

2023-2024 Spring
MAT124 Final
(04/06/2024)

1. Using Lagrange multipliers, find the closest point of the plane $x + z + 1 = 0$ to the point $(1, 2, 0)$.

2. Sketch the region and reverse the order of the double integral

$$\int_0^1 \int_0^{x^2/2} dy \, dx + \int_1^{\sqrt{2}} \int_{x^2-1}^{x^2/2} dy \, dx$$

3. Sketch the region and use a double integral in polar coordinates to find the area inside the cardioid $r = 1 - \cos \theta$ outside the circle $r = 1$.

4. Sketch the region and use a double integral to find the volume of the solid bounded above by the plane $x = z$ and below in the xy -plane by the part of the disk $x^2 + y^2 \leq 4$ in the fourth quadrant.

5. Find the surface area of the portion of the paraboloid $z = 25 - x^2 - y^2$ that lies above the xy -plane.

6. Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral

$$\iint_{\mathcal{R}} (x - y) \sin(x^2 - y^2) \, dy \, dx$$

where \mathcal{R} is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$.

7. Let R be the solid region bounded by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 9$. Let

$$I = \iiint_R (x^2 + y^2) \, dV$$

(i) Express (but do not evaluate) I as a triple integral in spherical coordinates.

(ii) Express (but do not evaluate) I as a triple integral in cylindrical coordinates.

2023-2024 Spring Final (04/06/2024) Solutions
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1. Let $g(x, y, z) = x + z + 1$ and $f(x, y, z) = D^2 = (x - 1)^2 + (y - 2)^2 + (z - 0)^2$. It is easier to work with the square of the distance. Finding the points will not change the result. Solve the system of equations below.

$$\left. \begin{array}{l} \nabla f = \lambda \nabla g \\ g(x, y, z) = 0 \end{array} \right\} \quad \begin{array}{l} \nabla f = \langle 2(x-1), 2(y-2), 2z \rangle = \lambda \langle 1, 0, 1 \rangle = \lambda \nabla g \\ \therefore y = 2, \quad \lambda = 2x - 2, \quad \lambda = 2z \end{array}$$

$$2x - 2 = 2z \implies z = x - 1 \quad (1)$$

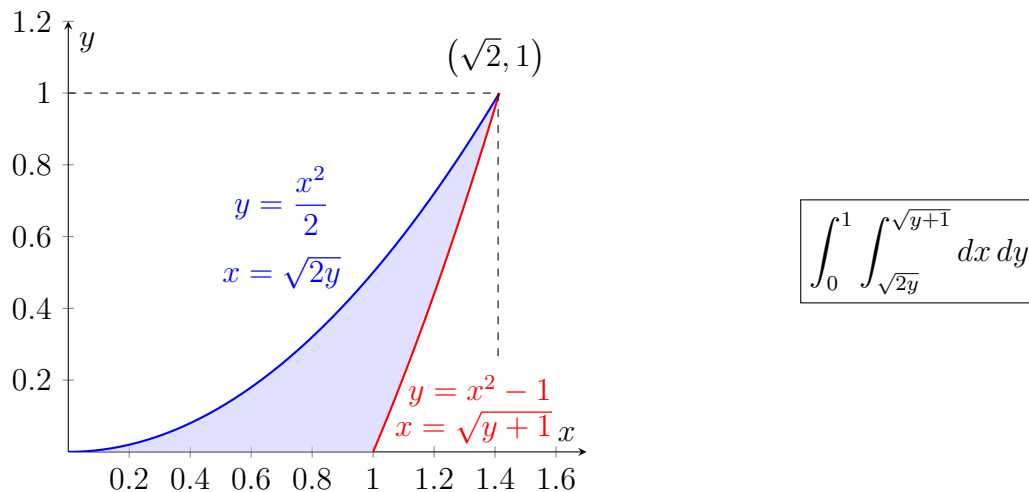
Substitute (1) into the constraint.

$$x + z + 1 = 0 \implies x + x - 1 + 1 = 0 \implies 2x = 0 \implies x = 0$$

$$\therefore 0 + z + 1 = 0 \implies z = -1$$

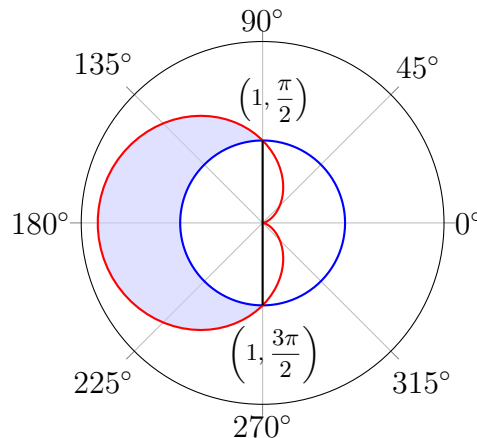
The closest point is then $\boxed{(0, 2, -1)}$.

2.



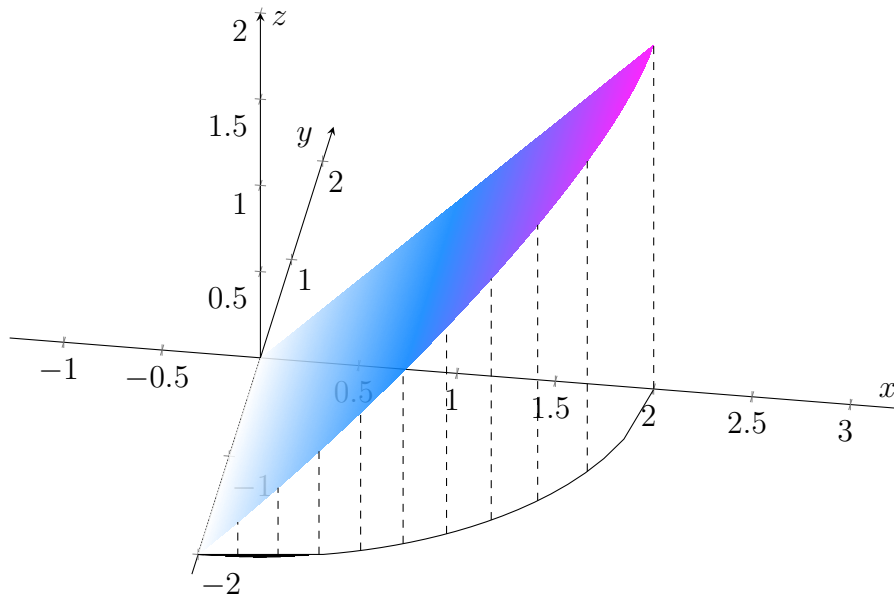
3. Find where these two curves intersect and then find the area.

$$1 = 1 - \cos \theta \implies \cos \theta = 0 \implies \theta = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}$$



$$\begin{aligned}
\text{Area} &= \int_{\pi/2}^{3\pi/2} \int_1^{1-\cos\theta} r \, dr \, d\theta = \int_{\pi/2}^{3\pi/2} \left[\frac{1}{2} r^2 \right]_{r=1}^{r=1-\cos\theta} d\theta = \frac{1}{2} \int_{\pi/2}^{3\pi/2} [(1-\cos\theta)^2 - 1^2] \, d\theta \\
&= \frac{1}{2} \int_{\pi/2}^{3\pi/2} (-2\cos\theta + \cos^2\theta) \, d\theta = \frac{1}{2} \int_{\pi/2}^{3\pi/2} \left(-2\cos\theta + \frac{\cos(2\theta) + 1}{2} \right) d\theta \\
&= \frac{1}{2} \left[-2\sin\theta + \frac{\sin(2\theta)}{4} + \frac{\theta}{2} \right]_{\pi/2}^{3\pi/2} = \frac{1}{2} \left[\left(2 + 0 + \frac{3\pi}{4} \right) - \left(-2 + 0 + \frac{\pi}{4} \right) \right] = \boxed{2 + \frac{\pi}{4}}
\end{aligned}$$

4.



$$\begin{aligned}
&\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ x^2 + y^2 &= r^2 \\ dA &= r \, dr \, d\theta \end{aligned} \quad \rightarrow \quad \begin{aligned} z = x &\implies z_{\text{upper}} = r \cos \theta \\ z = 0 &\implies z_{\text{lower}} = 0 \\ 0 &\leq \theta \leq \frac{\pi}{2} \end{aligned}
\end{aligned}$$

The volume of this solid can be evaluated with the following integral.

$$\begin{aligned}
\text{Volume} &= \int_0^{\pi/2} \int_0^2 (r \cos \theta - 0) \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 r^2 \cos \theta \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{r^3}{3} \right]_{r=0}^{r=2} \cos \theta \, d\theta \\
&= \frac{8}{3} \int_0^{\pi/2} \cos \theta \, d\theta = \frac{8}{3} \sin \theta \Big|_0^{\pi/2} = \frac{8}{3} (1 - 0) = \boxed{\frac{8}{3}}
\end{aligned}$$

5. For $z = 0$, we get the circle $x^2 + y^2 = 25$. Therefore, the domain is $x^2 + y^2 \leq 25$. Using the double integral below, we find the surface area.

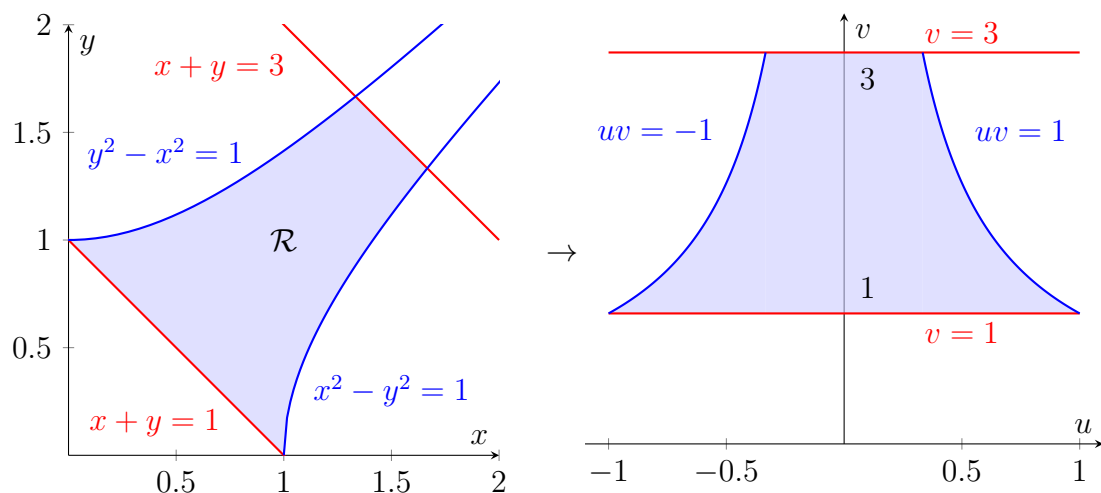
$$\begin{aligned}\text{Surface area} &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA \\ &= \int_{-5}^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{1 + (-2x)^2 + (-2y)^2} dy dx \\ &= \int_{-5}^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{1 + 4x^2 + 4y^2} dy dx\end{aligned}$$

If we switch to polar coordinates, we can easily evaluate the integral.

$$\begin{aligned}\text{Surface area} &= \int_0^{2\pi} \int_0^5 \sqrt{1 + 4r^2} r dr d\theta = \int_0^{2\pi} \left[\frac{1}{12} (1 + 4r^2)^{3/2} \right]_{r=0}^{r=5} d\theta \\ &= \frac{1}{12} \int_0^{2\pi} \left[(1 + 4 \cdot 25)^{3/2} - (1 + 4 \cdot 0)^{3/2} \right] d\theta = \frac{101^{3/2} - 1}{12} \int_0^{2\pi} d\theta \\ &= \boxed{\frac{101^{3/2} - 1}{6} \cdot \pi}\end{aligned}$$

6. Rewrite x and y in terms of u and v and sketch the regions in both coordinates.

$$\begin{aligned}\left. \begin{aligned} u &= x - y \\ v &= x + y \end{aligned} \right\} & x = \frac{u + v}{2}, \quad y = \frac{v - u}{2} \\ x + y = 3 &\implies \frac{u + v}{2} + \frac{v - u}{2} = 3 \implies v = 3 \\ x + y = 1 &\implies \frac{u + v}{2} + \frac{v - u}{2} = 1 \implies v = 1 \\ x^2 - y^2 = 1 &\implies \left(\frac{u + v}{2}\right)^2 - \left(\frac{v - u}{2}\right)^2 = \frac{1}{4} (u^2 + 2uv + v^2 - v^2 + 2uv - u^2) = 1 \\ &\implies u = \frac{1}{v} \\ y^2 - x^2 = 1 &\implies \left(\frac{v - u}{2}\right)^2 - \left(\frac{u + v}{2}\right)^2 = \frac{1}{4} (v^2 - 2uv + u^2 - u^2 - 2uv - v^2) = 1 \\ &\implies u = -\frac{1}{v}\end{aligned}$$



Calculate the Jacobian determinant to find the area in terms of u and v .

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \cdot \frac{1}{2} - \left(-\frac{1}{2} \cdot \frac{1}{2} \right) = \frac{1}{2}$$

The integral then becomes

$$\begin{aligned} I &= \iint_{\mathcal{R}} (x - y) \sin(x^2 - y^2) \, dy \, dx = \int_1^3 \int_{-1/v}^{1/v} u \cdot \sin(uv) \cdot \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv \\ &= \int_1^3 \int_{-1/v}^{1/v} u \cdot \sin(uv) \cdot \frac{1}{2} \, du \, dv \end{aligned}$$

Multiply each side by 2 to obviate the mess with the fraction $\frac{1}{2}$ and then change the order of integration.

$$\begin{aligned} 2I &= \int_{-1}^{-\frac{1}{3}} \int_1^{-\frac{1}{u}} u \sin(uv) \, dv \, du + \int_{-\frac{1}{3}}^{\frac{1}{3}} \int_1^3 u \sin(uv) \, dv \, du + \int_{\frac{1}{3}}^1 \int_1^{\frac{1}{u}} u \sin(uv) \, dv \, du \\ &= \int_{-1}^{-\frac{1}{3}} (-\cos(uv)) \Big|_{v=1}^{v=-\frac{1}{u}} \, du + \int_{-\frac{1}{3}}^{\frac{1}{3}} (-\cos(uv)) \Big|_{v=1}^{v=3} \, du + \int_{\frac{1}{3}}^1 (-\cos(uv)) \Big|_{v=1}^{v=\frac{1}{u}} \, du \\ &= \int_{-1}^{-\frac{1}{3}} [-\cos(-1) + \cos u] \, du + \int_{-\frac{1}{3}}^{\frac{1}{3}} [-\cos(3u) + \cos u] \, du + \int_{\frac{1}{3}}^1 [-\cos 1 + \cos u] \, du \\ &= \left[-u \cos(-1) + \sin u \right]_{-1}^{-\frac{1}{3}} + \left[-\frac{1}{3} \sin(3u) + \sin u \right]_{-\frac{1}{3}}^{\frac{1}{3}} + \left[-u \cos 1 + \sin u \right]_{\frac{1}{3}}^1 \end{aligned}$$

$$\begin{aligned}
2I &= \left[-\frac{2}{3} \cos(-1) + \sin\left(-\frac{1}{3}\right) - \sin(-1) \right] + \left[-\frac{1}{3} (\sin 1 - \sin(-1)) + \sin \frac{1}{3} - \sin\left(-\frac{1}{3}\right) \right] \\
&\quad + \left[-\frac{2}{3} \cos 1 + \sin 1 - \sin \frac{1}{3} \right] \\
&= \frac{4}{3} (\sin 1 - \cos 1)
\end{aligned}$$

This is the value of $2I$. Therefore, $I = \frac{2}{3} (\sin 1 - \cos 1)$.

7.

(i) For spherical coordinates, we have

$$\begin{aligned}
\begin{aligned} z &= \rho \cos \phi \\ r &= \rho \sin \phi \\ x^2 + y^2 &= r^2 \\ x^2 + y^2 + z^2 &= \rho^2 \\ dV &= \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{aligned} &\rightarrow &\begin{aligned} z &= \sqrt{3x^2 + 3y^2} \implies \rho \cos \phi = \sqrt{3} \rho \sin \phi \implies \phi = \frac{\pi}{6} \\ x^2 + y^2 &= r^2 = \rho^2 \sin^2 \phi \\ x^2 + y^2 + z^2 &= 9 \implies \rho^2 = 9 \implies \rho = 3 \\ 0 &\leq \theta \leq 2\pi \end{aligned}
\end{aligned}$$

The integral in spherical coordinates can be expressed as follows.

$$I = \int_0^{2\pi} \int_0^{\pi/6} \int_0^3 \rho^2 \sin^2 \phi \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/6} \int_0^3 \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta$$

(ii) For cylindrical coordinates, we have

$$\begin{aligned}
\begin{aligned} z &= z \\ r^2 &= x^2 + y^2 \\ dV &= r \, dz \, dr \, d\theta \end{aligned} &\rightarrow &\begin{aligned} z &= \sqrt{3x^2 + 3y^2} \implies z = r\sqrt{3} \\ x^2 + y^2 &= r^2 \\ x^2 + y^2 + z^2 &= 9 \implies z = \sqrt{9 - r^2} \\ 0 &\leq \theta \leq 2\pi \end{aligned}
\end{aligned}$$

Find where the curves intersect to find the upper limit of r .

$$r\sqrt{3} = \sqrt{9 - r^2} \implies 3r^2 = 9 - r^2 \implies r^2 = \frac{9}{4} \implies r = \frac{3}{2}$$

The integral in cylindrical coordinates can be expressed as follows.

$$I = \int_0^{2\pi} \int_0^{3/2} \int_{r\sqrt{3}}^{\sqrt{9-r^2}} r^2 \cdot r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{3/2} \int_{r\sqrt{3}}^{\sqrt{9-r^2}} r^3 \, dz \, dr \, d\theta$$