

# A generalized likelihood ratio test for multivariate analysis of variance in high dimension

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*Radarweg 29, Amsterdam*

*Elsevier Inc<sup>a,b</sup>, Global Customer Service<sup>b,\*</sup>*

<sup>a</sup>*1600 John F Kennedy Boulevard, Philadelphia*

<sup>b</sup>*360 Park Avenue South, New York*

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## Abstract

This paper considers in the high dimensional setting a canonical testing problem, namely testing the equality of multiple mean vectors of normal distribution. By a strategy similar to Roy's union-intersection test, we propose a generalized likelihood ratio test. The critical value is determined by permutation method, which produces an exact test. The limiting distribution of the test statistic is analysed under non-spiked and spiked covariance. Theoretical results and simulation studies show that the test is particularly powerful under spiked covariance.

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## 1. Introduction

Suppose we have independent observations  $X_{ij} \in \mathbb{R}^p$  ( $j = 1, \dots, n_k$ ;  $i = 1, \dots, K$ ) with distribution  $N_p(\mu_i, \Sigma)$ , where  $\mu_i$ ,  $i = 1, \dots, K$ , and  $\Sigma > 0$  are unknown. We would like to test

$$H : \mu_1 = \mu_2 = \dots = \mu_K \quad \text{v.s.} \quad K : \mu_i \neq \mu_j \text{ for some } i \neq j. \quad (1)$$

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<sup>\*</sup>Corresponding author

*Email address:* `support@elsevier.com` (Global Customer Service)

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The problem is known as one-way multivariate analysis of variance (MANOVA). There are four classical tests for hypothesis (1): Wilks' Lambda (which is also the LRT), Hotelling-Lawley trace, Pillai Trace and Roy's maximum root.

In some modern scientific applications, people would like to test hypothesis (1) in high dimensional setting, i.e.,  $p$  is greater than  $n = \sum_{i=1}^K n_i$ . See, for example, [1]. However, when  $p > n - K$ , the LRT for hypothesis (1) is not well defined. Researchers have done extensive work to study the testing problem (1) in high dimensional setting. So far, most tests in the literature are designed for two sample case, i.e.  $K = 2$ . See, for example, [2], [3], [4], [5] and [6]. For the multiple sample case, [7] modified the Dempster's trace test and proposed a test statistic

$$T_{SC} = \frac{1}{\sqrt{n-1}} \left( \frac{1}{K-1} \text{tr} \left( \sum_{i=1}^K n_i \bar{X}_i \bar{X}_i^T - n \bar{X} \bar{X}^T \right) - \frac{1}{n-K} \text{tr} \left( \sum_{i=1}^K \sum_{j=1}^{n_i} X_{ij} X_{ij}^T - \sum_{i=1}^K n_i \bar{X}_i \bar{X}_i^T \right) \right),$$

where  $\bar{X}_i = n_i^{-1} \sum_{j=1}^{n_i} X_{ij}$  and  $\bar{X} = n^{-1} \sum_{i=1}^K \sum_{j=1}^{n_i} X_{ij}$ . In another work, [8] proposed a test statistic

$$T_{CX} = \max_{1 \leq i \leq p} \sum_{1 \leq j < l \leq K} \frac{n_j n_l}{n_j + n_l} \frac{(\Omega(\bar{X}_j - \bar{X}_l))_i^2}{\omega_{ii}},$$

- 5 Where  $\Omega = (\omega)_{ij} = \Sigma^{-1}$  is the precision matrix. When  $\Omega$  is unknown, they substitute it by an estimator  $\hat{\Omega}$ . Statistics  $T_{SC}$  and  $T_{CX}$  are the representatives of two popular methodologies for high dimensional tests.  $T_{SC}$  is a so-called sum-of-squares type statistic as it is based on an estimation of squared Euclidean norm  $\sum_{i=1}^K n_i \|\mu_i - \bar{\mu}\|^2$ , where  $\bar{\mu} = n^{-1} \sum_{i=1}^K n_i \mu_i$ .  $T_{CX}$  is an extreme value  
10 type statistic.

Note that both sum-of-squares type statistic and extreme value type statistic are not based on likelihood function. It remains a problem how to construct likelihood-based tests in high dimensional setting. In a recent work, [9] proposed a generalized likelihood ratio test in the context of one-sample mean vector test.

- 15 Inspired by Roy's union-intersection tests ([10]), they write the null hypothesis as the intersection of a class of component hypotheses. For each component hypotheses, the likelihood ratio test is constructed. Using a least favorable

argument, they construct a test statistics based on these tests. Their simulation results showed that their test has particular good power performance when the variables are dependent.

Following [9]’s methodology, we proposed a generalized likelihood ratio test for hypothesis (1). Most existing tests for hypothesis (1) imposed conditions which prevent from large leading eigenvalues of  $\Sigma$ . However, when the correlations between variables are determined by a small number of factors,  $\Sigma$  is spiked in the sense that a few leading eigenvalues are much larger than the others. See, for example [11] and [12]. We derive the asymptotic distribution of the test statistic under both spiked and non-spiked covariance. Our theoretical results imply that the new test is particularly powerful under spiked covariance. We conduct a simulation study to examine the numerical performance of the test.

The rest of the paper is organized as follows.

Higher criticism CX are special case of UIT.

## 2. Methodology

### 2.1. Roy’s maximum root

Roy’s maximum root test statistic is derived in [10] as an example of Roy’s union intersection principle. The idea of Roy’s union intersection principle is to reduce testing problem to a class of pseudo-univariate problems. For  $a \in \mathbb{R}^p$  and  $a^T a = 1$ , define the hypothesis  $H_a$  and  $K_a$  as

$$H_a : a^T \mu_1 = a^T \mu_2 = \cdots = a^T \mu_k \quad \text{v.s.} \quad K : a^T \mu_i \neq a^T \mu_j \text{ for some } i \neq j.$$

Then the hypothesis (1)  $H = \cap_a H_a$ . The union-intersection principle tells that  $H$  is rejected if and only of any one of  $H_a$  is rejected. Let  $\mathbf{X}_i = (X_{i1}, \dots, X_{in_i})$  be the  $i$ th sample,  $i = 1, \dots, K$ . Let  $\mathbf{Z} = (\mathbf{X}_1, \dots, \mathbf{X}_k)$  be the pool sample. Note that the likelihood function based on  $a^T \mathbf{Z}$  is

$$f_a(a^T \mathbf{Z}; \mu_1, \dots, \mu_k, \Sigma) = (2\pi)^{-n/2} |a^T \Sigma a|^{-n/2} \exp \left( -\frac{1}{2a^T \Sigma a} \sum_{i=1}^K \sum_{j=1}^{n_i} (a^T X_{ij} - a^T \mu_i)^2 \right).$$

The likelihood ratio test statistic for  $H_a$  v.s.  $K_a$  (which is also uniformly most powerful unbiased test) is

$$\text{LRT}_a = \frac{\max_{\mu_1, \dots, \mu_k, \Sigma} f_a(a^T Z, \mu_1, \dots, \mu_k, \Sigma)}{\max_{\mu, \Sigma} f_a(a^T Z, \mu, \dots, \mu, \Sigma)} = \left(1 + \frac{a^T F a}{a^T G a}\right)^{n/2},$$

35 where  $G = \sum_{i=1}^K \sum_{j=1}^{n_i} (X_{ij} - \bar{\mathbf{X}}_i)(X_{ij} - \bar{\mathbf{X}}_i)^T$  and  $F = \sum_{i=1}^K n_i(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})^T$ . Follow the type II method of [10], the union intersection test statistic is  $\max_{a^T a=1} \text{LRT}_a = (1 + \lambda_{\max}(FG^{-1}))^{n/2}$  which is an increase function of  $\lambda_{\max}(FG^{-1})$ , Roy's maximum root test statistic. Note that here we need to assume  $p \leq n - K$ , or else  $G$  is singular.

## 40 2.2. The new test statistic

When  $p > n - K$ , Roy's maximum root test is not defined. Here we follow the strategy in [9].

The generalized likelihood ratio test statistic is defined as

$$T(Z) = \max_{a^T a=1, a^T G a=0} a^T F a \quad (2)$$

Let  $J = \text{diag}(n_1^{-1/2} \mathbf{1}_{n_1}, \dots, n_k^{-1/2} \mathbf{1}_{n_k})$ . Then  $G = Z(I_n - JJ^T)Z^T$  and

$$F = \sum_{i=1}^k n_i(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})^T = Z(JJ^T - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T)Z^T. \quad (3)$$

The matrix  $I_n - JJ^T$ ,  $JJ^T - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T$  and  $\frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T$  are all projection matrix and pairwise orthogonal with rank  $n - k$ ,  $k - 1$  and 1.

Let  $\tilde{J}$  be a  $n \times (n - k)$  matrix satisfied  $\tilde{J}\tilde{J}^T = I - JJ^T$ . Then  $S = Z\tilde{J}\tilde{J}^T Z^T$  and Note that

$$Z(JJ^T - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T)Z^T = ZJ(I_k - \frac{1}{n} J^T \mathbf{1}_n \mathbf{1}_n^T J)J^T Z^T.$$

45 Note that  $I_k - \frac{1}{n} J^T \mathbf{1}_n \mathbf{1}_n^T J$  is a projection matrix with rank  $k - 1$ . Let  $C$  be a  $k \times (k - 1)$  matrix satisfied  $CC^T = I_k - \frac{1}{n} J^T \mathbf{1}_n \mathbf{1}_n^T J$ .

In Proposition 1, letting  $A = Z\tilde{J}$  and  $B = ZJCC^T J^T Z^T$  yields

$$\begin{aligned} T(Z) &= \lambda_{\max}((I_p - Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1} \tilde{J}^T Z^T)ZJCC^T J^T Z^T(I_p - Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1} \tilde{J}^T Z^T)) \\ &= \lambda_{\max}(C^T J^T Z^T(I_p - Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1} \tilde{J}^T Z^T)ZJC). \end{aligned}$$

Note that

$$\begin{aligned} & \left( \begin{pmatrix} J^T \\ \tilde{J}^T \end{pmatrix} Z^T Z \begin{pmatrix} J & \tilde{J} \end{pmatrix} \right)^{-1} \\ &= \begin{pmatrix} J^T Z^T Z J & J^T Z^T Z \tilde{J} \\ \tilde{J}^T Z^T Z J & \tilde{J}^T Z^T Z \tilde{J} \end{pmatrix}^{-1} = \begin{pmatrix} J^T (Z^T Z)^{-1} J & J^T (Z^T Z)^{-1} \tilde{J} \\ \tilde{J}^T (Z^T Z)^{-1} J & \tilde{J}^T (Z^T Z)^{-1} \tilde{J} \end{pmatrix}. \end{aligned} \quad (4)$$

It follows that

$$\begin{aligned} & (J^T (Z^T Z)^{-1} J)^{-1} \\ &= J^T Z^T Z J - J^T Z^T Z \tilde{J} (\tilde{J}^T Z^T Z \tilde{J})^{-1} \tilde{J}^T Z^T Z J \\ &= J^T Z^T (I_p - Z \tilde{J} (\tilde{J}^T Z^T Z \tilde{J})^{-1} \tilde{J}^T Z^T) Z J \end{aligned} \quad (5)$$

It follows that

$$T(Z) = \lambda_{\max} \left( C^T (J^T (Z^T Z)^{-1} J)^{-1} C \right) \quad (6)$$

**Proposition 1.** Suppose  $A$  is a  $p \times r$  matrix with rank  $r$  and  $B$  is a  $p \times p$  non-zero semi-definite matrix. Let  $H_A = A(A^T A)^{-1} A^T$ . Then

$$\max_{a^T a=1, a^T A A^T a=0} a^T B a = \lambda_{\max}((I_p - H_A)B(I_p - H_A)). \quad (7)$$

*Proof.* Note that  $a^T A A^T a = 0$  is equivalent to  $A^T a = 0$  and is in turn equivalent to  $H_A a = 0$ . In this circumstance,  $a = (I_p - H_A)a$ . Then

$$\begin{aligned} \max_{a^T a=1, a^T A A^T a=0} a^T B a &= \max_{a^T a=1, H_A a=0} a^T B a \\ &= \max_{a^T a=1, H_A a=0} a^T (I_p - H_A)B(I_p - H_A)a. \end{aligned} \quad (8)$$

It's obvious that  $(8) \leq \lambda_{\max}((I - H_A)B(I - H_A))$ . On the other hand, let  $\alpha_1$  be one eigenvector corresponding to the largest eigenvalue of  $(I - H_A)B(I - H_A)$ . Note that the row of  $H_A$  are all eigenvectors of  $(I - H_A)B(I - H_A)$  corresponding to eigenvalue 0. It follows that  $H_A \alpha_1 = 0$ . Now that  $\alpha_1$  satisfies the constraint of (8), (8) is maximized when  $a = \alpha_1$ .

□

### 3. Schott's method

$$E = ZZ^T - \sum_{i=1}^k n_i \bar{X}_i \bar{X}_i^T.$$

$$H = \sum_{i=1}^k n_i \bar{X}_i \bar{X}_i^T - n \bar{X} \bar{X}^T.$$

$$\text{tr } E = \text{tr } Z^T Z - \text{tr } J^T Z^T Z J.$$

$$\text{tr } H = \text{tr } J^T Z^T Z J - \frac{1}{n} 1_n^T Z^T Z 1_n$$

$$T_{SC} = \frac{1}{\sqrt{n-1}} \left( \frac{1}{k-1} \text{tr } H - \frac{1}{n-k} \text{tr } E \right)$$

### 4. Theory

55 Let  $\Sigma = U\Lambda U^T$  be the eigenvalue decomposition of  $\Sigma$ , where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ . Let  $U = (U_1, U_2)$  where  $U_1$  is  $p \times r$  and  $U_2$  is  $p \times (p-r)$ . Let  $\Lambda_1 = \text{diag}(\lambda_1, \dots, \lambda_r)$  and  $\Lambda_2 = \text{diag}(\lambda_{r+1}, \dots, \lambda_p)$ . Then  $\Sigma = U_1 \Lambda_1 U_1^T + U_2 \Lambda_2 U_2^T$ .

Let  $Z\tilde{J} = U_{Z\tilde{J}} D_{Z\tilde{J}} V_{Z\tilde{J}}^T$  be the singular value decomposition of  $Z\tilde{J}$ . Let  $H_{Z\tilde{J}} = U_{Z\tilde{J}} U_{Z\tilde{J}}^T$ . Then  $T(Z) = \lambda_{\max}(C^T J^T Z^T (I_p - H_{Z\tilde{J}}) Z J C)$ . Note that

$$E(ZJC) = (\sqrt{n_1} \mu_1, \dots, \sqrt{n_k} \mu_k) C \stackrel{\text{def}}{=} \mu_f.$$

**Assumption 1.** Assume  $C \geq \lambda_{r+1} \geq \dots \geq \lambda_p \geq c$ , where  $c$  and  $C$  are absolute constant.

**Theorem 1.** Suppose Assumption (1) holds. Suppose

$$p/n \rightarrow \infty, \quad \text{and} \quad \frac{\lambda_1^2 p}{\lambda_r^2 n^2} \rightarrow 0. \quad (9)$$

Suppose

$$\frac{\lambda_r n}{p} \rightarrow \infty. \quad (10)$$

Suppose

$$\frac{1}{\sqrt{p}} \|\mu_f\|_F^2 = O(1). \quad (11)$$

Then

$$(\text{tr } \Lambda_2^2)^{-1/2} (C^T J^T Z^T (I_p - H_{Z\tilde{J}}) Z J C - (\text{tr } \Lambda_2) I_{k-1} - \mu_f^T (I_p - H_{Z\tilde{Z}}) \mu_f) \xrightarrow{\mathcal{L}} W_{k-1}, \quad (12)$$

60 where  $W_{k-1}$  is a  $(k-1) \times (k-1)$  symmetric random matrix whose entries above the main diagonal are i.i.d.  $N(0, 1)$  and the entries on the diagonal are i.i.d.  $N(0, 2)$ .

## 5. Simulation Results

In this section, we evaluate the numerical performance of the new test. For  
65 comparison, we also carried out simulation for the test of Tony Cai and Yin Xia and the test of Schott. These tests are denoted respectively by NEW, CX and SC.

In the simulations, we set  $k = 3$ . Note that the new test is invariant under orthogonal transformation. Without loss of generality, we only consider diagonal  $\Sigma$ . We set  $\Sigma = \text{diag}(p, 1, \dots, 1)$ . Define signal-to-noise ratio (SNR) as

$$\text{SNR} = \frac{\|\mu_f\|_F^2}{\sqrt{\sum_{i=2}^p \lambda_i(\Sigma)^2}}.$$

We use SNR to characterize the signal strength. We consider two alternative hypotheses: the non-sparse alternative and the sparse alternative. In the non-  
70 sparse case, we set  $\mu_1 = \kappa 1_p$ ,  $\mu_2 = -\kappa 1_p$  and  $\mu_3 = 0_p$ , where  $\kappa$  is selected to make the SNR equal to the given value. In the sparse case, we set  $\mu_1 = \kappa(1_{p/5}^T, 0_{4p/5}^T)^T$ ,  $\mu_2 = \kappa(0_{p/5}^T, 1_{p/5}^T, 0_{3p/5}^T)^T$  and  $\mu_3 = 0_p$ . Again,  $\kappa$  is selected to make the SNR equal to the given value.

## 6. Appendix

75 *Proof of Theorem 1.* It can be seen that  $ZJC$  is independent of  $Z\tilde{J}$ . Since  $E(Z\tilde{J}) = O_{p \times (n-k)}$ , we can write  $Z\tilde{J} = U\Lambda^{1/2}G_1$ , where  $G_1$  is a  $p \times (n-k)$

Table 1: Empirical powers of tests under non-sparse alternative with  $\alpha = 0.05$ ,  $k = 3$ ,  $n_1 = n_2 = n_3 = 10$ . Based on 1000 replications.

SNR	$p = 50$			$p = 75$			$p = 100$		
	CX	SC	NEW	CX	SC	NEW	CX	SC	NEW
0	0.035	0.048	0.052	0.057	0.052	0.057	0.053	0.048	0.045
1	0.060	0.049	0.096	0.081	0.050	0.092	0.063	0.062	0.085
2	0.100	0.058	0.140	0.073	0.045	0.169	0.086	0.055	0.171
3	0.145	0.066	0.234	0.119	0.070	0.266	0.117	0.056	0.307
4	0.126	0.064	0.317	0.121	0.059	0.380	0.122	0.061	0.402
5	0.179	0.072	0.392	0.178	0.068	0.541	0.141	0.071	0.579
6	0.198	0.070	0.513	0.189	0.071	0.639	0.143	0.066	0.717
7	0.249	0.085	0.629	0.227	0.084	0.777	0.206	0.073	0.822
8	0.268	0.092	0.685	0.252	0.084	0.822	0.217	0.078	0.894
9	0.324	0.100	0.786	0.256	0.090	0.911	0.246	0.074	0.949
10	0.342	0.115	0.828	0.303	0.097	0.937	0.270	0.075	0.973



Table 2: Empirical powers of tests under non-sparse alternative with  $\alpha = 0.05$ ,  $k = 3$ ,  $n_1 = n_2 = n_3 = 25$ . Based on 1000 replications.

SNR	$p = 100$			$p = 150$			$p = 200$		
	CX	SC	NEW	CX	SC	NEW	CX	SC	NEW
0	0.050	0.043	0.050	0.056	0.066	0.048	0.062	0.045	0.054
1	0.069	0.048	0.063	0.046	0.052	0.091	0.068	0.048	0.095
2	0.097	0.046	0.131	0.086	0.053	0.164	0.068	0.057	0.173
3	0.113	0.061	0.200	0.117	0.057	0.270	0.101	0.045	0.313
4	0.135	0.053	0.247	0.130	0.054	0.402	0.118	0.066	0.485
5	0.158	0.065	0.357	0.134	0.066	0.526	0.134	0.073	0.616
6	0.198	0.081	0.433	0.161	0.052	0.668	0.138	0.067	0.765
7	0.217	0.068	0.514	0.191	0.067	0.759	0.174	0.068	0.862
8	0.229	0.063	0.582	0.223	0.075	0.853	0.187	0.060	0.927
9	0.264	0.094	0.680	0.218	0.080	0.918	0.227	0.067	0.966
10	0.298	0.091	0.758	0.245	0.076	0.934	0.228	0.052	0.982

Table 3: Empirical powers of tests under sparse alternative with  $\alpha = 0.05$ ,  $k = 3$ ,  $n_1 = n_2 = n_3 = 10$ . Based on 1000 replications.

SNR	$p = 50$			$p = 75$			$p = 100$		
	CX	SC	NEW	CX	SC	NEW	CX	SC	NEW
0	0.063	0.056	0.052	0.048	0.049	0.048	0.057	0.047	0.042
1	0.087	0.058	0.071	0.069	0.044	0.096	0.076	0.051	0.080
2	0.091	0.066	0.116	0.113	0.037	0.133	0.080	0.058	0.139
3	0.155	0.065	0.177	0.131	0.062	0.228	0.113	0.058	0.218
4	0.184	0.065	0.246	0.174	0.076	0.308	0.144	0.061	0.310
5	0.225	0.081	0.337	0.214	0.075	0.386	0.176	0.083	0.417
6	0.270	0.088	0.425	0.266	0.085	0.507	0.228	0.071	0.508
7	0.364	0.080	0.501	0.307	0.078	0.571	0.302	0.087	0.629
8	0.405	0.105	0.549	0.381	0.080	0.698	0.362	0.089	0.721
9	0.470	0.121	0.634	0.408	0.078	0.774	0.391	0.070	0.797
10	0.547	0.128	0.702	0.484	0.109	0.819	0.415	0.088	0.877

Table 4: Empirical powers of tests under sparse alternative with  $\alpha = 0.05$ ,  $k = 3$ ,  $n_1 = n_2 = n_3 = 25$ . Based on 1000 replications.

SNR	$p = 100$			$p = 150$			$p = 200$		
	CX	SC	NEW	CX	SC	NEW	CX	SC	NEW
0	0.048	0.045	0.046	0.053	0.046	0.043	0.051	0.034	0.046
1	0.079	0.055	0.082	0.066	0.063	0.079	0.063	0.059	0.100
2	0.097	0.054	0.119	0.088	0.055	0.138	0.085	0.055	0.160
3	0.133	0.069	0.167	0.113	0.066	0.223	0.114	0.054	0.235
4	0.149	0.062	0.212	0.126	0.084	0.298	0.132	0.057	0.344
5	0.204	0.060	0.281	0.169	0.066	0.427	0.154	0.057	0.469
6	0.252	0.060	0.352	0.227	0.070	0.548	0.195	0.072	0.641
7	0.310	0.072	0.429	0.252	0.059	0.614	0.220	0.061	0.711
8	0.372	0.088	0.529	0.314	0.085	0.719	0.297	0.060	0.800
9	0.427	0.083	0.547	0.362	0.085	0.794	0.300	0.057	0.881
10	0.449	0.093	0.619	0.396	0.072	0.853	0.340	0.076	0.911

Table 5: Empirical powers of tests under non-sparse alternative with  $\alpha = 0.05$ ,  $k = 3$ ,  $n_1 = n_2 = n_3 = 25$ . The diagonal elements of  $\Sigma$  are generated from  $\text{sort}(\text{Unif}(1,100))$ . Based on 1000 replications.

SNR	$p = 100$			$p = 150$			$p = 200$		
	CX	SC	NEW	CX	SC	NEW	CX	SC	NEW
0	0.063	0.054	0.058	0.052	0.040	0.042	0.045	0.049	0.070
1	0.141	0.120	0.115	0.126	0.120	0.112	0.103	0.110	0.102
2	0.181	0.209	0.169	0.330	0.260	0.210	0.200	0.227	0.201
3	0.692	0.367	0.244	0.759	0.385	0.341	0.468	0.413	0.394
4	0.753	0.539	0.420	0.744	0.573	0.515	0.516	0.554	0.561
5	0.828	0.690	0.509	0.871	0.697	0.693	0.556	0.724	0.727
6	0.809	0.812	0.622	0.822	0.824	0.766	0.959	0.838	0.859
7	1.000	0.882	0.780	0.979	0.916	0.903	0.990	0.923	0.947
8	0.993	0.955	0.789	1.000	0.965	0.954	0.999	0.972	0.971
9	1.000	0.979	0.911	0.999	0.981	0.979	0.964	0.986	0.987
10	1.000	0.991	0.877	0.989	0.996	0.988	0.996	0.996	0.997

Table 6: Empirical powers of tests under sparse alternative with  $\alpha = 0.05$ ,  $k = 3$ ,  $n_1 = n_2 = n_3 = 25$ . The diagonal elements of  $\Sigma$  are generated from  $\text{sort}(\text{Unif}(1,100))$ . Based on 1000 replications.

SNR	$p = 100$			$p = 150$			$p = 200$		
	CX	SC	NEW	CX	SC	NEW	CX	SC	NEW
0	0.052	0.055	0.047	0.055	0.057	0.053	0.044	0.055	0.057
1	0.068	0.124	0.065	0.070	0.130	0.085	0.049	0.116	0.087
2	0.085	0.233	0.112	0.076	0.239	0.149	0.067	0.241	0.161
3	0.110	0.388	0.161	0.090	0.408	0.215	0.097	0.417	0.227
4	0.120	0.530	0.184	0.112	0.552	0.282	0.103	0.556	0.309
5	0.167	0.708	0.238	0.142	0.699	0.387	0.140	0.687	0.394
6	0.196	0.807	0.261	0.168	0.820	0.472	0.162	0.823	0.547
7	0.217	0.875	0.318	0.177	0.892	0.505	0.173	0.896	0.646
8	0.234	0.935	0.378	0.220	0.951	0.625	0.195	0.948	0.749
9	0.312	0.965	0.407	0.222	0.970	0.672	0.224	0.979	0.809
10	0.334	0.976	0.505	0.292	0.987	0.773	0.254	0.989	0.881

matrix with i.i.d.  $N(0, 1)$  entries. We write  $ZJC = \mu_f + U\Lambda^{1/2}G_2$ , where  $G_2$  is a  $p \times (k-1)$  matrix with i.i.d.  $N(0, 1)$  entries.

Then

$$\begin{aligned} C^T J^T Z^T (I_p - H_{Z\bar{j}}) ZJC &= G_2^T \Lambda^{1/2} U^T (I_p - H_{Z\bar{j}}) U \Lambda_{1/2} G_2 + \mu_f^T (I_p - H_{Z\bar{j}}) \mu_f + \\ &\quad \mu_f^T (I_p - H_{Z\bar{j}}) U \Lambda^{1/2} G_2 + G_2^T \Lambda^{1/2} U^T (I_p - H_{Z\bar{j}}) \mu_f. \end{aligned} \quad (13)$$

To deal the first term, we note that

$$G_2^T \Lambda^{1/2} U^T (I_p - H_{Z\bar{j}}) U \Lambda_{1/2} G_2 \sim \sum_{i=1}^p \lambda_i (\Lambda^{1/2} U^T (I_p - H_{Z\bar{j}}) U \Lambda^{1/2}) \xi_i \xi_i^T,$$

where  $\xi_i \stackrel{i.i.d.}{\sim} N(0, I_{k-1})$ . The key to its asymptotic behavior is the positive eigenvalues of  $\Lambda^{1/2} U^T (I_p - H_{Z\bar{j}}) U \Lambda^{1/2}$ , which in turn equal to the eigenvalues of  $(I_p - H_{Z\bar{j}}) U \Lambda U^T (I_p - H_{Z\bar{j}})$ . Write  $(I_p - H_{Z\bar{j}}) U \Lambda U^T (I_p - H_{Z\bar{j}})$  as the sum of two terms

$$\begin{aligned} &(I_p - H_{Z\bar{j}}) U \Lambda U^T (I_p - H_{Z\bar{j}}) \\ &= (I_p - H_{Z\bar{j}}) U_1 \Lambda_1 U_1^T (I_p - H_{Z\bar{j}}) + (I_p - H_{Z\bar{j}}) U_2 \Lambda_2 U_2^T (I_p - H_{Z\bar{j}}) \stackrel{def}{=} R_1 + R_2. \end{aligned}$$

Note that

$$\begin{aligned} \lambda_{\max}(R_1) &= \lambda_{\max}(\Lambda_1^{1/2} U_1^T (I_p - H_{Z\bar{j}}) U_1 \Lambda_1^{1/2}) \leq \lambda_{\max}(\Lambda_1^{1/2} U_1^T (I_p - U_{Z\bar{j}[1:r]} U_{Z\bar{j}[1:r]}^T) U_1 \Lambda_1^{1/2}) \\ &\leq \lambda_1 \lambda_{\max}(U_1^T (I_p - U_{Z\bar{j}[1:r]} U_{Z\bar{j}[1:r]}^T) U_1) = \lambda_1 \lambda_{\max}(I_r - U_1^T U_{Z\bar{j}[1:r]} U_{Z\bar{j}[1:r]}^T U_1). \end{aligned}$$

To investigate the behavior of  $U_{Z\bar{j}}$ , we need to investigate the behavior of  $D_{Z\bar{j}}$  first. Note that  $G_1^T \Lambda G_1 = \tilde{J}^T Z^T Z \tilde{J} = V_{Z\bar{j}} D_{Z\bar{j}}^2 V_{Z\bar{j}}^T$ , and  $G_1^T \Lambda G_1 = G_{1[1:r]}^T \Lambda_1 G_{1[1:r]} + G_{1[(r+1):p]}^T \Lambda_2 G_{1[(r+1):p]}$ . We have

$$V_{Z\bar{j}} D_{Z\bar{j}}^2 V_{Z\bar{j}}^T = G_{1[1:r]}^T \Lambda_1 G_{1[1:r]} + G_{1[(r+1):p]}^T \Lambda_2 G_{1[(r+1):p]}.$$

For  $i = 1, \dots, r$ ,

$$\begin{aligned} \lambda_i(G_{1[1:r]}^T \Lambda_1 G_{1[1:r]}) &\geq \lambda_i(G_{1[1:r]}^T \text{diag}(\lambda_i I_i, O_{(r-i) \times (r-i)}) G_{1[1:r]}) \\ &= \lambda_i \lambda_i (G_{1[1:i]}^T G_{1[1:i]}) = \lambda_i n(1 + o_P(1)), \end{aligned} \quad (14)$$

where the last equality holds since  $n^{-1}G_{1[1:i,]}G_{1[1:i,]}^T \xrightarrow{P} I_i$  by law of large numbers. On the other hand, for  $i = 1, \dots, r$ ,

$$\begin{aligned}
& \lambda_i(G_{1[1:r,]}^T \Lambda_1 G_{1[1:r,]}) \\
&= \lambda_i \left( G_{1[1:r,]}^T \left( \text{diag}(\lambda_1, \dots, \lambda_{i-1}, O_{(r-i+1) \times (r-i+1)}) + \text{diag}(O_{(i-1) \times (i-1)}, \lambda_i, \dots, \lambda_r) \right) G_{1[1:r,]} \right) \\
&\leq \lambda_1(G_{1[1:r,]}^T \text{diag}(O_{(i-1) \times (i-1)}, \lambda_i, \dots, \lambda_r) G_{1[1:r,]}) \leq \lambda_1(G_{1[1:r,]}^T \text{diag}(O_{(i-1) \times (i-1)}, \lambda_i I_{r-i+1}) G_{1[1:r,]}) \\
&= \lambda_i \lambda_1 (G_{1[i:r,]} G_{1[i:r,]}^T) = \lambda_i n(1 + o_P(1))
\end{aligned} \tag{15}$$

where the first inequality holds by Weyl's inequality. It follows from (14)

80 and (15) that  $\lambda_i(G_{1[1:r,]}^T \Lambda_1 G_{1[1:r,]}) = \lambda_i n(1 + o_P(1))$  for  $i = 1, \dots, r$ .

Note that  $\lambda_{\max}(G_{1[(r+1):p,]}^T \Lambda_2 G_{1[(r+1):p,]}) \leq C \lambda_{\max}(G_{1[(r+1):p,]}^T G_{1[(r+1):p,]}) = O_P(p)$  by Bai-Yin's law. By assumption  $\lambda_r n/p \rightarrow \infty$ , we can deduce that  $D_{Z\tilde{J}[i,i]}^2 = \lambda_i(G_1^T \Lambda G_1) = \lambda_i n(1 + o_P(1))$ ,  $i = 1, \dots, r$ .

Now we are ready to investigate the behavior of  $U_{Z\tilde{J}}$ . Since  $U \Lambda^{1/2} G_1 G_1^T \Lambda^{1/2} U^T = U_{Z\tilde{J}} D_{Z\tilde{J}}^2 U_{Z\tilde{J}}^T$ , we have  $G_1 G_1^T = \Lambda^{-1/2} U^T U_{Z\tilde{J}} D_{Z\tilde{J}}^2 U_{Z\tilde{J}}^T U \Lambda^{-1/2}$ , which further indicates

$$\begin{aligned}
& G_{1[(r+1):p,]} G_{1[(r+1):p,]}^T = \Lambda_2^{-1/2} U_{[(r+1):p,]}^T U_{Z\tilde{J}} D_{Z\tilde{J}}^2 U_{Z\tilde{J}}^T U_{[(r+1):p,]} \Lambda_2^{-1/2} \\
&\geq \Lambda_2^{-1/2} U_{[(r+1):p,]}^T U_{Z\tilde{J}[1:r,]} D_{Z\tilde{J}[1:r,]}^2 U_{Z\tilde{J}[1:r,]}^T U_{[(r+1):p,]} \Lambda_2^{-1/2} \\
&\geq D_{Z\tilde{J}[r,r]}^2 \Lambda_2^{-1/2} U_{[(r+1):p,]}^T U_{Z\tilde{J}[1:r,]} U_{Z\tilde{J}[1:r,]}^T U_{[(r+1):p,]} \Lambda_2^{-1/2}.
\end{aligned}$$

Thus,

$$\lambda_{\max}(U_{[(r+1):p,]}^T U_{Z\tilde{J}[1:r,]} U_{Z\tilde{J}[1:r,]}^T U_{[(r+1):p,]}) \leq \frac{C}{D_{Z\tilde{J}[r,r]}^2} \lambda_1(G_{1[(r+1):p,]} G_{1[(r+1):p,]}^T) = O_P\left(\frac{p}{\lambda_r n}\right).$$

Note that we have the simple relationship

$$\begin{aligned}
& \lambda_{\max}(U_{[(r+1):p,]}^T U_{Z\tilde{J}[1:r,]} U_{Z\tilde{J}[1:r,]}^T U_{[(r+1):p,]}) = \lambda_{\max}(U_{Z\tilde{J}[1:r,]}^T U_{[(r+1):p,]} U_{[(r+1):p,]}^T U_{Z\tilde{J}[1:r,]}) \\
&= \lambda_{\max}(U_{Z\tilde{J}[1:r,]}^T (I_p - U_1 U_1^T) U_{Z\tilde{J}[1:r,]}) = \lambda_{\max}(I_r - U_{Z\tilde{J}[1:r,]}^T U_1 U_1^T U_{Z\tilde{J}[1:r,]}) \\
&= 1 - \lambda_{\min}(U_{Z\tilde{J}[1:r,]}^T U_1 U_1^T U_{Z\tilde{J}[1:r,]}) = 1 - \lambda_{\min}(U_1^T U_{Z\tilde{J}[1:r,]} U_{Z\tilde{J}[1:r,]}^T U_1) \\
&= \lambda_{\max}(I_r - U_1^T U_{Z\tilde{J}[1:r,]} U_{Z\tilde{J}[1:r,]}^T U_1).
\end{aligned}$$

Therefore  $\lambda_{\max}(I_r - U_1^T U_{Z\tilde{J}[1:r,]} U_{Z\tilde{J}[1:r,]}^T U_1) = O_P(\frac{p}{\lambda_r n})$ , and we can conclude

$$85 \quad \lambda_{\max}(R_1) = O_P\left(\frac{\lambda_1 p}{\lambda_r n}\right).$$

We now deal with  $R_1 + R_2$ . For  $i = 1, \dots, r$ ,

$$\lambda_i(R_1 + R_2) \leq \lambda_1(R_1 + R_2) \leq \lambda_1(R_1) + \lambda_1(R_2) \leq O_P\left(\frac{\lambda_1 p}{\lambda_r n}\right) + C.$$

For  $i = r + 1, \dots, p - r$ ,

$$\lambda_i(R_1 + R_2) \leq \lambda_{i-r}(R_2) = \lambda_{i-r}(\Lambda_2^{1/2} U_2^T (I_p - H_{Z\bar{J}}) U_2 \Lambda_2^{1/2}) \leq \lambda_{i-r}(\Lambda_2) = \lambda_i.$$

On the other hand, for  $i = 1, \dots, p - r - n + k$ ,

$$\begin{aligned} \lambda_i(R_1 + R_2) &\geq \lambda_i(R_2) = \lambda_i(\Lambda_2^{1/2} U_2^T (I_p - H_{Z\bar{J}}) U_2 \Lambda_2^{1/2}) \\ &= \lambda_i(\Lambda_2 - \Lambda_2^{1/2} U_2^T H_{Z\bar{J}} U_2 \Lambda_2^{1/2}) \geq \lambda_{i+n-k}. \end{aligned}$$

The last equality holds since  $U_2^T H_{Z\bar{J}} U_2$  is at most of rank  $n - k$ .

As a consequence of these bounds, we have

$$\sum_{i=1}^{p-r-n+k} \lambda_{i+n-k}^2 \leq \text{tr}[(R_1 + R_2)^2] \leq r(O_P\left(\frac{\lambda_1 p}{\lambda_r n}\right) + C)^2 + \sum_{i=r+1}^{p-r} \lambda_i^2,$$

or

$$|\text{tr}[(R_1 + R_2)^2] - \sum_{i=r+1}^p \lambda_i^2| \leq \sum_{i=r+1}^{n-k} \lambda_i^2 + \sum_{i=p-r+1}^p \lambda_i^2 + r(O_P\left(\frac{\lambda_1 p}{\lambda_r n}\right) + C)^2.$$

Similarly,

$$|\text{tr}[(R_1 + R_2)] - \sum_{i=r+1}^p \lambda_i| \leq \sum_{i=r+1}^{n-k} \lambda_i + \sum_{i=p-r+1}^p \lambda_i + r(O_P\left(\frac{\lambda_1 p}{\lambda_r n}\right) + C).$$

These, combined with the assumptions, yield

$$\text{tr}[(R_1 + R_2)^2] = (1 + o_P(1)) \sum_{i=r+1}^p \lambda_i^2,$$

and

$$\text{tr}[(R_1 + R_2)] = \sum_{i=r+1}^p \lambda_i + O(n) + O_P\left(\frac{\lambda_1 p}{\lambda_r n}\right).$$

Now we have the Lyapunov condition

$$\frac{\lambda_1[(R_1 + R_2)^2]}{\text{tr}[(R_1 + R_2)^2]} = \frac{(O_P\left(\frac{\lambda_1 p}{\lambda_r n}\right) + C)^2}{(1 + o_P(1)) \sum_{i=r+1}^p \lambda_i^2} \xrightarrow{P} 0.$$



Apply Lyapunov central limit theorem conditioning on  $H_{Z\tilde{J}}$ , we have

$$\left(\text{tr}[(R_1 + R_2)^2]\right)^{-1/2} (G_2^T \Lambda^{1/2} U^T (I_p - H_{Z\tilde{J}}) U \Lambda_{1/2} G_2 - \text{tr}(R_1 + R_2) I_{k-1}) \xrightarrow{\mathcal{L}} W_{k-1}$$

where  $W_{k-1}$  is a  $(k-1) \times (k-1)$  symmetric random matrix whose entries above the main diagonal are i.i.d.  $N(0, 1)$  and the entries on the diagonal are i.i.d.  $N(0, 2)$ . By Slutsky's theorem, we have

$$\left(\sum_{i=r+1}^p \lambda_i^2\right)^{-1/2} (G_2^T \Lambda^{1/2} U^T (I_p - H_{Z\tilde{J}}) U \Lambda_{1/2} G_2 - \left(\sum_{i=r+1}^p \lambda_i\right) I_{k-1}) \xrightarrow{\mathcal{L}} W_{k-1}$$

As for the cross term of (13), we have

$$\begin{aligned} & \mathbb{E}[\|\mu_f^T (I_p - H_{Z\tilde{J}}) U \Lambda^{1/2} G_2\|_F^2 | Z, \tilde{J}] \\ &= (k-1) \text{tr}(\mu_f^T (I_p - H_{Z\tilde{J}}) U \Lambda U^T (I_p - H_{Z\tilde{J}}) \mu_f) \\ &\leq (k-1) \lambda_1 ((I_p - H_{Z\tilde{J}}) U \Lambda U^T (I_p - H_{Z\tilde{J}})) \|\mu_f\|_F^2 \\ &= (k-1) O_P\left(\frac{\lambda_1 p}{\lambda_r n}\right) \|\mu_f\|_F^2 \\ &= (k-1) O_P\left(\frac{\lambda_1 \sqrt{p}}{\lambda_r n}\right) \sqrt{p} \|\mu_f\|_F^2 = o_P(p) \end{aligned}$$

The last equality holds when we assume  $\frac{1}{\sqrt{p}} \|\mu_f\|_F^2 = O(1)$ . Hence  $\|\mu_f^T (I_p - H_{Z\tilde{J}}) U \Lambda^{1/2} G_2\|_F^2 = o_P(p)$ . This completes the proof of the theorem.  $\square$

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