Problem T3.1 (100 Points) We know by some "magical" calculation

$$I = \iint_{-1}^{1} u(x, y) dxdy \approx 3.156049594$$

where the integrand is

$$u(x, y) = e^{-(x^6+y^6)}$$

which looks like

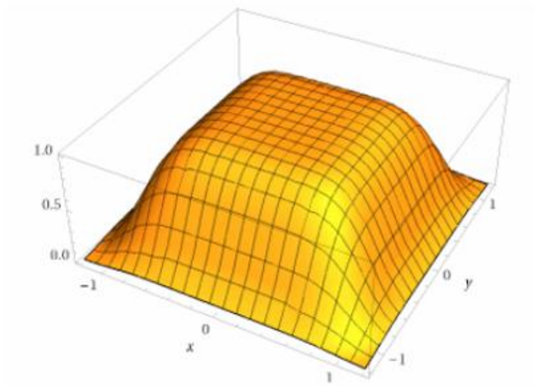

Figure 1. The integrand for this problem.

For this test, you write programs for two implementations of the Monte Carlo method with $N = 10^6$ samples (same for both implementations) and other given parameters/conditions to compute the integral. You need sufficient programming details in your implementations.

Method	Your Integral	Your Error	Credits to Give
Implementation 1: Using samples drawn randomly and uniformly in box $[-1, 1] \times [-1, 1]$			40
Implementation 2: Using samples drawn randomly with density proportional to $\|\nabla u(x, y)\|$ in box $[-1, 1] \times [-1, 1]$			60

Algorithm Description:

The programs estimate the value of a two-dimensional integral using two Monte Carlo integration techniques: uniform sampling and importance sampling.

In the uniform sampling method, random points (x,y) are generated uniformly within the square. The function u(x,y) is evaluated at each sampled point, and the average of these values is taken. This average is then multiplied by the area of the integration domain, which is 4, to obtain an estimate of the integral. This method is simple and unbiased but can be inefficient if the integrand is sharply concentrated in a small region of the domain, as is the case here.

To address this inefficiency, the importance sampling method is employed. In this approach, points are sampled from a bivariate Gaussian (normal) distribution centered at the origin with a specified standard deviation sigma, but only those within the [−1,1]^2 square are accepted. This sampling method concentrates more points in the region where the integrand is largest (near the origin), thereby reducing the variance of the estimate. Because the sampling distribution is no longer uniform, the integral estimate must account for the change in probability density. This is done by weighting each function value by the reciprocal of the sampling probability density function (PDF) at that point. The mean of these weighted values provides an importance-sampling-based estimate of the integral.

Result:

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Uniform Sampling Estimate: 3.15731898, Error: 1.26938671e-03
Importance Sampling Estimate: 3.23476103, Error: 7.87114340e-02
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