

# Integrated likelihood ratio test<sup>☆</sup>

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## Abstract

Likelihood ratio test (LRT) is the most widely used test procedure. However, it has some weaknesses. Likelihood is unbounded for some important models. Even when the likelihood is bounded, the maximum may be not easy to obtain if it is not convex in parameters. We propose a new test procedure called integrated likelihood ratio test (ILRT) which can overcome the above difficulties. Posterior Bayes factor is a special case of ILRT. We proof the Wilks phenomenon of ILRT and give the asymptotic local power.

*Keywords:*

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## 1. Introduction

Suppose that we have  $n$  observations  $\mathbf{X}^{(n)} = (X_1, \dots, X_n)$  which are independent identically distributed (i.i.d.) random variables with values in some space  $(\mathcal{X}; \mathcal{A})$ . Assume that there is a  $\sigma$ -finite measure  $\mu$  on  $\mathcal{X}$  and that the possible distribution  $P_\theta$  of  $X_i$  has a density  $p(X|\theta)$  with respect to  $\mu$ . The parameter  $\theta$  takes its values in some set  $\Theta$ .

Suppose we are interested in testing the hypotheses  $H_0 : \theta \in \Theta_0$  vs  $H_1 : \theta \in \Theta$  for a subset  $\Theta_0$  of  $\Theta$ . The well known likelihood ratio test (LRT) is defined as

$$\frac{\sup_{\Theta} p_n(\mathbf{X}^{(n)}|\theta)}{\sup_{\Theta_0} p_n(\mathbf{X}^{(n)}|\theta)}, \quad (1)$$

where  $p_n(\mathbf{X}^{(n)}|\theta) = \prod_{i=1}^n p(X_i|\theta)$  is the density of  $\mathbf{X}^{(n)}$  with respect to  $\mu^n$ , the  $n$ -fold product measure of  $\mu$ . LRT is the most widely used statistical method which enjoys many optimal properties. For example, by Neyman-Pearson lemma, it's the most powerful test (MPT) in simple null and simple alternative case (Lehmann J. P. R, 2005). In multi-dimensional parameter case, MPT does not exist. Nevertheless, the LRT is asymptotic optimal in the sense of Bahadur efficiency (Bahadur, 1971). However, even in some widely used models, likelihood may be unbounded. See Cam (1990) for some examples. In this case, LRT does not exist. Another weakness of LRT occurs when the likelihood is not convex in parameters. In this case, numerical algorithms for maximizing likelihood may trap in local maxima.

In Bayesian framework, Bayes factor is the most popular methodology. However, the frequency property of Bayes factor is not satisfactory. Several modifications of Bayes factor have been proposed. See, for

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example, xxxxxx. Among them, Aitkin (1991) proposed posterior Bayes factor (PBF). Where Gelfand D. K. D (1993) derived the null distribution of PBF. However, they didn't explicitly give the conditions needed. In fact, their proof relies on Laplace approximation, which assumes the existence of maximum likelihood estimator (MLE). Note that the existence of MLE implies the existence of LRT. Hence the scope of their method doesn't exceed that of classical LRT.

O'Hagan (1995) proposed the fractional Bayes factor (FBF). The idea of fractional likelihood is also adopted by Walker and Hjort (2001). We will see that FBF has a wider applicable scope than PBF.

Both PBF and FBF is a special case of the general ILRT.

Based on the proof of Bernstein-von Mises theorem (See der Vaart (2000) and Kleijn and Vaart (2012)), we give the proof of the Wilks phenomenon and local power of ILRT under fairly weak assumptions.

## 2. Integrated likelihood ratio test

The parameter space  $\Theta$  is an open subset of  $\mathbb{R}^p$ . Let  $\theta = (\nu^T, \xi^T)^T$ , where  $\nu$  is a  $p_0$  dimensional subvector, and  $\xi$  is a  $p - p_0$  dimensional subvector. The null space  $\Theta_0$  is a  $p_0$ -dimensional subspace of  $\Theta$  defined as

$$\Theta_0 = \{(\nu^T, \xi^T)^T : (\nu^T, \xi^T)^T \in \Theta, \xi = \xi_0\}. \quad (2)$$

We would like to test the hypothesis

$$H_0 : \xi = \xi_0. \quad (3)$$

Let  $\tilde{\Theta}_0 = \{\nu : (\nu^T, \xi^T)^T \in \Theta_0\}$ .

$$\Lambda_P(\mathbf{X}^{(n)}) = \frac{\int_{\Theta} p(\mathbf{X}^{(n)}|\theta) \pi(\theta|\mathbf{X}^{(n)}) d\theta}{\int_{\tilde{\Theta}_0} p(\mathbf{X}^{(n)}|\nu, \xi_0) \pi(\nu|\mathbf{X}^{(n)}) d\nu},$$

where  $\pi^*(\theta|\mathbf{X}^{(n)})$  and  $\pi(\theta|\mathbf{X}^{(n)})$  are the posterior densities under null hypotheses and alternative hypothesis. For  $t > 0$ , define  $L_t(\mathbf{X}^{(n)}) = \int_{\Theta} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta$ ,  $L_t^*(\mathbf{X}^{(n)}) = \int_{\Theta_0} [p_n(\mathbf{X}^{(n)}|\nu, \xi_0)]^t \pi(\nu) d\nu$ . Then PBF can be written as

$$\begin{aligned} \Lambda_P(\mathbf{X}^{(n)}) &= \frac{L_2(\mathbf{X}^{(n)})}{L_1(\mathbf{X}^{(n)})} \cdot \frac{L_1^*(\mathbf{X}^{(n)})}{L_2^*(\mathbf{X}^{(n)})}. \\ \Lambda_F(\mathbf{X}^{(n)}) &= \frac{L_{1/2}(\mathbf{X}^{(n)})}{L_{1/2}^*(\mathbf{X}^{(n)})} \cdot \frac{L_{1/2}^*(\mathbf{X}^{(n)})}{L_1^*(\mathbf{X}^{(n)})}. \end{aligned}$$

The posterior Bayes factor can be generalized to the integrated likelihood ratio test (ILRT) statistic, as follow

$$\Lambda(\mathbf{X}^{(n)}) = \frac{\int_{\Theta} [p_n(\mathbf{X}^{(n)}|\theta)]^a \pi(\theta; \mathbf{X}^{(n)}) d\theta}{\int_{\tilde{\Theta}_0} [p_n(\mathbf{X}^{(n)}|\nu, \xi_0)]^a \pi(\nu; \mathbf{X}^{(n)}) d\nu}, \quad (4)$$

where  $a > 0$  is a hyperparameter,  $\pi(\theta; X)$  and  $\pi^*(\theta; X)$  are weight functions which may be data dependent but does not need to be the posterior density of  $\theta$ .

If

$$\pi(\theta; \mathbf{X}^{(n)}) = \frac{[p_n(\mathbf{X}^{(n)}|\theta)]^b \pi(\theta)}{\int_{\Theta} [p_n(\mathbf{X}^{(n)}|\theta)]^b \pi(\theta) d\theta},$$

then

$$\Lambda(\mathbf{X}^{(n)}) = \frac{L_{a+b}(\mathbf{X}^{(n)})}{L_b(\mathbf{X}^{(n)})} \cdot \frac{L_b^*(\mathbf{X}^{(n)})}{L_{a+b}^*(\mathbf{X}^{(n)})}.$$

The case  $a = b = 1/2$  corresponds to the fractional Bayes factor (FBF) (O'Hagan, 1995). The case  $a = b = 1$  corresponds to the posterior Bayes factor (PBF).

Let  $\pi(\theta; \mathbf{X}^{(n)})$  and  $\pi(\nu; \mathbf{X}^{(n)})$  be the weight functions in  $\Theta$  and  $\tilde{\Theta}_0$ . The integrated likelihood ratio statistic is defined as

$$\Lambda(\mathbf{X}^{(n)}) = \frac{\int_{\Theta} [p_n(\mathbf{X}^{(n)}|\theta)]^a \pi(\theta; \mathbf{X}^{(n)}) d\theta}{\int_{\tilde{\Theta}_0} [p_n(\mathbf{X}^{(n)}|\nu, \xi_0)]^a \pi(\nu; \mathbf{X}^{(n)}) d\nu}, \quad (5)$$

### 3. Asymptotic behavior of FBF

In this section, we consider the general FBF

$$\Lambda_{a,b}(\mathbf{X}^{(n)}) = \frac{L_a(\mathbf{X}^{(n)})}{L_b(\mathbf{X}^{(n)})} \cdot \frac{L_b^*(\mathbf{X}^{(n)})}{L_a^*(\mathbf{X}^{(n)})},$$

where  $0 < b < a$ . Note that  $\Lambda_{2,1}(\mathbf{X}^{(n)})$  is PBF,  $\Lambda_{1,1/2}(\mathbf{X}^{(n)})$  is the conventional FBF.

We denote by  $\rightsquigarrow$  the weak convergence. Let  $\mathbf{X}^{(n)}$  denote the data.

Suppose  $\theta_0$  is the true parameter.

Denote by  $P_0$  the true distribution of  $\mathbf{X}$ . Let  $p_n(\mathbf{X}^{(n)}|\theta)$  be the density of  $P_\theta^n$  with respect to measure  $\mu^n$ .

Let

$$\dot{\ell}_{\theta_0}(X) = \frac{\partial}{\partial \theta} \log p(X|\theta) \Big|_{\theta=\theta_0}.$$

Let  $I_{\theta_0} = P_{\theta_0} \dot{\ell}_{\theta_0} \dot{\ell}_{\theta_0}^T$  be the Fisher information matrix at  $\theta_0$  and  $\Delta_{n,\theta_0} = \frac{1}{\sqrt{n}} \sum_{i=1}^n I_{\theta_0}^{-1} \dot{\ell}_{\theta_0}(X_i)$  be the ‘locally sufficient’ statistics. The corresponding quantities in the null space are

$$\dot{\ell}^*(X) = \frac{\partial}{\partial \nu} \log p(X|\nu, \xi_0) \Big|_{\nu=\nu_0}, \quad I_{\theta_0}^* = P_{\theta_0} \dot{\ell}_{\theta_0}^* \dot{\ell}_{\theta_0}^{*T}, \quad \Delta_{n,\theta_0}^* = \frac{1}{\sqrt{n}} \sum_{i=1}^n I_{\theta_0}^{*-1} \dot{\ell}_{\theta_0}^*(X_i).$$

**Assumption 1.** *The parameter space  $\Theta$  is an open subset of  $\mathbb{R}^p$ . Thue null space  $\tilde{\Theta}_0$  is an open subset of  $\mathbb{R}^{p_1}$ . The true parameter  $\theta_0$  is an inner point of  $\Theta$ ,  $\nu_0$  is an inner point of  $\tilde{\Theta}_0$ . The function  $\theta \mapsto \log p(X|\theta)$  is differentiable at  $\theta_0$   $P_0$ -a.s. with derivative  $\dot{\ell}_{\theta_0}(X)$ . There’s an open neighborhood  $V$  of  $\theta_0$  such that for every  $\theta_1, \theta_2 \in V$ ,*

$$|\log p(X|\theta_1) - \log p(X|\theta_2)| \leq m(X) \|\theta_1 - \theta_2\|,$$

where  $m(X)$  is a measurable function satisfying  $P_0 \exp[sm(X)] < \infty$  for some  $s > 0$ . The Fisher information matrix  $I_{\theta_0}$  is positive-definite and as  $\theta \rightarrow \theta_0$ ,

$$P_0 \log p(X|\theta) - P_0 \log p(X|\theta_0) = -\frac{1}{2}(\theta - \theta_0)^T I_{\theta_0} (\theta - \theta_0) + o(\|\theta - \theta_0\|^2).$$

Assumption 1 is a stand assumption for likelihood. See vaart (1998) and vaart (2012).

**Proposition 1.** *Under Assumption 1, we have  $\|\dot{\ell}_{\theta_0}(X)\| \leq m(X)$   $P_0$ -a.s.,  $P_0 \dot{\ell}_{\theta_0}(X) = 0$  and for every  $M > 0$*

$$\sup_{\|h\| \leq M} \left| \log \frac{p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} - h^T I_{\theta_0} \Delta_{n,\theta_0} + \frac{1}{2} h^T I_{\theta_0} h \right| \xrightarrow{P_0^n} 0.$$

(See der Vaart (2000) Theorem 5.23 or Kleijn and Vaart (2012) Lemma 2.1.)

For  $t > 0$ , We call  $L_t(\mathbf{X}^{(n)})$  consistent if for every  $M_n \rightarrow \infty$ ,

$$\frac{\int_{\{\theta: \|\theta - \theta_0\| \leq M_n/\sqrt{n}\}} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta}{\int_{\Theta} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta} \xrightarrow{P_{\theta_0}^n} 0.$$

Define

$$L_t(A) = \int_A [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta.$$

for  $t = 1$ , this condition is equivalent to the consistency of Posterior distribution.

**Theorem 1.** Suppose that Assumption 1 holds,  $L_a(\mathbf{X}^{(n)})$ ,  $L_b(\mathbf{X}^{(n)})$ ,  $L_a^*(\mathbf{X}^{(n)})$  and  $L_b^*(\mathbf{X}^{(n)})$  are consistent,  $\pi(\theta)$  is continuous at  $\theta_0$  with  $\pi(\theta_0) > 0$ ,  $\pi(\nu)$  is continuous at  $\nu_0$  with  $\pi(\nu_0) > 0$ , then for  $\{\theta_n\}$  such that  $\sqrt{n}(\theta_n - \theta_0) \rightarrow \eta$ ,

$$\log \Lambda_{a,b}(\mathbf{X}^{(n)}) \overset{P_{\theta_0}^n}{\rightsquigarrow} -\frac{p-p_1}{2} \log \frac{a}{b} + \frac{a-b}{2} \chi_{p-p_1}^2(\delta),$$

where  $\chi_{p-p_1}^2(\delta)$  is a random variable with chi-squared distribution with  $p-p_1$  degrees of freedom and non-centrality parameter  $\delta = \eta^T (I_{\theta_0} - I_{\theta_0} J (J^T I_{\theta_0} J)^{-1} J^T I_{\theta_0}) \eta$  and  $J = (I_{p_1}, 0_{p_1 \times (p-p_1)})^T$ .

**Proof of Theorem 1.** For fixed  $t > 0$  and  $M > 0$ , we have

$$\begin{aligned} & \log \int_{\{\theta: \|\theta - \theta_0\| \leq M/\sqrt{n}\}} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta \\ &= \log \int_{\{\theta: \|\theta - \theta_0\| \leq M/\sqrt{n}\}} [p_n(\mathbf{X}^{(n)}|\theta)]^t d\theta + \log \pi(\theta_0) + o_{P_{\theta_0}^n}(1) \\ &= \log \int_{\{h: \|h\| \leq M\}} \exp [t \log p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)] dh - \frac{p}{2} \log n + \log \pi(\theta_0) + o_{P_{\theta_0}^n}(1). \end{aligned}$$

By Proposition 1,

$$\begin{aligned} & \log \int_{\{h: \|h\| \leq M\}} \exp [t \log p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)] dh \\ &= \log \int_{\{h: \|h\| \leq M\}} \exp [t \log p_n(\mathbf{X}^{(n)}|\theta_0) + th^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{t}{2} h^T I_{\theta_0} h] dh + o_{P_{\theta_0}^n}(1) \\ &= \log \int_{\{h: \|h\| \leq M\}} \exp \left[ -\frac{t}{2} (h - \Delta_{n,\theta_0})^T I_{\theta_0} (h - \Delta_{n,\theta_0}) \right] dh + \frac{t}{2} \Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0} + t \log p_n(\mathbf{X}^{(n)}|\theta_0) + o_{P_{\theta_0}^n}(1). \end{aligned}$$

Thus

$$\begin{aligned} & \log \int_{\{\theta: \|\theta - \theta_0\| \leq M/\sqrt{n}\}} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta \\ &= \log \int_{\{h: \|h\| \leq M\}} \exp \left[ -\frac{t}{2} (h - \Delta_{n,\theta_0})^T I_{\theta_0} (h - \Delta_{n,\theta_0}) \right] dh \\ & \quad + \frac{t}{2} \Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0} + t \log p_n(\mathbf{X}^{(n)}|\theta_0) - \frac{p}{2} \log n + \log \pi(\theta_0) + o_{P_{\theta_0}^n}(1). \end{aligned}$$

This equality holds for every  $M > 0$  and hence also for some  $M_n \rightarrow \infty$ . Note that  $\Delta_{n,\theta_0}$  is bounded in probability. Hence

$$\begin{aligned} & \log \int_{\{h: \|h\| \leq M_n\}} \exp \left[ -\frac{t}{2} (h - \Delta_{n,\theta_0})^T I_{\theta_0} (h - \Delta_{n,\theta_0}) \right] dh \\ &= \log \int_{\mathbb{R}^p} \exp \left[ -\frac{t}{2} (h - \Delta_{n,\theta_0})^T I_{\theta_0} (h - \Delta_{n,\theta_0}) \right] dh + o_{P_{\theta_0}^n}(1) \\ &= \frac{p}{2} \log(2\pi) - \frac{p}{2} \log t - \frac{1}{2} \log |I_{\theta_0}| + o_{P_{\theta_0}^n}(1). \end{aligned}$$

Thus,

$$\begin{aligned} & \log \int_{\{\theta: \|\theta - \theta_0\| \leq M_n/\sqrt{n}\}} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta \\ &= \frac{p}{2} \log \left( \frac{2\pi}{n} \right) - \frac{p}{2} \log t - \frac{1}{2} \log |I_{\theta_0}| + \log \pi(\theta_0) + \frac{t}{2} \Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0} + t \log p_n(\mathbf{X}^{(n)}|\theta_0) + o_{P_{\theta_0}^n}(1). \end{aligned}$$

If  $L_t(\mathbf{X}^{(n)})$  is consistent, then

$$\begin{aligned}\log L_t(\mathbf{X}^{(n)}) &= \log \int_{\Theta} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta \\ &= \frac{p}{2} \log\left(\frac{2\pi}{n}\right) - \frac{p}{2} \log t - \frac{1}{2} \log |I_{\theta_0}| + \log \pi(\theta_0) + \frac{t}{2} \Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0} + t \log p_n(\mathbf{X}^{(n)}|\theta_0) + o_{P_{\theta_0}^n}(1).\end{aligned}$$

Similarly, if  $L_t^*(\mathbf{X}^{(n)})$  is consistent,

$$\begin{aligned}\log L_t^*(\mathbf{X}^{(n)}) &= \log \int_{\tilde{\Theta}_0} [p_n(\mathbf{X}^{(n)}|\nu, \xi_0)]^t \pi(\nu) d\nu \\ &= \frac{p_1}{2} \log\left(\frac{2\pi}{n}\right) - \frac{p_1}{2} \log t - \frac{1}{2} \log |I_{\theta_0}^*| + \log \pi(\nu_0) + \frac{t}{2} \Delta_{n,\theta_0}^{*T} I_{\theta_0}^* \Delta_{n,\theta_0}^* + t \log p_n(\mathbf{X}^{(n)}|\theta_0) + o_{P_{\theta_0}^n}(1).\end{aligned}$$

These expansions, combined with the mutually contiguity of  $P_{\theta_0}^n$  and  $P_{\theta_n}^n$ , yield

$$\begin{aligned}\log \Lambda_{a,b}(\mathbf{X}^{(n)}) &= \log L_a(\mathbf{X}^{(n)}) - \log L_b(\mathbf{X}^{(n)}) - \log L_a^*(\mathbf{X}^{(n)}) + \log L_b^*(\mathbf{X}^{(n)}) \\ &= -\frac{p-p_1}{2} \log \frac{a}{b} + \frac{a-b}{2} \left( \Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0} - \Delta_{n,\theta_0}^{*T} I_{\theta_0}^* \Delta_{n,\theta_0}^* \right) + o_{P_{\theta_n}^n}(1).\end{aligned}$$

Note that

$$I_{\theta_0}^* = J^T I_{\theta_0} J, \quad \Delta_{n,\theta_0}^* = (J^T I_{\theta_0} J)^{-1} J^T I_{\theta_0} \Delta_{n,\theta_0}.$$

Then

$$\Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0} - \Delta_{n,\theta_0}^{*T} I_{\theta_0}^* \Delta_{n,\theta_0}^* = \Delta_{n,\theta_0}^T I_{\theta_0}^{1/2} (I_p - I_{\theta_0}^{1/2} J (J^T I_{\theta_0} J)^{-1} J^T I_{\theta_0}^{1/2}) I_{\theta_0}^{1/2} \Delta_{n,\theta_0},$$

where  $I_p - I_{\theta_0}^{1/2} J (J^T I_{\theta_0} J)^{-1} J^T I_{\theta_0}^{1/2}$  is a projection matrix with rank  $p - p_1$ .

Now we need to derive the asymptotic distribution of  $\Delta_{n,\theta_0}$ . Let  $h_n = \sqrt{n}(\theta_n - \theta_0)$ . By Proposition 1 and CLT,

$$\begin{aligned}\begin{pmatrix} \frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) \\ \log \frac{p_n(\mathbf{X}^{(n)}|\theta_n)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) \\ \frac{1}{\sqrt{n}} \sum_{i=1}^n h_n^T \dot{\ell}_{\theta_0}(X_i) - \frac{1}{2} h_n^T I_{\theta_0} h_n \end{pmatrix} + o_{P_0^n}(1) \\ &\overset{P_0^n}{\rightsquigarrow} N \left( \begin{pmatrix} 0 \\ -\frac{1}{2} \eta^T I_{\theta_0} \eta \end{pmatrix}, \begin{pmatrix} I_{\theta_0} & I_{\theta_0} \eta \\ \eta^T I_{\theta_0} & \eta^T I_{\theta_0} \eta \end{pmatrix} \right).\end{aligned}$$

Hence by Le Cam's third lemma,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) \overset{P_{\theta_n}^n}{\rightsquigarrow} N(I_{\theta_0} \eta, I_{\theta_0}).$$

Consequently,  $\Delta_{n,\theta_0}$  weakly converges to  $N(\eta, I_{\theta_0}^{-1})$  in  $P_{\theta_n}^n$ . Hence

$$\Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0} - \Delta_{n,\theta_0}^{*T} I_{\theta_0}^* \Delta_{n,\theta_0}^* \overset{P_{\theta_n}^n}{\rightsquigarrow} \chi_{p-p_1}^2(\delta).$$

This completes the proof. □

The key assumption of 1 is the consistency of  $L_a(\mathbf{X}^{(n)})$ ,  $L_b(\mathbf{X}^{(n)})$ ,  $L_a^*(\mathbf{X}^{(n)})$  and  $L_b^*(\mathbf{X}^{(n)})$ . Next we consider the consistency of  $L_t(\mathbf{X}^{(n)})$ .

**Proposition 2.** *Under Assumption 1, for  $0 < b < a$ , if  $L_a$  is consistent, then  $L_b$  is consistent.*

We would like to investigate the asymptotic behavior of FBF in exponential family. Exponential family possesses many good properties. It can be seen that the full-rank exponential family fullfill Assumption 1.

**Proposition 3.** Suppose  $p(X|\theta) = \exp[\theta^T T(X) - A(\theta)]$ ,  $\Theta$  is an open subset of  $\mathbb{R}^p$ ,  $\theta_0$  is an interior point of  $\Theta$ ,

$$I_{\theta_0} = \frac{\partial^2}{\partial \theta \partial \theta^T} A(\theta_0) > 0.$$

Then  $L_t$  is consistent for  $t > 0$ .

*Proof.* By some algebra, we have

$$\Delta_{n,\theta_0} = n^{-1/2} \sum_{i=1}^n T(X_i) - \sqrt{n} \frac{\partial}{\partial \theta} A(\theta_0)$$

and

$$\log \frac{p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} = h^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{1}{2} h^T I_{\theta_0} h - g_n(h),$$

where

$$g_n(h) = n \left( A(\theta_0 + n^{-1/2}h) - A(\theta_0) - n^{-1/2}h \frac{\partial}{\partial \theta} A(\theta_0) - \frac{1}{2n} h^T I_{\theta_0} h \right).$$

Without loss of generality, we assume  $M_n \rightarrow \infty$  and  $M_n^3/\sqrt{n} \rightarrow 0$ . Then by Taylor's theorem and the continuity of the third derivative of  $A(\theta)$ ,

$$\max_{\{h: \|h\| \leq M_n\}} |g_n(h)| = O\left(\frac{M_n^3}{\sqrt{n}}\right) \rightarrow 0.$$

Then

$$\begin{aligned} \int_{\Theta} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta &\geq \int_{\{\theta: \|\theta - \theta_0\| \leq M_n/\sqrt{n}\}} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta \\ &= (1 + o_{P_0^n}(1)) n^{-p/2} \pi(\theta_0) [p_n(\mathbf{X}^{(n)}|\theta_0)]^t \int_{\{h: \|h\| \leq M_n\}} \exp[th^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{t}{2} h^T I_{\theta_0} h] dh \\ &= (1 + o_{P_0^n}(1)) n^{-p/2} \pi(\theta_0) [p_n(\mathbf{X}^{(n)}|\theta_0)]^t \int_{\mathbb{R}^p} \exp[th^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{t}{2} h^T I_{\theta_0} h] dh \\ &= (1 + o_{P_0^n}(1)) n^{-p/2} \pi(\theta_0) [p_n(\mathbf{X}^{(n)}|\theta_0)]^t \exp\left[-\frac{t}{2} \Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0}\right] (2\pi)^{p/2} t^{-p/2} |I_{\theta_0}|^{-1/2}. \end{aligned}$$

We have

$$\begin{aligned} \max_{\{\theta: \|\theta - \theta_0\| = M_n/\sqrt{n}\}} \log \frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} &= \max_{\{h: \|h\| = M_n\}} \log \frac{p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \\ &\leq \|I_{\theta_0} \Delta_{n,\theta_0}\| M_n - \frac{\lambda_{\min}(I_{\theta_0})}{2} M_n^2 + \max_{\{h: \|h\| = M_n\}} |g_n(h)|, \end{aligned}$$

where  $\lambda_{\min}(I_{\theta_0}) > 0$  is the minimum eigenvalue of  $I_{\theta_0}$ . Also note that  $I_{\theta_0} \Delta_{n,\theta_0}$  is bounded in probability. Hence with probability tending to 1,

$$\max_{\{\theta: \|\theta - \theta_0\| = M_n/\sqrt{n}\}} \log \frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \leq -\frac{\lambda_{\min}(I_{\theta_0})}{4} M_n^2.$$

By the concavity of  $\log p_n(\mathbf{X}^{(n)}|\theta)$ , for  $\|\theta - \theta_0\| \geq M_n/\sqrt{n}$ ,

$$\frac{M_n/\sqrt{n}}{\|\theta - \theta_0\|} \left( \log p_n(\mathbf{X}^{(n)}|\theta) - \log p_n(\mathbf{X}^{(n)}|\theta_0) \right) \leq \log p_n\left(\mathbf{X}^{(n)}\left|\theta_0 + \frac{M_n/\sqrt{n}}{\|\theta - \theta_0\|}(\theta - \theta_0)\right.\right) - \log p_n(\mathbf{X}^{(n)}|\theta_0).$$

Thus,

$$\begin{aligned}
\log \frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} &\leq \frac{\sqrt{n}\|\theta - \theta_0\|}{M_n} \log \frac{p_n\left(\mathbf{X}^{(n)}\left|\theta_0 + \frac{M_n/\sqrt{n}}{\|\theta - \theta_0\|}(\theta - \theta_0)\right.\right)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \\
&\leq \frac{\sqrt{n}\|\theta - \theta_0\|}{M_n} \left( -\frac{\lambda_{\min}(I_{\theta_0})}{4} M_n^2 \right) \\
&= -\frac{\lambda_{\min}(I_{\theta_0})}{4} \sqrt{n}\|\theta - \theta_0\| M_n.
\end{aligned}$$

For  $\epsilon > 0$  such that  $\sup_{\|\theta - \theta_0\| < \epsilon} \pi(\theta) \leq +\infty$ , we have

$$\begin{aligned}
&\int_{\{\theta: \|\theta - \theta_0\| > M_n/\sqrt{n}\}} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta \\
&\leq [p_n(\mathbf{X}^{(n)}|\theta_0)]^t \int_{\{\theta: \|\theta - \theta_0\| > M_n/\sqrt{n}\}} \exp\left[-\frac{t\lambda_{\min}(I_{\theta_0})}{4} \sqrt{n}\|\theta - \theta_0\| M_n\right] \pi(\theta) d\theta \\
&= [p_n(\mathbf{X}^{(n)}|\theta_0)]^t \left( \int_{\{\theta: M_n/\sqrt{n} \leq \|\theta - \theta_0\| \leq \epsilon\}} \exp\left[-\frac{t\lambda_{\min}(I_{\theta_0})}{4} \sqrt{n}\|\theta - \theta_0\| M_n\right] \pi(\theta) d\theta \right. \\
&\quad \left. + \int_{\{\theta: \|\theta - \theta_0\| > \epsilon\}} \exp\left[-\frac{t\lambda_{\min}(I_{\theta_0})}{4} \sqrt{n}\|\theta - \theta_0\| M_n\right] \pi(\theta) d\theta \right) \\
&\leq [p_n(\mathbf{X}^{(n)}|\theta_0)]^t \left( \left( \sup_{\|\theta - \theta_0\| < \epsilon} \pi(\theta) \right) \int_{\{\theta: \|\theta - \theta_0\| \geq M_n/\sqrt{n}\}} \exp\left[-\frac{t\lambda_{\min}(I_{\theta_0})}{4} \sqrt{n}\|\theta - \theta_0\| M_n\right] d\theta \right. \\
&\quad \left. + \exp\left[-\frac{t\lambda_{\min}(I_{\theta_0})}{4} \epsilon \sqrt{n} M_n\right] \right) \\
&= [p_n(\mathbf{X}^{(n)}|\theta_0)]^t \left( \left( \sup_{\|\theta - \theta_0\| < \epsilon} \pi(\theta) \right) n^{-p/2} \int_{\{h: \|h\| \geq M_n\}} \exp\left[-\frac{t\lambda_{\min}(I_{\theta_0})}{4} \|h\| M_n\right] dh \right. \\
&\quad \left. + \exp\left[-\frac{t\lambda_{\min}(I_{\theta_0})}{4} \epsilon \sqrt{n} M_n\right] \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\int_{\{\theta: \|\theta - \theta_0\| > M_n/\sqrt{n}\}} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta}{\int_{\Theta} [p_n(\mathbf{X}^{(n)}|\theta)]^t \pi(\theta) d\theta} \\
&= O_{P_{\theta_0}^n}(1) \left( \int_{\{h: \|h\| \geq M_n\}} \exp\left[-\frac{t\lambda_{\min}(I_{\theta_0})}{4} \|h\| M_n\right] dh + n^{p/2} \exp\left[-\frac{t\lambda_{\min}(I_{\theta_0})}{4} \epsilon \sqrt{n} M_n\right] \right) \\
&= O_{P_{\theta_0}^n}(1).
\end{aligned}$$

□

In general case, however, PBF is not good. In the general setting, it seems that FBF can be applied to wider problem. Consider the following example.

**Example 1.** Suppose  $X_1, \dots, X_n$  are i.i.d. from the density

$$p(x|\theta) = C|x - \theta|^{-1/2} \exp[-(x - \theta)^2],$$

where  $C$  is the normalizing constant. The parameter space  $\Theta$  is equal to  $\mathbb{R}$ . We would like to test the hypotheses  $H_0: \theta = 0$  vs  $H_1: \theta \neq 0$ . The likelihood is

$$p_n(\mathbf{X}^{(n)}|\theta) = C^n \left[ \prod_{i=1}^n |X_i - \theta| \right]^{-1/2} \exp\left[-\sum_{i=1}^n (X_i - \theta)^2\right].$$

Under the alternative hypothesis, the likelihood tends to infinity if  $\theta$  tends to  $X_i$ ,  $i = 1, \dots, n$ . Consequently, LRT fails in this model. To use FBF, we impose a prior  $\pi(\theta)$ . Suppose that  $\pi(\theta)$  is positive for all  $\theta$ . Then

$$L_t(\mathbf{X}^{(n)}) = \int_{-\infty}^{+\infty} \left[ \prod_{i=1}^n |X_i - \theta| \right]^{-t/2} \exp \left[ -t \sum_{i=1}^n (X_i - \theta)^2 \right] \pi(\theta) d\theta.$$

The likelihood will almost surely have no ties and consequently  $L_t(\mathbf{X}^{(n)}) = +\infty$  if and only if  $t \geq 2$ . While FBF is well defined, PBF is not defined.

This example motivates us that FBF is better than PBF. In general, the Assumption 4 can be removed for FBF. Now we consider the general case.

There are many works give Bernstein-von Mises type theorems, which assert that the posterior distribution of  $h = \sqrt{n}(\theta - \theta_0)$  converges to  $N(\Delta_{n,\theta_0}, I_{\theta_0}^{-1})$ , the normal distribution with mean  $\Delta_{n,\theta_0}$  and variance  $I_{\theta_0}^{-1}$ . However, most existing work consider the convergence under the total variation distance, that is

$$\int_{\mathbb{R}^p} |\pi(h|\mathbf{X}^{(n)}) - \phi(h|\Delta_{n,\theta^*}, \mathbf{V}_{\theta^*}^{-1})| dh \xrightarrow{P_{\theta}^n} 0.$$

Or Hellinger distance. Blabla.

**Proposition 4.** *If  $a < 1$ , then blabla*

For two parameters  $\theta_1$  and  $\theta_2$ , the Rényi divergence of order  $\alpha$  ( $0 < \alpha < 1$ ) of  $P_{\theta_1}$  from  $P_{\theta_2}$  is defined to be

$$D_\alpha(\theta_1 || \theta_2) = -\frac{1}{1-\alpha} \log \rho_\alpha(\theta_1, \theta_2),$$

where  $\rho_\alpha(\theta_1, \theta_2) = \int_{\mathcal{X}} p(X|\theta_1)^\alpha p(X|\theta_2)^{1-\alpha} d\mu$  is the so-called Hellinger integral.

**Assumption 2.** *There exist positive constants  $\delta$ ,  $\epsilon$  and  $C$  such that,  $D_\alpha(\theta, \theta_0) \geq C\|\theta - \theta_0\|^2$  for  $\|\theta - \theta_0\| \leq \delta$  and  $D_\alpha(\theta || \theta_0) \geq \epsilon$  for  $\|\theta - \theta_0\| > \delta$ .*

**Theorem 2.** *Suppose  $\theta_0$  is an interior of  $\Theta$ ,  $\pi(\theta)$  is continuous at  $\theta_0$  and  $\pi(\theta_0) > 0$ . Under Assumptions 1 and 2, for fixed  $a \in (0, 1)$  and any  $M_n \rightarrow \infty$ ,*

$$\frac{L_a(\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\})}{L_a} \xrightarrow{P_0^n} 0.$$

*Proof.* Without loss of generality, we assume  $M_n/\sqrt{n} \rightarrow 0$ . Note that

$$\frac{L_a(\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\})}{L_a(\Theta)} = \frac{\int_{\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\}} \left[ \frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \right]^a \pi(\theta) d\theta}{\int_{\Theta} \left[ \frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \right]^a \pi(\theta) d\theta}. \quad (6)$$

Consider the expectation of the numerator of 6. It follows from Fubini's theorem that

$$\begin{aligned} & P_0^n \int_{\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\}} \left[ \frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \right]^a \pi(\theta) d\theta \\ &= \int_{\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\}} \left\{ \int_{\mathcal{X}^n} [p_n(\mathbf{X}^{(n)}|\theta)]^a [p_n(\mathbf{X}^{(n)}|\theta_0)]^{1-a} d\mu^n \right\} \pi(\theta) d\theta \\ &= \int_{\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\}} [\rho_a(\theta, \theta_0)]^n \pi(\theta) d\theta \\ &= \int_{\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\}} \exp \left[ - (1-a)n D_a(\theta || \theta_0) \right] \pi(\theta) d\theta. \end{aligned}$$



Decompose the integral region into two parts  $\{\theta : \frac{M_n}{\sqrt{n}} \leq \|\theta - \theta_0\| \leq \delta\}$  and  $\{\theta : \|\theta - \theta_0\| > \delta\}$ ,

$$\begin{aligned} & \int_{\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\}} \exp[-(1-a)nD_a(\theta, \theta_0)] \pi(\theta) d\theta \\ &= \int_{\{\theta : \frac{M_n}{\sqrt{n}} \leq \|\theta - \theta_0\| \leq \delta\}} \exp[-(1-a)nD_a(\theta, \theta_0)] \pi(\theta) d\theta + \int_{\{\theta : \|\theta - \theta_0\| > \delta\}} \exp[-(1-a)nD_a(\theta, \theta_0)] \pi(\theta) d\theta \\ &\leq \max_{\|\theta - \theta_0\| \leq \delta} \pi(\theta) \int_{\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\}} \exp[-(1-a)Cn\|\theta - \theta_0\|^2] d\theta + \exp[-(1-a)\epsilon n]. \end{aligned}$$

Note that

$$\begin{aligned} & \int_{\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\}} \exp[-(1-a)Cn\|\theta - \theta_0\|^2] d\theta \\ &= (\sqrt{n}C_2)^{-p} \int_{\{t : \|t\| \geq C_2M_n\}} \exp\left[-\frac{t^2}{2}\right] dt = \left(\frac{\sqrt{2\pi}}{\sqrt{n}C_2}\right)^p P\{\chi_p^2 \geq C_2^2M_n^2\}, \end{aligned}$$

where  $\chi_p^2$  is a chi-squared random variable with  $p$  degrees of freedom. By Laurent and Massart (2000) Lemma 1, for  $M_n > \sqrt{2p}/C_2$ ,

$$P\{\chi_p^2 \geq C_2^2M_n^2\} \leq \exp\left[-\left(\frac{\sqrt{2}}{2}C_2M_n - \sqrt{p}\right)^2\right].$$

Thus, for large  $n$ ,

$$\begin{aligned} & P_0^n \int_{\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\}} \left[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\right]^{1/2} \pi(\theta) d\theta \\ &\leq \left[\max_{\|\theta - \theta_0\| \leq \delta} \pi(\theta)\right] \left(\frac{\sqrt{2\pi}}{\sqrt{n}C_2}\right)^p \exp\left[-\left(\frac{\sqrt{2}}{2}C_2M_n - \sqrt{p}\right)^2\right] + \exp\left[-\frac{n}{2}\epsilon^2\right]. \end{aligned} \tag{7}$$

Now we consider the denominator of (6).

$$\begin{aligned} & \int_{\Theta} \left[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\right]^t \pi(\theta) d\theta \geq \int_{\{\theta : \|\theta - \theta_0\| \leq n^{-1/2}\}} \left[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\right]^t \pi(\theta) d\theta \\ &\geq \left(\min_{\|\theta - \theta_0\| \leq n^{-1/2}} \left[\frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\right]^t \pi(\theta)\right) \int_{\{\theta : \|\theta - \theta_0\| \leq n^{-1/2}\}} 1 d\theta \\ &\geq \left(\exp\left[t \min_{\|\theta - \theta_0\| \leq n^{-1/2}} \log \frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}\right]\right) \left(\min_{\|\theta - \theta_0\| \leq n^{-1/2}} \pi(\theta)\right) n^{-p/2} \frac{2\pi^{p/2}}{\Gamma(p/2)}. \end{aligned}$$

By Proposition 1,

$$\min_{\|\theta - \theta_0\| \leq n^{-1/2}} \log \frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \geq -\|I_{\theta_0}\Delta_{n,\theta_0}\| - \frac{1}{2}\|I_{\theta_0}\| + o_{P_0^n}(1).$$

Since  $I_{\theta_0}\Delta_{n,\theta_0}$  is bounded in probability,

$$\min_{\|\theta - \theta_0\| \leq n^{-1/2}} \log \frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)}$$

is lower bounded in probability. Note that

$$\min_{\|\theta - \theta_0\| \leq n^{-1/2}} \pi(\theta) \rightarrow \pi(\theta_0) > 0.$$

Thus, for every  $\epsilon' > 0$ , there is a constant  $c > 0$  and an event  $B_{\epsilon'}$  with  $P_0^n B_{\epsilon'} > 1 - \epsilon'$  on which

$$\int_{\Theta} \left[ \frac{p_n(\mathbf{X}^{(n)}|\theta)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \right]^t \pi(\theta) d\theta \geq cn^{-p/2}.$$

Thus

$$\begin{aligned} P_0^n \left\{ \frac{L_{1/2}(\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\})}{L_{1/2}(\Theta)} \right\} &\leq P_0^n \left\{ \frac{L_{1/2}(\{\theta : \|\theta - \theta_0\| \geq \frac{M_n}{\sqrt{n}}\})}{L_{1/2}(\Theta)} \right\} + P_0^n B_{\epsilon'}^C \\ &\leq c^{-1} \left[ \max_{\|\theta - \theta_0\| \leq \delta} \pi(\theta) \right] \left( \frac{\sqrt{2\pi}}{C_2} \right)^p \exp \left[ - \left( \frac{\sqrt{2}}{2} C_2 M_n - \sqrt{p} \right)^2 \right] + c^{-1} n^{p/2} \exp \left[ - \frac{n}{2} \epsilon'^2 \right] + \epsilon' \rightarrow \epsilon'. \end{aligned}$$

Since  $\epsilon$  is arbitrary, the theorem follows.  $\square$

**Proposition 5.** *If  $a = 1$ , then blabla*

#### 4. Integrated likelihood ratio test

##### 4.1. The choice of the weight function

$L^1$  approximation of posterior by normal.

#### 5. Main results

We study the asymptotic behavior of the ILRT statistic around  $\theta_0$ . If there exists certain test, Bernstein von Mise theorem will be valid.

**Assumption 3.** *For every  $\epsilon > 0$ , there exists a sequence of tests  $\phi_n$  such that*

$$P_{\theta_0}^n \phi_n \rightarrow 0, \quad \sup_{\|\theta - \theta_0\| \geq \epsilon} P_{\theta}^n (1 - \phi_n) \rightarrow 0. \quad (8)$$

**Theorem 3.** *Under Assumptions 1 and 3, there exists for every  $M_n \rightarrow \infty$  a sequence of tests  $\phi_n$  and a constant  $\delta > 0$  such that, for every sufficiently large  $n$  and every  $\|\theta - \theta_0\| \geq M_n/\sqrt{n}$ ,*

$$P_0^n \phi_n \rightarrow 0, \quad P_{\theta}^n (1 - \phi_n) \leq \exp[-\delta n(\|\theta - \theta_0\|^2 \wedge 1)].$$

(See der Vaart (2000) Lemma 10.3., Kleijn and Vaart (2012))

Under Assumption 1 and 3, we have

$$\|\pi_n(h|\mathbf{X}^{(n)}) - \phi(h; \Delta_{n,\theta_0}, I_{\theta_0}^{-1})\| \xrightarrow{P_{\theta_0}^n} 0.$$

See Kleijn and Vaart (2012), Theorem 2.1. However, we may use more general weight function.

**Assumption 4.** *Let  $\pi_n(h; \mathbf{X}^{(n)})$  be a weight function satisfying*

$$\|\pi_n(h; \mathbf{X}^{(n)}) - \phi(h; \Delta_{n,\theta_0}, I_{\theta_0}^{-1})\| \xrightarrow{P_{\theta_0}^n} 0 \quad (9)$$

Furthermore, assume that for every  $\epsilon > 0$ , there's a Lebesgue integrable function  $T(h)$ , a  $K > 0$  and an  $A > 0$  such that

$$\lim_{n \rightarrow \infty} P_{\theta_0}^n \left( \sup_{\|h\| \geq K\sqrt{n}} (\pi_n(h; \mathbf{X}^{(n)}) - T(h)) \leq 0 \right) \geq 1 - \epsilon \quad (10)$$

$$\lim_{n \rightarrow \infty} P_{\theta_0}^n \left( \sup_{\|h\| \leq K\sqrt{n}} \pi_n(h; \mathbf{X}^{(n)}) \leq A \right) \geq 1 - \epsilon \quad (11)$$

The condition 10 assume there is a function controlling the tail of weight function. For a statistical model, the likelihood value makes no sense when  $\theta$  is far away from  $\theta_0$ , or  $\sqrt{n}h$  is large. To avoid the bad behavior of the likelihood function when  $\sqrt{n}h$  is large, many theoretical works impose assumptions to likelihood. Thanks to the flexibility of weight function, we can impose 10 to weight function instead. The condition 11 is satisfied in most usual case. Condition 10 and 11 will be satisfied, for example, when

$$\pi_n(h; X) = \min(\pi_n(h|X), M)1_{\|h\| \leq K\sqrt{n}}, \quad (12)$$

where  $M$  and  $K$  are user-specified constant and  $\pi_n(h|\mathbf{X}^{(n)})$  is the posterior density.

Our first theorem is

**Theorem 4.** *Under Assumptions 1-4, for bounded real numbers  $\eta_n$ , we have*

$$\left| \int_{\mathbb{R}^p} \frac{p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h; \mathbf{X}^{(n)}) dh - 2^{-\frac{p}{2}} \exp\left[\frac{1}{2} \Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0}\right] \right| \xrightarrow{P_{\eta_n}^n} 0. \quad (13)$$

**Proof of Theorem 4.** By contiguity, we only need to prove the convergence in  $P_0^n$ .

The proof consists of two steps. In the first part of the proof, let  $M$  be a fixed positive number. We prove

$$\left| \int_{\|h\| \leq M} \frac{p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h; \mathbf{X}^{(n)}) dh - \int_{\|h\| \leq M} \exp\left[h^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{1}{2} h^T I_{\theta_0} h\right] \phi(h; \Delta_{n,\theta_0}, I_{\theta_0}^{-1}) dh \right| \xrightarrow{P_0^n} 0 \quad (14)$$

By Theorem 1,

$$\sup_{\|h\| \leq M} \left| \log \frac{p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} - h^T I_{\theta_0} \Delta_{n,\theta_0} + \frac{1}{2} h^T I_{\theta_0} h \right| \xrightarrow{P_0^n} 0.$$

Hence we have

$$\int_{\|h\| \leq M} \frac{p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p_n(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h; \mathbf{X}^{(n)}) dh = \exp[o_{P_0^n}(1)] \int_{\|h\| \leq M} \exp\left[h^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{1}{2} h^T I_{\theta_0} h\right] \pi_n(h; \mathbf{X}^{(n)}) dh \quad (15)$$

So we only need to consider  $\int_{\|h\| \leq M} \exp\left[h^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{1}{2} h^T I_{\theta_0} h\right] \pi_n(h; \mathbf{X}^{(n)}) dh$ . By central limit theorem,  $\Delta_{n,\theta_0}$  weakly converges to a normal distribution. As a result,  $\sup_{\|h\| \leq M} \exp\left[h^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{1}{2} h^T I_{\theta_0} h\right]$  is bounded in probability. It follows that

$$\begin{aligned} & \int_{\|h\| \leq M} \exp\left[h^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{1}{2} h^T I_{\theta_0} h\right] |\pi_n(h; \mathbf{X}^{(n)}) - \phi(h; \Delta_{n,\theta_0}, I_{\theta_0}^{-1})| dh \\ & \leq \sup_{\|h\| \leq M} \exp\left[h^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{1}{2} h^T I_{\theta_0} h\right] \int_{\|h\| \leq M} |\pi_n(h; \mathbf{X}^{(n)}) - \phi(h; \Delta_{n,\theta_0}, I_{\theta_0}^{-1})| dh \xrightarrow{P_0^n} 0. \end{aligned}$$

Combining with (15), we can conclude that (14) holds. This is true for every  $M$  and hence also for some  $M_n \rightarrow \infty$ .

In the second part, we prove

$$\psi(M) \stackrel{def}{=} \frac{\int_{\|h\| > M} p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h) \pi_n(h; \mathbf{X}^{(n)}) dh}{\int p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h) \pi_n(h; \mathbf{X}^{(n)}) dh} \xrightarrow{P_0^n} 0. \quad (16)$$

Let  $\phi_n$  be a test function satisfying the conclusion of Theorem 3. We have

$$\psi(M) = \psi(M)\phi_n + \psi(M)(1 - \phi_n).$$

Since  $\psi(M) \leq 1$ ,  $\psi(M)\phi_n \leq \phi_n \xrightarrow{P_0^n} 0$ . So it's enough to prove

$$\psi(M)(1 - \phi_n) \xrightarrow{P_0^n} 0$$

Fix a positive number  $U$ . Then

$$\psi(M)(1 - \phi_n) \leq \frac{\int_{\|h\| > M_n} p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)\pi_n(h; \mathbf{X}^{(n)}) dh}{\int_{\|h\| \leq U} p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)\pi_n(h; \mathbf{X}^{(n)}) dh} (1 - \phi_n). \quad (17)$$

First we give a lower bound of  $\int_{\|h\| \leq U} p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)\pi_n(h; \mathbf{X}^{(n)}) dh$ . Note that

$$\begin{aligned} & \int_{\|h\| \leq U} p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)\pi_n(h; \mathbf{X}^{(n)}) dh \\ &= \exp[o_{P_0^n}(1)]p_n(\mathbf{X}^{(n)}|\theta_0) \int_{\|h\| \leq U} \exp[h^T I_{\theta_0} \Delta_{n, \theta_0} - \frac{1}{2}h^T I_{\theta_0} h] \pi_n(h; \mathbf{X}^{(n)}) dh \\ &\geq \exp[o_{P_0^n}(1)]p_n(\mathbf{X}^{(n)}|\theta_0) \left\{ \int_{\|h\| \leq U} \exp[h^T I_{\theta_0} \Delta_{n, \theta_0} - \frac{1}{2}h^T I_{\theta_0} h] \phi(h; \Delta_{n, \theta_0}, I_{\theta_0}^{-1}) dh \right. \\ &\quad \left. - \sup_{\|h\| \leq U} \exp[h^T I_{\theta_0} \Delta_{n, \theta_0} - \frac{1}{2}h^T I_{\theta_0} h] \int_{\|h\| \leq U} |\pi_n(h; \mathbf{X}^{(n)}) - \phi(h; \Delta_{n, \theta_0}, I_{\theta_0}^{-1})| dh \right\} \\ &= \exp[o_{P_0^n}(1)]p_n(\mathbf{X}^{(n)}|\theta_0) \left\{ \int_{\|h\| \leq U} \exp[h^T I_{\theta_0} \Delta_{n, \theta_0} - \frac{1}{2}h^T I_{\theta_0} h] \phi(h; \Delta_{n, \theta_0}, I_{\theta_0}^{-1}) dh - O_P(1)o_P(1) \right\}. \end{aligned}$$

Fix an  $\epsilon > 0$ . Since  $\Delta_{n, \theta_0}$  is uniformly tight, with probability at least  $1 - \epsilon/2$ ,  $|\Delta_{n, \theta_0}| \leq C$  for a constant  $C$ . If this event happens, we have

$$\int_{\|h\| \leq U} \exp[h^T I_{\theta_0} \Delta_{n, \theta_0} - \frac{1}{2}h^T I_{\theta_0} h] \phi(h; \Delta_{n, \theta_0}, I_{\theta_0}^{-1}) dh > 2c$$

for some  $c > 0$ . Thus, there is a  $c > 0$  and an event  $D_{1,n}$  with probability at least  $1 - \epsilon$  on which

$$\int_{\|h\| \leq U} p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)\pi_n(h; \mathbf{X}^{(n)}) dh \geq cp_n(\mathbf{X}^{(n)}|\theta_0)$$

for sufficiently large  $n$ .

On the other hand, by Assumption 4, there is a  $K > 0$ , a  $A > 0$  and an event  $D_{2,n}$  with probability at least  $1 - \epsilon$  on which

$$\sup_{\|h\| > K\sqrt{n}} (\pi_n(h; \mathbf{X}^{(n)}) - T(h)) \leq 0, \quad \sup_{\|h\| \leq K\sqrt{n}} \pi_n(h; \mathbf{X}^{(n)}) \leq A$$

for sufficiently large  $n$ .

Combining these bounds yields

$$\psi(M)(1 - \phi_n) \leq \frac{\int_{\|h\| > M_n} p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)(A\mathbf{1}_{M_n \leq \|h\| \leq K\sqrt{n}} + T(h)\mathbf{1}_{\|h\| > K\sqrt{n}}) dh}{cp_n(\mathbf{X}^{(n)}|\theta_0)} (1 - \phi_n) + \mathbf{1}_{\{D_{1,n}^C \cup D_{2,n}^C\}}.$$

Hence for sufficiently large  $n$ ,

$$\begin{aligned} & P_0^n \psi(M)(1 - \phi_n) \\ &\leq c^{-1} \int_{\mathcal{X}^n} \int_{\|h\| > M_n} (1 - \phi_n) p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h) (A\mathbf{1}_{M_n \leq \|h\| \leq K\sqrt{n}} + T(h)\mathbf{1}_{\|h\| > K\sqrt{n}}) dh d\mu^n + 2\epsilon \\ &= c^{-1} \int_{\|h\| > M_n} \left( \int_{\mathcal{X}^n} (1 - \phi_n) p_n(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h) d\mu^n \right) (A\mathbf{1}_{M_n \leq \|h\| \leq K\sqrt{n}} + T(h)\mathbf{1}_{\|h\| > K\sqrt{n}}) dh + 2\epsilon \\ &\leq c^{-1} \int_{\|h\| > M_n} \exp[-\delta(\|h\|^2 \wedge n)] (A\mathbf{1}_{M_n \leq \|h\| \leq K\sqrt{n}} + T(h)\mathbf{1}_{\|h\| > K\sqrt{n}}) dh + 2\epsilon. \end{aligned}$$

Note that  $\delta(\|h\|^2 \cap n) \geq \delta^*(\|h\|^2 \wedge K^2 n)$ , where  $\delta^* = \delta \min(1, K^{-2})$ . Hence

$$\begin{aligned} & \int_{\|h\| > M_n} \exp[-\delta(\|h\|^2 \wedge n)] (A \mathbf{1}_{M_n \leq \|h\| \leq K\sqrt{n}} + T(h) \mathbf{1}_{\|h\| > K\sqrt{n}}) dh \\ & \leq \int_{\|h\| > M_n} \exp[-\delta^*(\|h\|^2 \wedge K^2 n)] (A \mathbf{1}_{M_n \leq \|h\| \leq K\sqrt{n}} + T(h) \mathbf{1}_{\|h\| > K\sqrt{n}}) dh \\ & \leq A \int_{\|h\| \geq M_n} e^{-\delta^* \|h\|^2} dh + e^{-\delta^* K^2 n} \int_{\|h\| > K\sqrt{n}} T(h) dh \rightarrow 0. \end{aligned}$$

Therefore  $\psi(M) \xrightarrow{P_0^n} 0$ .

Finally we have

$$\begin{aligned} & \left| \int \frac{p(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h; \mathbf{X}^{(n)}) dh - 2^{-\frac{p}{2}} \exp\left[\frac{1}{2} \Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0}\right] \right| \\ & = \left| \int \frac{p(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h; \mathbf{X}^{(n)}) dh - \int_{\|h\| \leq M_n} \frac{p(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h; \mathbf{X}^{(n)}) dh \right| \\ & + \left| \int_{\|h\| \leq M_n} \frac{p(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h; \mathbf{X}^{(n)}) dh - \int_{\|h\| \leq M_n} \exp\left[h^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{1}{2} h^T I_{\theta_0} h\right] \phi(h; \Delta_{n,\theta_0}, I_{\theta_0}^{-1}) dh \right| \\ & + \left| \int_{\|h\| \leq M_n} \exp\left[h^T I_{\theta_0} \Delta_{n,\theta_0} - \frac{1}{2} h^T I_{\theta_0} h\right] \phi(h; \Delta_{n,\theta_0}, I_{\theta_0}^{-1}) dh - 2^{-\frac{p}{2}} \exp\left[\frac{1}{2} \Delta_{n,\theta_0}^T I_{\theta_0} \Delta_{n,\theta_0}\right] \right| \\ & = J_1 + J_2 + J_3 \end{aligned}$$

By the first step of the proof, we have  $J_2 \xrightarrow{P_0^n} 0$ . Hence

$$\int_{\|h\| \leq M_n} \frac{p(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h; \mathbf{X}^{(n)}) dh$$

is bounded in probability. Therefore

$$\begin{aligned} J_1 & = \int_{\|h\| \leq M_n} \frac{p(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h)}{p(\mathbf{X}^{(n)}|\theta_0)} \pi_n(h; \mathbf{X}^{(n)}) dh \left| \frac{\int_{\|h\| \leq M_n} p(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h) \pi_n(h; \mathbf{X}^{(n)}) dh}{\int_{\|h\| \leq M_n} p(\mathbf{X}^{(n)}|\theta_0 + n^{-1/2}h) \pi_n(h; \mathbf{X}^{(n)}) dh} - 1 \right| \\ & = O_{P_0^n}(1) o_{P_0^n}(1) \end{aligned}$$

And  $J_3$  converges to 0 for trivial reason.  $\square$

Based on Theorem 4, the asymptotic distribution of integrated likelihood ratio statistics under null hypothesis can be obtained. It can be used to determine the critical value of the test

**Theorem 5.** Suppose the assumptions of 4 are met for both  $\Theta_0$  and  $\Theta$ , the true parameter  $\theta_0$  is an interior point of  $\Theta$  and a relative interior point of  $\Theta_0$ , then we have

$$2 \log(\Lambda(X)) \xrightarrow{P_0^n} \chi_{p_2-p_1}^2 - (p_2 - p_1) \log(2) \quad (18)$$

**Proof of Theorem 5.** If the null hypothesis is true, the true parameter  $\theta_0$  is an interior point of  $\Theta$  and  $\theta_0$  is a relative interior point of  $\Theta_0$ . Then we can apply Theorem 4 to both the numerator and denominator of integrated likelihood ratio statistics with  $\eta_n = 0$ . By CLT,

$$I_{\theta_0} \Delta_{n,\theta_0} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) \xrightarrow{P_0^n} \xi, \quad (19)$$

where  $\xi \sim N(0, I_{\theta_0})$ .

$$I_{\theta_0}^* \Delta_{n, \theta_0}^* = \frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}^*(X_i) \overset{P_0^n}{\rightsquigarrow} \xi^*, \quad (20)$$

where  $\xi^*$  is the first  $p_1$  coordinates of  $\xi$ . Hence

$$\begin{aligned} \Lambda(X) &= \frac{2^{-\frac{p_2}{2}} \exp\{\frac{1}{2} \Delta_{n, \theta_0}^T I_{\theta_0} \Delta_{n, \theta_0}\} + o_{P_0^n}(1)}{2^{-\frac{p_1}{2}} \exp\{\frac{1}{2} \Delta_{n, \theta_0}^{*T} I_{\theta_0}^* \Delta_{n, \theta_0}^*\} + o_{P_0^n}(1)} \\ &\overset{P_0^n}{\rightsquigarrow} \frac{2^{-\frac{p_2}{2}} \exp\{\frac{1}{2} \xi^T I_{\theta_0}^{-1} \xi\}}{2^{-\frac{p_1}{2}} \exp\{\frac{1}{2} \xi^{*T} I_{\theta_0}^{*-1} \xi^*\}}. \end{aligned} \quad (21)$$

But

$$\xi^T I_{\theta_0}^{-1} \xi - \xi^{*T} I_{\theta_0}^{*-1} \xi^* = (I_{\theta_0}^{-\frac{1}{2}} \xi)^T \left( I_{p_2 \times p_2} - I_{\theta_0}^{\frac{1}{2}} \begin{pmatrix} I_{\theta_0}^{*-1} & 0 \\ 0 & 0 \end{pmatrix} I_{\theta_0}^{\frac{1}{2}} \right) (I_{\theta_0}^{-\frac{1}{2}} \xi). \quad (22)$$

$I_{\theta_0}^{-\frac{1}{2}} \xi$  is a  $p_2$ -dimensional standard normal distribution, The middle term is a projection matrix with rank  $p_2 - p_1$ . Hence we have

$$2 \log(\Lambda(X)) \overset{P_0^n}{\rightsquigarrow} \chi_{p_2 - p_1}^2 - (p_2 - p_1) \log(2). \quad (23)$$

□

We can obtain the asymptotic distribution of the integrated likelihood ratio test under local alternatives by Le Cam's third lemma.

**Theorem 6.** *Suppose the Assumptions of 5 are met. The true parameter  $\theta$  satisfies  $\eta_n = \sqrt{n}(\theta - \theta_0) \rightarrow \eta$ . If*

$$I_{\theta_0} = \begin{pmatrix} I_{\theta_0}^* & I_{12} \\ I_{21} & I_{22} \end{pmatrix}, \quad (24)$$

$I_{22 \cdot 1} = I_{22} - I_{21} I_{\theta_0}^{*-1} I_{12}$ , then we have

$$2 \log(\Lambda(X)) \overset{P_0^n}{\rightsquigarrow} \chi_{p_2 - p_1}^2(\delta) - (p_2 - p_1) \log(2) \quad (25)$$

where

$$\delta = \eta^T \begin{pmatrix} 0 & 0 \\ 0 & I_{22 \cdot 1} \end{pmatrix} \eta \quad (26)$$

The results can be explained by the limit experiment point of view. As  $h_n \rightarrow h$ , the 'locally sufficient' statistic  $\Delta_{n, \theta_0} \rightsquigarrow N(h, I_{\theta_0}^{-1})$ . In the limit experiment, we have one observation  $X \sim N(h, I_{\theta_0}^{-1})$ . In this case, the integrated likelihood ratio test statistics can be calculated easily whose distribution is exactly the same as 6.

**Proof of Theorem 6.** We note that  $h_n = \eta_n$  converges to  $\eta$ . By Proposition 1 and CLT,

$$\begin{aligned} \left( \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i)}{\log \frac{p_{\eta_n}(X)}{p_0(X)}} \right) &= \left( \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i)}{\frac{1}{\sqrt{n}} \sum_{i=1}^n \eta^T \dot{\ell}_{\theta_0}(X_i) - \frac{1}{2} \eta^T I_{\theta_0} \eta} \right) + o_{P_0^n}(1) \\ &\overset{P_0^n}{\rightsquigarrow} N\left( \begin{pmatrix} 0 \\ -\frac{1}{2} \eta^T I_{\theta_0} \eta \end{pmatrix}, \begin{pmatrix} I_{\theta_0} & I_{\theta_0} \eta \\ \eta^T I_{\theta_0} & \eta^T I_{\theta_0} \eta \end{pmatrix} \right). \end{aligned} \quad (27)$$

Hence by Le Cam's third lemma,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) \overset{P_{\eta_n}^n}{\rightsquigarrow} \xi \sim N(I_{\theta_0} \eta, I_{\theta_0}). \quad (28)$$

By Theorem 4, under  $P_{\eta_n}^n$ , we have (21). Hence

$$2 \log(\Lambda(X)) \overset{P_{\eta_n}^n}{\rightsquigarrow} \chi_{p_2-p_1}^2(\delta) - (p_2 - p_1) \log(2), \quad (29)$$

where noncentral parameter  $\delta$  can be obtained by substituting  $\xi$  by  $I_{\theta_0}\eta$  in (22):

$$\begin{aligned} \delta &= \eta^T (I_{\theta_0} - I_{\theta_0} \begin{pmatrix} I_{\theta_0}^{*-1} & 0 \\ 0 & 0 \end{pmatrix} I_{\theta_0}) \eta \\ &= \eta^T \begin{pmatrix} 0 & 0 \\ 0 & I_{22 \cdot 1} \end{pmatrix} \eta. \end{aligned} \quad (30)$$

□

### 5.1. Normal mixture

Although posterior Bayes factor can be used for some prior Aitkin et al. (1996). Posterior Bayes estimator can not be defined for certain prior distribution.

Fractional posterior Bayes factor (O'Hagan, 1995) can be defined.

## 6. Appendix

For two measure sequence  $P_n$  and  $Q_n$  on measurable spaces  $(\Omega_n, \mathcal{A}_n)$ , denote by  $P_n \triangleleft Q_n$  that  $P_n$  and  $Q_n$  are mutually contiguous. That is, for any statistics  $T_n: \Omega_n \mapsto \mathbb{R}^k$ , we have  $T_n \overset{P_n}{\rightsquigarrow} 0 \Leftrightarrow T_n \overset{Q_n}{\rightsquigarrow} 0$ .

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