_2) \\ &= \frac{1}{\tau p^\beta} \|V^T(\bar{X}_1 - \bar{X}_2)\|^2 + \frac{1}{\tau p^\beta} (\|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2 - \sigma^2 \tau (p - r) - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2) \\ &\quad - \frac{1}{\tau p^\beta} \left( \frac{1}{n_1} \text{tr } S_1 + \frac{1}{n_2} \text{tr } S_2 \right) + \frac{\sigma^2 (p - r)}{p^\beta} + \frac{1}{\tau p^\beta} \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 \\ &= \sum_{i=1}^r (\sqrt{\omega_i} Z_i + \zeta_i)^2 + \sqrt{2} \sigma^2 \delta_{\{\frac{1}{2}\}}(\beta) Z_0 - \left( \sum_{i=1}^r \omega_i + p^{1-\beta} \sigma^2 \right) + \frac{\sigma^2 (p - r)}{p^\beta} + \zeta^* + o_P(1) \\ &\xrightarrow{\mathcal{L}} \sum_{i=1}^r (\sqrt{\omega_i} Z_i + \zeta_i)^2 + \zeta^* + \sqrt{2} \sigma^2 \delta_{\{\frac{1}{2}\}}(\beta) Z_0 - \sum_{i=1}^r \omega_i. \end{aligned}$$

This implies the conclusions of Theorem 1 and Theorem 2.  $\square$

**Proof of Proposition 1.** Let  $U = (V, \tilde{V})$  and  $E = \text{diag}(\lambda_1 + \sigma^2, \dots, \lambda_r + \sigma^2, \sigma^2, \dots, \sigma^2)$ . Then  $\Sigma = UEU^T$  is the spectral decomposition of  $\Sigma$ . Denote by  $S = \hat{U}\hat{E}\hat{U}^T$  the spectral decomposition of  $S$ , where  $\hat{U} = (\hat{V}, \hat{\tilde{V}})$  and  $\hat{E} = \text{diag}(\hat{\lambda}_1, \dots, \hat{\lambda}_p)$ . Let  $\mathbf{Z}$  be a  $p \times (n-2)$  random matrix with all elements i.i.d. distributed as  $N(0, 1)$ . Denote  $\mathbf{Z} = (\mathbf{Z}_{(1)}^T, \mathbf{Z}_{(2)}^T)^T$ , where  $\mathbf{Z}_{(1)}$  and  $\mathbf{Z}_{(2)}$  are the first  $r$  rows and last  $p-r$  rows of  $\mathbf{Z}$ . We have

$$S \sim \frac{1}{n-2} UE^{1/2} \mathbf{Z} \mathbf{Z}^T E^{1/2} U^T.$$

It can be seen that  $\hat{\lambda}_i = \lambda_i(S) \sim (n-2)^{-1} \lambda_i(\mathbf{Z}^T E \mathbf{Z})$ ,  $i = 1, \dots, r$ . The equality  $\mathbf{Z}^T E \mathbf{Z} = \mathbf{Z}_{(1)}^T (\Lambda + \sigma^2 I_r) \mathbf{Z}_{(1)} + \sigma^2 \mathbf{Z}_{(2)}^T \mathbf{Z}_{(2)}$  leads to

$$\lambda_i(\mathbf{Z}_{(1)}^T (\Lambda + \sigma^2 I_r) \mathbf{Z}_{(1)}) \leq \lambda_i(\mathbf{Z}^T E \mathbf{Z}) \leq \lambda_i(\mathbf{Z}_{(1)}^T (\Lambda + \sigma^2 I_r) \mathbf{Z}_{(1)}) + \sigma^2 \lambda_{\max}(\mathbf{Z}_{(2)}^T \mathbf{Z}_{(2)}), \quad (16)$$

$i = 1, \dots, r$ . By Bai-Yin's law, we have

$$\lambda_{\max}(\mathbf{Z}_{(2)}^T \mathbf{Z}_{(2)}) = O_P(p). \quad (17)$$

By Weyl's inequality, we have

$$\begin{aligned} \lambda_i(\mathbf{Z}_{(1)}^T (\Lambda + \sigma^2 I_r) \mathbf{Z}_{(1)}) &\leq \lambda_{\max}(\mathbf{Z}_{(1)}^T \underbrace{\text{diag}(0, \dots, 0, \lambda_i + \sigma^2, \dots, \lambda_i + \sigma^2)}_{i-1, r-i+1} \mathbf{Z}_{(1)}) \\ &= (\lambda_i + \sigma^2) \lambda_{\max}(\mathbf{Z}_{(1)}^T \underbrace{\text{diag}(0, \dots, 0, 1, \dots, 1)}_{i-1, r-i+1} \mathbf{Z}_{(1)}) \\ &= (\lambda_i + \sigma^2) \lambda_{\max}(\underbrace{\text{diag}(0, \dots, 0, 1, \dots, 1)}_{i-1, r-i+1} \mathbf{Z}_{(1)} \mathbf{Z}_{(1)}^T \underbrace{\text{diag}(0, \dots, 0, 1, \dots, 1)}_{i-1, r-i+1}) \\ &= n(\lambda_i + \sigma^2)(1 + o_P(1)), \end{aligned}$$

Where the last inequality follows by law of large numbers. On the other hand, we have

$$\begin{aligned} \lambda_i(\mathbf{Z}_{(1)}^T (\Lambda + \sigma^2 I_r) \mathbf{Z}_{(1)}) &\geq \lambda_i(\mathbf{Z}_{(1)}^T \underbrace{\text{diag}(\lambda_i + \sigma^2, \dots, \lambda_i + \sigma^2, 0, \dots, 0)}_{i, r-i} \mathbf{Z}_{(1)}) \\ &= (\lambda_i + \sigma^2) \lambda_i(\mathbf{Z}_{(1)}^T \underbrace{\text{diag}(1, \dots, 1, 0, \dots, 0)}_{i, r-i} \mathbf{Z}_{(1)}) \\ &= (\lambda_i + \sigma^2) \lambda_i(\underbrace{\text{diag}(1, \dots, 1, 0, \dots, 0)}_{i, r-i} \mathbf{Z}_{(1)} \mathbf{Z}_{(1)}^T \underbrace{\text{diag}(1, \dots, 1, 0, \dots, 0)}_{i, r-i}) \\ &= n(\lambda_i + \sigma^2)(1 + o_P(1)). \end{aligned}$$

Hence, we have  $\lambda_i(\mathbf{Z}_{(1)}^T(\Lambda + \sigma^2 I_r)\mathbf{Z}_{(1)}) = n(\lambda_i + \sigma^2)(1 + o_P(1))$ . Combined with (16) and (17), we have

$$\frac{\lambda_i(S)}{\lambda_i} = 1 + o_P(1) + O_P\left(\frac{p}{n\lambda_i}\right).$$

The conclusion follows.  $\square$

**Proof of Proposition 2.** Note that

$$\begin{aligned} \|\tilde{V}^T(\bar{X}_1 - \bar{X}_2)\|^2 &= \|\tilde{V}^T(\mu_1 - \mu_2) + \tilde{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 \\ &= \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 + \|\tilde{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 + 2(\mu_1 - \mu_2)^T \tilde{V} \tilde{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)) \\ &= \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 + \|\tilde{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 + o_P\left(\frac{\sqrt{p}}{n}\right). \end{aligned} \quad (18)$$

The last equality holds since

$$\begin{aligned} 2(\mu_1 - \mu_2)^T \tilde{V} \tilde{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)) &\sim N(0, 4\sigma^2 \tau \|\tilde{V}^T(\mu_1 - \mu_2)\|^2) \\ &= O_P(\sqrt{\tau} \|\tilde{V}^T(\mu_1 - \mu_2)\|) = o_P\left(\frac{\sqrt{p}}{n}\right). \end{aligned}$$

For  $k = 1, 2$ , we have

$$\frac{1}{n_k} \text{tr}(\tilde{V}^T S_k \tilde{V}) \sim \frac{\sigma^2}{n_k(n_k - 1)} \chi_{(p-r)(n_k-1)}^2 = \sigma^2 \frac{p-r}{n_k} \left(1 + O_P\left(\frac{1}{\sqrt{(p-r)(n_k-1)}}\right)\right),$$

where the last equality comes from central limit theorem. It follows that

$$\frac{1}{n_1} \text{tr}(\tilde{V}^T S_1 \tilde{V}) + \frac{1}{n_2} \text{tr}(\tilde{V}^T S_2 \tilde{V}) = \sigma^2 \tau(p-r) + o_P\left(\frac{\sqrt{p}}{n}\right). \quad (19)$$

Equation (14) and (19) imply that

$$\frac{T_1 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2}{\sigma^2 \sqrt{2\tau^2 p}} = \frac{\|\bar{Y}_1 - \bar{Y}_2\|^2 - \sigma^2 \tau(p-r)}{\sigma^2 \sqrt{2\tau^2 p}} + o_P(1). \quad (20)$$

Since  $\|\tilde{V}^T(\bar{Z}_1 - \bar{Z}_2)\|^2 \sim \sigma^2 \tau \chi_{p-r}^2$ , the proposition follows by central limit theorem.  $\square$

**Proof of Proposition 3.** Note that  $(n-2)S \sim \text{Wishart}_p(n-2, \Sigma)$ . Denote by  $\Sigma = U E U^T$  the spectral decomposition of  $\Sigma$ , where  $U = (V, \tilde{V})$  is an orthogonal

matrix and  $E = \text{diag}(\lambda_1 + \sigma^2, \dots, \lambda_r + \sigma^2, \sigma^2, \dots, \sigma^2)$ . Let  $Z$  be a  $p \times (n-2)$  random matrix with all elements i.i.d. distributed as  $N(0, 1)$ , then

$$S \sim \frac{1}{n-2} U E^{1/2} Z Z^T E^{1/2} U^T.$$

Thus,

$$\hat{\sigma}^2 \sim \frac{1}{(p-r)(n-2)} \sum_{i=r+1}^p \lambda_i(U E^{1/2} Z Z^T E^{1/2} U^T) = \frac{1}{(p-r)(n-2)} \sum_{i=r+1}^{n-2} \lambda_i(Z^T E Z).$$

Denote  $Z = (Z_{(1)}^T, Z_{(2)}^T)^T$ , where  $Z_{(1)}$  and  $Z_{(2)}$  are the first  $r$  rows and last  $p-r$  rows of  $Z$ . We have

$$Z^T E Z = Z_{(1)}^T (\Lambda + \sigma^2 I_r) Z_{(1)} + \sigma^2 Z_{(2)}^T Z_{(2)}.$$

The first term is of rank  $r$ . By Weyl's inequality, we have

$$\sigma^2 \lambda_i(Z_{(2)}^T Z_{(2)}) \leq \lambda_i(Z^T E Z) \leq \sigma^2 \lambda_{i-r}(Z_{(2)}^T Z_{(2)}), \quad i = r+1, \dots, n-2.$$

Thus,

$$\sigma^2 \sum_{i=r+1}^{n-2} \lambda_i(Z_{(2)}^T Z_{(2)}) \leq \sum_{i=r+1}^{n-2} \lambda_i(Z^T E Z) \leq \sigma^2 \sum_{i=1}^{n-r-2} \lambda_i(Z_{(2)}^T Z_{(2)}).$$

It follows that

$$\begin{aligned} & \left| \frac{1}{(p-r)(n-2)} \sum_{i=r+1}^{n-2} \lambda_i(Z^T E Z) - \frac{1}{(p-r)(n-2)} \sigma^2 \sum_{i=1}^{n-2} \lambda_i(Z_{(2)}^T Z_{(2)}) \right| \\ & \leq r \sigma^2 \frac{1}{(p-r)(n-2)} \lambda_1(Z_{(2)}^T Z_{(2)}). \end{aligned}$$

By Corollary 2,  $\lambda_1(Z_{(2)}^T Z_{(2)}) = O_P(\max(n, p))$ . Thus,

$$\begin{aligned} & \frac{1}{(p-r)(n-2)} \sum_{i=r+1}^{n-2} \lambda_i(Z^T E Z) \\ &= \frac{1}{(p-r)(n-2)} \sigma^2 \sum_{i=1}^{n-2} \lambda_i(Z_{(2)}^T Z_{(2)}) + O_P\left(\frac{\max(n, p)}{np}\right) \\ &= \frac{1}{(p-r)(n-2)} \sigma^2 \text{tr}(Z_{(2)}^T Z_{(2)}) + O_P\left(\frac{\max(n, p)}{np}\right) \\ &= \sigma^2 + O_P\left(\frac{1}{\sqrt{np}}\right) + O_P\left(\frac{\max(n, p)}{np}\right). \end{aligned}$$



The last equality comes from central limit theorem. The theorem follows by noting that

$$O_P\left(\frac{1}{\sqrt{np}}\right) = O_P\left(\frac{\sqrt{np}}{np}\right) = O_P\left(\frac{\max(n, p)}{np}\right).$$

□

**Proof of Theorem 3.** Note that  $\text{tr}(\hat{V}^T S_k \hat{V}) = \sum_{i=r+1}^p \lambda_i(S_k)$ ,  $k = 1, 2$ . Similar to Proposition 3, we have  $\text{tr}(\hat{V}^T S_k \hat{V}) = (p - r)\sigma^2 + O_P(\max(n, p)/n)$ ,  $k = 1, 2$ . Then

$$\frac{T_2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2}{\sigma^2 \sqrt{2\tau^2 p}} = \frac{\|\hat{V}^T(\bar{X}_1 - \bar{X}_2)\|^2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 - \sigma^2 \tau(p - r)}{\sigma^2 \sqrt{2\tau^2 p}} + O_P\left(\frac{\max(n, p)}{n\sqrt{p}}\right).$$

By Assumption 2,  $n^{-1}p^{-1/2}\max(n, p) = \max(p^{-1/2}, p^{1/2}/n) \rightarrow 0$ . Note that

$$\begin{aligned} & \frac{\|\hat{V}^T(\bar{X}_1 - \bar{X}_2)\|^2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 - \sigma^2 \tau(p - r)}{\sigma^2 \sqrt{2\tau^2 p}} \\ &= \frac{1}{\sigma^2 \sqrt{2\tau^2 p}} \left( \|\hat{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 - \sigma^2 \tau(p - r) + \right. \\ & \quad \left. 2(\mu_1 - \mu_2)^T \hat{V} \hat{V}^T ((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)) + \|\hat{V}^T(\mu_1 - \mu_2)\|^2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2 \right). \end{aligned}$$

Let

$$\begin{aligned} P_1 &= \|\hat{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 - \sigma^2 \tau(p - r), \\ P_2 &= 2(\mu_1 - \mu_2)^T \hat{V} \hat{V}^T ((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)), \\ P_3 &= \|\hat{V}^T(\mu_1 - \mu_2)\|^2 - \|\tilde{V}^T(\mu_1 - \mu_2)\|^2. \end{aligned}$$

To prove the theorem, it suffices to show that

$$\frac{P_1}{\sigma^2 \sqrt{2\tau^2 p}} \xrightarrow{\mathcal{L}} N(0, 1), \quad \frac{P_2}{\sigma^2 \sqrt{2\tau^2 p}} \xrightarrow{P} 0 \quad \text{and} \quad \frac{P_3}{\sigma^2 \sqrt{2\tau^2 p}} \xrightarrow{P} 0.$$

We first deal with  $P_2$ . To prove the convergence in probability, we only need to prove the convergence in  $L^2$ . Note that  $\bar{X}_1$ ,  $\bar{X}_2$ , and  $S$  are mutually independent and  $\hat{V} \hat{V}^T$  only depends on  $S$ . Thus

$$\begin{aligned} \mathbb{E}P_2^2 &= \mathbb{E}[\mathbb{E}P_2^2 | S] = 4\tau \mathbb{E}[(\mu_1 - \mu_2)^T \hat{V} \hat{V}^T \Sigma \hat{V} \hat{V}^T (\mu_1 - \mu_2)] \\ &\leq 4\tau \mathbb{E}[\lambda_1(\hat{V}^T \Sigma \hat{V})(\mu_1 - \mu_2)^T \hat{V} \hat{V}^T (\mu_1 - \mu_2)] \leq 4\tau \|\mu_1 - \mu_2\|^2 \mathbb{E}[\lambda_1(\hat{V}^T \Sigma \hat{V})] \\ &= O\left(\frac{\sqrt{p}}{n^2}\right) \mathbb{E}[\lambda_1(\hat{V}^T (V \Lambda V^T + \sigma^2 I_p) \hat{V})] \leq O\left(\frac{\sqrt{p}}{n^2}\right) (\kappa p^\beta \mathbb{E}[\lambda_1(\hat{V}^T V V^T \hat{V})] + \sigma^2). \end{aligned}$$

By the relationship

$$\lambda_1(\hat{V}^T V V^T \hat{V}) \leq \text{tr}(\hat{V}^T V V^T \hat{V}) = \frac{1}{2} \|V V^T - \hat{V} \hat{V}^T\|_F^2$$

and Lemma 3, we have that

$$\mathbb{E} P_2^2 = O\left(\frac{\sqrt{p}}{n^2}\right) \left(O\left(\frac{p}{n}\right) + \sigma^2\right) = o\left(\frac{p}{n^2}\right).$$

Next we deal with  $P_3$ . To prove the convergence in probability, we prove the convergence in  $L^1$ .

$$\begin{aligned} \mathbb{E}|P_3| &= \mathbb{E}|(\mu_1 - \mu_2)^T (\hat{V} \hat{V}^T - \tilde{V} \tilde{V}^T)(\mu_1 - \mu_2)| \leq \|\mu_1 - \mu_2\|^2 \mathbb{E} \|\hat{V} \hat{V}^T - \tilde{V} \tilde{V}^T\| \\ &= \|\mu_1 - \mu_2\|^2 \mathbb{E} \|\hat{V} \hat{V}^T - V V^T\| \leq \|\mu_1 - \mu_2\|^2 \sqrt{\mathbb{E} \|\hat{V} \hat{V}^T - V V^T\|^2} \\ &\leq \|\mu_1 - \mu_2\|^2 \sqrt{\mathbb{E} \|\hat{V} \hat{V}^T - V V^T\|_F^2} = O\left(\frac{\sqrt{p}}{n}\right) \sqrt{O\left(\frac{p}{p^\beta n}\right)} = o\left(\frac{\sqrt{p}}{n}\right). \end{aligned}$$

Now we prove the asymptotic normality of  $P_1$ . To make clear the sense of convergence, we need a metric for weak convergence. For two distribution function  $F$  and  $G$ , the Levy metric  $\rho$  of  $F$  and  $G$  is defined as

$$\rho(F, G) = \inf\{\epsilon : F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon \text{ for all } x\}.$$

It's well known that  $\rho(F_n, F) \rightarrow 0$  if and only if  $F_n \xrightarrow{\mathcal{L}} F$ .

The conditional distribution of  $\hat{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))$  given  $S$  is  $N(0, \tau \hat{V}^T \Sigma \hat{V})$ .

It can be seen that

$$\tau^{-1} \|\hat{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 \sim \sum_{i=1}^{p-r} \lambda_i(\hat{V}^T \Sigma \hat{V}) \xi_i^2,$$

where  $\{\xi_i\}_{i=1}^{p-r}$  are i.i.d. standard normal random variables which are independent of  $\hat{V}$ . Note that

$$\lambda_1(\hat{V}^T \Sigma \hat{V}) \leq \frac{1}{2} \kappa p^\beta \|V V^T - \hat{V} \hat{V}^T\|_F^2 + \sigma^2.$$

Hence  $\lambda_i(\hat{V}^T \Sigma \hat{V}) = O_P(p/n + 1)$ ,  $i = 1, \dots, r$ . Moreover, by Weyl's inequality,  $\lambda_i(\hat{V}^T \Sigma \hat{V}) = \sigma^2$ ,  $i = r + 1, \dots, p - r$ . Therefore

$$\text{tr}(\hat{V}^T \Sigma \hat{V})^2 = \left(\frac{p}{n} + 1\right)^2 O_P(1) + (p - 2r) \sigma^4 = p \sigma^4 (1 + o_P(1)). \quad (21)$$

It follows that

$$\frac{\lambda_1^2(\hat{V}^T \Sigma \hat{V})}{\text{tr}(\hat{V}^T \Sigma \hat{V})^2} = O_P\left(\frac{(p/n+1)^2}{p}\right) = o_P(1). \quad (22)$$

Then for every subsequence of  $\{n\}$ , there's a further subsequence along which (22) holds almost surely. By Lemma 1, for every subsequence of  $\{n\}$ , there's a further subsequence along which

$$\rho(\mathcal{L}(Z_n|S), N(0, 1)) \xrightarrow{a.s.} 0, \quad (23)$$

where

$$Z_n = \frac{\|\hat{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 - \tau \text{tr}(\hat{V}^T \Sigma \hat{V})}{\sqrt{2\tau^2 \text{tr}(\hat{V}^T \Sigma \hat{V})^2}},$$

and  $\mathcal{L}(Z_n|S)$  is the conditional distribution of  $Z_n$  given  $S$ . By the definition of weak convergence, for every continuous bounded function  $f(\cdot)$ , we have  $E[f(Z_n)|S] \xrightarrow{a.s.} E[f(\epsilon)]$  along the subsequence, where  $\epsilon \sim N(0, 1)$ . By dominated convergence theorem, we have  $E[f(Z_n)] \rightarrow E[f(\epsilon)]$  along the subsequence. Thus,  $Z_n \xrightarrow{\mathcal{L}} N(0, 1)$  along the subsequence, or equivalently,  $\rho(\mathcal{L}(Z_n, N(0, 1))) \rightarrow 0$  along the subsequence. But this means  $Z_n \xrightarrow{\mathcal{L}} N(0, 1)$ , or

$$\frac{\|\hat{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 - \tau \text{tr}(\hat{V}^T \Sigma \hat{V})}{\sqrt{2\tau^2 \text{tr}(\hat{V}^T \Sigma \hat{V})^2}} \xrightarrow{\mathcal{L}} N(0, 1).$$

Similar to (21) we have

$$\text{tr}(\hat{V}^T \Sigma \hat{V}) = (p - r)\sigma^2\left(1 + O_P\left(\frac{1}{n} + \frac{1}{p}\right)\right). \quad (24)$$

By (21), (24) and Slutsky's theorem,

$$\frac{\|\hat{V}^T((\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2))\|^2 - \sigma^2\tau(p - r)}{\sigma^2\sqrt{2\tau^2 p}} \xrightarrow{\mathcal{L}} N(0, 1).$$

Now the desired asymptotic properties of  $P_1$ ,  $P_2$  and  $P_3$  are established, the theorem follows.  $\square$

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