A feasible high dimensional randomization test for the mean vector

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Abstract

The strength of randomization tests is that they are exact tests under certain symmetry assumption for distributions. In this paper, we propose a randomization test for the mean vector in high dimensional setting. We give an implementation of the proposed randomization test procedure, which has low computational complexity. So far, the asymptotic behaviors of randomization tests have only been studied in fixed dimension case. We investigate the asymptotic behavior of the proposed randomization test in high dimensional setting. It turns out that even if the symmetry assumption is violated, the proposed randomization test still has correct level asymptotically. The asymptotic power function is also given. A simulation study is carried out to verify our theoretical results.

Keywords: Asymptotic power function, High dimension, Randomization test, Symmetry assumption

1. Introduction

Suppose $X_1, \ldots, X_n$ are p-variate independent and identically distributed (iid) random vectors with mean vector μ and covariance matrix Σ. In this paper, we consider the problem of testing the hypotheses

$$H_0 : \mu = 0_p \quad \text{versus} \quad H_1 : \mu \neq 0_p. \tag{1}$$

A classical test statistic for hypotheses (1) is Hotelling's T^2, defined as $n\bar{X}^T S^{-1} \bar{X}$, where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ and $S = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})(X_i - \bar{X})^T$ are the sample mean vector and sample covariance matrix, respectively. Under normal distribution, Hotelling's T^2 is the likelihood ratio test and enjoys desirable properties in fixed p setting. See, for example, Anderson (2003). However, Hotelling's test can not be defined when $p > n - 1$ due to the singularity of S. In a seminal paper, Bai and Saranadasa (1996) considered two sample testing problem and proposed a statistic by removing S^{-1} from Hotelling's T^2 statistic. They studied the asymptotic properties of their test statistic when p/n tends to a positive constant. Many subsequent papers generalized the idea of Bai and Saranadasa (1996) to more general models (Srivastava and Du, 2008; Chen and Qin, 2010; Wang et al., 2015). The critical values of existing high dimensional tests are mostly determined by asymptotic distribution. We call it asymptotic method. In many real world problems, e.g., gene testing (Efron and Tibshirani, 2007), sample size n may be very small. In this case, the Type I error rate of the asymptotic method may be far away from nominal level.

The randomization test method is a tool to determine the critical value for a given test statistic. The idea of randomization tests dates back to Fisher (1935). See Romano (1990) for a general construction of randomization test. Its strength is in that the resulting test procedure has exact level under mild condition. There are many papers concerning the theoretical properties of randomization tests for fixed p case. See, for example, Romano (1990), Zhu (2000) and Chung and Romano (2016). In high dimensional setting, randomization tests are widely used in applied statistics (Subramanian et al., 2005; Efron and Tibshirani,

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2007; Ko et al., 2016). However, little is known about the theoretical properties of the randomization test in high dimensional setting.

In this paper, we consider the following randomization method. Suppose $T(X_1, \ldots, X_n)$ is certain test statistic for hypotheses (1). Let $\epsilon_1, \ldots, \epsilon_n$ be iid Rademacher variables ($\Pr(\epsilon_i = 1) = \Pr(\epsilon_i = -1) = 1/2$) which are independent of data. The randomization test rejects the null hypothesis when $T(X_1, \ldots, X_n)$ is greater than the $1 - \alpha$ quantile of the conditional distribution of $T(\epsilon_1 X_1, \ldots, \epsilon_i X_i, \ldots, \epsilon_n X_n)$ with respect to $X_1, \ldots, X_n$, and accepts the null hypothesis otherwise, where α is the significant level and the $1 - \alpha$ quantile of a distribution function $F(\cdot)$ is defined as $\inf\{y : F(y) \geq 1 - \alpha\}$. In fixed p setting, it's well known that randomization tests consume much more computing time than the asymptotic method, which historically hampered the use of randomization tests. The goal of this paper is to show that in high dimensional setting, randomization tests can be computationally feasible and have desirable statistic properties. Inspired by the work of Bai and Saranadasa (1996) and Chen and Qin (2010), we propose a randomization test for hypotheses (1). We give a fast implementation of the proposed randomization test, the computational complexity of which is low. When p is large, our method even consumes less computing time than the asymptotic method. We also investigate the asymptotic behavior of the test procedure. Our results show that even if the null distribution of X_1 is not symmetric, the randomization test is still asymptotically exact under mild assumptions. Hence the test procedure is robust. The local asymptotic power function is also given. To the best of our knowledge, this is the first work which gives the asymptotic behavior of randomization tests in high dimensional setting. A simulation study is carried out to examine the numerical performance of the proposed randomization test and compare with the asymptotic method. The simulation results show that the size of the proposed randomization test is closer to the nominal level than the asymptotic method while the power behaviors of the proposed randomization test and the asymptotic method are similar.

The rest of the paper is organized in the following way. In Section 2, we propose a randomization test and give a fast implementation. In Section 3, we investigate the asymptotic behavior of the proposed test. The simulation results are reported in Section 4. The technical proofs are presented in Appendix.

2. Test Procedure

Consider testing the hypotheses (1) in high dimensional setting. It is known that Hotelling's T^2 can not be defined when $p > n - 1$. Bai and Saranadasa (1996) removed S^{-1} from Hotelling's T^2 statistic and proposed a statistic which has good power behavior in high dimensional setting. Their idea can also be used for testing hypotheses (1) and the statistic becomes $\bar{X}^T \bar{X}$. The asymptotic properties of the statistic requires p/n tends to a positive constant. Chen and Qin (2010) found that the restriction on p and n can be considerably relaxed by removing the diagonal elements in the statistic of Bai and Saranadasa (1996). For hypotheses (1), their statistic is $\sum_{i \neq j} X_i^T X_j$. Inspired by the statistic of Bai and Saranadasa (1996) and Chen and Qin (2010), we consider the statistic

$$T(X_1, \ldots, X_n) = \sum_{j < i} X_i^T X_j. \tag{2}$$

Let $\epsilon_1, \ldots, \epsilon_n$ be iid Rademacher variables which are independent of data. Denote by

$$\mathcal{L}\big(T(\epsilon_1 X_1, \ldots, \epsilon_i X_i, \ldots, \epsilon_n X_n)|X_1, \ldots, X_n\big) \tag{3}$$

the distribution of $T(\epsilon_1 X_1, \ldots, \epsilon_i X_i, \ldots, \epsilon_n X_n)$ conditioning on $X_1, \ldots, X_n$. We propose a test procedure with test function $\phi(X_1, \ldots, X_n)$ to be equal to 1 if $T(X_1, \ldots, X_n)$ is greater than the $1 - \alpha$ quantile of the conditional distribution (3) and equal to 0 otherwise. Since $T(X_1, \ldots, X_n)$ equals to half of Chen and Qin (2010)'s statistic $\sum_{i \neq j} X_i^T X_j$, the test procedure $\phi(X_1, \ldots, X_n)$ is the randomization version of Chen and Qin (2010)'s test procedure. On the other hand, note that Bai and Saranadasa (1996)'s statistic $\bar{X}^T \bar{X}$ can be written as $n^{-2} \sum_{i=1}^n \sum_{j=1}^n X_i^T X_j$. Since $\sum_{i=1}^n X_i^T X_i$ is invariance under randomization, the test procedure $\phi(X_1, \ldots, X_n)$ is also the randomization version of Bai and Saranadasa (1996)'s test. Here we can see that the randomization test automatically removes the unwanted diagonal elements.

Under certain symmetric assumption, randomization tests are exact tests, which is a desirable property. See, for example, Lehmann and Romano (2005, Chapter 15). In our problem, the Type I error of $\phi(X_1, \ldots, X_n)$ is not larger than α provided that X_1 and $-X_1$ have the same distribution under null hypothesis. By refined definition of $\phi(X_1, \ldots, X_n)$ on the boundary of rejection region, one can obtain a test procedure with exact significant level. Such refinement only has minor effect on the test procedure and won't be considered in this paper.

The test procedure $\phi(X_1, \ldots, X_n)$ can be equivalently implemented by p-value. Define

$$p(X_1, \ldots, X_n) = \Pr(T(\epsilon_1 X_1, \ldots, \epsilon_n X_n) \geq T(X_1, \ldots, X_n)|X_1, \ldots, X_n). \tag{4}$$

Then our test procedure rejects the null hypothesis when $p(X_1, \ldots, X_n) \leq \alpha$.

The randomized statistic $T(\epsilon_1 X_1, \ldots, \epsilon_n X_n)$ is uniformly distributed on 2^n values conditioning on $X_1, \ldots, X_n$. To compute the exact quantile of (3) or the p-value (4), one needs to calculate at least 2^n values, which is not feasible even when n is moderate. In practice, randomization test is often realized through an approximation of p-value (4). More specifically, we sample $\epsilon_1^*, \ldots, \epsilon_n^*$ and compute the randomized statistic $T^* = T(\epsilon_1^* X_1, \ldots, \epsilon_n^* X_n)$. Repeat B times for a large B and we obtain $T_1^*, \ldots, T_B^*$. Let $T_0 = T(X_1, \ldots, X_n)$ be the original statistic and define

$$\tilde{p}(X_1, \ldots, X_n) = \frac{1}{B+1}\Big(1 + \sum_{i=1}^{B} \mathbf{1}_{\{T_i^* \geq T_0\}}\Big).$$

The null hypothesis is rejected when $\tilde{p}(X_1, \ldots, X_n) \leq \alpha$. Although $\tilde{p}(X_1, \ldots, X_n)$ is an approximation of $p(X_1, \ldots, X_n)$, the resulting test procedure can still control the significant level. In fact, we have $\Pr(\tilde{p}(X_1, \ldots, X_n) \leq u) \leq u$ for all $0 \leq u \leq 1$ and all positive integer B. See Lehmann and Romano (2005, Page 636). By Bernoulli's law of large numbers, $\tilde{p}(X_1, \ldots, X_n)$ tends to $p(X_1, \ldots, X_n)$ in probability as $B \to \infty$. Here we emphasis that the convergence rate of $\tilde{p}(X_1, \ldots, X_n)$ to $p(X_1, \ldots, X_n)$ only relies on $p(X_1, \ldots, X_n)$. Hence the choice of B can be independent of the sample size n and the dimension of data p.

Now we consider the implementation of the randomization test procedure. The computation of T_0 costs $O(n^2 p)$ operations. To obtain T_i^*, $i = 1, \ldots, B$, we need to generate $\epsilon_1, \ldots, \epsilon_n$ and compute

$$T(\epsilon_1 X_1, \ldots, \epsilon_n X_n) = \sum_{1 \leq j < i \leq n} X_i^T X_j \epsilon_i \epsilon_j.$$

Note that $X_i^T X_j$ $(1 \leq j < i \leq n)$ can be computed beforehand. Once we obtain $X_i^T X_j$, the computation of T_i^* costs $O(n^2)$ operations. Thus, the randomization test costs $O(n^2 p + n^2 B)$ operations in total. As mentioned earlier, the choice of B can be independent of n and p. Hence when p is large, the computational complexity of the entire procedure is $O(n^2 p)$. The randomization test doesn't need a variance estimator which is a necessary for the asymptotic method. Bai and Saranadasa (1996) used the following estimator for $\text{tr}(\Sigma^2)$:

$$\frac{(n-1)^2}{(n+1)(n-2)}\Big(\text{tr}(S_n^2) - \frac{1}{n-1}(\text{tr}\, S_n)^2\Big).$$

Even with a good implementation, the computation of this estimator still costs $O(np \min(n, p))$ operations. Chen and Qin (2010) used a more complicated estimator of $\text{tr}(\Sigma^2)$. Consequently, if p is large compared with n, the randomization test is very competitive compared with the asymptotic method in terms of computational complexity. This is different from the low dimensional setting where randomization tests usually consume much more computing time than the asymptotic methods.

If we only care about the decision (reject or accept) and the p-value is not needed, the computing time of the randomization test can be further reduced. In fact, the rejection region $\tilde{p}(X_1, \ldots, X_n) \leq \alpha$ can be written as

$$\sum_{i=1}^{B} (1 - \mathbf{1}_{\{T_i^* \geq T_0\}}) \geq B + 1 - (B+1)\alpha.$$

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Since the left hand side is a sum of non-negative values, we can reject the null hypothesis once $\sum_{i=1}^{B_0}(1 - \mathbf{1}_{\{T_i^* \geq T_0\}}) \geq B + 1 - (B+1)\alpha$ for some B_0. Similarly, the acceptance region can be written as

$$\sum_{i=1}^{B} \mathbf{1}_{\{T_i^* \geq T_0\}} > (B+1)\alpha - 1.$$

we can accept the null hypothesis once $\sum_{i=1}^{B_0} \mathbf{1}_{\{T_i^* \geq T_0\}} > (B+1)\alpha - 1$ for some B_0. Algorithm 1 summarizes our computing method.

Algorithm 1: Randomization Algorithm

    Data: Data $X_1, \ldots, X_n$
    Result: Reject or accept the null hypothesis
1   **for** $i \leftarrow 2$ **to** n **do**
2   **for** $j \leftarrow 1$ **to** $i - 1$ **do**
3         $D_{ij} \leftarrow X_i^T X_j$;
4   **end**
5   **end**
6   Compute $T_0 \leftarrow \sum_{1 \leq j < i \leq n} D_{ij}$;
7   Set $A \leftarrow 0$;
8   **for** $i = 1$ *to* B **do**
9   Generate $\epsilon_1, \ldots, \epsilon_n$ according to $\Pr(\epsilon_i = 1) = \Pr(\epsilon_i = -1) = \frac{1}{2}$;
10   **if** $\sum_{1 \leq j < i \leq n} D_{ij}\epsilon_i\epsilon_j \geq T_0$ **then**
11        $A \leftarrow A + 1$;
12   **if** $A > (B+1)\alpha - 1$ **then return** *Accept*;
13   **else**
14   **if** $i - A \geq B + 1 - (B+1)\alpha$ **then return** *Reject*;
15   **end**
16   **end**

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3. Asymptotic properties

As we have noted, the randomization test is exact provided that X_1 and $-X_1$ have the same distribution under null hypothesis. However, under this condition, all variables are symmetric about 0, which is unrealistic for many applications. In this section, we investigate the asymptotic properties of the test procedure $\phi(X_1, \ldots, X_n)$ without symmetric assumption.

We assume, like Chen and Qin (2010) and Bai and Saranadasa (1996), the following multivariate model:

$$X_i = \mu + \Gamma Z_i \text{ for } i = 1, \ldots, n, \tag{5}$$

where Γ is a $p \times m$ matrix for some $m \geq p$ such that $\Gamma\Gamma^T = \Sigma$ and $Z_1, \ldots, Z_n$ are m-variate iid random vectors satisfying $\mathrm{E}(Z_i) = 0$ and $\mathrm{Var}(Z_i) = I_m$, the $m \times m$ identity matrix. Write $Z_i = (z_{i1}, \ldots, z_{im})^T$. We assume $\mathrm{E}(z_{ij}^4) = 3 + \Delta < \infty$ and

$$\mathrm{E}(z_{il_1}^{\alpha_1} z_{il_2}^{\alpha_2} \cdots z_{il_q}^{\alpha_q}) = \mathrm{E}(z_{il_1}^{\alpha_1})\,\mathrm{E}(z_{il_2}^{\alpha_2}) \cdots \mathrm{E}(z_{il_q}^{\alpha_q}) \tag{6}$$

for a positive integer q such that $\sum_{l=1}^{q} \alpha_l \leq 8$ and $l_1 \neq l_2 \neq \cdots \neq l_q$.

Let $\lambda_i(\Sigma)$ be the ith largest eigenvalue of Σ. In Bai and Saranadasa (1996), a key assumption for the eigenvalues of Σ is

$$\frac{\lambda_1(\Sigma)}{\sqrt{\mathrm{tr}(\Sigma^2)}} \to 0. \tag{7}$$

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Correspondingly, Chen and Qin (2010) imposed the condition $\text{tr}(\Sigma^4) = o(\text{tr}^2(\Sigma^2))$. From the inequality

$$\frac{\lambda_1(\Sigma)^4}{(\sum_{i=1}^p \lambda_i(\Sigma)^2)^2} \leq \frac{\sum_{i=1}^p \lambda_i(\Sigma)^4}{(\sum_{i=1}^p \lambda_i(\Sigma)^2)^2} \leq \frac{\lambda_1(\Sigma)^2 \sum_{i=1}^p \lambda_i(\Sigma)^2}{(\sum_{i=1}^p \lambda_i(\Sigma)^2)^2}$$

we can see that these two conditions are equivalent. As pointed out by Katayama et al. (2013), however, the condition (7) excludes a typical situation where the population covariance matrix has spiked eigenvalues. To include such important cases, we consider a condition that is weaker than (7).

Suppose that there is a fixed integer $r \geq 0$ and positive constants $\kappa_1, \ldots, \kappa_r$ such that

$$\frac{\lambda_i(\Sigma)}{\sqrt{\text{tr}(\Sigma^2)}} \to \kappa_i \text{ for } i = 1, \ldots, r \quad \text{and} \quad \frac{\lambda_i(\Sigma)}{\sqrt{\text{tr}(\Sigma^2)}} \to 0 \text{ for } i > r. \tag{8}$$

The following proposition gives the asymptotic distribution of $T(X_1, \ldots, X_n)$ under (5), (6) and (8).

Theorem 1. *Under* (5), (6) *and* (8), *we have*

(i) *If* $\mu^T \Sigma \mu = o(n^{-1} \text{tr}(\Sigma^2))$,

$$\frac{T(X_1, \ldots, X_n) - \frac{n(n-1)}{2} \|\mu\|^2}{\sqrt{\frac{n(n-1)}{2} \text{tr}(\Sigma^2)}} \xrightarrow{w} \frac{\sqrt{2}}{2} \sum_{i=1}^r \kappa_i(\xi_i^2 - 1) + (1 - \sum_{i=1}^r \kappa_i^2)^{1/2} \xi_{r+1},$$

where $\xi_1, \ldots \xi_{r+1}$ *are iid standard normal random variables.*

(ii) *If* $n^{-1} \text{tr}(\Sigma^2) = o(\mu^T \Sigma \mu)$,

$$\frac{T(X_1, \ldots, X_n) - \frac{n(n-1)}{2} \|\mu\|^2}{(n-1)\sqrt{n \mu^T \Sigma \mu}} \xrightarrow{\mathcal{L}} N(0, 1).$$

Remark 1. A similar result has been provided by Katayama et al. (2013, Theorem 3.1). Compared with their result, Theorem 1 doesn't need the normal assumption.

If $r = 0$, Theorem 1 implies that $T(X_1, \ldots, X_n)$ is asymptotically normal distributed under the null hypothesis. This is a known result (Chen and Qin, 2010, Theorem 1). On the other hand, if $r > 0$, i.e. Σ has a few spiked eigenvalues, $T(X_1, \ldots, X_n)$ is not asymptotically normal distributed. In this case, the asymptotic methods of Bai and Saranadasa (1996) and Chen and Qin (2010) are not directly applicable.

Now we study the asymptotic properties of the randomization test. The conditional distribution

$$\mathcal{L}\left(\frac{T(\epsilon_1 X_1, \ldots, \epsilon_i X_i, \ldots, \epsilon_n X_n)}{\sqrt{\sum_{1 \leq j < i \leq n} (X_i^T X_j)^2}} \middle| X_1, \ldots, X_n\right)$$

plays an important role in our analysis, we shall call it randomization distribution. Let z_α be the $1 - \alpha$ quantile of the randomization distribution. Then it can be seen that the test function $\phi(X_1, \ldots, X_n)$ equals to 1 if

$$\frac{T(X_1, \ldots, X_n)}{\sqrt{\sum_{1 \leq j < i \leq n} (X_i^T X_j)^2}} > \xi_\alpha^*$$

and equals to 0 otherwise.

Since the randomization distribution itself is random, to study its asymptotic distribution, we need to define in what sense the convergence is. Let F and G be two distribution functions on $\mathbb{R}$, Levy metric ρ of F and G is defined as

$$\rho(F, G) = \inf\{\epsilon : F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon \text{ for all } x\}.$$

It's well known that $\rho(F_n, F) \to 0$ if and only if $F_n \xrightarrow{\mathcal{L}} F$. The following theorem shows that in high dimensional setting, the randomization distribution tends to a standard normal distribution.

Theorem 2. *Under* (5), (6), (7) *and*

$$\mu^T \mu = o\big(\sqrt{\operatorname{tr}(\Sigma^2)}\big), \tag{9}$$

we have that

$$\rho\bigg(\mathcal{L}\bigg(\frac{T(\epsilon_1 X_1, \ldots, \epsilon_n X_n)}{\sqrt{\sum_{1 \leq j < i \leq n}(X_i^T X_j)^2}}\bigg| X_1, \ldots, X_n\bigg), N(0,1)\bigg) \xrightarrow{P} 0.$$

It can be proved that

$$\frac{\sum_{j<i}(X_i^T X_j)^2}{\frac{n(n-1)}{2}\operatorname{tr}(\Sigma^2)} \xrightarrow{P} 1.$$

By Lemma **??**, under null hypothesis, we have that

$$\frac{T(X_1, \ldots, X_n)}{\sqrt{\sum_{1 \leq j < i \leq n}(X_i^T X_j)^2}} \xrightarrow{\mathcal{L}} N(0,1).$$

Compare this with Theorem 2, we can see that if $\mu^T \mu = o(\sqrt{\operatorname{tr}(\Sigma^2)})$, the randomization distribution mimics the actual null distribution. However, the behavior of randomization distribution is different when condition (9) is not valid. In fact, we have the following result.

Theorem 3. *Under* (5), (6), (7) *and*

$$\sqrt{\operatorname{tr}\Sigma^2} = o\big(\mu^T \mu\big), \tag{10}$$

we have that

$$\rho\bigg(\mathcal{L}\bigg(\frac{T(\epsilon_1 X_1, \ldots, \epsilon_n X_n)}{\sqrt{\sum_{1 \leq j < i \leq n}(X_i^T X_j)^2}}\bigg| X_1, \ldots, X_n\bigg), \frac{\sqrt{2}}{2}(\chi_1^2 - 1)\bigg) \xrightarrow{P} 0,$$

where χ_1^2 is the chi-squared distribution with freedom 1.

Once the limit of the randomization distribution is obtained, the asymptotic behavior of ξ_α^* can be derived immediately. Let $\Phi(\cdot)$ be the cumulative distribution function (CDF) of standard normal distribution, we have

Corollary 1. *Under the conditions of Theorem 2, we have*

$$\xi_\alpha^* \xrightarrow{P} \Phi^{-1}(1 - \alpha).$$

Corollary 2. *Under the conditions of Theorem 3,*

$$\xi_\alpha^* \xrightarrow{P} \frac{\sqrt{2}}{2}\Big(\big(\Phi^{-1}(1 - \tfrac{\alpha}{2})\big)^2 - 1\Big).$$

Now we are ready to derive the asymptotic power of the randomization test. The following two theorems give the power under (**??**) and (**??**), respectively.

Theorem 4. *Suppose conditions* (5), (6), (7) *and* (**??**) *holds. Then*

1. *If* (9) *holds,*

$$\Pr\bigg(\frac{T(X_1, \ldots, X_n)}{\sqrt{\sum_{1 \leq j < i \leq n}(X_i^T X_j)^2}} > \xi_\alpha^*\bigg) = \Phi\bigg(-\Phi^{-1}(1 - \alpha) + \frac{\sqrt{n(n-1)}\mu^T \mu}{\sqrt{2\operatorname{tr}(\Sigma^2)}}\bigg) + o(1). \tag{11}$$

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2. *If* (10) *holds,*

$$\Pr\left(\frac{T(X_1,\ldots,X_n)}{\sqrt{\sum_{1\le j<i\le n}\left(X_i^T X_j\right)^2}} > \xi_\alpha^*\right) \to 1.$$

Theorem 5. *Under* (5), (6), (7) (**??**) *and either* (9) *or* (10),

$$\Pr\left(\frac{T(X_1,\ldots,X_n)}{\sqrt{\sum_{1\le j<i\le n}\left(X_i^T X_j\right)^2}} > \xi_\alpha^*\right) = \Phi\left(\frac{\sqrt{n}\mu^T\mu}{2\sqrt{\mu^T\Sigma\mu}}\right) + o(1).$$

Remark 2. Neither (**??**) or (9) implies the other one. For example, suppose $\Sigma = I_p$, then (**??**) is equivalent to $\mu^T\mu = o(p/n)$ and (9) is equivalent to $\mu^T\mu = o(\sqrt{p})$. In this case, if $\sqrt{p}/n \to 0$, then (**??**) implies (9); conversely, if $\sqrt{p}/n \to \infty$, then (9) implies (**??**).

Theorem 4 implies that under (5), (6) and (7), the Type I error rate of the randomization test tends to the nominal level. Note that this result doesn't assume that the distribution of X_1 is symmetric under null hypothesis. This implies that our test procedure is robust when the symmetry assumption is break down. This property is not held by all randomization tests. See, for example, Romano (1990).

Use the method of Chen and Qin (2010), the asymptotic method rejects the null hypothesis when

$$\frac{\sum_{i\ne j} X_i^T X_j}{\sqrt{2n(n-1)\widehat{\mathrm{tr}(\Sigma^2)}}} > \Phi^{-1}(1-\alpha),$$

where

$$\widehat{\mathrm{tr}(\Sigma^2)} = \frac{1}{n(n-1)}\,\mathrm{tr}\left(\sum_{i\ne j}(X_i - \bar{X}_{(i,j)})X_i^T(X_j - \bar{X}_{(i,j)})X_j^T\right)$$

is a ratio consistent estimator of $\mathrm{tr}(\Sigma^2)$ and $\bar{X}_{(i,j)}$ is the sample mean after excluding X_i and X_j. Lemma **??** implies that the Type I error rate of the asymptotic method tends to the nominal level under (5), (6) and (7).

4. Simulation Studies

In this section, we report the simulation performance of the randomization test in various settings. For comparison purposes, we also carried out simulations for the asymptotic method of Chen and Qin (2010), the *p*-value of which is

$$p_{CQ}(X_1,\ldots,X_n) = 1 - \Phi\left(\frac{\sum_{i\ne j} X_i^T X_j}{\sqrt{2n(n-1)\widehat{\mathrm{tr}(\Sigma^2)}}}\right).$$

We consider two innovation structures: the moving average model and the factor model. The moving average model has the following structure:

$$X_{ij} = \sum_{l=0}^{k} \rho_l Z_{i,j+l},$$

$i = 1,\ldots,n$, $j = 1,\ldots,p$, where Z_{ij}'s are iid random variables with distribution F for $i = 1,\ldots,n$ and $j = 1,\ldots,p+k$. Like Chen and Qin (2010), we consider two different F. One is $N(0,1)$, and the other is $(\mathrm{Gamma}(4,1) - 4)/2$. We also consider different k. The ρ_i's are generated independently from $U(2,3)$ and are kept fixed throughout the simulation. The second model we consider is the factor model in Fan et al. (2007). In the simulation study of Fan et al. (2007), the factor model is used to reflect aspects of gene expression data. The model involves three group factor and one common factor among all p variables. We denote by $\{\xi_{ij}\}_{1\le i\le n, 1\le j\le p}$ a sequence of independent $N(0,1)$ random variables and by $\{\chi_{ij}\}_{1\le i\le n, 1\le j\le 4}$

a sequence of independent random variables with distribution $(\chi_6^2 - 6)/\sqrt{12}$. Note that χ_{ij} has mean 0, variance 1 and skewness $\sqrt{12}/3$. The data is generated by model

$$X_{ij} = \frac{a_{j1}\chi_{i1} + a_{j2}\chi_{i2} + a_{j3}\chi_{i3} + b_j\chi_{i4} + \xi_{ij}}{\left(1 + a_{j1}^2 + a_{j2}^2 + a_{j3}^2 + b_j^2\right)^{1/2}},$$

$i = 1, \ldots, n$, $j = 1, \ldots, p$, where $a_{jk} = 0$ except that $a_{j1} = a_j$ for $j = 1, \ldots, \frac{1}{3}p$, $a_{j2} = a_j$ for $\frac{1}{3}p + 1, \ldots, \frac{2}{3}p$ and $a_{j3} = a_j$ for $\frac{2}{3}p + 1, \ldots, p$. As in Fan et al. (2007), we consider two configurations of factor loadings. In case I we set $a_j = 0.25$ and $b_j = 0.1$ for $j = 1, \ldots, p$. In case II, a_i and b_i are generated independently from $U(0, 0.4)$ and $U(0, 0.2)$.

To control the significant level, the null distribution of a p-value should be close to $U(0, 1)$, the uniform distribution on $(0, 1)$. We simulate the p-values $\tilde{p}(X_1, \ldots, X_n)$ ($B = 1000$) and $p_{CQ}(X_1, \ldots, X_n)$ for 2000 times and Figure 1 plots the empirical distribution function (ECDF) of p-values. As shown by the plots, the p-values of the randomization test method are uniform distributed in all cases. For asymptotic method, the uniformity of p-values depends on model. Under the moving average model, the empirical distribution of p_{CQ} is close to uniform distribution for $k = 3$ but is far away from uniform distribution for $k = 500$. In factor model, the empirical distribution of p_{CQ} slightly deviates from uniform distribution.

In Theorem 2, we showed that the randomization distribution tends to a standard normal distribution under certain conditions. In Figure 2, we plot the histograms of the randomization distribution under null hypothesis. For comparison, we also plot the standard normal density. From the plots, we can see that the randomization distribution is very similar to the standard normal distribution in factor model and moving average model with $k = 3$. This verifies the Theorem 2. However, under moving average model with $k = 500$, the randomization distribution is far from standard normal distribution. In fact, in this case, $\lambda_1(\Sigma)/\sqrt{\text{tr}(\Sigma^2)}$ is not negligible and (7) is not satisfied. This implies the accuracy of normal approximation depends on the innovation model.

Now we simulate the empirical power and size of the randomization test and the asymptotic method. Let

$$\text{SNR} = \frac{\sqrt{n(n-1)}\mu^T\mu}{\sqrt{2\,\text{tr}\,\Sigma^2}}$$

be the signal to noise ratio (SNR). The theoretic asymptotic power is an increasing function of SNR. We scale μ to reach different levels of SNR. Our simulations consider two mean structure: dense mean and sparse mean. In the dense mean setting, we independently generate each coordinate of μ from $U(2, 3)$ and then scale μ to reach a given SNR. In the sparse mean setting, we randomly select 5% of μ's p coordinates to be non-zero. Each non-zero coordinate is again independently generated from $U(2, 3)$ and then scaled to reach a given SNR. We set $B = 1000$ for the randomization test method. The empirical power and size are computed based on 2000 simulations. The nominal level α is set to be 0.05.

Table 1 and Table 2 list the empirical power and size for the moving average model. It's not surprising that the randomization test can control the Type I error rate well in normal case. The results also show that the randomization method can control the Type I error rate well in gamma case even if gamma distribution is not symmetric under null. It justifies the robustness of the randomization method. On the other hand, the asymptotic method has small size when the correlations between variables are weak and has inflated size when the correlations are strong. The empirical power of the randomization method is similar to the asymptotic method. They are both similar to theoretical asymptotic power (11) when k is small and are both lower than theoretical asymptotic power when k is large. Table 3 lists the empirical power and size under factor model. Although the distribution is not symmetric, the results show that the level of the randomization method is close to nominal level while the asymptotic method suffers from level inflation. In summary, the simulation results show that the randomization method is robust and has similar power with asymptotic method.
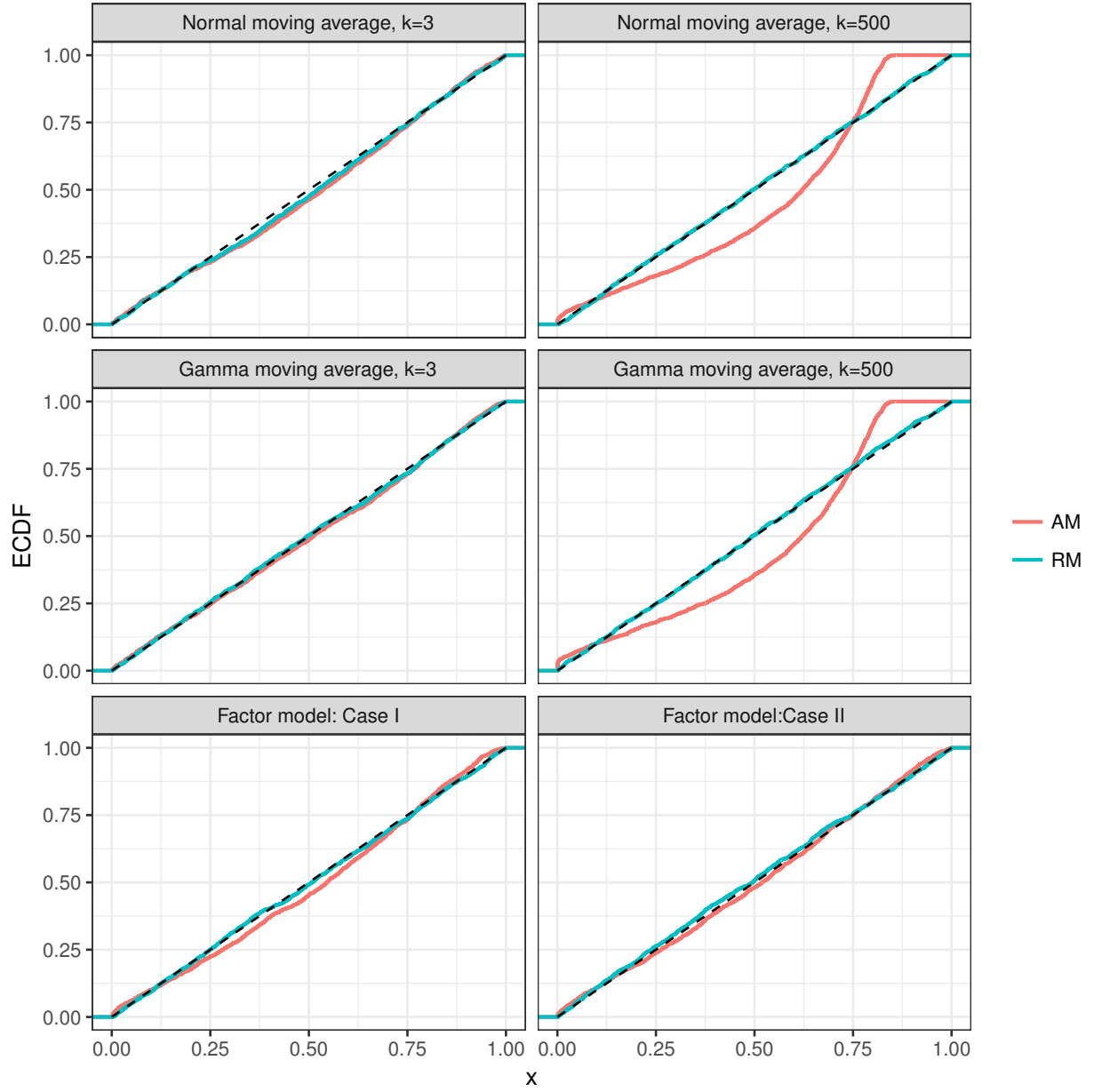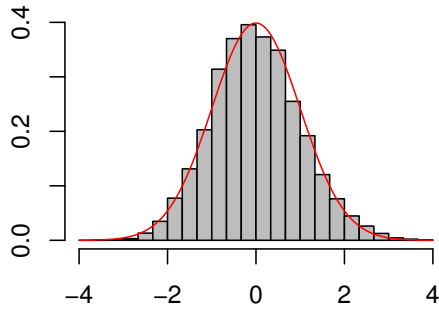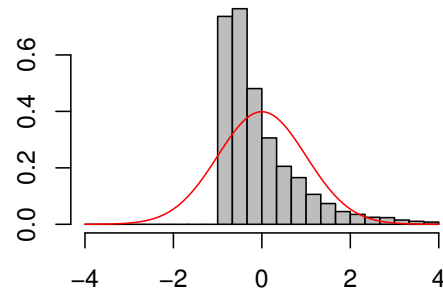
Figure 1: The ECDF of p-values for the asymptotic method (AM) and the randomization method (RM). $p = 600$, $n = 100$.

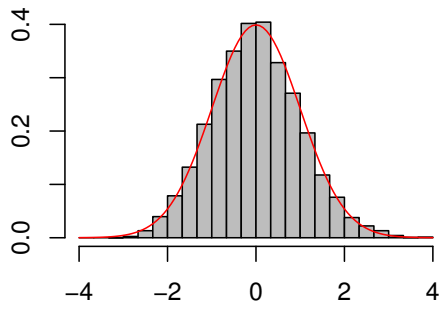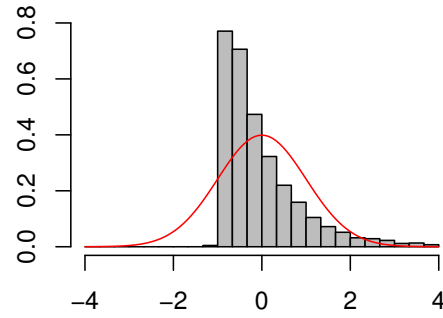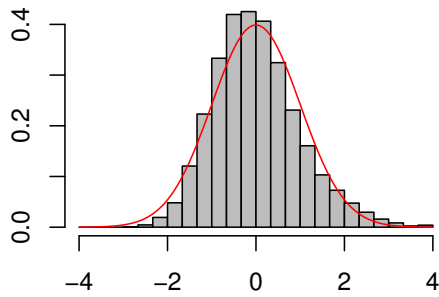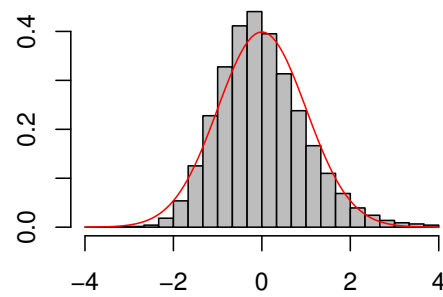Figure 2: The histograms of the randomization distribution. $p = 600$, $n = 100$.

Table 1: Empirical power and size of moving average model with normal innovation. TP represents the theoretical asymptotic power, RM represents the randomization method, AM represents the asymptotic method.

| | | | | Dense means | | | | Sparse means | | | |
| | | | | $k = 3$ | | $k = 500$ | | $k = 3$ | | $k = 500$ | |
n	p	SNR	TP	RM	AM	RM	AM	RM	AM	RM	AM
100	600	0.0 (size)	0.050	0.0445	0.0515	0.0530	0.0745	0.0500	0.0585	0.0515	0.0715
		0.5	0.126	0.1815	0.1940	0.1330	0.1685	0.1735	0.1875	0.0780	0.1130
		1.0	0.260	0.4075	0.4295	0.2250	0.2785	0.4060	0.4325	0.1505	0.2065
		1.5	0.442	0.6295	0.6535	0.3435	0.3920	0.6520	0.6755	0.2480	0.3355
		2.0	0.639	0.7895	0.8055	0.3935	0.4665	0.8575	0.8765	0.3850	0.5400
		2.5	0.804	0.9165	0.9215	0.4775	0.5425	0.9655	0.9695	0.6355	0.8155
		3.0	0.912	0.9640	0.9695	0.5445	0.6090	0.9910	0.9935	0.8720	0.9730
200	1000	0.0 (size)	0.050	0.0505	0.0550	0.0505	0.0695	0.0520	0.0575	0.0510	0.0700
		0.5	0.126	0.1885	0.2025	0.1410	0.1680	0.1745	0.1865	0.0855	0.1165
		1.0	0.260	0.3875	0.4050	0.2090	0.2620	0.4010	0.4160	0.1485	0.2075
		1.5	0.442	0.6425	0.6600	0.3180	0.3715	0.6725	0.6895	0.2355	0.3290
		2.0	0.639	0.8245	0.8355	0.4210	0.4760	0.8710	0.8845	0.3855	0.5310
		2.5	0.804	0.9210	0.9265	0.4885	0.5475	0.9690	0.9710	0.6290	0.8050
		3.0	0.912	0.9820	0.9830	0.6045	0.6530	0.9940	0.9960	0.8670	0.9780

5. Conclusion Remark

In this paper, we considered a randomization test for mean vector in high dimensional setting. A fast implementation was provided. We also derived some asymptotic properties of the test procedure. We showed that even if the symmetric assumption is violated, the randomization test also has correct level asymptotically. Hence the test procedure is robust. In fact, the algorithm and the proof method can also be applied to other quadratic based statistics.

In classical statistics, randomization test procedures are time consuming. Nevertheless, the computational complexity of our randomization test procedure is not affected by the data dimension p. Hence we have reason to believe that randomization tests may be generally suitable for high dimensional problems.

Our randomization test can be immediately generalized to the two sample problem for paired data (Konietschke and Pauly, 2014). However, maybe the most widely used randomization test for two sample problem is the permutation test. As pointed by Romano (1990), the asymptotic property of randomization tests depends heavily on the particular problem and the randomization method. The proof method used in this paper can not be applied to two sample permutation test. It is interesting to understand the behavior of permutation tests in high dimensional setting. We leave it for further work.

Appendices

Denote by $\Gamma^T \Gamma = Q \Lambda Q^T$ the spectral decomposition of $\Gamma^T \Gamma$, where Q is a $m \times p$ column orthogonal matrix and $\Lambda = \text{diag}(\lambda_1(\Sigma), \ldots, \lambda_p(\Sigma))$. Let $\Lambda_1 = \text{diag}(\lambda_1(\Sigma), \ldots, \lambda_r(\Sigma))$, $\Lambda_2 = \text{diag}(\lambda_{r+1}(\Sigma), \ldots, \lambda_p(\Sigma))$. Write

Table 2: Empirical power and size of moving average model with Gamma innovation. TP is the theoretical asymptotic power, RM represents the randomization method, AM represents the asymptotic method.

| | | | | Dense means | | | | Sparse means | | | |
| | | | | $k = 3$ | | $k = 500$ | | $k = 3$ | | $k = 500$ | |
n	p	SNR	TP	RM	AM	RM	AM	RM	AM	RM	AM
100	600	0.0 (size)	0.050	0.0450	0.0550	0.0475	0.0660	0.0405	0.0465	0.0505	0.0765
		0.5	0.126	0.1815	0.1975	0.1365	0.1750	0.1765	0.1870	0.0985	0.1345
		1.0	0.260	0.3825	0.4050	0.2375	0.2765	0.4130	0.4335	0.1550	0.2070
		1.5	0.442	0.6210	0.6465	0.2975	0.3490	0.6580	0.6745	0.2225	0.3135
		2.0	0.639	0.8180	0.8325	0.3920	0.4450	0.8645	0.8800	0.3890	0.5340
		2.5	0.804	0.9115	0.9260	0.4900	0.5465	0.9635	0.9665	0.6280	0.8200
		3.0	0.912	0.9710	0.9765	0.5505	0.6085	0.9940	0.9945	0.8600	0.9740
200	1000	0.0 (size)	0.050	0.0520	0.0555	0.0495	0.0715	0.0505	0.0555	0.0455	0.0690
		0.5	0.126	0.1740	0.1880	0.1355	0.1725	0.1725	0.1840	0.0780	0.1170
		1.0	0.260	0.3890	0.4045	0.2175	0.2595	0.4220	0.4415	0.1475	0.1950
		1.5	0.442	0.6470	0.6630	0.3240	0.3820	0.6605	0.6855	0.2550	0.3375
		2.0	0.639	0.8175	0.8285	0.4180	0.4755	0.8580	0.8750	0.3835	0.5205
		2.5	0.804	0.9295	0.9335	0.4870	0.5560	0.9600	0.9645	0.6075	0.7970
		3.0	0.912	0.9755	0.9765	0.5865	0.6505	0.9915	0.9935	0.8760	0.9790

$Q = (Q_1, Q_2)$ where Q_1 and Q_2 are the first r columns and last $p - r$ columns of Q respectively. Then $\Gamma^T \Gamma = Q_1 \Lambda_1 Q_1^T + Q_2 \Lambda_2 Q_2^T$. For nonzero constants $c \in \mathbb{R}$ and $t \in \mathbb{R}^r$, we denote

$$Y_{ij} = \big(\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2) \big)^{-1/2} Z_i^T Q_2 \Lambda_2 Q_2^T Z_j, \quad Y_i^* = n^{-1/2} Q_1^T Z_i.$$

Appendix A. Preparatory technical results

We state in this section some technical results that are used in the proofs of the main results.

Lemma 1. *Suppose $\{\eta_n\}_{n=1}^{\infty}$ is a sequence of random variables, weakly converges to η, a random variable with continuous distribution function. Then we have*

$$\sup_{x \in \mathbb{R}} | \operatorname{Pr}(\eta_n \leq x) - \operatorname{Pr}(\eta \leq x)| \to 0.$$

Lemma 2. *Under the condition* (6), *we have*

(i) *for any $m \times m$ positive semidefinite matrix B,*

$$\operatorname{E}(Z_1^T B Z_1)^2 \leq (3 + \Delta) \operatorname{tr}^2(B), \quad \operatorname{Var}(Z_1^T B Z_1) \leq (2 + \Delta) \operatorname{tr}(B^2); \tag{A.1}$$

(ii) *for any $m \times m$ positive semidefinite matrix B,*

$$\operatorname{E}(Z_1^T B Z_2)^4 \leq (3 + \Delta)^2 \operatorname{tr}^2(B^2); \tag{A.2}$$

Proof. The inequality (A.1) follows from (i) of Chen et al. (2010, Proposition A.1). The inequality (A.2) is obtained by using (A.1) twice:

$$\operatorname{E}(Z_1^T B Z_2)^4 = \operatorname{E} \operatorname{E} \big[(Z_1^T B Z_2 Z_2^T B Z_1)^2 | Z_2 \big]$$
$$\leq (3 + \Delta) \operatorname{E} \operatorname{tr}^2(B Z_2 Z_2^T B) = (3 + \Delta) \operatorname{E}(Z_2^T B^2 Z_2)^2 \leq (3 + \Delta)^2 \operatorname{tr}^2(B^2).$$

$\square$

Table 3: Empirical power and size of factor model innovation. TP is the theoretical asymptotic power, RM represents the randomization method, AM represents the asymptotic method.

| | | | | Dense means | | | | Sparse means | | | |
| | | | | Case I | | Case II | | Case I | | Case II | |
n	p	SNR	TP	RM	AM	RM	AM	RM	AM	RM	AM
100	600	0.0 (size)	0.050	0.0465	0.0610	0.0455	0.0590	0.0475	0.0625	0.0505	0.0615
		0.5	0.126	0.1315	0.1555	0.1465	0.1650	0.1200	0.1380	0.1080	0.1320
		1.0	0.260	0.2420	0.2780	0.2550	0.2780	0.1940	0.2250	0.2075	0.2400
		1.5	0.442	0.3635	0.3975	0.3555	0.3870	0.3670	0.4110	0.3740	0.4155
		2.0	0.639	0.4825	0.5165	0.4720	0.4975	0.5340	0.5930	0.5615	0.6015
		2.5	0.804	0.5860	0.6190	0.5825	0.6165	0.7040	0.7610	0.7120	0.7505
		3.0	0.912	0.6730	0.7060	0.6975	0.7210	0.8525	0.8815	0.8680	0.8920
200	1002	0.0 (size)	0.050	0.0520	0.0670	0.0530	0.0665	0.0485	0.0615	0.0490	0.0615
		0.5	0.126	0.1490	0.1895	0.1405	0.1675	0.0905	0.1115	0.0975	0.1230
		1.0	0.260	0.2450	0.2780	0.2485	0.2785	0.1835	0.2330	0.1920	0.2305
		1.5	0.442	0.3445	0.3915	0.3450	0.3745	0.3170	0.3745	0.3395	0.3935
		2.0	0.639	0.4605	0.5035	0.4775	0.5170	0.5080	0.5995	0.5365	0.5925
		2.5	0.804	0.5355	0.5810	0.5900	0.6320	0.7040	0.7775	0.7345	0.7850
		3.0	0.912	0.6280	0.6640	0.6660	0.6960	0.8670	0.9115	0.8655	0.9070

Appendix B. Proof of Theorem 1

Proposition 1. *Under* (5), (6), (8), *we have*

$$\frac{\sum_{j<i} Z_i^T \Gamma^T \Gamma Z_j}{\sqrt{\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2)}} \xrightarrow{w} \frac{\sqrt{2}}{2} \sum_{i=1}^{r} \kappa_i(\xi_i^2 - 1) + (1 - \sum_{i=1}^{r} \kappa_i^2)^{1/2}\xi_{r+1},$$

where $\xi_1, \ldots \xi_{r+1}$ *are iid standard normal random variables.*

Proof. Note that

$$\frac{\sum_{j<i} Z_i^T \Gamma^T \Gamma Z_j}{\sqrt{\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2)}} = \frac{\sum_{j<i} Z_i^T Q_1 \Lambda_1 Q_1^T Z_j}{\sqrt{\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2)}} + \frac{\sum_{j<i} Z_i^T Q_2 \Lambda_2 Q_2^T Z_j}{\sqrt{\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2)}}$$

$$= \left(2\frac{n-1}{n} \operatorname{tr}(\Sigma^2)\right)^{-1/2} \left(\sum_{i=1}^{n} Y_i^*\right)^T \Lambda_1 \sum_{i=1}^{n} Y_i^* + \sum_{j<i} Y_{ij} - \left(2\frac{n-1}{n} \operatorname{tr}(\Sigma^2)\right)^{-1/2} \sum_{i=1}^{n} Y_i^{*T} \Lambda_1 Y_i^*.$$

Next we show that

$$\left(\left(\sum_{i=1}^{n} Y_i^*\right)^T, \sum_{j<i} Y_{ij}\right)^T \xrightarrow{\mathcal{L}} N_{r+1}\left(0_{r+1}, \operatorname{diag}\left(I_r, 1 - \sum_{i=1}^{r} \kappa_i^2\right)\right). \tag{B.1}$$

To prove this, it suffices to show that for any nonzero constants $t \in \mathbb{R}^r$ and $c \in \mathbb{R}$,

$$t^T \sum_{i=1}^{n} Y_i^* + c \sum_{j<i} Y_{ij} \xrightarrow{\mathcal{L}} N\left(0, \|t\|^2 + c^2(1 - \sum_{i=1}^{r} \kappa_i^2)\right). \tag{B.2}$$

Write $t^T \sum_{i=1}^{n} Y_i^* + c \sum_{j<i} Y_{ij} = \sum_{i=1}^{n} U_{in}$, where $U_{in} = t^T Y_i^* + c \sum_{j:j<i} Y_{ij}$. Let $\mathcal{F}_{in}$ be the σ-field generated by $X_1, \ldots, X_i$, $k = 1, \ldots, n$. Then $\{U_{in}\}_{i=1}^{n}$ is a martingale difference array with respect to

$\{\mathcal{F}_{in}\}_{i=1}^{n}$. By the martingale central limit theorem (Pollard, 1984, Chapter VIII, Theorem 1), (B.2) holds provided

$$\sum_{i=1}^{n} \mathrm{E}(U_{in}^2 | \mathcal{F}_{i-1,n}) \xrightarrow{P} \|t\|^2 + c^2(1 - \sum_{i=1}^{r} \kappa_i^2), \tag{B.3}$$

and for every $\epsilon > 0$,

$$\sum_{i=1}^{n} \mathrm{E}\left(U_{in}^2 \{U_{in}^2 > \epsilon\} | \mathcal{F}_{i-1,n}\right) \xrightarrow{P} 0. \tag{B.4}$$

We first verify (B.3). By direct calculation, we have

$$\sum_{i=1}^{n} \mathrm{E}(U_{in}^2 | \mathcal{F}_{i-1,n}) = \|t\|^2 + c^2 \left(\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2)\right)^{-1} \sum_{i=1}^{n} \left(\left(\sum_{j:j<i} Z_j\right)^T Q_2 \Lambda_2^2 Q_2^T \sum_{j:j<i} Z_j\right).$$

Note that

$$\sum_{i=1}^{n} \left(\left(\sum_{j:j<i} Z_j\right)^T Q_2 \Lambda_2^2 Q_2^T \sum_{j:j<i} Z_j\right) = \sum_{i=1}^{n} \left(\sum_{j:j<i} Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_j + 2 \sum_{j,k:k<j<i} Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_k\right)$$

$$= \sum_{j=1}^{n} (n-j) Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_j + 2 \sum_{k<j} (n-j) Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_k. \tag{B.5}$$

The first term satisfies

$$\mathrm{E} \sum_{j=1}^{n} (n-j) Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_j = \frac{n(n-1)}{2} \operatorname{tr}(\Lambda_2^2)$$

and

$$\operatorname{Var}\left(\sum_{j=1}^{n} (n-j) Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_j\right) = \frac{(n-1)n(2n-1)}{6} \operatorname{Var}\left(Z_1^T Q_2 \Lambda_2^2 Q_2^T Z_1\right) \le (2+\Delta) n^3 \operatorname{tr}(\Lambda_2^4),$$

where the last inequality follows from (i) of Lemma 2. Hence

$$\sum_{j=1}^{n} (n-j) Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_j = \frac{n(n-1)}{2} \operatorname{tr}(\Lambda_2^2) + O_P\left(n^{3/2} \operatorname{tr}^{1/2}(\Lambda_2^4)\right)$$

$$= \left(\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2)\right) \left(\frac{\operatorname{tr}(\Lambda_2^2)}{\operatorname{tr}(\Sigma^2)} + o_P(1)\right) = \left(\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2)\right) \left(1 - \sum_{i=1}^{r} \kappa_i^2 + o_P(1)\right). \tag{B.6}$$

As for the second term of (B.5), we have

$$\mathrm{E}\left(\sum_{k<j} (n-j) Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_k\right)^2 = \sum_{k<j} (n-j)^2 \mathrm{E}\left(Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_k\right)^2 = \sum_{k<j} (n-j)^2 \operatorname{tr}(\Lambda_2^4)$$

$$\le n^4 \operatorname{tr}(\Lambda_2^4) \le n^4 \lambda_{r+1}^2(\Sigma) \operatorname{tr}(\Lambda_2^2) = o\left(\left(\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2)\right)^2\right).$$

Hence

$$\sum_{k<j} (n-j) Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_k = o_P\left(\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2)\right). \tag{B.7}$$

Combining (B.5), (B.6) and (B.7) yields

$$\left(\frac{n(n-1)}{2} \operatorname{tr}(\Sigma^2)\right)^{-1} \sum_{i=1}^{n} \left(\left(\sum_{j:j<i} Z_j\right)^T Q_2 \Lambda_2^2 Q_2^T \sum_{j:j<i} Z_j\right) = 1 - \sum_{i=1}^{r} \kappa_i^2 + o_P(1).$$

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This proves (B.3).

Now we verify (B.4). By Markov's inequality, it suffices to show $\sum_{i=1}^{n} \mathrm{E}\left(U_{in}^4\right) \to 0$. Note that

$$\mathrm{E}\left(U_{in}^4\right) \le 8n^{-2}\,\mathrm{E}\left(t^T Q_1^T Z_i\right)^4 + 8c^2\left(\frac{n(n-1)}{2}\operatorname{tr}(\Sigma^2)\right)^{-2}\mathrm{E}\left(Z_i^T Q_2 \Lambda_2 Q_2^T \sum_{j:j<i} Z_j\right)^4.$$

Applying (i) of Lemma 2 yields

$$\mathrm{E}\left(t^T Q_1^T Z_i\right)^4 \le \mathrm{E}\left(Z_i^T Q_1 t t^T Q_1^T Z_i\right)^2 \le (3+\Delta)\operatorname{tr}^2(Q_1 t t^T Q_1^T) = (3+\Delta)\|t\|^4.$$

Also by (i) of Lemma 2,

$$\mathrm{E}\left(Z_i^T Q_2 \Lambda_2 Q_2^T \sum_{j:j<i} Z_j\right)^4 = \mathrm{E}\left(Z_i^T Q_2 \Lambda_2 Q_2^T \left(\sum_{j:j<i} Z_j\right)\left(\sum_{j:j<i} Z_j\right)^T Q_2 \Lambda_2 Q_2^T Z_i\right)^2$$

$$\le (3+\Delta)\,\mathrm{E}\operatorname{tr}^2\left(Q_2 \Lambda_2 Q_2^T \left(\sum_{j:j<i} Z_j\right)\left(\sum_{j:j<i} Z_j\right)^T Q_2 \Lambda_2 Q_2^T\right)$$

$$= (3+\Delta)\,\mathrm{E}\left(\left(\sum_{j:j<i} Z_j\right)^T Q_2 \Lambda_2^2 Q_2^T \left(\sum_{j:j<i} Z_j\right)\right)^2$$

$$= (3+\Delta)\,\mathrm{E}\left(\sum_{j:j<i} Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_j + 2\sum_{j,k:k<j<i} Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_k\right)^2$$

$$= (3+\Delta)\,\mathrm{E}\left(\sum_{j:j<i} Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_j\right)^2 + 4(3+\Delta)\,\mathrm{E}\left(\sum_{j,k:k<j<i} Z_j^T Q_2 \Lambda_2^2 Q_2^T Z_k\right)^2$$

$$\le (3+\Delta)^2(i-1)^2 \operatorname{tr}^2(\Lambda_2^2) + 2(3+\Delta)(i-1)(i-2)\operatorname{tr}(\Lambda_2^4)$$

$$\le 2(3+\Delta)^2 i^2 \operatorname{tr}^2(\Lambda_2^2).$$

Thus,

$$\mathrm{E}\left(U_{in}^4\right) \le 8(3+\Delta)\|t\|^4 n^{-2} + 16c^2(3+\Delta)^2 i^2\left(\frac{n(n-1)}{2}\right)^{-2}.$$

It follows that $\sum_{i=1}^{n}\mathrm{E}\left(U_{in}^4\right) \to 0$. This completes the proof of (B.1).

It follows from (B.1) and Slutsky's theorem that

$$\left(2\frac{n-1}{n}\operatorname{tr}(\Sigma^2)\right)^{-1/2}\left(\sum_{i=1}^{n} Y_i^*\right)^T \Lambda_1 \sum_{i=1}^{n} Y_i^* + \sum_{j<i} Y_{ij} \xrightarrow{w} \frac{\sqrt{2}}{2}\sum_{i=1}^{r}\kappa_i \xi_i^2 + \left(1-\sum_{i=1}^{r}\kappa_i^2\right)^{1/2}\xi_{r+1}.$$

It remains to show that

$$\left(2\frac{n-1}{n}\operatorname{tr}(\Sigma^2)\right)^{-1/2}\sum_{i=1}^{n} Y_i^{*T}\Lambda_1 Y_i^* \xrightarrow{P} \frac{\sqrt{2}}{2}\sum_{i=1}^{r}\kappa_i.$$

This follows from

$$\mathrm{E}\left(2\frac{n-1}{n}\operatorname{tr}(\Sigma^2)\right)^{-1/2}\sum_{i=1}^{n} Y_i^{*T}\Lambda_1 Y_i^* = \left(2\frac{n-1}{n}\operatorname{tr}(\Sigma^2)\right)^{-1/2}\operatorname{tr}(\Lambda_1) = \frac{\sqrt{2}}{2}\sum_{i=1}^{r}\kappa_i + o(1)$$

and

$$\operatorname{Var}\left(\left(2\frac{n-1}{n}\operatorname{tr}(\Sigma^2)\right)^{-1/2}\sum_{i=1}^{n} Y_i^{*T}\Lambda_1 Y_i^*\right) = \left(2\frac{n-1}{n}\operatorname{tr}(\Sigma^2)\right)^{-1}\frac{(2+\Delta)\operatorname{tr}(\Lambda_1^2)}{n} = o(1).$$

This completes the proof.

$\square$

Proof of Theorem 1. Note that

$$T(X_1,\ldots,X_n) - \frac{n(n-1)}{2}\|\mu\|^2 = \sum_{j<i} Z_i^T \Gamma^T \Gamma Z_j + (n-1)\sum_{i=1}^n \mu^T \Gamma Z_i. \tag{B.8}$$

By CLT,

$$\frac{\sum_{i=1}^n \mu^T \Gamma Z_i}{\sqrt{n\mu^T \Sigma \mu}} \xrightarrow{\mathcal{L}} N(0,1). \tag{B.9}$$

If $\mu^T \Sigma \mu = o(n^{-1} \operatorname{tr}(\Sigma^2))$, the first term of (B.8) dominates the second term of (B.8), and the theorem follows from Proposition 1. On the other hand, if $n^{-1} \operatorname{tr}(\Sigma^2) = o(\mu^T \Sigma \mu)$, the second term of (B.8) dominates the first term of (B.8), and the theorem follows from (B.9).

$\square$

Appendix C. Proof of Theorem xxx

A CLT for quadratic form of Rademacher variables. The proof of the Theorem 2 is based on a CLT of the quadratic form of Rademacher variables. Such a CLT can be also used to study the asymptotic behavior of many other randomization test. Let $\epsilon_1,\ldots,\epsilon_n$ be independent Rademacher variables. Consider the quadratic form $W_n = \sum_{1\le j<i\le n} a_{ij}\epsilon_i\epsilon_j + \sum_{i=1}^n b_i\epsilon_i$, where $\{a_{ij}\}_{1\le j<i\le n}$ and $\{b_i\}_{1\le i\le n}$ are nonrandom numbers. Here $\{\epsilon_i\}$, $\{a_{ij}\}$ and $\{b_i\}$ may depend on n, a parameter we suppress. By direct calculation, we have $\mathrm{E}(W_n) = 0$ and $\operatorname{Var}(W_n) = \sum_{1\le j<i\le n} a_{ij}^2 + \sum_{i=1}^n b_i^2$.

Proposition 2. *A sufficient condition for*

$$\frac{W_n}{\sqrt{\sum_{1\le j<i\le n} a_{ij}^2 + \sum_{i=1}^n b_i^2}} \xrightarrow{\mathcal{L}} N(0,1)$$

is that

$$\sum_{k<j}\Big(\sum_{i:i>j} a_{ij}a_{ik}\Big)^2 + \sum_{j<i} a_{ij}^4 + \sum_{k<j<i} a_{ij}^2 a_{ik}^2 + \sum_{j=1}^n \Big(\sum_{i:i>j} a_{ij}b_i\Big)^2 + \sum_{i=1}^n b_i^4 = o\Big(\Big(\sum_{j<i} a_{ij}^2 + \sum_{i=1}^n b_i^2\Big)^2\Big). \tag{C.1}$$

Proof. Note that we have the decomposition $W_n = \sum_{i=1}^n U_{in}$, where $U_{in} = \epsilon_i(\sum_{j:j<i} a_{ij}\epsilon_j + b_i)$, $i = 1,\ldots,n$. Let $\mathcal{F}_{in}$ be the σ-field generated by $\epsilon_1,\ldots,\epsilon_i$, $i = 1,\ldots,n$. Then $\{U_{in}\}_{i=1}^n$ is a martingale difference array with respect to $\{\mathcal{F}_{in}\}_{i=1}^n$. By the martingale central limit theorem (Pollard, 1984, Theorem 1 of Chapter VIII), the conclusion holds if the following two conditions are satisfied:

$$\frac{\sum_{i=2}^n \mathrm{E}(U_{in}^2|\mathcal{F}_{i-1,n})}{\sum_{j<i} a_{ij}^2 + \sum_{i=1}^n b_i^2} \xrightarrow{P} 1, \tag{C.2}$$

and

$$\frac{\sum_{i=2}^n \mathrm{E}\left(U_{in}^2\{U_{in}^2 > \epsilon(\sum_{j<i} a_{ij}^2 + \sum_{i=1}^n b_i^2)\}\big|\mathcal{F}_{i-1,n}\right)}{\sum_{j<i} a_{ij}^2 + \sum_{i=1}^n b_i^2} \xrightarrow{P} 0, \tag{C.3}$$

for every $\epsilon > 0$.

Proof of (C.2). By direct calculation, we have

$$\sum_{i=1}^n \mathrm{E}(U_{in}^2|\mathcal{F}_{i-1,n}) = \sum_{i=1}^n \Big(\sum_{j:j<i} a_{ij}\epsilon_j + b_i\Big)^2$$

$$= \sum_{j<i} a_{ij}^2 + \sum_{i=1}^n b_i^2 + 2\sum_{k<j}\sum_{i:i>j}(a_{ij}a_{ik})\epsilon_j\epsilon_k + 2\sum_{j=1}^n\Big(\sum_{i:i>j} a_{ij}b_i\Big)\epsilon_j.$$

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By assumption, we have

$$\mathrm{E}\,\Big(\sum_{k<j}(\sum_{i:i>j}a_{ij}a_{ik})\epsilon_j\epsilon_k\Big)^2 = \sum_{k<j}(\sum_{i:i>j}a_{ij}a_{ik})^2 = o\big((\sum_{j<i}a_{ij}^2)^2 + \sum_{i=1}^{n}b_i^2\big)$$

and

$$\mathrm{E}\,\Big(\sum_{j=1}^{n}(\sum_{i:i>j}a_{ij}b_i)\epsilon_j\Big)^2 = \sum_{j=1}^{n}(\sum_{i:i>j}a_{ij}b_i)^2 = o\big((\sum_{j<i}a_{ij}^2)^2 + \sum_{i=1}^{n}b_i^2\big)$$

Hence (C.2) holds.

Proof of (C.3). By Markov inequality, it's sufficient to prove

$$\frac{\sum_{i=1}^{n}\mathrm{E}\,\big(U_{in}^4\big|\mathcal{F}_{i-1,n}\big)}{\big(\sum_{j<i}a_{ij}^2 + \sum_{i=1}^{n}b_i^2\big)^2} \xrightarrow{P} 0. \tag{C.4}$$

Since the relevant random variables are all positive, we only need to prove (C.4) converges to 0 in mean. But

$$\sum_{i=1}^{n}\mathrm{E}\,U_{in}^4 = \sum_{i=1}^{n}\mathrm{E}(\sum_{j:j<i}a_{ij}\epsilon_j + b_i)^4 \le 8\sum_{i=1}^{n}\mathrm{E}(\sum_{j:j<i}a_{ij}\epsilon_j)^4 + 8\sum_{i=1}^{n}b_i^4.$$

By assumption, we have

$$\sum_{i=1}^{n}b_i^4 = o\Big(\big(\sum_{j<i}a_{ij}^2 + \sum_{i=1}^{n}b_i^2\big)^2\Big)$$

and

$$\sum_{i=1}^{n}\mathrm{E}(\sum_{j:j<i}a_{ij}\epsilon_j)^4 = \sum_{i=1}^{n}\mathrm{E}\,\big(\sum_{j:j<i}a_{ij}^2 + 2\sum_{j,k:k<j<i}a_{ij}a_{ik}\epsilon_j\epsilon_k\big)^2$$

$$= \sum_{i=1}^{n}\big((\sum_{j:j<i}a_{ij}^2)^2 + 4\,\mathrm{E}(\sum_{j,k:k<j<i}a_{ij}a_{ik}\epsilon_j\epsilon_k)^2\big) = \sum_{i=1}^{n}(\sum_{j:j<i}a_{ij}^4 + 6\sum_{j,k:k<j<i}a_{ij}^2a_{ik}^2)$$

$$= \sum_{j<i}a_{ij}^4 + 6\sum_{k<j<i}a_{ij}^2a_{ik}^2 = o\Big(\big(\sum_{j<i}a_{ij}^2 + \sum_{i=1}^{n}b_i^2\big)^2\Big).$$

Hence (C.3) holds. This completes the proof. □

Proposition 3. *Suppose* (6) *and* (8) *holds. For nonzero constants* $c \in \mathbb{R}$ *and* $t \in \mathbb{R}^r$, *we have*

$$\sum_{k<j}(\sum_{i:i>j}Y_{ij}Y_{ik})^2 + \sum_{j<i}Y_{ij}^4 + \sum_{k<j<i}Y_{ij}^2Y_{ik}^2 + \sum_{j=1}^{n}\big(\sum_{i:i>j}Y_{ij}t^TY_i^*\big)^2 + \sum_{i=1}^{n}(t^TY_i^*)^4 = o_P(1), \tag{C.5}$$

$$c^2\sum_{j<i}Y_{ij}^2 + \sum_{i=1}^{n}(t^TY_i^*)^2 = c^2(1 - \sum_{i=1}^{r}\kappa_i^2) + \|t\|^2 + o_P(1). \tag{C.6}$$

Proof. To prove (C.5), we only need to show that the left hand side of (C.5) converges to 0 in mean. We have

$$\mathrm{E}\sum_{j<k}\big(\sum_{i:i>k}Y_{ij}Y_{ik}\big)^2 = \mathrm{E}\sum_{j<k}\Big(\sum_{i:i>k}Y_{ij}^2Y_{ik}^2 + 2\sum_{i_1,i_2:i_1>i_2>k}Y_{i_1j}Y_{i_1k}Y_{i_2j}Y_{i_2k}\Big)$$

$$= \mathrm{E}\sum_{j<k<i}Y_{ij}^2Y_{ik}^2 + 2\,\mathrm{E}\sum_{j<k<i_2<i_1}Y_{i_1j}Y_{i_1k}Y_{i_2j}Y_{i_2k}.$$

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By (ii) of Lemma 2, we have

$$\mathrm{E} \sum_{j<k<i} a_{ij}^2 a_{ik}^2 \le \frac{n(n-1)(n-2)}{6} \mathrm{E}\, a_{12}^4 \le c^4 \frac{2(3+\Delta)^2(n-2)\operatorname{tr}^2(\Lambda_2^2)}{3n(n-1)\operatorname{tr}^2(\Sigma^2)} = o(1).$$

By direct calculation, the second term equals

$$\mathrm{E} \sum_{j<k<i_2<i_1} a_{i_1 j} a_{i_1 k} a_{i_2 j} a_{i_2 k} = \frac{c^4(n-2)(n-3)\operatorname{tr}(\Lambda_2^4)}{6n(n-1)\operatorname{tr}^2(\Sigma^2)} \le \frac{c^4 \lambda_{r+1}^2(\Sigma)}{6 \operatorname{tr}(\Sigma^2)} = o(1).$$

This proves (C.5).

It follows from (ii) of Lemma 2 that

$$\mathrm{E} \sum_{j<i} a_{ij}^4 = \frac{n(n-1)}{2} \mathrm{E}\, a_{12}^4 \le c^4 \frac{2(3+\Delta)^2 \operatorname{tr}^2(\Lambda_2^2)}{n(n-1)\operatorname{tr}^2(\Sigma^2)} = o(1).$$

Hence (**??**) holds. And (**??**) follows similarly.

Now we prove (**??**). Note that

$$\sum_{j=1}^n \Big(\sum_{i:i>j} a_{ij} b_i \Big)^2 = \sum_{j=1}^n \Big(\sum_{i:i>j} a_{ij}^2 b_i^2 + 2 \sum_{i,k:i>k>j} a_{ij} a_{kj} b_i b_k \Big)$$

$$= \sum_{j<i} a_{ij}^2 b_i^2 + 2 \sum_{j<k<i} a_{ij} a_{kj} b_i b_k.$$

We have

$$\mathrm{E}\, a_{21}^2 b_2^2 = \mathrm{E}\, \mathrm{E}[a_{21}^2 b_2^2 | Z_2] = c^2 \Big(\frac{n^2(n-1)}{2} \operatorname{tr}(\Sigma^2) \Big)^{-1} \mathrm{E}[Z_2^T Q_2 \Lambda_2^2 Q_2^T Z_2 Z_2^T Q_1 t t^T Q_1^T Z_2]$$

$$\le c^2 \Big(\frac{n^2(n-1)}{2} \operatorname{tr}(\Sigma^2) \Big)^{-1} \big(\mathrm{E}(Z_2^T Q_2 \Lambda_2^2 Q_2^T Z_2)^2 \big)^{1/2} \big(\mathrm{E}(Z_2^T Q_1 t t^T Q_1^T Z_2)^2 \big)^{1/2}$$

$$\le (3+\Delta) c^2 \|t\|^2 \frac{2}{n^2(n-1)} \frac{\operatorname{tr}(\Lambda_2^2)}{\operatorname{tr}(\Sigma^2)}.$$

On the other hand, $\mathrm{E}[a_{31} a_{21} b_3 b_2] = \mathrm{E}\big[a_{21} b_2 \mathrm{E}[a_{31} b_3 | Z_1, Z_2] \big] = 0$. Thus,

$$\mathrm{E} \Big(\sum_{j<i} a_{ij}^2 b_i^2 + 2 \sum_{j<k<i} a_{ij} a_{kj} b_i b_k \Big) \le \frac{(3+\Delta) c^2 \|t\|^2}{n} \frac{\operatorname{tr}(\Lambda_2^2)}{\operatorname{tr}(\Sigma^2)} = o(1).$$

Finally, (**??**) follows from

$$\mathrm{E} \Big(\sum_{i=1}^n b_i^4 \Big) = n \mathrm{E}(b_1^4) = \frac{1}{n} \mathrm{E}(Z_1^T Q_1 t t^T Q_1^T Z_1)^2 \le \frac{(3+\Delta)\|t\|^4}{n} = o(1).$$

First we deal with (C.6). For the first term of (C.6), we have

$$\mathrm{E} \Big(\sum_{j<i} Y_{ij}^2 \Big) = \frac{\operatorname{tr}(\Lambda_2^2)}{\operatorname{tr}(\Sigma^2)} = \Big(1 - \sum_{i=1}^r \kappa_i^2 \Big) + o(1).$$

We need to show $\operatorname{Var}\big(\sum_{j<i} Y_{ij}^2 \big) = o(1)$. Write $\big(\sum_{j<i} Y_{ij}^2 \big)^2 = \big(\sum_{j<i} Y_{ij}^2 \big) \big(\sum_{l<k} Y_{kl}^2 \big)$. Collect terms according to $\operatorname{card}(\{i,j\} \cap \{k,l\}) = 0, 1, 2$, we have

$$\Big(\sum_{j<i} Y_{ij}^2 \Big)^2 = \sum_{j<i, l<k: \{i,j\} \cap \{k,l\} = \phi} Y_{ij}^2 Y_{kl}^2 + 2 \sum_{j<i<k} \Big(Y_{ij}^2 Y_{ki}^2 + Y_{ij}^2 Y_{kj}^2 + Y_{kj}^2 Y_{ki}^2 \Big) + \sum_{j<i} Y_{ij}^4.$$

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There are $n(n-1)(n-2)(n-3)/4$, $n(n-1)(n-2)/6$ and $n(n-1)/2$ terms in each sum, respectively. For distinct i,j,k,l,

$$\mathrm{E}\, a_{ij}^2 a_{kl}^2 = (\mathrm{E}\, a_{12}^2)^2 = c^4 \frac{4\,\mathrm{tr}^2(\Lambda_2^2)}{n^2(n-1)^2\,\mathrm{tr}^2(\Sigma^2)}.$$

Otherwise, by (ii) of Lemma 2,

$$\mathrm{E}\, a_{ij}^2 a_{kl}^2 \leq \mathrm{E}\, a_{12}^4 \leq c^4 \frac{4(3+\Delta)^2\,\mathrm{tr}^2(\Lambda_2^2)}{n^2(n-1)^2\,\mathrm{tr}^2(\Sigma^2)}.$$

It follows that

$$\mathrm{E}\,\Big(\sum_{j<i} a_{ij}^2\Big)^2 = c^4\frac{(n-2)(n-3)}{n(n-1)}\frac{\mathrm{tr}^2(\Lambda_2^2)}{\mathrm{tr}^2(\Sigma^2)} + O(1)\Big(n(n-1)(n-2) + \frac{n(n-1)}{2}\Big)c^4\frac{4(3+\Delta)^2\,\mathrm{tr}^2(\Lambda_2^2)}{n^2(n-1)^2\,\mathrm{tr}^2(\Sigma^2)}$$

$$= c^4(1-\sum_{i=1}^{r}\kappa_i^2)^2 + o(1).$$

Hence $\mathrm{Var}\,\big(\sum_{j<i} a_{ij}^2\big) = \mathrm{E}\,\big(\sum_{j<i} a_{ij}^2\big)^2 - \big(\mathrm{E}\sum_{j<i} a_{ij}^2\big)^2 = o(1)$. Thus,

$$\sum_{j<i} a_{ij}^2 = \mathrm{E}\,\big(\sum_{j<i} a_{ij}^2\big) + o_P(1) = c^2(1-\sum_{i=1}^{r}\kappa_i^2) + o_P(1). \tag{C.7}$$

Note that $\mathrm{E}\,\big(\sum_{i=1}^{n} b_i^2\big) = \|t\|^2$ and

$$\mathrm{Var}\,\Big(\sum_{i=1}^{n} b_i^2\Big) = \frac{1}{n}\,\mathrm{Var}(Z_1^T Q_1 t t^T Q_1^T Z_1) \leq \frac{1}{n}(2+\Delta)\|t\|^4,$$

where the last inequality follows from (i) of Lemma 2. Hence

$$\sum_{i=1}^{n} b_i^2 = \|t\|^2 + o_P(1). \tag{C.8}$$

Then (C.6) follows from (C.7) and (C.8).

$\square$

Proof of Theorem 2. By a standard subsequence argument, we only need to prove

$$\rho\Big(\mathcal{L}\Big(\frac{T_2(\epsilon_1 X_1,\ldots,\epsilon_i X_i,\ldots,\epsilon_n X_n)}{\sqrt{\sum_{1\leq j<i\leq n}(X_i^T X_j)^2}}\Big|X_1,\ldots,X_n\Big), N(0,1)\Big) \xrightarrow{a.s.} 0 \tag{C.9}$$

along a subsequence. But there exists a subsequence $\{n(k)\}$ along which (C.5), (**??**) and (**??**) holds almost surely. By Proposition 2, we have

$$\mathcal{L}\Big(\frac{T_2(\epsilon_1 X_1,\ldots,\epsilon_i X_i,\ldots,\epsilon_n X_n)}{\sqrt{\sum_{1\leq j<i\leq n}(X_i^T X_j)^2}}\Big|X_1,\ldots,X_n\Big) \xrightarrow{\mathcal{L}} N(0,1)$$

almost surely along $\{n(k)\}$, which means that (C.9) holds along $\{n(k)\}$.

$\square$

Proposition 4.

$$\sum_{j<i} Z_i^T \Gamma^T \Gamma Z_j \epsilon_i \epsilon_j$$

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Proof. We would like to prove

$$\rho\Big(\mathcal{L}\Big(\begin{pmatrix} n^{-1/2}Q_1^T \sum_{i=1}^n Z_i\epsilon_i \\ \big(\frac{n(n-1)}{2}\operatorname{tr}(\Sigma^2)\big)^{-1/2}\sum_{j<i} Z_i^T Q_2\Lambda_2 Q_2^T Z_j\epsilon_i\epsilon_j \end{pmatrix}\Big|Z_1,\ldots,Z_n\Big), N_{r+1}\big(0_{r+1},\operatorname{diag}(I_r, 1-\sum_{i=1}^r \kappa_i^2))\big)\Big) \xrightarrow{P} 0.$$

It suffices to show that for every nonzero constants $t \in \mathbb{R}^r$ and $c \in \mathbb{R}$,

$$\rho\Big(\mathcal{L}\Big(n^{-1/2}t^T Q_1^T \sum_{i=1}^n Z_i\epsilon_i + c\big(\frac{n(n-1)}{2}\operatorname{tr}(\Sigma^2)\big)^{-1/2}\sum_{j<i} Z_i^T Q_2\Lambda_2 Q_2^T Z_j\epsilon_i\epsilon_j\Big|Z_1,\ldots,Z_n\Big),$$

$$N\big(0, \|t\|^2 + c^2(1-\sum_{i=1}^r \kappa_i^2))\big)\Big) \xrightarrow{P} 0.$$

To prove this, we would like to apply Proposition 2. Let

$$a_{ij} = c\big(\frac{n(n-1)}{2}\operatorname{tr}(\Sigma^2)\big)^{-1/2} Z_i^T Q_2\Lambda_2 Q_2^T Z_j, \quad b_i = n^{-1/2}t^T Q_1^T Z_i.$$

We prove (C.1) holds in probability.

$\square$

Proof of Theorem 3

Proof.

$$\sum_{j<i} X_i^T X_j\epsilon_i\epsilon_j$$

$$= \sum_{j<i} Z_i^T \Gamma^T\Gamma Z_j\epsilon_i\epsilon_j + \sum_{j<i}\mu^T\Gamma Z_i\epsilon_i\epsilon_j + \sum_{j<i}\mu^T\Gamma Z_j\epsilon_i\epsilon_j + \mu^T\mu\sum_{j<i}\epsilon_i\epsilon_j$$

$$\stackrel{def}{=} C_1 + C_2 + C_3 + C_4.$$

Term C_4 plays a major role. Note that

$$C_4 = \frac{n}{2}\mu^T\mu\Big(\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_i\Big)^2 - 1\Big),$$

which does not depend on $X_1,\ldots,X_n$. By central limit theorem, we have

$$\rho\Big(\mathcal{L}\Big(\frac{C_4}{\frac{n}{2}\mu^T\mu}\Big|X_1,\ldots,X_n\Big), \chi_1^2 - 1\Big) \xrightarrow{a.s.} 0. \tag{C.10}$$

Now we show that

$$\operatorname{E}\Big(\frac{C_i}{\frac{n}{2}\mu^T\mu}\Big)^2 \to 0, \quad i = 1,2,3. \tag{C.11}$$

By direct calculation, we have

$$\operatorname{E}(C_1^2) = \operatorname{E}\operatorname{E}(C_1^2|X_1,\ldots,X_n) = \sum_{j<i}\operatorname{E}\big(Z_i^T\Gamma^T\Gamma Z_j\big)^2 = \frac{n(n-1)}{2}\operatorname{tr}\Sigma^2,$$

and

$$\operatorname{E}(C_2^2) = \operatorname{E}(C_3^2) = \frac{n(n-1)}{2}\mu^T\Sigma\mu \le \frac{n(n-1)}{2}\sqrt{\operatorname{tr}\Sigma^2}\mu^T\mu.$$

Thus (C.11) follows by the assumption (10). By Markov's inequality, we have

$$\operatorname{E}\Big(\Big(\frac{C_i}{\frac{n}{2}\mu^T\mu}\Big)^2\Big|X_1,\ldots,X_n\Big) \xrightarrow{P} 0, \quad i = 1,2,3.$$

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This, combined with (C.10) and a standard subsequence argument, yields

$$\rho\Big(\mathcal{L}\Big(\frac{T(\epsilon_1 X_1, \ldots, \epsilon_i X_i, \ldots, \epsilon_n X_n)}{\frac{n}{2}\mu^T\mu}\Big|X_1, \ldots, X_n\Big), \chi_1^2 - 1\Big) \xrightarrow{P} 0 \tag{C.12}$$

Then the theorem follows by (C.12), Lemma **??**, the assumption (10) and Slutsky's theorem. $\square$

Proof of Corollaries 1 and 2

Proof. For every subsequence, there is a further subsequence along which

$$\rho\Big(\mathcal{L}\Big(\frac{T_2(\epsilon_1 X_1, \ldots, \epsilon_i X_i, \ldots, \epsilon_n X_n)}{\sqrt{\sum_{1 \le j < i \le n}\left(X_i^T X_j\right)^2}}\Big|X_1, \ldots, X_n\Big), N(0, 1)\Big)$$

tends to 0 almost surely. By the property of convergence in law, $\xi_\alpha^* \to \Phi^{-1}(1 - \alpha)$ almost surely along this subsequence. That is, For every subsequence, there is a further subsequence along which $\xi_\alpha^* \to \Phi^{-1}(1 - \alpha)$ almost surely. This is equivalent to $\xi_\alpha^* \xrightarrow{P} \Phi^{-1}(1 - \alpha)$. The proof of Corollary 2 is similar. $\square$

Proof of Theorem 4

Proof. Note that

$$\Pr\Big(\frac{T(X_1, \ldots, X_n)}{\sqrt{\sum_{1 \le j < i \le n}\left(X_i^T X_j\right)^2}} > \xi_\alpha^*\Big)$$

$$= \Pr\Big(\frac{T(X_1, \ldots, X_n) - \frac{n(n-1)}{2}\mu^T\mu}{\sqrt{\sum_{1 \le j < i \le n}\left(X_i^T X_j\right)^2}} > \xi_\alpha^* - \frac{\frac{n(n-1)}{2}\mu^T\mu}{\sqrt{\sum_{1 \le j < i \le n}\left(X_i^T X_j\right)^2}}\Big).$$

If (9) holds, by Lemma **??**, we have

$$\frac{\sum_{1 \le j < i \le n}(X_i^T X_j)^2}{\frac{n(n-1)}{2}\operatorname{tr}\Sigma^2} \xrightarrow{P} 1.$$

By Corollary 1, we have $\xi_\alpha^* \xrightarrow{P} \Phi(1 - \alpha)$. Thus,

$$\Pr\Big(\frac{T(X_1, \ldots, X_n)}{\sqrt{\sum_{1 \le j < i \le n}\left(X_i^T X_j\right)^2}} > \xi_\alpha^*\Big)$$

$$= \Pr\Big(\frac{T(X_1, \ldots, X_n) - \frac{n(n-1)}{2}\mu^T\mu}{\sqrt{\frac{n(n-1)}{2}\operatorname{tr}\Sigma^2}} - \frac{\sqrt{\sum_{1 \le j < i \le n}\left(X_i^T X_j\right)^2}}{\sqrt{\frac{n(n-1)}{2}\operatorname{tr}\Sigma^2}}\xi_\alpha^* > -\frac{\sqrt{n(n-1)}\mu^T\mu}{\sqrt{2\operatorname{tr}\Sigma^2}}\Big)$$

$$= \Pr\Big(N(0, 1) - \Phi(1 - \alpha) > -\frac{\sqrt{n(n-1)}\mu^T\mu}{\sqrt{2\operatorname{tr}\Sigma^2}}\Big) + o(1)$$

$$= \Phi\big(-\Phi(1 - \alpha) + \frac{\sqrt{n(n-1)}\mu^T\mu}{\sqrt{2\operatorname{tr}\Sigma^2}}\big) + o(1),$$

where the last two equality holds by Lemma **??**, Slutsky's theorem and Lemma 1.

If (10) holds, by Lemma **??**, we have

$$\frac{\sum_{1 \le j < i \le n}(X_i^T X_j)^2}{\frac{n(n-1)}{2}(\mu^T\mu)^2} \xrightarrow{P} 1.$$

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Thus

$$\frac{T(X_1,\ldots,X_n) - \frac{n(n-1)}{2}\mu^T\mu}{\sqrt{\sum_{1\leq j<i\leq n}\left(X_i^T X_j\right)^2}}$$

$$=\frac{T(X_1,\ldots,X_n) - \frac{n(n-1)}{2}\mu^T\mu}{\sqrt{\frac{n(n-1)}{2}\operatorname{tr}\Sigma^2}}\frac{\sqrt{\frac{n(n-1)}{2}\operatorname{tr}\Sigma^2}}{\sqrt{\frac{n(n-1)}{2}(\mu^T\mu)^2}}\frac{\sqrt{\frac{n(n-1)}{2}(\mu^T\mu)^2}}{\sqrt{\sum_{1\leq j<i\leq n}\left(X_i^T X_j\right)^2}}\xrightarrow{P} 0.$$

By Corollary 2, $\xi_\alpha^* \xrightarrow{P} \frac{\sqrt{2}}{2}\left(\left(\Phi^{-1}(1-\frac{\alpha}{2})\right)^2 - 1\right)$. And

$$\frac{\frac{n(n-1)}{2}\mu^T\mu}{\sqrt{\sum_{1\leq j<i\leq n}\left(X_i^T X_j\right)^2}} = \sqrt{\frac{n(n-1)}{2}}\frac{\sqrt{\frac{n(n-1)}{2}(\mu^T\mu)^2}}{\sqrt{\sum_{1\leq j<i\leq n}\left(X_i^T X_j\right)^2}} \xrightarrow{P} +\infty.$$

As a result,

$$\Pr\left(\frac{T(X_1,\ldots,X_n)}{\sqrt{\sum_{1\leq j<i\leq n}\left(X_i^T X_j\right)^2}} > \xi_\alpha^*\right) \to 1.$$

$\square$

Proof of Theorem 5

Proof. Note that

$$\Pr\left(\frac{T(X_1,\ldots,X_n)}{\sqrt{\sum_{1\leq j<i\leq n}\left(X_i^T X_j\right)^2}} > \xi_\alpha^*\right)$$

$$=\Pr\left(\frac{T(X_1,\ldots,X_n) - \frac{n(n-1)}{2}\mu^T\mu}{\sqrt{(n-1)^2 n\mu^T\Sigma\mu}} > \frac{\sqrt{\sum_{1\leq j<i\leq n}\left(X_i^T X_j\right)^2}}{\sqrt{(n-1)^2 n\mu^T\Sigma\mu}}\xi_\alpha^* - \frac{\frac{n(n-1)}{2}\mu^T\mu}{\sqrt{(n-1)^2 n\mu^T\Sigma\mu}}\right). \qquad (C.13)$$

If (9) holds, the theorem follows by Lemma **??** and the fact that if (9) holds, the coefficient of ξ_α^* in (C.13) tends to 0.

If (10) holds, the theorem follows by noting that

$$\Pr\left(\frac{T(X_1,\ldots,X_n)}{\sqrt{\sum_{1\leq j<i\leq n}\left(X_i^T X_j\right)^2}} > \xi_\alpha^*\right)$$

$$=\Pr\left(\frac{T(X_1,\ldots,X_n) - \frac{n(n-1)}{2}\mu^T\mu}{\sqrt{(n-1)^2 n\mu^T\Sigma\mu}} > -(1+o_P(1))\frac{\frac{n(n-1)}{2}\mu^T\mu}{\sqrt{(n-1)^2 n\mu^T\Sigma\mu}}\right).$$

$\square$

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