

2019-2020 Spring  
MAT124 Resit  
(01/07/2020)

1. The temperature  $T$  at the point  $(x, y, z)$  in a region of space is given by the formula  $T = 100 - xy - xz - yz$ . Find the lowest temperature on the plane  $x + y + z = 10$ .

2. Show that if  $z = f(r, \theta)$ , where  $r$  and  $\theta$  are defined as functions of  $x$  and  $y$  by the equations  $x = r \cos \theta$  and  $y = r \sin \theta$ , then the equation  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$  becomes

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r} = 0.$$

3. Evaluate the integral  $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1-x^3} \, dx \, dy$ .

4. Evaluate

$$\int_2^4 \int_2^y dx \, dy + \int_4^8 \int_2^{16/y} dx \, dy$$

by reversing the order of integration.

5. Use a double integral to find the area inside the circle  $r = \cos \theta$  and outside the cardioid  $r = 1 - \cos \theta$ .

6. Use polar coordinates to evaluate the double integral

$$\iint_D \sin(x^2 + y^2) \, dA,$$

where  $D$  is the region bounded by the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$  and the lines  $y = 0$ ,  $x = \sqrt{3}y$ .

7. Using cylindrical coordinates, evaluate

$$\iiint_D \frac{dV}{x^2 + y^2 + z^2},$$

where  $D$  is the solid region bounded below by the paraboloid  $2z = x^2 + y^2$  and above by the sphere  $x^2 + y^2 + z^2 = 8$ .

8. Using spherical coordinates, evaluate the triple integral

$$\iiint_D \sqrt{x^2 + y^2 + z^2},$$

where  $D$  is the portion of the solid sphere  $x^2 + y^2 + z^2 \leq 1$  that lies in the first octant.

1) Let  $g(x, y, z) = x + y + z - 10$  and then, solve the system of equations below using the method of Lagrange multipliers.

$$\left. \begin{array}{l} \nabla T = \lambda \nabla g \\ g(x, y, z) = 0 \end{array} \right\} \quad \nabla T = \langle -y - z, -x - z, -x - y \rangle = \lambda \langle 1, 1, 1 \rangle = \lambda \nabla g$$

$$\begin{aligned} T_x + T_y + T_z &= (-y - z) + (-x - z) + (-x - y) = -2x - 2y - 2z \\ &= \lambda + \lambda + \lambda = 3\lambda \implies x + y + z = \frac{-3\lambda}{2} \end{aligned}$$

Use the constraint to find the value of  $\lambda$ .

$$g(x, y, z) = 0 \implies \frac{-3\lambda}{2} - 10 = 0 \implies \lambda = -\frac{20}{3}$$

So far, we have the equations below. Solve the system of equations and find the values of  $x, y, z$  one by one.

$$\left. \begin{array}{ll} -y - z = -\frac{20}{3} & (1) \\ -x - z = -\frac{20}{3} & (2) \\ -x - y = -\frac{20}{3} & (3) \end{array} \right\} \begin{array}{l} (1) \& (2) \rightarrow x - y = 0 \quad (4) \\ (3) \& (4) \rightarrow y = \frac{10}{3} \quad (5) \\ \therefore z = \frac{10}{3}, \quad x = \frac{10}{3} \end{array}$$

We now have all the values. Substitute in  $f(x, y, z)$  to find the minimum value of the temperature.

$$f_{\min} = f\left(\frac{10}{3}, \frac{10}{3}, \frac{10}{3}\right) = 100 - \left(\frac{10}{3}\right)^2 - \left(\frac{10}{3}\right)^2 - \left(\frac{10}{3}\right)^2 = \boxed{\frac{200}{3}}$$

2) We have  $x = r \cos \theta$  and  $y = r \sin \theta$ .

$$x^2 = r^2 \cos^2 \theta, \quad y^2 = r^2 \sin^2 \theta \implies x^2 + y^2 = r^2 (\cos^2 \theta + \sin^2 \theta) = r^2, \quad \therefore r = \sqrt{x^2 + y^2}$$

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta \implies \theta = \tan^{-1} \frac{y}{x}$$

Compute the first-order partial derivatives.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos \theta, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \sin \theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2} = \frac{-\sin \theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} = \frac{\cos \theta}{r}$$

Rewrite  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ .

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \cdot \frac{x}{\sqrt{x^2 + y^2}} + \frac{\partial z}{\partial \theta} \cdot \frac{-y}{x^2 + y^2}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r} \cdot \frac{y}{\sqrt{x^2 + y^2}} + \frac{\partial z}{\partial \theta} \cdot \frac{x}{x^2 + y^2}$$

Compute the second-order partial derivatives.

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial r} \cdot \frac{x}{\sqrt{x^2 + y^2}} + \frac{\partial z}{\partial \theta} \cdot \frac{-y}{x^2 + y^2} \right) \\ \frac{\partial^2 z}{\partial x^2} &= \left[ \left( \frac{\partial^2 z}{\partial r^2} \cdot \frac{\partial r}{\partial x} + \frac{\partial^2 z}{\partial \theta \partial r} \cdot \frac{\partial \theta}{\partial x} \right) \cdot \frac{x}{\sqrt{x^2 + y^2}} + \frac{\partial z}{\partial r} \cdot \frac{1 \cdot \sqrt{x^2 + y^2} - x \cdot \frac{x}{\sqrt{x^2 + y^2}}}{\left(\sqrt{x^2 + y^2}\right)^2} \right] \\ &\quad + \left[ \left( \frac{\partial^2 z}{\partial \theta^2} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial^2 z}{\partial r \partial \theta} \cdot \frac{\partial r}{\partial x} \right) \cdot \frac{-y}{x^2 + y^2} + \frac{\partial z}{\partial \theta} \cdot \frac{y}{(x^2 + y^2)^2} \cdot 2x \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial r} \cdot \frac{y}{\sqrt{x^2 + y^2}} + \frac{\partial z}{\partial \theta} \cdot \frac{x}{x^2 + y^2} \right) \\ \frac{\partial^2 z}{\partial y^2} &= \left[ \left( \frac{\partial^2 z}{\partial r^2} \cdot \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial \theta \partial r} \cdot \frac{\partial \theta}{\partial y} \right) \cdot \frac{y}{\sqrt{x^2 + y^2}} + \frac{\partial z}{\partial r} \cdot \frac{1 \cdot \sqrt{x^2 + y^2} - y \cdot \frac{y}{\sqrt{x^2 + y^2}}}{\left(\sqrt{x^2 + y^2}\right)^2} \right] \\ &\quad + \left[ \left( \frac{\partial^2 z}{\partial \theta^2} \cdot \frac{\partial \theta}{\partial y} + \frac{\partial^2 z}{\partial r \partial \theta} \cdot \frac{\partial r}{\partial y} \right) \cdot \frac{x}{x^2 + y^2} + \frac{\partial z}{\partial \theta} \cdot \frac{-x}{(x^2 + y^2)^2} \cdot 2y \right] \end{aligned}$$

Add the second-order partial derivatives and set to 0. The last terms eliminate each other. Write  $x$  and  $y$  in terms of  $r$  and  $\theta$ .

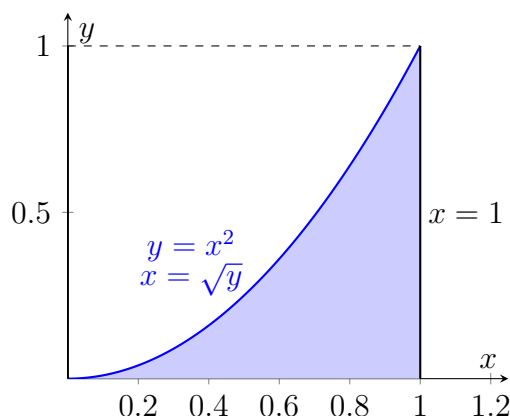
$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= \left[ \left( \frac{\partial^2 z}{\partial r^2} \cdot \cos \theta + \frac{\partial^2 z}{\partial \theta \partial r} \cdot \frac{-\sin \theta}{r} \right) \cdot \cos \theta + \frac{\partial z}{\partial r} \cdot \frac{\sin^2 \theta}{r} \right] \\ &\quad + \left[ \left( \frac{\partial^2 z}{\partial \theta^2} \cdot \frac{-\sin \theta}{r} + \frac{\partial^2 z}{\partial r \partial \theta} \cdot \cos \theta \right) \cdot \frac{-\sin \theta}{r} \right] \\ &\quad + \left[ \left( \frac{\partial^2 z}{\partial r^2} \cdot \sin \theta + \frac{\partial^2 z}{\partial \theta \partial r} \cdot \frac{\cos \theta}{r} \right) \cdot \sin \theta + \frac{\partial z}{\partial r} \cdot \frac{\cos^2 \theta}{r} \right] \\ &\quad + \left[ \left( \frac{\partial^2 z}{\partial \theta^2} \cdot \frac{\cos \theta}{r} + \frac{\partial^2 z}{\partial r \partial \theta} \cdot \sin \theta \right) \cdot \frac{\cos \theta}{r} \right] = 0 \end{aligned}$$

Inspect the terms that add up to 0. Recall  $\sin^2 \theta + \cos^2 \theta = 1$ , then the equation reduces to

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= \frac{\partial^2 z}{\partial r^2} \cdot (\cos^2 \theta + \sin^2 \theta) + \frac{\partial z}{\partial r} \cdot \frac{\sin^2 \theta + \cos^2 \theta}{r} + \frac{\partial^2 z}{\partial \theta^2} \cdot (\sin^2 \theta + \cos^2 \theta) \\ &= \frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial z}{\partial r} + \frac{\partial^2 z}{\partial \theta^2} = 0,\end{aligned}$$

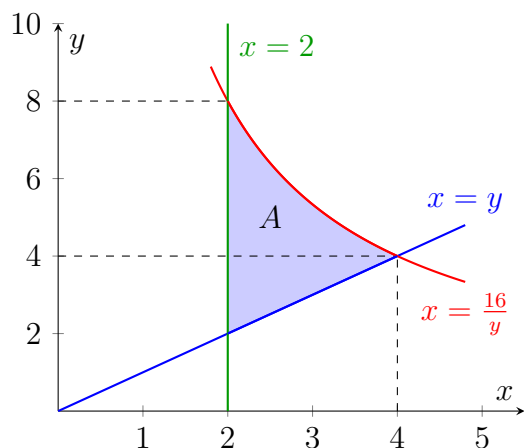
which we set out to demonstrate.

3) Change the order of integration using the graph below and then evaluate the integral.



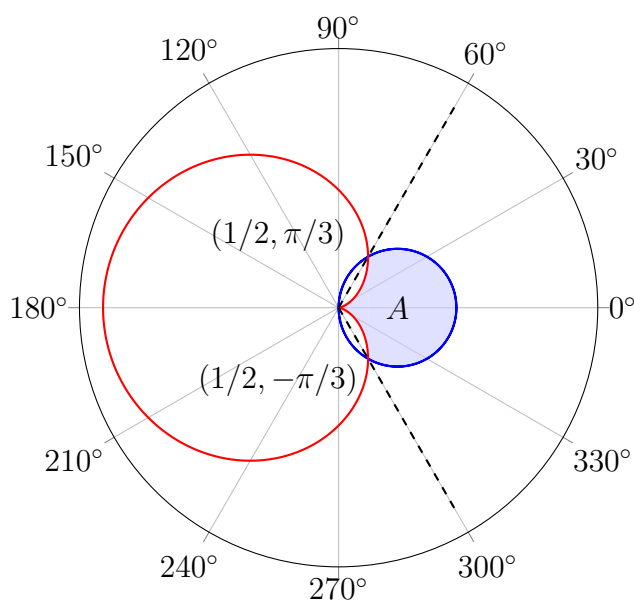
$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1-x^3} \, dx \, dy &= \int_0^1 \int_0^{x^2} \sqrt{1-x^3} \, dy \, dx = \int_0^1 x^2 \sqrt{1-x^3} \, dx \left[ \begin{array}{l} u = 1-x^3 \\ du = -3x^2 \, dx \end{array} \right] \\ &= \int \frac{\sqrt{u}}{-3} \, du = -\frac{2}{9} u^{3/2} + c = -\frac{2}{9} (1-x^3)^{3/2} \Big|_0^1 = 0 - \left[ -\frac{2}{9} \right] = \boxed{\frac{2}{9}}\end{aligned}$$

4)



$$\begin{aligned}A &= \int_2^4 \int_2^y dx \, dy + \int_4^8 \int_2^{16/y} dx \, dy \\ &= \int_2^4 \int_x^{16/x} dy \, dx = \int_2^4 \left( \frac{16}{x} - x \right) dx \\ &= \left[ 16 \ln |x| - \frac{x^2}{2} \right]_2^4 \\ &= \left[ \left( 16 \ln 4 - \frac{4^2}{2} \right) - \left( 16 \ln 2 - \frac{2^2}{2} \right) \right] \\ &= \boxed{16 \ln 2 - 6}\end{aligned}$$

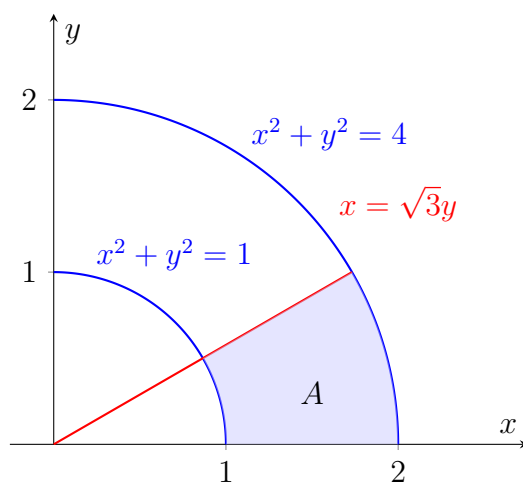
5)



$$A = \int_{-\pi/3}^{\pi/3} \int_{1-\cos\theta}^{\cos\theta} r \, dr \, d\theta = \frac{1}{2} \int_{-\pi/3}^{\pi/3} [\cos^2\theta - (1 - \cos\theta)^2] \, d\theta = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (2\cos\theta - 1) \, d\theta$$

$$= \frac{1}{2} \left[ 2\sin\theta - \theta \right]_{-\pi/3}^{\pi/3} = \frac{1}{2} \left[ \left( 2\sin\frac{\pi}{3} - \frac{\pi}{3} \right) - \left( 2\sin\left(-\frac{\pi}{3}\right) + \frac{\pi}{3} \right) \right] = \boxed{\sqrt{3} - \frac{\pi}{3}}$$

6)



Use the following transformation to switch to polar coordinates.

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ x^2 + y^2 &= r^2 \\ \theta &= \tan^{-1} \frac{y}{x} \\ dA &= r \, dr \, d\theta \end{aligned} \quad \rightarrow \quad \begin{aligned} x^2 + y^2 = 1 &\implies r^2 = 1 \implies r = 1 \\ x^2 + y^2 = 4 &\implies r^2 = 4 \implies r = 2 \\ y = 0 &\implies \theta = 0 \\ x = \sqrt{3}y &\implies \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{1}{\sqrt{3}} \implies \theta = \frac{\pi}{6} \end{aligned}$$

$$\therefore 1 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{6}$$

$$\begin{aligned}\int_0^{\pi/6} \int_1^2 \sin(r^2) r \, dr \, d\theta &= \int_0^{\pi/6} \left[ -\frac{1}{2} \cos r^2 \right]_1^2 d\theta = -\frac{1}{2} \int_0^{\pi/6} (\cos 4 - \cos 1) \, d\theta \\ &= \frac{1}{2} (\cos 1 - \cos 4) \cdot \theta \Big|_0^{\pi/6} = \boxed{\frac{\pi}{12} (\cos 1 - \cos 4)}\end{aligned}$$

7) For cylindrical coordinates, we have

$$\begin{aligned}z &= z \\ r^2 &= x^2 + y^2 & \rightarrow & \quad 2z = x^2 + y^2 \rightarrow z = \frac{r^2}{2} \\ dV &= r \, dz \, dr \, d\theta & \quad & \quad x^2 + y^2 + z^2 = 8 \rightarrow z = \sqrt{8 - r^2}\end{aligned}$$

Find where the surfaces  $2z = x^2 + y^2$  and  $x^2 + y^2 + z^2 = 8$  intersect to determine the limits of  $r$ .

$$\begin{aligned}x^2 + y^2 + z^2 = 8 &\implies 2z + z^2 = 8 \implies (z + 4)(z - 2) = 0 \implies z = 2 \\ &\implies 4 = x^2 + y^2 = r^2 \implies r = 2\end{aligned}$$

The lower limit of  $r$  is apparently 0. The region in the  $xy$ -plane is circular if we project the domain. Therefore,  $0 \leq \theta \leq 2\pi$ . Now, set up the triple integral in polar coordinates.

$$\begin{aligned}I &= \iiint_D \frac{dV}{x^2 + y^2 + z^2} = \int_0^{2\pi} \int_0^2 \int_{r^2/2}^{\sqrt{8-r^2}} \frac{r}{r^2 + z^2} \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \int_{r^2/2}^{\sqrt{8-r^2}} \frac{1}{1 + \left(\frac{z}{r}\right)^2} \cdot \frac{1}{r} \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \left[ \arctan\left(\frac{z}{r}\right) \right]_{r^2/2}^{z=\sqrt{8-r^2}} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \arctan\left(\frac{\sqrt{8-r^2}}{r}\right) \, dr \, d\theta - \int_0^{2\pi} \int_0^2 \arctan\left(\frac{r}{2}\right) \, dr \, d\theta\end{aligned}$$

$\theta$  is independent of  $r$ . Therefore, we can write the following.

$$I = 2\pi \int_0^2 \arctan\left(\frac{\sqrt{8-r^2}}{r}\right) \, dr - 2\pi \int_0^2 \arctan\left(\frac{r}{2}\right) \, dr \quad (1)$$

Now, use integration by parts for the left-hand integral in (1). Apply the chain rule and the quotient rule rigorously.

$$\begin{aligned}u = \arctan\left(\frac{\sqrt{8-r^2}}{r}\right) &\rightarrow du = \frac{1}{1 + \left(\frac{\sqrt{8-r^2}}{r}\right)^2} \cdot \frac{\frac{1}{2\sqrt{8-r^2}} \cdot (-2r) \cdot r - \sqrt{8-r^2} \cdot 1}{r^2} \, dr \\ dv &= dr \rightarrow v = r\end{aligned}$$

Notice that we have an improper integral, where we need to use limits.

$$\begin{aligned}\int_0^2 \arctan\left(\frac{\sqrt{8-r^2}}{r}\right) dr &= \lim_{T \rightarrow 0^+} \left[ r \cdot \arctan\left(\frac{\sqrt{8-r^2}}{r}\right) \Big|_T^2 - \int_T^2 r \cdot \frac{-1}{\sqrt{8-r^2}} dr \right] \\ &= \lim_{T \rightarrow 0^+} \left[ r \cdot \arctan\left(\frac{\sqrt{8-r^2}}{r}\right) - \sqrt{8-r^2} \right]_T^2\end{aligned}\quad (2)$$

Compute the other integral in (1) using integration by parts.

$$\begin{aligned}u = \arctan\left(\frac{r}{2}\right) &\rightarrow du = \frac{1}{1 + \left(\frac{r}{2}\right)^2} \cdot \frac{1}{2} dr \\ dv = dr &\rightarrow v = r\end{aligned}$$

$$\begin{aligned}\int_0^2 \arctan\left(\frac{r}{2}\right) dr &= r \cdot \arctan\left(\frac{r}{2}\right) \Big|_0^2 - \int_0^2 r \cdot \frac{2}{4+r^2} dr \\ &= \left[ r \cdot \arctan\left(\frac{r}{2}\right) - \ln|4+r^2| \right]_0^2\end{aligned}\quad (3)$$

Rewrite (1) using (2) and (3).

$$\begin{aligned}I &= 2\pi \lim_{T \rightarrow 0^+} \left[ r \cdot \arctan\left(\frac{\sqrt{8-r^2}}{r}\right) - \sqrt{8-r^2} \right]_T^2 - 2\pi \left[ r \cdot \arctan\left(\frac{r}{2}\right) - \ln|4+r^2| \right]_0^2 \\ I &= 2\pi (2 \cdot \arctan 1 - 2) - 2\pi \lim_{T \rightarrow 0^+} \left[ T \cdot \arctan \frac{\sqrt{8-T^2}}{T} - \sqrt{8-T^2} \right] \\ &\quad - 2\pi [(2 \cdot \arctan 1 - \ln 8) - (0 - \ln 4)] \\ I &= \pi \left( \ln 4 - 4 + 2 \lim_{T \rightarrow 0^+} \left( \sqrt{8-T^2} \right) \right) - 2\pi \lim_{T \rightarrow 0^+} \left( T \cdot \arctan \frac{\sqrt{8-T^2}}{T} \right)\end{aligned}\quad (4)$$

We need to evaluate the limit on the right side in (4) using the squeeze theorem.

$$\begin{aligned}-\frac{\pi}{2} &\leq \arctan\left(\frac{\sqrt{8-T^2}}{T}\right) \leq \frac{\pi}{2} \\ -\frac{T \cdot \pi}{2} &\leq T \cdot \arctan\left(\frac{\sqrt{8-T^2}}{T}\right) \leq \frac{T \cdot \pi}{2} \\ \lim_{T \rightarrow 0^+} \frac{-T \cdot \pi}{2} &= \lim_{T \rightarrow 0^+} \frac{T \cdot \pi}{2} = 0 \implies \lim_{T \rightarrow 0^+} \left( T \cdot \arctan \frac{\sqrt{8-T^2}}{T} \right) = 0\end{aligned}$$

The limit on the left side in (4) is simply equal to  $2\sqrt{2}$ . The value of the integral is then

$$\boxed{I = \pi \left( \ln 4 - 4 + 4\sqrt{2} \right)}$$

8) For spherical coordinates, we have

$$\begin{array}{ll} \begin{array}{l} z = \rho \cos \phi \\ r = \rho \sin \phi \\ x^2 + y^2 + z^2 = \rho^2 \\ dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{array} & \rightarrow \begin{array}{l} x^2 + y^2 + z^2 \leq 1 \implies \rho^2 \leq 1 \implies 0 \leq \rho \leq 1 \\ \sqrt{x^2 + y^2 + z^2} = 1 \rightarrow \sqrt{\rho^2} = 1 \implies \rho = 1 \\ \therefore 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2} \end{array} \end{array}$$

Set up the integral and then evaluate.

$$\begin{aligned} I &= \iiint_D \sqrt{x^2 + y^2 + z^2} \, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{\pi/2} \int_0^{\pi/2} \left[ \frac{\rho^4}{4} \right]_{\rho=0}^{\rho=1} \sin \phi \, d\phi \, d\theta = \frac{1}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin \phi \, d\phi \, d\theta = \frac{1}{4} \int_0^{\pi/2} \left[ -\cos \phi \right]_0^{\pi/2} d\theta \\ &= \frac{1}{4} \int_0^{\pi/2} d\theta = \boxed{\frac{\pi}{8}} \end{aligned}$$