Statistica Sinica

Least Favorable Direction Test for Multivariate Analysis of Variance in High Dimension

Rui Wang, Xingzhong Xu

Beijing Institute of Technology

Abstract: This ~~paper~~ study considers ~~the problem of~~ multivariate analysis of variance for normal samples in ~~the high dimension~~ a high-dimensional medium sample size setting. When the sample dimension is larger than the sample size, the classical likelihood ratio test is not defined ~~since~~ , because the likelihood function is unbounded. Based on ~~the~~ this unboundedness ~~of the likelihood function~~, we propose a new test called the least favorable direction test. The asymptotic distributions of the test statistic are derived ~~under~~ for both nonspiked and spiked covariances. The local asymptotic power function of the test is also given. The results for the asymptotic power function and simulations show that the proposed test is particularly powerful under a spiked covariance.

Note 1

Key words and phrases: High dimensional data, least favorable direction test, multivariate analysis of variance, principal component analysis, spiked covariance.

1. Introduction

Suppose there are k ($k \geq 2$) independent samples of p-dimensional data. Within the ith sample ($1 \leq i \leq k$), the observations $\{X_{ij}\}_{j=1}^{n_i}$ are independent and identically distributed (~~iid~~ i.i.d.) as $\mathcal{N}_p(\theta_i, \boldsymbol{\Sigma})$, ~~the~~ which is a p-dimensional normal distribution with mean vector θ_i and common variance matrix $\boldsymbol{\Sigma}$. We ~~would like to~~ test the following hypotheses: (Insert a colon)

(Insert a comma)

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_k \quad \cancel{\text{v.s.}}\ \text{vs.} \quad H_1 : \theta_i \neq \theta_j, \text{ for some } i \neq j. \tag{1.1}$$

This testing problem is known as a one-way multivariate analysis of variance (MANOVA), and has been well studied when p is small ~~compared with~~ relative to N, where $N = \sum_{i=1}^{k} n_i$ is the total sample size.

Let $\mathbf{H} = \sum_{i=1}^{k} n_i(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})^\top$ be the sum-of-squares between groups, and let $\mathbf{G} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{\mathbf{X}}_i)(X_{ij} - \bar{\mathbf{X}}_i)^\top$ be the sum-of-squares within groups, where $\bar{\mathbf{X}}_i = n_i^{-1} \sum_{j=1}^{n_i} X_{ij}$ is the sample mean of group i, and $\bar{\mathbf{X}} = N^{-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij}$ is the pooled sample mean. There are four classical test statistics for hypotheses (1.1), ~~which are all~~ all of which are based on the eigenvalues of $\mathbf{H}\mathbf{G}^{-1}$.

Wilks' Lambda:	$\lvert\mathbf{G} + \mathbf{H}\rvert / \lvert\mathbf{G}\rvert$
Pillai trace:	$\mathrm{tr}[\mathbf{H}(\mathbf{G} + \mathbf{H})^{-1}]$
(En-dash) Hotelling–Lawley trace:	$\mathrm{tr}[\mathbf{H}\mathbf{G}^{-1}]$
Roy's maximum root:	$\lambda_1(\mathbf{H}\mathbf{G}^{-1})$

In some modern scientific applications, ~~people~~ researchers would like to test hypotheses (1.1) in a high-dimensional setting~~, i.e.,~~ that is, where p is greater than N~~. See, e.g.,~~ ; see, for example, Verstynen et al. (2005) and Tsai and Chen (2009). However, none of the four classical test statistics are defined when $p \geq N$~~, the four classical test statistics are all not defined. Researchers have done extensive work to study~~. As a result, extensive research has been done on the testing problem (1.1) in high-dimensional ~~setting. So~~ settings. Thus far, numerous tests have been proposed for the case $k = 2$~~. See, e.g.,~~ ; see, for example, Bai and Saranadasa (1996), Srivastava (2007), Chen and Qin (2010), Cai et al. (2014); and Feng et al. (2015). ~~Some tests~~ Tests have also been ~~introduced~~ proposed for the general case of ~~general~~ $k \geq 2$. Schott (2007) modified the Hotelling–Lawley trace and proposed the following test statistic:

$$T_{Sc} = \frac{1}{\sqrt{N-1}} \left(\frac{1}{k-1} \operatorname{tr}(\mathbf{H}) - \frac{1}{N-k} \operatorname{tr}(\mathbf{G}) \right).$$

~~Statistic~~ Here, T_{Sc} is a ~~representative~~ member of the so-called sum-of-squares-type statistics ~~as~~ , because it is based on an estimation of the squared Euclidean norm $\sum_{i=1}^{k} n_i \|\theta_i - \bar{\theta}\|^2$, where $\bar{\theta} = N^{-1} \sum_{i=1}^{k} n_i \theta_i$. See Srivastava and Kubokawa (2013), Yamada and Himeno (2015), Hu et al. (2017), Zhang et al. (2017), Zhou et al. (2017); and Cao et al. (2019) for ~~some~~ other sum-of-squares-type test statistics for ~~general~~ $k \geq 2$. ~~It is known that the sum of squares type~~ Sum-of-squares tests are known to be particularly powerful ~~against~~ in the case of dense alternatives. In another work, Cai and

Xia (2014) proposed ~~a~~ the test statistic

$$T_{CX} = \max_{1 \le i \le p} \sum_{1 \le j < l \le k} \frac{n_j n_l}{n_j + n_l} \frac{(\Omega(\bar{\mathbf{X}}_j - \bar{\mathbf{X}}_l))_i^2}{\omega_{ii}},$$

where $\Omega = (\omega)_{ij} = \mathbf{\Sigma}^{-1}$ is the precision matrix. When Ω is unknown, it is substituted by an estimator. Unlike T_{Sc}, ~~the test statistic~~ T_{CX} is an extreme- value ~~type one~~ test statistic, and is ~~very~~ powerful ~~against~~ in the case of sparse alternatives.

Most existing sum-of-squares ~~type~~ test procedures require the condition $\operatorname{tr}(\mathbf{\Sigma}^4)/\operatorname{tr}^2(\mathbf{\Sigma}^2) \to 0$, which is equivalent to

$$\frac{\boldsymbol{\lambda}_1}{\sqrt{\operatorname{tr}(\mathbf{\Sigma}^2)}} \to 0, \tag{1.2}$$

where $\boldsymbol{\lambda}_i$ is the ith largest eigenvalue of $\mathbf{\Sigma}$, for $i = 1, \ldots, p$. In fact, the equivalence of these two conditions can be seen from the following inequalities:

$$\frac{\boldsymbol{\lambda}_1^4}{\operatorname{tr}^2(\mathbf{\Sigma}^2)} \le \frac{\operatorname{tr}(\mathbf{\Sigma}^4)}{\operatorname{tr}^2(\mathbf{\Sigma}^2)} \le \frac{\boldsymbol{\lambda}_1^2 \operatorname{tr}(\mathbf{\Sigma}^2)}{\operatorname{tr}^2(\mathbf{\Sigma}^2)} = \frac{\boldsymbol{\lambda}_1^2}{\operatorname{tr}(\mathbf{\Sigma}^2)}.$$

Condition (1.2) is reasonable if $\mathbf{\Sigma}$ is nonspiked, in the sense that it does not have significantly large eigenvalues. ~~In some important situations, however,~~ However, in practice, variables ~~are~~ may be heavily correlated with common factors, ~~and~~ in which case, the covariance matrix $\mathbf{\Sigma}$ is ~~thus~~ spiked, in the sense that a few eigenvalues of $\mathbf{\Sigma}$ are significantly larger than the others (Fan et al., 2013; Cai et al., 2015; Wang and Fan, 2017). In such cases, condition (1.2) can be violated~~, and~~ and, consequently, existing sum-of-squares type tests may not have the correct level. ~~Some~~ Adjusted

~~adjusted~~ sum-of-squares ~~type~~ test procedures have been proposed to solve ~~the~~ this problem~~. See, e.g.,~~ ; see, for example, Katayama et al. (2013), Ma et al. (2015), Zhang et al. (2017), and Wang and Xu (2019). However, the power behavior of these corrected tests may not be satisfactory.

Recently, Aoshima and Yata (2018) and Wang and Xu (2018) considered a two-sample-mean testing problem under ~~the~~ a spiked covariance model. These tests have better power behavior ~~compared with~~ than that of sum-of-squares-type tests. However, both ~~papers~~ studies imposed strong conditions on the magnitude of p. For example, under the approximate factor model in Fan et al. (2013), the test in Aoshima and Yata (2018) requires $p/n \to 0$, ~~while~~ whereas the test in Wang and Xu (2018) requires that $p/n^2 \to 0$ and that the small eigenvalues of $\mathbf{\Sigma}$ are all equal.

The likelihood ratio test (LRT) method has been very successful in leading to satisfactory procedures in many specific problems. However, the LRT statistic for hypotheses (1.1), ~~i.e.~~ that is, Wilks' ~~Lambda~~ lambda statistic, is not defined for $p > N - k$. In a high-dimensional setting, ~~both~~ neither the sum-of-squares ~~type statistics and~~ nor the extreme-value ~~type~~ statistics are ~~not~~ based on a likelihood function. This motivates us to construct a likelihood-based test in a high-dimensional setting. In a recent work, Zhao and Xu (2016) proposed a generalized likelihood ratio test in the context of a one-sample-mean vector test.

They used a least favorable argument to construct a generalized likelihood ratio test statistic. Their simulation results showed that their test ~~has~~ exhibits good power performance, especially when the variables are correlated. However, ~~this phenomenon is not theoretically proved~~ they do not provide a theoretical proof.

~~In this paper, we~~ We propose a generalized likelihood ratio test statistic for hypotheses (1.1), called the least favorable direction (LFD) test statistic, which is a generalization of the test in Zhao and Xu (2016). We give the asymptotic distributions of the test statistic under both nonspiked and spiked covariances. An adaptive LFD test procedure is constructed by consistently detecting the unknown covariance structure and estimating the unknown parameters. The asymptotic local power function of the LFD test is also given. Our theoretical results show that the LFD test is particularly powerful under ~~the~~ a spiked covariance. This explains the simulation results of Zhao and Xu (2016). ~~Compared with~~ Extending the work of Zhao and Xu (2016), our main contribution is that we ~~give~~ provide a thorough theoretical analysis of the LFD test. ~~Our theoretical~~ This analysis ~~fall into the high dimension~~ falls within a high-dimensional medium sample size setting, where both $n, p \to \infty$, but $p/n \to \infty$ (see Aoshima et al. (2018), Section 5). To prove our main results, we carefully study the high-order asymptotic behavior of the eigenvalues and eigenspaces of the sample covariance matrix. These results are also of independent ~~interests~~ interest. We further compare the proposed test procedure with existing tests ~~by~~ using simulations. ~~It is shown~~ Here, we show that the LFD test ~~has comparable~~ exhibits behavior ~~to~~ comparable with that of existing sum-of-squares tests under ~~the~~ a nonspiked covariance, while significantly ~~outperforms~~ outperforming competing tests under ~~the~~ a spiked covariance.

The rest of the paper is organized as follows. In Section 2, we propose the LFD test statistic and derive its explicit forms. The asymptotic distributions of the LFD test statistic under ~~both~~ nonspiked and spiked covariances are given in Section 3. Based on these theoretical results, an adaptive LFD test procedure is proposed. Section 4 complements our study with numerical simulations. ~~In~~ Section 5~~, we give a short discussion~~ concludes the paper. Finally, the proofs are gathered in the ~~supplementary material~~ Supplementary Material.

2. Least favorable direction test

We ~~introduce some notations~~ first introduce some necessary notation. Define the $p \times N$ pooled sample matrix $\mathbf{X}$ as

$$\mathbf{X} = (X_{11}, X_{12}, \ldots, X_{1n_1}, X_{21}, X_{22}, \ldots, X_{2n_2}, \ldots, X_{k1}, X_{k2}, \ldots, X_{kn_k}).$$

The sum-of-squares within-groups $\mathbf{G}$ can be written as $\mathbf{G} = \mathbf{X}(\mathbf{I}_N - \mathbf{J}\mathbf{J}^\top)\mathbf{X}^\top$,

where

$$\mathbf{J} = \begin{pmatrix} \frac{1}{\sqrt{n_1}}\mathbf{1}_{n_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{n_2}}\mathbf{1}_{n_2} & \mathbf{0} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \frac{1}{\sqrt{n_k}}\mathbf{1}_{n_k} \end{pmatrix}$$

is an $N \times k$ matrix, and $\mathbf{1}_{n_i}$ is an n_i-dimensional vector with all elements equal to ~~1,~~ one, for $i = 1, \ldots, k$. Let $n = N - k$ be the degrees of freedom of $\mathbf{G}$. Construct an $N \times n$ matrix $\tilde{\mathbf{J}}$ as

$$\tilde{\mathbf{J}} = \begin{pmatrix} \tilde{\mathbf{J}}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{J}}_2 & \mathbf{0} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{J}}_k \end{pmatrix},$$

where $\tilde{\mathbf{J}}_i$ is an $n_i \times (n_i - 1)$ matrix defined as

$$\tilde{\mathbf{J}}_i = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{(n_i-2)(n_i-1)}} & \frac{1}{\sqrt{(n_i-1)n_i}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{(n_i-2)(n_i-1)}} & \frac{1}{\sqrt{(n_i-1)n_i}} \\ 0 & -\frac{2}{\sqrt{6}} & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & -\frac{n_i-2}{\sqrt{(n_i-2)(n_i-1)}} & \frac{1}{\sqrt{(n_i-1)n_i}} \\ 0 & 0 & \cdots & 0 & -\frac{n_i-1}{\sqrt{(n_i-1)n_i}} \end{pmatrix}.$$

The matrix $\tilde{\mathbf{J}}$ is a column orthogonal matrix satisfying $\tilde{\mathbf{J}}^\top \tilde{\mathbf{J}} = \mathbf{I}_n$ and $\tilde{\mathbf{J}}\tilde{\mathbf{J}}^\top = \mathbf{I}_N - \mathbf{J}\mathbf{J}^\top$. Define $\mathbf{Y} = \mathbf{X}\tilde{\mathbf{J}}$. Then $\mathbf{G}$ can be written as

$$\mathbf{G} = \mathbf{Y}\mathbf{Y}^\top.$$

The sum-of-squares between groups $\mathbf{H}$ can be written as

$$\mathbf{H} = \mathbf{X}(\mathbf{J}\mathbf{J}^\top - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top)\mathbf{X}^\top = \mathbf{X}\mathbf{J}(\mathbf{I}_k - \frac{1}{N}\mathbf{J}^\top\mathbf{1}_N\mathbf{1}_N^\top\mathbf{J})\mathbf{J}^\top\mathbf{X}^\top.$$

By some matrix algebra, we have $\mathbf{I}_k - N^{-1}\mathbf{J}^\top\mathbf{1}_N\mathbf{1}_N^\top\mathbf{J} = \mathbf{C}\mathbf{C}^\top$ where $\mathbf{C}$ is a $k \times (k-1)$ matrix defined as $\mathbf{C} = \mathbf{C}_1\mathbf{C}_2$, and

$$\mathbf{C}_1 = \begin{pmatrix} \sqrt{n_1} & \sqrt{n_1} & \cdots & \sqrt{n_1} & \sqrt{n_1} \\ -\frac{n_1}{\sqrt{n_2}} & \sqrt{n_2} & \cdots & \sqrt{n_2} & \sqrt{n_2} \\ 0 & -\frac{n_1+n_2}{\sqrt{n_3}} & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & -\frac{\sum_{i=1}^{k-2} n_i}{\sqrt{n_{k-1}}} & \sqrt{n_{k-1}} \\ 0 & 0 & \cdots & 0 & -\frac{\sum_{i=1}^{k-1} n_i}{\sqrt{n_k}} \end{pmatrix},$$

$$\mathbf{C}_2 = \begin{pmatrix} \frac{n_1(n_1+n_2)}{n_2} & 0 & \cdots & 0 \\ 0 & \frac{(\sum_{i=1}^{2} n_i)(\sum_{i=1}^{3} n_i)}{n_3} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{(\sum_{i=1}^{k-1} n_i)(\sum_{i=1}^{k} n_i)}{n_k} \end{pmatrix}^{-\frac{1}{2}}.$$

Then $\mathbf{H}$ can be written as

$$\mathbf{H} = \mathbf{X}\mathbf{J}\mathbf{C}\mathbf{C}^\top\mathbf{J}^\top\mathbf{X}^\top.$$

Define $\boldsymbol{\Theta} = (\sqrt{n_1}\theta_1, \ldots, \sqrt{n_k}\theta_k)$. Then, the null hypothesis H_0 is equivalent to $\boldsymbol{\Theta}\mathbf{C} = \mathbf{O}_{p\times(k-1)}$, where $\mathbf{O}_{p\times(k-1)}$ is a $p \times (k-1)$ matrix with all entries zero ~~0 entries~~. Thus, the hypotheses (1.1) are equivalent to

$$H_0 : \boldsymbol{\Theta}\mathbf{C} = \mathbf{O}_{p\times(k-1)} \quad \overset{\text{vs.}}{\cancel{\text{v.s.}}} \quad H_1 : \boldsymbol{\Theta}\mathbf{C} \neq \mathbf{O}_{p\times(k-1)}.$$

The ~~In low-dimensional setting, the~~ testing problem (1.1) is well studied for low-dimensional settings. A classical test statistic is Roy's maximum root, ~~which is~~ constructed by Roy (1953) using his well-known union intersection principle. The key idea is to decompose ~~data~~ $\mathbf{X}$ into a set of univariate data $\{\mathbf{X}_a = a^\top \mathbf{X} : a \in \mathbb{R}^p, a^\top a = 1\}$. This induces the following decompositions ~~a decomposition~~ of the null ~~hypothesis~~ and ~~the~~ alternative hypotheses ~~hypothesis~~:

$$H_0 = \bigcap_{a\in\mathbb{R}^p, a^\top a=1} H_{0a} \quad \overset{\text{vs.}}{\cancel{\text{v.s.}}} \quad H_1 = \bigcup_{a\in\mathbb{R}^p, a^\top a=1} H_{1a},$$

where $H_{0a} : a^\top \boldsymbol{\Theta}\mathbf{C} = \mathbf{O}_{1\times(k-1)}$ and $H_{1a} : a^\top \boldsymbol{\Theta}\mathbf{C} \neq \mathbf{O}_{1\times(k-1)}$. Let $L_0(a)$ and $L_1(a)$ be the maximum likelihood of $\mathbf{X}_a$ under H_{0a} and H_{1a}, respectively. For each a satisfying $a^\top a = 1$, the component LRT statistic

$$\frac{L_1(a)}{L_0(a)} = \Big(\frac{a^\top(\mathbf{G}+\mathbf{H})a}{a^\top \mathbf{G} a}\Big)^{N/2}$$

can be used to test H_{0a} versus ~~v.s.~~ H_{1a}. Using the union intersection principle, Roy proposed the test statistic $\max_{a^\top a=1} L_1(a)/L_0(a) = (1 + \lambda_1(\mathbf{H}\mathbf{G}^{-1}))^{N/2}$, where $\lambda_i(\cdot)$ denotes ~~means~~ the ith largest eigenvalue. This statistic is an increasing function of Roy's maximum root.

From a likelihood point of view, the log likelihood ratio is an estimator of the Kullback-Leibler divergence between the true distribution and the null distribution. Hence, the component LRT statistic $L_1(a)/L_0(a)$ characterizes the discrepancy between the true ~~distribution~~ and the null distribution along the direction a. This motivates us to consider the direction

$$a^* = \arg\max_{a^\top a=1} \frac{L_1(a)}{L_0(a)}, \tag{2.1}$$

which ~~can~~ hopefully ~~achieve~~ yields the largest discrepancy between the true ~~distribution~~ and the null distribution. Thus, H_{0a^*} is the component null hypothesis ~~most~~ least likely to be ~~not~~ true. We ~~shall~~ call a^* the least favorable direction. Note that Roy's maximum root is ~~in fact~~ the component LRT statistic along the least favorable direction.

Unfortunately, Roy's maximum root can only be defined when $n \geq p$, ~~hence can not~~ and hence cannot be used in a high-dimensional setting. In what follows, we assume $p > n$. In this case, the set

$$\mathcal{A} \stackrel{def}{=} \{a : L_1(a) = +\infty,\ a^\top a = 1\} = \{a : a^\top \mathbf{G} a = 0,\ a^\top a = 1\}$$

is not empty ~~since~~ because $\mathbf{G}$ is singular. Consequently, the right-hand side of (2.1) is not well defined ~~since~~ because the ratio involves infinity. Hence, we need a new definition for the LFD in a high-dimensional setting. Define

$$\mathcal{B} = \{a : L_0(a) = +\infty,\ a^\top a = 1\} = \{a : a^\top (\mathbf{G}+\mathbf{H}) a = 0,\ a^\top a = 1\}.$$

~~It can be seen~~ Note that $\mathcal{B} \subset \mathcal{A}$. Moreover, by the independence of $\mathbf{G}$ and $\mathbf{H}$, with probability ~~1~~ one, we have $\mathcal{A} \cap \mathcal{B}^c \neq \emptyset$. Then, for any direction a, there are three possible scenarios: $L_1(a) < +\infty$ and $L_0(a) < +\infty$; $L_1(a) = +\infty$ and $L_0(a) < +\infty$; and $L_1(a) = +\infty$ and $L_0(a) = +\infty$. To ~~maximizes~~ maximize the discrepancy between $L_1(a)$ and $L_0(a)$, one may consider the direction a, such that $L_1(a) = +\infty$ and $L_0(a) < +\infty$. This suggests that the least favorable direction a^*, which hopefully maximizes the discrepancy between $L_1(a)$ and $L_0(a)$, should be defined as $a^* = \arg\min_{a \in \mathcal{A} \cap \mathcal{B}^c} L_0(a)$. Equivalently,

$$a^* = \arg\min_{a \in \mathcal{A} \cap \mathcal{B}^c} L_0(a) = \arg\max_{a^\top a = 1, a^\top G a = 0} a^\top \mathbf{H} a.$$

Based on a^* and the likelihood $L_0(a)$, we propose a new test statistic,

$$T(\mathbf{X}) = a^{*T} \mathbf{H} a^* = \max_{a^\top a = 1, a^\top \mathbf{G} a = 0} a^\top \mathbf{H} a.$$

The null hypothesis is rejected when $T(\mathbf{X})$ is sufficiently large ~~enough~~. We ~~shall~~ call $T(\mathbf{X})$ the LFD test statistic. ~~Since~~ Because the least favorable direction a^* is obtained from the component likelihood function, the statistic $T(\mathbf{X})$ is also a generalized likelihood ratio test statistic.

Now, we derive the explicit forms of the LFD test statistic. Let $\mathbf{Y} = \mathbf{U}_\mathbf{Y} \mathbf{D}_\mathbf{Y} \mathbf{V}_\mathbf{Y}^\top$ be the singular value decomposition of $\mathbf{Y}$, where $\mathbf{U}_\mathbf{Y}$ and $\mathbf{V}_\mathbf{Y}$ are $p \times \min(n,p)$ and $n \times \min(n,p)$ column orthogonal matrices, respectively, and $\mathbf{D}_\mathbf{Y}$ is a $\min(n,p) \times \min(n,p)$ diagonal matrix ~~whose~~ , with diagonal elements

~~are~~ comprising the non-increasingly ordered singular values of $\mathbf{Y}$. If $p > n$, let $\mathbf{P}_{\mathbf{Y}} = \mathbf{U}_{\mathbf{Y}}\mathbf{U}_{\mathbf{Y}}^{\top}$ be the projection matrix onto the column space of $\mathbf{Y}$. Then, Lemma 1 in ~~supplementary material~~ the Supplementary Material implies that, for $p > n$,

$$T(\mathbf{X}) = \lambda_1\big(\mathbf{C}^{\top}\mathbf{J}^{\top}\mathbf{X}^{\top}(\mathbf{I}_p - \mathbf{P}_{\mathbf{Y}})\mathbf{X}\mathbf{J}\mathbf{C}\big). \tag{2.2}$$

~~While~~ Although (2.2) is convenient for a theoretical analysis, it is not convenient for computation. When $p > N$, another simple form of $T(\mathbf{X})$ can be used for computation. If $p > N$, then $\mathbf{X}^{\top}\mathbf{X}$ is invertible. By the relationship

$$\begin{aligned}
\begin{pmatrix} \mathbf{J}^{\top}\mathbf{X}^{\top}\mathbf{X}\mathbf{J} & \mathbf{J}^{\top}\mathbf{X}^{\top}\mathbf{X}\tilde{\mathbf{J}} \\ \tilde{\mathbf{J}}^{\top}\mathbf{X}^{\top}\mathbf{X}\mathbf{J} & \tilde{\mathbf{J}}^{\top}\mathbf{X}^{\top}\mathbf{X}\tilde{\mathbf{J}} \end{pmatrix}^{-1} &= \left(\begin{pmatrix} \mathbf{J}^{\top} \\ \tilde{\mathbf{J}}^{\top} \end{pmatrix} \mathbf{X}^{\top}\mathbf{X} \begin{pmatrix} \mathbf{J} & \tilde{\mathbf{J}} \end{pmatrix} \right)^{-1} \\
&= \begin{pmatrix} \mathbf{J}^{\top}(\mathbf{X}^{\top}\mathbf{X})^{-1}\mathbf{J} & \mathbf{J}^{\top}(\mathbf{X}^{\top}\mathbf{X})^{-1}\tilde{\mathbf{J}} \\ \tilde{\mathbf{J}}^{\top}(\mathbf{X}^{\top}\mathbf{X})^{-1}\mathbf{J} & \tilde{\mathbf{J}}^{\top}(\mathbf{X}^{\top}\mathbf{X})^{-1}\tilde{\mathbf{J}} \end{pmatrix}
\end{aligned}$$

and the matrix inverse formula, we have that

$$\begin{aligned}
\big(\mathbf{J}^{\top}(\mathbf{X}^{\top}\mathbf{X})^{-1}\mathbf{J}\big)^{-1} =& \mathbf{J}^{\top}\mathbf{X}^{\top}\mathbf{X}\mathbf{J} - \mathbf{J}^{\top}\mathbf{X}^{\top}\mathbf{X}\tilde{\mathbf{J}}(\tilde{\mathbf{J}}^{\top}\mathbf{X}^{\top}\mathbf{X}\tilde{\mathbf{J}})^{-1}\tilde{\mathbf{J}}^{\top}\mathbf{X}^{\top}\mathbf{X}\mathbf{J} \\
=& \mathbf{J}^{\top}\mathbf{X}^{\top}(\mathbf{I}_p - \mathbf{P}_{\mathbf{Y}})\mathbf{X}\mathbf{J}.
\end{aligned}$$

Thus,

$$T(\mathbf{X}) = \lambda_1\Big(\mathbf{C}^{\top}\big(\mathbf{J}^{\top}(\mathbf{X}^{\top}\mathbf{X})^{-1}\mathbf{J}\big)^{-1}\mathbf{C}\Big). \tag{2.3}$$

Compared with (2.2), the expression in (2.3) ~~doesn't~~ does not involve $\mathbf{P}_{\mathbf{Y}}$ and, thus, is more convenient for computation.

In the case of $k = 2$, it can be seen that the least favorable direction is proportional to $(\mathbf{I}_p - \mathbf{P}_{\mathbf{Y}})(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)$, and the LFD test statistic has expression

$$T(\mathbf{X}) = \frac{n_1 n_2}{n_1 + n_2} \|(\mathbf{I}_p - \mathbf{P}_{\mathbf{Y}})(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)\|^2.$$

In this case, the least favorable direction coincides with the maximal data piling direction proposed by Ahn and Marron (2010).

3. Theoretical analysis

We now ~~turn to the analysis of~~ analyze the asymptotic distributions of the LFD test statistic. The normality of the observations is an important assumption for our results, and ~~will be~~ is assumed throughout this section. We ~~shall give~~ present theoretical results ~~under~~ for both nonspiked and spiked covariances. Based on these results, we construct an adaptive test with an asymptotically correct level ~~can be constructed~~. ~~Also~~ In addition, these results allow us to derive the local asymptotic power function of the LFD test.

3.1 Nonspiked covariance

In this subsection, we establish the asymptotic distribution of $T(\mathbf{X})$ under ~~the~~ a nonspiked covariance. Let $\mathbf{W}_{k-1}$ be a $(k-1) \times (k-1)$ symmetric random matrix ~~whose~~ in which the entries above the main diagonal are ~~iid~~ i.i.d. $\mathcal{N}(0, 1)$ random variables, and the entries on the diagonal are ~~iid~~ i.i.d. $\mathcal{N}(0, 2)$ random variables.

The following theorem establishes the asymptotic distribution of the LFD test statistic.

Theorem 1. *Suppose as $n, p \to \infty$, condition (1.2) holds. Furthermore, suppose ~~that~~ $n\boldsymbol{\lambda}_1/\operatorname{tr}(\boldsymbol{\Sigma}) \to 0$ and $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_p = O(n^{-1}\sqrt{\operatorname{tr}(\boldsymbol{\Sigma}^2)})$. Then, under the local alternative hypothesis $\|\mathbf{C}^\top \boldsymbol{\Theta}^\top \boldsymbol{\Theta} \mathbf{C}\| = O(\sqrt{\operatorname{tr}(\boldsymbol{\Sigma}^2)})$,*

$$\frac{T(\mathbf{X}) - (\operatorname{tr}(\boldsymbol{\Sigma}) - n \operatorname{tr}(\boldsymbol{\Sigma}^2)/\operatorname{tr}(\boldsymbol{\Sigma}))}{\sqrt{\operatorname{tr}(\boldsymbol{\Sigma}^2)}} \sim \lambda_1\left(\mathbf{W}_{k-1} + \frac{\mathbf{C}^\top \boldsymbol{\Theta}^\top \boldsymbol{\Theta} \mathbf{C}}{\sqrt{\operatorname{tr}(\boldsymbol{\Sigma}^2)}}\right) + o_P(1),$$

where $\sim$ means having the same distribution.

Remark 1. The condition $n\boldsymbol{\lambda}_1/\operatorname{tr}(\boldsymbol{\Sigma}) \to 0$ implies $p/n \to \infty$. Hence, $T(\mathbf{X})$ is well defined for large n. The condition $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_p = O(n^{-1}\sqrt{\operatorname{tr}(\boldsymbol{\Sigma}^2)})$ requires that the range of the eigenvalues of $\boldsymbol{\Sigma}$ ~~is not~~ not be too large.

To centralize $T(\mathbf{X})$ under the conditions of Theorem 1, we need to estimate the parameters $\operatorname{tr}(\boldsymbol{\Sigma})$ and $\operatorname{tr}(\boldsymbol{\Sigma}^2)$ ~~should be estimated~~. Let $\hat{\boldsymbol{\Sigma}} = n^{-1}\mathbf{G} = n^{-1}\mathbf{Y}\mathbf{Y}^\top$ be the sample covariance matrix. We use the following simple estimators:

$$\widehat{\operatorname{tr}(\boldsymbol{\Sigma})} = \operatorname{tr}(\hat{\boldsymbol{\Sigma}}), \quad \widehat{\operatorname{tr}(\boldsymbol{\Sigma}^2)} = \operatorname{tr}(\hat{\boldsymbol{\Sigma}}^2) - n^{-1}\operatorname{tr}^2(\hat{\boldsymbol{\Sigma}}).$$

Define

$$Q_1 = \frac{T(\mathbf{X}) - \left(\widehat{\operatorname{tr}(\boldsymbol{\Sigma})} - n\widehat{\operatorname{tr}(\boldsymbol{\Sigma}^2)}/\widehat{\operatorname{tr}(\boldsymbol{\Sigma})}\right)}{\sqrt{\widehat{\operatorname{tr}(\boldsymbol{\Sigma}^2)}}}.$$

Let $F_1(x)$ be the cumulative distribution function of $\lambda_1(\mathbf{W}_{k-1})$. Then, we reject the null hypothesis if $Q_1 > F_1^{-1}(1-\alpha)$. The following corollary gives the asymptotic local power function of the proposed test under ~~the~~ a nonspiked covariance.

Corollary 1. *Under the conditions of Theorem 1,*

$$\begin{aligned}
&\Pr\left(Q_1 > F_1^{-1}(1-\alpha)\right)\\
&= \Pr\left(\lambda_1\left(\mathbf{W}_{k-1} + \frac{\mathbf{C}^\top\mathbf{\Theta}^\top\mathbf{\Theta}\mathbf{C}}{\sqrt{\operatorname{tr}(\mathbf{\Sigma}^2)}}\right) > F_1^{-1}(1-\alpha)\right) + o(1).
\end{aligned}$$

Corollary 1 shows that under ~~the~~ a nonspiked covariance, the LFD test ~~has similar~~ exhibits power behavior ~~to~~ similar to that of existing sum-of-squares ~~type~~ tests. In fact, if $k = 2$, the asymptotic local power function given by Corollary 1 is equal to the asymptotic local power function of the tests in Bai and Saranadasa (1996) and Chen and Qin (2010).

3.2 Spiked covariance

Now, we derive the asymptotic results under ~~the~~ a spiked covariance, which ~~are much~~ is more involved than the nonspiked case. Let $\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$ denote the eigenvalue decomposition of $\mathbf{\Sigma}$, where $\mathbf{\Lambda} = \operatorname{diag}(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_p)$ and $\mathbf{U}$ is an orthogonal matrix. Suppose that $\mathbf{\Sigma}$ has r spiked eigenvalues, where $1 \leq r \leq p$ can also vary as $n, p \to \infty$. We ~~shall~~ first assume the spiked number r is known. We later adapt the analysis for ~~Adaptation to~~ unknown r ~~will be considered latter~~. Denote $\mathbf{\Lambda}_1 = \operatorname{diag}(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_r)$ and $\mathbf{\Lambda}_2 = \operatorname{diag}(\boldsymbol{\lambda}_{r+1}, \ldots, \boldsymbol{\lambda}_p)$. Correspondingly, we denote $\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2)$, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are the first r columns and the last $p - r$ columns, respectively, of $\mathbf{U}$. Then, $\boldsymbol{\Sigma} = \mathbf{U}_1 \boldsymbol{\Lambda}_1 \mathbf{U}_1^\top + \mathbf{U}_2 \boldsymbol{\Lambda}_2 \mathbf{U}_2^\top$.

First, we ~~shall~~ derive the asymptotic properties of the eigenvalues and eigenspaces of the sample covariance matrix $\hat{\boldsymbol{\Sigma}}$, because these ~~since they~~ play a key role in our later ~~latter~~ analysis. The following proposition gives the asymptotic behavior of $\lambda_1(\hat{\boldsymbol{\Sigma}}), \ldots, \lambda_r(\hat{\boldsymbol{\Sigma}})$ and $\sum_{i=r+1}^{n} \lambda_i(\hat{\boldsymbol{\Sigma}})$.

Proposition 1. *Suppose ~~that~~ $r \leq n$. Then, uniformly for $i = 1, \ldots, r$,*

$$\lambda_i(\hat{\boldsymbol{\Sigma}}) = \boldsymbol{\lambda}_i + n^{-1} \operatorname{tr}(\boldsymbol{\Lambda}_2) + O_P\left(\boldsymbol{\lambda}_i \sqrt{\frac{r}{n}} + \sqrt{\frac{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}{n}} + \boldsymbol{\lambda}_{r+1} \right)$$

and

$$\sum_{i=r+1}^{n} \lambda_i(\hat{\boldsymbol{\Sigma}}) = \left(1 - \frac{r}{n}\right) \operatorname{tr}(\boldsymbol{\Lambda}_2) + O_P\left(r \sqrt{\frac{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}{n}} + r \boldsymbol{\lambda}_{r+1} \right).$$

Remark 2. Recent works have examined ~~Recently,~~ the asymptotic behavior of the spiked eigenvalues of the sample covariance matrix ; see, for example, ~~is actively studied. See, e.g.,~~ Yata and Aoshima (2013), Shen et al. (2016), Wang and Fan (2017), and Cai et al. (2019). An important improvement of Proposition 1 over existing results is that Proposition 1 does not impose any conditions on ~~condition for~~ the structure of $\boldsymbol{\Sigma}$, but ~~while~~ still gives the correct convergence rate.

Based on Proposition 1, we propose the following estimators of $\operatorname{tr}(\boldsymbol{\Lambda}_2)$

and $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_r$:

$$\widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2)} = \left(1 - \frac{r}{n}\right)^{-1} \sum_{i=r+1}^{n} \lambda_i(\hat{\boldsymbol{\Sigma}}), \quad \hat{\boldsymbol{\lambda}}_i = \lambda_i(\hat{\boldsymbol{\Sigma}}) - n^{-1}\widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2)}, \quad i = 1, \dots, r.$$

Moreover, ~~our latter analysis requires an~~ we propose the following estimator of $\operatorname{tr}(\boldsymbol{\Lambda}_2^2)$, which we use in our later analysis:~~. We propose the following estimator of $\operatorname{tr}(\boldsymbol{\Lambda}_2^2)$,~~

$$\widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)} = \sum_{i=r+1}^{n} \left(\lambda_i(\hat{\boldsymbol{\Sigma}}) - n^{-1}\widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2)}\right)^2.$$

The following proposition gives the convergence rate of these estimators.

Proposition 2. *Suppose ~~that~~ $r = o(n)$. Then, uniformly for $i = 1, \dots, r$,*

$$\hat{\boldsymbol{\lambda}}_i = \boldsymbol{\lambda}_i + O_P\left(\boldsymbol{\lambda}_i\sqrt{\frac{r}{n}} + \sqrt{\frac{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}{n}} + \boldsymbol{\lambda}_{r+1}\right)$$

and

$$\widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2)} = \operatorname{tr}(\boldsymbol{\Lambda}_2) + O_P\left(r\sqrt{\frac{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}{n}} + r\boldsymbol{\lambda}_{r+1}\right),$$
$$\widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)} = \operatorname{tr}(\boldsymbol{\Lambda}_2^2) + O_P\left(\frac{r\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}{n} + r\boldsymbol{\lambda}_{r+1}^2\right).$$

Remark 3. Our estimators of $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_r$ and $\operatorname{tr}(\boldsymbol{\Lambda}_2)$ are similar to some existing estimators, ~~e.g.,~~ including the noise-reduction estimators ~~in~~ of Yata and Aoshima (2012) and the estimators ~~in~~ of Wang and Fan (2017). However, their theoretical results require that r is fixed, p is not large, and $\boldsymbol{\Sigma}$ satisfies certain spiked covariance models.

Remark 4. The estimation of $\operatorname{tr}(\mathbf{\Lambda}_2^2)$ is relatively unexplored. Recently, Aoshima and Yata (2018) proposed an estimator of $\operatorname{tr}(\mathbf{\Lambda}_2^2)$ based on ~~by using~~ the cross-data-matrix methodology. They also proved the consistency of their estimator. However, their ~~Their~~ method relies~~, however,~~ on an arbitrary split of the data into two samples of equal size.

Next, we consider the asymptotic behavior of the eigenspaces of $\hat{\mathbf{\Sigma}}$. Let $\mathbf{U}_{\mathbf{Y},1}$ denote the first r columns of $\mathbf{U}_{\mathbf{Y}}$. Then, the columns of $\mathbf{U}_{\mathbf{Y},1}$ are the principal eigenvectors of $\hat{\mathbf{\Sigma}}$, and $\mathbf{P}_{\mathbf{Y},1} = \mathbf{U}_{\mathbf{Y},1}\mathbf{U}_{\mathbf{Y},1}^\top$ is the projection matrix onto the rank r principal subspace of $\hat{\mathbf{\Sigma}}$. The properties of $\mathbf{P}_{\mathbf{Y},1}$ and the individual principal eigenvectors have been studied extensively ~~studied~~. See Cai et al. (2015), Shen et al. (2016), and Wang and Fan (2017), and the references therein. Existing ~~The existing~~ results include the consistency of the principal subspace and the high-order asymptotic behavior of the individual principal eigenvectors. However, these results are not sufficient ~~enough~~ for our ~~latter~~ analysis. The following proposition gives the high-order asymptotic behavior of $\mathbf{P}_{\mathbf{Y},1}$. To the best of our knowledge, this is a novel result ~~such result has never appeared~~ in the literature ~~before~~.

Write $\mathbf{Y} = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{Z}$, where $\mathbf{Z}$ is a $p \times n$ random matrix with i.i.d. ~~iid~~ $\mathcal{N}(0,1)$ entries. Then, $\mathbf{Y} = \mathbf{U}_1\mathbf{\Lambda}_1^{1/2}\mathbf{Z}_1 + \mathbf{U}_2\mathbf{\Lambda}_2^{1/2}\mathbf{Z}_2$, where $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are the first r rows and the last $p-r$ rows, respectively, of $\mathbf{Z}$.

Proposition 3. *Suppose ~~that~~ $r = o(n)$, $\operatorname{tr}(\mathbf{\Lambda}_2)/(n\boldsymbol{\lambda}_r) \to 0$, and $r\boldsymbol{\lambda}_{r+1}/\operatorname{tr}(\mathbf{\Lambda}_2) \to$*

0. *Then,*

$$\left\|\mathbf{P}_{\mathbf{Y},1} - \mathbf{P}^{\dagger}_{\mathbf{Y},1}\right\| = O_P\left(\frac{\operatorname{tr}(\mathbf{\Lambda}_2)}{n\boldsymbol{\lambda}_r} + \frac{\boldsymbol{\lambda}_{r+1}}{\boldsymbol{\lambda}_r}\right),$$

where $\|\cdot\|$ *is the spectral norm,* $\mathbf{P}^{\dagger}_{\mathbf{Y},1} = \mathbf{U}_1\mathbf{U}_1^\top + \mathbf{U}_1\mathbf{Q}^\top\mathbf{U}_2^\top + \mathbf{U}_2\mathbf{Q}\mathbf{U}_1^\top$, *and*

$\mathbf{Q} = \mathbf{\Lambda}_2^{1/2}\mathbf{Z}_2\mathbf{Z}_1^\top(\mathbf{Z}_1\mathbf{Z}_1^\top)^{-1}\mathbf{\Lambda}_1^{-1/2}$.

Remark 5. The condition $\operatorname{tr}(\mathbf{\Lambda}_2)/(n\boldsymbol{\lambda}_r) \to 0$ is commonly adopted in ~~the study of the~~ studies on principal subspaces. In fact, when this condition is violated, the principal subspace ~~will lose~~ loses its relation to the rank-r eigenspace of $\mathbf{\Sigma}$; ~~See, e.g.,~~ see, for example, Nadler (2008).

Remark 6. ~~Recently, some~~ Several high-order Davis–Kahan theorems ~~are~~ have been established, ~~e.g.,~~ for example, Lemma 2 in Koltchinskii and Lounici (2016) and Lemma 2 in Fan et al. (2019). These general results explicitly ~~characterizes~~ characterize the linear term and the high-order error on the rank-r eigenspace ~~due~~, owing to matrix perturbation. ~~By applying~~ Applying these results to $\hat{\mathbf{\Sigma}}$ and $\mathbf{\Sigma}$, ~~one can~~ we obtain similar results to those given in Proposition 3~~. Compared with Proposition 3, however, the results so obtained~~; however, the above results are slightly weaker and ~~requires~~ require stronger conditions.

If $p > n$, let $\mathbf{U}_{\mathbf{Y},2}$ be the $r+1$ to nth columns of $\mathbf{U}_{\mathbf{Y}}$. Then, $\mathbf{P}_{\mathbf{Y},2} = \mathbf{U}_{\mathbf{Y},2}\mathbf{U}_{\mathbf{Y},2}^\top$ is the projection matrix onto the eigenspace spanned by the $r+1$ to nth eigenvectors of $\hat{\mathbf{\Sigma}}$. Our ~~latter~~ later analysis also requires the asymptotic properties of $\mathbf{P}_{\mathbf{Y},2}$, which ~~has~~ have not been considered in the literature. Let

$\mathbf{V}_{\mathbf{Z}_1} = \mathbf{Z}_1^\top(\mathbf{Z}_1\mathbf{Z}_1^\top)^{-1/2}$. Then, $\mathbf{V}_{\mathbf{Z}_1}\mathbf{V}_{\mathbf{Z}_1}^\top = \mathbf{Z}_1^\top(\mathbf{Z}_1\mathbf{Z}_1^\top)^{-1}\mathbf{Z}_1$ is the projection matrix onto the row space of $\mathbf{Z}_1$. Let $\tilde{\mathbf{V}}_{\mathbf{Z}_1}$ be ~~a~~ an $n\times(n-r)$-column orthogonal matrix ~~which~~ that satisfies $\tilde{\mathbf{V}}_{\mathbf{Z}_1}\tilde{\mathbf{V}}_{\mathbf{Z}_1}^\top = \mathbf{I}_n - \mathbf{V}_{\mathbf{Z}_1}\mathbf{V}_{\mathbf{Z}_1}^\top$. The following proposition gives the asymptotic behavior of $\mathbf{P}_{\mathbf{Y},2}$.

Proposition 4. *Suppose ~~that~~ $r = o(n)$, $\operatorname{tr}(\mathbf{\Lambda}_2)\boldsymbol{\lambda}_1/(n\boldsymbol{\lambda}_r^2) \to 0$, and $n\boldsymbol{\lambda}_{r+1}/\operatorname{tr}(\mathbf{\Lambda}_2) \to 0$. Then,*

$$\left\|\mathbf{P}_{\mathbf{Y},2} - \mathbf{P}_{\mathbf{Y},2}^\dagger\right\| = O_P\left(\sqrt{\frac{\operatorname{tr}(\mathbf{\Lambda}_2)\boldsymbol{\lambda}_1}{n\boldsymbol{\lambda}_r^2}} + \sqrt{\frac{n\boldsymbol{\lambda}_{r+1}}{\operatorname{tr}(\mathbf{\Lambda}_2)}}\right),$$

where $\mathbf{P}_{\mathbf{Y},2}^\dagger = (\operatorname{tr}(\mathbf{\Lambda}_2))^{-1}\,\mathbf{U}_2\mathbf{\Lambda}_2^{1/2}\mathbf{Z}_2\tilde{\mathbf{V}}_{\mathbf{Z}_1}\tilde{\mathbf{V}}_{\mathbf{Z}_1}^\top\mathbf{Z}_2^\top\mathbf{\Lambda}_2^{1/2}\mathbf{U}_2^\top$.

Remark 7. The condition $\operatorname{tr}(\mathbf{\Lambda}_2)\boldsymbol{\lambda}_1/(n\boldsymbol{\lambda}_r^2) \to 0$ is stronger than the condition $\operatorname{tr}(\mathbf{\Lambda}_2)/(n\boldsymbol{\lambda}_r) \to 0$ in Proposition 3. These two conditions are equivalent if $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_r$ are of the same order.

Now, we are ready to derive the asymptotic properties of $T(\mathbf{X})$ under ~~the~~ a spiked covariance. Let $\mathbf{W}_{k-1}^*$ be a $(k-1)\times(k-1)$ symmetric random matrix, distributed as $\text{Wishart}(r, \mathbf{I}_{k-1})$ and ~~is~~ independent of $\mathbf{W}_{k-1}$, where $\text{Wishart}(m, \mathbf{\Psi})$ is the Wishart distribution with parameter $\mathbf{\Psi}$ and m degrees of freedom. The following theorem gives the asymptotic distribution of $T(\mathbf{X})$ under the null and ~~the~~ local alternative ~~hypothesis~~ hypotheses.

Theorem 2. *Suppose ~~that~~ $r = o(\sqrt{n})$, $r\operatorname{tr}(\mathbf{\Lambda}_2)\boldsymbol{\lambda}_1/(n\boldsymbol{\lambda}_r^2) \to 0$, $rn\boldsymbol{\lambda}_{r+1}/\operatorname{tr}(\mathbf{\Lambda}_2) \to 0$, $r\boldsymbol{\lambda}_{r+1}/\sqrt{\operatorname{tr}(\mathbf{\Lambda}_2^2)} \to 0$, and $\boldsymbol{\lambda}_{r+1} - \boldsymbol{\lambda}_p = O(n^{-1}\sqrt{\operatorname{tr}(\mathbf{\Lambda}_2^2)})$. Then,* *(i) under the null hypothesis* $\mathbf{\Theta C} = \mathbf{O}_{p\times(k-1)}$,

$$\begin{aligned}&\frac{T(\mathbf{X}) - ((1+r/n)\operatorname{tr}(\mathbf{\Lambda}_2) - n\operatorname{tr}(\mathbf{\Lambda}_2^2)/\operatorname{tr}(\mathbf{\Lambda}_2))}{\sqrt{rn^{-2}\operatorname{tr}^2(\mathbf{\Lambda}_2)+\operatorname{tr}(\mathbf{\Lambda}_2^2)}}\\ &\sim\lambda_1\left(\frac{n^{-1}\operatorname{tr}(\mathbf{\Lambda}_2)}{\sqrt{rn^{-2}\operatorname{tr}^2(\mathbf{\Lambda}_2)+\operatorname{tr}(\mathbf{\Lambda}_2^2)}}(\mathbf{W}^*_{k-1} - r\mathbf{I}_{k-1})\right.\\ &\qquad\left. + \frac{\sqrt{\operatorname{tr}(\mathbf{\Lambda}_2^2)}}{\sqrt{rn^{-2}\operatorname{tr}^2(\mathbf{\Lambda}_2)+\operatorname{tr}(\mathbf{\Lambda}_2^2)}}\mathbf{W}_{k-1}\right) + o_P(1);\end{aligned}$$

(ii) if $r\to\infty$ *or* $\operatorname{tr}(\mathbf{\Lambda}_2)/(n\sqrt{\operatorname{tr}(\mathbf{\Lambda}_2^2)})\to 0$, *then under the local alternative hypothesis* $\|\mathbf{C}^\top\mathbf{\Theta}^\top\mathbf{\Theta C}\| = O(\sqrt{rn^{-2}\operatorname{tr}^2(\mathbf{\Lambda}_2)+\operatorname{tr}(\mathbf{\Lambda}_2^2)})$,

$$\begin{aligned}&\frac{T(\mathbf{X}) - ((1+r/n)\operatorname{tr}(\mathbf{\Lambda}_2) - n\operatorname{tr}(\mathbf{\Lambda}_2^2)/\operatorname{tr}(\mathbf{\Lambda}_2))}{\sqrt{rn^{-2}\operatorname{tr}^2(\mathbf{\Lambda}_2)+\operatorname{tr}(\mathbf{\Lambda}_2^2)}}\\ &\sim\lambda_1\left(\frac{n^{-1}\operatorname{tr}(\mathbf{\Lambda}_2)}{\sqrt{rn^{-2}\operatorname{tr}^2(\mathbf{\Lambda}_2)+\operatorname{tr}(\mathbf{\Lambda}_2^2)}}(\mathbf{W}^*_{k-1} - r\mathbf{I}_{k-1})\right.\\ &\qquad + \frac{\sqrt{\operatorname{tr}(\mathbf{\Lambda}_2^2)}}{\sqrt{rn^{-2}\operatorname{tr}^2(\mathbf{\Lambda}_2)+\operatorname{tr}(\mathbf{\Lambda}_2^2)}}\mathbf{W}_{k-1}\\ &\qquad\left. + \frac{\mathbf{C}^\top\mathbf{\Theta}^\top\mathbf{U}_2\mathbf{U}_2^\top\mathbf{\Theta C}}{\sqrt{rn^{-2}\operatorname{tr}^2(\mathbf{\Lambda}_2)+\operatorname{tr}(\mathbf{\Lambda}_2^2)}}\right) + o_P(1).\end{aligned}$$

Remark 8. Suppose the approximate factor model in Fan et al. (2013) holds. That is, r is fixed, $\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r$ diverge at rate $O(p)$, and $\boldsymbol{\lambda}_{r+1},\dots,\boldsymbol{\lambda}_p$ are bounded. Then, the conditions of Theorem 2 become $p/n\to\infty$ and $\boldsymbol{\lambda}_{r+1}-\boldsymbol{\lambda}_p = O(\sqrt{p}/n)$. Hence, Theorem 2 holds for ~~ultra-high-dimensional~~ ultrahigh-dimensional data. In contrast, ~~recently proposed~~ recent tests ~~under~~ for the spiked covariance model can only be used for lower-dimensional data. In fact, under the approximate factor model in Fan et al. (2013), Aoshima and Yata (2018) requires $p/n\to$

0, and ~~while~~ Wang and Xu (2018) requires $p/n^2 \to 0$ and $\boldsymbol{\lambda}_{r+1} = \cdots = \boldsymbol{\lambda}_p$. Note ~~We note~~ that if $k = 2$ and $p/n^2 \to 0$, then the coefficient of $\mathbf{W}_{k-1}^* - r\mathbf{I}_{k-1}$ is negligible and, as a result, ~~, and consequently,~~ $T(\mathbf{X})$ is asymptotically normally ~~normal~~ distributed. Thus, Theorem 2 gives the high-order behavior of $T(\mathbf{X})$.

Now, we formulate a test procedure with an asymptotically correct level. Define the standardized statistic as

$$Q_2 = \frac{T(\mathbf{X}) - \left((1 + r/n)\widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2)} - n\widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}/\widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2)}\right)}{\sqrt{rn^{-2}(\widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2)})^2 + \widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}}}.$$

Let $F_2(x; \operatorname{tr}(\boldsymbol{\Lambda}_2), \operatorname{tr}(\boldsymbol{\Lambda}_2^2))$ be the cumulative distribution function of

$$\lambda_1 \left(\frac{n^{-1}\operatorname{tr}(\boldsymbol{\Lambda}_2)}{\sqrt{rn^{-2}\operatorname{tr}^2(\boldsymbol{\Lambda}_2) + \operatorname{tr}(\boldsymbol{\Lambda}_2^2)}} (\mathbf{W}_{k-1}^* - r\mathbf{I}_{k-1}) + \frac{\sqrt{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}}{\sqrt{rn^{-2}\operatorname{tr}^2(\boldsymbol{\Lambda}_2) + \operatorname{tr}(\boldsymbol{\Lambda}_2^2)}} \mathbf{W}_{k-1} \right).$$

Then, we reject the null hypothesis if

$$Q_2 > F_2^{-1}\left(1 - \alpha; \widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2)}, \widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}\right).$$

The following corollary shows that this test procedure has an asymptotically correct level, as well as giving ~~and also gives~~ the asymptotic local power function.

Corollary 2. *Suppose the conditions of Theorem 2 hold. Then,*

(i) under the null hypothesis $\boldsymbol{\Theta}\mathbf{C} = \mathbf{O}_{p\times(k-1)}$*,*

$$\Pr\left(Q_2 > F_2^{-1}\left(1 - \alpha; \widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2)}, \widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}\right)\right) = \alpha + o(1);$$

(ii) if $r \to \infty$ *or* $\operatorname{tr}(\boldsymbol{\Lambda}_2)/(n\sqrt{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}) \to 0$, *then under the local alternative hypothesis* $\|\mathbf{C}^\top \boldsymbol{\Theta}^\top \boldsymbol{\Theta} \mathbf{C}\| = O(\sqrt{rn^{-2}\operatorname{tr}^2(\boldsymbol{\Lambda}_2) + \operatorname{tr}(\boldsymbol{\Lambda}_2^2)})$,

$$
\begin{aligned}
&\Pr\left(Q_2 > F_2^{-1}\left(1-\alpha; \widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2)}, \widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}\right)\right) \\
&= \Pr\Bigg(\lambda_1\Bigg(\frac{n^{-1}\operatorname{tr}(\boldsymbol{\Lambda}_2)}{\sqrt{rn^{-2}\operatorname{tr}^2(\boldsymbol{\Lambda}_2) + \operatorname{tr}(\boldsymbol{\Lambda}_2^2)}}(\mathbf{W}_{k-1}^* - r\mathbf{I}_{k-1}) \\
&\qquad + \frac{\sqrt{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)}}{\sqrt{rn^{-2}\operatorname{tr}^2(\boldsymbol{\Lambda}_2) + \operatorname{tr}(\boldsymbol{\Lambda}_2^2)}}\mathbf{W}_{k-1} \\
&\qquad + \frac{\mathbf{C}^\top \boldsymbol{\Theta}^\top \mathbf{U}_2 \mathbf{U}_2^\top \boldsymbol{\Theta} \mathbf{C}}{\sqrt{rn^{-2}\operatorname{tr}^2(\boldsymbol{\Lambda}_2) + \operatorname{tr}(\boldsymbol{\Lambda}_2^2)}}\Bigg) \\
&\qquad > F_2^{-1}\left(1-\alpha; \operatorname{tr}(\boldsymbol{\Lambda}_2), \operatorname{tr}(\boldsymbol{\Lambda}_2^2)\right)\Bigg) + o(1).
\end{aligned}
$$

To gain some insight into the asymptotic behavior of $T(\mathbf{X})$, we consider $k = 2$ and compare the results of the LFD test with ~~the tests in~~ those of Bai and Saranadasa (1996) and Chen and Qin (2010). Corollary 2 implies that if

$$
\liminf_{n\to\infty} \frac{\mathbf{C}^\top \boldsymbol{\Theta}^\top \mathbf{U}_2 \mathbf{U}_2^\top \boldsymbol{\Theta} \mathbf{C}}{\sqrt{rn^{-2}\operatorname{tr}^2(\boldsymbol{\Lambda}_2) + \operatorname{tr}(\boldsymbol{\Lambda}_2^2)}} > 0,
$$

then the LFD test has nontrivial power, asymptotically. In contrast, if

$$
\limsup_{n\to\infty} \frac{\mathbf{C}^\top \boldsymbol{\Theta}^\top \boldsymbol{\Theta} \mathbf{C}}{\sqrt{\operatorname{tr}(\boldsymbol{\Sigma}^2)}} = 0,
$$

then the tests in Bai and Saranadasa (1996) and Chen and Qin (2010) ~~has~~ exhibit trivial power, asymptotically. To compare $\mathbf{C}^\top \boldsymbol{\Theta}^\top \mathbf{U}_2 \mathbf{U}_2^\top \boldsymbol{\Theta} \mathbf{C}$ and $\mathbf{C}^\top \boldsymbol{\Theta}^\top \boldsymbol{\Theta} \mathbf{C}$, we temporarily place a prior on $\boldsymbol{\Theta}$. Suppose ~~that~~ $\sqrt{n_i}\theta_i$ has prior distribution $\mathcal{N}_p(\mathbf{0}_p, \psi \mathbf{I}_p)$, for $i = 1, 2$. Then, $\psi^{-1}\mathbf{C}^\top \boldsymbol{\Theta}^\top \boldsymbol{\Theta} \mathbf{C}$ ~~is distributed as~~ follows a χ^2 distribution with p degrees of freedom. On the other hand, $\psi^{-1}\mathbf{C}^\top \boldsymbol{\Theta}^\top \mathbf{U}_2 \mathbf{U}_2^\top \boldsymbol{\Theta} \mathbf{C}$

~~is distributed as~~ follows a χ^2 distribution with $p - r$ degrees of freedom. ~~Then~~ Thus, we have

$$\frac{\mathbf{C}^\top \mathbf{\Theta}^\top \mathbf{U}_2 \mathbf{U}_2^\top \mathbf{\Theta} \mathbf{C}}{\mathbf{C}^\top \mathbf{\Theta}^\top \mathbf{\Theta} \mathbf{C}} \xrightarrow{P} 1.$$

~~So in~~ Therefore, on average, the signal contained in $\mathbf{C}^\top \mathbf{\Theta}^\top \mathbf{U}_2 \mathbf{U}_2^\top \mathbf{\Theta} \mathbf{C}$ is roughly the same as that in $\mathbf{C}^\top \mathbf{\Theta}^\top \mathbf{\Theta} \mathbf{C}$. Now, we compare the asymptotic variance. It is not hard to see that

$$\frac{rn^{-2} \operatorname{tr}^2(\mathbf{\Lambda}_2) + \operatorname{tr}(\mathbf{\Lambda}_2^2)}{\operatorname{tr}(\mathbf{\Sigma}^2)} \to 0.$$

That is, the asymptotic variance of $T(\mathbf{X})$ is much smaller than those of the tests in Bai and Saranadasa (1996) and Chen and Qin (2010). To appreciate this, note ~~phenomenon, we note~~ that in the expression (2.2), $(\mathbf{I}_p - \mathbf{P}_\mathbf{Y})\mathbf{XJC}|\mathbf{P}_\mathbf{Y} \sim \mathcal{N}_p(\mathbf{0}_p, (\mathbf{I}_p - \mathbf{P}_\mathbf{Y})\mathbf{\Sigma}(\mathbf{I}_p - \mathbf{P}_\mathbf{Y}))$. ~~But~~ However, $\mathbf{I}_p - \mathbf{P}_\mathbf{Y}$ tends to be orthogonal to $\mathbf{U}_1 \mathbf{U}_1^\top$, which is the projection matrix onto the eigenspace corresponding to the leading eigenvalues of $\mathbf{\Sigma}$. Hence, the projection by $\mathbf{I}_p - \mathbf{P}_\mathbf{Y}$ helps reduce the variance of $\mathbf{XJC}$.

Thus, if $\mathbf{\Theta}$ satisfies

$$\liminf_{n\to\infty} \frac{\mathbf{C}^\top \mathbf{\Theta}^\top \mathbf{\Theta} \mathbf{C}}{\sqrt{rn^{-2} \operatorname{tr}^2(\mathbf{\Lambda}_2) + \operatorname{tr}(\mathbf{\Lambda}_2^2)}} > 0, \quad \limsup_{n\to\infty} \frac{\mathbf{C}^\top \mathbf{\Theta}^\top \mathbf{\Theta} \mathbf{C}}{\sqrt{\operatorname{tr}(\mathbf{\Sigma}^2)}} = 0,$$

then the LFD test has nontrivial power ~~while~~, whereas the tests in Bai and Saranadasa (1996) and Chen and Qin (2010) ~~has~~ exhibit trivial power. Hence, the LFD test tends to be more powerful than ~~the tests in~~ those of Bai and Saranadasa (1996) and Chen and Qin (2010).

In practice, ~~one~~ we may not know ~~weather~~ whether the covariance matrix is spiked. Furthermore, even if we know that it ~~Even if it is known that the covariance matrix~~ is spiked, the spike number r may be unknown. ~~So we would like to~~ Therefore, we propose an adaptive test procedure. Note that Theorem 1 requires $n\boldsymbol{\lambda}_1/\operatorname{tr}(\boldsymbol{\Sigma}) \to 0$ ~~while~~ , and Theorem 2 requires $\operatorname{tr}(\boldsymbol{\Lambda}_2)/n\boldsymbol{\lambda}_r \to 0$ and $n\boldsymbol{\lambda}_{r+1}/\operatorname{tr}(\boldsymbol{\Lambda}_2) \to 0$. This motivates us to consider the following adaptive test procedure. Let $\tau > 1$ be a hyperparameter. If

$$\frac{n\lambda_1(\hat{\boldsymbol{\Sigma}})}{\operatorname{tr}(\hat{\boldsymbol{\Sigma}})} < \tau,$$

then we reject the null hypothesis if $Q_1 > F^{-1}(1-\alpha)$. Otherwise, we reject the null hypothesis if $Q_2 > F_2^{-1}(1-\alpha; \widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2,)}, \widehat{\operatorname{tr}(\boldsymbol{\Lambda}_2^2)})$, where the unknown r is substituted by the estimator

$$\hat{r} = \min\left\{1 \le i < n : \frac{n\lambda_{i+1}(\hat{\boldsymbol{\Sigma}})}{\sum_{j=i+1}^{n} \lambda_j(\hat{\boldsymbol{\Sigma}})} < \tau\right\}.$$

We have the following proposition.

Proposition 5. *Let $\tau > 1$ be a constant.*

(i) Under the conditions of Theorem 1,

$$\Pr\left(\frac{n\lambda_1(\hat{\boldsymbol{\Sigma}})}{\operatorname{tr}(\hat{\boldsymbol{\Sigma}})} < \tau\right) \to 1;$$

(ii) Under the conditions of Theorem 2,

$$\Pr\left(\frac{n\lambda_1(\hat{\boldsymbol{\Sigma}})}{\operatorname{tr}(\hat{\boldsymbol{\Sigma}})} < \tau\right) \to 0, \quad \Pr(\hat{r} = r) \to 1.$$

Proposition 5 implies that the spiked covariance structure can be detected consistently ~~detected~~. ~~So~~ Therefore, the proposed adaptive LFD test procedure can indeed adapt to the unknown covariance structure.

4. Numerical study

In this section, we compare the numerical performance of the adaptive LFD test procedure with that of ~~some existing tests, including~~ the MANOVA tests in Schott (2007), Cai and Xia (2014), Hu et al. (2017), and Zhang et al. (2017). These competing tests are denoted by Sc, CX, HBWW, and ZGZ, respectively. Throughout the simulations, we take the nominal test level $\alpha = 0.05$ and the group number $k = 3$. For the adaptive LFD test, we take $\tau = 5$. For CX, we use their oracle procedure. All ~~the~~ simulation results are based on 5000 replications.

First, we simulate the empirical level and power under various models of $\mathbf{\Sigma}$ and $\mathbf{\Theta}$. To characterize the signal strength, we define the signal-to-noise ratio (SNR) as

$$\mathrm{SNR} = \frac{\mathbf{C}^\top \mathbf{\Theta}^\top \mathbf{\Theta} \mathbf{C}}{\sqrt{\mathrm{tr}(\mathbf{\Sigma}^2)}}.$$

We consider four models for $\mathbf{\Sigma}$, where the first two ~~of them~~ are nonspiked, and the last two ~~of them~~ are spiked.

- Model I: $\mathbf{\Sigma} = \mathbf{I}_p$.

- Model II: $\mathbf{\Sigma} = (\sigma_{ij})$, where $\sigma_{ij} = 0.6^{|i-j|}$.

- Model III: $\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$, where $\mathbf{U}$ is a $p \times p$ orthogonal matrix generated from a Haar distribution and $\mathbf{\Lambda} = \mathrm{diag}(3p, 2p, p, 1, \ldots, 1)$.

- Model IV: $\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top + \mathbf{A}\mathbf{A}^\top$, where $\mathbf{U}$ is a $p \times p$ orthogonal matrix generated from a Haar distribution, $\mathbf{\Lambda} = \mathrm{diag}(p, p, 1, \ldots, 1)$, and $\mathbf{A}$ is a $p \times p$ matrix ~~whose~~ , the elements of which are independently generated from a Bernoulli distribution with success probability 0.01.

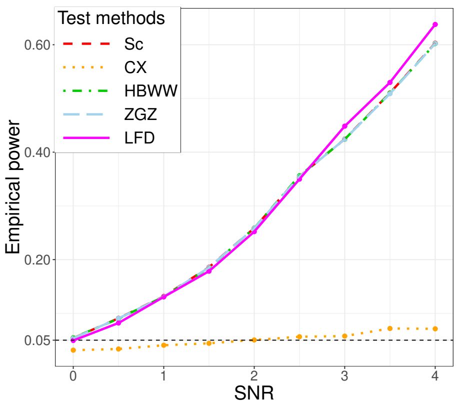
Under the null hypothesis, we ~~shall~~ always take $\theta_1 = \cdots = \theta_k = \mathbf{0}_p$. We consider two ~~different~~ structures ~~of~~ for the alternative hypotheses: the ~~non-sparse~~ nonsparse alternative, and the sparse alternative. In the ~~non-sparse~~ nonsparse case, we take $\theta_1 = \kappa \mathbf{1}_p$, $\theta_2 = -\kappa \mathbf{1}_p$, and $\theta_3 = \mathbf{0}_p$, where κ is selected to make the SNR equal to specific values. In the sparse case, we take $\theta_1 = \kappa(\mathbf{1}_{p/5}^\top, \mathbf{0}_{4p/5}^\top)^\top$, $\theta_2 = \kappa(\mathbf{0}_{p/5}^\top, \mathbf{1}_{p/5}^\top, \mathbf{0}_{3p/5}^\top)^\top$, and $\theta_3 = \mathbf{0}_p$. Again, κ is selected to make the SNR equal to specific values. The simulation results are summarized in Figures ~~1–4. It can be seen~~ 1–4, and show that in all scenarios, the empirical sizes of the LFD test are reasonably close to the nominal level 0.05. Under model I and model II, where the covariance matrices are nonspiked, the empirical power of the LFD test is slightly lower than that of the sum-of-squares ~~type~~ tests, but is higher than that of the CX test. Under model III and model IV, where the covariance matrices are spiked, the empirical power of the LFD test is significantly higher than that of the sum-of-squares ~~type~~ tests. ~~Also~~ In addition, the LFD test ~~offers~~ exhibits higher empirical power than that of the CX test in most cases, except for model IV with sparse means. These simulation results verify our theoretical results that the LFD test is particularly powerful under ~~the~~ a spiked covariance.

In our second simulation study, we ~~would like to~~ investigate the effect of correlations between the variables. We consider ~~the~~ a compound symmetry structure; that is, the diagonal elements of $\boldsymbol{\Sigma}$ are ~~1~~ one, and the off-diagonal elements are ρ, with $0 \leq \rho < 1$. The parameter ρ characterizes the correlations between the variables. We take $\theta_1 = \kappa(\mathbf{1}_{p/5}^\top, \mathbf{0}_{4p/5}^\top)^\top$, $\theta_2 = \kappa(\mathbf{0}_{p/5}^\top, \mathbf{1}_{p/5}^\top, \mathbf{0}_{3p/5}^\top)^\top$, and $\theta_3 = \mathbf{0}_p$, where κ is selected such that $\mathbf{C}^\top\boldsymbol{\Theta}^\top\boldsymbol{\Theta}\mathbf{C}/(\sum_{i=2}^p \boldsymbol{\lambda}_i^2)^{1/2} = 5$. Figure 5 plots the empirical ~~powers of~~ power for various tests versus ρ. We can see that the empirical power of the LFD test ~~holds~~ remains nearly constant as ρ varies, ~~while~~ whereas the empirical ~~powers of~~ power of the competing sum-of-squares ~~type~~ tests ~~decrease~~ decreases rapidly as ρ increases. When ρ is ~~non-zero~~ nonzero, the LFD test outperforms the competing tests significantly.

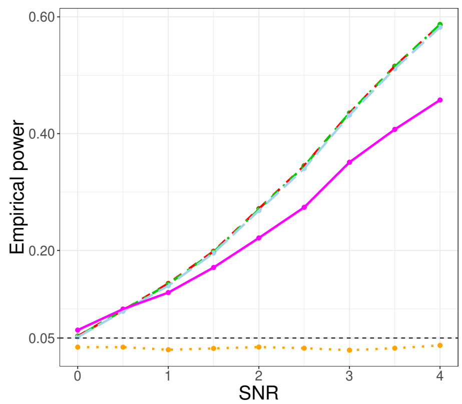
5. Concluding remarks

~~In this paper, using~~ Using the idea of a least favorable direction, we have proposed ~~the~~ an LFD test for MANOVA in a high-dimensional setting. We have derived the asymp-

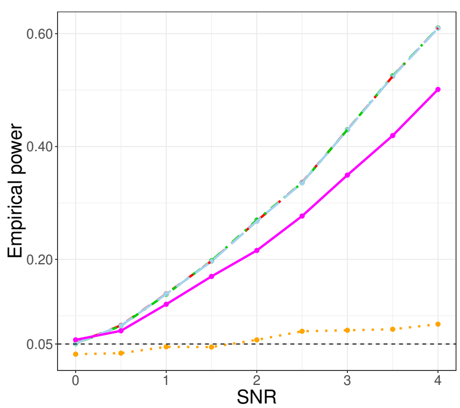
(a) Model I, ~~non-sparse~~ nonsparse case



(b) Model II, ~~non-sparse~~ nonsparse case



(c) Model I, sparse case



(d) Model II, sparse case

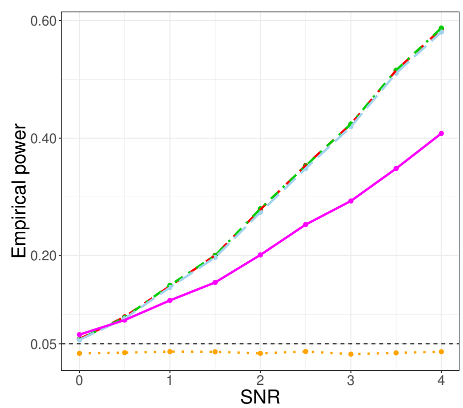
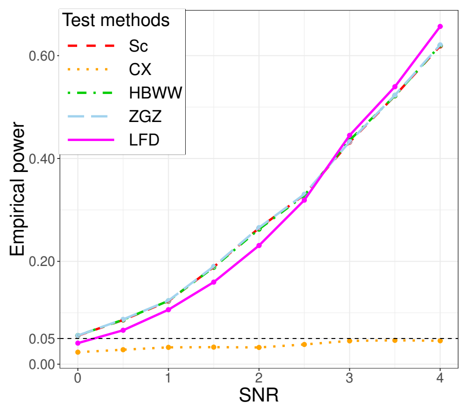
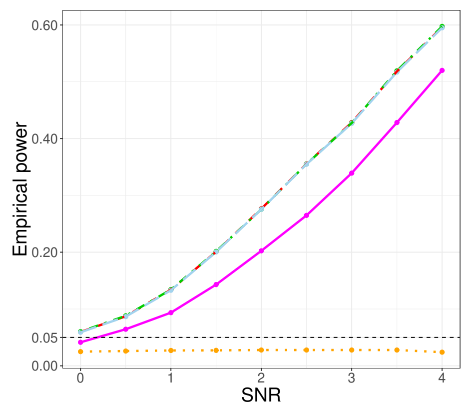


Figure 1: Empirical ~~sizes~~ size and ~~powers~~ power of tests under model I and model II; $n_1 = n_2 = n_3 = 20$, $p = 300$.

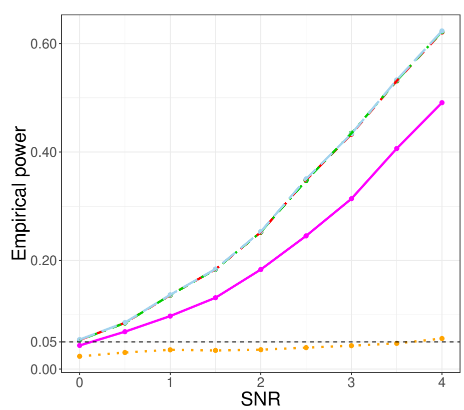
(a) Model I, ~~non-sparse~~ nonsparse case



(b) Model II, ~~non-sparse~~ nonsparse case



(c) Model I, sparse case



(d) Model II, sparse case

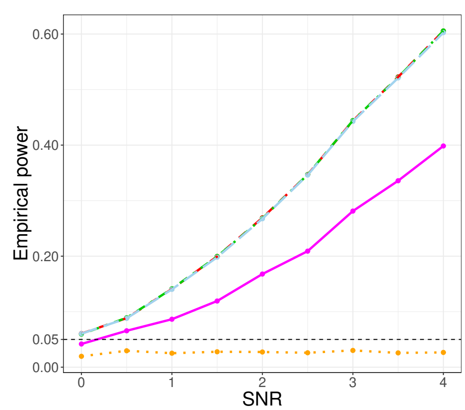
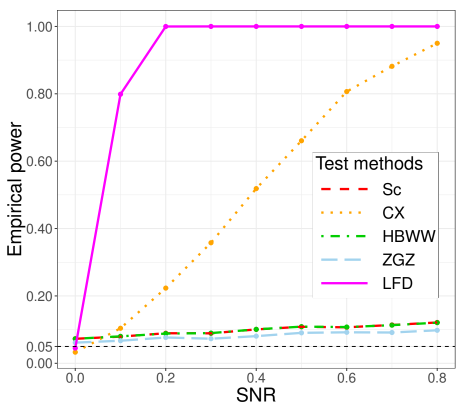


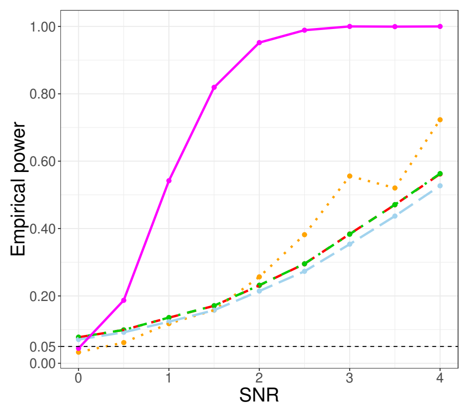
Figure 2: Empirical ~~sizes~~ size and ~~powers~~ power of tests under model I and model II;

$n_1 = n_2 = n_3 = 25$, $p = 800$.

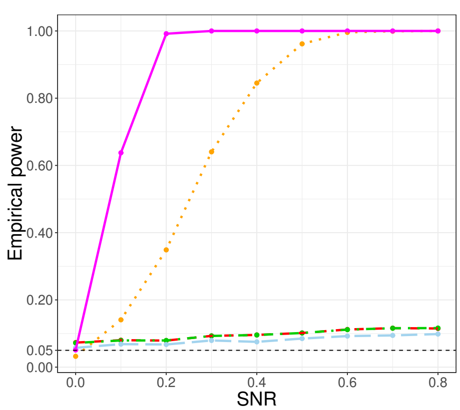
nonsparse
(a) Model III, ~~non-sparse~~ case



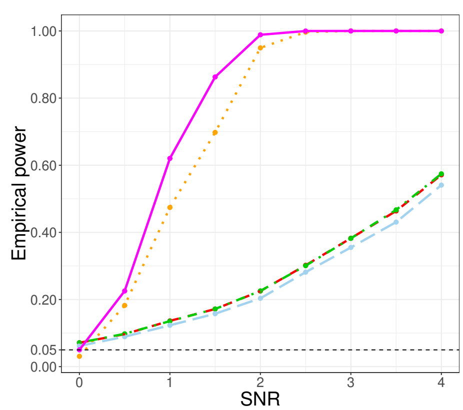
nonsparse
(b) Model IV, ~~non-sparse~~ case



(c) Model III, sparse case

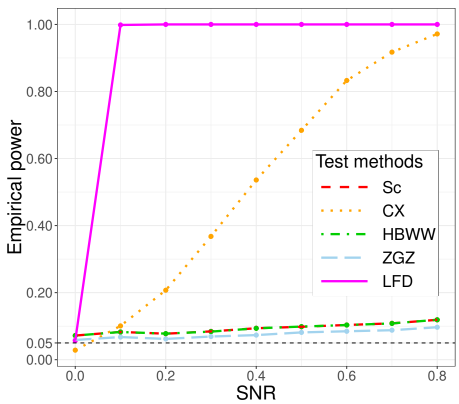


(d) Model IV, sparse case

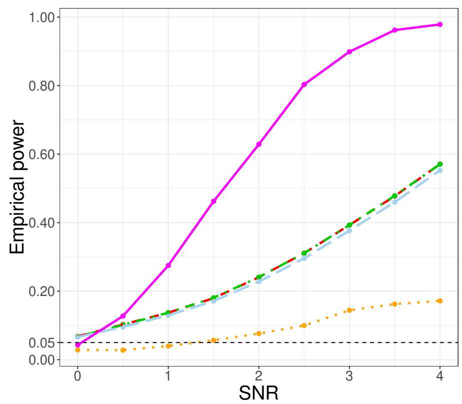


size power
Figure 3: Empirical ~~sizes~~ and ~~powers~~ of tests under model III and model IV. $n_1 = n_2 = n_3 = 20$, $p = 300$.

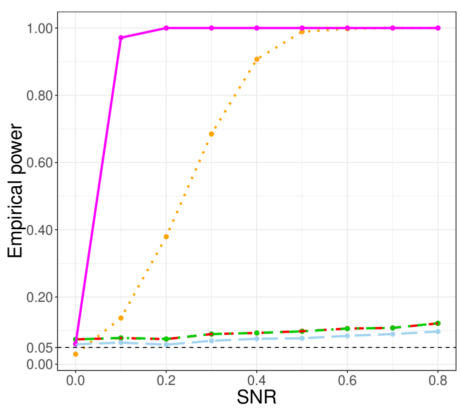
(a) Model III, ~~non-sparse~~ nonsparse case



(b) Model IV, ~~non-sparse~~ nonsparse case



(c) Model III, sparse case



(d) Model IV, sparse case

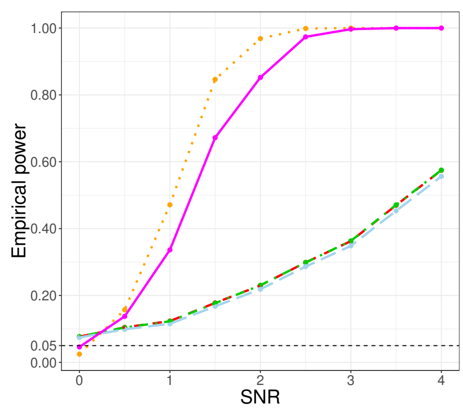


Figure 4: Empirical ~~sizes~~ size and ~~powers~~ power of tests under model III and model; IV. $n_1 = n_2 = n_3 = 25$, $p = 800$.

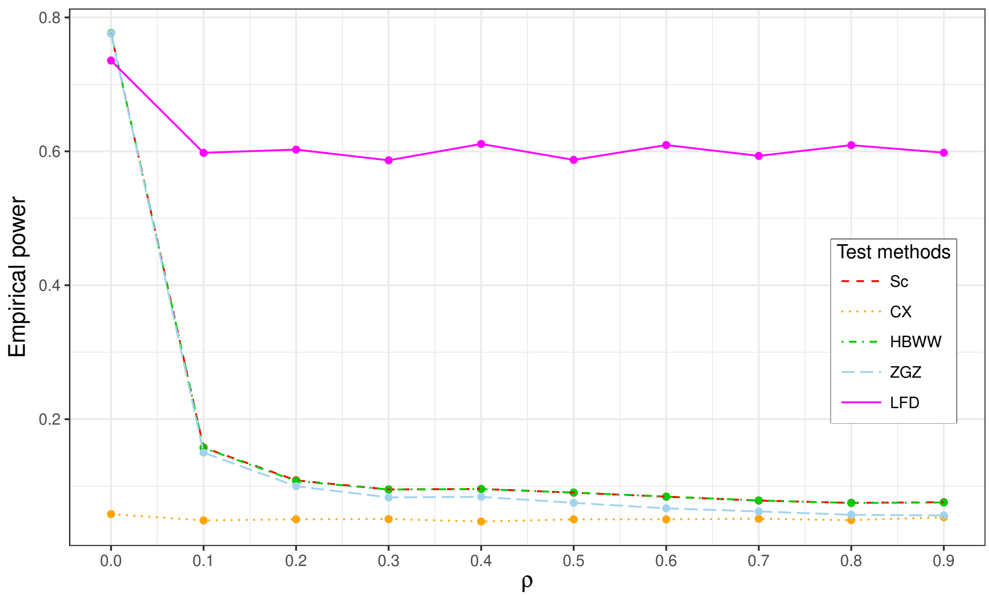


Figure 5: Empirical ~~powers~~ power of tests; $n_1 = n_2 = n_3 = 35$, $p = 1000$.

totic distribution of the LFD test statistic ~~under~~ for both nonspiked and spiked covariances. The asymptotic local power functions are also given. ~~From~~ The results of our ~~theoretic~~ theoretical results and simulation studies~~, it is seen~~ show that the LFD test ~~has~~ exhibits ~~comparable~~ power behavior ~~to~~ comparable with that of existing tests when the covariance matrix is nonspiked, ~~while~~ and tends to be much more powerful than existing tests when the covariance matrix is spiked.

~~There are several~~ Several interesting, but challenging problems ~~yet to be solved~~ remain. First, for the case ~~where the~~ of an unknown covariance structure ~~is unknown~~, we proposed an adaptive LFD test procedure by consistently detecting the unknown covariance structure and estimating the unknown r. However, this procedure relies on a hyperparameter τ. Determining ~~How to choose~~ an optimal τ remains an interesting problem. Second, our theoretical results rely on the normality of the observations. In fact, our proofs use ~~utilize~~ the independence of $\mathbf{XJC}$ and $\mathbf{Y}$. Note that $\mathbf{XJC}$ and $\mathbf{Y} = \mathbf{X}\tilde{\mathbf{J}}$ are both ~~the~~ linear combinations of independent ~~indepedent~~ random vectors X_{ij}, where this independence is known to characterize ~~. It is known that the independence of linear combinations of independent random variables essentially characterizes~~ the normality of the variables (see, e.g., Kagan et al. (1973), Section 3.1). Hence our strategy is not feasible without the normality assumption. It is unclear whether the conclusions ~~conclutions~~ of our theorems hold without this ~~normal~~ assumption. Third, our theoretical results require $p/n \to \infty$. In fact, the asymptotic behavior of $T(\mathbf{X})$ will be different in the regime where $p/n \to$ constant. Random matrix theory may be useful to investigate the asymptotic behavior of $T(\mathbf{X})$ in this regime. We leave these topics for future research.

Supplementary Material ~~Materials~~

The online Supplementary Material ~~supplementary material~~ presents proofs of the propositions and theorems.

Acknowledgments ~~**Acknowledgements**~~ We ~~would like to~~ thank the associate editor and two anonymous reviewers for their helpful comments and suggestions ~~that~~

~~improve the paper significantly~~. This work was supported by the National Natural Science Foundation of China under Grant Nos. 11471035, and 11471030.

References

Ahn, J. and J. S. Marron (2010). The maximal data piling direction for discrimination. *Biometrika 97*(1), 254–259.

Aoshima, M., D. Shen, H. Shen, K. Yata, Y.-H. Zhou, and J. S. Marron (2018). A survey of high dimension low sample size asymptotics. *Australian & New Zealand Journal of Statistics 60*(1), 4–19.

Aoshima, M. and K. Yata (2018). Two-sample tests for high-dimension, strongly spiked eigenvalue models. *Statistica Sinica 28*(1), 43–62.

Bai, Z. and H. Saranadasa (1996). Effect of high dimension: By an example of a two sample problem. *Statistica Sinica 6*(2), 311–329.

Cai, T., X. Han, and G. Pan (2019). Limiting Laws for Divergent Spiked Eigenvalues and Largest Non-spiked Eigenvalue of Sample Covariance Matrices. *The Annals of Statistics In press*.

Cai, T., Z. Ma, and Y. Wu (2015). Optimal estimation and rank detection for sparse spiked covariance matrices. *Probability Theory & Related Fields 161*(3-4), 781–815.

Cai, T. T., W. Liu, and Y. Xia (2014). Two-sample test of high dimensional means under dependence. *Journal of the Royal Statistical Society: Series B (Statistical Methodology) 76*(2), 349–372.

Cai, T. T. and Y. Xia (2014). High-dimensional sparse MANOVA. *Journal of Multivariate Analysis 131*, 174–196.

Cao, M.-X., J. Park, and D.-J. He (2019). A test for the k sample behrensfisher problem in high dimensional data. *Journal of Statistical Planning and Inference 201*, 86–102.

Chen, S. X. and Y.-L. Qin (2010). A two-sample test for high-dimensional data with applications to gene-set testing. *The Annals of Statistics 38*(2), 808–835.

Fan, J., D. Wang, K. Wang, and Z. Zhu (2019). Distributed Estimation of Principal Eigenspaces. *The Annals of Statistics In press*.

Fan, J., L. Yuan, and M. Mincheva (2013). Large covariance estimation by thresholding principal orthogonal complements. *Journal of the Royal Statistical Society: Series B (Statistical Methodology) 75*(4), 603–680.

Feng, L., C. Zou, Z. Wang, and L. Zhu (2015). Two-sample behrens-fisher problem for high-dimensional data. *Statistica Sinica 25*(4), 1297–1312.

Hu, J., Z. Bai, C. Wang, and W. Wang (2017). On testing the equality of high dimensional mean vectors with unequal covariance matrices. *Annals of the Institute of Statistical Mathematics 69*(2), 365–387.

Kagan, A., Y. Linnik, and C. Rao (1973). *Characterization Problems in Mathematical Statistics*

(1st ed.). New York: Wiley.

Katayama, S., Y. Kano, and M. S. Srivastava (2013). Asymptotic distributions of some test criteria for the mean vector with fewer observations than the dimension. *Journal of Multivariate Analysis 116*, 410–421.

Koltchinskii, V. and K. Lounici (2016). Asymptotics and concentration bounds for bilinear forms of spectral projectors of sample covariance. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques 52*(4), 1976–2013.

Ma, Y., W. Lan, and H. Wang (2015). A high dimensional two-sample test under a low dimensional factor structure. *Journal of Multivariate Analysis 140*, 162–170.

Nadler, B. (2008). Finite sample approximation results for principal component analysis: A matrix perturbation approach. *The Annals of Statistics 36*(6), 2791–2817.

Roy, S. N. (1953). On a heuristic method of test construction and its use in multivariate analysis. *The Annals of Mathematical Statistics 24*(2), 220–238.

Schott, J. R. (2007). Some high-dimensional tests for a one-way MANOVA. *Journal of Multivariate Analysis 98*(9), 1825–1839.

Shen, D., H. Shen, and J. S. Marron (2016). A general framework for consistency of principal component analysis. *Journal of Machine Learning Research 17*(150), 1–34.

Srivastava, M. S. (2007). Multivariate theory for analyzing high dimensional data. *Journal of the Japan Statistical Society 37*(1), 53–86.

Srivastava, M. S. and T. Kubokawa (2013). Tests for multivariate analysis of variance in high dimension under non-normality. *Journal of Multivariate Analysis 115*, 204–216.

Tsai, C.-A. and J. J. Chen (2009). Multivariate analysis of variance test for gene set analysis. *Bioinformatics 25*(7), 897–903.

Verstynen, T., J. Diedrichsen, N. Albert, P. Aparicio, and R. Ivry (2005). Ipsilateral motor cortex activity during unimanual hand movements relates to task complexity. *Journal of Neurophysiology 93*(3), 1209–1222.

Wang, R. and X. Xu (2018). On two-sample mean tests under spiked covariances. *Journal of Multivariate Analysis 167*, 225 – 249.

Wang, R. and X. Xu (2019). A feasible high dimensional randomization test for the mean vector. *Journal of Statistical Planning and Inference 199*, 160–178.

Wang, W. and J. Fan (2017). Asymptotics of empirical eigenstructure for high dimensional spiked covariance. *The Annals of Statistics 45*(3), 1342–1374.

Yamada, T. and T. Himeno (2015). Testing homogeneity of mean vectors under heteroscedasticity in high-dimension. *Journal of Multivariate Analysis 139*, 7–27.

Yata, K. and M. Aoshima (2012). Effective pca for high-dimension, low-sample-size data with noise reduction via geometric representations. *Journal of Multivariate Analysis 105*(1), 193 – 215.

Yata, K. and M. Aoshima (2013). Pca consistency for the power spiked model in high-dimensional settings. *Journal of Multivariate Analysis 122*, 334 – 354.

Zhang, J.-T., J. Guo, and B. Zhou (2017). Linear hypothesis testing in high-dimensional one-way manova. *Journal of Multivariate Analysis 155*, 200 – 216.

Zhao, J. and X. Xu (2016). A generalized likelihood ratio test for normal mean when p is greater than n. *Computational Statistics & Data Analysis 99*, 91–104.

Zhou, B., J. Guo, and J.-T. Zhang (2017). High-dimensional general linear hypothesis testing under heteroscedasticity. *Journal of Statistical Planning and Inference 188*, 36–54.

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081,China

E-mail: wangruiphd@bit.edu.cn

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081,China

and Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing 100081,China

E-mail: xuxz@bit.edu.cn