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Abstract

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1. GLRT

Suppose $\{X_{i1}, \dots, X_{in_i}\}$ are i.i.d. distributed as $N(\mu_i, \Sigma)$ for $1 \le i \le K$. Let $\mathbf{X}_i = (X_{i1}, \dots, X_{in_i})$ for $i = 1, \dots, k$. The k samples are independent. μ_i, $i = 1 \dots, k$ and $\Sigma > 0$ are unknown. An interesting problem in multivariate analysis is to test the hypotheses

$$H : \mu_1 = \mu_2 = \cdots = \mu_k \quad v.s. \quad K : \mu_i \neq \mu_j \text{ for some } i \neq j. \tag{1}$$

Let $\mathbf{Z} = (X_1, \dots, X_k)$.

$$f(Z; \mu_1, \dots, \mu_k, \Sigma) = \prod_{i=1}^{k} \Big[(2\pi)^{-n_i p/2} |\Sigma|^{-n_i/2} \exp(-\frac{1}{2} \mathrm{tr}\Sigma^{-1} \sum_{j=1}^{n_i} (x_{ij} - \mu_i)(x_{ij} - \mu_i)^T) \Big].$$

Assume $n = \sum_{i=1}^{p} n_i < p$. Let $a \in \mathbb{R}^p$ be a vector satisfying $a^T a = 1$. Then

$$f_a(a^T Z; \mu_1, \dots, \mu_k, \Sigma) = (2\pi)^{-n/2} |a^T \Sigma a|^{-n/2} \exp\Big(-\frac{1}{2a^T \Sigma a} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (a^T x_{ij} - a^T \mu_i)^2 \Big)$$

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$$\max_{\mu_1,\ldots,\mu_k,\Sigma} f_a(a^T Z, \mu_1, \ldots, \mu_k, \Sigma) = (2\pi)^{-n/2}\big(\sum_{i=1}^{k}\sum_{j=1}^{n_i}(a^T x_{ij} - a^T \bar{\mathbf{X}}_i)^2\big)^{-n/2} e^{-n/2} \tag{2}$$

Let $S_i = \sum_{j=1}^{n_i}(x_{ij} - \bar{\mathbf{X}}_i)(x_{ij} - \bar{\mathbf{X}}_i)^T$ and $S = \sum_{i=1}^{k} S_i$.

Under H, we have

$$\max_{\mu,\Sigma} f_a(a^T Z, \mu, \ldots, \mu, \Sigma) = (2\pi)^{-n/2}\big(\sum_{i=1}^{k}\sum_{j=1}^{n_i}(a^T x_{ij} - a^T \bar{\mathbf{X}})^2\big)^{-n/2} e^{-n/2} \tag{3}$$

The generalized likelihood ratio test statistic is defined as

$$T(Z) = \max_{a^T a=1, a^T S a=0} a^T \sum_{i=1}^{k} n_i(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})^T a \tag{4}$$

Let $J = \operatorname{diag}(n_1^{-1/2}\mathbf{1}_{n_1}, \ldots, n_k^{-1/2}\mathbf{1}_{n_k})$. Then $S = Z(I_n - JJ^T)Z^T$ and

$$\sum_{i=1}^{k} n_i(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})^T = Z(JJ^T - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T)Z^T. \tag{5}$$

The matrix $I_n - JJ^T$, $JJ^T - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$ and $\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$ are all projection matrix and pairwise orthogonal with rank $n-k$, $k-1$ and 1.

Let $\tilde{J}$ be a $n \times (n-k)$ matrix satisfied $\tilde{J}\tilde{J}^T = I - JJ^T$. Then $S = Z\tilde{J}\tilde{J}^T Z^T$ and Note that

$$Z(JJ^T - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T)Z^T = ZJ(I_k - \frac{1}{n}J^T\mathbf{1}_n\mathbf{1}_n^T J)J^T Z^T.$$

Note that $I_k - \frac{1}{n}J^T\mathbf{1}_n\mathbf{1}_n^T J$ is a projection matrix with rank $k-1$. Let C be a $k \times (k-1)$ matrix satisfied $CC^T = I_k - \frac{1}{n}J^T\mathbf{1}_n\mathbf{1}_n^T J$.

In Proposition 1, letting $A = Z\tilde{J}$ and $B = ZJCC^TJ^TZ^T$ yields

$$\begin{aligned} T(Z) &= \lambda_{max}\big((I_p - Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1}\tilde{J}^T Z^T)ZJCC^TJ^TZ^T(I_p - Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1}\tilde{J}^T Z^T)\big) \\ &= \lambda_{max}\big(C^TJ^TZ^T(I_p - Z\tilde{J}(\tilde{J}^T Z^T Z\tilde{J})^{-1}\tilde{J}^T Z^T)ZJC\big). \end{aligned}$$

Note that

$$\begin{aligned} &\Big(\begin{pmatrix} J^T \\ \tilde{J}^T \end{pmatrix} Z^T Z \begin{pmatrix} J & \tilde{J} \end{pmatrix}\Big)^{-1} \\ &= \begin{pmatrix} J^TZ^TZJ & J^TZ^TZ\tilde{J} \\ \tilde{J}^TZ^TZJ & \tilde{J}^TZ^TZ\tilde{J} \end{pmatrix}^{-1} = \begin{pmatrix} J^T(Z^TZ)^{-1}J & J^T(Z^TZ)^{-1}\tilde{J} \\ \tilde{J}^T(Z^TZ)^{-1}J & \tilde{J}^T(Z^TZ)^{-1}\tilde{J} \end{pmatrix}. \end{aligned} \tag{6}$$

It follows that

$$
\begin{aligned}
&\left(J^T(Z^TZ)^{-1}J\right)^{-1} \\
=&J^TZ^TZJ - J^TZ^TZ\tilde{J}(\tilde{J}^TZ^TZ\tilde{J})^{-1}\tilde{J}^TZ^TZJ \\
=&J^TZ^T(I_p - Z\tilde{J}(\tilde{J}^TZ^TZ\tilde{J})^{-1}\tilde{J}^TZ^T)ZJ
\end{aligned}
\tag{7}
$$

It follows that

$$
T(Z) = \lambda_{\max}\Big(C^T\left(J^T(Z^TZ)^{-1}J\right)^{-1}C\Big) \tag{8}
$$

Proposition 1. *Suppose A is a $p \times r$ matrix with rank r and B is a $p \times p$ non-zero semi-definite matrix. Let $H_A = A(A^TA)^{-1}A^T$. Then*

$$
\max_{a^Ta=1, a^TAA^Ta=0} a^TBa = \lambda_{\max}\big((I_p - H_A)B(I_p - H_A)\big). \tag{9}
$$

Proof. Note that $a^TAA^Ta = 0$ is equivalent to $A^Ta = 0$ and is in turn equivalent to $H_Aa = 0$. In this circumstance, $a = (I_p - H_A)a$. Then

$$
\begin{aligned}
\max_{a^Ta=1, a^TAA^Ta=0} a^TBa &= \max_{a^Ta=1, H_Aa=0} a^TBa \\
&= \max_{a^Ta=1, H_Aa=0} a^T(I_p - H_A)B(I_p - H_A)a.
\end{aligned}
\tag{10}
$$

It's obvious that (10) $\le \lambda_{\max}\big((I - H_A)B(I - H_A)\big)$. On the other hand, let α_1 be one eigenvector corresponding to the largest eigenvalue of $(I - H_A)B(I - H_A)$. Note that the row of H_A are all eigenvetors of $(I - H_A)B(I - H_A)$ corresponding to eigenvalue 0. It follows that $H_A\alpha_1 = 0$. Now that α_1 satisfies the constraint of (10), (10) is maximized when $a = \alpha_1$.

□

2. Schott's method

$$
E = ZZ^T - \sum_{i=1}^{k} n_i\bar{X}_i\bar{X}_i^T.
$$

$$
H = \sum_{i=1}^{k} n_i\bar{X}_i\bar{X}_i^T - n\bar{X}\bar{X}^T.
$$

$$\operatorname{tr} E = \operatorname{tr} Z^T Z - \operatorname{tr} J^T Z^T Z J.$$

$$\operatorname{tr} H = \operatorname{tr} J^T Z^T Z J - \frac{1}{n} 1_n^T Z^T Z 1_n$$

$$T_{SC} = \frac{1}{\sqrt{n-1}} (\frac{1}{k-1} \operatorname{tr} H - \frac{1}{n-k} \operatorname{tr} E)$$

References