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Abstract

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1. Main

Suppose (X_1^T, Y_1),...,(X_n^T, Y_n) are i.i.d. from $N_{p+1}(\mu, \Sigma)$, where $X_i \in \mathbb{R}^p$ and $Y_i \in \mathbb{R}$. Denote $X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_n)^T$.

Write $Y = \beta_0 \mathbf{1}_n + X^T\beta + \epsilon$, where $\mathbf{1}_n$ is n dimensional vector with all elements equal to 1. ϵ has distribution $N(0, \sigma^2 I_n)$.

The problem is to test hypotheses $H : \beta = 0$.

The test statistic is

$$T = \frac{(\mathbf{1}_n^T(X^TX)^{-1}Q_nY)^2}{\hat{\sigma}^2\mathbf{1}_n^T(X^TX)^{-1}Q_n(X^TX)^{-1}\mathbf{1}_n}.$$

where $Q_n = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$ and

$$\hat{\sigma}^2 = \frac{1}{n-2}Y^TQ_n\Big[I_n - \frac{(X^TX)^{-1}\mathbf{1}_n\mathbf{1}_n^T(X^TX)^{-1}}{\mathbf{1}_n^T(X^TX)^{-1}Q_n(X^TX)^{-1}\mathbf{1}_n}\Big]Q_nY$$

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[1]Since 1880.

Let $Q_n = WW^T$ be the rank decomposition of Q_n, where W_n is a $n \times n-1$ matrix with $W^T W = I_{n-1}$. The new test statistic is

$$T = \frac{y^T Q_n y}{y^T W (W^T X^T X W)^{-1} W^T y}$$

or equivalently

$$\frac{y^T Q_n y}{y^T Q_n (X^T X)^{-1} Q_n y - (y^T Q_n (X^T X)^{-1} \mathbf{1}_n)^2 / (\mathbf{1}_n (X^T X)^{-1} \mathbf{1}_n)}$$

Let $\tilde{y} = W^T y$, $\tilde{X} = XW$, $\tilde{\epsilon} = W^T \epsilon$. Then

$$\tilde{y} = \tilde{X}^T \beta + \tilde{\epsilon}$$

and

$$T = \frac{\tilde{y}^T \tilde{y}}{\tilde{y}^T \left(\tilde{X}^T \tilde{X}\right)^{-1} \tilde{y}}$$

Next we derive another form of of T. We follow the similar technique of Hotelling's T^2.

Let R be an $(n-1) \times (n-1)$ orthogonal matrix satisfies

$$R\tilde{y} = \begin{pmatrix} \|\tilde{y}\| \\ 0 \\ \dots \\ 0 \end{pmatrix}.$$

We can write

$$T = \frac{\|\tilde{y}\|^2}{\tilde{y} R^T \left(R \tilde{X}^T \tilde{X} R^T\right)^{-1} R\tilde{y}} \tag{1}$$

Denote by $B = R\tilde{X}^T \tilde{X} R^T$, then

$$T = \frac{1}{(B^{-1})_{11}}.$$

Let

$$B = \begin{pmatrix} b_{11} & b_{(1)}^T \\ b_{(1)} & B_{22} \end{pmatrix},$$

and apply the matrix inverse formula, we have $(B^{-1})_{11} = 1/(b_{11} - b_{(1)}^T B_{22}^{-1} b_{(1)})$. Hence

$$T = b_{11} - b_{(1)}^T B_{22}^{-1} b_{(1)}.$$

2. Asymptotic distribution

Note that conditioning on $\tilde{y}$, R is a constant orthogonal matrix. And $\tilde{y}$ is independent of $\tilde{X}$ under null hypotheses. So $B|\tilde{y}$ has the same distribution with $\tilde{X}^T\tilde{X}$ under null hypotheses. Hence B is independent of $\tilde{y}$ and can be written as

$$B = \sum_{i=1}^{p} \lambda_i z_i z_i^T \tag{2}$$

where z_i's are i.i.d. $n-1$ dimensional random vectors distributed as $N(0, I_{n-1})$, $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_p > 0$ are eigenvalues of Σ_X. Denote by $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)$, $Z = (Z_1, \ldots, Z_p)$. Let $Z_{(1)}$ and $Z_{(2)}$ be the first 1 row and last $n-2$ rows of Z, that is

$$Z = \begin{pmatrix} Z_{(1)} \\ Z_{(2)} \end{pmatrix}.$$

Then

$$\begin{aligned} B &= Z\Lambda Z^T \\ &= \begin{pmatrix} Z_{(1)}\Lambda Z_{(1)}^T & Z_{(1)}\Lambda Z_{(2)}^T \\ Z_{(2)}\Lambda Z_{(1)}^T & Z_{(2)}\Lambda Z_{(2)}^T \end{pmatrix}. \end{aligned} \tag{3}$$

Hence

$$\begin{aligned} T &= Z_{(1)}\Lambda Z_{(1)}^T - Z_{(1)}\Lambda Z_{(2)}^T \left(Z_{(2)}\Lambda Z_{(2)}^T\right)^{-1} Z_{(2)}\Lambda Z_{(1)}^T \\ &= Z_{(1)}\left(\Lambda - \Lambda Z_{(2)}^T \left(Z_{(2)}\Lambda Z_{(2)}^T\right)^{-1} Z_{(2)}\Lambda\right) Z_{(1)}^T. \end{aligned} \tag{4}$$

But

$$\begin{aligned} \text{rank}(\Lambda Z_{(2)}^T \left(Z_{(2)}\Lambda Z_{(2)}^T\right)^{-1} Z_{(2)}\Lambda) &= \text{rank}(\Lambda^{\frac{1}{2}} Z_{(2)}^T \left(Z_{(2)}\Lambda Z_{(2)}^T\right)^{-1} Z_{(2)}\Lambda^{\frac{1}{2}}) \\ &= \text{rank}(I_{n-2}) = n-2, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \text{rank}(\Lambda - \Lambda Z_{(2)}^T \left(Z_{(2)}\Lambda Z_{(2)}^T\right)^{-1} Z_{(2)}\Lambda) &= \text{rank}(I_p - \Lambda^{\frac{1}{2}} Z_{(2)}^T \left(Z_{(2)}\Lambda Z_{(2)}^T\right)^{-1} Z_{(2)}\Lambda^{\frac{1}{2}}) \\ &= p - n + 2. \end{aligned} \tag{6}$$

Hence

$$T \sim \sum_{i=1}^{p-n+2} \lambda_i(\Lambda - \Lambda Z_{(2)}^T \left(Z_{(2)}\Lambda Z_{(2)}^T\right)^{-1} Z_{(2)}\Lambda)\chi_1^2$$

By Weyl's inequality, we have for $1 \le i \le p-n+2$

$$\lambda_i(\Lambda - \Lambda Z_{(2)}^T (Z_{(2)} \Lambda Z_{(2)}^T)^{-1} Z_{(2)} \Lambda) \le \lambda_i(\Lambda), \tag{7}$$

and

$$\begin{aligned} &\lambda_i(\Lambda - \Lambda Z_{(2)}^T (Z_{(2)} \Lambda Z_{(2)}^T)^{-1} Z_{(2)} \Lambda) \\ \ge &\lambda_{i+n-2}(\Lambda) + \lambda_{p-n+2}(-\Lambda Z_{(2)}^T (Z_{(2)} \Lambda Z_{(2)}^T)^{-1} Z_{(2)} \Lambda) \\ = &\lambda_{i+n-2}. \end{aligned} \tag{8}$$

Hence

$$\sum_{i=n-1}^{p} \lambda_i \chi_1^2 \le T \le \sum_{i=1}^{p-n+2} \lambda_i \chi_1^2$$

Note that under condition $\mathrm{tr}\Sigma^4/(\mathrm{tr}\Sigma^2)^2 \to 0$, we have by Liapounoff central limit theorem that

$$\frac{\sum_{i=1}^p \lambda_i \chi_1^2 - \mathrm{tr}\Sigma_X}{\sqrt{\mathrm{tr}(\Sigma_X^2)}} \xrightarrow{\mathcal{L}} N(0,1).$$

And

$$\frac{T - \mathrm{tr}\Sigma_X}{\sqrt{\mathrm{tr}(\Sigma_X^2)}} - \frac{\sum_{i=1}^p \lambda_i \chi_1^2 - \mathrm{tr}\Sigma_X}{\sqrt{\mathrm{tr}(\Sigma_X^2)}} = \frac{T - \sum_{i=1}^p \lambda_i \chi_1^2}{\sqrt{\mathrm{tr}(\Sigma_X^2)}}, \tag{9}$$

To prove (9) $\xrightarrow{P} 0$, we only need to prove

$$\mathrm{E}\Big(\frac{\sum_{i=1}^{n-2} \lambda_i \chi_1^2}{\sqrt{\mathrm{tr}(\Sigma_X^2)}}\Big) \to 0,$$

that is

$$\mathrm{E}\Big(\frac{\sum_{i=1}^{n-2} \lambda_i}{\sqrt{\mathrm{tr}(\Sigma_X^2)}}\Big) \to 0. \tag{10}$$

If λ_i's are bounded below and above, then (10) is equivalent to

$$n/\sqrt{p} \to 0, \tag{11}$$

or $p/n^2 \to \infty$. We thus obtain the following theorem.

Theorem 1. *Suppose*

$$E\Big(\frac{\sum_{i=1}^{n-2} \lambda_i}{\sqrt{tr(\Sigma_X^2)}}\Big) \to 0,$$

and

$$\frac{tr\Sigma^4}{(tr\Sigma^2)^2} \to 0.$$

Then under null hypotheses, we have

$$\frac{T - tr\Sigma_X}{\sqrt{tr(\Sigma_X^2)}} \xrightarrow{\mathcal{L}} N(0,1).$$

3. Simulation Results

n	p	$\|\beta\|^2$	Chen	New
40	310	0.00	0.05	0.06
40	310	0.04	0.05	1.00
80	550	0.00	0.05	0.01
80	550	0.04	0.05	1.00

Table 1: Non-sparse case, $T = 1$

n	p	$\|\beta\|^2$	Chen	New
40	310	0.00	0.05	0.00
40	310	0.04	0.35	1.00
80	550	0.00	0.05	0.00
80	550	0.04	0.29	1.00

Table 2: Non-sparse case, $T = 10$

n	p	$\|\beta\|^2$	Chen	New
40	310	0.00	0.05	0.00
40	310	0.04	0.88	1.00
80	550	0.00	0.05	0.00
80	550	0.04	1.00	1.00

Table 3: Non-sparse case, $T = 20$

References

n	p	$\|\beta\|^2$	Chen	New
40	310	0.00	0.05	0.06
40	310	0.04	0.05	0.88
80	550	0.00	0.05	0.01
80	550	0.04	0.05	0.98

Table 4: Sparse case, $T = 1$

n	p	$\|\beta\|^2$	Chen	New
40	310	0.00	0.05	0.00
40	310	0.04	0.06	0.20
80	550	0.00	0.05	0.00
80	550	0.04	0.06	0.38

Table 5: Sparse case, $T = 10$

n	p	$\|\beta\|^2$	Chen	New
40	310	0.00	0.05	0.00
40	310	0.04	0.12	0.04
80	550	0.00	0.05	0.00
80	550	0.04	0.13	0.00

Table 6: Sparse case, $T = 20$