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Abstract

This paper considers in the high dimensional setting a canonical testing problem, namely testing the equality of multiple mean vectors of normal distribution. By a strategy similar to Roy's union-intersection test, we propose a generalized likelihood ratio test. The critical value is determined by permutation method, which produces an exact test. The limiting distribution of the test statistic is analysed under non-spiked and spiked covariance. Theoretical results and simulation studies show that the test is particularly powerful under spiked covariance.

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1. Introduction

In high dimensional setting, it is known that the likelihood ratio test, which has a dominated position in classical multivariate analysis, is not well defined for many testing problems. It remains a problem how to construct likelihood-based tests in high dimensional setting.

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2. GLRT

Suppose $\{X_{i1}, \dots, X_{in_i}\}$ are i.i.d. distributed as $N(\mu_i, \Sigma)$ for $1 \le i \le K$. Let $\mathbf{X}_i = (X_{i1}, \dots, X_{in_i})$ for $i = 1, \dots, k$. The k samples are independent. μ_i, $i = 1 \dots, k$ and $\Sigma > 0$ are unknown. An interesting problem in multivariate analysis is to test the hypotheses

$$H : \mu_1 = \mu_2 = \cdots = \mu_k \quad v.s. \quad K : \mu_i \neq \mu_j \text{ for some } i \neq j. \tag{1}$$

Let $\mathbf{Z} = (X_1, \dots, X_k)$.

$$f(Z; \mu_1, \dots, \mu_k, \Sigma) = \prod_{i=1}^{k} \Big[(2\pi)^{-n_i p/2} |\Sigma|^{-n_i/2} \exp(-\frac{1}{2} \mathrm{tr} \Sigma^{-1} \sum_{j=1}^{n_i} (x_{ij} - \mu_i)(x_{ij} - \mu_i)^T) \Big].$$

Assume $n = \sum_{i=1}^{p} n_i < p$. Let $a \in \mathbb{R}^p$ be a vector satisfying $a^T a = 1$. Then

$$f_a(a^T Z; \mu_1, \dots, \mu_k, \Sigma) = (2\pi)^{-n/2} |a^T \Sigma a|^{-n/2} \exp\Big(-\frac{1}{2a^T \Sigma a} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (a^T x_{ij} - a^T \mu_i)^2 \Big)$$

$$\max_{\mu_1, \dots, \mu_k, \Sigma} f_a(a^T Z, \mu_1, \dots, \mu_k, \Sigma) = (2\pi)^{-n/2} \big(\sum_{i=1}^{k} \sum_{j=1}^{n_i} (a^T x_{ij} - a^T \bar{\mathbf{X}}_i)^2 \big)^{-n/2} e^{-n/2} \tag{2}$$

Let $S_i = \sum_{j=1}^{n_i} (x_{ij} - \bar{\mathbf{X}}_i)(x_{ij} - \bar{\mathbf{X}}_i)^T$ and $S = \sum_{i=1}^{k} S_i$.

Under H, we have

$$\max_{\mu, \Sigma} f_a(a^T Z, \mu, \dots, \mu, \Sigma) = (2\pi)^{-n/2} \big(\sum_{i=1}^{k} \sum_{j=1}^{n_i} (a^T x_{ij} - a^T \bar{\mathbf{X}})^2 \big)^{-n/2} e^{-n/2} \tag{3}$$

The generalized likelihood ratio test statistic is defined as

$$T(Z) = \max_{a^T a = 1, a^T S a = 0} a^T \sum_{i=1}^{k} n_i (\bar{\mathbf{X}}_i - \bar{\mathbf{X}})(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})^T a \tag{4}$$

Let $J = \mathrm{diag}(n_1^{-1/2} \mathbf{1}_{n_1}, \dots, n_k^{-1/2} \mathbf{1}_{n_k})$. Then $S = Z(I_n - JJ^T) Z^T$ and

$$\sum_{i=1}^{k} n_i (\bar{\mathbf{X}}_i - \bar{\mathbf{X}})(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})^T = Z(JJ^T - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T) Z^T. \tag{5}$$

The matrix $I_n - JJ^T$, $JJ^T - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T$ and $\frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T$ are all projection matrix and pairwise orthogonal with rank $n - k$, $k - 1$ and 1.

Let $\tilde{J}$ be a $n \times (n-k)$ matrix satisfied $\tilde{J}\tilde{J}^T = I - JJ^T$. Then $S = Z\tilde{J}\tilde{J}^T Z^T$ and Note that

$$Z(JJ^T - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T)Z^T = ZJ(I_k - \frac{1}{n}J^T\mathbf{1}_n\mathbf{1}_n^T J)J^T Z^T.$$

Note that $I_k - \frac{1}{n}J^T\mathbf{1}_n\mathbf{1}_n^T J$ is a projection matrix with rank $k-1$. Let C be a $k \times (k-1)$ matrix satisfied $CC^T = I_k - \frac{1}{n}J^T\mathbf{1}_n\mathbf{1}_n^T J$.

In Proposition 1, letting $A = Z\tilde{J}$ and $B = ZJCC^T J^T Z^T$ yields

$$\begin{aligned} T(Z) &= \lambda_{max}\big((I_p - Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1}\tilde{J}^T Z^T)ZJCC^T J^T Z^T(I_p - Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1}\tilde{J}^T Z^T)\big) \\ &= \lambda_{max}\big(C^T J^T Z^T(I_p - Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1}\tilde{J}^T Z^T)ZJC\big). \end{aligned}$$

Note that

$$\begin{aligned} &\Big(\begin{pmatrix} J^T \\ \tilde{J}^T \end{pmatrix} Z^T Z \begin{pmatrix} J & \tilde{J} \end{pmatrix}\Big)^{-1} \\ =&\begin{pmatrix} J^T Z^T ZJ & J^T Z^T Z\tilde{J} \\ \tilde{J}^T Z^T ZJ & \tilde{J}^T Z^T Z\tilde{J} \end{pmatrix}^{-1} = \begin{pmatrix} J^T(Z^T Z)^{-1}J & J^T(Z^T Z)^{-1}\tilde{J} \\ \tilde{J}^T(Z^T Z)^{-1}J & \tilde{J}^T(Z^T Z)^{-1}\tilde{J} \end{pmatrix}. \end{aligned} \tag{6}$$

It follows that

$$\begin{aligned} &\big(J^T(Z^T Z)^{-1}J\big)^{-1} \\ =&J^T Z^T ZJ - J^T Z^T Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1}\tilde{J}^T Z^T ZJ \\ =&J^T Z^T(I_p - Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1}\tilde{J}^T Z^T)ZJ \end{aligned} \tag{7}$$

It follows that

$$T(Z) = \lambda_{\max}\Big(C^T\big(J^T(Z^T Z)^{-1}J\big)^{-1}C\Big) \tag{8}$$

Proposition 1. *Suppose A is a $p \times r$ matrix with rank r and B is a $p \times p$ non-zero semi-definite matrix. Let $H_A = A(A^T A)^{-1}A^T$. Then*

$$\max_{a^T a=1, a^T AA^T a=0} a^T Ba = \lambda_{\max}\big((I_p - H_A)B(I_p - H_A)\big). \tag{9}$$

Proof. Note that $a^T AA^T a = 0$ is equivalent to $A^T a = 0$ and is in turn equivalent to $H_A a = 0$. In this circumstance, $a = (I_p - H_A)a$. Then

$$\begin{aligned} \max_{a^T a=1, a^T AA^T a=0} a^T Ba &= \max_{a^T a=1, H_A a=0} a^T Ba \\ &= \max_{a^T a=1, H_A a=0} a^T(I_p - H_A)B(I_p - H_A)a. \end{aligned} \tag{10}$$

It's obvious that (10) $\leq \lambda_{\max}\big((I-H_A)B(I-H_A)\big)$. On the other hand, let α_1 be one eigenvector corresponding to the largest eigenvalue of $(I - H_A)B(I - H_A)$. Note that the row of H_A are all eigenvetors of $(I-H_A)B(I-H_A)$ corresponding to eigenvalue 0. It follows that $H_A\alpha_1 = 0$. Now that α_1 satisfies the constraint of (10), (10) is maximized when $a = \alpha_1$.

$\square$

3. Schott's method

$$E = ZZ^T - \sum_{i=1}^{k} n_i \bar{X}_i \bar{X}_i^T.$$

$$H = \sum_{i=1}^{k} n_i \bar{X}_i \bar{X}_i^T - n\bar{X}\bar{X}^T.$$

$$\operatorname{tr} E = \operatorname{tr} Z^T Z - \operatorname{tr} J^T Z^T Z J.$$

$$\operatorname{tr} H = \operatorname{tr} J^T Z^T Z J - \frac{1}{n} 1_n^T Z^T Z 1_n$$

$$T_{SC} = \frac{1}{\sqrt{n-1}}\left(\frac{1}{k-1}\operatorname{tr} H - \frac{1}{n-k}\operatorname{tr} E\right)$$

4. Theory

Let $\Sigma = U\Lambda U^T$ be the eigenvalue decomposition of Σ, where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_p)$. Let $U = (U_1, U_2)$ where U_1 is $p \times r$ and U_2 is $p \times (p-r)$. Let $\Lambda_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$ and $\Lambda_2 = \operatorname{diag}(\lambda_{r+1}, \ldots, \lambda_p)$. Then $\Sigma = U_1\Lambda_1 U_1^T + U_2\Lambda_2 U_2^T$.

Let $Z\tilde{J} = U_{Z\tilde{J}} D_{Z\tilde{J}} V_{Z\tilde{J}}^T$ be the singular value decomposition of $Z\tilde{J}$. Let $H_{Z\tilde{J}} = U_{Z\tilde{J}} U_{Z\tilde{J}}^T$. Then $T(Z) = \lambda_{\max}(C^T J^T Z^T (I_p - H_{Z\tilde{J}}) Z J C)$. Note that

$$\mathrm{E}(ZJC) = (\sqrt{n_1}\mu_1, \ldots, \sqrt{n_k}\mu_k)C \stackrel{def}{=} \mu_f.$$

Assumption 1. *Assume $C \geq \lambda_{r+1} \geq \ldots \geq \lambda_p \geq c$, where c and C are absolute constant.*

Theorem 1. *Suppose Assumption* (1) *holds. Suppose*

$$p/n \to \infty, \quad \textit{and} \quad \frac{\lambda_1^2 p}{\lambda_r^2 n^2} \to 0. \tag{11}$$

Suppose

$$\frac{\lambda_r n}{p} \to \infty. \tag{12}$$

Suppose

$$\frac{1}{\sqrt{p}}\|\mu_f\|_F^2 = O(1). \tag{13}$$

Then

$$(\operatorname{tr}\Lambda_2^2)^{-1/2}\big(C^T J^T Z^T (I_p - H_{Z\tilde{J}}) ZJC - (\operatorname{tr}\Lambda_2) I_{k-1} - \mu_f^T (I_p - H_{Z\tilde{Z}})\mu_f\big) \xrightarrow{\mathcal{L}} W_{k-1}, \tag{14}$$

where W_{k-1} is a $(k-1)\times(k-1)$ symmetric random matrix whose entries above the main diagonal are i.i.d. $N(0,1)$ and the entries on the diagonal are i.i.d. $N(0,2)$.

5. Simulation Results

In this section, we evaluate the numerical performance of the new test. For comparison, we also carried out simulation for the test of Tony Cai and Yin Xia and the test of Schott. These tests are denoted respectively by NEW, CX and SC.

In the simulations, we set $k = 3$. Note that the new test is invariant under orthogonal transformation. Without loss of generality, we only consider diagonal Σ. We set $\Sigma = \operatorname{diag}(p, 1, \ldots, 1)$. Define signal-to-noise ratio (SNR) as

$$\mathrm{SNR} = \frac{\|\mu_f\|_F^2}{\sqrt{\sum_{i=2}^p \lambda_i(\Sigma)^2}}.$$

We use SNR to characterize the signal strength. We consider two alternative hypotheses: the non-sparse alternative and the sparse alternative. In the non-sparse case, we set $\mu_1 = \kappa 1_p$, $\mu_2 = -\kappa 1_p$ and $\mu_3 = 0_p$, where κ is selected to make the SNR equal to the given value. In the sparse case, we set $\mu_1 = \kappa(1_{p/5}^T, 0_{4p/5}^T)^T$, $\mu_2 = \kappa(0_{p/5}^T, 1_{p/5}^T, 0_{3p/5}^T)^T$ and $\mu_3 = 0_p$. Again, κ is selected to make the SNR equal to the given value.

Table 1: Empirical powers of tests under non-sparse alternative with $\alpha = 0.05$, $k = 3$, $n_1 = n_2 = n_3 = 10$. Based on 1000 replications.

SNR	$p = 50$			$p = 75$			$p = 100$		
	SC	CX	NEW	SC	CX	NEW	SC	CX	NEW
0	0.035	0.048	0.052	0.057	0.052	0.057	0.053	0.048	0.045
1	0.060	0.049	0.096	0.081	0.050	0.092	0.063	0.062	0.085
2	0.100	0.058	0.140	0.073	0.045	0.169	0.086	0.055	0.171
3	0.145	0.066	0.234	0.119	0.070	0.266	0.117	0.056	0.307
4	0.126	0.064	0.317	0.121	0.059	0.380	0.122	0.061	0.402
5	0.179	0.072	0.392	0.178	0.068	0.541	0.141	0.071	0.579
6	0.198	0.070	0.513	0.189	0.071	0.639	0.143	0.066	0.717
7	0.249	0.085	0.629	0.227	0.084	0.777	0.206	0.073	0.822
8	0.268	0.092	0.685	0.252	0.084	0.822	0.217	0.078	0.894
9	0.324	0.100	0.786	0.256	0.090	0.911	0.246	0.074	0.949
10	0.342	0.115	0.828	0.303	0.097	0.937	0.270	0.075	0.973

Table 2: $n_1 = n_2 = n_3 = 25$, non-sparse

SNR	$p = 100$			$p = 150$			$p = 200$		
	SC	CX	NEW	SC	CX	NEW	SC	CX	NEW
0	2.000	3.000	4.000	5.000	6.000	7.000	8.000	9.000	10.000
0.1									
0.2									

Table 3: $n_1 = n_2 = n_3 = 10$, sparse

SNR	$p = 50$			$p = 75$			$p = 100$		
	SC	CX	NEW	SC	CX	NEW	SC	CX	NEW
0	2.000	3.000	4.000	5.000	6.000	7.000	8.000	9.000	10.000
0.1									
0.2									

6. Appendix

Proof of Theorem 1. It can be seen that ZJC is independent of $Z\tilde{J}$. Since $\mathrm{E}(Z\tilde{J}) = O_{p\times(n-k)}$, we can write $Z\tilde{J} = U\Lambda^{1/2}G_1$, where G_1 is a $p\times(n-k)$ matrix with i.i.d. $N(0,1)$ entries. We write $ZJC = \mu_f + U\Lambda^{1/2}G_2$, where G_2 is a $p\times(k-1)$ matrix with i.i.d. $N(0,1)$ entries.

Then

$$\begin{aligned} C^TJ^TZ^T(I_p - H_{Z\tilde{J}})ZJC =& G_2^T\Lambda^{1/2}U^T(I_P - H_{Z\tilde{J}})U\Lambda_{1/2}G_2 + \mu_f^T(I_p - H_{Z\tilde{J}})\mu_f + \\ & \mu_f^T(I_p - H_{Z\tilde{J}})U\Lambda^{1/2}G_2 + G_2^T\Lambda^{1/2}U^T(I_P - H_{Z\tilde{J}})\mu_f. \end{aligned} \tag{15}$$

To deal the first term, we note that

$$G_2^T\Lambda^{1/2}U^T(I_p - H_{Z\tilde{J}})U\Lambda_{1/2}G_2 \sim \sum_{i=1}^{p}\lambda_i(\Lambda^{1/2}U^T(I_p - H_{Z\tilde{J}})U\Lambda^{1/2})\xi_i\xi_i^T,$$

where $\xi_i \overset{i.i.d.}{\sim} N(0, I_{k-1})$. The key to its asymptotic behavior is the positive eigenvalues of $\Lambda^{1/2}U^T(I_p - H_{Z\tilde{J}})U\Lambda^{1/2}$, which in turn equal to the eigenvalues of $(I_p - H_{Z\tilde{J}})U\Lambda U^T(I_p - H_{Z\tilde{J}})$. Write $(I_p - H_{Z\tilde{J}})U\Lambda U^T(I_p - H_{Z\tilde{J}})$ as the sum of two terms

$$\begin{aligned} &(I_p - H_{Z\tilde{J}})U\Lambda U^T(I_p - H_{Z\tilde{J}}) \\ =&(I_p - H_{Z\tilde{J}})U_1\Lambda_1U_1^T(I_p - H_{Z\tilde{J}}) + (I_p - H_{Z\tilde{J}})U_2\Lambda_2U_2^T(I_p - H_{Z\tilde{J}}) \overset{def}{=} R_1 + R_2. \end{aligned}$$

Table 4: $n_1 = n_2 = n_3 = 25$, sparse

SNR	$p = 100$			$p = 150$			$p = 200$		
	SC	CX	NEW	SC	CX	NEW	SC	CX	NEW
0	2.000	3.000	4.000	5.000	6.000	7.000	8.000	9.000	10.000
0.1									
0.2									

Note that

$$\lambda_{\max}\big(R_1\big) = \lambda_{\max}\big(\Lambda_1^{1/2} U_1^T (I_p - H_{Z\tilde{J}}) U_1 \Lambda_1^{1/2}\big) \le \lambda_{\max}\big(\Lambda_1^{1/2} U_1^T (I_p - U_{Z\tilde{J}[,1:r]} U_{Z\tilde{J}[,1:r]}^T) U_1 \Lambda_1^{1/2}\big)$$
$$\le \lambda_1 \lambda_{\max}\big(U_1^T (I_p - U_{Z\tilde{J}[,1:r]} U_{Z\tilde{J}[,1:r]}^T) U_1\big) = \lambda_1 \lambda_{\max}\big(I_r - U_1^T U_{Z\tilde{J}[,1:r]} U_{Z\tilde{J}[,1:r]}^T U_1\big).$$

To investigate the behavior of $U_{Z\tilde{J}}$, we need to investigate the behavior of $D_{Z\tilde{J}}$ first. Note that $G_1^T \Lambda G_1 = \tilde{J}^T Z^T Z \tilde{J} = V_{Z\tilde{J}} D_{Z\tilde{J}}^2 V_{Z\tilde{J}}^T$, and $G_1^T \Lambda G_1 = G_{1[1:r,]}^T \Lambda_1 G_{1[1:r,]} + G_{1[(r+1):p,]}^T \Lambda_2 G_{1[(r+1):p,]}$. We have

$$V_{Z\tilde{J}} D_{Z\tilde{J}}^2 V_{Z\tilde{J}}^T = G_{1[1:r,]}^T \Lambda_1 G_{1[1:r,]} + G_{1[(r+1):p,]}^T \Lambda_2 G_{1[(r+1):p,]}.$$

For $i = 1, \ldots, r$,

$$\begin{aligned} &\lambda_i(G_{1[1:r,]}^T \Lambda_1 G_{1[1:r,]}) \ge \lambda_i(G_{1[1:r,]}^T \operatorname{diag}(\lambda_i I_i, O_{(r-i)\times(r-i)}) G_{1[1:r,]}) \\ =&\lambda_i \lambda_i(G_{1[1:i,]} G_{1[1:i,]}^T) = \lambda_i n(1 + o_P(1)), \end{aligned} \tag{16}$$

where the last equality holds since $n^{-1} G_{1[1:i,]} G_{1[1:i,]}^T \xrightarrow{P} I_i$ by law of large numbers. On the other hand, for $i = 1, \ldots, r$,

$$\begin{aligned} &\lambda_i(G_{1[1:r,]}^T \Lambda_1 G_{1[1:r,]}) \\ =&\lambda_i\Big(G_{1[1:r,]}^T \big(\operatorname{diag}(\lambda_1, \ldots, \lambda_{i-1}, O_{(r-i+1)\times(r-i+1)}) + \operatorname{diag}(O_{(i-1)\times(i-1)}, \lambda_i, \ldots, \lambda_r)\big) G_{1[1:r,]}\Big) \\ \le&\lambda_1(G_{1[1:r,]}^T \operatorname{diag}(O_{(i-1)\times(i-1)}, \lambda_i, \ldots, \lambda_r) G_{1[1:r,]}) \le \lambda_1(G_{1[1:r,]}^T \operatorname{diag}(O_{(i-1)\times(i-1)}, \lambda_i I_{r-i+1}) G_{1[1:r,]}) \\ =&\lambda_i \lambda_1(G_{1[i:r,]} G_{1[i:r,]}^T) = \lambda_i n(1 + o_P(1)) \end{aligned} \tag{17}$$

where the first inequality holds by Weyl's inequality. It follows from (16) and (17) that $\lambda_i(G_{1[1:r,]}^T \Lambda_1 G_{1[1:r,]}) = \lambda_i n(1 + o_P(1))$ for $i = 1, \ldots, r$.

Note that $\lambda_{\max}(G_{1[(r+1):p,]}^T \Lambda_2 G_{1[(r+1):p,]}) \le C \lambda_{\max}(G_{1[(r+1):p,]}^T G_{1[(r+1):p,]}) = O_P(p)$ by Bai-Yin's law. By assumption $\lambda_r n/p \to \infty$, we can deduce that $D_{Z\tilde{J}[i,i]}^2 = \lambda_i(G_1^T \Lambda G_1) = \lambda_i n(1 + o_P(1))$, $i = 1, \ldots, r$.

Now we are ready to investigate the behavior of $U_{Z\tilde{J}}$. Since $U \Lambda^{1/2} G_1 G_1^T \Lambda^{1/2} U^T = U_{Z\tilde{J}} D_{Z\tilde{J}}^2 U_{Z\tilde{J}}^T$, we have $G_1 G_1^T = \Lambda^{-1/2} U^T U_{Z\tilde{J}} D_{Z\tilde{J}}^2 U_{Z\tilde{J}}^T U \Lambda^{-1/2}$, which further indicates

$$\begin{aligned} &G_{1[(r+1):p,]} G_{1[(r+1):p,]}^T = \Lambda_2^{-1/2} U_{[,(r+1):p]}^T U_{Z\tilde{J}} D_{Z\tilde{J}}^2 U_{Z\tilde{J}}^T U_{[,(r+1):p]} \Lambda_2^{-1/2} \\ \ge&\Lambda_2^{-1/2} U_{[,(r+1):p]}^T U_{Z\tilde{J}[,1:r]} D_{Z\tilde{J}[1:r,1:r]}^2 U_{Z\tilde{J}[,1:r]}^T U_{[,(r+1):p]} \Lambda_2^{-1/2} \\ \ge&D_{Z\tilde{J}[r,r]}^2 \Lambda_2^{-1/2} U_{[,(r+1):p]}^T U_{Z\tilde{J}[,1:r]} U_{Z\tilde{J}[,1:r]}^T U_{[,(r+1):p]} \Lambda_2^{-1/2}. \end{aligned}$$

Thus,

$$\lambda_{\max}(U^T_{[,(r+1):p]}U_{Z\tilde{J}[,1:r]}U^T_{Z\tilde{J}[,1:r]}U_{[,(r+1):p]}) \le \frac{C}{D^2_{Z\tilde{J}[r,r]}}\lambda_1(G_{1[(r+1):p,]}G^T_{1[(r+1):p,]}) = O_P(\frac{p}{\lambda_r n}).$$

Note that we have the simple relationship

$$\begin{aligned}
&\lambda_{\max}(U^T_{[,(r+1):p]}U_{Z\tilde{J}[,1:r]}U^T_{Z\tilde{J}[,1:r]}U_{[,(r+1):p]}) = \lambda_{\max}(U^T_{Z\tilde{J}[,1:r]}U_{[,(r+1):p]}U^T_{[,(r+1):p]}U_{Z\tilde{J}[,1:r]})\\
=&\lambda_{\max}(U^T_{Z\tilde{J}[,1:r]}(I_p - U_1U_1^T)U_{Z\tilde{J}[,1:r]}) = \lambda_{\max}(I_r - U^T_{Z\tilde{J}[,1:r]}U_1U_1^TU_{Z\tilde{J}[,1:r]})\\
=&1 - \lambda_{\min}(U^T_{Z\tilde{J}[,1:r]}U_1U_1^TU_{Z\tilde{J}[,1:r]}) = 1 - \lambda_{\min}(U_1^TU_{Z\tilde{J}[,1:r]}U^T_{Z\tilde{J}[,1:r]}U_1)\\
=&\lambda_{\max}(I_r - U_1^TU_{Z\tilde{J}[,1:r]}U^T_{Z\tilde{J}[,1:r]}U_1).
\end{aligned}$$

Therefore $\lambda_{\max}(I_r - U_1^TU_{Z\tilde{J}[,1:r]}U^T_{Z\tilde{J}[,1:r]}U_1) = O_P(\frac{p}{\lambda_r n})$, and we can conclude $\lambda_{\max}(R_1) = O_P(\frac{\lambda_1 p}{\lambda_r n})$.

We now deal with $R_1 + R_2$. For $i = 1, \ldots, r$,

$$\lambda_i(R_1 + R_2) \le \lambda_1(R_1 + R_2) \le \lambda_1(R_1) + \lambda_1(R_2) \le O_P(\frac{\lambda_1 p}{\lambda_r n}) + C.$$

For $i = r + 1, \ldots, p - r$,

$$\lambda_i(R_1 + R_2) \le \lambda_{i-r}(R_2) = \lambda_{i-r}\big(\Lambda_2^{1/2}U_2^T(I_p - H_{Z\tilde{J}})U_2\Lambda_2^{1/2}\big) \le \lambda_{i-r}(\Lambda_2) = \lambda_i.$$

On the other hand, for $i = 1, \ldots, p - r - n + k$,

$$\begin{aligned}
&\lambda_i(R_1 + R_2) \ge \lambda_i(R_2) = \lambda_i\big(\Lambda_2^{1/2}U_2^T(I_p - H_{Z\tilde{J}})U_2\Lambda_2^{1/2}\big)\\
=&\lambda_i\big(\Lambda_2 - \Lambda_2^{1/2}U_2^TH_{Z\tilde{J}}U_2\Lambda_2^{1/2}\big) \ge \lambda_{i+n-k}.
\end{aligned}$$

The last equality holds since $U_2^TH_{Z\tilde{J}}U_2$ is at most of rank $n - k$.

As a consequence of these bounds, we have

$$\sum_{i=1}^{p-r-n+k}\lambda^2_{i+n-k} \le \operatorname{tr}[(R_1 + R_2)^2] \le r(O_P(\frac{\lambda_1 p}{\lambda_r n}) + C)^2 + \sum_{i=r+1}^{p-r}\lambda_i^2,$$

or

$$|\operatorname{tr}[(R_1 + R_2)^2] - \sum_{i=r+1}^{p}\lambda_i^2| \le \sum_{i=r+1}^{n-k}\lambda_i^2 + \sum_{i=p-r+1}^{p}\lambda_i^2 + r(O_P(\frac{\lambda_1 p}{\lambda_r n}) + C)^2.$$

Similarly,

$$|\operatorname{tr}[(R_1 + R_2)] - \sum_{i=r+1}^{p}\lambda_i| \le \sum_{i=r+1}^{n-k}\lambda_i + \sum_{i=p-r+1}^{p}\lambda_i + r(O_P(\frac{\lambda_1 p}{\lambda_r n}) + C).$$

These, conbined with the assumptions, yield

$$\mathrm{tr}[(R_1+R_2)^2] = (1+o_P(1))\sum_{i=r+1}^{p}\lambda_i^2,$$

and

$$\mathrm{tr}[(R_1+R_2)] = \sum_{i=r+1}^{p}\lambda_i + O(n) + O_P(\frac{\lambda_1 p}{\lambda_r n}).$$

Now we have the Lyapunov condition

$$\frac{\lambda_1[(R_1+R_2)^2]}{\mathrm{tr}[(R_1+R_2)^2]} = \frac{\big(O_P(\frac{\lambda_1 p}{\lambda_r n})+C\big)^2}{(1+o_P(1))\sum_{i=r+1}^{p}\lambda_i^2} \xrightarrow{P} 0.$$

Apply Lyapunov central limit theorem conditioning on $H_{Z\tilde{J}}$, we have

$$\big(\,\mathrm{tr}[(R_1+R_2)^2]\big)^{-1/2}\big(G_2^T\Lambda^{1/2}U^T(I_p-H_{Z\tilde{J}})U\Lambda_{1/2}G_2 - \mathrm{tr}(R_1+R_2)I_{k-1}\big) \xrightarrow{\mathcal{L}} W_{k-1}$$

where W_{k-1} is a $(k-1)\times(k-1)$ symmetric random matrix whose entries above the main diagonal are i.i.d. $N(0,1)$ and the entries on the diagonal are i.i.d. $N(0,2)$. By Slutsky's theorem, we have

$$(\sum_{i=r+1}^{p}\lambda_i^2)^{-1/2}\big(G_2^T\Lambda^{1/2}U^T(I_p-H_{Z\tilde{J}})U\Lambda_{1/2}G_2 - (\sum_{i=r+1}^{p}\lambda_i)I_{k-1}\big) \xrightarrow{\mathcal{L}} W_{k-1}$$

As for the cross term of (15), we have

$$\begin{aligned}
&\mathrm{E}[\|\mu_f^T(I_p-H_{Z\tilde{J}})U\Lambda^{1/2}G_2\|_F^2|Z\tilde{J}]\\
=&(k-1)\,\mathrm{tr}(\mu_f^T(I_p-H_{Z\tilde{J}})U\Lambda U^T(I_p-H_{Z\tilde{J}})\mu_f)\\
\le&(k-1)\lambda_1\big((I_p-H_{Z\tilde{J}})U\Lambda U^T(I_p-H_{Z\tilde{J}})\big)\|\mu_f\|_F^2\\
=&(k-1)O_P(\frac{\lambda_1 p}{\lambda_r n})\|\mu_f\|_F^2\\
=&(k-1)O_P(\frac{\lambda_1\sqrt{p}}{\lambda_r n})\sqrt{p}\|\mu_f\|_F^2 = o_P(p)
\end{aligned}$$

The last equality holds when we assume $\frac{1}{\sqrt{p}}\|\mu_f\|_F^2 = O(1)$. Hence $\|\mu_f^T(I_p - H_{Z\tilde{J}})U\Lambda^{1/2}G_2\|_F^2 = o_P(p)$. This completes the proof of the theorem.

□

References