Integrated likelihood ratio test[☆]

author[1]

Radarweg 29, Amsterdam

Elsevier Inc[a,b], Global Customer Service[b,]*

[a]*1600 John F Kennedy Boulevard, Philadelphia*
[b]*360 Park Avenue South, New York*

Abstract

This paper is concerned with the Bayesian hypothesis testing problem. A general methodology called integrated likelihood ratio test is proposed which takes posterior Bayes factor and fractional Bayes fator as special cases. Our methodology also includes the statistics produced by approximation computation. The frequentist properties of the integrated likelihood ratio test are investigated. We find that the integrated likelihood ratio shares the same asymptotic local power as that of likelihood ratio test.

Keywords: Bayes consistency, Bayes factor, hypothesis testing

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1. Introduction

The Bayes factor, proposed by Jeffreys (1931), is the conventional tool for Bayesian hypothesis testing and has been widely used by practitioners (See Kass and Raftery (1995) for a review). Compared with the methods in other Beyasian inference problem, such as point estimation and credible sets, Bayes factor is developed on its own ground and thus has its own nature. A notable feature of Bayes factor is that it can not be obtained solely from the posterior distribution of paramters. There are two consequence of this feature. First, the computation of Bayes factor is highly nontrivial. See Kass and Raftery (1995), Han and Carlin (2001), Raftery et al. (2006) and the references therein. Second, Bayes factor is sensitive to the choice of prior distribution even in the large sample setting. In contrast, it is well known that the posterior distribution tends to become independent of the prior distribution as the sample size increases.

Several modifications of Bayes factor have been proposed. Aitkin (1991) proposed the posterior Bayes factor (PBF) which integrated the likelihood with respect to the posterior distribution. Another approach uses a portion of data as training sample. A posterior is computed using the training sample and then be used to calculate Bayes factor. Berger and Pericchi (1996) proposed the intrinsic Bayes factor by using all possible training samples of minimal size and averaging the resulting Bayes factor. O'Hagan (1995) found that the training sample approximates to the full likelihood raised to a fractional power. They called the resulting statistic an fractional Bayes factor (FBF). Compared with the Bayes factor, these testing methods are less sensitive to the prior distribution.

The computations of PBF and FBF are easier than that of Bayes factor since the PBF and FBF can be computed by sampling the likelihood according to posterior distribution or fractional posterior distribution.

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[*]Corresponding author
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For moderately complex model, however, sampling from posterior may be difficult and hence some approximation methods may be used in practice. Variational inference is a popular method for approximating intractable posterior distribution. See Blei et al. (2017) and the references therein.

In this paper, we are interested in the frequentist evaluations of Bayesian hypothesis testing methods. We propose a flexible methodology called integrated likelihood ratio test (ILRT) which takes PBF and FBF as special examples. ILRT also includes methos that are produced by approximation computation. Under certain regular conditions, we rigorously derive the asymptotic behavior of ILRT statistic.

The frequentist properties of Bayesian methods have drawn much attention in recent years. See Ghosal et al. (2000), Shen and Wasserman (2001), van der Vaart and Ghosal (2007), Kleijn and Vaart (2012) and the references therein. However, existing research is largely concerned with the consistency and asymptotic normality of the posetrior distribution and can not be directly used to study the asymptotic behavior of ILRT.

The paper is organized as follow. In Section 2, we investegate the asymptotic properties of the generalzied FBF. Section 3 consider the ILRT with general weight function. Section 4 concludes the paper. All technical proves are in Apppendix.

2. Integrated likelihood ratio test

2.1. The test statistic

Let $\mathbf{X}^{(n)} = (X_1, \ldots, X_n)$ be independent identically distributed (i.i.d.) observations with values in some space space $(\mathcal{X}; \mathscr{A})$. Suppose that there is a σ-finite measure μ on $\mathcal{X}$ and that the possible distribution P_θ of X_i has a density $p(X|\theta)$ with respect to μ. Denote by P_θ^n the joint distribution of $\mathbf{X}^{(n)}$. Let $p_n(\mathbf{X}^{(n)}|\theta) = \prod_{i=1}^n p(X_i|\theta)$ denote the density of P_θ^n with respect to the n-fold product measure μ^n. The parameter θ takes its values in Θ, a subset of $\mathbb{R}^p$. Suppose $\theta = (\nu^T, \xi^T)^T$, where ν is a p_0 dimensional subvector and ξ is a $p - p_0$ dimensional subvector. We would like to test the nested hypotheses

$$H_0 : \theta \in \Theta_0 \quad \text{v.s.} \quad \theta \in \Theta,$$

where the null space Θ_0 is a p_0-dimensional subspace of Θ defined as

$$\Theta_0 = \{(\nu^T, \xi^T)^T : (\nu^T, \xi^T)^T \in \Theta, \, \xi = \xi_0\}.$$

If the null hypothesis is true, we denote by $\theta_0 = (\nu_0^T, \xi_0^T)^T$ the true parameter which generates the data.

In Bayesian hypothesis testing framework, one puts prior $\pi(\nu)$ and $\pi(\theta)$ on parameters under the null and alternative hypotheses, respectively. The conventional Bayes factor is defined as

$$\frac{\int_\Theta p_n(\mathbf{X}^{(n)}|\theta)\pi(\theta)\, d\theta}{\int_{\tilde{\Theta}_0} p_n(\mathbf{X}^{(n)}|\nu, \xi_0)\pi(\nu)\, d\nu},$$

where $\tilde{\Theta}_0 = \{\nu : (\nu^T, \xi^T)^T \in \Theta_0\}$. However, Bayes factor is sensitive to the specification of prior, which may cause difficulties in the absense of a well-formulated subjective prior. See, for example, Shafer (1982). To deal with this problem, Aitkin (1991) proposed PBF which is defined to be

$$\frac{\int_\Theta p_n(\mathbf{X}^{(n)}|\theta)\pi(\theta|\mathbf{X}^{(n)})\, d\theta}{\int_{\tilde{\Theta}_0} p_n(\mathbf{X}^{(n)}|\nu, \xi_0)\pi(\nu|\mathbf{X}^{(n)})\, d\nu},$$

where $\pi(\nu|\mathbf{X}^{(n)})$ and $\pi(\theta|\mathbf{X}^{(n)})$ are the posterior densities under the null and alternative hypothesis, respectively. O'Hagan (1995) proposed FBF which is defiend to be

$$\frac{L_1}{L_b} \cdot \frac{L_b^*}{L_1^*} \quad \text{for} \quad 0 < b < 1,$$

where for $t > 0$,

$$L_t = \int_\Theta \left[p_n(\mathbf{X}^{(n)}|\theta)\right]^t \pi(\theta)\,d\theta, \quad L_t^* = \int_{\Theta_0} \left[p_n(\mathbf{X}^{(n)}|\nu,\xi_0)\right]^t \pi(\nu)\,d\nu.$$

We generalize the PBF and FBF and propose the ILRT statistic as

$$\frac{\int_\Theta \left[p_n(\mathbf{X}^{(n)}|\theta)\right]^a \pi(\theta;\mathbf{X}^{(n)})\,d\theta}{\int_{\tilde\Theta_0} \left[p_n(\mathbf{X}^{(n)}|\nu,\xi_0)\right]^a \pi(\nu;\mathbf{X}^{(n)})\,d\nu}, \tag{1}$$

where $a > 0$ are hyperparameters, the weight functions $\pi(\theta;\mathbf{X}^{(n)})$ and $\pi(\nu;\mathbf{X}^{(n)})$ are probability density functions in Θ and $\tilde\Theta_0$ respectively. Note that $\pi(\theta;\mathbf{X}^{(n)})$ and $\pi(\nu;\mathbf{X}^{(n)})$ may be data dependent but does not need to be the posterior density. If we take the weight function as

$$\pi(\theta;\mathbf{X}^{(n)}) = \frac{\left[p_n(\mathbf{X}^{(n)}|\theta)\right]^b \pi(\theta)}{\int_\Theta \left[p_n(\mathbf{X}^{(n)}|\theta)\right]^b \pi(\theta)\,d\theta}, \quad \pi(\nu;\mathbf{X}^{(n)}) = \frac{\left[p_n(\mathbf{X}^{(n)}|\nu,\xi_0)\right]^b \pi(\nu)}{\int_{\tilde\Theta_0} \left[p_n(\mathbf{X}^{(n)}|\nu,\xi_0)\right]^b \pi(\nu)\,d\nu}, \tag{2}$$

then the ILRT statistic equals

$$\Lambda_{a,b}(\mathbf{X}^{(n)}) = \frac{L_{a+b}}{L_b}\cdot\frac{L_b^*}{L_{a+b}^*}.$$

We shall call $\Lambda_{a,b}(\mathbf{X}^{(n)})$ the generalized FBF throughout the paper. The FBF and PBF are both the special cases of the generalized FBF. In fact, the FBF is equal to $\Lambda_{1,b}(\mathbf{X}^{(n)})$, the PBF is equal to $\Lambda_{2,1}(\mathbf{X}^{(n)})$.

The computation of the generalized FBF is simple. We can independently generate $\theta_1,\ldots,\theta_m$ and $\nu_1,\ldots,\nu_m$ according to (2) for a large m. Then $\Lambda_{a,b}(\mathbf{X}^{(n)})$ can be approximated by

$$\frac{\sum_{i=1}^m \left[p_n(\mathbf{X}^{(n)}|\theta)\right]^a}{\sum_{i=1}^m \left[p_n(\mathbf{X}^{(n)}|\nu,\xi_0)\right]^a}.$$

For some moderately complex models, (2) may be complicated. Consequently, sampling from (2) may be intractable. In this case, one may use some simple form weight function to approximate (2). A popular method for approximating (2) is variational inference. See. See, for example, Blei et al. (2017). In this case, the weight function in (1) is equals to the variational apprixomation of (2). The ILRT methodology also includes such approximate method.

2.2. Generalized FBF

In this section, we investigate the asymptotic behavior of the generalized FBF.

The following assumption is adapted from Kleijn and Vaart (2012).

Assumption 1. *The parameter space Θ is an open subset of $\mathbb{R}^p$. The null space $\tilde\Theta_0$ is an open subset of $\mathbb{R}^{p_0}$. The parameter θ_0 is an inner point of Θ, ν_0 is an inner point of $\tilde\Theta_0$. The function $\theta\mapsto\log p(X|\theta)$ is differentiable at θ_0 P_{θ_0}-a.s. with derivative*

$$\dot\ell_{\theta_0}(X) = \frac{\partial}{\partial\theta}\log p(X|\theta)\Big|_{\theta=\theta_0}.$$

There's an open neighborhood V of θ_0 such that for every $\theta_1,\theta_2\in V$,

$$|\log p(X|\theta_1) - \log p(X|\theta_2)| \le m(X)\|\theta_1-\theta_2\|,$$

where $m(X)$ is a measurable function satisfying $P_0\exp[sm(X)]<\infty$ for some $s>0$. The Fisher information matrix $I_{\theta_0} = P_{\theta_0}\dot\ell_{\theta_0}\dot\ell_{\theta_0}^T$ is positive-definite and as $\theta\to\theta_0$,

$$P_{\theta_0}\log\frac{p(X|\theta)}{\log(X|\theta_0)} = -\frac{1}{2}(\theta-\theta_0)^T I_{\theta_0}(\theta-\theta_0) + o(\|\theta-\theta_0\|^2).$$

Assumption 1 is satisfied by many common models, it ensures a local asymptotic normality expansion of likelihood. See Lemma 1 in Appendix.

For $t > 0$, we say L_t is $\sqrt{n}$-consistent if for every $M_n \to \infty$,

$$\frac{L_t(\{\theta : \|\theta - \theta_0\| > M_n/\sqrt{n}\})}{L_t} \xrightarrow{P_{\theta_0}^n} 0,$$

where for a set $A \subset \Theta$,

$$L_t(A) = \int_A \Big[p_n(\mathbf{X}^{(n)}|\theta)\Big]^t \pi(\theta)\, d\theta.$$

The $\sqrt{n}$-consistency of L_t^* is defined similarly. Note that the consistency of L_1 is equivalent to the consistency of the posterior distribution. In Kleijn and Vaart (2012), the $\sqrt{n}$-consistency of posterior distribution is a key assumption to prove Bernstein-von Mises theorem. Likewise, the $\sqrt{n}$-consistency of L_t is a key assumption of the following theorem.

Theorem 1. *Suppose that Assumption 1 holds, L_{a+b}, L_b, L_{a+b}^* and L_b^* are $\sqrt{n}$-consistent, $\pi(\theta)$ is continuous at θ_0 with $\pi(\theta_0) > 0$, $\pi(\nu)$ is continuous at ν_0 with $\pi(\nu_0) > 0$, then for $\{\theta_n\}$ such that $\sqrt{n}(\theta_n - \theta_0) \to \eta$,*

$$2\log \Lambda_{a,b}(\mathbf{X}^{(n)}) \overset{P_{\theta_n}^n}{\rightsquigarrow} -(p - p_0)\log(1 + \frac{a}{b}) + a\chi^2_{p-p_0}(\delta),$$

where $\chi^2_{p-p_0}(\delta)$ is a noncentral chi-squared random variable with $p - p_0$ degrees of freedom and noncentrality parameter $\delta = \eta^T\big(I_{\theta_0} - I_{\theta_0}J(J^T I_{\theta_0} J)^{-1} J^T I_{\theta_0}\big)\eta$ and $J = (I_{p_0}, 0_{p_0\times(p-p_0)})^T$, "$\rightsquigarrow$" means weak convergence.

Theorem 1 gives the asymptotic distribution of $2\log\Lambda_{a,b}(\mathbf{X}^{(n)})$ under the null hypothesis and the local alternative hypothesis. To obtain a test with asymptotic type I error rate α, the critical value of $2\log\Lambda_{a,b}(\mathbf{X}^{(n)})$ can be defined to be $-(p-p_0)\log(1+a/b)+a\chi^2_{p-p_0,1-\alpha}$, where $\chi^2_{p-p_0,1-\alpha}$ is the $1-\alpha$ quantile of a chi-squared random variable with $p-p_0$ degrees of freedom. By Theorem 1, the resulting test has local asymptotic power

$$\Pr\big(\chi^2_{p-p_0}(\delta) > \chi^2_{p-p_0,1-\alpha}\big). \tag{3}$$

It is known that, under certain regular conditions, (3) is also the local asymptotic power of the likelihood ratio test. In this view, $\Lambda_{a,b}(\mathbf{X}^{(n)})$ enjoys good frequentist properties.

The $\sqrt{n}$-consistency of L_t is a key assumption of Theorem 1. Hence we would like to give sufficient conditions for the $\sqrt{n}$-consistency of L_t. First we consider the exponential family. The following proposition shows that for full-rank exponential family, L_t is $\sqrt{n}$-consistent for all $t > 0$.

Proposition 1. *Suppose $p(X|\theta) = \exp\big[\theta^T T(X) - A(\theta)\big]$, Θ is an open subset of $\mathbb{R}^p$, θ_0 is an interior point of Θ,*

$$I_{\theta_0} = \frac{\partial^2}{\partial\theta\partial\theta^T} A(\theta_0) > 0.$$

Then L_t is consistent for $t > 0$.

In the general setting, however, the $\sqrt{n}$-consistency of L_t needs further conditions. It turns out that the behavior of L_t for $t > 1$ and $t \le 1$ are different. Note that L_1 is well defined $P_{\theta_0}^n$-a.s. since it has finite integral:

$$\int_{\mathcal{X}^n} L_1\, d\mu^n = \int_\Theta \Big(\int_{\mathcal{X}^n} p_n(\mathbf{X}^{(n)}|\theta)\, d\mu^n\Big)\pi(\theta)\, d\theta = 1.$$

By hölder's inequality, L_t is also well defined $P_{\theta_0}^n$-a.s. for $0 < t < 1$. However, the following example shows that L_t is not always well defined for $t > 1$.

Example 1. *Suppose $X_1, \dots, X_n$ are i.i.d. from the density*

$$p(x|\theta) = C|x - \theta|^{-1/2} \exp\big[-(x-\theta)^2\big],$$

where C is the normalizing constant. The parameter space Θ is equal to $\mathbb{R}$. We would like to test the hypotheses $H_0 : \theta = 0$ vs $H_1 : \theta \neq 0$. The likelihood is

$$p_n(\mathbf{X}^{(n)}|\theta) = C^n \Big[\prod_{i=1}^n |X_i - \theta| \Big]^{-1/2} \exp \big[- \sum_{i=1}^n (X_i - \theta)^2 \big].$$

Under the alternative hypothesis, the likelihood tends to infinity if θ tends to X_i, $i = 1, \ldots, n$. Consequently, LRT fails in this model. We impose a prior $\pi(\theta)$. Suppose that $\pi(\theta)$ is positive for all θ. Then

$$L_t(\mathbf{X}^{(n)}) = \int_{-\infty}^{+\infty} \Big[\prod_{i=1}^n |X_i - \theta| \Big]^{-t/2} \exp \big[- t \sum_{i=1}^n (X_i - \theta)^2 \big] \pi(\theta)\, d\theta.$$

The likelihood will almost surely have no ties and consequently $L_t(\mathbf{X}^{(n)}) = +\infty$ if and only if $t \geq 2$.

Because of the bad behavior of L_t for $t > 1$, next we consider L_t for $t \leq 1$. For $t = 1$, the $\sqrt{n}$-consistency of L_t is equivalent to the $\sqrt{n}$-consistency of posterior distribution. The consistency of posterior distribution has drawn much attention in the literature. See, for example, Ghosal et al. (2000), Shen and Wasserman (2001), van der Vaart and Ghosal (2007). A popular and convenient way of establishing the consistency of posterior is through the condition that suitable test sequences exist. This approach is adopted by Ghosal et al. (2000), van der Vaart and Ghosal (2007) and Kleijn and Vaart (2012).

Assumption 2. *For every $\epsilon > 0$, there exists a sequence of tests ϕ_n such that*

$$P_{\theta_0}^n \phi_n \to 0, \qquad \sup_{\|\theta - \theta_0\| \geq \epsilon} P_\theta^n (1 - \phi_n) \to 0.$$

Assumption 2 is satisfied when the parameter space is compact and the model is suitably continuous. See Theorem 3.2 of Kleijn and Vaart (2012).

Proposition 2 (Kleijn and Vaart (2012), Theorem 3.1)**.** *Suppose θ_0 is an interior of Θ, $\pi(\theta)$ is continuous at θ_0 and $\pi(\theta_0) > 0$. Under Assumptions 1 and 2, L_1 is consistent.*

The consistency of L_t for $0 < t < 1$ is different from $t = 1$. Walker and Hjort (2001) considered the Hellinger consistency of $L_{1/2}$. However, they only consider $t = 1/2$ and didn't consider the $\sqrt{n}$-convergence result. We shall prove the consisency of L_t for $0 < t < 1$ under certain conditions on the Rényi divergence between distributions in the family $\{P_\theta : \theta \in \Theta\}$.

For two parameters θ_1 and θ_2, the α order Rényi divergence ($0 < \alpha < 1$) of P_{θ_1} from P_{θ_2} is defined to be

$$D_\alpha(\theta_1 || \theta_2) = -\frac{1}{1 - \alpha} \log \rho_\alpha(\theta_1, \theta_2),$$

where $\rho_\alpha(\theta_1, \theta_2) = \int_{\mathcal{X}} p(X|\theta_1)^\alpha p(X|\theta_2)^{1-\alpha}\, d\mu$ is the so-called Hellinger integral. The following assumption is needed for our $\sqrt{n}$-consistency result.

Assumption 3. *For some $\alpha \in (0, 1)$, there exist positive constancts δ, ϵ and C such that, $D_\alpha(\theta || \theta_0) \geq C \|\theta - \theta_0\|^2$ for $\|\theta - \theta_0\| \leq \delta$ and $D_\alpha(\theta || \theta_0) \geq \epsilon$ for $\|\theta - \theta_0\| > \delta$.*

Remark 1. A remarkable property of Rényi divergence is the equivalence of all D_α: If $0 < \alpha < \beta < 1$, then

$$\frac{\alpha}{1 - \alpha} \frac{1 - \beta}{1 - \alpha} D_\beta(\theta_1 || \theta_2) \leq D_\alpha(\theta_1 || \theta_2) \leq D_\beta(\theta_1 || \theta_2).$$

See, for example, Bobkov et al. (2016). As a result, if Assumption 3 holds for some $\alpha \in (0, 1)$, then it will hold for every $\alpha \in (0, 1)$.

To appreciate Assumption 3, suppose, for example, that $D_\alpha(\theta||\theta_0)$ is twice continuously differentiable in θ. Since $\theta=\theta_0$ is a minumum point of $D_\alpha(\theta||\theta_0)$, the first order derivative of $D_\alpha(\theta||\theta)$ at $\theta=\theta_0$ is zero and the second order derivative at $\theta=\theta_0$ is positive semidefinite. By Taylor theorem, in a small neighbourhood of θ_0,

$$D_\alpha(\theta||\theta_0)=\frac{1}{2}(\theta-\theta_0)^T\frac{\partial^2}{\partial\theta\partial\theta^T}D_\alpha(\theta||\theta_0)\Big|_{\theta=\theta^*}(\theta-\theta_0),$$

where θ^* is between θ_0 and θ. If we further assume the second order derivative is positive definite at $\theta=\theta_0$, then in a small neighbourhood of θ_0, there is a positive constant C such that $D_\alpha(\theta||\theta_0)\geq C\|\theta-\theta_0\|^2$. Thus, Assumption 3 is a fairly weak condition.

Proposition 3. *Suppose θ_0 is an interior of Θ, $\pi(\theta)$ is continuous at θ_0 and $\pi(\theta_0)>0$. Under Assumptions 1 and 3, for fixed $t\in(0,1)$, L_t is consistent.*

Compared with Assumption 2, it appears that Assumption 3 is easier to verify. Note that the asymptotic power of $\Lambda_{a,b}(\mathbf{X}^{(n)})$ is independent of a,b. Hence it can be recommanded to use the generalized FBF with $a+b<1$.

2.3. General weight function

For some moderately complexi models, the densities (2) are not easy to calculate. In this case, we can use simpler weight functions to approximate (2).

Let $h=\sqrt{n}(\theta-\theta_0)$. For two densities $q_1(h)$ and $q_2(h)$, let $\|q_1(h)-q_2(h)\|=\int|p(h)-q(h)|\,dh$ be the total variation distance between $q_1(h)$ and $q_2(h)$. Theorem 2.1 of Kleijn and Vaart (2012) states that under Assumptions 1, 2,

$$\|\pi_n(h|\mathbf{X}^{(n)})-\phi(h;\Delta_{n,\theta_0},I_{\theta_0}^{-1})\|\overset{P_{\theta_0}^n}{\to}0,$$

where $\Delta_{n,\theta_0}=n^{-1/2}\sum_{i=1}^n I_{\theta_0}^{-1}\dot{\ell}_{\theta_0}(X_i)$. We shall assume that the weight function inherits this property.

Assumption 4. *Let $\pi_n(h;\mathbf{X}^{(n)})$ be a weight function satisfying*

$$\|\pi_n(h;\mathbf{X}^{(n)})-\phi(h;\Delta_{n,\theta_0},b^{-1}I_{\theta_0}^{-1})\|\overset{P_{\theta_0}^n}{\to}0 \tag{4}$$

Furthermore, assume that for every $\epsilon>0$, there's a Lebesgue integrable function $T(h)$, a $K>0$ and an $A>0$ such that

$$\lim_{n\to\infty}P_{\theta_0}^n(\sup_{\|h\|\geq K\sqrt{n}}(\pi_n(h;\mathbf{X}^{(n)})-T(h))\leq 0)\geq 1-\epsilon \tag{5}$$

$$\lim_{n\to\infty}P_{\theta_0}^n(\sup_{\|h\|\leq K\sqrt{n}}\pi_n(h;\mathbf{X}^{(n)})\leq A)\geq 1-\epsilon \tag{6}$$

The condition 5 assumes there is a function controlling the tail of weight function. For a statistical model, the likelihood value makes no sense when $\|\theta-\theta_0\|=n^{-1/2}h$ is large. The bad behavior of the tail of likelihood function may affect the behavior of posterior distribution. To avoid the bad behavior of the likelihood function for large $n^{-1/2}h$, we impose 5 on weight function instead. The condition 6 is satisfied in most usual case.

Theorem 2. *Suppose that Assumptions 1 and 4 hold, the true parameter θ_0 is an interior point of Θ, ν is a relative interior point of $\tilde{\Theta}_0$. If $a+b=1$, we assume Assumption 2 holds. If $a+b<1$, we assume Assumption 3 holds. Then, for bounded real numbers η_n, we have*

$$2\log\Lambda_{a,b}(\mathbf{X}^{(n)})\overset{P_{\theta_n}^n}{\rightsquigarrow}-(p-p_0)\log(1+\frac{a}{b})+a\chi^2_{p-p_0}(\delta).$$

A practical method to obtain simple form weight function $\pi_n(h; \mathbf{X}^{(n)})$ is the variational inference. See, for example, Blei et al. (2017). The following example shows that the weight function obtained from Rényi divergence variational inference satisfies Assumption 4.

Example 2. *Suppose $\pi_n(h; \mathbf{X}^{(n)})$ is obtain from Rényi divergence variational inference (Li and Turner, 2016):*

$$\pi_n(h; \mathbf{X}^{(n)}) = \min_{q \in \mathcal{Q}} -\frac{1}{1-\alpha} \log \int_{\mathcal{X}} q(h)^{\alpha} \pi(h|\mathbf{X}^{(n)})^{1-\alpha} \, d\mu,$$

where $\mathcal{Q}$ is the family of all p dimensional normal distribution. Since

$$-\frac{1}{1-\alpha} \log \int_{\mathcal{X}} \pi(h; \mathbf{X}^{(n)})^{\alpha} \pi(h|\mathbf{X}^{(n)})^{1-\alpha} \, d\mu \leq -\frac{1}{1-\alpha} \log \int_{\mathcal{X}} \phi(h; \Delta_{n,\theta_0}, I_{\theta_0}^{-1})^{\alpha} \pi(h|\mathbf{X}^{(n)})^{1-\alpha} \, d\mu. \tag{7}$$

By the equivalence of Rényi divergence and total variation distance and Bernstein-von Mise theorem, the right hand side of (7) tends to 0. Again by the equivalence of Rényi divegence and total variation distance, (4) holds. Since $\pi_n(h; \mathbf{X}^{(n)})$ is a normal density, (4) implies the mean and covariance parameter of $\pi_n(h; \mathbf{X}^{(n)})$ converges to Δ_{n,θ_0} and $I_{\theta_0}^{-1}$ respectively. Then (5) and (6) hold.

3. Normal mixture

In this section, we apply ILRT to the normal mixture model. is the first example of unbounded likelihood given in Cam (1990).

Suppose $X_1, \ldots, X_n$ are i.i.d. distributed as a mixture of normal distributions

$$p(X|\theta) = \frac{1-\alpha}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(X-\mu)^2\right\} + \frac{\alpha}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{(X-\mu)^2}{\sigma^2}\right\}, \tag{8}$$

where α is a known constant. Suppose the parameter space is

$$\Theta = \{\theta = (\mu, \sigma^2)^T : \mu \in (\infty, \infty), \sigma^2 \in (0, M)\}, \tag{9}$$

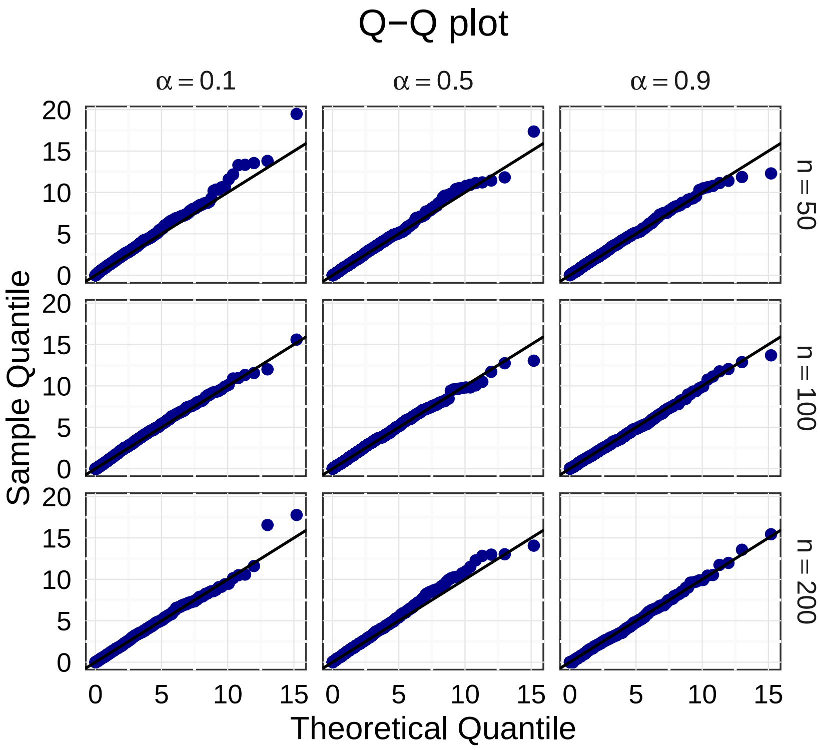
where M is a sufficiently large parameter. Cam (1990) pointed out that the likelihood of the model is unbounded. In fact, let $\mu = X_1$ and let $\sigma^2 \to 0$, then the likelihood tends to infinity.

Under the model, we consider testing the hypotheses $H_0 : \mu = 0, \sigma = 2$ vs $H_1 : \theta \in \Theta$. Although LRT fails in this model, ILRT can still be used. To use ILRT, we need a weight function. First let the weight function be the posterior density of parameters. To make the posterior density bounded from infinity, we consider the so-called zero-avoiding prior. See Gelman et al. (2013) section 13.2. When $\mu = X_1$, as $\sigma^2 \to 0$, the likelihood tends to infinity at the rate of $1/\sigma$. The rate can be hedged by the density of χ_3^2. Hence we adopt the following prior distribution

$$\phi(\mu) \times d\chi_3^2(\sigma^2), \tag{10}$$

where $d\chi_3^2(\sigma^2)$ represents the density of χ^2 distribution with freedom 3 taking value at σ^2. Because the σ^2 is limited in $(0, M)$, we also truncate prior of σ^2 at M.

We take sample size $n = 50, 100, 200$ and $\alpha = 0.1, 0.5, 0.9$. In every combination, we repeat 1000 samples and obtain 1000 ILRT statistics. We expect the empirical distribution of $2 \log \Lambda_{a,b}(\mathbf{X}^{(n)})$ is similar to that of $-2\log 2 + \chi_2^2$. We plot the QQ-plot of empirical distribution relative to $-2\log 2 + \chi_2^2$ distribution, it can be seen that ILRT can be well approximated by $\chi^2(2)$.



Next we consider another weight function $\pi(\theta; X) = N(\hat{\theta}, \frac{1}{n}\hat{I}_{\hat{\theta}}^{-1})$. Lt $\hat{\theta}$ be the highest probability density estimator. And

$$\hat{I}_{\theta}^{-1} = \sum_{i=1}^{n} \begin{bmatrix} -\frac{\partial^2 \log p_\theta(x_i)}{\partial \mu^2} & -\frac{\partial^2 \log p_\theta(x_i)}{\partial \mu \partial(\sigma^2)} \\ -\frac{\partial^2 \log p_\theta(x_i)}{\partial \mu \partial(\sigma^2)} & -\frac{\partial^2 \log p_\theta(x_i)}{\partial (\sigma^2)^2} \end{bmatrix}$$

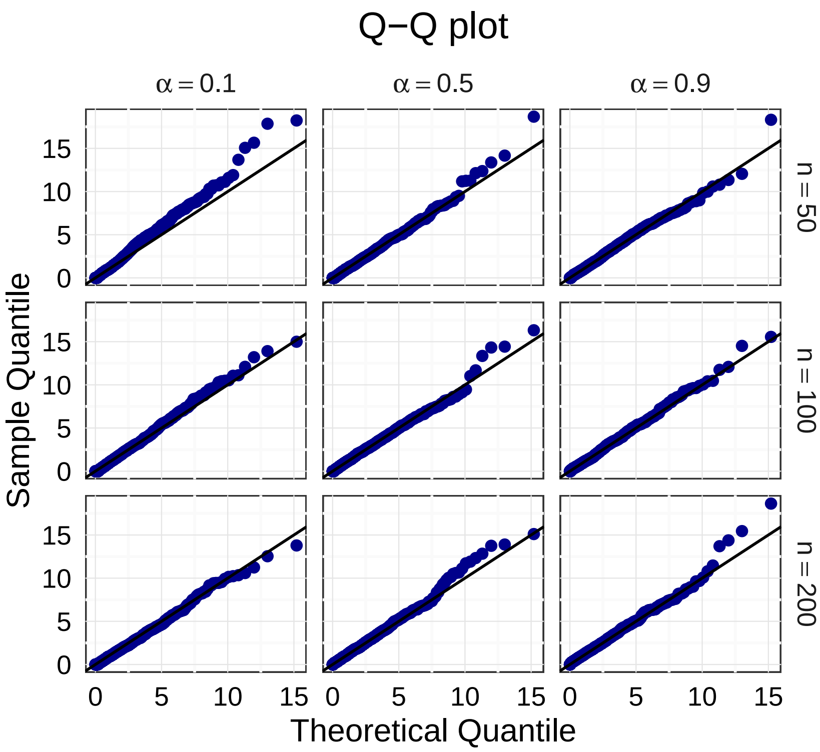
where

$$\begin{aligned} \frac{\partial^2 \log p_\theta(x)}{\partial \mu^2} =& \frac{(1-\alpha)((x-\mu)^2 - 1)dN(\mu,1)(x) + \alpha((x-\mu)^2/\sigma^4 - \sigma^{-2})dN(\mu,\sigma^2)(x)}{p_\theta(x)} - \\ & \Big(\frac{(1-\alpha)(x-\mu)dN(\mu,1)(x) + \alpha(x-\mu)/\sigma^2 dN(\mu,\sigma^2)(x)}{p_\theta(x)}\Big)^2, \end{aligned} \tag{11}$$

$$\begin{aligned} \frac{\partial^2 \log p_\theta(x)}{\partial \mu \partial(\sigma^2)} =& \frac{(\frac{3\alpha(\mu-x)}{2\sigma^4} - \frac{\alpha(\mu-x)^3}{2\sigma^6})dN(\mu,\sigma^2)(x)}{p_\theta(x)} - \\ & \frac{\alpha(\frac{(\mu-x)^2}{2\sigma^4} - \frac{1}{2\sigma^2})dN(\mu,\sigma^2)(x)\big((1-\alpha)(x-\mu)dN(\mu,1)(x) + \alpha(x-\mu)/\sigma^2 d(\mu,\sigma^2)(x)\big)}{p_\theta(x)^2}, \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{\partial^2 \log p_\theta(x)}{\partial (\sigma^2)^2} =& \frac{\alpha\big(\frac{3}{4\sigma^4} - \frac{3(x-\mu)^2}{2\sigma^6} + \frac{(x-\mu)^4}{4\sigma^8}\big)dN(\mu,\sigma^2)(x)}{p_\theta(x)} - \\ & \Big(\frac{\alpha\big(\frac{(x-\mu)^2}{2\sigma^4} - \frac{1}{2\sigma^2}\big)dN(\mu,\sigma^2)(x)}{p_\theta(x)}\Big)^2. \end{aligned} \tag{13}$$

We do the same simulation as above and the QQ-plot is given. It can be seen that the distribution of ILRT statistic is still close to the theoretical distribution in this case. For mixture model, sampling from posterior distribution is troublesome. The computing burden will be reduce by normal approximation. This is an advantage of normal weight ILRT. From this example, we can see that ILRT is more flexible than posterior Bayes factor.



4. Conclusion

In this paper, we proposed a flexible methodology ILRT which includes some existing method as special cases. We gave the asymptotic distribution of the generalized FPF, which is a special case of ILRT. We also investigates the asymptotic behavior of ILRT for general weight functions. This allows one to use a simple form approximation of the posterior distribution as weight function. In particular, we show that the weight function can be obtained from Rényi divergence variational inference.

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Appendices

Define

$$\dot{\ell}^*(X) = \frac{\partial}{\partial \nu} \log p(X|\nu, \xi_0)\Big|_{\nu=\nu_0}, \quad I^*_{\theta_0} = P_{\theta_0} \dot{\ell}^*_{\theta_0} \dot{\ell}^{*T}_{\theta_0}, \quad \Delta^*_{n,\theta_0} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} I^{*-1}_{\theta_0} \dot{\ell}^*_{\theta_0}(X_i).$$

Lemma 1 (Kleijn and Vaart (2012), Lemma 2.1.). *Under Assumption 1, we have* $\|\dot{\ell}_{\theta_0}(X)\| \le m(X)$ P_0*-a.s.,* $P_0\dot{\ell}_{\theta_0}(X) = 0$ *and for every* $M > 0$

$$\sup_{\|h\|\le M}\Big|\log\frac{p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} - h^T I_{\theta_0}\Delta_{n,\theta_0} + \frac{1}{2}h^T I_{\theta_0}h\Big| \xrightarrow{P_0^n} 0.$$

Lemma 2. *Under Assumptions 1 and 2, there exists for every* $M_n \to \infty$ *a sequence of tests* ϕ_n *and a constant* $\delta > 0$ *such that, for every sufficiently large* n *and every* $\|\theta-\theta_0\| \ge M_n/\sqrt{n}$,

$$P_0^n\phi_n \to 0, \quad P_\theta^n(1-\phi_n) \le \exp[-\delta n(\|\theta-\theta\|^2 \wedge 1)].$$

(See der Vaart (2000) Lemma 10.3., Kleijn and Vaart (2012))

Appendix A Proofs in Section 2

Proof of Theorem 1. For fixed $t > 0$ and $M > 0$, we have

$$\begin{aligned}
&\log\int_{\{\theta:\|\theta-\theta_0\|\le M/\sqrt{n}\}}\big[p_n(\mathbf{X}^{(n)}|\theta)\big]^t\pi(\theta)\,d\theta\\
=&\log\int_{\{\theta:\|\theta-\theta_0\|\le M/\sqrt{n}\}}\big[p_n(\mathbf{X}^{(n)}|\theta)\big]^t\,d\theta + \log\pi(\theta_0) + o_{P_{\theta_0}^n}(1)\\
=&\log\int_{\{h:\|h\|\le M\}}\exp\big[t\log p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)\big]\,dh - \frac{p}{2}\log n + \log\pi(\theta_0) + o_{P_{\theta_0}^n}(1).
\end{aligned}$$

By Proposition 1,

$$\begin{aligned}
&\log\int_{\{h:\|h\|\le M\}}\exp\big[t\log p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)\big]\,dh\\
=&\log\int_{\{h:\|h\|\le M\}}\exp\big[t\log p_n(\mathbf{X}^{(n)}|\theta_0) + th^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{t}{2}h^T I_{\theta_0}h\big]\,dh + o_{P_{\theta_0}^n}(1)\\
=&\log\int_{\{h:\|h\|\le M\}}\exp\big[-\frac{t}{2}(h-\Delta_{n,\theta_0})^T I_{\theta_0}(h-\Delta_{n,\theta_0})\big]\,dh + \frac{t}{2}\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0} + t\log p_n(\mathbf{X}^{(n)}|\theta_0) + o_{P_{\theta_0}^n}(1).
\end{aligned}$$

Thus

$$\begin{aligned}
&\log\int_{\{\theta:\|\theta-\theta_0\|\le M/\sqrt{n}\}}\big[p_n(\mathbf{X}^{(n)}|\theta)\big]^t\pi(\theta)\,d\theta\\
=&\log\int_{\{h:\|h\|\le M\}}\exp\big[-\frac{t}{2}(h-\Delta_{n,\theta_0})^T I_{\theta_0}(h-\Delta_{n,\theta_0})\big]\,dh\\
&+\frac{t}{2}\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0} + t\log p_n(\mathbf{X}^{(n)}|\theta_0) - \frac{p}{2}\log n + \log\pi(\theta_0) + o_{P_{\theta_0}^n}(1).
\end{aligned}$$

This equality holds for every $M > 0$ and hence also for some $M_n \to \infty$. Note that Δ_{n,θ_0} is bounded in probability. Hence

$$\begin{aligned}
&\log\int_{\{h:\|h\|\le M_n\}}\exp\big[-\frac{t}{2}(h-\Delta_{n,\theta_0})^T I_{\theta_0}(h-\Delta_{n,\theta_0})\big]\,dh\\
=&\log\int_{\mathbb{R}^p}\exp\big[-\frac{t}{2}(h-\Delta_{n,\theta_0})^T I_{\theta_0}(h-\Delta_{n,\theta_0})\big]\,dh + o_{P_{\theta_0}^n}(1)\\
=&\frac{p}{2}\log(2\pi) - \frac{p}{2}\log t - \frac{1}{2}\log|I_{\theta_0}| + o_{P_{\theta_0}^n}(1).
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\log \int_{\{\theta:\|\theta-\theta_0\|\le M_n/\sqrt{n}\}} \left[p_n(\mathbf{X}^{(n)}|\theta)\right]^t \pi(\theta)\,d\theta \\
=&\frac{p}{2}\log\big(\frac{2\pi}{n}\big) - \frac{p}{2}\log t - \frac{1}{2}\log|I_{\theta_0}| + \log\pi(\theta_0) + \frac{t}{2}\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0} + t\log p_n(\mathbf{X}^{(n)}|\theta_0) + o_{P_{\theta_0}^n}(1).
\end{aligned}
$$

If $L_t(\mathbf{X}^{(n)})$ is consistent, then

$$
\begin{aligned}
&\log L_t(\mathbf{X}^{(n)}) = \log \int_{\Theta} \left[p_n(\mathbf{X}^{(n)}|\theta)\right]^t \pi(\theta)\,d\theta \\
=&\frac{p}{2}\log\big(\frac{2\pi}{n}\big) - \frac{p}{2}\log t - \frac{1}{2}\log|I_{\theta_0}| + \log\pi(\theta_0) + \frac{t}{2}\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0} + t\log p_n(\mathbf{X}^{(n)}|\theta_0) + o_{P_{\theta_0}^n}(1).
\end{aligned}
$$

Similarly, if $L_t^*(\mathbf{X}^{(n)})$ is consistent,

$$
\begin{aligned}
&\log L_t^*(\mathbf{X}^{(n)}) = \log \int_{\tilde{\Theta}_0} \left[p_n(\mathbf{X}^{(n)}|\nu,\xi_0)\right]^t \pi(\nu)\,d\nu \\
=&\frac{p_1}{2}\log\big(\frac{2\pi}{n}\big) - \frac{p_1}{2}\log t - \frac{1}{2}\log|I_{\theta_0}^*| + \log\pi(\nu_0) + \frac{t}{2}\Delta_{n,\theta_0}^{*T} I_{\theta_0}^*\Delta_{n,\theta_0}^* + t\log p_n(\mathbf{X}^{(n)}|\theta_0) + o_{P_{\theta_0}^n}(1).
\end{aligned}
$$

These expansions, combined with the mutually contiguity of $P_{\theta_0}^n$ and $P_{\theta_n}^n$, yield

$$
\begin{aligned}
\log\Lambda_{a,b}(\mathbf{X}^{(n)}) =& \log L_a(\mathbf{X}^{(n)}) - \log L_b(\mathbf{X}^{(n)}) - \log L_a^*(\mathbf{X}^{(n)}) + \log L_b^*(\mathbf{X}^{(n)}) \\
=& -\frac{p-p_1}{2}\log\frac{a}{b} + \frac{a-b}{2}\Big(\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0} - \Delta_{n,\theta_0}^{*T} I_{\theta_0}^*\Delta_{n,\theta_0}^*\Big) + o_{P_{\theta_n}^n}(1).
\end{aligned}
$$

Note that

$$
I_{\theta_0}^* = J^T I_{\theta_0} J, \quad \Delta_{n,\theta_0}^* = (J^T I_{\theta_0} J)^{-1} J^T I_{\theta_0}\Delta_{n,\theta_0}.
$$

Then

$$
\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0} - \Delta_{n,\theta_0}^{*T} I_{\theta_0}^*\Delta_{n,\theta_0}^* = \Delta_{n,\theta_0}^T I_{\theta_0}^{1/2}\big(I_p - I_{\theta_0}^{1/2} J (J^T I_{\theta_0} J)^{-1} J^T I_{\theta_0}^{1/2}\big) I_{\theta_0}^{1/2}\Delta_{n,\theta_0},
$$

where $I_p - I_{\theta_0}^{1/2} J (J^T I_{\theta_0} J)^{-1} J^T I_{\theta_0}^{1/2}$ is a projection matrix with rank $p-p_1$.

Now we need to derive the asymptotic distribution of Δ_{n,θ_0}. Let $h_n = \sqrt{n}(\theta_n - \theta_0)$. By Proposition 1 and CLT,

$$
\begin{aligned}
\begin{pmatrix} \frac{1}{\sqrt{n}}\sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) \\ \log\frac{p_n(\mathbf{X}^{(n)}|\theta_n)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{n}}\sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) \\ \frac{1}{\sqrt{n}}\sum_{i=1}^n h_n^T\dot{\ell}_{\theta_0}(X_i) - \frac{1}{2}h_n^T I_{\theta_0} h_n \end{pmatrix} + o_{P_0^n}(1) \\
&\overset{P_0^n}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ -\frac{1}{2}\eta^T I_{\theta_0}\eta \end{pmatrix}, \begin{pmatrix} I_{\theta_0} & I_{\theta_0}\eta \\ \eta^T I_{\theta_0} & \eta^T I_{\theta_0}\eta \end{pmatrix}\right).
\end{aligned}
$$

Hence by Le Cam's third lemma,

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) \overset{P_{\theta_n}^n}{\rightsquigarrow} N(I_{\theta_0}\eta, I_{\theta_0}).
$$

Consequently, Δ_{n,θ_0} weakly converges to $N(\eta, I_{\theta_0}^{-1})$ in $P_{\theta_n}^n$. Hence

$$
\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0} - \Delta_{n,\theta_0}^{*T} I_{\theta_0}^*\Delta_{n,\theta_0}^* \overset{P_{\eta_n}^n}{\rightsquigarrow} \chi_{p-p_1}^2(\delta).
$$

This completes the proof.

□

Proof of Proposition 1. By some algebra, we have

$$\Delta_{n,\theta_0} = n^{-1/2}\sum_{i=1}^{n} T(X_i) - \sqrt{n}\frac{\partial}{\partial\theta}A(\theta_0)$$

and

$$\log\frac{p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} = h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0}h - g_n(h),$$

where

$$g_n(h) = n\Big(A(\theta_0 + n^{-1/2}h) - A(\theta_0) - n^{-1/2}h\frac{\partial}{\partial\theta}A(\theta_0) - \frac{1}{2n}h^T I_{\theta_0}h\Big).$$

Without loss of generality, we assume $M_n \to \infty$ and $M_n^3/\sqrt{n} \to 0$. Then by Taylor's theorem and the continuity of the third derivative of $A(\theta)$,

$$\max_{\{h:\|h\|\le M_n\}} |g_n(h)| = O\Big(\frac{M_n^3}{\sqrt{n}}\Big) \to 0.$$

Then

$$\begin{aligned}
&\int_\Theta \big[p_n(\mathbf{X}^{(n)}|\theta)\big]^t \pi(\theta)\, d\theta \ge \int_{\{\theta:\|\theta-\theta_0\|\le M_n/\sqrt{n}\}} \big[p_n(\mathbf{X}^{(n)}|\theta)\big]^t \pi(\theta)\, d\theta \\
&= (1 + o_{P_0^n}(1))n^{-p/2}\pi(\theta_0)\big[p_n(\mathbf{X}^{(n)}|\theta_0)\big]^t \int_{\{h:h\le M_n\}} \exp\big[th^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{t}{2}h^T I_{\theta_0}h\big]\, dh \\
&= (1 + o_{P_0^n}(1))n^{-p/2}\pi(\theta_0)\big[p_n(\mathbf{X}^{(n)}|\theta_0)\big]^t \int_{\mathbb{R}^p} \exp\big[th^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{t}{2}h^T I_{\theta_0}h\big]\, dh \\
&= (1 + o_{P_0^n}(1))n^{-p/2}\pi(\theta_0)\big[p_n(\mathbf{X}^{(n)}|\theta_0)\big]^t \exp\big[-\frac{t}{2}\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0}\big](2\pi)^{p/2}t^{-p/2}|I_{\theta_0}|^{-1/2}.
\end{aligned}$$

We have

$$\begin{aligned}
&\max_{\{\theta:\|\theta-\theta_0\|=M_n/\sqrt{n}\}} \log\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} = \max_{\{h:\|h\|=M_n\}} \log\frac{p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \\
&\le \|I_{\theta_0}\Delta_{n,\theta_0}\|M_n - \frac{\lambda_{\min}(I_{\theta_0})}{2}M_n^2 + \max_{\{h:\|h\|=M_n\}} |g_n(h)|,
\end{aligned}$$

where $\lambda_{\min}(I_{\theta_0}) > 0$ is the minimum eigenvalue of I_{θ_0}. Also note that $I_{\theta_0}\Delta_{n,\theta_0}$ is bounded in probability. Hence with probability tending to 1,

$$\max_{\{\theta:\|\theta-\theta_0\|=M_n/\sqrt{n}\}} \log\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \le -\frac{\lambda_{\min}(I_{\theta_0})}{4}M_n^2.$$

By the concavity of $\log p_n(\mathbf{X}^{(n)}|\theta)$, for $\|\theta-\theta_0\| \ge M_n/\sqrt{n}$,

$$\frac{M_n/\sqrt{n}}{\|\theta-\theta_0\|}\Big(\log p_n(\mathbf{X}^{(n)}|\theta) - \log p_n(\mathbf{X}^{(n)}|\theta_0)\Big) \le \log p_n\Big(\mathbf{X}^{(n)}\Big|\theta_0 + \frac{M_n/\sqrt{n}}{\|\theta-\theta_0\|}(\theta-\theta_0)\Big) - \log p_n(\mathbf{X}^{(n)}|\theta_0).$$

Thus,

$$\begin{aligned}
\log\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} &\le \frac{\sqrt{n}\|\theta-\theta_0\|}{M_n}\log\frac{p_n\Big(\mathbf{X}^{(n)}\Big|\theta_0 + \frac{M_n/\sqrt{n}}{\|\theta-\theta_0\|}(\theta-\theta_0)\Big)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \\
&\le \frac{\sqrt{n}\|\theta-\theta_0\|}{M_n}\Big(-\frac{\lambda_{\min}(I_{\theta_0})}{4}M_n^2\Big) \\
&= -\frac{\lambda_{\min}(I_{\theta_0})}{4}\sqrt{n}\|\theta-\theta_0\|M_n.
\end{aligned}$$

For $\epsilon > 0$ such that $\sup_{\|\theta-\theta_0\|<\epsilon} \pi(\theta) \leq +\infty$, we have

$$
\begin{aligned}
&\int_{\{\theta:\|\theta-\theta_0\|>M_n/\sqrt{n}\}} \big[p_n(\mathbf{X}^{(n)}|\theta)\big]^t \pi(\theta)\, d\theta \\
\leq&\big[p_n(\mathbf{X}^{(n)}|\theta_0)\big]^t \int_{\{\theta:\|\theta-\theta_0\|>M_n/\sqrt{n}\}} \exp\Big[-\frac{t\lambda_{\min}(I_{\theta_0})}{4}\sqrt{n}\|\theta-\theta_0\|M_n\Big]\pi(\theta)\, d\theta \\
=&\big[p_n(\mathbf{X}^{(n)}|\theta_0)\big]^t \Big(\int_{\{\theta:M_n/\sqrt{n}\leq\|\theta-\theta_0\|\leq\epsilon\}} \exp\Big[-\frac{t\lambda_{\min}(I_{\theta_0})}{4}\sqrt{n}\|\theta-\theta_0\|M_n\Big]\pi(\theta)\, d\theta \\
&+\int_{\{\theta:\|\theta-\theta_0\|>\epsilon\}} \exp\Big[-\frac{t\lambda_{\min}(I_{\theta_0})}{4}\sqrt{n}\|\theta-\theta_0\|M_n\Big]\pi(\theta)\, d\theta\Big) \\
\leq&\big[p_n(\mathbf{X}^{(n)}|\theta_0)\big]^t \Big(\big(\sup_{\|\theta-\theta_0\|<\epsilon}\pi(\theta)\big)\int_{\{\theta:\|\theta-\theta_0\|\geq M_n/\sqrt{n}\}} \exp\Big[-\frac{t\lambda_{\min}(I_{\theta_0})}{4}\sqrt{n}\|\theta-\theta_0\|M_n\Big]\, d\theta \\
&+\exp\Big[-\frac{t\lambda_{\min}(I_{\theta_0})}{4}\epsilon\sqrt{n}M_n\Big]\Big) \\
=&\big[p_n(\mathbf{X}^{(n)}|\theta_0)\big]^t \Big(\big(\sup_{\|\theta-\theta_0\|<\epsilon}\pi(\theta)\big)n^{-p/2}\int_{\{h:\|h\|\geq M_n\}} \exp\Big[-\frac{t\lambda_{\min}(I_{\theta_0})}{4}\|h\|M_n\Big]\, dh \\
&+\exp\Big[-\frac{t\lambda_{\min}(I_{\theta_0})}{4}\epsilon\sqrt{n}M_n\Big]\Big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{\int_{\{\theta:\|\theta-\theta_0\|>M_n/\sqrt{n}\}} \big[p_n(\mathbf{X}^{(n)}|\theta)\big]^t \pi(\theta)\, d\theta}{\int_\Theta \big[p_n(\mathbf{X}^{(n)}|\theta)\big]^t \pi(\theta)\, d\theta} \\
=&O_{P^n_{\theta_0}}(1)\Big(\int_{\{h:\|h\|\geq M_n\}} \exp\Big[-\frac{t\lambda_{\min}(I_{\theta_0})}{4}\|h\|M_n\Big]\, dh + n^{p/2}\exp\Big[-\frac{t\lambda_{\min}(I_{\theta_0})}{4}\epsilon\sqrt{n}M_n\Big]\Big) \\
=&o_{P^n_{\theta_0}}(1).
\end{aligned}
$$

□

Proof of Proposition 3. Note that

$$
\frac{L_t(\{\theta:\|\theta-\theta_0\|\geq\frac{M_n}{\sqrt{n}}\})}{L_t} = \frac{\int_{\big\{\theta:\|\theta-\theta_0\|\geq\frac{M_n}{\sqrt{n}}\big\}} \Big[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\Big]^t \pi(\theta)\, d\theta}{\int_\Theta \Big[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\Big]^t \pi(\theta)\, d\theta}. \tag{14}
$$

Without loss of generality, we assume $M_n/\sqrt{n} \to 0$.

Consider the expectation of the numerator of 14. It follows from Fubini's theorem that

$$
\begin{aligned}
&P^n_0 \int_{\{\theta:\|\theta-\theta_0\|\geq\frac{M_n}{\sqrt{n}}\}} \Big[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\Big]^t \pi(\theta)\, d\theta \\
=&\int_{\{\theta:\|\theta-\theta_0\|\geq\frac{M_n}{\sqrt{n}}\}} \Big\{\int_{\mathcal{X}^n} \big[p_n(\mathbf{X}^{(n)}|\theta)\big]^t \big[p_n(\mathbf{X}^{(n)}|\theta_0)\big]^{1-t}\, d\mu^n\Big\}\pi(\theta)\, d\theta \\
=&\int_{\{\theta:\|\theta-\theta_0\|\geq\frac{M_n}{\sqrt{n}}\}} \big[\rho_t(\theta,\theta_0)\big]^n \pi(\theta)\, d\theta \\
=&\int_{\{\theta:\|\theta-\theta_0\|\geq\frac{M_n}{\sqrt{n}}\}} \exp\big[-(1-t)nD_t(\theta||\theta_0)\big]\pi(\theta)\, d\theta.
\end{aligned}
$$

Decompose the integral region into two parts $\{\theta : \frac{M_n}{\sqrt{n}} \le \|\theta-\theta_0\| \le \delta\}$ and $\{\theta : \|\theta-\theta_0\| > \delta\}$,

$$
\begin{aligned}
&\int_{\{\theta:\|\theta-\theta_0\|\ge \frac{M_n}{\sqrt{n}}\}} \exp\big[-(1-t)nD_a(\theta||\theta_0)\big]\pi(\theta)\,d\theta \\
=&\int_{\{\theta:\frac{M_n}{\sqrt{n}}\le\|\theta-\theta_0\|\le\delta\}} \exp\big[-(1-t)nD_t(\theta||\theta_0)\big]\pi(\theta)\,d\theta + \int_{\{\theta:\|\theta-\theta_0\|>\delta\}} \exp\big[-(1-t)nD_t(\theta||\theta_0)\big]\pi(\theta)\,d\theta \\
\le& \max_{\|\theta-\theta_0\|\le\delta}\pi(\theta)\int_{\left\{\theta:\|\theta-\theta_0\|\ge\frac{M_n}{\sqrt{n}}\right\}} \exp\big[-(1-t)Cn\|\theta-\theta_0\|^2\big]\,d\theta + \exp\big[-(1-t)\epsilon n\big] \\
=&\big(\max_{\|\theta-\theta_0\|\le\delta}\pi(\theta)\big)n^{-p/2}\int_{\left\{h:\|h\|\ge M_n\right\}} \exp\big[-(1-t)C\|h\|^2\big]\,d\theta + \exp\big[-(1-t)\epsilon n\big].
\end{aligned}
$$

Now we consider the denominator of (14).

$$
\begin{aligned}
&\int_\Theta \left[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\right]^t \pi(\theta)\,d\theta \ge \int_{\{\theta:\|\theta-\theta_0\|\le n^{-1/2}\}} \left[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\right]^t \pi(\theta)\,d\theta \\
\ge&\Big(\min_{\|\theta-\theta_0\|\le n^{-1/2}} \left[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\right]^t \pi(\theta)\Big)\int_{\{\theta:\|\theta-\theta_0\|\le n^{-1/2}\}} 1\,d\theta \\
\ge&\Big(\exp\Big[t\min_{\|\theta-\theta_0\|\le n^{-1/2}}\log\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\Big]\Big)\Big(\min_{\|\theta-\theta_0\|\le n^{-1/2}}\pi(\theta)\Big)n^{-p/2}\frac{2\pi^{p/2}}{\Gamma(p/2)}.
\end{aligned}
$$

By Proposition 1,

$$
\min_{\|\theta-\theta_0\|\le n^{-1/2}}\log\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \ge -\|I_{\theta_0}\Delta_{n,\theta_0}\| - \frac{1}{2}\|I_{\theta_0}\| + o_{P_0^n}(1).
$$

Since $I_{\theta_0}\Delta_{n,\theta_0}$ is bounded in probability,

$$
\min_{\|\theta-\theta_0\|\le n^{-1/2}}\log\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}
$$

is lower bounded in probability. Note that

$$
\min_{\|\theta-\theta_0\|\le n^{-1/2}}\pi(\theta)\to\pi(\theta_0)>0.
$$

Then for every $\epsilon'>0$, there is a constant $c>0$ such that with probability at least $1-\epsilon'$,

$$
\int_\Theta \left[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\right]^t \pi(\theta)\,d\theta \ge cn^{-p/2}.
$$

Combining the upper bound and the lower bound yields that with probability at least $1-\epsilon'$,

$$
\begin{aligned}
&\frac{L_t(\{\theta:\|\theta-\theta_0\|\ge\frac{M_n}{\sqrt{n}}\})}{L_t} \\
\le& c^{-1}\big(\max_{\|\theta-\theta_0\|\le\delta}\pi(\theta)\big)\int_{\left\{h:\|h\|\ge M_n\right\}} \exp\big[-(1-t)C\|h\|^2\big]\,d\theta + c^{-1}n^{p/2}\exp\big[-(1-t)\epsilon n\big]\to 0.
\end{aligned}
$$

Since ϵ is arbitrary, the theorem follows. □

Appendix B Proofs in Section 3

Proof of Theorem 2. By contiguity, we only need to prove the convergence in P_0^n.

The proof consists of two steps. In the first part of the proof, let M be a fixed positive number. We prove

$$\left| \int_{\|h\|\le M} \frac{p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h;\mathbf{X}^{(n)})\,dh - \int_{\|h\|\le M} \exp\left[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\right]\phi(h;\Delta_{n,\theta_0}, I_{\theta_0}^{-1})\,dh \right| \xrightarrow{P_0^n} 0 \tag{15}$$

Propostion 1 implies that

$$\int_{\|h\|\le M} \frac{p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h;\mathbf{X}^{(n)})\,dh = \exp[o_{p_0^n}(1)] \int_{\|h\|\le M} \exp\left[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\right]\pi_n(h;\mathbf{X}^{(n)})\,dh \tag{16}$$

So we only need to consider $\int_{\|h\|\le M} \exp\left[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\right]\pi_n(h;\mathbf{X}^{(n)})\,dh$. By central limit theorem, Δ_{n,θ_0} weakly converges to a normal distribution. As a result, $\sup_{\|h\|\le M} \exp\left[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\right]$ is bounded in probability. It follows that

$$\begin{aligned}
&\int_{\|h\|\le M} \exp\left[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\right]\left|\pi_n(h;\mathbf{X}^{(n)}) - \phi(h;\Delta_{n,\theta_0}, I_{\theta_0}^{-1})\right|dh \\
\le &\sup_{\|h\|\le M} \exp\left[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\right] \int_{\|h\|\le M} \left|\pi_n(h;\mathbf{X}^{(n)}) - \phi(h;\Delta_{n,\theta_0}, I_{\theta_0}^{-1})\right|dh \xrightarrow{P_0^n} 0.
\end{aligned}$$

This, combined with (16), proves (15). This is true for every M and hence also for some $M_n \to \infty$.

In the second part, we prove

$$\psi(M) \stackrel{def}{=} \frac{\int_{\|h\|>M} p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)\pi_n(h;\mathbf{X}^{(n)})\,dh}{\int p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)\pi_n(h;\mathbf{X}^{(n)})\,dh} \xrightarrow{P_0^n} 0. \tag{17}$$

Let ϕ_n be a test function satisfying the conclusion of Lemma 2. We have

$$\psi(M) = \psi(M)\phi_n + \psi(M)(1-\phi_n).$$

Since $\psi(M)\le 1$, $\psi(M)\phi_n \le \phi_n \xrightarrow{P_0^n} 0$. So it's enough to prove

$$\psi(M)(1-\phi_n) \xrightarrow{P_0^n} 0$$

Fix a positive number U. Then

$$\psi(M)(1-\phi_n) \le \frac{\int_{\|h\|>M_n} p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)\pi_n(h;\mathbf{X}^{(n)})\,dh}{\int_{\|h\|\le U} p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)\pi_n(h;\mathbf{X}^{(n)})\,dh}(1-\phi_n). \tag{18}$$

First we give a lower bound of $\int_{\|h\|\le U} p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)\pi_n(h;\mathbf{X}^{(n)})\,dh$. Note that

$$\begin{aligned}
&\int_{\|h\|\le U} p_n(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)\pi_n(h;\mathbf{X}^{(n)})\,dh \\
=&\exp[o_{P_0^n}(1)]p_n(\mathbf{X}^{(n)}|\theta_0)\int_{\|h\|\le U} \exp\left[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\right]\pi_n(h;\mathbf{X}^{(n)})\,dh \\
\ge&\exp[o_{P_0^n}(1)]p_n(\mathbf{X}^{(n)}|\theta_0)\Big\{\int_{\|h\|\le U} \exp\left[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\right]\phi(h;\Delta_{n,\theta_0}, I_{\theta_0}^{-1})\,dh \\
&- \sup_{\|h\|\le U} \exp\left[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\right]\int_{\|h\|\le U}\left|\pi_n(h;\mathbf{X}^{(n)}) - \phi(h;\Delta_{n,\theta_0}, I_{\theta_0}^{-1})\right|dh\Big\} \\
=&\exp[o_{P_0^n}(1)]p_n(\mathbf{X}^{(n)}|\theta_0)\Big\{\int_{\|h\|\le U} \exp\left[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\right]\phi(h;\Delta_{n,\theta_0}, I_{\theta_0}^{-1})\,dh - O_P(1)o_P(1)\Big\}.
\end{aligned}$$

Fix an $\epsilon > 0$. Since Δ_{n,θ_0} is uniformly tight, with probability at least $1-\epsilon/2$, $|\Delta_{n,\theta_0}| \le C$ for a constant C. If this event happens, we have

$$\int_{\|h\|\le U} \exp\big[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0} h\big]\phi(h;\Delta_{n,\theta_0}, I_{\theta_0}^{-1})\,dh > 2c$$

for some $c > 0$. Thus, there is a $c > 0$ and an event $D_{1,n}$ with probability at least $1-\epsilon$ on which

$$\int_{\|h\|\le U} p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)\pi_n(h;\mathbf{X}^{(n)})\,dh \ge c p_n(\mathbf{X}^{(n)}|\theta_0)$$

for sufficiently large n.

On the other hand, by Assumption 4, there is a $K > 0$, a $A > 0$ and an event $D_{2,n}$ with probability at least $1-\epsilon$ on which

$$\sup_{\|h\|>K\sqrt{n}} (\pi_n(h;\mathbf{X}^{(n)}) - T(h)) \le 0, \qquad \sup_{\|h\|\le K\sqrt{n}} \pi_n(h;\mathbf{X}^{(n)}) \le A$$

for sufficiently large n.

Combining these bounds yields

$$\psi(M)(1-\phi_n) \le \frac{\int_{\|h\|>M_n} p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)\big(A\mathbf{1}_{M_n\le\|h\|\le K\sqrt{n}} + T(h)\mathbf{1}_{\|h\|>K\sqrt{n}}\big)\,dh}{c p_n(\mathbf{X}^{(n)}|\theta_0)}(1-\phi_n) + \mathbf{1}\{D_{1,n}^C \cup D_{2,n}^C\}.$$

Hence for sufficiently large n,

$$\begin{aligned}
&P_0^n\psi(M)(1-\phi_n) \\
\le& c^{-1}\int_{\mathcal{X}^n}\int_{\|h\|>M_n} (1-\phi_n)p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)\big(A\mathbf{1}_{M_n\le\|h\|\le K\sqrt{n}} + T(h)\mathbf{1}_{\|h\|>K\sqrt{n}}\big)\,dh\,d\mu^n + 2\epsilon \\
=& c^{-1}\int_{\|h\|>M_n}\Big(\int_{\mathcal{X}^n}(1-\phi_n)p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)\,d\mu^n\Big)\big(A\mathbf{1}_{M_n\le\|h\|\le K\sqrt{n}} + T(h)\mathbf{1}_{\|h\|>K\sqrt{n}}\big)\,dh + 2\epsilon \\
\le& c^{-1}\int_{\|h\|>M_n} \exp\big[-\delta(\|h\|^2\wedge n)\big]\big(A\mathbf{1}_{M_n\le\|h\|\le K\sqrt{n}} + T(h)\mathbf{1}_{\|h\|>K\sqrt{n}}\big)\,dh + 2\epsilon.
\end{aligned}$$

Note that $\delta(\|h\|^2 \cap n) \ge \delta^*(\|h\|^2\wedge K^2 n)$, where $\delta^* = \delta\min(1, K^{-2})$. Hence

$$\begin{aligned}
&\int_{\|h\|>M_n} \exp\big[-\delta(\|h\|^2\wedge n)\big]\big(A\mathbf{1}_{M_n\le\|h\|\le K\sqrt{n}} + T(h)\mathbf{1}_{\|h\|>K\sqrt{n}}\big)\,dh \\
\le& \int_{\|h\|>M_n} \exp\big[-\delta^*(\|h\|^2\wedge K^2 n)\big]\big(A\mathbf{1}_{M_n\le\|h\|\le K\sqrt{n}} + T(h)\mathbf{1}_{\|h\|>K\sqrt{n}}\big)\,dh \\
\le& A\int_{\|h\|\ge M_n} e^{-\delta^*\|h\|^2}\,dh + e^{-\delta^* K^2 n}\int_{\|h\|>K\sqrt{n}} T(h)\,dh \to 0.
\end{aligned}$$

Therefore $\psi(M) \xrightarrow{P_0^n} 0$.

Finally we have

$$
\begin{aligned}
&\left|\int \frac{p(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)}\pi_n(h;\mathbf{X}^{(n)})\,dh - 2^{-\frac{p}{2}}\exp\big[\frac{1}{2}\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0}\big]\right| \\
&= \left|\int \frac{p(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)}\pi_n(h;\mathbf{X}^{(n)})\,dh - \int_{\|h\|\le M_n} \frac{p(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)}\pi_n(h;\mathbf{X}^{(n)})\,dh\right| \\
&+ \left|\int_{\|h\|\le M_n} \frac{p(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)}\pi_n(h;\mathbf{X}^{(n)})\,dh - \int_{\|h\|\le M_n}\exp\big[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0}h\big]\phi(h;\Delta_{n,\theta_0},I_{\theta_0}^{-1})\,dh\right| \\
&+ \left|\int_{\|h\|\le M_n}\exp\big[h^T I_{\theta_0}\Delta_{n,\theta_0} - \frac{1}{2}h^T I_{\theta_0}h\big]\phi(h;\Delta_{n,\theta_0},I_{\theta_0}^{-1})\,dh - 2^{-\frac{p}{2}}\exp\big[\frac{1}{2}\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0}\big]\right| \\
&= J_1 + J_2 + J_3
\end{aligned}
$$

By the first step of the proof, we have $J_2 \xrightarrow{P_0^n} 0$. Hence

$$\int_{\|h\|\le M_n} \frac{p(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)}\pi_n(h;\mathbf{X}^{(n)})\,dh$$

is bounded in probability. Therefore

$$
\begin{aligned}
J_1 &= \int_{\|h\|\le M_n} \frac{p(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)}\pi_n(h;\mathbf{X}^{(n)})\,dh \left|\frac{\int p(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)\pi_n(h;\mathbf{X}^{(n)})\,dh}{\int_{\|h\|\le M_n} p(\mathbf{X}^{(n)}|\theta_0+n^{-1/2}h)\pi_n(h;\mathbf{X}^{(n)})\,dh} - 1\right| \\
&= O_{P_0^n}(1)o_{P_0^n}(1)
\end{aligned}
$$

And J_3 convenges to 0 for trivial reason.

Then we can apply the argument to both the numerator and denominator of integrated likelihood ratio statistics. By CLT,

$$I_{\theta_0}\Delta_{n,\theta_0} = \frac{1}{\sqrt{n}}\sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) \overset{P_0^n}{\rightsquigarrow} \xi, \tag{19}$$

where $\xi \sim N(0, I_{\theta_0})$.

$$I_{\theta_0}^*\Delta_{n,\theta_0}^* = \frac{1}{\sqrt{n}}\sum_{i=1}^n \dot{\ell}_{\theta_0}^*(X_i) \overset{P_0^n}{\rightsquigarrow} \xi^*, \tag{20}$$

where ξ^* is the first p_1 coordinates of ξ. Hence

$$
\begin{aligned}
\Lambda(X) &= \frac{2^{-\frac{p_2}{2}}\exp\{\frac{1}{2}\Delta_{n,\theta_0}^T I_{\theta_0}\Delta_{n,\theta_0}\} + o_{P_0^n}(1)}{2^{-\frac{p_1}{2}}\exp\{\frac{1}{2}\Delta_{n,\theta_0}^{*T} I_{\theta_0}^*\Delta_{n,\theta_0}^*\} + o_{P_0^n}(1)} \\
&\overset{P_0^n}{\rightsquigarrow} \frac{2^{-\frac{p_2}{2}}\exp\{\frac{1}{2}\xi^T I_{\theta_0}^{-1}\xi\}}{2^{-\frac{p_1}{2}}\exp\{\frac{1}{2}\xi^{*T} I_{\theta_0}^{*-1}\xi^*\}}.
\end{aligned}
\tag{21}
$$

But

$$\xi^T I_{\theta_0}^{-1}\xi - \xi^{*T} I_{\theta_0}^{*-1}\xi^* = (I_{\theta_0}^{-\frac{1}{2}}\xi)^T\Big(I_{p_2\times p_2} - I_{\theta_0}^{\frac{1}{2}}\begin{pmatrix} I_{\theta_0}^{*-1} & 0 \\ 0 & 0\end{pmatrix} I_{\theta_0}^{\frac{1}{2}}\Big)(I_{\theta_0}^{-\frac{1}{2}}\xi). \tag{22}$$

$I_{\theta_0}^{-\frac{1}{2}}\xi$ is a p_2-dimensional standard normal distribution, The middle term is a projection matrix with rank $p_2 - p_1$. Hence we have

$$2\log(\Lambda(X)) \overset{P_0^n}{\rightsquigarrow} \chi^2_{p_2-p_1} - (p_2-p_1)\log(2). \tag{23}$$

$\square$

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