2019-2020 Fall
MAT123 Midterm
(04/11/2019)

1) Evaluate

$$\lim_{x\to 3^+} \cos(x-3)^{\ln\left(\frac{2x}{3}-2\right)}$$

2) Find constants a and b such that $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{\tan ax}{\tan bx}, & \text{if } x < 0 \\ 4, & \text{if } x = 0 \\ ax + b, & \text{if } x > 0 \end{cases}$$

will be continuous ~~for all $x \in \mathbb{R}$~~ at the point $x = 0$.

3) Use the Intermediate Value Theorem to show that the equation $1 - 2x = \sin x$ has at least one real solution. Then use Rolle's Theorem to show it has no more than one solution.

4) Ship A is 60 miles north of point O and moving in the north direction at 20 miles per hour. Ship B is 80 miles east of point O and moving west at 25 miles per hour. How fast is the distance between the ships changing at this moment?

5) Sketch the graph of

$$f(x) = \frac{e^x}{x}$$

6) Evaluate the following integrals.
(a) $\displaystyle\int x^2\sqrt{9+x^2}\,dx$

(b) $\displaystyle\int \tan x \cdot \sec^6 x\,dx$

(c) $\displaystyle\int_4^8 \frac{1}{(x-4)^3}\,dx$

(d) $\displaystyle\int e^{2x}\sin e^x\,dx$

(e) $\displaystyle\int \frac{dx}{\sin x - \cos x}$

7) Let us consider the area A of the region bounded by the curves $x = e^y$, $x = y^2 - 2$ and the straight lines $y = 1$, $y = -1$. Write an integral (but don't evaluate) corresponding to the area A

(i) with respect to the y and
(ii) with respect to the x.

Solutions (Last update: 7/15/25 (15th of July) 2:36 AM)

1) Let L be the value of the limit.

$$L = \lim_{x \to 3^+} \cos(x-3)^{\ln\left(\frac{2x}{3}-2\right)}$$
$$\ln(L) = \ln\left(\lim_{x \to 3^+} \cos(x-3)^{\ln\left(\frac{2x}{3}-2\right)}\right)$$

Since $\cos(x-3)^{\ln(2x/3-2)}$ is continuous for $x > 3$, we can take the logarithm function inside the limit. Using the property of logarithms, we get:

$$\ln(L) = \lim_{x \to 3^+} \left[\ln\left(\cos(x-3)^{\ln\left(\frac{2x}{3}-2\right)}\right)\right] = \lim_{x \to 3^+} \left[\ln\left(\cos(x-3)\right)\ln\left(\frac{2x}{3}-2\right)\right] \quad [0 \cdot \infty]$$

We can rearrange the limit to obtain an indeterminate form. Afterwards, we may apply L'Hôpital's rule.

$$\lim_{x \to 3^+} \left[\ln\left(\cos(x-3)\right)\ln\left(\frac{2x}{3}-2\right)\right] = \lim_{x \to 3^+} \frac{\ln\left(\frac{2x}{3}-2\right)}{1/\ln\left(\cos(x-3)\right)} \quad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x \to 3^+} \frac{\dfrac{1}{\frac{2x}{3}-2} \cdot \dfrac{2}{3}}{\left(-\ln^{-2}\cos(x-3)\right) \cdot \dfrac{1}{\cos(x-3)} \cdot \left(-\sin(x-3)\right)}$$

$$= \lim_{x \to 3^+} \frac{\ln^2(\cos(x-3))}{(x-3)\tan(x-3)} \quad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x \to 3^+} \frac{2\ln(\cos(x-3)) \cdot \dfrac{1}{\cos(x-3)} \cdot (-\sin(x-3))}{\tan(x-3) + (x-3)\sec^2(x-3)}$$

$$= \lim_{x \to 3^+} \frac{-2\ln(\cos(x-3))\sin(x-3)}{\sin(x-3) + (x-3)\sec(x-3)} \overset{u=x-3}{=} \lim_{u \to 0^+} \left(\frac{-2\ln(\cos(u))\sin(u)}{\sin(u) + u\sec(u)} \cdot \frac{u}{u}\right)$$

$$= \frac{\lim\limits_{u \to 0^+}\left[-2\ln(\cos(u))\right] \cdot \lim\limits_{u \to 0^+} \dfrac{\sin(u)}{u}}{\lim\limits_{u \to 0^+} \sec(u) + \lim\limits_{u \to 0^+} \dfrac{\sin(u)}{u}} \quad \left[\lim_{u \to 0} \frac{\sin(u)}{u} = 1\right]$$

$$= \frac{\lim\limits_{u \to 0^+}\left[-2\ln(\cos(u))\right]}{\lim\limits_{u \to 0^+} \sec(u) + 1} = \frac{-2\ln(\cos(0))}{1+1} = \frac{-2\ln(1)}{2} = 0$$

We found out that $\ln(L) = 0$. Therefore, $\boxed{L = 1}$.

2) **Remark**: There is misinformation about the continuity in the original question, and I replaced it with continuity at $x = 0$.

To ensure continuity at $x = 0$, the one-sided limit values must be equal to the value of the function at that point.

$$\lim_{x \to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x \to 0^+} (ax + b) = f(0) = 4$$

The easy part is that we can calculate the limit from the right.

$$\lim_{x \to 0^+} (ax + b) = 0 + b = b$$

Hence, $b = 4$. To calculate from the left, we need another technique.

$$\lim_{x \to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x \to 0^-} \left(\frac{\sin ax}{\cos ax} \cdot \frac{\cos bx}{\sin bx} \cdot \frac{bx}{bx} \cdot \frac{ax}{ax} \right)$$

$$= \lim_{x \to 0^-} \left(\frac{\sin ax}{ax} \right) \cdot \lim_{x \to 0^-} \left(\frac{1}{\frac{\sin bx}{bx}} \right) \cdot \lim_{x \to 0^-} \left(\frac{\cos(bx) \cdot ax}{\cos(ax) \cdot bx} \right)$$

$$= 1 \cdot \frac{1}{\lim_{x \to 0^-} \frac{1}{\frac{\sin bx}{bx}}} \cdot \lim_{x \to 0^-} \left(\frac{\cos(bx) \cdot a}{\cos(ax) \cdot b} \right) = 1 \cdot 1 \cdot \left(\frac{\cos(0) \cdot a}{\cos(0) \cdot b} \right)$$

$$= \frac{a}{b}$$

Now, set $\frac{a}{b} = b \to a = 16$. $\boxed{a = 16,\ b = 4}$

3) $-1 \le \sin x \le 1$, and $\sin x$ is continuous $\forall x \in \mathbb{R}$. $1 - 2x$ is continuous everywhere and takes any value in $\mathbb{R}$. Therefore, the equation $\sin x = 1 - 2x$ must have at least one real solution by IVT, and the y-intercept is on the interval $[-1, 1]$.

Let $f(x) = \sin x - 1 + 2x$ and x_1 be one solution to the equation. Then, the root must satisfy $|1 - 2x| \le 1$. To disprove the existence of another root, we assume that x_2 is another distinct root. Since $f(x_1) = f(x_2) = 0$ and $f(x)$ is differentiable everywhere, by Rolle's theorem, there must exist a point c such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

Take the first derivative and calculate $f'(c)$

$$f'(c) = \cos c + 2$$

$-1 \le \cos x \le 1$. Therefore, there is no such c that satisfies $f'(c) = 0$. This is a contradiction. By Rolle's theorem, there is *only* one root satisfying $\sin x = 1 - 2x$.

4) Let $f(t)$ and $g(t)$ represent the distance between Ship A and point O, and the distance between Ship B and point O, respectively. The distance between the ships can be represented using the Pythagorean theorem as follows:

$$D^2(t) = f^2(t) + g^2(t)$$

Take the derivative of both sides.

$$2D\frac{dD}{dt} = 2f(t)f'(t) + 2g(t)g'(t)$$

Solve for $\dfrac{dD}{dt}$.

$$\frac{dD}{dt} = \frac{f(t)f'(t) + g(t)g'(t)}{D}$$

For $t = t_0$, we have $f(t_0) = 60$, $g(t_0) = 80$, $f'(t_0) = 20$, $g'(t_0) = -25$, $D(t_0) = \sqrt{60^2 + 80^2} = 100$. We may now find the rate of change of the distance at that time.

$$\left.\frac{dD}{dt}\right|_{t=t_0} = \frac{60 \cdot 20 - 80 \cdot 25}{100} = \boxed{-8\,\text{miles/hour}}$$

5) First off, find the domain. The expression is undefined when the denominator is zero. Therefore, $x \neq 0$. The only vertical asymptote occurs at $x = 0$.

$$\mathcal{D} = \mathbb{R} - \{0\}$$

Let us find the limit at infinity and the limit at negative infinity.

$$\lim_{x\to\infty} \frac{\mathrm{e}^x}{x} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{\mathrm{e}^x}{1} = \infty$$

$$\lim_{x\to-\infty} \frac{\mathrm{e}^x}{x} = 0$$

The horizontal asymptote occurs only at $y = 0$.

Take the first derivative by applying the quotient rule.

$$y' = \frac{\mathrm{e}^x \cdot x - \mathrm{e}^x \cdot 1}{x^2} = \frac{\mathrm{e}^x(x-1)}{x^2}$$

y' is undefined for $x = 0$, and $y' = 0$ for $x = 1$. Since 0 is not in the domain, the *only* critical point is $x = 1$.

Take the second derivative.

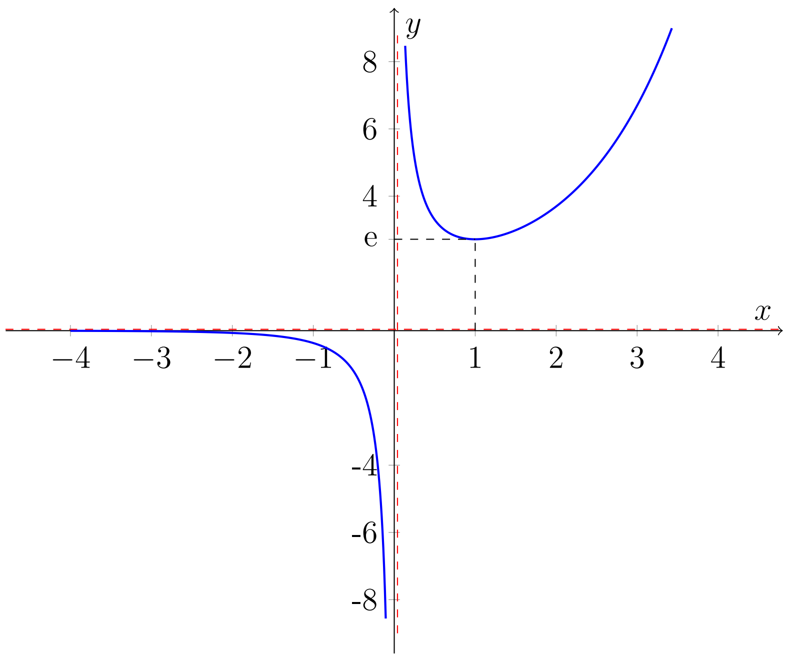
$$y'' = \frac{[\mathrm{e}^x(x-1) + \mathrm{e}^x]x^2 - \mathrm{e}^x(x-1) \cdot 2x}{x^4} = \frac{\mathrm{e}^x(x^2 - 2x + 2)}{x^3}$$

There is no inflection point.

Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(1) = \frac{\mathrm{e}^1}{1} = \mathrm{e}$$

x	$(-\infty, 0)$	$(0, 1]$	$[1, \infty)$
y	$(-\infty, 0)$	$(\infty, \mathrm{e}]$	$[e, \infty)$
y' sign	-	-	+
y'' sign	-	+	+



6)

(a) Let $x = 3\tan u$, then $dx = 3\sec^2 u\, du$.

$$\mathrm{I} = \int x^2\sqrt{9+x^2}\,dx = \int (3\tan u)^2 \cdot \sqrt{9+(3\tan u)^2} \cdot 3\sec^2(u)\,du \quad [1+\tan^2 u = \sec^2 u]$$

$$= 81\int \tan^2 u \cdot \sec^3 u\,du = 81\int \frac{\sin^2 u}{\cos^5 u}\,du = 81\int \frac{1-\cos^2 u}{\cos^5 u}\,du$$

$$= 81\int \sec^5 u\,du - 81\int \sec^3 u\,du \tag{1}$$

Find the left-hand integral in (1) with integration by parts.

$$w = \sec^3 u \;\rightarrow\; dw = 3\sec^3 u \tan u\,du$$
$$dz = \sec^2 u\,du \;\rightarrow\; z = \tan u$$

$$\int \sec^5 u\,du = \tan u \cdot \sec^3 u - 3\int \tan^2 u \cdot \sec^3 u\,du = \tan u \cdot \sec^3 u - 3\int (\sec^5 u - \sec^3 u)\,du$$

The integral we want to evaluate appears on the right side. After a little algebra, we get:

$$\int \sec^5 u\,du = \frac{1}{4} \cdot \tan u \cdot \sec^3 u + \frac{3}{4}\int \sec^3 u\,du$$

Rewrite (1) and calculate the other integral in (1) with integration by parts.

$$\mathrm{I} = \frac{81}{4} \cdot \tan u \cdot \sec^3 u - \frac{81}{4}\int \sec^3 u\,du \tag{2}$$

$$\begin{aligned} w = \sec u \;&\rightarrow\; dw = \sec u \tan u\,du \\ dz = \sec^2 u\,du \;&\rightarrow\; z = \tan u \end{aligned}$$

$$\int \sec^3 u\,du = \tan u \cdot \sec u - \int \tan^2 u \sec u\,du = \tan u \cdot \sec u - \int \frac{1-\cos^2 u}{\cos^3 u}\,du$$

$$= \tan u \cdot \sec u - \int \sec^3 u\,du + \int \sec u\,du$$

We encountered a similar case when calculating $\int \sec^5 u\,du$. So,

$$\int \sec^3 u\,du = \frac{1}{2} \cdot \tan u \cdot \sec u + \frac{1}{2} \cdot \int \sec u\,du$$

The integral of $\sec u$ with respect to u is as follows. One can derive it with particular methods.

$$\int \sec u\,du = \ln|\tan u + \sec u| + c_1,\ c_1 \in \mathbb{R} \tag{3}$$

Rewrite (2) using (3).

$$\mathrm{I} = \frac{81}{4} \cdot \tan u \cdot \sec^3 u - \frac{81}{8} \cdot \tan u \cdot \sec u - \frac{81}{8} \cdot \ln|\tan u + \sec u| + c$$

Recall that $x = 3\tan u$, then $x^2 = 9\tan^2 u = 9\sec^2 u - 9 \rightarrow \sec u = \dfrac{\sqrt{x^2+9}}{3}$. The result is then

$$\boxed{\mathrm{I} = \frac{x\sqrt{x^2+9}}{8}\left(2x^2+9\right) - \frac{81}{8}\ln\left|x + \sqrt{x^2+9}\right| + c,\ c \in \mathbb{R}}$$

(b) Rewrite the expression. Then, let $u = \tan^2 x + 1$. So, $du = 2\tan x \sec^2 x\,dx$

$$\mathrm{I} = \int \tan x \cdot \sec^6 x \, dx \quad \left[\tan^2 x + 1 = \sec^2 x\right]$$

$$= \int \tan x \cdot \sec^2 x \cdot (1 + \tan^2 x)^2 \, dx = \frac{1}{2} \int u^2 \, du = \frac{u^3}{6} + c$$

$$\boxed{\mathrm{I} = \frac{(\tan^2 x + 1)^3}{6} + c, \; c \in \mathbb{R}}$$

(c) This is an improper integral; we need to make use of the limit concept. The expression is undefined for $x = 4$.

$$\mathrm{I} = \int_4^8 \frac{1}{(x-4)^3} \, dx = \lim_{\mathrm{R} \to 4^+} \int_{\mathrm{R}}^8 \frac{1}{(x-4)^3} \, dx = \lim_{\mathrm{R} \to 4^+} \left[-\frac{1}{2(x-4)^2} \right]_{\mathrm{R}}^8$$

$$= \lim_{\mathrm{R} \to 4^+} \left[-\frac{1}{32} + \frac{1}{(R-4)^2} \right] = \boxed{\infty}$$

(d) We'll use integration by parts.

$$u = e^x \;\rightarrow\; du = \mathrm{e}^x \, dx$$
$$dv = \mathrm{e}^x \sin \mathrm{e}^x \, dx \;\rightarrow\; v = -\cos \mathrm{e}^x$$

$$\int \mathrm{e}^{2x} \sin \mathrm{e}^x \, dx = (-\cos \mathrm{e}^x) \cdot \mathrm{e}^x - \int \mathrm{e}^x \cdot (-\cos \mathrm{e}^x) \, dx$$

$$= \boxed{-e^x \cos e^x + \sin e^x + c, \; c \in \mathbb{R}}$$

(e) We may utilize the tangent half-angle substitution, which is sometimes called the Weierstrass substitution. Let $t = \tan\left(\dfrac{x}{2}\right)$. After some mathematical operations, we get the following. One can later derive the formulas.

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2} \, dt$$

Rewrite the integral.

$$\mathrm{I} = \int \frac{dx}{\sin x - \cos x} = \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} \, dt = \int \frac{2}{t^2 + 2t - 1} \, dt = 2 \int \frac{1}{(t+1)^2 - (\sqrt{2})^2} \, dt$$

$$= 2 \int \left(\frac{A}{t + 1 + \sqrt{2}} + \frac{B}{t + 1 - \sqrt{2}} \right) dt \quad \text{[partial fraction decomposition]} \tag{4}$$

$$
\begin{aligned}
A(t+1+\sqrt{2}) + B(t+1-\sqrt{2}) &= 1 \\
t(A+B) + A + B + \sqrt{2}(A-B) &= 1 \\
\therefore A+B = 0 \quad [\text{eliminate } t] \;\rightarrow\; A-B &= \frac{1}{\sqrt{2}}
\end{aligned}
$$

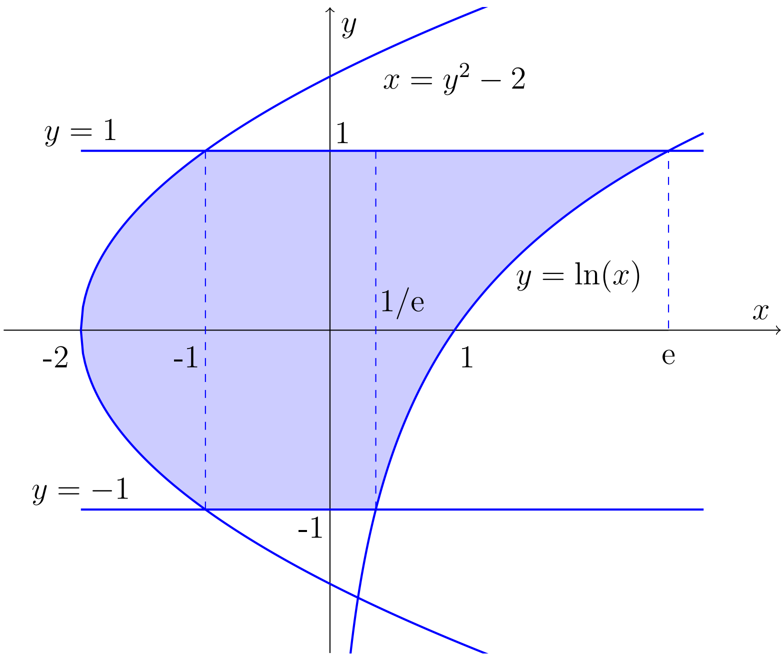
$$
\left.\begin{aligned} A+B &= 0 \\ A-B &= \frac{1}{\sqrt{2}} \end{aligned}\right\} \quad A = \frac{1}{2\sqrt{2}}, \quad B = -\frac{1}{2\sqrt{2}}
$$

Plug the values of A and B into (4).

$$
\mathrm{I} = \frac{\sqrt{2}}{2} \int \left(\frac{1}{t+1+\sqrt{2}} - \frac{1}{t+1-\sqrt{2}} \right) dx = \frac{\sqrt{2}}{2} \ln \left(\frac{|t+1+\sqrt{2}|}{|t+1-\sqrt{2}|} \right) + c,\ c \in \mathbb{R}
$$

$$
\boxed{\mathrm{I} = \frac{\sqrt{2}}{2} \ln \left(\frac{\left|\tan\left(\frac{x}{2}\right) + 1 + \sqrt{2}\right|}{\left|\tan\left(\frac{x}{2}\right) + 1 - \sqrt{2}\right|} \right) + c,\ c \in \mathbb{R}}
$$

7)



(i) The variable is y. Hence, the limits are $-1, 1$, respectively.

$$
A = \int_{-1}^{1} \left[e^y - (y^2 - 2) \right] dy
$$

(ii) We have three different regions. This leads us to take three different integrals.

$$
A = \int_{-2}^{-1} \left[\sqrt{x+2} - (-\sqrt{x+2}) \right] dx + \int_{-1}^{1/e} [1 - (-1)]\, dx + \int_{1/e}^{e} (1 - \ln x)\, dx
$$