

2024-2025 Fall  
MAT123 Midterm  
(02/12/2024)

1) Let

$$f(x) = \begin{cases} \frac{\tan ax}{\tan bx}, & \text{if } x < 0 \\ 4, & \text{if } x = 0 \\ ax + b, & \text{if } x > 0 \end{cases}$$

Determine the values of  $a$  and  $b$  such that  $f$  is continuous at the point  $x = 0$ .

2. Use differential to approximate  $3\sqrt[3]{66} + 2\sqrt{66}$ .

3. (a) Without using L'Hôpital's rule, evaluate  $\lim_{x \rightarrow 0} \frac{5 - 6 \cos x + \cos^2 x}{x \sin x}$ .

(b) Prove that  $\lim_{x \rightarrow -3} \sqrt{-x - 2} = 1$  by using the formal definition of limit.

(c) Evaluate  $\lim_{x \rightarrow 1^+} (\sqrt{x})^{\ln(x-1)}$ .

4. Coffee is draining out of a conical filter at a rate of  $2.25 \text{ in.}^3/\text{min}$ . If the cone is 5 in. tall and has a radius of 2 in., how fast is the coffee level dropping when the coffee is 3 in. deep?

5. Using the mean value theorem, show that  $\ln(x + 1) < x$  for  $x > 0$ .

6. Let  $f(x) = \frac{x^2 - 2}{(x - 1)^2}$

(a) Determine the interval of increase, decrease and concavity of  $f$ .

(b) Construct a table.

(c) Sketch the graph of  $f$ .

1) To ensure continuity at  $x = 0$ , the one-sided limit values must be equal to the value of the function at that point.

$$\lim_{x \rightarrow 0^-} \frac{\tan ax}{\tan bx} = \lim_{x \rightarrow 0^+} (ax + b) = f(0) = 4$$

The easy part is that we can calculate the limit from the right.

$$\lim_{x \rightarrow 0^+} (ax + b) = 0 + b = b$$

Hence,  $b = 4$ . To calculate from the left, we need another technique.

$$\begin{aligned} \lim_{x \rightarrow 0^-} \frac{\tan ax}{\tan bx} &= \lim_{x \rightarrow 0^-} \left( \frac{\sin ax}{\cos ax} \cdot \frac{\cos bx}{\sin bx} \cdot \frac{bx}{bx} \cdot \frac{ax}{ax} \right) \\ &= \lim_{x \rightarrow 0^-} \left( \frac{\sin ax}{ax} \right) \cdot \lim_{x \rightarrow 0^-} \left( \frac{1}{\frac{\sin bx}{bx}} \right) \cdot \lim_{x \rightarrow 0^-} \left( \frac{\cos(bx) \cdot ax}{\cos(ax) \cdot bx} \right) \\ &= 1 \cdot \frac{1}{\lim_{x \rightarrow 0^-} \frac{\sin bx}{bx}} \cdot \lim_{x \rightarrow 0^-} \left( \frac{\cos(bx) \cdot a}{\cos(ax) \cdot b} \right) = 1 \cdot 1 \cdot \left( \frac{\cos(0) \cdot a}{\cos(0) \cdot b} \right) \\ &= \frac{a}{b} \end{aligned}$$

Now, set  $\frac{a}{b} = b \rightarrow a = 16$ .  $a = 16, b = 4$

2) Let  $f(x) = x^{1/3}$  and  $g(x) = x^{1/2}$ . Using the differential approximation, we get

$$\begin{aligned} f(x + \Delta x) &\approx f(x) + f'(x)\Delta x = x^{1/3} + \frac{1}{3}x^{-2/3}\Delta x \\ g(x + \Delta x) &\approx g(x) + g'(x)\Delta x = x^{1/2} + \frac{1}{2}x^{-1/2}\Delta x \end{aligned}$$

Set  $x = 64$  and  $\Delta x = 2$ .

$$\begin{aligned} 3\sqrt[3]{66} + 2\sqrt{66} &\approx 3 \left( 64^{1/3} + \frac{1}{3} \cdot 64^{-2/3} \cdot 2 \right) + 2 \left( 64^{1/2} + \frac{1}{2} \cdot 64^{-1/2} \cdot 2 \right) \\ &= 3 \left( 4 + \frac{1}{24} \right) + 2 \left( 8 + \frac{1}{8} \right) = \boxed{28.375} \end{aligned}$$

3)

(a)

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{5 - 6 \cos x + \cos^2 x}{x \sin x} &= \lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - 5)}{x \sin x} \\
&= \lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - 5)(\cos x + 1)}{(x \sin x)(\cos x + 1)} = \lim_{x \rightarrow 0} \left( -\frac{\sin^2 x \cdot (\cos x - 5)}{x \sin x \cdot (\cos x + 1)} \right) \\
&= \lim_{x \rightarrow 0} \left( -\frac{\sin x \cdot (\cos x - 5)}{x \cdot (\cos x + 1)} \right) = -\lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\cos x - 5}{\cos x + 1} = -1 \cdot \frac{\cos 0 - 5}{\cos 0 + 1} = \boxed{2}
\end{aligned}$$

(b) For all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that

$$0 < |x + 3| < \delta \implies |f(x) - 1| < \epsilon$$

$$\begin{aligned}
|f(x) - 1| &= \left| \sqrt{-x - 2} - 1 \right| = \left| \sqrt{-x - 2} - 1 \cdot \frac{\sqrt{-x - 2} + 1}{\sqrt{-x - 2} + 1} \right| \\
&= \left| \frac{-x - 3}{\sqrt{-x - 2} + 1} \right| \leq |-x - 3| \quad [\sqrt{-x - 2} + 1 \geq 0 + 1 = 1] \\
&= |x + 3| < \delta = \epsilon
\end{aligned}$$

If we set  $\delta = \epsilon$ , the proof is complete.

(c) Let  $L$  be the value of the limit. Then, take the logarithm of both sides. Since the expression is continuous for  $x > 1$ , we can take the logarithm function inside the limit.

$$\begin{aligned}
L &= \lim_{x \rightarrow 1^+} (\sqrt{x})^{\ln(x-1)} \quad [1^{-\infty}] \\
\ln(L) &= \ln \left[ \lim_{x \rightarrow 1^+} (\sqrt{x})^{\ln(x-1)} \right]
\end{aligned}$$

$$\begin{aligned}
\ln(L) &= \lim_{x \rightarrow 1^+} \ln [(\sqrt{x})^{\ln(x-1)}] = \lim_{x \rightarrow 1^+} [\ln(x-1) \cdot \ln(\sqrt{x})] \quad [\infty \cdot 0] \\
&= \lim_{x \rightarrow 1^+} \left[ \frac{\ln(x-1)}{\frac{1}{\ln(\sqrt{x})}} \right] \quad \left[ \frac{\infty}{\infty} \right] \\
&= \lim_{x \rightarrow 1^+} \left[ \frac{\frac{1}{x-1}}{\frac{1}{-\ln^2(\sqrt{x})} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}} \right] = \lim_{x \rightarrow 1^+} \left[ -\frac{\ln^2(\sqrt{x}) \cdot 2x}{x-1} \right] \quad \left[ \frac{0}{0} \right] \\
&= \lim_{x \rightarrow 1^+} \left( -\frac{2 \ln(\sqrt{x}) \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \cdot 2x + \ln^2(\sqrt{x}) \cdot 2}{1} \right) = \lim_{x \rightarrow 1^+} [2 \ln(\sqrt{x}) + 2 \ln^2(\sqrt{x})] \\
&= [2 \ln(\sqrt{1}) + 2 \ln^2(\sqrt{1})] = 0
\end{aligned}$$

$\ln(L) = 0$ . Therefore,  $\boxed{L = 1}$ .

4) Let  $f(x)$  represent the volume of coffee in the cone in cubic inches. The coffee in the cone will have a conical shape while draining. We may set up the equation below using the formula of the volume of a cone.

$$f(t) = \frac{1}{3} \cdot h(t) \cdot \pi r^2(t)$$

$h(t), r(t)$  represent the height and radius of the circular area that coffee forms, respectively, in inches. We can eliminate  $r$  to proceed with  $h$ .  $r$  and  $h$  are proportional.

$$\frac{r}{h} = \frac{2}{5} \rightarrow r = \frac{2h}{5}$$

$$f(t) = \frac{4\pi h^3(t)}{75}$$

Take the derivative of both sides.

$$f'(t) = \frac{4\pi}{25} \cdot h^2(t) \cdot h'(t)$$

Given that at  $t = t_0$ ,  $f'(t_0) = -2.25$ ,  $h(t_0) = 3$ . We may now find  $h'(t_0)$ . Solve for  $h'(t_0)$ .

$$h'(t_0) = \frac{25f'(t_0)}{4\pi h^2(t_0)} = \frac{25 \cdot (-2.25)}{4\pi \cdot (3)^2} = \boxed{-\frac{1.5625}{\pi} \text{ inches/minute}}$$

5) Let  $f(x) = \ln(1+x) - x$ . We have  $f(0) = \ln(1+0) - 0 = 0$ . The mean value theorem (MVT) states that if a function  $g(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists a point  $c$  such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$

$f$  is continuous on  $[0, x]$  and differentiable on  $(0, x)$ . By MVT,  $\frac{f(x) - f(0)}{x - 0} = f'(c)$  provided for some point  $c$  such that  $0 < c < x$ .

$$\begin{aligned} f'(c) &= \frac{1}{c+1} - 1 = \frac{\ln(x+1) - x}{x} = \frac{f(x) - f(0)}{x - 0} \\ \frac{1}{c+1} &= \frac{\ln(x+1)}{x} \rightarrow c+1 = \frac{x}{\ln(x+1)} \\ c &= \frac{x - \ln(x+1)}{\ln(x+1)} \end{aligned}$$

From the inequality  $0 < c < x$ ,

$$\begin{aligned} 0 &< \frac{x - \ln(x+1)}{\ln(x+1)} \\ 0 &< x - \ln(x+1) \\ \ln(x+1) &< x \end{aligned}$$

6)

(a) First off, find the domain. The expression is undefined when the denominator is zero. Therefore,  $(x-1)^2 \neq 0 \rightarrow x \neq 1$ . The only vertical asymptote occurs at  $x = 1$ .

$$\mathcal{D} = \mathbb{R} - \{1\}$$

Let us find the limit at infinity.

$$\lim_{x \rightarrow \infty} \frac{x^2 - 2}{(x-1)^2} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{2x}{2(x-1)} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{2}{2} = 1$$

Similarly,

$$\lim_{x \rightarrow -\infty} \frac{x^2 - 2}{(x-1)^2} = 1$$

The horizontal asymptote occurs only at  $y = 0$ .

Take the first derivative by applying the quotient rule.

$$y' = \frac{(2x) \cdot (x-1)^2 - (x^2 - 2) \cdot 2(x-1)}{(x-1)^4} = \frac{4 - 2x}{(x-1)^3}$$

$y'$  is undefined for  $x = 1$ , and  $y' = 0$  for  $x = 2$ . Since 1 is not in the domain, the *only* critical point is  $x = 2$ .

Take the second derivative.

$$y'' = \frac{(-2) \cdot (x-1)^3 - (4-2x) \cdot 3(x-1)^2}{(x-1)^6} = \frac{4x-10}{(x-1)^4}$$

The only inflection point occurs at  $x = \frac{5}{2}$ .

(b) Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(-\sqrt{2}) = f(\sqrt{2}) = 0, f(0) = -2, f(2) = 2, f(5/2) = 17/9$$

$x$	$(-\infty, -\sqrt{2}]$	$(-\sqrt{2}, 0]$	$[0, 1)$	$(1, \sqrt{2}]$	$[\sqrt{2}, 2]$	$[2, \frac{5}{2}]$	$[\frac{5}{2}, \infty)$
$y$	$(1, 0]$	$[-2, 0)$	$(-\infty, -2]$	$(-\infty, 0]$	$[0, 2]$	$[2, \frac{17}{9}]$	$[\frac{17}{9}, 1]$
$y'$ sign	-	-	-	+	+	-	-
$y''$ sign	-	-	-	-	-	-	+

(c)

