2022-2023 Fall
MAT123-02,05 Final
(11/01/2023)

1. The radius R of a spherical ball is measured as 14 in.

(a) Use differentials to estimate the maximum propagated error in computing the volume V if R is measured with a maximum error of 1/8 inches.

(b) With what accuracy must the radius R be measured to guarantee and error of at most 2 in^3 in the calculated volume?

2. Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{\sqrt{x}\,(\sqrt{x}+2)}$

(b) $\cancel{\displaystyle\int \frac{\sin x}{x^2+1}\,dx}$

(c) $\displaystyle\int \frac{dx}{\sqrt{3-x^2}}$

(d) $\displaystyle\int \frac{dx}{2+\cos x}$

3. Use the Shell Method and then the Washer Method to set up an integral (but do not evaluate) the volume of the solid generated by revolving the region R about the y-axis, where R is bounded by the curve $y = x^2$ and the line $y = -x + 1$.

4. Find the area of the surface obtained by rotating the arc of the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $[1/2, 1]$ about the x-axis.

5. Using the Integral Test, determine whether the series

$$\sum_{n=1}^{\infty} \frac{2}{3n+5}$$

converges or diverges.

6. Find the Maclaurin series for $f(x) = \dfrac{1}{x^2 - 5x + 6}$.

2022-2023 Final (11/01/2023) Solutions
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1.

(a) The volume of a sphere with radius r is

$$V = \frac{4}{3}\pi r^3$$

The differential of V is

$$dV = 4\pi r^2\, dr$$

The maximum error is known to be 1/8 inches. So, $|dr| \leq 1/8$. The maximum propagated error is then

$$dV = 4\pi \cdot 14^2 \cdot \frac{1}{8} = \boxed{98\pi \text{ in}^3.}$$

(b) $|dV| = 2$ at most. Solve the differential form for dr.

$$dr = \frac{dV}{4\pi r^2} = \frac{2}{4\pi \cdot 14^2} = \boxed{\frac{1}{392\pi} \text{ inches}}$$

2.

(a) Let $x = u^2$, then $u = \sqrt{x}$ and $dx = 2u\, du$.

$$\int \frac{dx}{\sqrt{x}\,(\sqrt{x}+2)} = \int \frac{2u}{u(u+2)}\, du = 2\int \frac{du}{u+2} = 2\ln|u+2| + c$$

$$= \boxed{2\ln\left|\sqrt{x}+2\right| + c,\ c \in \mathbb{R}}$$

(b) This question is beyond the scope of the curriculum, and students are not expected to solve it using the knowledge they have acquired in this course.

(c) Let $x = \sqrt{3}\sin u$ for $-\frac{\pi}{2} < u < \frac{\pi}{2}$, then $dx = \sqrt{3}\cos u\, du$.

$$\int \frac{dx}{\sqrt{3-x^2}} = \int \frac{\sqrt{3}\cos u}{\sqrt{3-3\sin^2 u}}\, du = \int \frac{\cos u}{\sqrt{\cos^2 u}}\, du = \int \frac{\cos u}{|\cos u|}\, du$$

$$= \int \frac{\cos u}{\cos u}\, du \quad [\cos u > 0]$$

$$= \int du = u + c$$

If $x = \sqrt{3}\sin u$, then $\sin u = \dfrac{x}{\sqrt{3}} \implies u = \arcsin\left(\dfrac{x}{\sqrt{3}}\right)$. The answer is then

$$\boxed{\arcsin\left(\frac{x}{\sqrt{3}}\right) + c, \quad c \in \mathbb{R}}$$

(d) We may utilize the tangent half-angle substitution, which is sometimes called the Weierstrass substitution. Let $t = \tan\left(\dfrac{x}{2}\right)$. Then

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2}\,dt$$

Rewrite the integral.

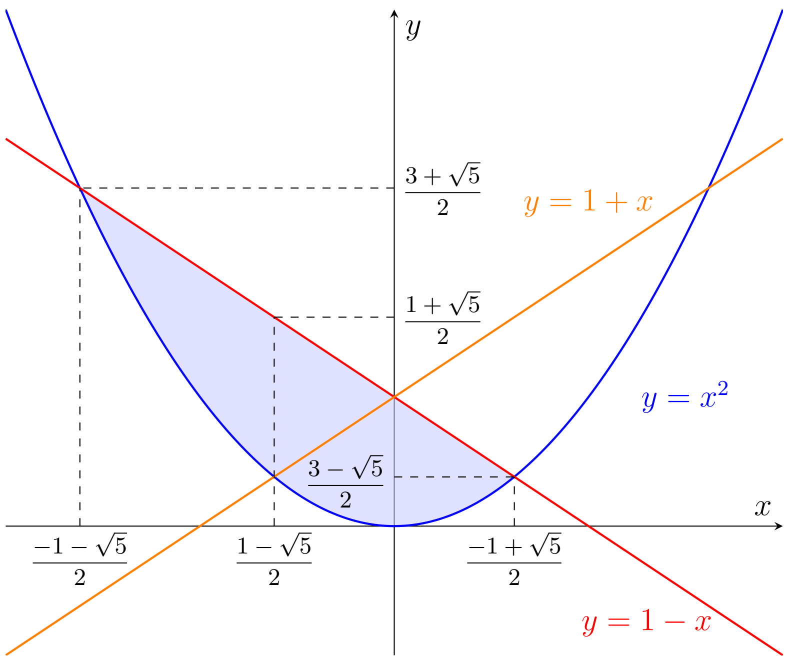
$$\int \frac{dx}{2+\cos x} = \int \frac{\frac{2}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}\,dt = \int \frac{2}{3+t^2}\,dt = \int \frac{2}{3\left(1+\frac{t^2}{3}\right)}\,dt = \frac{2}{3}\int \frac{1}{1+\left(\frac{t}{\sqrt{3}}\right)^2}\,dt$$

Let $u = \dfrac{t}{\sqrt{3}}$, then $\sqrt{3}\,du = dt$.

$$\frac{2}{3}\int \frac{dt}{1+\left(\frac{t}{\sqrt{3}}\right)^2} = \frac{2\sqrt{3}}{3}\int \frac{du}{1+u^2} = \frac{2\sqrt{3}}{3}\arctan u + c = \frac{2\sqrt{3}}{3}\arctan\frac{t}{\sqrt{3}} + c$$

$$= \boxed{\frac{2\sqrt{3}}{3}\arctan\left(\frac{1}{\sqrt{3}}\tan\left(\frac{x}{2}\right)\right) + c, \quad c \in \mathbb{R}}$$

3.



We have the symmetry of the region that is bounded to the right of the y-axis. Therefore, it is not necessary to apply the method to the symmetrical region on the left. The volume of this solid is

$$\boxed{\begin{aligned}&\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{5}}{2}} 2\pi(-x)\left[(1-x)-\left(x^2\right)\right]\,dx\\ &+\int_{\frac{1-\sqrt{5}}{2}}^{0} 2\pi(-x)\left[(1-x)-(1+x)\right]\,dx+\int_{0}^{\frac{-1+\sqrt{5}}{2}} 2\pi(x)\left[(1-x)-\left(x^2\right)\right]\,dx\end{aligned}}$$

4. If the function $y=f(x)\geq 0$ is continuously differentiable on $[a,b]$, the area of the surface generated by revolving the graph of $y=f(x)$ about the x-axis is

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$\dfrac{dy}{dx}=\dfrac{x^2}{2}-\dfrac{1}{2x^2}$. Set $a=1/2,\ b=1$ and then evaluate the integral.

$$S=\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$$

$$=\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{1+\frac{x^4}{4}-\frac{1}{2}+\frac{1}{4x^4}}\,dx=\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{\frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}}\,dx$$

$$=\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2}\,dx=\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)\,dx$$

$$=\int_{1/2}^{1} 2\pi\left(\frac{x^5}{12}+\frac{x}{12}+\frac{x}{4}+\frac{1}{4x^3}\right)\,dx=2\pi\left[\frac{x^6}{72}+\frac{x^2}{24}+\frac{x^2}{8}-\frac{1}{8x^2}\right]_{1/2}^{1}$$

$$=2\pi\left[\left(\frac{1}{72}+\frac{1}{24}+\frac{1}{8}-\frac{1}{8}\right)-\left(\frac{1}{4608}+\frac{1}{96}+\frac{1}{32}-\frac{1}{2}\right)\right]=\boxed{\frac{2367\pi}{2304}}$$

5. Take the corresponding function $f(x)=\dfrac{2}{3x+5}$. The function is continuous for $x\geq 1$ because the denominator is a first-degree polynomial whose root is $x_0=-\dfrac{5}{3}<1$. f is also positive and increasing for $x\geq 1$. Since the criteria hold, we may apply the Integral Test. Handle the improper integral by taking the limit.

$$\int_1^{\infty}\frac{2}{3x+5}\,dx=\lim_{R\to\infty}\int_1^R\frac{2}{3x+5}\,dx=\lim_{R\to\infty}\frac{2}{3}\ln|3x+5|\Big|_1^R=\frac{2}{3}\lim_{R\to\infty}\left(\ln|3R+5|-\ln 8\right)$$

$$=\infty$$

Since the integral diverges, by the Integral Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n+5}$ also diverges.

6. Recall the equality $\dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$.

$$f(x) = \frac{1}{x^2 - 5x + 6} = \frac{1}{(x-3)(x-2)} = \frac{1}{x-3} - \frac{1}{x-2}$$

The Maclaurin series of f is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0), f^{(4)}(0)$ to look for the pattern.

$$f'(x) = -\frac{1}{(x-3)^2} + \frac{1}{(x-2)^2}, \quad f''(x) = \frac{2}{(x-3)^3} - \frac{2}{(x-2)^3}$$
$$f'''(x) = -\frac{6}{(x-3)^4} + \frac{6}{(x-2)^4}, \quad f^{(4)}(x) = \frac{24}{(x-3)^5} - \frac{24}{(x-2)^5}$$

$$f(0) = -\frac{1}{3} + \frac{1}{2}, \quad f'(0) = -\frac{1}{9} + \frac{1}{4}, \quad f''(0) = -\frac{2}{27} + \frac{2}{8}$$
$$f'''(0) = -\frac{6}{81} + \frac{6}{16}, \quad f^{(4)}(0) = -\frac{24}{243} + \frac{24}{32}$$

This is a sequence where each term is defined by the following.

$$f^{(k)}(0) = (k!) \cdot \left(-\frac{1}{3^{k+1}} + \frac{1}{2^{k+1}}\right)$$

Rewrite the sum.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{(k!) \cdot x^k}{k!} \cdot \left(-\frac{1}{3^{k+1}} + \frac{1}{2^{k+1}}\right)$$

$$= \boxed{\sum_{k=0}^{\infty} x^k \left(\frac{1}{2^{k+1}} - \frac{1}{3^{k+1}}\right)}$$