

A Bayesian-motivated test for linear model in high-dimensional setting

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1 Introduction

The proposed test is the limit of Bayes factors.

Fixed design

Suppose we would like to test the hypotheses:

$$\begin{aligned}\mathcal{H}_0 : \mathbf{y} &= \mathbf{X}_a \boldsymbol{\beta}_a + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}_n(0, \phi^{-1} \mathbf{I}_n), \\ \mathcal{H}_1 : \mathbf{y} &= \mathbf{X}_a \boldsymbol{\beta}_a + \mathbf{X}_b \boldsymbol{\beta}_b + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}_n(0, \phi^{-1} \mathbf{I}_n).\end{aligned}$$

Here $\boldsymbol{\beta}_a$ is q dimensional and $\boldsymbol{\beta}_b$ is p dimensional. We assume that as n tends to infinity, q is fixed while $p/n \rightarrow \infty$. This assumption is reasonable. We assume \mathbf{X}_a has full column rank and \mathbf{X}_b has full row rank. In practice, p_0 is often 1 and \mathbf{X}_a is $\mathbf{1}_n$.

As Goeman et al. (2006) pointed out, if $\boldsymbol{\beta}_b \neq 0$ but $\mathbf{X}_b \boldsymbol{\beta}_b = 0$, no test has any power. Goeman et al. (2006) used Bayesian method. Their idea is to choose an ‘unbiased’ distribution of $\boldsymbol{\beta}_b$. As they noticed, their test has negligible power for many alternatives, and is not unbiased.

The following proposition implies that there is no nontrivial unbiased test.

Proposition 1. *Suppose $\mathbf{y} \sim \mathcal{N}_n(\mu, \phi^{-1} \mathbf{I}_n)$. We test $H_0 : \mu = \mathbf{X}_a \boldsymbol{\beta}_a, \boldsymbol{\beta}_a \in \mathbb{R}^q$ versus $H_1 : \mu \in \mathbb{R}^n$, where \mathbf{X}_a is an $n \times q$ matrix with full column rank, $q < n$. Let $\varphi(\mathbf{y})$ be a test function, that is, a Borel measurable function, $0 \leq \varphi(\mathbf{y}) \leq 1$. If $\int \varphi(\mathbf{y}) \mathcal{N}_n(\mathbf{X}_a \boldsymbol{\beta}_a, \phi^{-1} \mathbf{I}_n)(d\mathbf{y}) = \alpha$ for $\boldsymbol{\beta}_a \in \mathbb{R}^q$, $\phi > 0$ and $\int \varphi(\mathbf{y}) \mathcal{N}_n(\mu, \phi^{-1} \mathbf{I}_n)(d\mathbf{y}) \geq \alpha$ for $\mu \in \mathbb{R}^n$, $\phi > 0$, then $\varphi(\mathbf{y}) = \alpha$, a.s.*

So we can not find a universally good test. Instead, we would like to find a test with good average behaviour. So Bayesian methods are natural choices in this case.

Bayes hypothesis testing use the Bayes factor.

$$B_{10} = \frac{\int f_1(\mathbf{y} | \boldsymbol{\beta}_b, \boldsymbol{\beta}_a, \phi) \pi_1(\boldsymbol{\beta}_b, \boldsymbol{\beta}_a, \phi) d\boldsymbol{\beta}_b d\boldsymbol{\beta}_a d\phi}{\int f_0(\mathbf{y} | \boldsymbol{\beta}_a, \phi) \pi_0(\boldsymbol{\beta}_a, \phi) d\boldsymbol{\beta}_a d\phi}.$$

There have been several extensions of g -priors to $p > n$ case: Maruyama and George (2011), Shang and Clayton (2011).

Under H_0 , we impose the reference prior $\pi_0(\boldsymbol{\beta}_a, \phi) = c/\phi$. Note that under H_1 , the posterior corresponding to the reference prior is proper if and only if $\text{Rank}(\mathbf{X}_a, \mathbf{X}_b) = q + p$ and $n > q + p$. That is, the minimal training sample size is $q + p + 1$. So we cannot impose the reference prior under H_1 provided $q + p \geq n$. We temporarily impose the conditional prior $\boldsymbol{\beta}_b|\boldsymbol{\beta}_a, \phi \sim \mathcal{N}_p(0, \kappa^{-1}\phi^{-1}\mathbf{I}_p)$. There are extensive literature consider the choice of κ . Kass and Wasserman (1995) choose κ such that the amount of information about the parameter equal to the amount of information contained in one observation. Thus, under H_1 , we put the prior

$$\pi_1(\boldsymbol{\beta}_b|\boldsymbol{\beta}_a, \phi) = \mathcal{N}_p\left(0, \frac{1}{\kappa\phi}\mathbf{I}_p\right)(\boldsymbol{\beta}_b), \quad \pi_1(\boldsymbol{\beta}_a, \phi) = \frac{c}{\phi}.$$

$$\begin{aligned} m_0(\mathbf{y}; \kappa, \tau) &:= \int f_0^\tau(\mathbf{y}|\boldsymbol{\beta}_a, \phi)\pi_0(\boldsymbol{\beta}_a, \phi)d\boldsymbol{\beta}_ad\phi \\ &= \frac{c_0\Gamma\left(\frac{\tau n - q}{2}\right)}{\pi^{\frac{\tau n - q}{2}}\tau^{\frac{\tau n}{2}}|\mathbf{X}_a^\top\mathbf{X}_a|^{\frac{1}{2}}\|(\mathbf{I}_n - \mathbf{P}_a)\mathbf{y}\|^{\tau n - q}}. \end{aligned}$$

$$\begin{aligned} m_1(\mathbf{y}; \kappa, \tau) &:= \int f_1^\tau(\mathbf{y}|\boldsymbol{\beta}_b, \boldsymbol{\beta}_a, \phi)\pi_1(\boldsymbol{\beta}_b|\boldsymbol{\beta}_a, \phi)\pi_1(\boldsymbol{\beta}_a, \phi)d\boldsymbol{\beta}_ad\boldsymbol{\beta}_bd\phi \\ &= \frac{c_1\kappa^{\frac{p}{2}}\Gamma\left(\frac{\tau n - q}{2}\right)}{\pi^{\frac{\tau n - q}{2}}\tau^{\frac{\tau n + p}{2}}|\mathbf{X}_a^\top\mathbf{X}_a|^{\frac{1}{2}}|\mathbf{X}_b^{*\top}\mathbf{X}_b^* + \frac{\kappa}{\tau}\mathbf{I}_p|^{\frac{1}{2}}}\frac{1}{[\mathbf{y}^{*\top}\mathbf{y}^* - \mathbf{y}^{*\top}\mathbf{X}_b^*(\mathbf{X}_b^{*\top}\mathbf{X}_b^* + \frac{\kappa}{\tau}\mathbf{I}_p)^{-1}\mathbf{X}_b^{*\top}\mathbf{y}^*]^{\frac{\tau n - q}{2}}}. \\ \frac{m_1(\mathbf{y}; \kappa, \tau)}{m_0(\mathbf{y}; \kappa, \tau)} &= \frac{c_1\kappa^{\frac{p}{2}}}{c_0\tau^{\frac{p}{2}}|\mathbf{X}_b^{*\top}\mathbf{X}_b^* + \frac{\kappa}{\tau}\mathbf{I}_p|^{\frac{1}{2}}}\left(\frac{\mathbf{y}^{*\top}\mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^* - \mathbf{y}^{*\top}\mathbf{X}_b^*(\mathbf{X}_b^{*\top}\mathbf{X}_b^* + \frac{\kappa}{\tau}\mathbf{I}_p)^{-1}\mathbf{X}_b^{*\top}\mathbf{y}^*}\right)^{\frac{\tau n - q}{2}} \end{aligned}$$

It is straightforward to show that the Bayes factor associated with these priors is

$$\begin{aligned} B_{10}^\kappa &= \frac{\kappa^{p/2}}{|\mathbf{X}_b^\top(\mathbf{I}_n - \mathbf{P}_a)\mathbf{X}_b + \kappa\mathbf{I}_p|^{1/2}} \\ &\quad \left(\frac{\mathbf{y}^\top(\mathbf{I}_n - \mathbf{P}_a)\mathbf{y}}{\mathbf{y}^\top(\mathbf{I}_n - \mathbf{P}_a)\mathbf{y} - \mathbf{y}^\top(\mathbf{I}_n - \mathbf{P}_a)\mathbf{X}_b(\mathbf{X}_b^\top(\mathbf{I} - \mathbf{P}_a)\mathbf{X}_b + \kappa\mathbf{I}_p)^{-1}\mathbf{X}_b^\top(\mathbf{I}_n - \mathbf{P}_a)\mathbf{y}}\right)^{(n-q)/2}. \end{aligned}$$

Thus,

$$\begin{aligned} 2\log B_{10}^\kappa &= p\log \kappa - \log |\mathbf{X}_b^\top(\mathbf{I}_n - \mathbf{P}_a)\mathbf{X}_b + \kappa\mathbf{I}_p| \\ &\quad - (n - q)\log \left(1 - \frac{\mathbf{y}^\top(\mathbf{I}_n - \mathbf{P}_a)\mathbf{X}_b(\mathbf{X}_b^\top(\mathbf{I} - \mathbf{P}_a)\mathbf{X}_b + \kappa\mathbf{I}_p)^{-1}\mathbf{X}_b^\top(\mathbf{I}_n - \mathbf{P}_a)\mathbf{y}}{\mathbf{y}^\top(\mathbf{I}_n - \mathbf{P}_a)\mathbf{y}}\right). \end{aligned}$$

Denote by $\mathbf{I}_n - \mathbf{P}_a = \tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top$ the rank decomposition of $\mathbf{I}_n - \mathbf{P}_a$, where $\tilde{\mathbf{U}}_a$ is a $n \times (n - q)$ column orthogonal matrix. Let $\mathbf{X}_b^* = \tilde{\mathbf{U}}_a^\top\mathbf{X}_b$, $\mathbf{y}^* = \tilde{\mathbf{U}}_a^\top\mathbf{y}$. Let γ_i be the i th largest eigenvalue of $\mathbf{X}_b^*\mathbf{X}_b^{*\top}$, $i = 1, \dots, n - q$. Denote by $\mathbf{X}_b^* = \mathbf{U}_b^*\mathbf{D}_b^*\mathbf{V}_b^{*\top}$ the singular value decomposition of \mathbf{X}_b^* , where \mathbf{U}_b^* , \mathbf{V}_b^* are $(n - q) \times (n - q)$ and $p \times (n - q)$ column orthogonal matrices, respectively, and $\mathbf{D}_b^* = \text{diag}(\sqrt{\gamma_1}, \dots, \sqrt{\gamma_{n-q}})$. Then

$$\begin{aligned}
2 \log B_{10}^\kappa &= p \log \kappa - \sum_{i=1}^{n-q} \log(\gamma_i + \kappa) - (p - (n - q)) \log \kappa \\
&\quad - (n - q) \log \left(1 - \frac{\mathbf{y}^{*\top} \mathbf{X}_b^* (\mathbf{X}_b^{*\top} \mathbf{X}_b^* + \kappa \mathbf{I}_p)^{-1} \mathbf{X}_b^{*\top} \mathbf{y}^*}{\mathbf{y}^{*\top} \mathbf{y}^*} \right) \\
&= - \sum_{i=1}^{n-q} \log(\gamma_i + \kappa) + (n - q) \log \left(\frac{\mathbf{y}^{*\top} \mathbf{y}^*}{\mathbf{y}^{*\top} \mathbf{U}_b^* \left[\frac{1}{\kappa} (\mathbf{I}_{n-q} - \mathbf{D}_b^* (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1} \mathbf{D}_b^*) \right] \mathbf{U}_b^{*\top} \mathbf{y}^*} \right) \\
&= (n - q) \log \kappa - \sum_{i=1}^{n-q} \log(\gamma_i + \kappa) - (n - q) \log \left(1 - \frac{\mathbf{y}^{*\top} \mathbf{U}_b^* \mathbf{D}_b^* (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1} \mathbf{D}_b^* \mathbf{U}_b^{*\top} \mathbf{y}^*}{\mathbf{y}^{*\top} \mathbf{y}^*} \right).
\end{aligned}$$

The main part of $2 \log B_{10}^\kappa$ is

$$T_n^\kappa = \frac{\mathbf{y}^{*\top} \mathbf{U}_b^* \mathbf{D}_b^* (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1} \mathbf{D}_b^* \mathbf{U}_b^{*\top} \mathbf{y}^*}{\mathbf{y}^{*\top} \mathbf{y}^*}.$$

A large value of T_n^κ supports the alternative hypothesis. Under the null hypothesis,

$$\mathbb{E} T_n^\kappa = \frac{1}{n - q} \text{tr} (\mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1}).$$

Under the alternative hypothesis, consider $\beta_b = c \beta_b^\dagger$ where $\beta_b^\dagger \neq 0$ is a fixed direction and $c > 0$.

As $c \rightarrow \infty$,

$$T_n^\kappa \rightarrow \frac{\beta_b^{\dagger\top} \mathbf{V}_b^* \mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1} \mathbf{D}_b^{*2} \mathbf{V}_b^{*\top} \beta_b^\dagger}{\beta_b^{\dagger\top} \mathbf{V}_b^* \mathbf{D}_b^{*2} \mathbf{V}_b^{*\top} \beta_b^\dagger}.$$

We say T_n^κ is consistent along the direction β_b^\dagger if

$$\frac{\beta_b^{\dagger\top} \mathbf{V}_b^* \mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1} \mathbf{D}_b^{*2} \mathbf{V}_b^{*\top} \beta_b^\dagger}{\beta_b^{\dagger\top} \mathbf{V}_b^* \mathbf{D}_b^{*2} \mathbf{V}_b^{*\top} \beta_b^\dagger} > \frac{1}{n - q} \text{tr} (\mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1}),$$

or equivalently

$$\beta_b^{\dagger\top} \mathbf{V}_b^* \left[\mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1} \mathbf{D}_b^{*2} - \frac{1}{n - q} \text{tr} (\mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1}) \mathbf{D}_b^{*2} \right] \mathbf{V}_b^{*\top} \beta_b^\dagger > 0.$$

Let k_κ be the number of positive eigenvalues of

$$\mathbf{V}_b^* \left[\mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1} \mathbf{D}_b^{*2} - \frac{1}{n - q} \text{tr} (\mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1}) \mathbf{D}_b^{*2} \right] \mathbf{V}_b^{*\top}.$$

Let \mathcal{S}_κ be the linear space spanned by the first k_κ columns of \mathbf{V}_b^* . Denote by \mathcal{S}_κ^\perp the orthogonal complement space of \mathcal{S}_κ . We have $\mathbb{R}^p = \mathcal{S}_\kappa \oplus \mathcal{S}_\kappa^\perp$. If $\beta_b^\dagger \in \mathcal{S}_\kappa$,

$$\mathbf{V}_b^* \left[\mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1} \mathbf{D}_b^{*2} - \frac{1}{n - q} \text{tr} (\mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1}) \mathbf{D}_b^{*2} \right] \mathbf{V}_b^{*\top} > 0.$$

On the other hand, if $\beta_b^\dagger \in \mathcal{S}_\kappa^\perp$,

$$\mathbf{V}_b^* \left[\mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1} \mathbf{D}_b^{*2} - \frac{1}{n-q} \text{tr} (\mathbf{D}_b^{*2} (\mathbf{D}_b^{*2} + \kappa \mathbf{I}_{n-q})^{-1}) \mathbf{D}_b^{*2} \right] \mathbf{V}_b^{*\top} \leq 0.$$

We would like to choose a hyperparameter κ which consists the most consistent directions. To achieve this, we maximize k_κ with respect to κ .

Proposition 2. *For $\kappa_2 > \kappa_1 > 0$, we have $k_{\kappa_1} \geq k_{\kappa_2}$. That is, k_κ ($\kappa > 0$) is decreasing in κ .*

The proposition implies that we should put κ as small as possible. This motivates us to consider $B_{10}^0 = \lim_{\kappa \rightarrow 0} B_{10}^\kappa$. It is straightforward to show that

$$2 \log B_{10}^0 = - \sum_{i=1}^{n-q} \log(\gamma_i) + (n-q) \log \left(\frac{\mathbf{y}^{*\top} \mathbf{y}^*}{\mathbf{y}^{*\top} (\mathbf{X}_b^* \mathbf{X}_b^{*\top})^{-1} \mathbf{y}^*} \right).$$

B_{10}^0 can be regarded as the Bayes factor with respect to noninformative prior.

Define

$$T_n = \frac{\mathbf{y}^{*\top} (\mathbf{X}_b^* \mathbf{X}_b^{*\top})^{-1} \mathbf{y}^*}{\mathbf{y}^{*\top} \mathbf{y}^*}.$$

Then we reject the null hypothesis if T_n is small. It can be seen that under the null hypothesis,

$$T_n \sim \frac{\sum_{i=1}^{n-q} \gamma_i^{-1} Z_i^2}{\sum_{i=1}^{n-q} Z_i^2},$$

where γ_i is the i th eigenvalue of $\mathbf{X}_b^* \mathbf{X}_b^{*\top}$, $i = 1, \dots, n-q$, and Z_1, \dots, Z_{n-q} are iid $\mathcal{N}(0, 1)$ random variables.

2 Asymptotic results

Let $\boldsymbol{\varepsilon} = (\epsilon_1, \dots, \epsilon_n)^\top$, where ϵ_i 's are iid random variable. Denote $\mu_k = \mathbb{E} \epsilon_1^k$. Then $\mu_1 = 0$, $\mu_2 = \phi^{-1}$.

Assumption 1. *Suppose*

Lemma 1. *If $\phi^2 \mu_4 = o(n-q)$,*

$$\mathbf{y}^{*\top} \mathbf{y}^* = (1 + o_P(1)) \left(\beta_b^\top \mathbf{X}_b^\top (\mathbf{I}_n - \mathbf{P}_a) \mathbf{X}_b \beta_b + \phi^{-1} (n-q) \right).$$

Proof.

$$\mathbf{y}^{*\top} \mathbf{y}^* = \beta_b^\top \mathbf{X}_b^\top (\mathbf{I}_n - \mathbf{P}_a) \mathbf{X}_b \beta_b + 2\boldsymbol{\varepsilon}^\top (\mathbf{I}_n - \mathbf{P}_a) \mathbf{X}_b \beta_b + \boldsymbol{\varepsilon}^\top (\mathbf{I}_n - \mathbf{P}_a) \boldsymbol{\varepsilon}.$$

$$\mathbb{E} \left(\mathbf{y}^{*\top} \mathbf{y}^* \right) = \beta_b^\top \mathbf{X}_b^\top (\mathbf{I}_n - \mathbf{P}_a) \mathbf{X}_b \beta_b + \phi^{-1} (n-q).$$

$$\text{Var} \left(\mathbf{y}^{*\top} \mathbf{y}^* \right) \leq 2 \text{Var} \left(2\boldsymbol{\varepsilon}^\top (\mathbf{I}_n - \mathbf{P}_a) \mathbf{X}_b \beta_b \right) + 2 \text{Var} \left(\boldsymbol{\varepsilon}^\top (\mathbf{I}_n - \mathbf{P}_a) \boldsymbol{\varepsilon} \right)$$

From (i) of (Chen et al., 2010, Proposition A.1),

$$\text{Var}\left(\varepsilon^\top (\mathbf{I}_n - \mathbf{P}_a) \varepsilon\right) = \phi^{-2} \left((\phi^2 \mu_4 - 3) \sum_{i=1}^n ((\mathbf{I}_n - \mathbf{P}_a)_{i,i})^2 + 2(n-q) \right) \leq \phi^{-2} (2 + \phi^2 \mu_4) (n-q).$$

Then

$$\text{Var}\left(\mathbf{y}^{*\top} \mathbf{y}^*\right) \leq 8\phi^{-1} \boldsymbol{\beta}_b^\top \mathbf{X}_b^\top (\mathbf{I}_n - \mathbf{P}_a) \mathbf{X}_b \boldsymbol{\beta}_b + 2\phi^{-2} (2 + \phi^2 \mu_4) (n-q)$$

Thus, if $\phi^2 \mu_4 = o(n-q)$, we have

$$\frac{\text{Var}\left(\mathbf{y}^{*\top} \mathbf{y}^*\right)}{(\mathbb{E}(\mathbf{y}^{*\top} \mathbf{y}^*))^2} \rightarrow 0,$$

and consequently $\mathbf{y}^{*\top} \mathbf{y}^* = (1 + o_P(1)) \mathbb{E}(\mathbf{y}^{*\top} \mathbf{y}^*)$.

□

Note that under the normality, $T_n - \text{tr}((\mathbf{X}_b^* \mathbf{X}_b^{*\top})^{-1})/(n-q)$ has zero mean.

Theorem 1. *Let \mathbf{A}_n be an $(n-q) \times (n-q)$ symmetric matrix.*

$$\left(\boldsymbol{\beta}_b^\top \mathbf{X}_b^\top (\mathbf{I}_n - \mathbf{P}_a) \mathbf{X}_b \boldsymbol{\beta}_b + \phi^{-1} (n-q) \right) \left(\frac{\mathbf{y}^{*\top} \mathbf{A}_n \mathbf{y}^*}{\mathbf{y}^{*\top} \mathbf{y}^*} - \frac{\text{tr}(\mathbf{A}_n)}{n-q} \right) \rightsquigarrow \mathcal{N}(0, 1).$$

Proof.

$$\frac{\mathbf{y}^{*\top} \mathbf{A}_n \mathbf{y}^*}{\mathbf{y}^{*\top} \mathbf{y}^*} - \frac{\text{tr}(\mathbf{A}_n)}{n-q} = \frac{(\phi^{1/2} \mathbf{y})^\top \left(\tilde{\mathbf{U}}_a \mathbf{A}_n \tilde{\mathbf{U}}_a^\top - \frac{\text{tr}(\mathbf{A}_n)}{n-q} \tilde{\mathbf{U}}_a \tilde{\mathbf{U}}_a^\top \right) (\phi^{1/2} \mathbf{y})}{\phi \mathbf{y}^{*\top} \mathbf{y}^*}.$$

$$\begin{aligned} & (\phi^{1/2} \mathbf{y})^\top \left(\tilde{\mathbf{U}}_a \mathbf{A}_n \tilde{\mathbf{U}}_a^\top - \frac{\text{tr}(\mathbf{A}_n)}{n-q} \tilde{\mathbf{U}}_a \tilde{\mathbf{U}}_a^\top \right) (\phi^{1/2} \mathbf{y}) \\ &= (\phi^{1/2} \boldsymbol{\varepsilon})^\top \left(\tilde{\mathbf{U}}_a \mathbf{A}_n \tilde{\mathbf{U}}_a^\top - \frac{\text{tr}(\mathbf{A}_n)}{n-q} \tilde{\mathbf{U}}_a \tilde{\mathbf{U}}_a^\top \right) (\phi^{1/2} \boldsymbol{\varepsilon}) + \\ & \quad 2\phi^{1/2} (\phi^{1/2} \boldsymbol{\varepsilon})^\top \left(\tilde{\mathbf{U}}_a \mathbf{A}_n \tilde{\mathbf{U}}_a^\top - \frac{\text{tr}(\mathbf{A}_n)}{n-q} \tilde{\mathbf{U}}_a \tilde{\mathbf{U}}_a^\top \right) \mathbf{X}_b \boldsymbol{\beta}_b + \\ & \quad \phi \boldsymbol{\beta}_b^\top \mathbf{X}_b^\top \left(\tilde{\mathbf{U}}_a \mathbf{A}_n \tilde{\mathbf{U}}_a^\top - \frac{\text{tr}(\mathbf{A}_n)}{n-q} \tilde{\mathbf{U}}_a \tilde{\mathbf{U}}_a^\top \right) \mathbf{X}_b \boldsymbol{\beta}_b \end{aligned}$$

From (Jiang, 1996, Theorem 5.1),

□

As in Vershynin (2018), sub-gaussian norm of a sub-gaussian random variable is defined as

$$\|X\|_{\psi_2} = \inf\{t > 0 : \mathbb{E} \exp(X^2/t^2) \leq 2\}.$$

A random vector $Z \in \mathbb{R}^p$ is called sub-gaussian if $z^\top Z$ are sub-gaussian random variables for all $z \in \mathbb{R}^p$. The sub-gaussian norm of Z is defined as

$$\|Z\|_{\psi_2} = \sup_{z \in S^{p-1}} \|z^\top Z\|_{\psi_2},$$

where S^{p-1} is the unit sphere in \mathbb{R}^p .

Suppose $\mathbf{X}_b = \mathbf{Z}_b \Gamma + \mathbf{1}_n \mu_b^\top$, where the rows of \mathbf{Z}_b are iid sub-gaussian random vectors with identity covariance matrix.

The following lemma is a simple extension of Theorem 4.6.1 of Vershynin (2018).

Lemma 2. *Let \mathbf{Z} be an $N \times n$ random matrix whose columns Z_i are independent sub-gaussian random vectors with $\mathbb{E}(Z_i) = 0$, $\text{Var}(Z_i) = \mathbf{I}_n$. Suppose $K := \max_i \|Z_i\|_{\psi_2}$ is uniformly bounded. Write $Z_i = (z_{i1}, \dots, z_{iN})^\top$. Assume that $\mathbb{E}(z_{i\ell}^4) = 3 + \Delta < \infty$ and for any intergers $\ell_v \geq 0$ with $\sum_{v=1}^s \ell_v \leq 4$,*

$$\mathbb{E}(Z_{ij_1}^{\ell_1} Z_{ij_2}^{\ell_2} \cdots Z_{ij_s}^{\ell_s}) = \mathbb{E}(Z_{ij_1}^{\ell_1}) \mathbb{E}(Z_{ij_2}^{\ell_2}) \cdots \mathbb{E}(Z_{ij_s}^{\ell_s})$$

Let \mathbf{W} be a nonrandom $N \times N$ symmetric matrix. Then

$$\|\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{tr}(\mathbf{W}) \mathbf{I}_n\| = O_P(\sqrt{n} \|\mathbf{W}\|_F + n \|\mathbf{W}\|).$$

TO BE DONE:

$$\|U \mathbf{Z}^\top \mathbf{W} \mathbf{Z} U^\top - \text{tr}(\mathbf{W}) \mathbf{I}_n\|$$

Next we verify the following. Let $\Sigma = \Gamma^\top \Gamma$.

Proof. Let

$$\mathbf{B} = \mathbf{X}_b^* \mathbf{X}_b^{*\top} = \tilde{\mathbf{U}}_a^\top \mathbf{X}_b \mathbf{X}_b^\top \tilde{\mathbf{U}}_a = \tilde{\mathbf{U}}_a^\top \mathbf{Z}_b \Gamma \Gamma^\top \mathbf{Z}_b^\top \tilde{\mathbf{U}}_a$$

Note that Lemma 2 implies that

$$\|\mathbf{B} - \text{tr}(\Sigma) \mathbf{I}_{n-q}\| = O_P(\sqrt{n} \|\Sigma\|_F + n \|\Sigma\|).$$

That is, uniformly for $i = 1, \dots, n - q$,

$$\frac{\lambda_i(\mathbf{B})}{\text{tr}(\Sigma)} = 1 + O_P\left(\frac{\sqrt{n} \|\Sigma\|_F}{\text{tr}(\Sigma)} + \frac{n \|\Sigma\|}{\text{tr}(\Sigma)}\right)$$

Define

$$\begin{aligned} \delta_i &= \frac{\lambda_i(\mathbf{B})}{\text{tr}(\Sigma)} - 1 \\ \eta &= \frac{\sqrt{n} \|\Sigma\|_F}{\text{tr}(\Sigma)} + \frac{n \|\Sigma\|}{\text{tr}(\Sigma)} \end{aligned}$$

We assume $\eta \rightarrow 0$.

Thus, we need to verify

$$\tilde{\mathbf{U}}_a \mathbf{B}^{-1} \tilde{\mathbf{U}}_a^\top - \frac{\text{tr}(\mathbf{B}^{-1})}{n - q} \tilde{\mathbf{U}}_a \tilde{\mathbf{U}}_a^\top$$

satisfies that

$$\left\| \mathbf{B}^{-1} - \frac{\text{tr}(\mathbf{B}^{-1})}{n - q} \mathbf{I}_{n-q} \right\|^2 \Bigg/ \text{tr} \left(\mathbf{B}^{-1} - \frac{\text{tr}(\mathbf{B}^{-1})}{n - q} \mathbf{I}_{n-q} \right)^2 \rightarrow 0. \quad (1)$$

Note that

$$\text{tr} \left(\mathbf{B}^{-1} - \frac{\text{tr}(\mathbf{B}^{-1})}{n-q} \mathbf{I}_{n-q} \right)^2 = \sum_{i=1}^{n-q} \frac{1}{\lambda_i^2(\mathbf{B})} - \frac{1}{n-q} \left(\sum_{i=1}^{n-q} \frac{1}{\lambda_i(\mathbf{B})} \right)^2.$$

By Taylor's theorem, uniformly for $i = 1, \dots, n$, we have

$$\begin{aligned} \frac{1}{\lambda_i(\mathbf{B})} &= \frac{1}{\text{tr}(\mathbf{\Sigma})} \frac{1}{1 + \delta_i} = \frac{1}{\text{tr}(\mathbf{\Sigma})} (1 - \delta_i + \delta_i^2 + O_P(\eta^3)), \\ \frac{1}{\lambda_i^2(\mathbf{B})} &= \frac{1}{\text{tr}^2(\mathbf{\Sigma})} \frac{1}{(1 + \delta_i)^2} = \frac{1}{\text{tr}^2(\mathbf{\Sigma})} (1 - 2\delta_i + 3\delta_i^2 + O_P(\eta^3)). \end{aligned}$$

Thus,

$$\begin{aligned} &\text{tr} \left(\mathbf{B}^{-1} - \frac{\text{tr}(\mathbf{B}^{-1})}{n-q} \mathbf{I}_{n-q} \right)^2 \\ &= \sum_{i=1}^{n-q} \frac{1}{\text{tr}^2(\mathbf{\Sigma})} (1 - 2\delta_i + 3\delta_i^2 + O_P(\eta^3)) - \frac{1}{n-q} \left(\sum_{i=1}^{n-q} \frac{1}{\text{tr}(\mathbf{\Sigma})} (1 - \delta_i + \delta_i^2 + O_P(\eta^3)) \right)^2 \\ &= \frac{1}{\text{tr}^2(\mathbf{\Sigma})} \left(n - q - 2 \sum_{i=1}^{n-q} \delta_i + 3 \sum_{i=1}^{n-q} \delta_i^2 + O_P(n\eta^3) - \frac{1}{n-q} \left(n - q - \sum_{i=1}^{n-q} \delta_i + \sum_{i=1}^{n-q} \delta_i^2 + O_P(n\eta^3) \right)^2 \right) \\ &= \frac{1}{\text{tr}^2(\mathbf{\Sigma})} \left(n - q - 2 \sum_{i=1}^{n-q} \delta_i + 3 \sum_{i=1}^{n-q} \delta_i^2 + O_P(n\eta^3) - (n-q) \left(1 - \frac{1}{n-q} \sum_{i=1}^{n-q} \delta_i + \frac{1}{n-q} \sum_{i=1}^{n-q} \delta_i^2 + O_P(\eta^3) \right)^2 \right) \\ &= \frac{1}{\text{tr}^2(\mathbf{\Sigma})} \left(n - q - 2 \sum_{i=1}^{n-q} \delta_i + 3 \sum_{i=1}^{n-q} \delta_i^2 + O_P(n\eta^3) \right. \\ &\quad \left. - (n-q) \left(1 + \left(\frac{1}{n-q} \sum_{i=1}^{n-q} \delta_i \right)^2 - \frac{2}{n-q} \sum_{i=1}^{n-q} \delta_i + \frac{2}{n-q} \sum_{i=1}^{n-q} \delta_i^2 + O_P(\eta^3) \right) \right) \\ &= \frac{1}{\text{tr}^2(\mathbf{\Sigma})} \left(\sum_{i=1}^{n-q} \delta_i^2 - \frac{1}{n-q} \left(\sum_{i=1}^{n-q} \delta_i \right)^2 + O_P(n\eta^3) \right) \\ &= \frac{1}{\text{tr}^2(\mathbf{\Sigma})} \left(\text{tr} \left(\frac{1}{\text{tr}(\mathbf{\Sigma})} \mathbf{B} - \mathbf{I}_{n-q} \right)^2 - \frac{1}{n-q} \left(\frac{\text{tr}(\mathbf{B})}{\text{tr}(\mathbf{\Sigma})} - (n-q) \right)^2 + O_P(n\eta^3) \right) \\ &= \frac{1}{\text{tr}^2(\mathbf{\Sigma})} \left(\frac{1}{\text{tr}^2(\mathbf{\Sigma})} \left(\text{tr}(\mathbf{B}^2) - \frac{1}{n-q} \text{tr}^2(\mathbf{B}) \right) + O_P(n\eta^3) \right). \end{aligned}$$

□

Lemma 3. Let \mathbf{Z} be an $n \times m$ random matrix whose rows Z_i are independent random vectors with $\mathbb{E}(Z_i) = 0$, $\text{Var}(Z_i) = \mathbf{I}_m$. Write $Z_i = (z_{i1}, \dots, z_{im})^\top$. Assume that $\mathbb{E}(z_{i\ell}^4) = 3 + \Delta < \infty$ and for any intergers $\ell_v \geq 0$ with $\sum_{v=1}^s \ell_v \leq 8$. Let \mathbf{B} be a $m \times m$ symmetric matrix. Let \mathbf{Q} be a $n \times n$ projection maitrx with rank r . Then

$$\text{tr}(\mathbf{QZBZ}^\top \mathbf{Q})^2 - \frac{1}{r} \left(\text{tr}(\mathbf{QZBZ}^\top \mathbf{Q}) \right)^2$$

Proof. Let $\mathbf{Q} = \mathbf{U}\mathbf{U}^\top$, where \mathbf{U} is a $n \times r$ column orthogonal matrix. Then

$$\begin{aligned} & \text{tr}(\mathbf{Q}\mathbf{Z}\mathbf{B}\mathbf{Z}^\top\mathbf{Q})^2 - \frac{1}{r} \left(\text{tr}(\mathbf{Q}\mathbf{Z}\mathbf{B}\mathbf{Z}^\top\mathbf{Q}) \right)^2 \\ &= \text{tr}(\mathbf{U}^\top\mathbf{Z}\mathbf{B}\mathbf{Z}^\top\mathbf{U})^2 - \frac{1}{r} \left(\text{tr}(\mathbf{U}^\top\mathbf{Z}\mathbf{B}\mathbf{Z}^\top\mathbf{U}) \right)^2 = \end{aligned}$$

$$\text{tr}(\mathbf{Z}\mathbf{B}\mathbf{Z}^\top\mathbf{Q}) = \sum_{i=1}^n \sum_{j=1}^n q_{i,j} Z_i^\top \mathbf{B} Z_j$$

$$\text{tr}^2(\mathbf{Z}\mathbf{B}\mathbf{Z}^\top\mathbf{Q}) = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n q_{i,j} q_{k,l} Z_i^\top \mathbf{B} Z_j Z_k^\top \mathbf{B} Z_l$$

And

$$\text{tr}(\mathbf{Z}\mathbf{B}\mathbf{Z}^\top\mathbf{Q})^2 = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n q_{j,k} q_{l,i} Z_i^\top \mathbf{B} Z_j Z_k^\top \mathbf{B} Z_l$$

Thus,

$$\text{tr}(\mathbf{Z}\mathbf{B}\mathbf{Z}^\top\mathbf{Q})^2 - r^{-1} \text{tr}^2(\mathbf{Z}\mathbf{B}\mathbf{Z}^\top\mathbf{Q}) = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n (q_{i,l} q_{j,k} - r^{-1} q_{i,j} q_{k,l}) Z_i^\top \mathbf{B} Z_j Z_k^\top \mathbf{B} Z_l$$

Thus,

$$e(i, j, k, l) := q_{i,l} q_{j,k} - r^{-1} q_{i,j} q_{k,l}$$

satisfies,

$$e(i_1, i_2, i_3, i_4) = e(i_2, i_1, i_4, i_3) = e(i_3, i_4, i_1, i_2) = e(i_4, i_3, i_2, i_1)$$

□

Lemma 4. Let $e(i, j; k, l)$ satisfy:

$$e(i_1, i_2, i_3, i_4) = e(i_2, i_1, i_4, i_3) = e(i_3, i_4, i_1, i_2) = e(i_4, i_3, i_2, i_1)$$

Then

$$\sum_{i,j,k,l=1}^n e(i, j, k, l) Z_i^\top \mathbf{B} Z_j Z_k^\top \mathbf{B} Z_l$$

Proof. Let $f(i, j, k, l) = e(i, j, k, l) Z_i^\top \mathbf{B} Z_j Z_k^\top \mathbf{B} Z_l$. Then f also satisfies

$$f(i_1, i_2, i_3, i_4) = f(i_2, i_1, i_4, i_3) = f(i_3, i_4, i_1, i_2) = f(i_4, i_3, i_2, i_1)$$

$$\begin{aligned}
& \sum_{i,j,k,l=1}^n f(i,j,k,l) \\
&= \sum_{i=j=1}^n \sum_{k=l=1}^n + \sum_{i=j=1}^n \sum_{k,l}^* + \sum_{i,j}^* \sum_{k=l=1}^n + \sum_{i,j}^* \sum_{k,l}^* \\
&= \sum_{i=1}^n \sum_{j=1}^n f(i,i,j,j) + \sum_{i=1}^n \sum_{j,k}^* f(i,i,j,k) + \sum_{i=1}^n \sum_{j,k}^* f(j,k,i,i) + \sum_{i,j}^* \sum_{k,l}^* f(i,j,k,l) \\
&= \sum_{i=1}^n \sum_{j=1}^n f(i,i,j,j) + 2 \sum_{i=1}^n \sum_{j,k}^* f(i,i,j,k) + \sum_{i,j}^* \sum_{k,l}^* f(i,j,k,l)
\end{aligned}$$

For the first term,

$$\sum_{i=1}^n \sum_{j=1}^n f(i,i,j,j) = \sum_{i=1}^n f(i,i,i,i) + \sum_{i,j}^* f(i,i,j,j).$$

For the second term,

$$\begin{aligned}
& \sum_{i=1}^n \sum_{j,k}^* f(i,i,j,k) \\
&= \sum_{i=1}^n \sum_{j \neq i}^n f(i,i,i,j) + \sum_{i=1}^n \sum_{j \neq i}^n f(i,i,j,i) + \sum_{i,j,k}^* f(i,i,j,k) \\
&= 2 \sum_{i,j}^* f(i,i,i,j) + \sum_{i,j,k}^* f(i,i,j,k)
\end{aligned}$$

For the third term,

$$\begin{aligned}
& \sum_{i,j}^* \sum_{k,l}^* f(i,j,k,l) \\
&= \sum_{i,j}^* \sum_{k=i,l=j}^* + \sum_{i,j}^* \sum_{k=j,l=i}^* + \sum_{i,j}^* \sum_{k=i,l \neq \{i,j\}}^* + \sum_{i,j}^* \sum_{k=j,l \neq \{i,j\}}^* + \sum_{i,j}^* \sum_{l=i,k \neq \{i,j\}}^* + \sum_{i,j}^* \sum_{l=j,k \neq \{i,j\}}^* + \sum_{i,j,k,l}^* \\
&= \sum_{i,j}^* (f(i,j,i,j) + f(i,j,j,i)) + 2 \sum_{i,j,k}^* (f(i,j,i,k) + f(i,j,k,i)) + \sum_{i,j,k,l}^* f(i,j,k,l)
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{i,j,k,l=1}^n f(i,j,k,l) &= \sum_{i=1}^n f(i,i,i,i) + \sum_{i,j}^* f(i,i,j,j) \\
&\quad + 4 \sum_{i,j}^* f(i,i,i,j) + 2 \sum_{i,j,k}^* f(i,i,j,k) \\
&\quad + \sum_{i,j}^* (f(i,j,i,j) + f(i,j,j,i)) + 2 \sum_{i,j,k}^* (f(i,j,i,k) + f(i,j,k,i)) + \sum_{i,j,k,l}^* f(i,j,k,l) \\
&=: \sum_{i=1}^7 A_i
\end{aligned}$$

Note that for distinct i, j, k, l ,

$$\mathbb{E} f(i, i, i, i) = e(i, i, i, i) \mathbb{E}(Z_i^\top \mathbf{B} Z_i)^2 = e(i, i, i, i) (\text{tr}^2(\mathbf{B}) + 2 \text{tr}(\mathbf{B}^2) + \Delta \text{tr}(\mathbf{B} \circ \mathbf{B}))$$

$$\mathbb{E} f(i, i, j, j) = e(i, i, j, j) \mathbb{E}(Z_i^\top \mathbf{B} Z_i)(Z_i^\top \mathbf{B} Z_i) = e(i, i, j, j) \text{tr}^2(\mathbf{B})$$

$$\mathbb{E} f(i, i, i, j) = 0$$

$$\mathbb{E} (f(i, j, i, j) + f(i, j, j, i)) = (e(i, j, i, j) + e(i, j, j, i)) \mathbb{E}(Z_i^\top \mathbf{B} Z_j)^2 = (e(i, j, i, j) + e(i, j, j, i)) \mathbb{E}(\mathbf{B}^2)$$

$$\mathbb{E} (f(i, j, i, k) + f(i, j, k, i)) = 0$$

$$\mathbb{E} f(i, j, k, l) = 0$$

Let $h(i, j, k, l) = f(i, j, k, l) - \mathbb{E} f(i, j, k, l)$. Then

$$\begin{aligned} \text{Var} \left(\sum_{i,j,k,l=1}^n e(i, j, k, l) Z_i^\top \mathbf{B} Z_j Z_k^\top \mathbf{B} Z_l \right) &= \mathbb{E} \left(\sum_{i,j,k,l=1}^n h(i, j, k, l) \right)^2 \\ &= \sum_{i=1}^7 \mathbb{E}(A_i - \mathbb{E} A_i)^2 + 2 \sum_{1 \leq i < j \leq 7} \mathbb{E}(A_i - \mathbb{E} A_i)(A_j - \mathbb{E} A_j) \end{aligned}$$

We have

$$\mathbb{E}(A_1 - \mathbb{E} A_1)^2 = \sum_{i=1}^n e(i, i, i, i)^2 \mathbb{E} \left((Z_1^\top \mathbf{B} Z_1)^2 - \mathbb{E}(Z_1^\top \mathbf{B} Z_1)^2 \right)^2$$

□

Lemma 5.

$$\mathbb{E} \left((Z^\top \mathbf{B} Z)^2 - \mathbb{E}(Z^\top \mathbf{B} Z)^2 \right)^2 =$$

Proof.

$$\begin{aligned} (Z^\top \mathbf{B} Z)^2 &= \left(\sum_{i=1}^m b_{i,i} z_i^2 + \sum_{i,j}^* b_{i,j} z_i z_j \right)^2 \\ &= \left(\sum_{i=1}^m b_{i,i} z_i^2 \right)^2 + \left(\sum_{i,j}^* b_{i,j} z_i z_j \right)^2 + 2 \left(\sum_{i=1}^m b_{i,i} z_i^2 \right) \left(\sum_{i,j}^* b_{i,j} z_i z_j \right) \\ &= \sum_{i=1}^m b_{i,i}^2 z_i^4 + \sum_{i,j}^* b_{i,i} b_{j,j} z_i^2 z_j^2 \\ &\quad + \sum_{i,j}^* \sum_{k,l}^* b_{i,j} b_{k,l} z_i z_j z_k z_l + 2 \sum_{k=1}^m \sum_{i,j}^* b_{i,j} b_{k,k} z_i z_j z_k^2 \end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{i,j}^* \sum_{k,l}^* b_{i,j} b_{k,l} z_i z_j z_k z_l \\
&= \sum_{i,j}^* \sum_{k=i,l=j}^* + \sum_{i,j}^* \sum_{k=i,l \neq j}^* + \sum_{i,j}^* \sum_{k=j,l=i}^* + \sum_{i,j}^* \sum_{k=j,l \neq i}^* + \sum_{i,j}^* \sum_{k \notin \{i,j\}, l=i}^* + \sum_{i,j}^* \sum_{k \notin \{i,j\}, l=j}^* + \sum_{i,j}^* \sum_{k \notin \{i,j\}, l \notin \{i,j\}}^* \\
&= 2 \sum_{i,j}^* b_{i,j}^2 z_i^2 z_j^2 + 4 \sum_{i,j,k}^* b_{i,j} b_{i,k} z_i^2 z_j z_k + \sum_{i,j,k,l}^* b_{i,j} b_{k,l} z_i z_j z_k z_l
\end{aligned}$$

On the other hand

$$\begin{aligned}
& \sum_{k=1}^m \sum_{i,j}^* b_{i,j} b_{k,k} z_i z_j z_k^2 = \sum_{i,j}^* \sum_{k=1}^m b_{i,j} b_{k,k} z_i z_j z_k^2 \\
&= \sum_{i,j}^* b_{i,i} b_{i,i} z_i^3 z_j + \sum_{i,j}^* b_{i,i} b_{j,j} z_i^2 z_j^3 + \sum_{i,j,k}^* b_{i,i} b_{k,k} z_i z_j z_k^2 = \sum_{i,j}^* b_{i,i} b_{i,j} z_i^3 z_j + \sum_{i,j,k}^* b_{i,j} b_{k,k} z_i z_j z_k^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
(Z^\top \mathbf{B} Z)^2 &= \sum_{i=1}^m b_{i,i}^2 z_i^4 + \sum_{i,j}^* b_{i,i} b_{j,j} z_i^2 z_j^2 + \sum_{i,j}^* \sum_{k,l}^* b_{i,j} b_{k,l} z_i z_j z_k z_l + 2 \sum_{k=1}^m \sum_{i,j}^* b_{i,j} b_{k,k} z_i z_j z_k^2 \\
&= \sum_{i=1}^m b_{i,i}^2 z_i^4 + \sum_{i,j}^* b_{i,i} b_{j,j} z_i^2 z_j^2 + 2 \sum_{i,j}^* b_{i,j}^2 z_i^2 z_j^2 + 4 \sum_{i,j,k}^* b_{i,j} b_{i,k} z_i^2 z_j z_k + \sum_{i,j,k,l}^* b_{i,j} b_{k,l} z_i z_j z_k z_l \\
&\quad + 2 \sum_{i,j}^* b_{i,i} b_{i,j} z_i^3 z_j + 2 \sum_{i,j,k}^* b_{i,j} b_{k,k} z_i z_j z_k^2. \\
&= \sum_{i=1}^m b_{i,i}^2 z_i^4 + \sum_{i,j}^* (b_{i,i} b_{j,j} + 2b_{i,j}^2) z_i^2 z_j^2 + 2 \sum_{i,j}^* b_{i,i} b_{i,j} z_i^3 z_j \\
&\quad + \sum_{i,j,k}^* (2b_{i,i} b_{j,k} + 4b_{i,j} b_{i,k}) z_i^2 z_j z_k + \sum_{i,j,k,l}^* b_{i,j} b_{k,l} z_i z_j z_k z_l
\end{aligned}$$

Firstly,

$$\text{Var} \left(\sum_{i=1}^m b_{i,i}^2 z_i^4 \right) = O \left(\sum_{i=1}^4 b_{i,i}^4 \right)$$

Secondly,

$$\mathbb{E} \sum_{i,j}^* (b_{i,i} b_{j,j} + 2b_{i,j}^2) z_i^2 z_j^2 = \sum_{i,j}^* (b_{i,i} b_{j,j} + 2b_{i,j}^2) = \left(\sum_{i=1}^m b_{i,i} \right)^2 - \sum_{i=1}^m b_{i,i}^2 + 2 \sum_{i,j=1}^n b_{i,j}^2 - 2 \sum_{i=1}^n b_{i,i}^2$$

$$\begin{aligned}
& \left(\sum_{i,j}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2) z_i^2 z_j^2 \right)^2 \\
&= \sum_{i,j}^* \sum_{k,l}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{k,k}b_{l,l} + 2b_{k,l}^2) z_i^2 z_j^2 z_k^2 z_l^2 \\
&= \sum_{i,j}^* \sum_{k=i,l=j}^* + \sum_{i,j}^* \sum_{k=i,l \neq j}^* + \sum_{i,j}^* \sum_{k=j,l=i}^* + \sum_{i,j}^* \sum_{k=j,l \neq i}^* + \sum_{i,j}^* \sum_{k \notin \{i,j\}, l=i}^* + \sum_{i,j}^* \sum_{k \notin \{i,j\}, l=j}^* + \sum_{i,j}^* \sum_{k \notin \{i,j\}, l \notin \{i,j\}}^* \\
&= 2 \sum_{i,j}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)^2 z_i^4 z_j^4 + \sum_{i,j,l}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{i,i}b_{l,l} + 2b_{i,l}^2) z_i^4 z_j^2 z_l^2 \\
&\quad + \sum_{i,j,l}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{j,j}b_{l,l} + 2b_{j,l}^2) z_i^2 z_j^4 z_l^2 \\
&\quad + \sum_{i,j,k}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{k,k}b_{i,i} + 2b_{k,i}^2) z_i^4 z_j^2 z_k^2 \\
&\quad + \sum_{i,j,k}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{k,k}b_{j,j} + 2b_{k,j}^2) z_i^2 z_j^4 z_k^2 \\
&\quad + \sum_{i,j,k,l}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{k,k}b_{l,l} + 2b_{k,l}^2) z_i^2 z_j^2 z_k^2 z_l^2 \\
&= 2 \sum_{i,j}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)^2 z_i^4 z_j^4 + 4 \sum_{i,j,k}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{i,i}b_{k,k} + 2b_{i,k}^2) z_i^4 z_j^2 z_k^2 \\
&\quad + \sum_{i,j,k,l}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{k,k}b_{l,l} + 2b_{k,l}^2) z_i^2 z_j^2 z_k^2 z_l^2
\end{aligned}$$

Hence

$$\begin{aligned}
& \mathbb{E} \left(\sum_{i,j}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2) z_i^2 z_j^2 \right)^2 \\
&= 2\mu_4^2 \sum_{i,j}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)^2 + 4\mu_4 \sum_{i,j,k}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{i,i}b_{k,k} + 2b_{i,k}^2) \\
&\quad + \sum_{i,j,k,l}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{k,k}b_{l,l} + 2b_{k,l}^2) \\
&= 2(\mu_4^2 - 1) \sum_{i,j}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)^2 + 4(\mu_4 - 1) \sum_{i,j,k}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2)(b_{i,i}b_{k,k} + 2b_{i,k}^2) \\
&\quad + \left(\sum_{i,j}^* (b_{i,i}b_{j,j} + 2b_{i,j}^2) \right)^2
\end{aligned}$$

Note that

$$\sum_{i,j}^* b_{i,i}^2 b_{j,j}^2 = \left(\sum_{i=1}^m b_{i,i}^2 \right)^2 - \sum_{i=1}^m b_{i,i}^4$$

$$\begin{aligned}
\sum_{i,j,k}^* b_{i,i}^2 b_{j,j} b_{k,k} &= \sum_{i=1}^m \sum_{j \neq i} \sum_{k \notin \{i,j\}} b_{i,i}^2 b_{j,j} b_{k,k} \\
&= \sum_{i=1}^m \sum_{j \neq i} \sum_{k \neq i} b_{i,i}^2 b_{j,j} b_{k,k} - \sum_{i=1}^m \sum_{j \neq i} b_{i,i}^2 b_{j,j}^2 \\
&= 2 \sum_{i=1}^m b_{i,i}^4 - \left(\sum_{i=1}^m b_{i,i}^2 \right)^2 - 2 \left(\sum_{i=1}^m b_{i,i}^3 \right) \left(\sum_{i=1}^m b_{i,i} \right) + \left(\sum_{i=1}^m b_{i,i}^2 \right) \left(\sum_{i=1}^m b_{i,i} \right)^2
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E} \left(\sum_{i,j}^* b_{i,i} b_{j,j} z_i^2 z_j^2 \right)^2 \\
&= 2\mu_4^2 \sum_{i,j}^* b_{i,i}^2 b_{j,j}^2 + 4\mu_4 \sum_{i,j,k}^* b_{i,i}^2 b_{j,j} b_{k,k} + \sum_{i,j,k,l}^* b_{i,i} b_{j,j} b_{k,k} b_{l,l} \\
&= 2(\mu_4^2 - 1) \sum_{i,j}^* b_{i,i}^2 b_{j,j}^2 + 4(\mu_4 - 1) \sum_{i,j,k}^* b_{i,i}^2 b_{j,j} b_{k,k} + \left(\sum_{i,j}^* b_{i,i} b_{j,j} \right)^2 \\
&= 2(\mu_4^2 - 1) \left(\left(\sum_{i=1}^m b_{i,i}^2 \right)^2 - \sum_{i=1}^m b_{i,i}^4 \right) \\
&\quad + 4(\mu_4 - 1) \left(2 \sum_{i=1}^m b_{i,i}^4 - \left(\sum_{i=1}^m b_{i,i}^2 \right)^2 - 2 \left(\sum_{i=1}^m b_{i,i}^3 \right) \left(\sum_{i=1}^m b_{i,i} \right) + \left(\sum_{i=1}^m b_{i,i}^2 \right) \left(\sum_{i=1}^m b_{i,i} \right)^2 \right) \\
&\quad + \left(\sum_{i=1}^m b_{i,i}^2 \right)^2 + \left(\sum_{i=1}^m b_{i,i} \right)^4 - 2 \left(\sum_{i=1}^m b_{i,i}^2 \right) \left(\sum_{i=1}^m b_{i,i} \right)^2 \\
&= \left(\sum_{i=1}^m b_{i,i} \right)^4 + (2\mu_4^2 - 4\mu_4 + 3) \left(\sum_{i=1}^m b_{i,i}^2 \right)^2 + (-2\mu_4^2 + 8\mu_4 - 6) \sum_{i=1}^m b_{i,i}^4 \\
&\quad + (-8\mu_4 + 8) \left(\sum_{i=1}^m b_{i,i}^3 \right) \left(\sum_{i=1}^m b_{i,i} \right) + (4\mu_4 - 6) \left(\sum_{i=1}^m b_{i,i}^2 \right) \left(\sum_{i=1}^m b_{i,i} \right)^2
\end{aligned}$$

Thus,

$$\begin{aligned}
& \text{Var}\left(\sum_{i,j}^* b_{i,i} b_{j,j} z_i^2 z_j^2\right) \\
&= \left(\sum_{i=1}^m b_{i,i}\right)^4 + (2\mu_4^2 - 4\mu_4 + 3)\left(\sum_{i=1}^m b_{i,i}^2\right)^2 + (-2\mu_4^2 + 8\mu_4 - 6) \sum_{i=1}^m b_{i,i}^4 \\
&\quad + (-8\mu_4 + 8)\left(\sum_{i=1}^m b_{i,i}^3\right)\left(\sum_{i=1}^m b_{i,i}\right) + (4\mu_4 - 6)\left(\sum_{i=1}^m b_{i,i}^2\right)\left(\sum_{i=1}^m b_{i,i}\right)^2 \\
&\quad - \left(\sum_{i=1}^m b_{i,i}\right)^4 - \left(\sum_{i=1}^m b_{i,i}^2\right)^2 + 2\left(\sum_{i=1}^m b_{i,i}\right)^2\left(\sum_{i=1}^m b_{i,i}^2\right) \\
&= (2\mu_4^2 - 4\mu_4 + 2)\left(\sum_{i=1}^m b_{i,i}^2\right)^2 + (-2\mu_4^2 + 8\mu_4 - 6) \sum_{i=1}^m b_{i,i}^4 \\
&\quad + (-8\mu_4 + 8)\left(\sum_{i=1}^m b_{i,i}^3\right)\left(\sum_{i=1}^m b_{i,i}\right) + (4\mu_4 - 4)\left(\sum_{i=1}^m b_{i,i}^2\right)\left(\sum_{i=1}^m b_{i,i}\right)^2
\end{aligned}$$

□

Appendices

Appendix A haha1

Proof of Proposition 1. We assume $0 < \alpha < 1$ since the case $\alpha = 0$ or 1 is trivial. Note that the condition implies $\int[\varphi(\mathbf{y}) - \alpha]\mathcal{N}_n(0, \phi^{-1}\mathbf{I}_n)(d\mathbf{y}) = 0$. Hence it suffices to prove $\varphi(\mathbf{y}) \geq \alpha$, a.s. We prove this by contradiction. Suppose $\lambda(\{\mathbf{y} : \varphi(\mathbf{y}) < \alpha\}) > 0$. Then there exists a $\eta > 0$, such that $\lambda(\{\mathbf{y} : \varphi(\mathbf{y}) < \alpha - \eta\}) > 0$. We denote $E = \{\mathbf{y} : \varphi(\mathbf{y}) < \alpha - \eta\}$. From Lebesgue density theorem (Cohn, 2013, Corollary 6.2.6), there exists a point $z \in E$, such that, for each $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that

$$\left| \frac{\lambda(E^c \cap C_\epsilon)}{\lambda(C_\epsilon)} \right| < \epsilon,$$

where $C_\epsilon = \prod_{i=1}^n [z_i - \delta_\epsilon, z_i + \delta_\epsilon]$. We put

$$\epsilon = \left(\frac{\sqrt{\pi}}{\sqrt{2}\Phi^{-1}\left(1 - \frac{\eta}{6n}\right)} \right)^n \frac{\eta}{3}.$$

Then for any $\phi > 0$,

$$\begin{aligned}
\alpha &\leq \int_{\mathbb{R}^n} \varphi(\mathbf{y}) \mathcal{N}_n(z, \phi^{-1} \mathbf{I}_n)(d\mathbf{y}) \\
&= \int_{E \cap C_\epsilon} \varphi(\mathbf{y}) \mathcal{N}_n(z, \phi^{-1} \mathbf{I}_n)(d\mathbf{y}) + \int_{E^c \cap C_\epsilon} \varphi(\mathbf{y}) \mathcal{N}_n(z, \phi^{-1} \mathbf{I}_n)(d\mathbf{y}) + \int_{C_\epsilon^c} \varphi(\mathbf{y}) \mathcal{N}_n(z, \phi^{-1} \mathbf{I}_n)(d\mathbf{y}) \\
&\leq \alpha - \eta + \int_{E^c \cap C_\epsilon} \mathcal{N}_n(z, \phi^{-1} \mathbf{I}_n)(d\mathbf{y}) + \int_{C_\epsilon^c} \mathcal{N}_n(z, \phi^{-1} \mathbf{I}_n)(d\mathbf{y}) \\
&\leq \alpha - \eta + \left(\frac{\phi}{2\pi} \right)^{n/2} \lambda(E^c \cap C_\epsilon) + 2n \left(1 - \Phi(\sqrt{\phi} \delta_\epsilon) \right) \\
&\leq \alpha - \eta + \left(\frac{\phi}{2\pi} \right)^{n/2} \epsilon (2\delta_\epsilon)^n + 2n \left(1 - \Phi(\sqrt{\phi} \delta_\epsilon) \right) \\
&= \alpha - \eta + \left(\frac{\sqrt{\phi} \delta_\epsilon}{\Phi^{-1}\left(1 - \frac{\eta}{6n}\right)} \right)^n \frac{\eta}{3} + 2n \left(1 - \Phi(\sqrt{\phi} \delta_\epsilon) \right).
\end{aligned}$$

Putting

$$\phi = \left(\frac{\Phi^{-1}\left(1 - \frac{\eta}{6n}\right)}{\delta_\epsilon} \right)^2$$

yields the contradiction $\alpha \leq \alpha - (2/3)\eta$. This completes the proof. \square

Proof of Proposition 2. For positive integer m , define $[m] = \{1, \dots, m\}$. For a set A , denote by $|A|$ its cardinality. We have

$$\begin{aligned}
k_\kappa &= \left| \left\{ i \in [n-q] : \frac{\gamma_i^2}{\gamma_i + \kappa} - \frac{1}{n-q} \sum_{j=1}^{n-q} \frac{\gamma_j \gamma_i}{\gamma_j + \kappa} > 0 \right\} \right| \\
&= \left| \left\{ i \in [n-q] : \frac{\gamma_i}{\gamma_i + \kappa} > \frac{1}{n-q} \sum_{j=1}^{n-q} \frac{\gamma_j}{\gamma_j + \kappa} \right\} \right|.
\end{aligned}$$

Let X be a random variable uniformly distributed on $\{\gamma_1, \dots, \gamma_{n-q}\}$. That is, $\Pr(X = \gamma_i) = 1/(n-q)$, $i = 1, \dots, n-q$. Then it can be seen that

$$k_\kappa = (n-q) \Pr \left(\frac{X}{X + \kappa} > \mathbb{E} \left[\frac{X}{X + \kappa} \right] \right).$$

Hence we only need to verify

$$\Pr \left(\frac{X}{X + \kappa_1} > \mathbb{E} \left[\frac{X}{X + \kappa_1} \right] \right) \geq \Pr \left(\frac{X}{X + \kappa_2} > \mathbb{E} \left[\frac{X}{X + \kappa_2} \right] \right). \quad (2)$$

Let $Y = X/(X + \kappa_2)$. Then

$$\frac{X}{(X + \kappa_1)} = \frac{\kappa_2 Y}{\kappa_1 + (\kappa_2 - \kappa_1) Y} := f(Y).$$

Note that $f(Y)$ is increasing for $Y \geq 0$. Then the inequality (2) is equivalent to

$$\Pr(Y > f^{-1}(E f(Y))) \geq \Pr(Y > E Y).$$

Hence we only need to verify $f^{-1}(E f(Y)) \leq E Y$, or equivalently, $E f(Y) \leq f(E Y)$. But the last inequality is a direct consequence of the concavity of $f(Y)$. This completes the proof. \square

Lemma 6. *Let \mathbf{W} be an $N \times N$ positive semi-definite matrix. Let Z be an N dimensional sub-gaussian random vector with $E Z = 0$, $\text{Var}(Z) = \mathbf{I}_n$ and $\|Z\|_{\psi_2} \leq K$. For all $t > 0$,*

$$\Pr\left\{Z^\top \mathbf{W} Z > t\right\} \leq e^{-t}.$$

Remark 1. This lemma is adapted from Hsu et al. (2012) Theorem 2.1 with minor modifications. Indeed, their result did not track the variance of Z .

Proof. Let $\mathbf{A} = \sqrt{\mathbf{W}}$. Let Z^* be a vector of N independent standard Gaussian random variables which are independent of Z .

$$E[\exp(\lambda Z^{*\top} \mathbf{A} Z)] = E E[\exp(\lambda Z^{*\top} \mathbf{A} Z) | Z] = E[\exp(\frac{\lambda^2}{2} Z^\top \mathbf{W} Z)]$$

\square

Lemma 7.

Proof.

\square

Proof of Lemma 2.

$$\|\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{tr}(\mathbf{W}) \mathbf{I}_n\| \leq \|\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z})\| + \|\text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z}) - \text{tr}(\mathbf{W}) \mathbf{I}_n\|$$

We have

$$\|\text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z}) - \text{tr}(\mathbf{W}) \mathbf{I}_n\| = \max_{1 \leq i \leq n} |Z_i^\top \mathbf{W} Z_i - \text{tr}(\mathbf{W})| \leq \sqrt{\sum_{i=1}^n (Z_i^\top \mathbf{W} Z_i - \text{tr}(\mathbf{W}))^2}$$

From (Chen et al., 2010, Proposition A.1),

$$E \left[\sum_{i=1}^n \left(Z_i^\top \mathbf{W} Z_i - \text{tr}(\mathbf{W}) \right)^2 \right] = 2n \text{tr}(\mathbf{W}^2) + \Delta n \text{tr}(\mathbf{W} \circ \mathbf{W}) \leq (2 + \Delta) n \text{tr}(\mathbf{W}^2).$$

Hence

$$\|\text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z}) - \text{tr}(\mathbf{W}) \mathbf{I}_n\| = O_P(\sqrt{n} \|\mathbf{W}\|_F).$$

Next we deal with

$$\|\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z})\|$$

From (Vershynin, 2018, Lemma 5.2), there is a $1/4$ -net \mathcal{C} of the unit sphere S^{n-1} such that $|\mathcal{C}| \leq 9^n$. By (Vershynin, 2018, Exercise 4.4.3),

$$\|\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z})\| \leq 2 \sup_{x \in \mathcal{C}} \left| x^\top \left(\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z}) \right) x \right|.$$

Fix $x \in \mathcal{C}$. Then

$$\left| x^\top \left(\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z}) \right) x \right| = \left| \sum_{i=1}^n \sum_{j \neq i}^n x_i x_j Z_i^\top \mathbf{W} Z_j \right|$$

Now we bound the moment generating function of $\sum_{i=1}^n \sum_{j \neq i}^n x_i x_j Z_i^\top \mathbf{W} Z_j$. We apply the decoupling technique in Vershynin (2018), Section 6.1. Let $\delta_1, \dots, \delta_n$ be independent Bernoulli random variables with $\Pr\{\delta_i = 0\} = \Pr\{\delta_i = 1\} = 1/2$. For any $\lambda \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E} \exp \left\{ \lambda \sum_{i=1}^n \sum_{j \neq i}^n x_i x_j Z_i^\top \mathbf{W} Z_j \right\} &= \mathbb{E} \exp \left\{ \mathbb{E} \left(4\lambda \sum_{i=1}^n \sum_{j=1}^n \delta_i (1 - \delta_j) x_i x_j Z_i^\top \mathbf{W} Z_j \middle| \mathbf{Z} \right) \right\} \\ &\leq \mathbb{E} \exp \left\{ 4\lambda \sum_{i=1}^n \sum_{j=1}^n \delta_i (1 - \delta_j) x_i x_j Z_i^\top \mathbf{W} Z_j \right\} \\ &= \mathbb{E} \exp \left\{ 4\lambda \left(\sum_{i: \delta_i=1} x_i Z_i \right)^\top \mathbf{W} \left(\sum_{j: \delta_j=0} x_j Z_j \right) \right\} \\ &\leq \max_{I \subset [n]} \mathbb{E} \exp \left\{ 4\lambda \left(\sum_{i \in I} x_i Z_i \right)^\top \mathbf{W} \left(\sum_{j \notin I} x_j Z_j \right) \right\}, \end{aligned}$$

where the first inequality follows from Jensen's inequality. Fix an $I \subset [n]$. From Vershynin (2018), Proposition 2.6.1, $\|\sum_{i \in I} x_i Z_i\|_{\psi_2} \leq C_1 K$, $\|\sum_{j \notin I} x_j Z_j\|_{\psi_2} \leq C_1 K$ for some absolute constant C_1 . Then Vershynin (2018), Lemma 6.2.2 and Lemma 6.2.3 imply that there exist absolute constants C_2, C_3 such that,

$$\mathbb{E} \exp \left\{ 4\lambda \left(\sum_{i \in I} x_i Z_i \right)^\top \mathbf{W} \left(\sum_{j \notin I} x_j Z_j \right) \right\} \leq \exp \left\{ C_2 K^4 \|\mathbf{W}\|_F^2 \lambda^2 \right\}$$

for all $|\lambda| \leq C_3/(K^2 \|\mathbf{W}\|)$. Note that this bound does not depend on $I \subset [n]$. It follows that

$$\mathbb{E} \exp \left\{ \lambda \sum_{i=1}^n \sum_{j \neq i}^n x_i x_j Z_i^\top \mathbf{W} Z_j \right\} \leq \exp \left\{ C_2 K^4 \|\mathbf{W}\|_F^2 \lambda^2 \right\},$$

for all $|\lambda| \leq C_3/(K^2 \|\mathbf{W}\|)$. Then applying Chernoff bound yields that, for any $t > 0$,

$$\begin{aligned} \Pr \left(\left| \sum_{i=1}^n \sum_{j \neq i}^n x_i x_j Z_i^\top \mathbf{W} Z_j \right| > t \right) &\leq \inf_{0 < \lambda \leq \frac{C_3}{K^2 \|\mathbf{W}\|}} 2 \exp \left\{ -\lambda t + C_2 K^4 \|\mathbf{W}\|_F^2 \lambda^2 \right\} \\ &\leq 2 \exp \left\{ -\min \left(\frac{t^2}{4C_2 K^4 \|\mathbf{W}\|_F^2}, \frac{C_3 t}{2K^2 \|\mathbf{W}\|} \right) \right\}. \end{aligned}$$

This inequality, combined with union bound, yields

$$\begin{aligned} \Pr\left(\|\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z})\| > t\right) &\leq \Pr\left(2 \sup_{x \in \mathcal{C}} \left| x^\top \left(\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z}) \right) x \right| > t\right) \\ &\leq 2 \cdot 9^n \exp\left\{-\min\left(\frac{t^2}{16C_2 K^4 \|\mathbf{W}\|_F^2}, \frac{C_3 t}{4K^2 \|\mathbf{W}\|}\right)\right\}. \end{aligned}$$

Thus, there exists a large $C > 0$ such that for every $t > 0$,

$$\Pr\left(\|\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z})\| > C(K^2(\sqrt{n} + t)\|\mathbf{W}\|_F + K^2(n + t^2)\|\mathbf{W}\|)\right) \leq 2 \exp\{-t^2\}.$$

Consequently, $\|\mathbf{Z}^\top \mathbf{W} \mathbf{Z} - \text{diag}(\mathbf{Z}^\top \mathbf{W} \mathbf{Z})\| = O_P(K^2(\sqrt{n}\|\mathbf{W}\|_F + n\|\mathbf{W}\|))$. This completes the proof. \square

Appendix B haha2

Theorem 2. Let ζ_1, \dots, ζ_d be iid random variables with mean 0 and variance 1, and assume $\mu_k := \mathbb{E}(\zeta_1^k)$ is finite for $k \leq 8$. Let $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_d)^\top \in \mathbb{R}^d$. For $k = 1, \dots, K$, let $\mathbf{Q}_k = (q_{ij}^{(k)})$ be a $d \times d$ symmetric matrix and let $\check{\mathbf{Q}}_k = \text{diag}(q_{11}^{(k)}, \dots, q_{dd}^{(k)})$, $\hat{\mathbf{Q}}_k = \mathbf{I}_d - \check{\mathbf{Q}}_k$. Define $\hat{w}_k = \boldsymbol{\zeta}^\top \hat{\mathbf{Q}}_k \boldsymbol{\zeta}$, $\check{w}_k = \boldsymbol{\zeta}^\top \check{\mathbf{Q}}_k \boldsymbol{\zeta} - \text{tr}(\mathbf{Q}_k)$, and

$$W = \begin{pmatrix} \hat{w}_1 \\ \check{w}_1 \\ \vdots \\ \hat{w}_K \\ \check{w}_K \end{pmatrix} = \begin{pmatrix} \boldsymbol{\zeta}^\top \hat{\mathbf{Q}}_1 \boldsymbol{\zeta} \\ \boldsymbol{\zeta}^\top \check{\mathbf{Q}}_1 \boldsymbol{\zeta} - \text{tr}(\mathbf{Q}_1) \\ \vdots \\ \boldsymbol{\zeta}^\top \hat{\mathbf{Q}}_K \boldsymbol{\zeta} \\ \boldsymbol{\zeta}^\top \check{\mathbf{Q}}_K \boldsymbol{\zeta} - \text{tr}(\mathbf{Q}_K) \end{pmatrix} \in \mathbb{R}^{2K}.$$

Finally, let $Z \sim \mathcal{N}_{2K}(0, \mathbf{I}_{2K})$ and $\mathbf{V} = \text{Cov}(W)$. There is an absolute constant $0 < C < \infty$ such that

haha

Proof. Let $f : \mathbb{R}^{2K} \rightarrow \mathbb{R}$ be a four-times differentiable function. From xxx, there is a 4-times differentiable function $g : \mathbb{R}^{2K} \rightarrow \mathbb{R}$ satisfying the Stein identity

$$\mathbb{E}[f(W)] - \mathbb{E}[f(\mathbf{V}^{1/2}W)] = \mathbb{E}[\nabla^\top \mathbf{V} \nabla g(W) - W^\top \nabla g(W)]$$

and

$$\left| \frac{\partial^k g(\mathbf{x})}{\prod_{j=1}^k \partial x_{i_j}} \right| \leq \frac{1}{k} \left| \frac{\partial^k f(\mathbf{x})}{\prod_{j=1}^k \partial x_{i_j}} \right| \quad \text{for all } \mathbf{x} = (x_1, \dots, x_{2K})^\top \in \mathbb{R}^{2K}, k = 1, 2, 3, \text{ and } i_j \in \{1, \dots, 2K\}.$$

To prove the theorem, we bound

$$S = \mathbb{E}[\nabla^\top \mathbf{V} \nabla g(W) - W^\top \nabla g(W)].$$

Next, we use exchangeability. Let $\zeta' = (\zeta'_1, \dots, \zeta'_d)^\top$ be an independent copy of ζ , and let $\underline{i} \in \{1, \dots, d\}$ be an independent and uniformly distributed random index. Define the vector $W' \in \mathbb{R}^{2K}$ exactly as we defined W , except that $\zeta_{\underline{i}}$ is replaced with $\zeta'_{\underline{i}}$ throughout. More precisely, let $e_i \in \mathbb{R}^d$ be the i th standard basis vector in \mathcal{R}^d and define

$$\begin{aligned}\hat{w}'_k &= (\zeta + (\zeta'_{\underline{i}} - \zeta_{\underline{i}})e_{\underline{i}})^\top \hat{\mathbf{Q}}_k (\zeta + (\zeta'_{\underline{i}} - \zeta_{\underline{i}})e_{\underline{i}}) \\ &= \hat{w}_k + 2(\zeta'_{\underline{i}} - \zeta_{\underline{i}})e_{\underline{i}}^\top \hat{\mathbf{Q}}_k \zeta,\end{aligned}$$

$$\begin{aligned}\check{w}'_k &= (\zeta + (\zeta'_{\underline{i}} - \zeta_{\underline{i}})e_{\underline{i}})^\top \check{\mathbf{Q}}_k (\zeta + (\zeta'_{\underline{i}} - \zeta_{\underline{i}})e_{\underline{i}}) - \text{tr}(\mathbf{Q}_k) \\ &= \check{w}_k + e_{\underline{i}}^\top \check{\mathbf{Q}}_k e_{\underline{i}} ((\zeta'_{\underline{i}})^2 - \zeta_{\underline{i}}^2),\end{aligned}$$

for $k = 1, \dots, K$. Then $W' = (\hat{w}'_1, \check{w}'_1, \dots, \hat{w}'_K, \check{w}'_K)^\top \in \mathbb{R}^{2K}$. Its straightforward to verify that

$$\mathbb{E}(\hat{w}'_k - \hat{w}_k | \zeta) = -\frac{2}{d}\hat{w}_k, \quad \mathbb{E}(\check{w}'_k - \check{w}_k | \zeta) = -\frac{1}{d}\check{w}_k.$$

Then

$$\mathbb{E}(W' - W | \zeta) = -\Lambda_K W,$$

where

$$\Lambda_1 = \begin{pmatrix} \frac{2}{d} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}, \quad \Lambda_K = \begin{pmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \Lambda_1 \end{pmatrix} \in \mathbb{R}^{2K \times 2K}.$$

By exchangeability, we have

$$\begin{aligned}0 &= \frac{1}{2} \mathbb{E}[(W' - W)^\top \Lambda_K^{-\top} (\nabla g(W') + \nabla g(W))] \\ &= \mathbb{E}[(W' - W)^\top \Lambda_K^{-\top} \nabla g(W)] + \frac{1}{2} \mathbb{E}[(W' - W)^\top \Lambda_K^{-\top} (\nabla g(W') - \nabla g(W))] \\ &= -\mathbb{E}[W^\top \nabla g(W)] + \frac{1}{2} \mathbb{E}[(W' - W)^\top \Lambda_K^{-\top} (\nabla g(W') - \nabla g(W))].\end{aligned}$$

That is,

$$\mathbb{E}[W^\top \nabla g(W)] = \frac{1}{2} \mathbb{E}[(W' - W)^\top \Lambda_K^{-\top} (\nabla g(W') - \nabla g(W))].$$

Apply Taylor's theorem,

$$\begin{aligned}
& W^\top \nabla g(W) \\
&= \frac{1}{2} \sum_{i,j=1}^{2K} \Lambda_{K,ii}^{-1} D^{ij} g(W) (w'_i - w_i)(w'_j - w_j) + \frac{1}{4} \sum_{i,j,k=1}^{2K} \Lambda_{K,ii}^{-1} D^{ijk} g(W) (w'_i - w_i)(w'_j - w_j)(w'_k - w_k) \\
&\quad + \frac{1}{12} \sum_{i,j,k,l=1}^{2K} \Lambda_{K,ii}^{-1} D^{ijkl} g(t^*(W' - W) + W) (w'_i - w_i)(w'_j - w_j)(w'_k - w_k)(w'_l - w_l) \\
&= \frac{1}{2} \text{tr}[(W' - W)(W' - W)^\top \Lambda_K^{-\top} \nabla^2 g(W)] + \frac{1}{4} \sum_{i,j,k=1}^{2K} \Lambda_{K,ii}^{-1} D^{ijk} g(W) (w'_i - w_i)(w'_j - w_j)(w'_k - w_k) \\
&\quad + \frac{1}{12} \sum_{i,j,k,l=1}^{2K} \Lambda_{K,ii}^{-1} D^{ijkl} g(t^*(W' - W) + W) (w'_i - w_i)(w'_j - w_j)(w'_k - w_k)(w'_l - w_l),
\end{aligned} \tag{3}$$

where $t^* \in [0, 1]$. Also by exchangeability,

$$\mathbb{E}[(W' - W)(W' - W)^\top] = 2 \mathbb{E}[W(W - W')^\top] = 2 \mathbb{E}[WW^\top \Lambda_K^\top] = 2\mathbf{V}\Lambda_K^\top.$$

It follows that

$$\mathbb{E}[\nabla^\top \mathbf{V} \nabla g(W)] = \mathbb{E} \text{tr}[\mathbf{V} \nabla^2 g(W)] = \frac{1}{2} \mathbb{E} \text{tr}[\mathbb{E}[(W' - W)(W' - W)^\top] \Lambda_K^{-\top} \nabla^2 g(W)]$$

Thus,

$$\begin{aligned}
S &= \mathbb{E}[\nabla^\top \mathbf{V} \nabla g(W) - W^\top \nabla g(W)] \\
&= \frac{1}{2} \mathbb{E} \text{tr}[\mathbb{E}[(W' - W)(W' - W)^\top] \Lambda_K^{-\top} \nabla^2 g(W)] - \frac{1}{2} \mathbb{E} \text{tr}[(W' - W)(W' - W)^\top \Lambda_K^{-\top} \nabla^2 g(W)] \\
&\quad - \frac{1}{4} \mathbb{E} \sum_{i,j,k=1}^{2K} \Lambda_{K,ii}^{-1} D^{ijk} g(W) (w'_i - w_i)(w'_j - w_j)(w'_k - w_k) \\
&\quad - \frac{1}{12} \mathbb{E} \sum_{i,j,k,l=1}^{2K} \Lambda_{K,ii}^{-1} D^{ijkl} g(t^*(W' - W) + W) (w'_i - w_i)(w'_j - w_j)(w'_k - w_k)(w'_l - w_l).
\end{aligned}$$

□

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