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A Bayesian-motivated test for high-dimensional linear regression models with fixed design matrix

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Abstract This paper considers testing regression coefficients in high-dimensional linear model with fixed design matrix. We prove that there does not exist nontrivial unbiased test for this problem. Moreover, no test can guarantee nontrivial power even when the true model is largely deviate from the null hypothesis. Nevertheless, Bayesian methods can still produce tests with good average power behavior. We propose a new test statistic which is the limit of Bayes factors under normal distribution. The null distribution of the proposed test statistic is approximated by Lindeberg's replacement trick. Under certain conditions, the global asymptotic power function of the proposed test is also given. The finite sample performance of the proposed test is demonstrated via simulation studies.

Keywords Fixed design matrix · High-dimensional test · Lindeberg method · Linear model · Unbiasedness

Mathematics Subject Classification (2010) 62H15 · 62J05

1 Introduction

Consider linear regression model of the form

$$\mathbf{y} = \mathbf{X}_a\boldsymbol{\beta}_a + \mathbf{X}_b\boldsymbol{\beta}_b + \boldsymbol{\epsilon}, \tag{1}$$

where $\mathbf{y} \in \mathbb{R}^n$ is the response, $\mathbf{X}_a$, $\mathbf{X}_b$ are $n \times q$ and $n \times p$ design matrices, respectively, $\boldsymbol{\beta}_a \in \mathbb{R}^q$, $\boldsymbol{\beta}_b \in \mathbb{R}^p$ are unknown regression coefficients, and $\boldsymbol{\epsilon} \in$

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$\mathbb{R}^n$ is the vector of random errors. Here the design matrix $\mathbf{X}_a$ contains the predictors that are known to have effects on the response, and our interest is to test if $\mathbf{X}_b$ contains any useful predictors. That is, we would like to test the hypotheses

$$\mathcal{H}_0 : \boldsymbol{\beta}_b = 0, \quad \text{v.s.} \quad \mathcal{H}_1 : \boldsymbol{\beta}_b \neq 0. \tag{2}$$

The conventional test for hypotheses (2) is the F-test which is also the likelihood ratio test under normal errors. However, the F-test is not well defined in high-dimensional setting. In fact, if $\boldsymbol{\epsilon}$ is normal distributed and $\mathrm{Rank}[\mathbf{X}_a; \mathbf{X}_b] = n$, then the likelihood is unbounded under the alternative hypothesis. Thus, new test methodology is required in high-dimensional setting.

For the problem of testing hypotheses (2), two different high-dimensional settings have been extensively considered in the literature. One is the small p, large q setting. An important example of this setting is testing individual coefficients of a high-dimensional regression. See Bühlmann (2013); Zhang and Zhang (2014); Lan et al. (2016b) for testing procedures in this setting. In this paper, however, we focus on the other setting, namely the large p, small q setting. In this case, there are just a few covariates, namely $\mathbf{X}_a$, are known to have effects on the response, while there remain a large number of covariates, namely $\mathbf{X}_b$, to be tested.

Many test procedures have been proposed in the large p, small q setting. Based on an empirical Bayes model, Goeman et al. (2006) and Goeman et al. (2011) proposed a score test statistic as well as a method to determine the critical value. This score test was further investigated by Lan et al. (2014) and Lan et al. (2016a). Based on U-statistics, Zhong and Chen (2011) proposed a test for the case where $\mathbf{X}_a = \mathbf{1}_n$. Their method was then extended and improved by Wang and Cui (2015) and Cui et al. (2018), respectively. To accommodate outlying observations and heavy-tailed distributions, Feng et al. (2013) proposed a rank-based test for the entire coefficients. Xu (2016) modified the test of Feng et al. (2013) and proposed a scalar invariant rank-based test. Janson et al. (2016) utilized a convex optimization program to obtain an unbiased estimator of a signal to noise ratio with small variance, based on which a test is constructed. Apart from the aforementioned tests, there is another line of research utilizing desparsified Lasso estimator; see, e.g., Dezeure et al. (2017) Zhang and Cheng (2017).

Most existing high-dimensional tests adopted the random design assumption, that is, the rows of $\mathbf{X}_b$ are considered as being generated from a super population. Often in practice, however, at least a part of the design matrix is designed by the experimenter; see Draper and Pukelsheim (1996) and the references therein. In this case, assuming a random design is not appropriate. Hence in this paper, we consider the fixed design setting. Of course, our test method is still valid for random design from a conditional inference perspective. As noted by Lei et al. (2018), assuming a fixed design or a random design could lead to qualitatively different inferential results. In frequentist hypotheses testing, unbiasedness and minimaxity are two important considerations of test methods; see, e.g., Lehmann and Romano (2005). Under the random design assumption, nontrivial unbiased test does exist and the testing problem can be considered from the minimax perspective; see, e.g., Ingster et al. (2010). Surprisingly, these two frequentist considerations can not be applied in the fixed design setting. In fact, in Section 2, we shall show that in the fixed design setting, the only unbiased test and minimax test is the trivial constant function $\varphi(\mathbf{y}) \equiv \alpha$. This negative result of frequentist considerations motivates us to propose a frequentist test with the aid of Bayesian methods.

In this paper, we propose a new test statistic for hypotheses (2) which is the limit of Bayes factors as the prior magnitude of $\boldsymbol{\beta}_b$ tend to infinity. The proposed test statistic is a ratio of quadratic forms of $\mathbf{y}$. We give an approximation of the distribution of quadratic form using Lindeberg's replacement trick. Based on this approximated distribution, the critical value of the proposed test statisitc is calculated by a one step iteration. We prove that the proposed test procedure is valid under weak conditions. In particular, the validity of the proposed test procedure does not require that the test statistic is asymptotically normally distributed. The proposed test statistic is closely related to the test statistic of Goeman et al. (2006) which is also derived by a Bayesian argument. However, Goeman et al. (2006) let the prior magnitude of $\boldsymbol{\beta}_b$ tend to zero, which is contrary to our strategy. Under certain conditions, we derive the asymptotic power function of the proposed test and the test of Goeman et al. (2006). It is shown that the proposed test can detect the signals from more directions than the test of Goeman et al. (2006).

This paper has made three main contributions for testing hypotheses (2) in large p, small q setting with fixed design matrix. First, we prove that no test can detect all large deviations from the null hypothesis. Second, we propose a novel Bayesian-motivated test, which can be regarded as an extention of the likelihood ratio test in high-dimensional setting. Third, we propose to use Lindeberg's replacement trick to approximate the distribution of quadratic forms.

The rest of the paper is organized as follows. In Section 2, we prove the nonexistence of unbiased test for hypotheses (2) in large p, small q setting. In Section 3, we propose a Bayesian-motivated test and study the theoretical properties of the proposed test procedure. The asymptotic power function is also derived. Section 4 contains the simulation results. Section 5 concludes the paper. The technical proofs are presented in Appendix.

2 Nonexistence of unbiased test

In this paper, we consider testing hypotheses (2) in large p, small q setting. More specifically, we make the following assumption throughout the paper.

Assumption 1 *In the linear model* (1)*, suppose* $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)^\top$ *where* $\epsilon_1, \ldots, \epsilon_n$ *are independent indentically distributed (iid) with* $\mathrm{E}(\epsilon_1) = 0$ *and* $\mathrm{Var}(\epsilon_1) = \phi^{-1}$*. Furthermore, suppose* $q < n$*,* $\mathrm{Rank}(\mathbf{X}_a) = q$ *and* $\mathrm{Rank}([\mathbf{X}_a; \mathbf{X}_b]) = n$*.*

Testing hypotheses (2) in large p, small q setting is a challenging problem. Goeman et al. (2006) noticed that their test has negligible power for many alternatives and consequently is not an unbiased test. Biased tests are often regarded as problematic in classical statistics. However, the situation is very different for our problem. In fact, we shall show that under normal assumption, there is no nontrivial unbiased test in large p, small q setting.

Let $\mathbf{P}_a = \mathbf{X}_a(\mathbf{X}_a^\top \mathbf{X}_a)^{-1}\mathbf{X}_a^\top$ be the projection matrix onto the column space of $\mathbf{X}_a$ and denote $\tilde{\mathbf{P}}_a = \mathbf{I}_n - \mathbf{P}_a$. Define $\boldsymbol{\theta} = (\boldsymbol{\beta}_a^\top, \boldsymbol{\beta}_b^\top, \phi)^\top$ and

$$\begin{aligned}
\Theta_0 &= \left\{ \boldsymbol{\theta} = (\boldsymbol{\beta}_a^\top, \boldsymbol{\beta}_b^\top, \phi)^\top : \boldsymbol{\beta}_a \in \mathbb{R}^q, \boldsymbol{\beta}_b = 0, \phi > 0 \right\}, \\
\Theta &= \left\{ \boldsymbol{\theta} = (\boldsymbol{\beta}_a^\top, \boldsymbol{\beta}_b^\top, \phi)^\top : \boldsymbol{\beta}_a \in \mathbb{R}^q, \boldsymbol{\beta}_b \in \mathbb{R}^p, \phi > 0 \right\}.
\end{aligned}$$

A test function $\varphi(\mathbf{y})$ of level α is a Borel measurable function from $\mathbb{R}^n$ to $\mathbb{R}$ such that $0 \le \varphi(\mathbf{y}) \le 1$ and $\mathrm{E}_{\boldsymbol{\theta}}[\varphi(\mathbf{y})] \le \alpha$ for $\boldsymbol{\theta} \in \Theta_0$. Let $\lambda(\cdot)$ be the Lebesgue measure on $\mathbb{R}^n$. We have the following theorem.

Theorem 1 *Suppose Assumption 1 holds. Furthermore, assume $\boldsymbol{\epsilon} \sim \mathcal{N}_n(0, \phi^{-1}\mathbf{I}_n)$. Let $\varphi(\mathbf{y})$ be a test function of level α, $0 < \alpha < 1$. If $\lambda(\{\mathbf{y} : \varphi(\mathbf{y}) \ne \alpha\}) > 0$, then for every $M \ge 0$, there exists a parameter $\boldsymbol{\theta} = (\boldsymbol{\beta}_a^\top, \boldsymbol{\beta}_b^\top, \phi)^\top \in \Theta$ such that $\phi \boldsymbol{\beta}_b^\top \mathbf{X}_b^\top \tilde{\mathbf{P}}_a \mathbf{X}_b \boldsymbol{\beta}_b > M$ and $\mathrm{E}_{\boldsymbol{\theta}}[\varphi(\mathbf{y})] < \alpha$.*

In Theorem 1, the quantity $\phi \boldsymbol{\beta}_b^\top \mathbf{X}_b^\top \tilde{\mathbf{P}}_a \mathbf{X}_b \boldsymbol{\beta}_b$ can be regarded as a signal to noise ratio. In any conventional sense, it is expected that the power of a reasonable test should tend to 1 as $\phi \boldsymbol{\beta}_b^\top \mathbf{X}_b^\top \tilde{\mathbf{P}}_a \mathbf{X}_b \boldsymbol{\beta}_b \to \infty$. Indeed, such a test can be easily constructed in low dimensional setting. However, Theorem 1 claims that it is impossible to find such a test in large p, small q setting. Moreover, Theorem 1 implies that the trivial test $\varphi(\mathbf{y}) \equiv \alpha$ (a.e. λ) is the only unbiased test and is also the only minimax test for testing $\mathcal{H}_0$ against $\mathcal{H}_1^M : \phi \boldsymbol{\beta}_b^\top \mathbf{X}_b^\top \tilde{\mathbf{P}}_a \mathbf{X}_b \boldsymbol{\beta}_b > M$ for any $M > 0$. This is somewhat surprising since nontrivial unbiased test and minimax test do exist in random design setting; see, e.g., Ingster et al. (2010).

Note that unbiasedness and minimaxity are two key considerations of frequentist hypotheses testing; see, e.g., Lehmann and Romano (2005). In our problem, however, these considerations can not be applied and there is no test with guaranteed power. Nevertheless, it is still feasible to propose a test with good average power behavior. Note that from the viewpoint of the decision theory, Bayesian methods aim at minimizing the average risk; see, e.g., Lehmann and Romano (2005), Section 1.6. This motivates us to propose a frequentist test with the aid of Bayesian methods.

3 A Bayesian-motivated test

In Bayesian literature, Bayes factor (see, e.g., Kass and Raftery (1995)) is a fundamental tool for hypotheses testing and model comparison. So far, many Bayesian tests have been proposed for hypothesis (2) in low-dimensional setting; see Javier Girón et al. (2006); Goddard and Johnson (2016); Zhou and

Guan (2018) and the references therein. However, most existing Bayesian tests are not applicable in large p, small q setting. In this section, we shall propose a new frequentist test statistic in large p, small q setting which is the limit of Bayesian factors. In principle, using Bayes factor as a frequentist test statistic in high-dimensional setting is a good strategy for at least two reasons. First, the Bayes factors corresponding to proper priors are always well defined, even if the likelihood is unbounded. Second, under mild conditions, tests based on Bayes factors are admissible (See, e.g., Lehmann and Romano (2005), Theorem 6.7.2).

3.1 Test statistic

The new test statistic will be derived under normal assumption. Later we will show that the proposed test method can be applied for general distribution. Suppose $\boldsymbol{\epsilon} \sim \mathcal{N}_n(0, \phi^{-1}\mathbf{I}_n)$, then the Bayes factor for hypotheses (2) is

$$B_{10} = \frac{\int d\mathcal{N}_n(\mathbf{X}_a\boldsymbol{\beta}_a + \mathbf{X}_b\boldsymbol{\beta}_b, \phi^{-1}\mathbf{I}_n)(\mathbf{y})\pi_1(\boldsymbol{\beta}_b, \boldsymbol{\beta}_a, \phi)\, \mathrm{d}\boldsymbol{\beta}_b \mathrm{d}\boldsymbol{\beta}_a \mathrm{d}\phi}{\int d\mathcal{N}_n(\mathbf{X}_a\boldsymbol{\beta}_a, \phi^{-1}\mathbf{I}_n)(\mathbf{y})\pi_0(\boldsymbol{\beta}_a, \phi)\, \mathrm{d}\boldsymbol{\beta}_a \mathrm{d}\phi},$$

where $d\mathcal{N}_n(\mu, \boldsymbol{\Sigma})(\mathbf{y})$ is the density function of a $\mathcal{N}_n(\mu, \boldsymbol{\Sigma})$ random vector evaluated at $\mathbf{y}$, $\pi_0(\boldsymbol{\beta}_a, \phi)$ and $\pi_1(\boldsymbol{\beta}_b, \boldsymbol{\beta}_a, \phi)$ are the prior densities under the null and alternative hypotheses, respectively. If B_{10} is large, the alternative hypothesis is preferred. It is known that if the true parameters indeed come from the specified prior distribution, then the Bayes factor is the likelihood ratio statistic and consequently, the test based on the Bayes factor is the most powerful test. We notice that in the frequentist framework, the Bayes factor can be treated as a test statisitc provided the critical value can be determined to preserve the test level.

The behavior of a Bayes factor largely depends on the choice of priors. In Bayesian literature, many priors have been considered for testing the coefficients of linear model. Popular priors include g-priors (Liang et al., 2008) and intrinsic priors (Casella and Moreno, 2006). Unfortunately, these priors are not well defined in large p, small q setting. Note that under the null hypothesis $\mathcal{H}_0$, the model is low-dimensional. This allows us to impose the reference prior $\pi_0(\boldsymbol{\beta}_a, \phi) = c/\phi$, where c is a constant. Under $\mathcal{H}_1$, write $\pi_1(\boldsymbol{\beta}_b, \boldsymbol{\beta}_a, \phi) = \pi_1(\boldsymbol{\beta}_b|\boldsymbol{\beta}_a, \phi)\pi_1(\boldsymbol{\beta}_a, \phi)$. For parameters $\boldsymbol{\beta}_a$ and ϕ, we consider the same prior as in $\mathcal{H}_0$, that is $\pi_1(\boldsymbol{\beta}_a, \phi) = \pi_0(\boldsymbol{\beta}_a, \phi)$. For parameter $\boldsymbol{\beta}_b$, however, imposing the improper reference prior would not produce valid marginal density of $\mathbf{y}$. To make the marginal density of $\mathbf{y}$ well defined, we consider the simple normal prior $p_1(\boldsymbol{\beta}_b|\boldsymbol{\beta}_a, \phi) = d\mathcal{N}_p(0, \kappa^{-1}\phi^{-1}\mathbf{I}_p)(\boldsymbol{\beta}_b)$, where $\kappa > 0$ is a hyperparameter. That is, we put the following priors,

$$\pi_0(\boldsymbol{\beta}_a, \phi) = \frac{c}{\phi}, \quad \pi_1(\boldsymbol{\beta}_b, \boldsymbol{\beta}_a, \phi) = \frac{c}{\phi} d\mathcal{N}_p\left(0, \frac{1}{\kappa\phi}\mathbf{I}_p\right)(\boldsymbol{\beta}_b). \tag{3}$$

Let $B_{10,\kappa}$ be the Bayes factor corresponding to the priors (3). It is straightforward to show that

$$\begin{aligned}
&2\log(B_{10,\kappa})\\
=&p\log\kappa-\log|\mathbf{X}_b^\top\tilde{\mathbf{P}}_a\mathbf{X}_b+\kappa\mathbf{I}_p|\\
&-(n-q)\log\left(1-\frac{\mathbf{y}^\top\tilde{\mathbf{P}}_a\mathbf{X}_b\left(\mathbf{X}_b^\top\tilde{\mathbf{P}}_a\mathbf{X}_b+\kappa\mathbf{I}_p\right)^{-1}\mathbf{X}_b^\top\tilde{\mathbf{P}}_a\mathbf{y}}{\mathbf{y}^\top\tilde{\mathbf{P}}_a\mathbf{y}}\right).
\end{aligned}\tag{4}$$

Denote by $\tilde{\mathbf{P}}_a=\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top$ the rank decomposition of $\tilde{\mathbf{P}}_a$, where $\tilde{\mathbf{U}}_a$ is an $n\times(n-q)$ column orthogonal matrix. Let $\mathbf{X}_b^*=\tilde{\mathbf{U}}_a^\top\mathbf{X}_b$, $\mathbf{y}^*=\tilde{\mathbf{U}}_a^\top\mathbf{y}$. Let γ_i be the ith largest eigenvalue of $\mathbf{X}_b^*\mathbf{X}_b^{*\top}$, $i=1,\ldots,n-q$. Denote by $\mathbf{X}_b^*=\mathbf{U}_b^*\mathbf{D}_b^*\mathbf{V}_b^{*\top}$ the singular value decomposition of $\mathbf{X}_b^*$, where $\mathbf{U}_b^*$, $\mathbf{V}_b^*$ are $(n-q)\times(n-q)$ and $p\times(n-q)$ column orthogonal matrices, respectively, and $\mathbf{D}_b^*=\mathrm{diag}(\sqrt{\gamma_1},\ldots,\sqrt{\gamma_{n-q}})$. Then we have

$$\begin{aligned}
&2\log(B_{10,\kappa})\\
=&(n-q)\log\kappa-\sum_{i=1}^{n-q}\log(\gamma_i+\kappa)\\
&-(n-q)\log\left(1-\frac{\mathbf{y}^{*\top}\mathbf{U}_b^*\mathbf{D}_b^*\left(\mathbf{D}_b^{*2}+\kappa\mathbf{I}_{n-q}\right)^{-1}\mathbf{D}_b^*\mathbf{U}_b^{*\top}\mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^*}\right).
\end{aligned}$$

The main part of the above expression is

$$T_\kappa=\frac{\mathbf{y}^{*\top}\mathbf{U}_b^*\mathbf{D}_b^*\left(\mathbf{D}_b^{*2}+\kappa\mathbf{I}_{n-q}\right)^{-1}\mathbf{D}_b^*\mathbf{U}_b^{*\top}\mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^*}.$$

A large value of T_κ supports the alternative hypothesis. Hence T_κ can be regarded as a frequentist test statistic. It remains to choose an appropriate hyperparameter κ. As Goeman et al. (2006) noted, the priors should place most probability on the alternatives which are perceived as more interesting to detect. Their strategy is to let the prior magnitude of $\boldsymbol{\beta}_b$ tend to zero to obtain a test with good power behavior under local alternatives, that is, $\|\boldsymbol{\beta}_b\|$ is small. Note that the prior magnitude of $\boldsymbol{\beta}_b$ decreases with the increase of κ. In fact, if we let κ tends to infinity, the limit

$$\lim_{\kappa\to\infty}\kappa T_\kappa=\frac{\mathbf{y}^{*\top}\mathbf{X}_b^*\mathbf{X}_b^{*\top}\mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^*}$$

is exactly the test statistic of Goeman et al. (2006).

As implied by Theorem 1, however, testing hypotheses (2) in large p, small q setting is such a difficult problem that no test can guarantee reasonable power, even if the true model is largely deviate from the null hypothesis. In this case, the global power behavior may worth more consideration than the local power behavior. Thus, contrary to the strategy of Goeman et al. (2006), we let κ tend to 0 to obtain a test with good average power behavior for large $\|\boldsymbol{\beta}_b\|$. Note that the statistic T_κ degenerates to 1 as $\kappa \to 0$. Nevertheless, the scaled statistic $(T_\kappa - 1)/\kappa$ has a proper limit as $\kappa \to 0$. Thus, we proposed the following test statistic

$$T = \lim_{\kappa \to 0} \frac{T_\kappa - 1}{\kappa} = -\frac{\mathbf{y}^{*\top}(\mathbf{X}_b^* \mathbf{X}_b^{*\top})^{-1}\mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^*}.$$

The null hypothesis will be rejected if T is large.

Note that the test of Goeman et al. (2006) can be used in both high-dimensional setting and low-dimensional setting. In comparison, the proposed test statistic T can only be used in high-dimensional setting as it involves the inverse of $\mathbf{X}_b^* \mathbf{X}_b^{*\top}$. Nevertheless, if we apply the proposed methodology in low-dimensional setting, that is, letting $\kappa \to 0$ in (4), then the resulting test statistic is exactly the likelihood ratio test. In this view, the proposed test is an extension of the likelihood ratio test to the high-dimensional setting.

3.2 Critical value

To formulate a valid frequentist test, we need to determine the critical value of T. If $\boldsymbol{\epsilon}$ were indeed normally distributed, then under the null hypothesis, $T \sim -(\sum_{i=1}^{n-q} \gamma_i^{-1} z_i^2)/(\sum_{i=1}^{n-q} z_i^2)$, where $z_1, \ldots, z_{n-q}$ are iid $\mathcal{N}(0,1)$ random variables. In this case, the exact critical value can be easily obtained. In practice, however, normal distribution often serves as an approximation rather than the true distribution of data. Hence it would be better to give a critical value of T_n which can preserve the test level under general distributions of $\boldsymbol{\epsilon}$. If the design matrix admits certain structures, the critical value can be well determined by permutation methods and bootstrap methods; see Arboretti et al. (2018); Baltagi et al. (2013) and the references therein. To deal with general design matrix, we shall employ a direct approach which approximates the distribution of the test statistic under the null hypothesis.

Under the null hypothesis,

$$T = -\frac{(\sqrt{\phi}\boldsymbol{\epsilon})^\top \tilde{\mathbf{U}}_a (\mathbf{X}_b^* \mathbf{X}_b^{*\top})^{-1} \tilde{\mathbf{U}}_a^\top (\sqrt{\phi}\boldsymbol{\epsilon})}{(\sqrt{\phi}\boldsymbol{\epsilon})^\top \tilde{\mathbf{U}}_a \tilde{\mathbf{U}}_a^\top (\sqrt{\phi}\boldsymbol{\epsilon})}.$$

The numerator and the denominator of the above expression are both quadratic forms of iid standardized random variables. Hence the key step towards the goal is to approximate the distribution of quadratic forms. The asymptotics of quadratic forms have been extensively studied; see, e.g., Bai et al. (2018); Bentkus and Götze (1996); Dicker and Erdogdu (2017); Götze and Tikhomirov (2002); Jiang (1996); Jong (1987). Most existing work focused on the asymptotic normality of quadratic forms under certain conditions. However, normal distribution is just one of the possible limit distributions of quadratic forms. See Sevast'yanov (1961) for a full characterization of the limit distributions of quadratic forms of normal random variables. We would like to approximate the distribution of quadratic forms under general conditions, not confined to the asymptotic normality case.

A general method for distribution approximation is Lindeberg's replacement trick, the technique used in the Lindeberg's original proof of central limit theorem. The idea of Lindeberg's replacement trick is to subsequently replace the random variables by suitable normal random variables; see, e.g., Pollard (1984), Section III.4. Recently, this technique shows great power for many difficult problems; see Chatterjee (2006) and the references therein. To approximate the distribution of quadratic forms, we shall apply Lindeberg's replacement trick to the squared terms and the cross-products terms, respectivey. Let $\mathscr{C}^4(\mathbb{R})$ denote the class of all bounded real functions on $\mathbb{R}$ having bounded, continuous kth derivatives, $0 \leq k \leq 4$. It is known that if $\mathrm{E}\, f(Z_n) \to \mathrm{E}\, f(Z)$ for every $f \in \mathscr{C}^4(\mathbb{R})$ then $Z_n \rightsquigarrow Z$; see, e.g., Pollard (1984), Theorem 12 of Chapter III. We have the following approximation theorem.

Theorem 2 *Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^\top$, where ξ_i's are iid random variable with $\mathrm{E}\,\xi_1 = 0$, $\mathrm{Var}(\xi_1) = 1$. Furthermore, suppose the distribution of ξ_1 is symmetric about 0 and has finite eighth moments. Let $\mathbf{A}$ be an $n \times n$ symmetric matrix with elements $a_{i,j}$. Define*

$$S = \frac{\boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi} - \mathrm{tr}(\mathbf{A})}{\sqrt{\mathrm{Var}(\boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi})}}.$$

Let $z_1, \ldots, z_n$ be iid random variables with distribution $\mathcal{N}(0,1)$ and $\check{z}_1, \ldots, \check{z}_n$ be iid random variables with distribution $\mathcal{N}(0,1)$ which are independent of $\xi_1, \ldots, \xi_n$. Let τ be a real number. Define

$$S_\tau^* = \frac{\tau \sum_{i=1}^n a_{i,i} \check{z}_i + 2 \sum_{1 \leq i < j \leq n} a_{i,j} z_i z_j}{\sqrt{\mathrm{Var}(\boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi})}}.$$

Then for every $f \in \mathscr{C}^4(\mathbb{R})$,

$$\begin{aligned}
&\left|\mathrm{E}\, f(S) - \mathrm{E}\, f(S_\tau^*)\right| \\
\leq &\frac{\left|\mathrm{E}(\xi_1^4) - 1 - \tau^2\right|}{2} \|f^{(2)}\|_\infty \frac{\sum_{i=1}^n a_{i,i}^2}{\mathrm{Var}(\boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi})} \\
&+ \frac{\max\left(\left|\mathrm{E}(\xi_1^2 - 1)^3\right|, 12(\mathrm{E}(\xi_1^4) - 1)\right)}{6} \|f^{(3)}\|_\infty \frac{\sum_{l=1}^n \left(|a_{l,l}| \sum_{i=1}^n a_{i,l}^2\right)}{\left(\mathrm{Var}(\boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi})\right)^{3/2}} \qquad (5) \\
&+ \frac{16\,\mathrm{E}(\xi_1^8) + 80\,\mathrm{E}(\xi_1^4) + 3\tau^4 + 96}{24} \|f^{(4)}\|_\infty \frac{\sum_{l=1}^n \left(\sum_{i=1}^n a_{i,l}^2\right)^2}{\left(\mathrm{Var}(\boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi})\right)^2}.
\end{aligned}$$

Remark 1 If $\tau^2 = \mathrm{E}(\xi_1^4) - 1$, the first term of the right hand side of (5) disappear. In practice, however, the quantity $\mathrm{E}(\xi_1^4)$ is often unknown and τ^2 should be chosen as an estimator of $\mathrm{E}(\xi_1^4) - 1$.

Remark 2 As noted in Chatterjee (2008), Section 3.1, an almost necessary condition for the asymptotic normality of S is

$$\frac{\operatorname{tr}(\mathbf{A}^4)}{\left(\operatorname{Var}(\boldsymbol{\xi}^\top \mathbf{A}\boldsymbol{\xi})\right)^2} \to 0. \tag{6}$$

On the other hand, it can be seen that the right hand side of (5) converges to 0 provided $\tau^2 = \mathrm{E}(\xi_1^4) - 1$ and

$$\frac{\sum_{l=1}^n \left(\sum_{i=1}^n a_{i,l}^2\right)^2}{\left(\operatorname{Var}(\boldsymbol{\xi}^\top \mathbf{A}\boldsymbol{\xi})\right)^2} \to 0. \tag{7}$$

It can be seen that (7) is much weaker than (6). For example, if $a_{i,j} = 1$, $i = 1, \ldots, n$, $j = 1, \ldots, n$ and $\mathrm{E}(\xi_1^4) = 3$, then the condition (7) holds but the condition (6) does not hold.

We now apply Theorem 2 to approximate the null distribution of the proposed statistic T. For $n \times n$ matrix $\mathbf{A}$ and real number τ, let $F(x; \mathbf{A}, \tau)$ be the cumulative distribution function of $\tau \sum_{i=1}^n a_{i,i}\check{z}_i + 2\sum_{1\leq i<j\leq n} a_{i,j} z_i z_j$, where $z_1, \ldots, z_n, \check{z}_1, \ldots, \check{z}_n$ are iid $\mathcal{N}(0,1)$ random variables.

Under the null hypothesis, Theorem 2 implies that

$$\begin{aligned}
&\Pr(T > x)\\
=&\Pr\left((\sqrt{\phi}\boldsymbol{\epsilon})^\top \left(-\tilde{\mathbf{U}}_a(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1}\tilde{\mathbf{U}}_a^\top - x\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top\right)(\sqrt{\phi}\boldsymbol{\epsilon}) > 0\right)\\
\approx&1 - F\Bigg(\operatorname{tr}\left((\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1}\right) + (n-q)x;\\
&\qquad -\tilde{\mathbf{U}}_a(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1}\tilde{\mathbf{U}}_a^\top - x\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top, \sqrt{\phi^2\,\mathrm{E}(\epsilon_1^4) - 1}\Bigg).
\end{aligned}$$

Thus, in order to obtain a test with level α asymptotically, the ideal strategy is to find a consistent estimator $\hat{\tau}$ of $\sqrt{\phi^2\,\mathrm{E}(\epsilon_1^4) - 1}$ and solve the critical value x from the equation

$$F\left(\operatorname{tr}(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1} + (n-q)x; -\tilde{\mathbf{U}}_a(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1}\tilde{\mathbf{U}}_a^\top - x\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top, \hat{\tau}\right) = 1 - \alpha.$$

However, solving this equation is not an easy task. For ease of implementation, we propose a one step iteration algorithm. We set the start point as

$$x^{(0)} = -\frac{\operatorname{tr}\left((\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1}\right)}{n-q},$$

which is chosen such that $\operatorname{tr}(-\tilde{\mathbf{U}}_a(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1}\tilde{\mathbf{U}}_a^\top - x^{(0)}\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top) = 0$. Then the critical value obtained from the one step iteration is

$$x^{(1)} = \frac{F^{-1}\left(1-\alpha; -\tilde{\mathbf{U}}_a(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1}\tilde{\mathbf{U}}_a^\top - x^{(0)}\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top, \hat{\tau}\right) - \operatorname{tr}(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1}}{n-q}.$$

As noted in Remark 1, here $\hat{\tau}^2$ should be a consistent estimator of $\phi^2\operatorname{E}(\epsilon_1^4) - 1$ under the null hypothesis.

Theorem 3 *Suppose Assumption 1 holds. Furthermore, suppose the distribution of ϵ_1 is symmetric about 0 and has finite eighth moments. Suppose $\operatorname{E}\epsilon_1^4 > \phi^{-2}$. Let*

$$\mathbf{A} = -\tilde{\mathbf{U}}_a(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1}\tilde{\mathbf{U}}_a^\top - x^{(0)}\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top.$$

Suppose as $n \to \infty$,

$$\frac{\max_{i,j} a_{i,j}^2}{\operatorname{tr}(\mathbf{A})^2} \to 0. \tag{8}$$

Let $\hat{\tau}^2$ be an consistent estimator of $\phi^2\operatorname{E}(\epsilon_1^4) - 1$ based on $\mathbf{X}$, $\mathbf{y}$. Then

$$\Pr\left(T > x^{(1)}\right) \to \alpha.$$

It remains to consistently estimate $\phi^2\operatorname{E}(\epsilon_1^4) - 1$. Bai et al. (2018) gave a consistent estimator of $\phi^2\operatorname{E}(\epsilon_1^4) - 1$ based on the standardized residuals. Here we use a slightly different estimator which is based on the ordinary least squares residuals $\tilde{\boldsymbol{\epsilon}} = (\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_n)^\top = \tilde{\mathbf{P}}_a\mathbf{y}$. From Bai et al. (2018), Theorem 2.1,

$$\begin{aligned}
&\operatorname{E}\left(\tilde{\boldsymbol{\epsilon}}^\top\tilde{\mathbf{P}}_a\tilde{\boldsymbol{\epsilon}}\right) = (n-q)\phi^{-1},\\
&\operatorname{E}\left(\sum_{i=1}^n \tilde{\epsilon}_i^4\right) = 3\phi^{-2}\operatorname{tr}(\tilde{\mathbf{P}}_a^{\circ 2}) + \left(\operatorname{E}(\epsilon_1^4) - 3\phi^{-2}\right)\operatorname{tr}\left(\tilde{\mathbf{P}}_a^{\circ 2}\right)^2,
\end{aligned}$$

where $\tilde{\mathbf{P}}_a^{\circ 2} = \tilde{\mathbf{P}}_a \circ \tilde{\mathbf{P}}_a$ and $\circ$ means Hadamard product. Then a moment estimator of $\phi^2\operatorname{E}(\epsilon_1^4) - 1$ is

$$\hat{\tau}^2 = \frac{\dfrac{(n-q)^2\sum_{i=1}^n \tilde{\epsilon}_i^4}{\left(\tilde{\boldsymbol{\epsilon}}^\top\tilde{\mathbf{P}}_a\tilde{\boldsymbol{\epsilon}}\right)^2} - 3\operatorname{tr}(\tilde{\mathbf{P}}_a^{\circ 2})}{\operatorname{tr}\left(\tilde{\mathbf{P}}_a^{\circ 2}\right)^2} + 2.$$

Proposition 1 *Suppose the assumptions of Theorem 3 hold. Furthermore, suppose $q/n \to 0$. Then under the null hypothesis, $\hat{\tau}^2 \xrightarrow{P} \phi^2\operatorname{E}(\epsilon_1^4) - 1$.*

We reject the null hypothesis if

$$T > x^{(1)}.$$

This test procedure is asymptotically exact of size α under the conditions of Theorem 3 and Proposition 1.

3.3 Power analysis

In this section, we investigate the asymptotic power of the proposed test procedure as well as the test of Goeman et al. (2006). To make the expression of the asymptotic power functions tractable, we shall assume further conditions so that the test statistics are asymptotically normally distributed. Also, $\boldsymbol{\epsilon}$ is assumed to be normally distributed so that we can obtain the global power function rather than only local power function.

To derive the asymptotic power of the proposed test and the test of Goeman et al. (2006) simultaneously, we consider the general statistic $\mathbf{y}^{*\top}(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^k\mathbf{y}^*/\mathbf{y}^{*\top}\mathbf{y}^*$. In fact, the proposed test statistic corresponds to $k=-1$ while the test statistic of Goeman et al. (2006) corresponds to $k=1$. Note that for any $x \in \mathbb{R}$,

$$\Pr\left(\frac{\mathbf{y}^{*\top}(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^k\mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^*} \le x\right) = \Pr\left(\mathbf{y}^{*\top}\left((\mathbf{X}_b^*\mathbf{X}_b^{*\top})^k - x\mathbf{I}_{n-q}\right)\mathbf{y}^* \le 0\right).$$

Hence the asymptotic behavior of noncentral quadratic form will play a key role in our investigation. We have the following proposition.

Proposition 2 *Let $Z = (z_1, \ldots, z_n)^\top$, where z_i's are iid $\mathcal{N}(0,1)$ random variables. Let $\mathbf{A}$ be an $n \times n$ symmetric matrix with elements $a_{i,j}$. Let $\mathbf{b} = (b_1, \ldots, b_n)^\top$ be an n dimensional vector. If $\operatorname{tr}(\mathbf{A}^4)/\operatorname{tr}^2(\mathbf{A}^2) \to 0$, then*

$$\frac{Z^\top\mathbf{A}Z + b^\top Z - \operatorname{tr}(\mathbf{A})}{\sqrt{2\operatorname{tr}(\mathbf{A}^2) + \|\mathbf{b}\|^2}} \rightsquigarrow \mathcal{N}(0,1).$$

Remark 3 Proposition 2 does not impose any condition on $\mathbf{b}$. This allows us to give the global asymptotic power function of tests. As the cost of this flexibility, we have to make the normal assumption.

Now we investigate the asymptotic behavior of $\mathbf{y}^{*\top}(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^k\mathbf{y}^*/\mathbf{y}^{*\top}\mathbf{y}^*$. Let $w_i = (\mathbf{V}_b^{*\top}\boldsymbol{\beta}_b)_i$ be the coordinate of β_b along the ith principal component direction of $\mathbf{X}_b^{*\top}\mathbf{X}_b^*$, $i = 1, \ldots, n-q$. Let I be a random variable with uniform distribution on $\{1, \ldots, n-q\}$, that is, $\Pr(I = i) = i/(n-q)$, $i = 1, \ldots, n-q$. It turns out that many quantities involved can be conveniently represented as the expectations with respect to I. For example, $\operatorname{tr}(\mathbf{X}_b^*\mathbf{X}_b^{*\top})^k/(n-q) = \sum_{i=1}^{n-q}\gamma_i^k/(n-q) = \mathrm{E}(\gamma_I^k)$.

Theorem 4 *Suppose model* (1) *holds with $\boldsymbol{\epsilon} \sim \mathcal{N}_n(0, \phi^{-1}\mathbf{I}_n)$. Let $k \neq 0$ be a fixed number. Suppose as $n \to \infty$, $n-q \to \infty$ and*

$$\frac{\max_{1\le i\le n-q}\left(\gamma_i^k - \mathrm{E}(\gamma_I^k)\right)^2}{(n-q)\operatorname{Var}(\gamma_I^k)} \to 0. \tag{9}$$

Then for any $x \in \mathbb{R}$,

$$
\begin{aligned}
&\Pr\left(\frac{\mathbf{y}^{*\top}\left(\mathbf{X}_b^*\mathbf{X}_b^{*\top}\right)^k\mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^*} \le \mathrm{E}(\gamma_I^k) + \sqrt{\frac{2\operatorname{Var}\left(\gamma_I^k\right)}{n-q}}x\right) \\
=&\Phi\left(\frac{\left(\phi\,\mathrm{E}(\gamma_I w_I^2)+1\right)x - \sqrt{n-q}\,\phi\frac{\operatorname{Cov}\left(\gamma_I^k,\gamma_I w_I^2\right)}{\sqrt{2\operatorname{Var}\left(\gamma_I^k\right)}}}{\sqrt{1+2\phi\,\mathrm{E}\left[\left(\frac{\gamma_I^k-\mathrm{E}(\gamma_I^k)}{\sqrt{\operatorname{Var}\left(\gamma_I^k\right)}} - \sqrt{\frac{2}{n-q}}x\right)^2\gamma_I w_I^2\right]}}\right) + o(1),
\end{aligned}
$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable.

Under the conditions of Theorem 4, the proposed test should reject the null hypothesis when

$$
\frac{\mathbf{y}^{*\top}\left(\mathbf{X}_b^*\mathbf{X}_b^{*\top}\right)^{-1}\mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^*} \le \mathrm{E}(\gamma_I^{-1}) + \sqrt{\frac{2\operatorname{Var}\left(\gamma_I^{-1}\right)}{n-q}}\Phi^{-1}(\alpha),
$$

and the asymptotic power function of the proposed test is

$$
\Phi\left(\frac{\left(\phi\,\mathrm{E}(\gamma_I w_I^2)+1\right)\Phi^{-1}(\alpha) + \sqrt{n-q}\,\phi\frac{\operatorname{Cov}\left(-\gamma_I^{-1},\gamma_I w_I^2\right)}{\sqrt{2\operatorname{Var}(\gamma_I^{-1})}}}{\sqrt{1+2\phi\,\mathrm{E}\left[\left(\frac{\gamma_I^{-1}-\mathrm{E}(\gamma_I^{-1})}{\sqrt{\operatorname{Var}\left(\gamma_I^{-1}\right)}} - \sqrt{\frac{2}{n-q}}\Phi^{-1}(\alpha)\right)^2\gamma_I w_I^2\right]}}\right).
$$

On the other hand, the test of Goeman et al. (2006) should reject the null hypothesis when

$$
\frac{\mathbf{y}^{*\top}\mathbf{X}_b^*\mathbf{X}_b^{*\top}\mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^*} > \mathrm{E}(\gamma_I) + \sqrt{\frac{2\operatorname{Var}(\gamma_I)}{n-q}}\Phi^{-1}(1-\alpha),
$$

and the asymptotic power function of their test is

$$
\Phi\left(\frac{\left(\phi\,\mathrm{E}(\gamma_I w_I^2)+1\right)\Phi^{-1}(\alpha) + \sqrt{n-q}\,\phi\frac{\operatorname{Cov}\left(\gamma_I,\gamma_I w_I^2\right)}{\sqrt{2\operatorname{Var}(\gamma_I)}}}{\sqrt{1+2\phi\,\mathrm{E}\left[\left(\frac{\gamma_I-\mathrm{E}(\gamma_I)}{\sqrt{\operatorname{Var}(\gamma_I)}} + \sqrt{\frac{2}{n-q}}\Phi^{-1}(\alpha)\right)^2\gamma_I w_I^2\right]}}\right).
$$

Although the global asymptotic power functions are obtained, their forms are complicated, which makes it hard to compare the two tests in general. To facilitate the analysis, we would like to further simplify the forms of the power functions by imposing more conditions. It turns out that some terms in the power functions are negligible under the local alternative hypothesis, as the following proposition shows.

Proposition 3 *Suppose the conditions of Theorem 4 hold. Furthermore, suppose* $\mathrm{E}(\gamma_I w_I^2) = O((n-q)^{-1/2})$ *and* $\max_{i\in\{1,\ldots,n-q\}}(\gamma_i w_i^2) = o(1)$. *Then for any* $x \in \mathbb{R}$,

$$\Pr\left(\frac{\mathbf{y}^{*\top}\left(\mathbf{X}_b^*\mathbf{X}_b^{*\top}\right)^k\mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^*} \le \mathrm{E}(\gamma_I^k) + \sqrt{\frac{2\,\mathrm{Var}\left(\gamma_I^k\right)}{n-q}}x\right)$$

$$=\Phi\left(x - \sqrt{n-q}\phi\frac{\mathrm{Cov}\left(\gamma_I^k, \gamma_I w_I^2\right)}{\sqrt{2\,\mathrm{Var}\left(\gamma_I^k\right)}}\right) + o(1).$$

Under the conditions of Proposition 3, the asymptotic power function of the proposed test is

$$\Phi\left(\Phi^{-1}(\alpha) + \sqrt{n-q}\phi\frac{\mathrm{Cov}(-\gamma_I^{-1}, \gamma_I w_I^2)}{\sqrt{2\,\mathrm{Var}(\gamma_I^{-1})}}\right).$$

On the other hand, the asymptotic power function of the test of Goeman et al. (2006) is

$$\Phi\left(\Phi^{-1}(\alpha) + \sqrt{n-q}\phi\frac{\mathrm{Cov}(\gamma_I, \gamma_I w_I^2)}{\sqrt{2\,\mathrm{Var}(\gamma_I)}}\right).$$

It can be seen that the local power of the proposed test mainly depends on $\mathrm{Cov}(-\gamma_I^{-1}, \gamma_I w_I^2)/\sqrt{\mathrm{Var}(\gamma_I^{-1})}$ while the power of the test of Goeman et al. (2006) mainly depends on $\mathrm{Cov}(\gamma_I, \gamma_I w_I^2)/\sqrt{\mathrm{Var}(\gamma_I)}$. Unfortunately, neither of these two quantities is positive definite. This fact is not surprising in view of Theorem 1. Nevertheless, $\mathrm{Cov}(-\gamma_I^{-1}, \gamma_I w_I^2)$ and $\mathrm{Cov}(\gamma_I, \gamma_I w_I^2)$ are positive definite if $\boldsymbol{\beta}_b$ is restricted in certain subspaces of $\mathbb{R}^p$.

Let d_1 be the maximum i such that $\gamma_i^{-1} < \mathrm{E}(\gamma_I^{-1})$. Let d_2 be the maximum i such that $\gamma_i > \mathrm{E}(\gamma_I)$. Then it can be seen that

$$\mathrm{Cov}(-\gamma_I^{-1}, \gamma_I w_I^2) = \frac{1}{n-q}\sum_{i=1}^{n-q}\left(\mathrm{E}(\gamma_I^{-1}) - \gamma_i^{-1}\right)\gamma_i w_i^2$$

is positive definite if $w_{d_1+1} = \cdots = w_{n-q} = 0$, while

$$\mathrm{Cov}(\gamma_I, \gamma_I w_I^2) = \frac{1}{n-q}\sum_{i=1}^{n-q}(\gamma_i - \mathrm{E}\,\gamma_I)\gamma_i w_i^2$$

is positive definite if $w_{d_2+1} = \cdots = w_{n-q} = 0$. In other words, $\mathrm{Cov}(-\gamma_I^{-1}, \gamma_I w_I^2)$ is positive definite if $\boldsymbol{\beta}_b$ belongs to the rank d_1 principal component subspace of $\mathbf{X}_b^{*\top}\mathbf{X}_b^*$, and $\mathrm{Cov}(\gamma_I, \gamma_I w_I^2)$ is positive definite if $\boldsymbol{\beta}_b$ belongs to the rank d_2 principal component subspace of $\mathbf{X}_b^{*\top}\mathbf{X}_b^*$. Note that $\gamma_i > \mathrm{E}(\gamma_I)$ implies that $\gamma_i^{-1} < (\mathrm{E}(\gamma_I))^{-1} \le \mathrm{E}(\gamma_I^{-1})$, where the last inequality follows from Jensen's inequality. It follows that $d_1 \ge d_2$. Hence the proposed test can always detect the signals from more directions than the test of Goeman et al. (2006).

4 Numerical results

In this section, we conduct simulations to examine the empirical size and power of the proposed test (abbreviated as NEW) and compare it with the global test proposed by Goeman et al. (2006) (abbreviated as GT), the EigenPrism test proposed by Janson et al. (2016) (abbreviated as EP) and the desparsifying lasso test proposed by Zhang and Cheng (2017) (abbreviated as DL).

The original test statistic of Zhang and Cheng (2017) is $\sqrt{n}|\breve{\boldsymbol{\beta}}_b|_\infty$, where $(\breve{\boldsymbol{\beta}}_a^\top, \breve{\boldsymbol{\beta}}_b^\top)^\top$ is the desparsifying estimator of $(\boldsymbol{\beta}_a^\top, \boldsymbol{\beta}_b^\top)^\top$. However, the computation of their test procedure is rather time-consuming. Nevertheless, the behavior of their test statistic is asymptotically equivalent to a simple one. In fact, Zhang and Cheng (2017), Theorem 2.1 implies that under certain regular conditions,

$$\sqrt{n}(\breve{\boldsymbol{\beta}}_b - \boldsymbol{\beta}_b) = [\mathbf{0}_{p\times q}; \mathbf{I}_p]\hat{\Omega}[\mathbf{X}_a; \mathbf{X}_b]^\top \epsilon/\sqrt{n} + \Delta, \quad \|\Delta\|_\infty = o_P(1),$$

where $\hat{\Omega}$ is the approximation for the inverse of $[\mathbf{X}_a; \mathbf{X}_b]^\top [\mathbf{X}_a; \mathbf{X}_b]/n$ given by the nodewise lasso regression. Hence in simulations, we use the oracle version of their test which reject the null hypothesis when

$$\left\| \sqrt{n}\boldsymbol{\beta}_b + [\mathbf{0}_{p\times q}; \mathbf{I}_p]\hat{\Omega}[\mathbf{X}_a; \mathbf{X}_b]^\top \epsilon/\sqrt{n} \right\|_\infty$$

is large.

Throughout the simulations, we take $q = 10$, $p = 1000$, $\alpha = 0.05$ and the empirical results are obtained from 3000 replications. We consider two distributions of ϵ_1, namely the chi-squared distribution $\epsilon_1 \sim (\chi^2(4) - 4)/\sqrt{8}$ and the Student's t distribution $\epsilon_1 \sim t_9$. Define the signal to noise ratio (SNR) as $\mathrm{SNR} = \sqrt{(n-q)\operatorname{Var}(\gamma_I)}\phi\|\boldsymbol{\beta}_b\|^2/p$. We consider two structures of $\boldsymbol{\beta}_b$: dense $\boldsymbol{\beta}_b$ and sparse $\boldsymbol{\beta}_b$. In dense $\boldsymbol{\beta}_b$ setting, the coordinates of $\boldsymbol{\beta}_b$ are independently generated from the uniform distribution $U(-c, c)$ where c is selected to reach certain SNR. In sparse $\boldsymbol{\beta}_b$ setting, we randomly select 5% of the coordinates of $\boldsymbol{\beta}_b$ to be non-zero and the non-zero coordinates are independently generated from $U(-c, c)$ where c is selected to reach certain SNR. The design matrices $\mathbf{X}_a$ and $\mathbf{X}_b$ are randomly generated beforehand and are fixed during each simulation. In all simulations, the elements of $\mathbf{X}_a$ are iid from standard normal distribution. We consider four models of $\mathbf{X}_b$, as follows.

- Model I: the rows of $\mathbf{X}_b$ are iid from $\mathcal{N}(0, \boldsymbol{\Sigma}_b)$, where $\boldsymbol{\Sigma}_b = \Gamma\Gamma^\top$ and Γ is a $p \times p$ matrix whose elements are iid generated from $\mathcal{N}(0, 1)$.
- Model II: the rows of $\mathbf{X}_b$ are iid from $\mathcal{N}(0, \boldsymbol{\Sigma}_b)$, where $(\boldsymbol{\Sigma}_b)_{i,j} = 0.1$ if $i \neq j$ and $(\boldsymbol{\Sigma}_b)_{i,j} = 1$ otherwise, $i, j = 1, \ldots, p$.
- Model III: $\mathbf{X}_b$ is generated as Model II but the observed design matrix is $\mathbf{F}\mathbf{B}^\top + \mathbf{X}_b$, where $\mathbf{F}$ is an $n \times 2$ random matrix with iid $\mathcal{N}(0, 1)$ entries and $\mathbf{B}$ is a $p \times 2$ loading matrix with iid $\mathcal{N}(0, 1)$ entries.
- Model IV: the columns of $\mathbf{X}_b$ are iid from $\mathcal{N}(0, \boldsymbol{\Sigma}_n)$, where $(\boldsymbol{\Sigma}_n)_{i,j} = 0.1$ if $i \neq j$ and $(\boldsymbol{\Sigma}_n)_{i,j} = 1$ otherwise, $i, j = 1, \ldots, n$.

The simulation results are illustrated in Figures 1-4. From the results, it can be seen that all four tests can well maintain the significant level, although the EP test is a little conservative in some cases. Note that for $n = 50$, the powers of tests typically do not converge to 1 as SNR increases. This phenomenon, which is not surprising in view of Theorem 1, is eased when the sample size rises to 100. In the dense $\boldsymbol{\beta}_b$ setting, the proposed test shows relatively good power behavior compared with its competitors. However, in the sparse $\boldsymbol{\beta}_b$ setting of Model II and Model III, the proposed test is not as powerful as the DL test. In fact, as an extreme value type test, the DL test is designed for sparse linear models. On the basis of the simulations, we recommend using the proposed test when there is no prior knowledge of sparsity.

5 Conclusions

This paper is concerned with testing regression coefficients in large p, small q setting. It was proved that nontrivial unbiased test does not exists in this setting. We have proposed a Bayesian-motivated test statistic for high-dimensional linear model with fixed design. We proposed an approximation of the null distribution of the proposed test statistic which is then used to determine the critical value of the test statistic. Under weak conditions, we proved the proposed test procedure is asymptotically level α. Under certain conditions, we also derived the asymptotic power function of the proposed test. It was shown that the proposed test can detect the signals from a wide range of directions.

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Appendix

Lemma 1 *Under the assumptions of Theorem 1, if there exists a Borel set $G \subset \mathbb{R}^n$ and a number $M \geq 0$ such that*

$$\int_{\mathbb{R}^n} \varphi(\mathbf{y}) d\mathcal{N}_n(\mu, \phi^{-1}\mathbf{I}_n)(\mathbf{y})\,\mathrm{d}\mathbf{y} \geq \alpha \quad \text{for all } \mu \in G \text{ and } \phi > M,$$

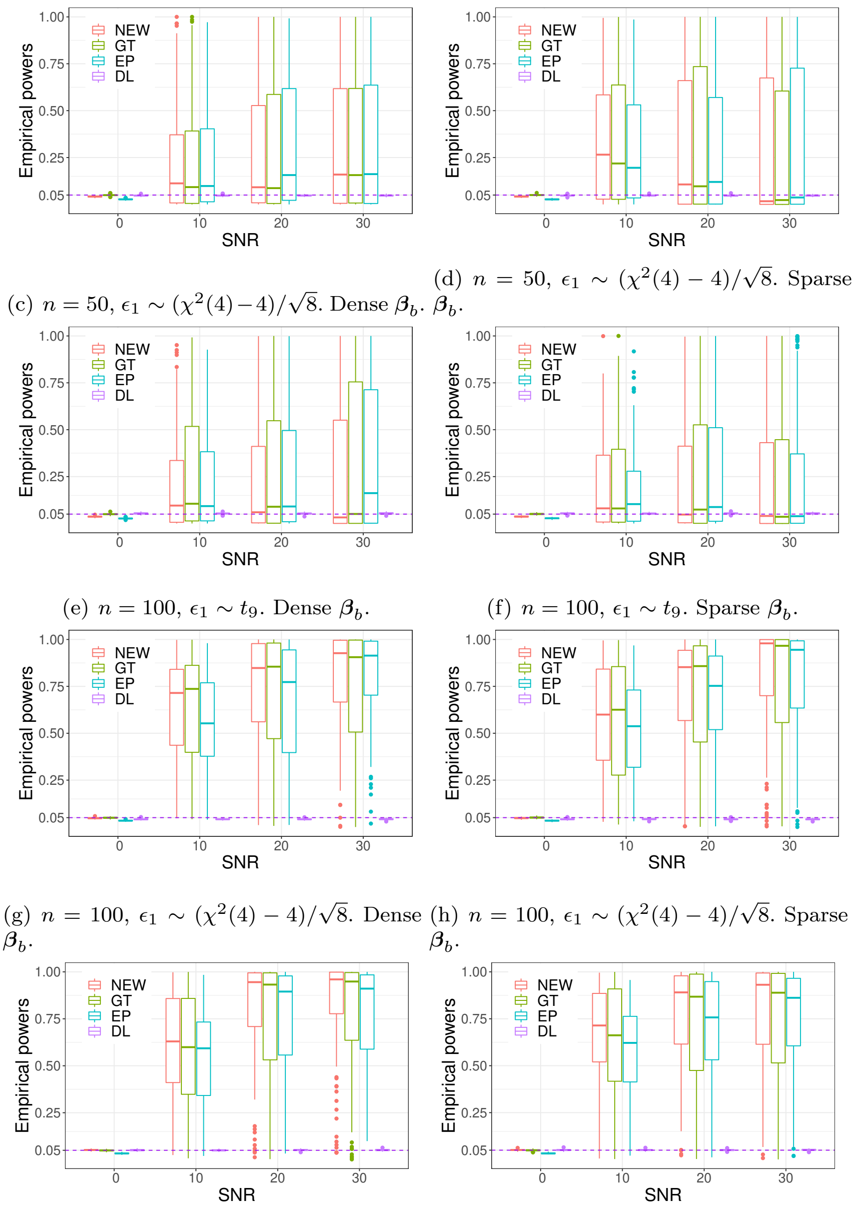
then $\varphi(\mathbf{y})\mathbf{1}_G(\mathbf{y}) \geq \alpha \mathbf{1}_G(\mathbf{y})$ a.e. λ.

Proof We prove the claim by contradiction. Suppose $\lambda(\{\mathbf{y} : \varphi(\mathbf{y}) < \alpha\} \cap G) > 0$. Then there exists a sufficiently small $0 < \eta < \alpha$, such that $\lambda(\{\mathbf{y} : \varphi(\mathbf{y}) < \alpha - \eta\} \cap G) > 0$. We denote $E := \{\mathbf{y} : \varphi(\mathbf{y}) < \alpha - \eta\} \cap G$. From Lebesgue density theorem (Cohn, 2013, Corollary 6.2.6), there exists a point $z \in E$, such that, for any $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for any $0 < \delta' < \delta_\varepsilon$,

$$\left| \frac{\lambda(E^{\complement} \cap C_{\delta'})}{\lambda(C_{\delta'})} \right| < \varepsilon,$$

where $C_{\delta'} = \prod_{i=1}^n [z_i - \delta', z_i + \delta']$. We put

$$\varepsilon = \left(\frac{\sqrt{\pi}}{\sqrt{2}\Phi^{-1}\left(1 - \frac{\eta}{6n}\right)} \right)^n \frac{\eta}{3}.$$



(a) $n = 50$, $\epsilon_1 \sim t_9$. Dense $\boldsymbol{\beta}_b$.

(b) $n = 50$, $\epsilon_1 \sim t_9$. Sparse $\boldsymbol{\beta}_b$.

(c) $n = 50$, $\epsilon_1 \sim (\chi^2(4)-4)/\sqrt{8}$. Dense $\boldsymbol{\beta}_b$.

(d) $n = 50$, $\epsilon_1 \sim (\chi^2(4)-4)/\sqrt{8}$. Sparse $\boldsymbol{\beta}_b$.

(e) $n = 100$, $\epsilon_1 \sim t_9$. Dense $\boldsymbol{\beta}_b$.

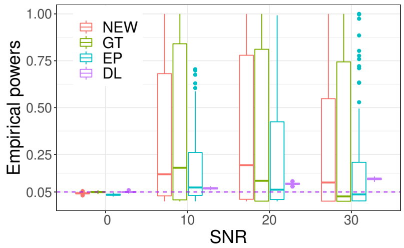
(f) $n = 100$, $\epsilon_1 \sim t_9$. Sparse $\boldsymbol{\beta}_b$.

(g) $n = 100$, $\epsilon_1 \sim (\chi^2(4)-4)/\sqrt{8}$. Dense $\boldsymbol{\beta}_b$.

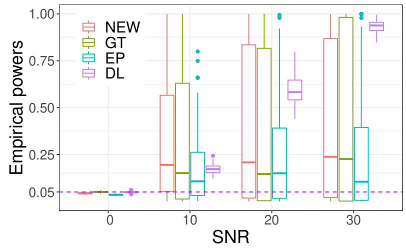
(h) $n = 100$, $\epsilon_1 \sim (\chi^2(4)-4)/\sqrt{8}$. Sparse $\boldsymbol{\beta}_b$.

Fig. 1 Box plots of the empirical powers based on 100 independently generated $\boldsymbol{\beta}_b$. $\mathbf{X}_b$ is generated by Model I.

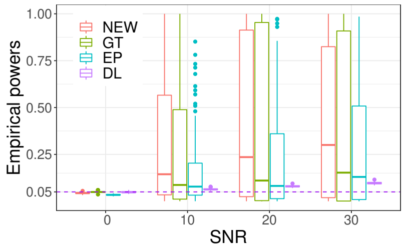
(a) $n = 50$, $\epsilon_1 \sim t_9$. Dense $\boldsymbol{\beta}_b$.



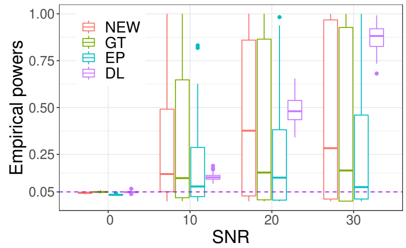
(b) $n = 50$, $\epsilon_1 \sim t_9$. Sparse $\boldsymbol{\beta}_b$.



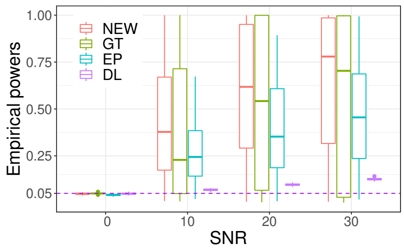
(c) $n = 50$, $\epsilon_1 \sim (\chi^2(4) - 4)/\sqrt{8}$. Dense $\boldsymbol{\beta}_b$.



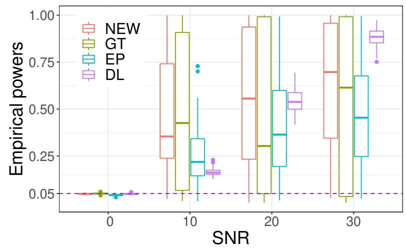
(d) $n = 50$, $\epsilon_1 \sim (\chi^2(4) - 4)/\sqrt{8}$. Sparse $\boldsymbol{\beta}_b$.



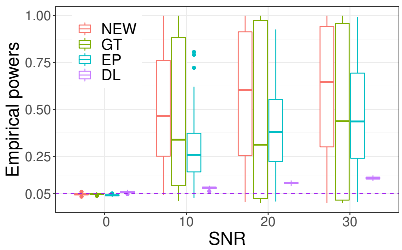
(e) $n = 100$, $\epsilon_1 \sim t_9$. Dense $\boldsymbol{\beta}_b$.



(f) $n = 100$, $\epsilon_1 \sim t_9$. Sparse $\boldsymbol{\beta}_b$.



(g) $n = 100$, $\epsilon_1 \sim (\chi^2(4) - 4)/\sqrt{8}$. Dense $\boldsymbol{\beta}_b$.



(h) $n = 100$, $\epsilon_1 \sim (\chi^2(4) - 4)/\sqrt{8}$. Sparse $\boldsymbol{\beta}_b$.

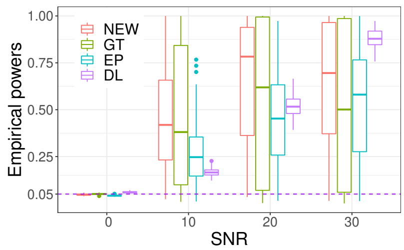
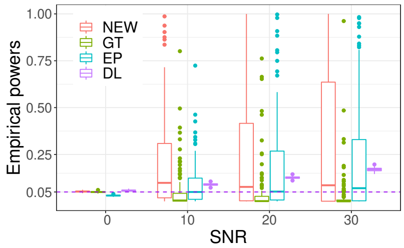
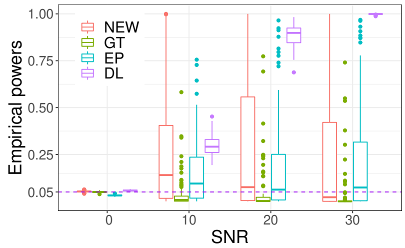


Fig. 2 Box plots of the empirical powers based on 100 independently generated $\boldsymbol{\beta}_b$. $\mathbf{X}_b$ is generated by Model II.

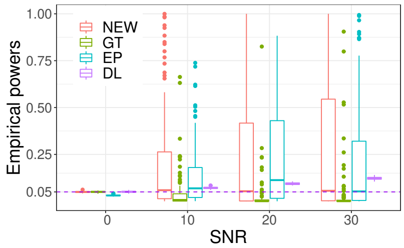
(a) $n = 50$, $\epsilon_1 \sim t_9$. Dense $\boldsymbol{\beta}_b$.



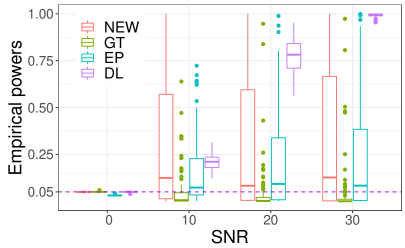
(b) $n = 50$, $\epsilon_1 \sim t_9$. Sparse $\boldsymbol{\beta}_b$.



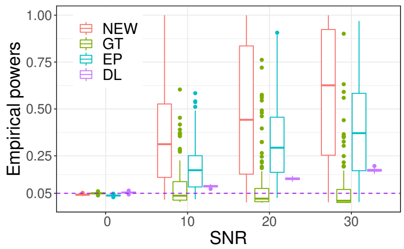
(c) $n = 50$, $\epsilon_1 \sim (\chi^2(4) - 4)/\sqrt{8}$. Dense $\boldsymbol{\beta}_b$.



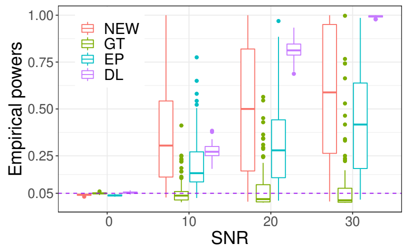
(d) $n = 50$, $\epsilon_1 \sim (\chi^2(4) - 4)/\sqrt{8}$. Sparse $\boldsymbol{\beta}_b$.



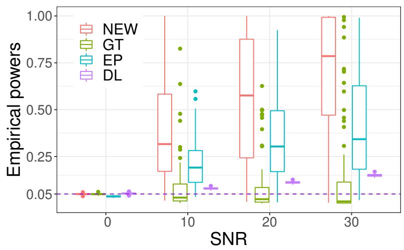
(e) $n = 100$, $\epsilon_1 \sim t_9$. Dense $\boldsymbol{\beta}_b$.



(f) $n = 100$, $\epsilon_1 \sim t_9$. Sparse $\boldsymbol{\beta}_b$.



(g) $n = 100$, $\epsilon_1 \sim (\chi^2(4) - 4)/\sqrt{8}$. Dense $\boldsymbol{\beta}_b$.



(h) $n = 100$, $\epsilon_1 \sim (\chi^2(4) - 4)/\sqrt{8}$. Sparse $\boldsymbol{\beta}_b$.

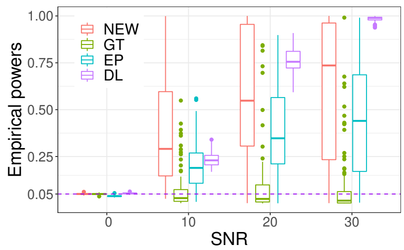
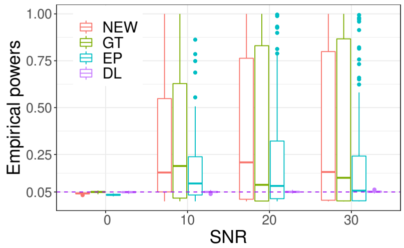
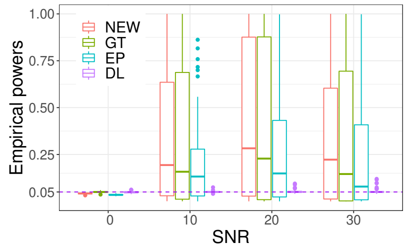


Fig. 3 Box plots of the empirical powers based on 100 independently generated $\boldsymbol{\beta}_b$. $\mathbf{X}_b$ is generated by Model III.

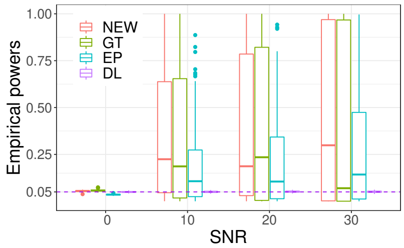
(a) $n = 50$, $\epsilon_1 \sim t_9$. Dense $\boldsymbol{\beta}_b$.



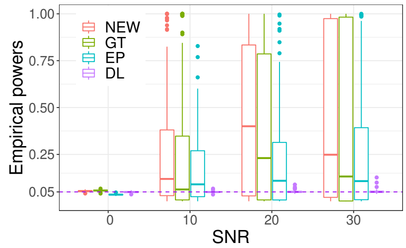
(b) $n = 50$, $\epsilon_1 \sim t_9$. Sparse $\boldsymbol{\beta}_b$.



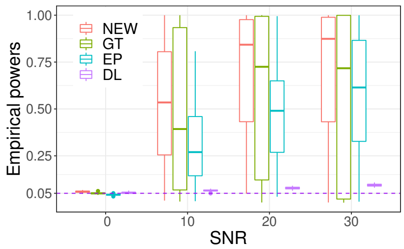
(c) $n = 50$, $\epsilon_1 \sim (\chi^2(4)-4)/\sqrt{8}$. Dense $\boldsymbol{\beta}_b$.



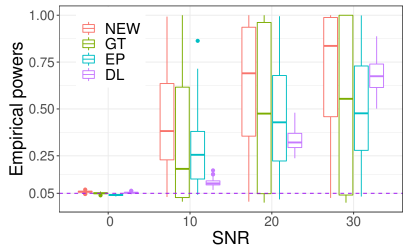
(d) $n = 50$, $\epsilon_1 \sim (\chi^2(4)-4)/\sqrt{8}$. Sparse $\boldsymbol{\beta}_b$.



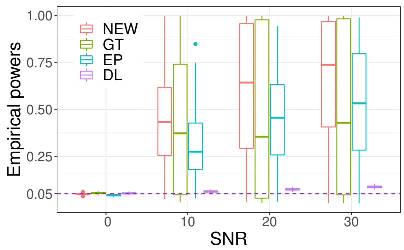
(e) $n = 100$, $\epsilon_1 \sim t_9$. Dense $\boldsymbol{\beta}_b$.



(f) $n = 100$, $\epsilon_1 \sim t_9$. Sparse $\boldsymbol{\beta}_b$.



(g) $n = 100$, $\epsilon_1 \sim (\chi^2(4)-4)/\sqrt{8}$. Dense $\boldsymbol{\beta}_b$.



(h) $n = 100$, $\epsilon_1 \sim (\chi^2(4)-4)/\sqrt{8}$. Sparse $\boldsymbol{\beta}_b$.

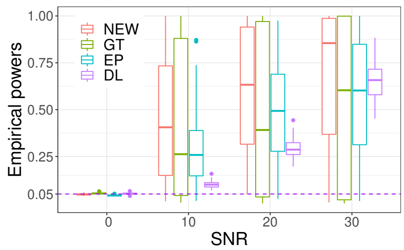


Fig. 4 Box plots of the empirical powers based on 100 independently generated $\boldsymbol{\beta}_b$. $\mathbf{X}_b$ is generated by Model IV.

Then for any $\phi > M$ and $0 < \delta' < \delta_\varepsilon$,

$$\begin{aligned}
\alpha \leq & \int_{\mathbb{R}^n} \varphi(\mathbf{y}) d\mathcal{N}_n(z, \phi^{-1}\mathbf{I}_n)(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
= & \int_{E\cap C_{\delta'}} \varphi(\mathbf{y}) d\mathcal{N}_n(z, \phi^{-1}\mathbf{I}_n)(\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_{E^{\complement}\cap C_{\delta'}} \varphi(\mathbf{y}) d\mathcal{N}_n(z, \phi^{-1}\mathbf{I}_n)(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
& + \int_{C_{\delta'}^{\complement}} \varphi(\mathbf{y}) d\mathcal{N}_n(z, \phi^{-1}\mathbf{I}_n)(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
\leq & \alpha - \eta + \int_{E^{\complement}\cap C_{\delta'}} d\mathcal{N}_n(z, \phi^{-1}\mathbf{I}_n)(\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_{C_{\delta'}^{\complement}} d\mathcal{N}_n(z, \phi^{-1}\mathbf{I}_n)(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
\leq & \alpha - \eta + \left(\frac{\phi}{2\pi}\right)^{n/2} \lambda(E^{\complement}\cap C_{\delta'}) + 2n\left(1 - \Phi(\sqrt{\phi}\delta')\right) \\
\leq & \alpha - \eta + \left(\frac{\phi}{2\pi}\right)^{n/2} \varepsilon(2\delta')^n + 2n\left(1 - \Phi(\sqrt{\phi}\delta')\right) \\
= & \alpha - \eta + \left(\frac{\sqrt{\phi}\delta'}{\Phi^{-1}\left(1 - \frac{\eta}{6n}\right)}\right)^n \frac{\eta}{3} + 2n\left(1 - \Phi(\sqrt{\phi}\delta')\right).
\end{aligned}$$

In the last inequality, we put δ' small enough such that

$$\left(\frac{\Phi^{-1}\left(1 - \frac{\eta}{6n}\right)}{\delta'}\right)^2 > M,$$

and put

$$\phi = \left(\frac{\Phi^{-1}\left(1 - \frac{\eta}{6n}\right)}{\delta'}\right)^2.$$

Then we obtain the contradiction $\alpha \leq \alpha - (2/3)\eta$. This completes the proof.

Proof ***(Proof of Theorem 1)***
We prove the claim by contradiction. Suppose there exists an $M \geq 0$ such that

$$\int_{\mathbb{R}^n} \varphi(\mathbf{y}) d\mathcal{N}_n\left(\mathbf{X}_a\boldsymbol{\beta}_a + \mathbf{X}_b\boldsymbol{\beta}_b, \phi^{-1}\mathbf{I}_n\right)(\mathbf{y})\,\mathrm{d}\mathbf{y} \geq \alpha$$

for every $\boldsymbol{\beta}_a \in \mathbb{R}^q$, $\boldsymbol{\beta}_b \in \mathbb{R}^p$, $\phi > 0$ satisfying $\phi\boldsymbol{\beta}_b^\top\mathbf{X}_b^\top\tilde{\mathbf{P}}_a\mathbf{X}_b\boldsymbol{\beta}_b > M$. Note that for any $h > 0$,

$$\left\{(\boldsymbol{\beta}_b^\top, \phi)^\top : \boldsymbol{\beta}_b^\top\mathbf{X}_b^\top\tilde{\mathbf{P}}_a\mathbf{X}_b\boldsymbol{\beta}_b > h\sqrt{M}, \phi > h^{-1}\sqrt{M}\right\}$$

is a subset of

$$\left\{(\boldsymbol{\beta}_b^\top, \phi)^\top : \phi\boldsymbol{\beta}_b^\top\mathbf{X}_b^\top\tilde{\mathbf{P}}_a\mathbf{X}_b\boldsymbol{\beta}_b > M\right\}.$$

Then Lemma 1 implies that for any $h > 0$, $\varphi(\mathbf{y})\mathbf{1}_{G_h}(\mathbf{y}) \geq \alpha\mathbf{1}_{G_h}(\mathbf{y})$ a.e. λ, where

$$G_h = \left\{\mathbf{X}_a\boldsymbol{\beta}_a + \mathbf{X}_b\boldsymbol{\beta}_b : \boldsymbol{\beta}_a \in \mathbb{R}^q, \boldsymbol{\beta}_b^\top\mathbf{X}_b^\top\tilde{\mathbf{P}}_a\mathbf{X}_b\boldsymbol{\beta}_b > h\sqrt{M}\right\}.$$

It can be seen that $\lambda(\{\cup_{n=1}^\infty G_{1/n}\}^{\complement}) = 0$. Hence $\varphi(\mathbf{y}) \geq \alpha$ a.e. λ. On the other hand, since $\varphi(\mathbf{y})$ is a level α test, for every $\phi > 0$,

$$\int_{\mathbb{R}^n} [\varphi(\mathbf{y}) - \alpha] d\mathcal{N}_n(0, \phi^{-1}\mathbf{I}_n)(\mathbf{y})\,\mathrm{d}\mathbf{y} \leq 0. \tag{10}$$

Note that the integrand of (10) is nonnegative. It follows that $\varphi(\mathbf{y}) = \alpha$ a.s. λ, a contradiction. This completes the proof.

Proof (***Proof of Theorem 2***) We note that $\mathrm{Var}(\boldsymbol{\xi}^\top \mathbf{A}\boldsymbol{\xi}) = 2\,\mathrm{tr}(\mathbf{A}^2) + (\mathrm{E}(\xi_1^4) - 3)\,\mathrm{tr}(\mathbf{A}^{\circ 2})$. Let

$$\tilde{a}_{i,j} := \frac{a_{i,j}}{\sqrt{2\,\mathrm{tr}(\mathbf{A}^2) + (\mathrm{E}(\xi_1^4) - 3)\,\mathrm{tr}(\mathbf{A}^{\circ 2})}}.$$

Then

$$S = \sum_{i=1}^{n} \tilde{a}_{i,i}(\xi_i^2 - 1) + 2 \sum_{1\le i<j\le n} \tilde{a}_{i,j}\xi_i\xi_j, \quad S_\tau^* = \tau \sum_{i=1}^{n} \tilde{a}_{i,i}\check{z}_i + 2 \sum_{1\le i<j\le n} \tilde{a}_{i,j} z_i z_j.$$

For $l = 1, \ldots, n$, define

$$\begin{aligned}
S_l =& \sum_{i=1}^{l-1} \tilde{a}_{i,i}(\xi_i^2 - 1) + \tau \sum_{i=l+1}^{n} \tilde{a}_{i,i}\check{z}_i \\
&+ 2 \sum_{1\le i<j\le l-1} \tilde{a}_{i,j}\xi_i\xi_j + 2 \sum_{i=1}^{l-1} \sum_{j=l+1}^{n} \tilde{a}_{i,j}\xi_i z_j + 2 \sum_{l+1\le i<j\le n} \tilde{a}_{i,j} z_i z_j, \\
h_l =& \tilde{a}_{l,l}(\xi_l^2 - 1) + 2 \sum_{i=1}^{l-1} \tilde{a}_{i,l}\xi_i\xi_l + 2 \sum_{i=l+1}^{n} \tilde{a}_{i,l} z_i \xi_l, \\
g_l =& \tau \tilde{a}_{l,l}\check{z}_l + 2 \sum_{i=1}^{l-1} \tilde{a}_{i,l}\xi_i z_l + 2 \sum_{i=l+1}^{n} \tilde{a}_{i,l} z_i z_l.
\end{aligned}$$

It can be seen that for $l = 2, \ldots, n$, $S_{l-1} + h_{l-1} = S_l + g_l$, and $S = S_n + h_n$, $S_1 + g_1 = S_\tau^*$.

Thus, for any $f \in \mathscr{C}^4(\mathbb{R})$,

$$\begin{aligned}
&|\mathrm{E}\, f(S) - \mathrm{E}\, f(S_\tau^*)| \\
=& |\mathrm{E}\, f(S_n + h_n) - \mathrm{E}\, f(S_1 + g_1)| \\
=& \left| \sum_{l=2}^{n} (\mathrm{E}\, f(S_l + h_l) - \mathrm{E}\, f(S_{l-1} + h_{l-1})) + \mathrm{E}\, f(S_1 + h_1) - \mathrm{E}\, f(S_1 + g_1) \right| \\
=& \left| \sum_{l=1}^{n} \mathrm{E}\, f(S_l + h_l) - \mathrm{E}\, f(S_l + g_l) \right|.
\end{aligned}$$

Apply Taylor's theorem, for $l = 1, \ldots, n$,

$$\begin{aligned}
f(S_l + h_l) =& f(S_l) + \sum_{k=1}^{3} \frac{1}{k!} h_l^k f^{(k)}(S_l) + \frac{1}{24} h_l^4 f^{(4)}(S_l + \theta_1 h_l), \\
f(S_l + g_l) =& f(S_l) + \sum_{k=1}^{3} \frac{1}{k!} g_l^k f^{(k)}(S_l) + \frac{1}{24} g_l^4 f^{(4)}(S_l + \theta_2 g_l),
\end{aligned}$$

where $\theta_1, \theta_2 \in [0, 1]$. Thus,

$$\begin{aligned}
&|\mathrm{E}\, f(S_l + h_l) - \mathrm{E}\, f(S_l + g_l)| \\
\le& \left| \sum_{k=1}^{3} \frac{1}{k!} \mathrm{E}\, f^{(k)}(S_l)\, \mathrm{E}_l(h_l^k - g_l^k) \right| + \frac{1}{24} \|f^{(4)}\|_\infty \left(\mathrm{E}(h_l^4) + \mathrm{E}(g_l^4) \right),
\end{aligned}$$

where E_l denotes taking expectation with respect to $\xi_l, z_l, \check{z}_l$. It is straightforward to show that

$$\begin{aligned}
&\mathrm{E}_l(h_l - g_l) = 0, \\
&\mathrm{E}_l(h_l^2 - g_l^2) = \left(\mathrm{E}(\xi_1^4) - 1 - \tau^2 \right) \tilde{a}_{l,l}^2, \\
&\mathrm{E}_l(h_l^3 - g_l^3) = \mathrm{E}(\xi_1^2 - 1)^3 \tilde{a}_{l,l}^3 + 12(\mathrm{E}(\xi_1^4) - 1)\tilde{a}_{l,l} \left(\sum_{i=1}^{l-1} \tilde{a}_{i,l}\xi_i + \sum_{i=l+1}^{n} \tilde{a}_{i,l} z_i \right)^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&|\mathrm{E}\, f(S_l+h_l)-\mathrm{E}\, f(S_l+g_l)| \\
\le&\frac{1}{2}\|f^{(2)}\|_\infty \left|\mathrm{E}(\xi_1^4)-1-\tau^2\right| \tilde{a}_{l,l}^2 \\
&+\frac{1}{6}\|f^{(3)}\|_\infty \Bigg(\left|\mathrm{E}(\xi_1^2-1)^3\right| |\tilde{a}_{l,l}^3| \\
&\qquad + 12(\mathrm{E}(\xi_1^4)-1)|\tilde{a}_{l,l}|\,\mathrm{E}\left(\sum_{i=1}^{l-1}\tilde{a}_{i,l}\xi_i+\sum_{i=l+1}^{n}\tilde{a}_{i,l}z_i\right)^2\Bigg) \\
&+\frac{1}{24}\|f^{(4)}\|_\infty \left(\mathrm{E}(h_l^4)+\mathrm{E}(g_l^4)\right) \\
\le&\frac{\left|\mathrm{E}(\xi_1^4)-1-\tau^2\right|}{2}\|f^{(2)}\|_\infty \tilde{a}_{l,l}^2 \\
&+\frac{\max\left(\left|\mathrm{E}(\xi_1^2-1)^3\right|, 12(\mathrm{E}(\xi_1^4)-1)\right)}{6}\|f^{(3)}\|_\infty |\tilde{a}_{l,l}|\sum_{i=1}^{n}\tilde{a}_{i,l}^2 \\
&+\frac{1}{24}\|f^{(4)}\|_\infty \left(\mathrm{E}(h_l^4)+\mathrm{E}(g_l^4)\right).
\end{aligned} \tag{11}$$

Now we bound $\mathrm{E}(h_l^4)$ and $\mathrm{E}(g_l^4)$. By direct calculation,

$$\begin{aligned}
\mathrm{E}(h_l^4)=&\mathrm{E}(\xi_1^2-1)^4\tilde{a}_{l,l}^4+24\,\mathrm{E}[\xi_1^2(\xi_1^2-1)^2]\tilde{a}_{l,l}^2\,\mathrm{E}\left(\sum_{i=1}^{l-1}\tilde{a}_{i,l}\xi_i+\sum_{i=l+1}^{n}\tilde{a}_{i,l}z_i\right)^2 \\
&+16\,\mathrm{E}(\xi_1^4)\,\mathrm{E}\left(\sum_{i=1}^{l-1}\tilde{a}_{i,l}\xi_i+\sum_{i=l+1}^{n}\tilde{a}_{i,l}z_i\right)^4 \\
=&\mathrm{E}(\xi_1^2-1)^4\tilde{a}_{l,l}^4+24\,\mathrm{E}[\xi_1^2(\xi_1^2-1)^2]\tilde{a}_{l,l}^2\left((\sum_{i=1}^{n}\tilde{a}_{i,l}^2)-\tilde{a}_{l,l}^2\right) \\
&+16\,\mathrm{E}(\xi_1^4)\left(\left(\mathrm{E}(\xi_1^4)-3\right)\sum_{i=1}^{l-1}\tilde{a}_{i,l}^4+3\left((\sum_{i=1}^{n}\tilde{a}_{i,l}^2)-\tilde{a}_{l,l}^2\right)^2\right).
\end{aligned}$$

To upper bound the above quantity, we use the facts $24\,\mathrm{E}[\xi_1^2(\xi_1^2-1)^2]\le 2(16\,\mathrm{E}(\xi_1^2-1)^4+(9/4)\,\mathrm{E}(\xi_1^4))$, $\mathrm{E}(\xi_1^2-1)^4\le\mathrm{E}(\xi_1^8)$ and

$$\left(\mathrm{E}(\xi_1^4)-3\right)\sum_{i=1}^{l-1}\tilde{a}_{i,l}^4\le\left(\mathrm{E}(\xi_1^4)-1\right)\sum_{i=1}^{l-1}\tilde{a}_{i,l}^4\le\left(\mathrm{E}(\xi_1^4)-1\right)\left((\sum_{i=1}^{n}\tilde{a}_{i,l}^2)-\tilde{a}_{l,l}^2\right)^2.$$

Then we obtain the bound

$$\mathrm{E}(h_l^4)\le\left(16\,\mathrm{E}(\xi_1^8)+32\,\mathrm{E}(\xi_1^4)\right)\left(\sum_{i=1}^{n}\tilde{a}_{i,l}^2\right)^2. \tag{12}$$

Similarly, we have

$$\mathrm{E}(g_l^4)\le\left(48\,\mathrm{E}(\xi_1^4)+3\tau^4+96\right)\left(\sum_{i=1}^{n}\tilde{a}_{i,l}^2\right)^2. \tag{13}$$

Combining (11), (12) and (13) yields

$$
\begin{aligned}
&\sum_{l=1}^{n} |\operatorname{E} f(S_l + h_l) - \operatorname{E} f(S_l + g_l)| \\
\leq & \frac{\left|\operatorname{E}(\xi_1^4) - 1 - \tau^2\right|}{2} \|f^{(2)}\|_\infty \sum_{l=1}^{n} \tilde{a}_{l,l}^2 \\
& + \frac{\max\left(\left|\operatorname{E}(\xi_1^2 - 1)^3\right|, 12(\operatorname{E}(\xi_1^4) - 1)\right)}{6} \|f^{(3)}\|_\infty \sum_{l=1}^{n} \left(|\tilde{a}_{l,l}| \sum_{i=1}^{n} \tilde{a}_{i,l}^2 \right) \\
& + \frac{16 \operatorname{E}(\xi_1^8) + 80 \operatorname{E}(\xi_1^4) + 3\tau^4 + 96}{24} \|f^{(4)}\|_\infty \sum_{l=1}^{n} \left(\sum_{i=1}^{n} \tilde{a}_{i,l}^2 \right)^2 .
\end{aligned}
$$

This completes the proof.

Proof (***Proof of Theorem 3***) Throughout the proof, we use the similar notations as in Theorem 2 and define

$$
S = \frac{(\sqrt{\phi}\boldsymbol{\epsilon})^\top \mathbf{A} \sqrt{\phi}\boldsymbol{\epsilon} - \operatorname{tr}(\mathbf{A})}{\sqrt{2\operatorname{tr}(\mathbf{A}^2) + (\phi^2 \operatorname{E}(\epsilon_1^4) - 3)\operatorname{tr}(\mathbf{A}^{\circ 2})}}
$$

and

$$
S_{\hat{\tau}}^* = \frac{\hat{\tau}\sum_{i=1}^n a_{i,i}\check{z}_i + 2\sum_{1\leq i<j\leq n} a_{i,j} z_i z_j}{\sqrt{2\operatorname{tr}(\mathbf{A}^2) + (\phi^2 \operatorname{E}(\epsilon_1^4) - 3)\operatorname{tr}(\mathbf{A}^{\circ 2})}},
$$

where $z_1, \ldots, z_n, \check{z}_1, \ldots, \check{z}_n$ are iid $\mathcal{N}(0,1)$ random variables and are independent of $\hat{\tau}^2$.

By a standard subsequence argument, we only need to prove the theorem along a subsequence of $\{n\}$. Hence, without loss of generality, we assume $\hat{\tau}^2 \xrightarrow{a.s.} \phi^2 \operatorname{E}(\epsilon_1^4) - 1$. Write

$$
\begin{aligned}
S_{\hat{\tau}}^* = & \frac{\sqrt{\phi^2 \operatorname{E}(\epsilon_1^4) - 1}\sum_{i=1}^n a_{i,i}\check{z}_i + 2\sum_{1\leq i<j\leq n} a_{i,j} z_i z_j}{\sqrt{2\operatorname{tr}(\mathbf{A}^2) + (\phi^2 \operatorname{E}(\epsilon_1^4) - 3)\operatorname{tr}(\mathbf{A}^{\circ 2})}} \\
& + \frac{(\hat{\tau} - \sqrt{\phi^2 \operatorname{E}(\epsilon_1^4) - 1})\sum_{i=1}^n a_{i,i}\check{z}_i}{\sqrt{2\operatorname{tr}(\mathbf{A}^2) + (\phi^2 \operatorname{E}(\epsilon_1^4) - 3)\operatorname{tr}(\mathbf{A}^{\circ 2})}} \\
=: & S_{\hat{\tau},1}^* + S_{\hat{\tau},2}^*.
\end{aligned}
$$

Note that $S_{\hat{\tau},1}^*$ is independent of $\hat{\tau}$. Since $\operatorname{E}(S_{\hat{\tau},1}^{*2}) = 1$, the distributions $\mathcal{L}(S_{\hat{\tau},1}^*)$ are tight as $n \to \infty$. Hence, without loss of generality, we assume $\mathcal{L}(S_{\hat{\tau},1}^*)$ weakly converges to a limit distribution with distribution function $F^\dagger(x)$. Let $S^\dagger$ be a random variable with distribution function $F^\dagger(x)$. By some algebra (See, e.g., Chen et al. (2010), Proposition A.1.(iii)), it can be shown that $\operatorname{E}(S_{\hat{\tau},1}^{*4})$ is uniformly bounded. Then $\mathcal{L}(S_{\hat{\tau},1}^{*2})$ is uniformly integrable. Hence $\operatorname{E}(S^{\dagger 2}) = 1$ and $F^\dagger(x)$ can not concentrate on a single point. Consequently, $F^\dagger(x)$ is continuous and is strict increasing for $x \in \{x : 0 < F(x) < 1\}$; see Sevast'yanov (1961) as well the remark made by A. N. Kolmogorov on that paper.

The condition (8) implies that $\operatorname{E}[S_{\hat{\tau},2}^{*2}|\hat{\tau}] \to 0$ almost surely. Then almost surely, $\mathcal{L}(S_{\hat{\tau}}^*|\hat{\tau}) \rightsquigarrow \mathcal{L}(S^\dagger)$. Consequently, for every $f \in \mathscr{C}^4(\mathbb{R})$, we have $|\operatorname{E}[f(S_{\hat{\tau}}^*)|\hat{\tau}] - \operatorname{E} f(S^\dagger)| \to 0$ almost surely. On the other hand, Theorem 2 and the condition (8) imply $|\operatorname{E} f(S) - \operatorname{E}[f(S_{\hat{\tau}}^*)|\hat{\tau}]| \to 0$ almost surely. Thus, $|\operatorname{E} f(S) - \operatorname{E} f(S^\dagger)| \to 0$. That is, $\mathcal{L}(S) \rightsquigarrow \mathcal{L}(S^\dagger)$.

Note that

$$
x^{(1)} = \frac{F^{-1}(1-\alpha; \mathbf{A}, \hat{\tau}) - \operatorname{tr}(\mathbf{X}_b^* \mathbf{X}_b^{*\top})^{-1}}{n-q}.
$$

We need to deal with $F^{-1}(1-\alpha;\mathbf{A},\hat{\tau})$. Since $\mathcal{L}(S^*_{\hat{\tau}}|\hat{\tau}) \rightsquigarrow \mathcal{L}(S^\dagger)$ almost surely, the fact

$$\Pr\left(S^*_{\hat{\tau}} > \frac{F^{-1}(1-\alpha;\mathbf{A},\hat{\tau})}{\sqrt{2\operatorname{tr}(\mathbf{A}^2)+(\phi^2\operatorname{E}(\epsilon_1^4)-3)\operatorname{tr}(\mathbf{A}^{\circ 2})}}\middle|\hat{\tau}\right)=\alpha$$

implies that almost surely,

$$\frac{F^{-1}(1-\alpha;\mathbf{A},\hat{\tau})}{\sqrt{2\operatorname{tr}(\mathbf{A}^2)+(\phi^2\operatorname{E}(\epsilon_1^4)-3)\operatorname{tr}(\mathbf{A}^{\circ 2})}} \to F^{\dagger-1}(1-\alpha). \tag{14}$$

We also need the fact that

$$(\sqrt{\phi}\boldsymbol{\epsilon})^\top\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top(\sqrt{\phi}\boldsymbol{\epsilon})=(1+o_p(1))(n-q), \tag{15}$$

which is a consequence of

$$\operatorname{E}\left((\sqrt{\phi}\boldsymbol{\epsilon})^\top\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top(\sqrt{\phi}\boldsymbol{\epsilon})\right)=n-q,\quad \operatorname{Var}\left((\sqrt{\phi}\boldsymbol{\epsilon})^\top\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top(\sqrt{\phi}\boldsymbol{\epsilon})\right)=O(n-q).$$

The fact $S \rightsquigarrow S^\dagger$, the equations (14), (15) and Slutsky's theorem lead to

$$\begin{aligned}
&\Pr\left(T > x^{(1)}\right)\\
=&\Pr\left(T > \frac{F^{-1}(1-\alpha;\mathbf{A},\hat{\tau})-\operatorname{tr}\left((\mathbf{X}_b^*\mathbf{X}_b^{*\top})^{-1}\right)}{n-q}\right)\\
=&\Pr\left((\sqrt{\phi}\boldsymbol{\epsilon})^\top\mathbf{A}(\sqrt{\phi}\boldsymbol{\epsilon}) > \frac{(\sqrt{\phi}\boldsymbol{\epsilon})^\top\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top(\sqrt{\phi}\boldsymbol{\epsilon})}{n-q}F^{-1}(1-\alpha;\mathbf{A},\hat{\tau})\right)\\
=&\Pr\left(S > \frac{(\sqrt{\phi}\boldsymbol{\epsilon})^\top\tilde{\mathbf{U}}_a\tilde{\mathbf{U}}_a^\top(\sqrt{\phi}\boldsymbol{\epsilon})}{n-q}\frac{F^{-1}(1-\alpha;\mathbf{A},\hat{\tau})}{\sqrt{2\operatorname{tr}(\mathbf{A}^2)+(\phi^2\operatorname{E}(\epsilon_1^4)-3)\operatorname{tr}(\mathbf{A}^{\circ 2})}}\right)\\
=&\Pr\left(S > (1+o_P(1))F^{-1}(1-\alpha)\right)\\
\to&\alpha.
\end{aligned}$$

This proves the theorem.

Proof ***(Proof of Proposition 1)*** From Bai et al. (2018), Theorem 2.1, one can obtain the explicit forms of $\operatorname{Var}\left(\tilde{\boldsymbol{\epsilon}}^\top\left(\tilde{\mathbf{P}}_a\right)\tilde{\boldsymbol{\epsilon}}\right)$ and $\operatorname{Var}\left(\sum_{i=1}^n\tilde{\epsilon}_i^4\right)$ which involves the traces of certain matrices. Using Horn and Johnson (1991), Theorem 5.5.1, one can see that the eigenvalues of these matrices are all bounded. Hence it can be deduced that $\operatorname{Var}(\tilde{\boldsymbol{\epsilon}}^\top\tilde{\mathbf{P}}_a\tilde{\boldsymbol{\epsilon}})=O(n)$ and $\operatorname{Var}\left(\sum_{i=1}^n\tilde{\epsilon}_i^4\right)=O(n)$. Thus,

$$\begin{aligned}
&\tilde{\boldsymbol{\epsilon}}^\top\tilde{\mathbf{P}}_a\tilde{\boldsymbol{\epsilon}}=(n-q)\phi^{-1}+O_P(\sqrt{n}),\\
&\sum_{i=1}^n\tilde{\epsilon}_i^4=3\phi^{-2}\operatorname{tr}(\tilde{\mathbf{P}}_a^{\circ 2})+\left(\operatorname{E}(\epsilon_1^4)-3\phi^{-2}\right)\operatorname{tr}\left(\tilde{\mathbf{P}}_a^{\circ 2}\right)^2+O_P(\sqrt{n}).
\end{aligned}$$

It follows that

$$\hat{\tau}^2=\phi^2\operatorname{E}(\epsilon_1^4)-1+O_P\left(\frac{\sqrt{n}}{\operatorname{tr}\left(\tilde{\mathbf{P}}_a^{\circ 2}\right)^2}\right).$$

Let $\delta_{i,j} = 1$ if $i = j$ and 0 if $i \neq j$. We have

$$
\begin{aligned}
n =& \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{i,j}^4 \\
=& \sum_{i=1}^{n} \sum_{j=1}^{n} \left((\tilde{\mathbf{P}}_a)_{i,j} + (\mathbf{P}_a)_{i,j} \right)^4 \\
\leq& 8 \sum_{i=1}^{n} \sum_{j=1}^{n} \left((\tilde{\mathbf{P}}_a)_{i,j} \right)^4 + 8 \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathbf{P}_a)_{i,j}^4 \\
\leq& 8 \sum_{i=1}^{n} \sum_{j=1}^{n} \left((\tilde{\mathbf{P}}_a)_{i,j} \right)^4 + 8 \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathbf{P}_a)_{i,j}^2 \\
=& 8 \operatorname{tr} \left(\tilde{\mathbf{P}}_a^{\circ 2} \right)^2 + 8q.
\end{aligned}
$$

Then

$$
\frac{\sqrt{n}}{\operatorname{tr} \left(\tilde{\mathbf{P}}_a^{\circ 2} \right)^2} = O\left(\frac{1}{\sqrt{n}} \right).
$$

This completes the proof.

Proof ***(Proof of Proposition 2)***
Without loss of generality, we assume $\mathbf{A}$ is a diagonal matrix and $|b_1| \geq \cdots \geq |b_n|$. By a standard subsequence argument, we only need to prove the result along a subsequence. Hence we can assume $\lim_{n\to\infty} \|b\|^2/\operatorname{tr}(\mathbf{A}^2) = c \in [0, +\infty]$. If $c = 0$, Lyapunov central limit theorem implies that

$$
\frac{Z^\top \mathbf{A} Z + b^\top Z - \operatorname{tr}(\mathbf{A})}{\sqrt{2\operatorname{tr}(\mathbf{A}^2) + \|\mathbf{b}\|^2}} = (1 + o_P(1)) \frac{Z^\top \mathbf{A} Z - \operatorname{tr}(\mathbf{A})}{\sqrt{2\operatorname{tr}(\mathbf{A}^2)}} + o_P(1) \rightsquigarrow \mathcal{N}(0,1).
$$

If $c = +\infty$,

$$
\frac{Z^\top \mathbf{A} Z + b^\top Z - \operatorname{tr}(\mathbf{A})}{\sqrt{2\operatorname{tr}(\mathbf{A}^2) + \|\mathbf{b}\|^2}} = (1 + o_P(1)) \frac{b^\top Z}{\|\mathbf{b}\|} + o_P(1) \rightsquigarrow \mathcal{N}(0,1).
$$

In what follows, we assume $c \in (0, +\infty)$. By Helly selection theorem, we can assume $\lim_{n\to\infty} |b_i|/\|\mathbf{b}\| = b_i^* \in [0,1]$, $i = 1, 2, \ldots$. From Fatou's lemma, we have $\sum_{i=1}^\infty (b_i^*)^2 \leq 1$. Consequently, $\lim_{i\to\infty} b_i^* = 0$.

Note that the condition $\operatorname{tr}(\mathbf{A}^4)/\operatorname{tr}^2(\mathbf{A}^2) \to 0$ is equivalent to $\lambda_1(\mathbf{A}^2)/\operatorname{tr}(\mathbf{A}^2) \to 0$. Then for every fixed integer $r > 0$,

$$
\frac{\sum_{i=1}^r a_{i,i}^2}{\sum_{i=1}^n a_{i,i}^2} \leq \frac{r \max_{1\leq i\leq n} a_{i,i}^2}{\sum_{i=1}^n a_{i,i}^2} \to 0.
$$

Then there exists a sequence of positive integers $r(n) \to \infty$ such that $\left(\sum_{i=1}^{r(n)} a_{i,i}^2\right)/\left(\sum_{i=1}^n a_{i,i}^2\right) \to 0$ and $r(n)/n \to 0$. Write

$$
Z^\top \mathbf{A} Z + b^\top Z - \operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{r(n)} a_{i,i}(z_i^2 - 1) + \sum_{i=1}^{r(n)} b_i z_i + \sum_{i=r(n)+1}^{n} \left(a_{i,i}(z_i^2 - 1) + b_i z_i \right),
$$

which is a sum of independent random variables. The first term is negligible since $\operatorname{Var}(\sum_{i=1}^{r(n)} a_{i,i}(z_i^2 - 1)) = o(\sum_{i=1}^n a_{i,i}^2)$. Now we deal with the third term. From Berry-Esseen inequality (See, e.g., DasGupta (2008), Theorem 11.2), there exists an absolute constant $C^* > 0$, such that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left| \Pr\left(\frac{\sum_{i=r(n)+1}^{n} \left(a_{i,i}(z_i^2-1)+b_i z_i\right)}{\sqrt{2\sum_{i=r(n)+1}^{n} a_{i,i}^2 + \sum_{i=r(n)+1}^{n} b_i^2}} \le x \right) - \Phi(x) \right| \\
\le & C^* \frac{\sum_{i=r(n)+1}^{n} \mathrm{E}\left|a_{i,i}(z_i^2-1)+b_i z_i\right|^3}{\left(2\sum_{i=r(n)+1}^{n} a_{i,i}^2 + \sum_{i=r(n)+1}^{n} b_i^2\right)^{3/2}}.
\end{aligned}$$

By some simple algebra, there exist absolute constants $C_1^*, C_2^* > 0$ such that for sufficiently large n,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left| \Pr\left(\frac{\sum_{i=r(n)+1}^{n} \left(a_{i,i}(z_i^2-1)+b_i z_i\right)}{\sqrt{2\sum_{i=r(n)+1}^{n} a_{i,i}^2 + \sum_{i=r(n)+1}^{n} b_i^2}} \le x \right) - \Phi(x) \right| \\
\le & C_1^* \frac{\max_{1\le i\le n} |a_{i,i}|}{\sqrt{\sum_{i=1}^{n} a_{i,i}^2}} + C_2^* \frac{|b_{r(n)+1}|}{\|\mathbf{b}\|}.
\end{aligned}$$

Since the right hand side tends to 0, we have

$$\frac{\sum_{i=r(n)+1}^{n} \left(a_{i,i}(z_i^2-1)+b_i z_i\right)}{\sqrt{2\sum_{i=r(n)+1}^{n} a_{i,i}^2 + \sum_{i=r(n)+1}^{n} b_i^2}} \rightsquigarrow \mathcal{N}(0,1).$$

Note that $\sum_{i=r(n)+1}^{n} \left(a_{i,i}(z_i^2-1)+b_i z_i\right)$ is independent of $\sum_{i=1}^{r(n)} b_i z_i$ and $\sum_{i=1}^{r(n)} b_i z_i \sim \mathcal{N}(0, \sum_{i=1}^{r(n)} b_i^2)$. Thus,

$$\frac{\sum_{i=1}^{r(n)} b_i z_i + \sum_{i=r(n)+1}^{n} \left(a_{i,i}(z_i^2-1)+b_i z_i\right)}{\sqrt{2\sum_{i=1}^{n} a_{i,i}^2 + \sum_{i=1}^{n} b_i^2}} \rightsquigarrow \mathcal{N}(0,1).$$

This completes the proof.

Note that under the normality, $T_n - \mathrm{tr}((\mathbf{X}_b^* \mathbf{X}_b^{*\top})^{-1})/(n-q)$ has zero mean.

Proof ***(Proof of Theorem 4)***
We note that

$$\begin{aligned}
&\Pr\left(\frac{\mathbf{y}^{*\top} \left(\mathbf{X}_b^* \mathbf{X}_b^{*\top}\right)^k \mathbf{y}^*}{\mathbf{y}^{*\top}\mathbf{y}^*} \le \mathrm{E}(\gamma_I^k) + \sqrt{\frac{2\,\mathrm{Var}\left(\gamma_I^k\right)}{n-q}} x \right) \\
=& \Pr\left(\mathbf{y}^{*\top} \left(\mathbf{X}_b^* \mathbf{X}_b^{*\top}\right)^k \mathbf{y}^* \le \left(\mathrm{E}(\gamma_I^k) + \sqrt{\frac{2\,\mathrm{Var}\left(\gamma_I^k\right)}{n-q}} x \right) \mathbf{y}^{*\top}\mathbf{y}^* \right) \\
=& \Pr\left(\mathbf{y}^{*\top} \mathbf{B} \mathbf{y}^* \le 0 \right),
\end{aligned} \tag{16}$$

where

$$\mathbf{B} = \left(\mathbf{X}_b^* \mathbf{X}_b^{*\top}\right)^k - \left(\mathrm{E}(\gamma_I^k) + \sqrt{\frac{2\,\mathrm{Var}\left(\gamma_I^k\right)}{n-q}} x \right) \mathbf{I}_{n-q}.$$

Since $\mathbf{y}^{*\top}\mathbf{B}\mathbf{y}^* = \boldsymbol{\epsilon}^\top \tilde{\mathbf{U}}_a \mathbf{B} \tilde{\mathbf{U}}_a^\top \boldsymbol{\epsilon} + 2\boldsymbol{\epsilon}^\top \tilde{\mathbf{U}}_a \mathbf{B}\mathbf{X}_b^* \boldsymbol{\beta}_b + \boldsymbol{\beta}_b^\top \mathbf{X}_b^{*\top}\mathbf{B}\mathbf{X}_b^*\boldsymbol{\beta}_b$, we have

$$\begin{aligned}
&\Pr\left(\mathbf{y}^{*\top}\mathbf{B}\mathbf{y}^* \le 0\right) \\
=&\Pr\left(\frac{\boldsymbol{\epsilon}^\top \tilde{\mathbf{U}}_a \mathbf{B} \tilde{\mathbf{U}}_a^\top \boldsymbol{\epsilon} + 2\boldsymbol{\epsilon}^\top \tilde{\mathbf{U}}_a \mathbf{B}\mathbf{X}_b^* \boldsymbol{\beta}_b - \phi^{-1}\operatorname{tr}(\mathbf{B})}{\sqrt{2\phi^{-2}\operatorname{tr}(\mathbf{B}^2) + 4\phi^{-1}\boldsymbol{\beta}_b^\top \mathbf{X}_b^{*\top}\mathbf{B}^2\mathbf{X}_b^*\boldsymbol{\beta}_b}}\right. \\
&\qquad \left.\le \frac{-\boldsymbol{\beta}_b^\top \mathbf{X}_b^{*\top}\mathbf{B}\mathbf{X}_b^*\boldsymbol{\beta}_b - \phi^{-1}\operatorname{tr}(\mathbf{B})}{\sqrt{2\phi^{-2}\operatorname{tr}(\mathbf{B}^2) + 4\phi^{-1}\boldsymbol{\beta}_b^\top \mathbf{X}_b^{*\top}\mathbf{B}^2\mathbf{X}_b^*\boldsymbol{\beta}_b}}\right).
\end{aligned}$$

To apply proposition 2, we need to verify the condition $\lambda_1\left(\mathbf{B}^2\right)/\operatorname{tr}\left(\mathbf{B}^2\right) \to 0$. It is straightforward to show that $\operatorname{tr}(\mathbf{B}^2) = (n-q+2x^2)\operatorname{Var}(\gamma_I^k)$. On the other hand,

$$\begin{aligned}
\lambda_1\left(\mathbf{B}^2\right) =& \max_{1\le i\le n-q}\left(\gamma_i^k - \operatorname{E}(\gamma_I^k) - \sqrt{\frac{2\operatorname{Var}\left(\gamma_I^k\right)}{n-q}}x\right)^2 \\
\le& 2\max_{1\le i\le n-q}\left(\gamma_i^k - \operatorname{E}(\gamma_I^k)\right)^2 + 4\frac{\operatorname{Var}\left(\gamma_I^k\right)}{n-q}x^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\lambda_1(\mathbf{B}^2)}{\operatorname{tr}(\mathbf{B}^2)} \le& 2\frac{\max_{1\le i\le n-q}\left(\gamma_i^k - \operatorname{E}(\gamma_I^k)\right)^2}{(n-q+2x^2)\operatorname{Var}(\gamma_I^k)} + 4\frac{x^2}{(n-q)(n-q+x^2)} \\
\le& 2\frac{\max_{1\le i\le n-q}\left(\gamma_i^k - \operatorname{E}(\gamma_I^k)\right)^2}{(n-q)\operatorname{Var}(\gamma_I^k)} + \frac{4}{(n-q)},
\end{aligned}$$

which tends to 0 by the condition (9). Hence Proposition 2 implies that

$$\Pr\left(\mathbf{y}^{*\top}\mathbf{B}\mathbf{y}^* \le 0\right) = \Phi\left(\frac{-\boldsymbol{\beta}_b^\top \mathbf{X}_b^{*\top}\mathbf{B}\mathbf{X}_b^*\boldsymbol{\beta}_b - \phi^{-1}\operatorname{tr}(\mathbf{B})}{\sqrt{2\phi^{-2}\operatorname{tr}(\mathbf{B}^2) + 4\phi^{-1}\boldsymbol{\beta}_b^\top \mathbf{X}_b^{*\top}\mathbf{B}^2\mathbf{X}_b^*\boldsymbol{\beta}_b}}\right) + o(1). \tag{17}$$

Then the conclusion follows from (16), (17) and the following facts

$$\begin{aligned}
\operatorname{tr}(\mathbf{B}) =& -(n-q)\sqrt{\frac{2\operatorname{Var}\left(\gamma_I^k\right)}{n-q}}x, \\
\operatorname{tr}(\mathbf{B}^2) =& (1+o(1))(n-q)\operatorname{Var}(\gamma_I^k), \\
\boldsymbol{\beta}_b^\top \mathbf{X}_b^{*\top}\mathbf{B}\mathbf{X}_b^*\boldsymbol{\beta}_b =& (n-q)\left(\operatorname{Cov}\left(\gamma_I^k, \gamma_I w_I^2\right) - \operatorname{E}(\gamma_I w_I^2)\sqrt{\frac{2\operatorname{Var}\left(\gamma_I^k\right)}{n-q}}x\right), \\
\boldsymbol{\beta}_b^\top \mathbf{X}_b^{*\top}\mathbf{B}^2\mathbf{X}_b^*\boldsymbol{\beta}_b =& (n-q)\operatorname{E}\left[\left(\gamma_I^k - \operatorname{E}(\gamma_I^k) - \sqrt{\frac{2\operatorname{Var}(\gamma_I^k)}{n-q}}x\right)^2\gamma_I w_I^2\right].
\end{aligned}$$

Proof ***(Proof of Proposition 3)*** Fix an $x \in \mathbb{R}$. In view of Theorem 4, we only need to show that $\operatorname{E}(\gamma_I w_I^2) = o(1)$ and

$$\operatorname{E}\left[\left(\frac{\gamma_I^k - \operatorname{E}(\gamma_I^k)}{\sqrt{\operatorname{Var}\left(\gamma_I^k\right)}} - \sqrt{\frac{2}{n-q}}x\right)^2\gamma_I w_I^2\right] = o(1).$$

The former one is a consequence of the assumption $\mathrm{E}(\gamma_I w_I^2) = O((n-q)^{-1/2})$. For the latter one, we have

$$
\begin{aligned}
&\mathrm{E}\left[\left(\frac{\gamma_I^k - \mathrm{E}(\gamma_I^k)}{\sqrt{\mathrm{Var}\left(\gamma_I^k\right)}} - \sqrt{\frac{2}{n-q}x}\right)^2 \gamma_I w_I^2\right] \\
\leq& 2\,\mathrm{E}\left[\left(\frac{\gamma_I^k - \mathrm{E}(\gamma_I^k)}{\sqrt{\mathrm{Var}\left(\gamma_I^k\right)}}\right)^2 \gamma_I w_I^2\right] + \frac{4}{n-q}x^2\,\mathrm{E}\left[\gamma_I w_I^2\right] \\
\leq& 2\left(\max_{i\in\{1,\ldots,n-q\}} \gamma_i w_i^2\right)\mathrm{E}\left(\frac{\gamma_I^k - \mathrm{E}(\gamma_I^k)}{\sqrt{\mathrm{Var}\left(\gamma_I^k\right)}}\right)^2 + \frac{4}{n-q}x^2\,\mathrm{E}\left[\gamma_I w_I^2\right] \\
=& 2\left(\max_{i\in\{1,\ldots,n-q\}} \gamma_i w_i^2\right) + \frac{4}{n-q}x^2\,\mathrm{E}\left[\gamma_I w_I^2\right] \\
\to& 0.
\end{aligned}
$$

This completes the proof.

References

Arboretti R, Ceccato R, Corain L, Ronchi F, Salmaso L (2018) Multivariate small sample tests for two-way designs with applications to industrial statistics. Stat Pap 59(4):1483-1503

Bai Z, Pan G, Yin Y (2018) A central limit theorem for sums of functions of residuals in a high-dimensional regression model with an application to variance homoscedasticity test. TEST 27(4):896–920

Baltagi BH, Kao C, Na S (2013) Testing for cross-sectional dependence in a panel factor model using the wild bootstrap F test. Stat Pap 54(4):1067–1094

Bentkus V, Götze F (1996) Optimal rates of convergence in the CLT for quadratic forms. Ann Probab 24(1):466–490

Bühlmann P (2013) Statistical significance in high-dimensional linear models. Bernoulli 19(4):1212–1242

Casella G, Moreno E (2006) Objective Bayesian variable selection. J Amer Statist Assoc 101(473):157–167

Chatterjee S (2006) A generalization of the Lindeberg principle. Ann Probab 34(6):2061–2076

Chatterjee S (2008) A new method of normal approximation. Ann Probab 36(4):1584–1610

Chen SX, Zhang LX, Zhong PS (2010) Tests for high-dimensional covariance matrices. J Amer Statist Assoc 105(490):810–819

Cohn DL (2013) Measure Theory, 2nd edn. Birkhäuser, New York

Cui H, Guo W, Zhong W (2018) Test for high-dimensional regression coefficients using refitted cross-validation variance estimation. Ann Statist 46(3):958–988

DasGupta A (2008) Asymptotic Theory of Statistics and Probability, 1st edn. Springer, New York

Dezeure R, Bühlmann P, Zhang CH (2017) High-dimensional simultaneous inference with the bootstrap. TEST 26(4):685–719

Dicker LH, Erdogdu MA (2017) Flexible results for quadratic forms with applications to variance components estimation. Ann Statist 45(1):386–414

Draper NR, Pukelsheim F (1996) An overview of design of experiments. Stat Pap 37(1):1–32

Fan J, Yuan L, Mincheva M (2013) Large covariance estimation by thresholding principal orthogonal complements. J R Stat Soc Ser B Stat Methodol 75(4):603–680

Feng L, Zou C, Wang Z, Chen B (2013) Rank-based score tests for high-dimensional regression coefficients. Electron J Stat 7(0):2131–2149

Goddard SD, Johnson VE (2016) Restricted most powerful Bayesian tests for linear models. Scand J Stat 43(4):1162–1177

Goeman JJ, van de Geer SA, van Houwelingen HC (2006) Testing against a high dimensional alternative. J R Stat Soc Ser B Stat Methodol 68(3):477–493

Goeman JJ, van Houwelingen HC, Finos L (2011) Testing against a high-dimensional alternative in the generalized linear model: asymptotic type I error control. Biometrika 98(2):381–390

Götze F, Tikhomirov A (2002) Asymptotic distribution of quadratic forms and applications. J Theoret Probab 15(2):423–475

Horn RA, Johnson CR (1991) Topics in matrix analysis, 1st edn. Cambridge University Press, New York

Ingster YI, Tsybakov AB, Verzelen N (2010) Detection boundary in sparse regression. Electron J Stat 4(0):1476-1526

Janson L, Barber RF, Candès E (2016) EigenPrism: inference for high dimensional signal-to-noise ratios. J R Stat Soc Ser B Stat Methodol 79(4):1037–1065

Javier Girón F, Lina Martínez M, Moreno E, Torres F (2006) Objective testing procedures in linear models: Calibration of the p-values. Scand J Stat 33(4):765–784

Jiang J (1996) Reml estimation: asymptotic behavior and related topics. Ann Statist 24(1):255–286

Jong PD (1987) A central limit theorem for generalized quadratic forms. Probab Theory Related Fields 75(2):261–277

Kass RE, Raftery AE (1995) Bayes factors. J Amer Statist Assoc 90(430):773–795

Lan W, Wang H, Tsai CL (2014) Testing covariates in high-dimensional regression. Ann Inst Statist Math 66(2):279–301

Lan W, Ding Y, Fang Z, Fang K (2016a) Testing covariates in high dimension linear regression with latent factors. J Multivariate Anal 144:25–37

Lan W, Zhong PS, Li R, Wang H, Tsai CL (2016b) Testing a single regression coefficient in high dimensional linear models. J Econometrics 195(1):154–168

Lehmann EL, Romano JP (2005) Testing statistical hypotheses, 3rd edn. Springer, New York

Lei L, Bickel PJ, Karoui NE (2018) Asymptotics for high dimensional regression M-estimates: fixed design results. Probab Theory Related Fields 172(3-4):983–1079

Liang F, Paulo R, Molina G, Clyde MA, Berger JO (2008) Mixtures of g priors for Bayesian variable selection. J Amer Statist Assoc 103(481):410–423

Pollard D (1984) Convergence of stochastic processes, 1st edn. Springer, New York

Sevast'yanov BA (1961) A class of limit distributions for quadratic forms of stochastic random variables. Theory Probab Appl 6(3):337–340

Wang S, Cui H (2015) A new test for part of high dimensional regression coefficients. J Multivariate Anal 137:187–203

Xu K (2016) A new nonparametric test for high-dimensional regression coefficients. J Stat Comput Simul 87(5):855–869

Zhang X, Cheng G (2017) Simultaneous inference for high-dimensional linear models, J Amer Statist Assoc 112(518):757–768

Zhang CH, Zhang SS (2014) Confidence intervals for low dimensional parameters in high dimensional linear models. J R Stat Soc Ser B Stat Methodol 76(1):217–242

Zhong PS, Chen SX (2011) Tests for high-dimensional regression coefficients with factorial designs. J Amer Statist Assoc 106(493):260–274

Zhou Q, Guan Y (2018) On the null distribution of bayes factors in linear regression. J Amer Statist Assoc 113(523):1362–1371