

Elsevier L^AT_EX template[☆]

Elsevier¹

Radarweg 29, Amsterdam

Elsevier Inc^{a,b}, Global Customer Service^{b,}*

^a1600 John F Kennedy Boulevard, Philadelphia

^b360 Park Avenue South, New York

Abstract

This template helps you to create a properly formatted L^AT_EX manuscript.

Keywords:

1. Main

Suppose $(X_1^T, Y_1), \dots, (X_n^T, Y_n)$ are i.i.d. from $N_{p+1}(\mu, \Sigma)$, where $X_i \in \mathbb{R}^p$ and $Y_i \in \mathbb{R}$. Denote $X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_n)^T$.

Write $Y = \beta_0 \mathbf{1}_n + X^T \beta + \epsilon$, where $\mathbf{1}_n$ is n dimensional vector with all elements equal to 1. ϵ has distribution $N(0, \sigma^2 I_n)$.

The problem is to test hypotheses $H : \beta = 0$.

Let $Q_n = WW^T$ be the rank decomposition of Q_n , where W_n is a $n \times n - 1$ matrix with $W^T W = I_{n-1}$. The new test statistic is

$$T = \frac{y^T Q_n y}{y^T W (W^T X^T X W)^{-1} W^T y}$$

or equivalently

$$\frac{y^T Q_n y}{y^T Q_n (X^T X)^{-1} Q_n y - (y^T Q_n (X^T X)^{-1} \mathbf{1}_n)^2 / (\mathbf{1}_n^T (X^T X)^{-1} \mathbf{1}_n)}$$

[☆]Fully documented templates are available in the elsarticle package on CTAN.

^{*}Corresponding author

Email address: support@elsevier.com (Global Customer Service)

URL: www.elsevier.com (Elsevier Inc)

¹Since 1880.

Let $\tilde{y} = W^T y$, $\tilde{X} = XW$, $\tilde{\epsilon} = W^T \epsilon$. Then

$$\tilde{y} = \tilde{X}^T \beta + \tilde{\epsilon}$$

and

$$T = \frac{\tilde{y}^T \tilde{y}}{\tilde{y}^T (\tilde{X}^T \tilde{X})^{-1} \tilde{y}}$$

2. Asymptotic Results

$$\Sigma_X = P \Lambda P^T$$

Non-spike: there's no principal component ($r = 0$). That is, $\lambda_1 = \dots = \lambda_p$.

Spike: there's r principal components. That is, $\lambda_1 \geq \lambda_2 \geq \dots \lambda_r \geq \lambda_{r+1} = \dots = \lambda_p$. Denote by P_1 the first r column of P and P_2 the last $p - r$ column of P .

$$\begin{aligned} Y &= \beta_0 \mathbf{1}_n + X^T \beta + \epsilon \\ &= \beta_0 \mathbf{1}_n + X^T P_1 P_1^T \beta + X^T P_2 P_2^T \beta + \epsilon \end{aligned} \tag{1}$$

In either case, let λ be $\lambda = \lambda_{r+1} = \dots = \lambda_p$.

PCR try to do regression between Y and (estimated) $X^T P_1$. If P_1 is observed, then the problem is reduced to testing an ordinary regression model. However, it's not the case.

Simply estimating P_1 and invoke classical testing procedure may not be a good idea since the estimation may not be consistent in high dimension. In fact, there may be even no principal component!

In this paper, testing PCR means testing:

H_0 : There's no principal component or there's r principal components but $P_1^T \beta = 0$.

H_1 : There's r principal components and $P_1^T \beta \neq 0$.

Next we consider:

1. There's no PC.
2. There's r principal components but $P_1 \beta = 0$.

Assumption 1. X and ϵ are normal distribution.

$$\begin{aligned}
T &= \frac{\beta^T \tilde{X} \tilde{X}^T \beta + 2\beta^T \tilde{X} \tilde{\epsilon} + \tilde{\epsilon}^T \tilde{\epsilon}}{\beta^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \beta + 2\beta^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon} + \tilde{\epsilon}^T (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon}} \\
&= \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3}
\end{aligned} \tag{2}$$

2.1. *circumstance 1*

Assumption 2. $r = 0$.

Assumption 3. $n^2/p \rightarrow 0$.

2.1.1. *Step 1*

Independent of data, generate a random p dimensional orthonormal matrix O with Haar invariant distribution. And

$$T = \frac{(O\beta)^T O\tilde{X}(O\tilde{X})^T O\beta + 2(O\beta)^T \tilde{X}\tilde{\epsilon} + \tilde{\epsilon}^T \tilde{\epsilon}}{(O\beta)^T O\tilde{X}((O\tilde{X})^T O\tilde{X})^{-1} (O\tilde{X})^T \beta + 2(O\beta)^T O\tilde{X}((O\tilde{X})^T O\tilde{X})^{-1} \tilde{\epsilon} + \tilde{\epsilon}^T ((O\tilde{X})^T O\tilde{X})^{-1} \tilde{\epsilon}} \tag{3}$$

Note that conditioning on O , $O\tilde{X}$ is a random matrix with each entry independently distributed as $N(0, \lambda)$. Hence O is independent of $O\tilde{X}$. Observe also that $O\beta/\|\beta\|$ is uniformly distributed on the unit ball. We can without loss of generality assuming that $\beta/\|\beta\|$ is uniformly distributed on the unit ball.

2.1.2. *Step 2*

Independent of data, generate $R > 0$ with R^2 distributed as χ_p^2 . Then $\xi = R\beta/\|\beta\|$ distributed as $N_p(0, I_p)$. Note that conditioning on \tilde{X} , $\eta = (\tilde{X}^T \tilde{X})^{-1/2} \tilde{X}^T \xi$ is distributed as $N_{n-1}(0, I_{n-1})$. Hence η is independent of \tilde{X} .

Then

$$\begin{aligned}
T &= \frac{(\|\beta\|/R)^2 \xi^T \tilde{X} \tilde{X}^T \xi + 2(\|\beta\|/R) \xi^T \tilde{X} \tilde{\epsilon} + \tilde{\epsilon}^T \tilde{\epsilon}}{(\|\beta\|/R)^2 \xi^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \xi + 2(\|\beta\|/R) \xi^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon} + \tilde{\epsilon}^T (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon}} \\
&= \frac{(\|\beta\|/R)^2 \eta^T \tilde{X}^T \tilde{X} \eta + 2(\|\beta\|/R) \eta^T (\tilde{X}^T \tilde{X})^{1/2} \tilde{\epsilon} + \tilde{\epsilon}^T \tilde{\epsilon}}{(\|\beta\|/R)^2 \eta^T \eta + 2(\|\beta\|/R) \eta^T (\tilde{X}^T \tilde{X})^{-1/2} \tilde{\epsilon} + \tilde{\epsilon}^T (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon}} \\
&= \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3}
\end{aligned} \tag{4}$$

2.1.3. Step 3: CLT

Similar to the derivation of the distribution of Hotelling's T^2 statistic.

Now we deal with

$$\frac{A_3}{B_3} = \frac{\tilde{\epsilon}^T \tilde{\epsilon}}{\tilde{\epsilon}^T (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon}} \tag{5}$$

Let O be an $(n-1) \times (n-1)$ orthogonal matrix satisfies

$$O\tilde{\epsilon} = \begin{pmatrix} \|\tilde{\epsilon}\| \\ 0 \\ \dots \\ 0 \end{pmatrix}.$$

Then

$$\frac{A_3}{B_3} = \frac{(O\tilde{\epsilon})^T O\tilde{\epsilon}}{(O\tilde{\epsilon})^T ((\tilde{X}O^T)^T \tilde{X}O^T)^{-1} O\tilde{\epsilon}} \tag{6}$$

Note that $\tilde{X}O^T$ has the same distribution as \tilde{X} and is independent of O . We have

$$\frac{A_3}{B_3} \sim \frac{1}{((\tilde{X}^T \tilde{X})^{-1})_{11}}. \tag{7}$$

where $((\tilde{X}^T \tilde{X})^{-1})_{11}$ is the first element of $(\tilde{X}^T \tilde{X})^{-1}$. Apply the matrix inverse formula, we have

$$\frac{A_3}{B_3} \sim (\tilde{X}^T \tilde{X})_{11 \cdot 2}. \tag{8}$$

Since $\tilde{X}^T \tilde{X} \sim \text{Wishart}_{n-1}(\lambda I_{n-1}, p)$, $(\tilde{X}^T \tilde{X})_{11 \cdot 2} \sim \lambda \chi_{p-n+2}^2$. Hence by CLT,

$$\frac{A_3/B_3 - \lambda(p-n+2)}{\lambda\sqrt{2(p-n+2)}} \xrightarrow{\mathcal{L}} N(0, 1). \quad (9)$$

But

$$\begin{aligned} \frac{A_3/B_3 - \lambda(p-n+2)}{\lambda\sqrt{2(p-n+2)}} &= \frac{\sqrt{p}}{\sqrt{p-n+2}} \frac{A_3/B_3 - \lambda(p-n+2)}{\lambda\sqrt{2p}} \\ &= \frac{\sqrt{p}}{\sqrt{p-n+2}} \left(\frac{A_3/B_3 - \lambda p}{\lambda\sqrt{2p}} + \frac{(n-2)}{\sqrt{2p}} \right). \end{aligned} \quad (10)$$

By Slutsky Theorem, if $n^2/p \rightarrow 0$, we have

$$\frac{A_3/B_3 - \lambda p}{\lambda\sqrt{2p}} \xrightarrow{\mathcal{L}} N(0, 1) \quad (11)$$

Similar technique can deal with A_1/B_1 .

$$\frac{A_1}{B_1} = \frac{\eta^T \tilde{X}^T \tilde{X} \eta}{\eta^T \eta} \sim (\tilde{X}^T \tilde{X})_{11} \sim \lambda \chi_p^2 \quad (12)$$

Hence by CLT,

$$\frac{A_1/B_1 - \lambda p}{\lambda\sqrt{2p}} \xrightarrow{\mathcal{L}} N(0, 1). \quad (13)$$

2.1.4. step 4

It's obvious that $A_3 \asymp n$ and $B_1 \asymp \frac{n}{p} \|\beta\|^2$. We already have $A_1/B_1 \asymp p$ and $A_3/B_3 \asymp p$. It follows that $A_1 \asymp n \|\beta\|^2$ and $B_3 \asymp n/p$. And

$$\begin{aligned} A_2 &= O_P(\|\beta\|/\sqrt{p}) \eta^T (\tilde{X}^T \tilde{X})^{1/2} \tilde{\epsilon} \\ &= O_P(\|\beta\|/\sqrt{p}) \sqrt{\eta^T (\tilde{X}^T \tilde{X}) \eta} \\ &= O_P(\|\beta\|/\sqrt{p}) O_P(\sqrt{np}) \\ &= O_P(\sqrt{n} \|\beta\|), \end{aligned} \quad (14)$$

$$\begin{aligned} B_2 &= O_P(\|\beta\|/\sqrt{p}) \eta^T (\tilde{X}^T \tilde{X})^{-1/2} \tilde{\epsilon} \\ &= O_P(\|\beta\|/\sqrt{p}) \sqrt{\eta^T (\tilde{X}^T \tilde{X})^{-1} \eta} \\ &= O_P(\|\beta\|/\sqrt{p}) O_P(\sqrt{n/p}) \\ &= O_P\left(\frac{\sqrt{n}}{p} \|\beta\|\right). \end{aligned} \quad (15)$$

We can deduce that: If $\|\beta\|^2 \rightarrow 0$, then

$$T = \frac{\tilde{\epsilon}^T \tilde{\epsilon}}{\tilde{\epsilon}^T (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon}} (1 + o_P(1)). \quad (16)$$

Hence $T/(\lambda p) \rightarrow 1$. If $\|\beta\|^2 \rightarrow \infty$, then

$$T = \frac{\eta^T \tilde{X}^T \tilde{X} \eta}{\eta^T \eta} (1 + o_P(1)). \quad (17)$$

Hence $T/(\lambda p) \rightarrow 1$.

2.1.5. Step 5

If $\|\beta\|^2 \rightarrow 0$, we have

$$\begin{aligned} & \left| \frac{T - \lambda p}{\lambda \sqrt{2p}} - \frac{A_3/B_3 - \lambda p}{\lambda \sqrt{2p}} \right| \\ &= \frac{1}{\lambda \sqrt{2p}} \left| \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3} - \frac{A_3}{B_3} \right| \\ &= \frac{1}{\lambda \sqrt{2p}} \left| \frac{(A_1 + A_2)B_3 - (B_1 + B_2)A_3}{(B_1 + B_2 + B_3)B_3} \right| \\ &= \frac{O_P(1)}{\lambda \sqrt{2p}} \left| \frac{(O_P(n\|\beta\|^2) + O_P(\sqrt{n}\|\beta\|))O_P(\frac{n}{p}) - (O_P(\frac{n}{p}\|\beta\|^2) + O_P(\frac{\sqrt{n}}{p}\|\beta\|))O_P(n)}{n^2/p^2} \right| \\ &= O_P(\sqrt{p}\|\beta\|^2) + O_P(\frac{\sqrt{p}}{\sqrt{n}}\|\beta\|) \end{aligned} \quad (18)$$

Hence if $\sqrt{p}\|\beta\|^2 \rightarrow 0$ and $\frac{n}{p}\|\beta\|^2 \rightarrow 0$, CLT holds.

On the other hand. If $\|\beta\|^2 \rightarrow \infty$, we have

$$\begin{aligned} & \left| \frac{T - \lambda p}{\lambda \sqrt{2p}} - \frac{A_1/B_1 - \lambda p}{\lambda \sqrt{2p}} \right| \\ &= \frac{1}{\lambda \sqrt{2p}} \left| \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3} - \frac{A_1}{B_1} \right| \\ &= \frac{1}{\lambda \sqrt{2p}} \left| \frac{(A_2 + A_3)B_1 - (B_2 + B_3)A_1}{(B_1 + B_2 + B_3)B_1} \right| \\ &= \frac{O_P(1)}{\lambda \sqrt{2p}} \left| \frac{(O_P(\sqrt{n}\|\beta\|) + O_P(n))O_P(\frac{n}{p}\|\beta\|^2) - (O_P(\frac{\sqrt{n}}{p}\|\beta\|) + O_P(\frac{n}{p}))O_P(n\|\beta\|^2)}{\frac{n^2}{p^2}\|\beta\|^4} \right| \\ &= O_P(\frac{\sqrt{p}}{\sqrt{n}}\|\beta\|^{-1}) + O_P(\sqrt{p}\|\beta\|^{-2}) \end{aligned} \quad (19)$$

Hence if $\frac{n}{p}\|\beta\|^2 \rightarrow \infty$ and $\frac{1}{\sqrt{p}}\|\beta\|^2 \rightarrow \infty$, CLT holds.

2.2. circumstance 2

Assumption 4. $P_1^T \beta = 0$.

$$\begin{aligned}
T &= \frac{\beta^T \tilde{X} \tilde{X}^T \beta + 2\beta^T \tilde{X} \tilde{\epsilon} + \tilde{\epsilon}^T \tilde{\epsilon}}{\beta^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \beta + 2\beta^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon} + \tilde{\epsilon}^T (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon}} \\
&= \frac{\beta^T P_2 P_2^T \tilde{X} \tilde{X}^T P_2 P_2^T \beta + 2\beta^T P_2 P_2^T \tilde{X} \tilde{\epsilon} + \tilde{\epsilon}^T \tilde{\epsilon}}{\beta^T P_2 P_2^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T P_2 P_2^T \beta + 2\beta^T P_2 P_2^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon} + \tilde{\epsilon}^T (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon}} \\
&= \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3}
\end{aligned} \tag{20}$$

2.2.1. Step 1

Like before, we have $A_3/B_3 \sim (\tilde{X}^T \tilde{X})_{11,2}$. Denote by $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$. Let $Z = (Z_1, \dots, Z_p)$ be a $n-1 \times p$ matrix with all elements independently distributed as $N(0, 1)$. Let $Z_{(1)}$ and $Z_{(2)}$ be the first 1 row and last $n-2$ rows of Z , that is

$$Z = \begin{pmatrix} Z_{(1)} \\ Z_{(2)} \end{pmatrix}.$$

Then

$$\begin{aligned}
\tilde{X}^T \tilde{X} &\sim Z \Lambda Z^T \\
&= \begin{pmatrix} Z_{(1)} \Lambda Z_{(1)}^T & Z_{(1)} \Lambda Z_{(2)}^T \\ Z_{(2)} \Lambda Z_{(1)}^T & Z_{(2)} \Lambda Z_{(2)}^T \end{pmatrix}.
\end{aligned} \tag{21}$$

Hence

$$\begin{aligned}
T &\sim Z_{(1)} \Lambda Z_{(1)}^T - Z_{(1)} \Lambda Z_{(2)}^T (Z_{(2)} \Lambda Z_{(2)}^T)^{-1} Z_{(2)} \Lambda Z_{(1)}^T \\
&= Z_{(1)} \Lambda^{1/2} (I_p - \Lambda^{1/2} Z_{(2)}^T (Z_{(2)} \Lambda Z_{(2)}^T)^{-1} Z_{(2)} \Lambda^{1/2}) \Lambda^{1/2} Z_{(1)}^T \\
&\leq Z_{(1)} \Lambda^{1/2} (I_p - \hat{V} \hat{V}^T) \Lambda^{1/2} Z_{(1)}^T.
\end{aligned} \tag{22}$$

We require $p = o(n^2)$. The principal space is $V = (e_1, \dots, e_r)$. Then

$$Z_{(1)} \Lambda^{1/2} (V V^T - \hat{V} \hat{V}^T) \Lambda^{1/2} Z_{(1)}^T = o(\sqrt{p}) \tag{23}$$

Note that

$$Z_{(1)} \Lambda^{1/2} (I - V V^T) \Lambda^{1/2} Z_{(1)}^T \sim \lambda \chi_{p-r}^2 \tag{24}$$

Hence $T \leq \lambda \chi_{p-r}^2 + o(\sqrt{p})$.

On the other hand, the eigenvalues of $\Lambda^{1/2}(I_p - \Lambda^{1/2}Z_{(2)}^T(Z_{(2)}\Lambda Z_{(2)}^T)^{-1}Z_{(2)}\Lambda^{1/2})\Lambda^{1/2}$ is no less than $I_p - \Lambda^{1/2}Z_{(2)}^T(Z_{(2)}\Lambda Z_{(2)}^T)^{-1}Z_{(2)}\Lambda^{1/2}$. Hence $T \geq \lambda \chi_{p-n+2}^2$.

Hence $A_3/B_3 \asymp p$ if $p/n \rightarrow \infty$.

2.2.2. Step 2

Note that $P_2^T \tilde{X}$ is an $(p-r) \times (n-1)$ matrix with all elements independently distributed as $N(0, \lambda)$.

$$A_1 \asymp n \|P_2^T \beta\|^2, A_2 = O_P(\sqrt{n} \|P_2^T \beta\|), A_3 \asymp n.$$

$$B_3 \asymp n/p.$$

As for B_1 ,

$$\begin{aligned} B_1 &\leq \beta^T P_2 P_2^T \tilde{X} (\tilde{X}^T P_2 P_2^T \tilde{X})^{-1} \tilde{X}^T P_2 P_2^T \beta \\ &\asymp \frac{n-1}{p-r} \|P_2^T \beta\|^2 \end{aligned} \quad (25)$$

To get the lower bound, let $P_2^T \tilde{X} = U_2 D_2 V_2^T$ be the SVD of $P_2^T \tilde{X}$, where U_2 is a $(p-r) \times (n-1)$ orthonormal matrix, D_2 is a $(n-1) \times (n-1)$ diagonal matrix and V_2 is a $(n-1) \times (n-1)$ orthonormal matrix. Then

$$\begin{aligned} B_1 &= \beta^T P_2 P_2^T \tilde{X} (\tilde{X}^T P_1 P_1^T \tilde{X} + \tilde{X}^T P_2 P_2^T \tilde{X})^{-1} \tilde{X}^T P_2 P_2^T \beta \\ &= \beta^T P_2 U_2 D_2 V_2^T (\tilde{X}^T P_1 P_1^T \tilde{X} + V_2 D_2^2 V_2^T)^{-1} V_2 D_2 U_2^T P_2^T \beta \\ &= \beta^T P_2 U_2 (D_2^{-1} V_2^T \tilde{X}^T P_1 P_1^T \tilde{X} V_2 D_2^{-1} + I_{n-1})^{-1} U_2^T P_2^T \beta \end{aligned} \quad (26)$$

Note that U_2 is independent of $(V_2, D_2, P_1^T \tilde{X})$, and

$$(D_2^{-1} V_2^T \tilde{X}^T P_1 P_1^T \tilde{X} V_2 D_2^{-1} + I_{n-1})^{-1} \geq I_{n-1} - U^* U^{*T} \quad (27)$$

where U^* is the first r eigenvectors of $D_2^{-1} V_2^T \tilde{X}^T P_1 P_1^T \tilde{X} V_2 D_2^{-1}$ and is independent of U_2 . Note also that U_2 is of Haar distribution. Hence

$$\begin{aligned} B_1 &\geq \beta^T P_2 U_2 (I_{n-1} - U^* U^{*T}) U_2^T P_2^T \beta \\ &\asymp \frac{n-1-r}{p-r} \|P_2^T \beta\|^2 \end{aligned} \quad (28)$$

$$\begin{aligned} \text{Upper-Lower} &\leq \beta^T P_2 U_2 U^* U^{*T} U_2^T P_2^T \beta \\ &\asymp \frac{r}{p-r} \|P_2^T \beta\|^2 \end{aligned} \quad (29)$$

Hence

$$\begin{aligned} B_1 &= \beta^T P_2 P_2^T \tilde{X} (\tilde{X}^T P_2 P_2^T \tilde{X})^{-1} \tilde{X}^T P_2 P_2^T \beta + O_p\left(\frac{r}{p-r} \|P_2^T \beta\|^2\right) \\ &= \beta^T P_2 P_2^T \tilde{X} (\tilde{X}^T P_2 P_2^T \tilde{X})^{-1} \tilde{X}^T P_2 P_2^T \beta (1 + O_P(1/n)) \end{aligned} \quad (30)$$

$$\begin{aligned} B_2 &= O_P(1) \sqrt{\beta^T P_2 P_2^T \tilde{X} (\tilde{X}^T \tilde{X})^{-2} \tilde{X}^T P_2 P_2^T \beta} \\ &\leq \lambda_{\min}(\tilde{X}^T \tilde{X})^{-1/2} O_P(1) \sqrt{\beta^T P_2 P_2^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T P_2 P_2^T \beta} \end{aligned} \quad (31)$$

$$\lambda_{\min}(\tilde{X}^T \tilde{X}) \geq \lambda_{\min}(\tilde{X}^T P_2 P_2^T \tilde{X}) \asymp p-r \quad (32)$$

Hence $B_2 = O_P(\frac{\sqrt{n}}{p} \|P_2^T \beta\|)$.

Hence the similar law of large number and CLT holds.

2.2.3. Step 3

$$\frac{A_1}{B_1} \sim \frac{\chi_p^2}{1 + O_P(1/n)} = \lambda \chi_p^2 (1 + O_P(1/n)) \quad (33)$$

Hence if $\|P_2^T \beta\| \rightarrow \infty$ or $\|P_2^T \beta\| \rightarrow 0$,

$$\frac{T}{\lambda p} \xrightarrow{P} 1. \quad (34)$$

We have

$$\frac{A_1/B_1 - \lambda p}{\lambda \sqrt{2p}} \sim \frac{\chi_p^2 (1 + O_P(1/n)) - p}{\sqrt{2p}} \xrightarrow{\mathcal{L}} N(0, 1), \quad (35)$$

if $p = o(n^2)$.

Because $A_3/B_3 \leq \lambda \chi_{p-r}^2 + o(\sqrt{p}) \leq \lambda \chi_p^2 + o(\sqrt{p})$ (if $p = o(n^2)$). We have

$$\frac{A_3/B_3 - \lambda p}{\lambda \sqrt{2p}} \leq \frac{\chi_p^2 + o(\sqrt{p}) - p}{\sqrt{2p}} \xrightarrow{\mathcal{L}} N(0, 1). \quad (36)$$

If $\|P_2^T \beta\|^2 \rightarrow 0$, we have

$$\begin{aligned}
& \left| \frac{T - \lambda p}{\lambda \sqrt{2p}} - \frac{A_3/B_3 - \lambda p}{\lambda \sqrt{2p}} \right| \\
&= \frac{1}{\lambda \sqrt{2p}} \left| \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3} - \frac{A_3}{B_3} \right| \\
&= \frac{1}{\lambda \sqrt{2p}} \left| \frac{(A_1 + A_2)B_3 - (B_1 + B_2)A_3}{(B_1 + B_2 + B_3)B_3} \right| \\
&= \frac{O_P(1)}{\lambda \sqrt{2p}} \left| \frac{(O_P(n\|P_2^T \beta\|^2) + O_P(\sqrt{n}\|P_2^T \beta\|))O_P(\frac{n}{p}) - (O_P(\frac{n}{p}\|P_2^T \beta\|^2) + O_P(\frac{\sqrt{n}}{p}\|P_2^T \beta\|))O_P(n)}{n^2/p^2} \right| \\
&= O_P(\sqrt{p}\|P_2^T \beta\|^2) + O_P\left(\frac{\sqrt{p}}{\sqrt{n}}\|P_2^T \beta\|\right)
\end{aligned} \tag{37}$$

Hence if $\sqrt{p}\|P_2^T \beta\|^2 \rightarrow 0$ and $\frac{n}{p}\|P_2^T \beta\|^2 \rightarrow 0$, CLT holds.

On the other hand. If $\|P_2^T \beta\|^2 \rightarrow \infty$, we have

$$\begin{aligned}
& \left| \frac{T - \lambda p}{\lambda \sqrt{2p}} - \frac{A_1/B_1 - \lambda p}{\lambda \sqrt{2p}} \right| \\
&= \frac{1}{\lambda \sqrt{2p}} \left| \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3} - \frac{A_1}{B_1} \right| \\
&= \frac{1}{\lambda \sqrt{2p}} \left| \frac{(A_2 + A_3)B_1 - (B_2 + B_3)A_1}{(B_1 + B_2 + B_3)B_1} \right| \\
&= \frac{O_P(1)}{\lambda \sqrt{2p}} \left| \frac{(O_P(\sqrt{n}\|P_2^T \beta\|) + O_P(n))O_P(\frac{n}{p}\|P_2^T \beta\|^2) - (O_P(\frac{\sqrt{n}}{p}\|P_2^T \beta\|) + O_P(\frac{n}{p}))O_P(n\|P_2^T \beta\|^2)}{\frac{n^2}{p^2}\|P_2^T \beta\|^4} \right| \\
&= O_P\left(\frac{\sqrt{p}}{\sqrt{n}}\|P_2^T \beta\|^{-1}\right) + O_P(\sqrt{p}\|P_2^T \beta\|^{-2})
\end{aligned} \tag{38}$$

Hence if $\frac{n}{p}\|P_2^T \beta\|^2 \rightarrow \infty$ and $\frac{1}{\sqrt{p}}\|P_2^T \beta\|^2 \rightarrow \infty$, CLT holds.

2.3. Consistency of Test

β from normal distribution. Then consistency can be proved. Assume that $\beta \sim N(0, \sigma_\beta^2 I_p)$. Then $\gamma = (\tilde{X}^T \tilde{X})^{-1/2} \tilde{X}^T \beta \sim N(0, \sigma_\beta^2 I_{n-1})$.

$$\begin{aligned}
T &= \frac{\beta^T \tilde{X} \tilde{X}^T \beta + 2\beta^T \tilde{X} \tilde{\epsilon} + \tilde{\epsilon}^T \tilde{\epsilon}}{\beta^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \beta + 2\beta^T \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon} + \tilde{\epsilon}^T (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon}} \\
&= \frac{\gamma^T \tilde{X}^T \tilde{X} \gamma + 2\gamma^T (\tilde{X}^T \tilde{X})^{1/2} \tilde{\epsilon} + \tilde{\epsilon}^T \tilde{\epsilon}}{\gamma^T \gamma + 2\gamma^T (\tilde{X}^T \tilde{X})^{-1/2} \tilde{\epsilon} + \tilde{\epsilon}^T (\tilde{X}^T \tilde{X})^{-1} \tilde{\epsilon}} \\
&= \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3}
\end{aligned} \tag{39}$$

$$\begin{aligned}
A_1 &\sim \|\gamma\|^2 \sum_{i=1}^p \lambda_i \chi_1^2 \asymp \|\gamma\|^2 (p + \lambda_1) \asymp \sigma_\beta^2 n (p + \lambda_1). \quad A_2 = O_P(\sqrt{A_1}). \\
A_3 &\asymp n. \\
B_1 &\asymp \sigma_\beta^2 n. \quad B_3 \leq \tilde{\epsilon} (\tilde{X}^T P_2 P_2^T \tilde{X})^{-1} \tilde{\epsilon} \asymp n/p. \quad B_2 = O_P(\sqrt{B_3} \sigma_\beta). \\
A_1/B_1 &\sim \sum_{i=1}^p \lambda_i \chi_1^2. \quad \text{Hence} \\
\mathbb{P}\left(\frac{A_1/B_1 - (p-r)\lambda}{\lambda\sqrt{2(p-r)}} \geq \Phi^{-1}(1-\alpha)\right) &\sim \mathbb{P}\left(N(0,1) \geq \Phi^{-1}(1-\alpha) - \frac{\sum_{i=1}^r \lambda_i \chi_i^2}{\lambda\sqrt{2(p-r)}}\right) \\
&= \mathbb{E}\left[\Phi\left(-\Phi^{-1}(1-\alpha) + \frac{\sum_{i=1}^r \lambda_i \chi_i^2}{\lambda\sqrt{2(p-r)}}\right)\right] \\
&\quad (40)
\end{aligned}$$

And note that if $p\sigma_\beta^2 \rightarrow \infty$,

$$\begin{aligned}
&\left| \frac{T - \lambda p}{\lambda\sqrt{2p}} - \frac{A_1/B_1 - \lambda p}{\lambda\sqrt{2p}} \right| \\
&= \frac{1}{\lambda\sqrt{2p}} \left| \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3} - \frac{A_1}{B_1} \right| \\
&= \frac{1}{\lambda\sqrt{2p}} \left| \frac{(A_2 + A_3)B_1 - (B_2 + B_3)A_1}{(B_1 + B_2 + B_3)B_1} \right| \\
&= \frac{O_P(1)}{\lambda\sqrt{2p}} \left| \frac{(O_P(\sigma_\beta \sqrt{n(p+\lambda_1)}) + O_P(n))O_P(\sigma_\beta^2 n) - (O_P(\sigma_\beta \frac{\sqrt{n}}{\sqrt{p}}) + O_P(\frac{n}{p}))O_P(\sigma_\beta^2 n(p+\lambda_1))}{\sigma_\beta^4 n^2} \right| \\
&= O_P\left(\frac{p+\lambda_1}{\sigma_\beta \sqrt{np}}\right) + O_P\left(\frac{p+\lambda_1}{\sigma_\beta^2 p^{3/2}}\right) \\
&\quad (41)
\end{aligned}$$

Hence if

$$\frac{np^2 + p^{5/2} + \lambda_1 p^{3/2}}{(p + \lambda_1)^2} \sigma_\beta^2 \rightarrow \infty \quad (42)$$

Then Power function holds.

3. Simulation Results

References