

2019-2020 Fall
MAT123 Midterm
(04/11/2019)

1) Evaluate

$$\lim_{x \rightarrow 3^+} \cos(x-3)^{\ln(\frac{2x}{3}-2)}$$

2) Find constants a and b such that $f(x)$ defined by

$$f(x) = \begin{cases} \frac{\tan ax}{\tan bx}, & \text{if } x < 0 \\ 4, & \text{if } x = 0 \\ ax + b, & \text{if } x > 0 \end{cases}$$

will be continuous for all $x \in \mathbb{R}$ at the point $x = 0$.

3) Use the Intermediate Value Theorem to show that the equation $1 - 2x = \sin x$ has at least one real solution. Then use Rolle's Theorem to show it has no more than one solution.

4) Ship A is 60 miles north of point O and moving in the north direction at 20 miles per hour. Ship B is 80 miles east of point O and moving west at 25 miles per hour. How fast is the distance between the ships changing at this moment?

5) Sketch the graph of

$$f(x) = \frac{e^x}{x}$$

6) Evaluate the following integrals.

(a) $\int x^2 \sqrt{9+x^2} dx$

(b) $\int \tan x \cdot \sec^6 x dx$

(c) $\int_4^8 \frac{1}{(x-4)^3} dx$

(d) $\int e^{2x} \sin e^x dx$

(e) $\int \frac{dx}{\sin x - \cos x}$

7) Let us consider the area A of the region bounded by the curves $x = e^y$, $x = y^2 - 2$ and the straight lines $y = 1$, $y = -1$. Write an integral (but don't evaluate) corresponding to the area A

- (i) with respect to the y and
- (ii) with respect to the x .

1) Let L be the value of the limit.

$$L = \lim_{x \rightarrow 3^+} \cos(x-3)^{\ln(\frac{2x}{3}-2)}$$

$$\ln(L) = \ln \left(\lim_{x \rightarrow 3^+} \cos(x-3)^{\ln(\frac{2x}{3}-2)} \right)$$

Since $\cos(x-3)^{\ln(2x/3-2)}$ is continuous for $x > 3$, we can take the logarithm function inside the limit. Using the property of logarithms, we get:

$$\ln(L) = \lim_{x \rightarrow 3^+} \left[\ln \left(\cos(x-3)^{\ln(\frac{2x}{3}-2)} \right) \right] = \lim_{x \rightarrow 3^+} \left[\ln(\cos(x-3)) \ln \left(\frac{2x}{3} - 2 \right) \right] \quad [0 \cdot \infty]$$

We can rearrange the limit to obtain an indeterminate form. Afterwards, we may apply L'Hôpital's rule.

$$\lim_{x \rightarrow 3^+} \left[\ln(\cos(x-3)) \ln \left(\frac{2x}{3} - 2 \right) \right] = \lim_{x \rightarrow 3^+} \frac{\ln \left(\frac{2x}{3} - 2 \right)}{1/\ln(\cos(x-3))} \quad \left[\frac{0}{0} \right]$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 3^+} \frac{\frac{1}{\frac{2x}{3}-2} \cdot \frac{2}{3}}{\left(-\ln^{-2} \cos(x-3) \right) \cdot \frac{1}{\cos(x-3)} \cdot (-\sin(x-3))}$$

$$= \lim_{x \rightarrow 3^+} \frac{\ln^2(\cos(x-3))}{(x-3) \tan(x-3)} \quad \left[\frac{0}{0} \right]$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 3^+} \frac{2 \ln(\cos(x-3)) \cdot \frac{1}{\cos(x-3)} \cdot (-\sin(x-3))}{\tan(x-3) + (x-3) \sec^2(x-3)}$$

$$= \lim_{x \rightarrow 3^+} \frac{-2 \ln(\cos(x-3)) \sin(x-3)}{\sin(x-3) + (x-3) \sec(x-3)} \stackrel{u=x-3}{=} \lim_{u \rightarrow 0^+} \left(\frac{-2 \ln(\cos(u)) \sin(u)}{\sin(u) + u \sec(u)} \cdot \frac{u}{u} \right)$$

$$= \frac{\lim_{u \rightarrow 0^+} [-2 \ln(\cos(u))] \cdot \lim_{u \rightarrow 0^+} \frac{\sin(u)}{u}}{\lim_{u \rightarrow 0^+} \sec(u) + \lim_{u \rightarrow 0^+} \frac{\sin(u)}{u}} \quad \left[\lim_{u \rightarrow 0} \frac{\sin(u)}{u} = 1 \right]$$

$$= \frac{\lim_{u \rightarrow 0^+} [-2 \ln(\cos(u))]}{\lim_{u \rightarrow 0^+} \sec(u) + 1} = \frac{-2 \ln(\cos(0))}{1 + 1} = \frac{-2 \ln(1)}{2} = 0$$

We found out that $\ln(L) = 0$. Therefore, $\boxed{L = 1}$.

2) **Remark:** There is misinformation about the continuity in the original question, and I replaced it with continuity at $x = 0$.

To ensure continuity at $x = 0$, the one-sided limit values must be equal to the value of the function at that point.

$$\lim_{x \rightarrow 0^-} \frac{\tan ax}{\tan bx} = \lim_{x \rightarrow 0^+} (ax + b) = f(0) = 4$$

The easy part is that we can calculate the limit from the right.

$$\lim_{x \rightarrow 0^+} (ax + b) = 0 + b = b$$

Hence, $b = 4$. To calculate from the left, we need another technique.

$$\begin{aligned} \lim_{x \rightarrow 0^-} \frac{\tan ax}{\tan bx} &= \lim_{x \rightarrow 0^-} \left(\frac{\sin ax}{\cos ax} \cdot \frac{\cos bx}{\sin bx} \cdot \frac{bx}{bx} \cdot \frac{ax}{ax} \right) \\ &= \lim_{x \rightarrow 0^-} \left(\frac{\sin ax}{ax} \right) \cdot \lim_{x \rightarrow 0^-} \left(\frac{1}{\frac{\sin bx}{bx}} \right) \cdot \lim_{x \rightarrow 0^-} \left(\frac{\cos(bx) \cdot ax}{\cos(ax) \cdot bx} \right) \\ &= 1 \cdot \frac{1}{\lim_{x \rightarrow 0^-} \frac{\sin bx}{bx}} \cdot \lim_{x \rightarrow 0^-} \left(\frac{\cos(bx) \cdot a}{\cos(ax) \cdot b} \right) = 1 \cdot 1 \cdot \left(\frac{\cos(0) \cdot a}{\cos(0) \cdot b} \right) \\ &= \frac{a}{b} \end{aligned}$$

Now, set $\frac{a}{b} = b \rightarrow a = 16$. $a = 16, b = 4$

3) $-1 \leq \sin x \leq 1$, and $\sin x$ is continuous $\forall x \in \mathbb{R}$. $1 - 2x$ is continuous everywhere and takes any value in \mathbb{R} . Therefore, the equation $\sin x = 1 - 2x$ must have at least one real solution by IVT, and the y -intercept is on the interval $[-1, 1]$.

Let $f(x) = \sin x - 1 + 2x$ and x_1 be one solution to the equation. Then, the root must satisfy $|1 - 2x| \leq 1$. To disprove the existence of another root, we assume that x_2 is another distinct root. Since $f(x_1) = f(x_2) = 0$ and $f(x)$ is differentiable everywhere, by Rolle's theorem, there must exist a point c such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

Take the first derivative and calculate $f'(c)$

$$f'(c) = \cos c + 2$$

$-1 \leq \cos x \leq 1$. Therefore, there is no such c that satisfies $f'(c) = 0$. This is a contradiction. By Rolle's theorem, there is *only* one root satisfying $\sin x = 1 - 2x$.

4) Let $f(t)$ and $g(t)$ represent the distance between Ship A and point O, and the distance between Ship B and point O, respectively. The distance between the ships can be represented using the Pythagorean theorem as follows:

$$D^2(t) = f^2(t) + g^2(t)$$

Take the derivative of both sides.

$$2D \frac{dD}{dt} = 2f(t)f'(t) + 2g(t)g'(t)$$

Solve for $\frac{dD}{dt}$.

$$\frac{dD}{dt} = \frac{f(t)f'(t) + g(t)g'(t)}{D}$$

For $t = t_0$, we have $f(t_0) = 60$, $g(t_0) = 80$, $f'(t_0) = 20$, $g'(t_0) = -25$, $D(t_0) = \sqrt{60^2 + 80^2} = 100$. We may now find the rate of change of the distance at that time.

$$\left. \frac{dD}{dt} \right|_{t=t_0} = \frac{60 \cdot 20 - 80 \cdot 25}{100} = \boxed{-8 \text{ miles/hour}}$$

5) First off, find the domain. The expression is undefined when the denominator is zero. Therefore, $x \neq 0$. The only vertical asymptote occurs at $x = 0$.

$$\mathcal{D} = \mathbb{R} - \{0\}$$

Let us find the limit at infinity and the limit at negative infinity.

$$\lim_{x \rightarrow \infty} \frac{e^x}{x} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{1} = \infty$$

$$\lim_{x \rightarrow -\infty} \frac{e^x}{x} = 0$$

The horizontal asymptote occurs only at $y = 0$.

Take the first derivative by applying the quotient rule.

$$y' = \frac{e^x \cdot x - e^x \cdot 1}{x^2} = \frac{e^x(x - 1)}{x^2}$$

y' is undefined for $x = 0$, and $y' = 0$ for $x = 1$. Since 0 is not in the domain, the *only* critical point is $x = 1$.

Take the second derivative.

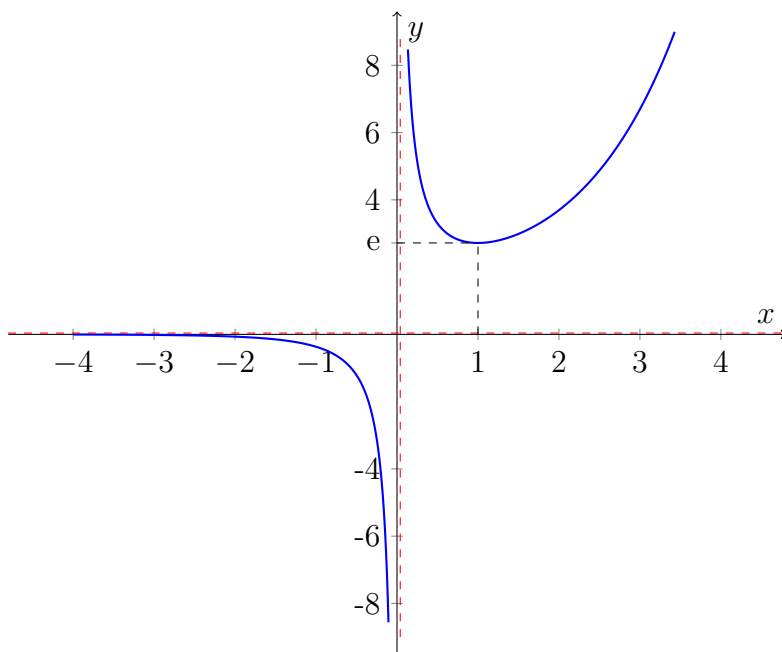
$$y'' = \frac{[e^x(x - 1) + e^x]x^2 - e^x(x - 1) \cdot 2x}{x^4} = \frac{e^x(x^2 - 2x + 2)}{x^3}$$

There is no inflection point.

Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(1) = \frac{e^1}{1} = e$$

x	$(-\infty, 0)$	$(0, 1]$	$[1, \infty)$
y	$(-\infty, 0)$	$(\infty, e]$	$[e, \infty)$
y' sign	-	-	+
y'' sign	-	+	+



6)

(a) Let $x = 3 \tan u$, then $dx = 3 \sec^2 u \, du$.

$$\begin{aligned}
 I &= \int x^2 \sqrt{9 + x^2} \, dx = \int (3 \tan u)^2 \cdot \sqrt{9 + (3 \tan u)^2} \cdot 3 \sec^2(u) \, du \quad [1 + \tan^2 u = \sec^2 u] \\
 &= 81 \int \tan^2 u \cdot \sec^3 u \, du = 81 \int \frac{\sin^2 u}{\cos^5 u} \, du = 81 \int \frac{1 - \cos^2 u}{\cos^5 u} \, du \\
 &= 81 \int \sec^5 u \, du - 81 \int \sec^3 u \, du \tag{1}
 \end{aligned}$$

Find the left-hand integral in (1) with integration by parts.

$$\begin{aligned}
 w &= \sec^3 u \rightarrow dw = 3 \sec^3 u \tan u \, du \\
 dz &= \sec^2 u \, du \rightarrow z = \tan u
 \end{aligned}$$

$$\int \sec^5 u \, du = \tan u \cdot \sec^3 u - 3 \int \tan^2 u \cdot \sec^3 u \, du = \tan u \cdot \sec^3 u - 3 \int (\sec^5 u - \sec^3 u) \, du$$

The integral we want to evaluate appears on the right side. After a little algebra, we get:

$$\int \sec^5 u \, du = \frac{1}{4} \cdot \tan u \cdot \sec^3 u + \frac{3}{4} \int \sec^3 u \, du$$

Rewrite (1) and calculate the other integral in (1) with integration by parts.

$$I = \frac{81}{4} \cdot \tan u \cdot \sec^3 u - \frac{81}{4} \int \sec^3 u \, du \quad (2)$$

$$\begin{aligned} w = \sec u &\rightarrow dw = \sec u \tan u \, du \\ dz = \sec^2 u \, du &\rightarrow z = \tan u \end{aligned}$$

$$\begin{aligned} \int \sec^3 u \, du &= \tan u \cdot \sec u - \int \tan^2 u \sec u \, du = \tan u \cdot \sec u - \int \frac{1 - \cos^2 u}{\cos^3 u} \, du \\ &= \tan u \cdot \sec u - \int \sec^3 u \, du + \int \sec u \, du \end{aligned}$$

We encountered a similar case when calculating $\int \sec^5 u \, du$. So,

$$\int \sec^3 u \, du = \frac{1}{2} \cdot \tan u \cdot \sec u + \frac{1}{2} \cdot \int \sec u \, du$$

The integral of $\sec u$ with respect to u is as follows. One can derive it with particular methods.

$$\int \sec u \, du = \ln |\tan u + \sec u| + c_1, \quad c_1 \in \mathbb{R} \quad (3)$$

Rewrite (2) using (3).

$$I = \frac{81}{4} \cdot \tan u \cdot \sec^3 u - \frac{81}{8} \cdot \tan u \cdot \sec u - \frac{81}{8} \cdot \ln |\tan u + \sec u| + c$$

Recall that $x = 3 \tan u$, then $x^2 = 9 \tan^2 u = 9 \sec^2 u - 9 \rightarrow \sec u = \frac{\sqrt{x^2 + 9}}{3}$. The result is then as follows. Furthermore, we can omit the constant part to simplify.

$$I = \frac{x\sqrt{x^2 + 9}}{8} (2x^2 + 9) - \frac{81}{8} \ln \left| \frac{x + \sqrt{x^2 + 9}}{3} \right| + c, \quad c \in \mathbb{R}$$

$$\boxed{I = \frac{x\sqrt{x^2 + 9}}{8} (2x^2 + 9) - \frac{81}{8} \ln |x + \sqrt{x^2 + 9}| + c, \quad c \in \mathbb{R}}$$

(b) Rewrite the expression. Then, let $u = \tan^2 x + 1$. So, $du = 2 \tan x \sec^2 x \, dx$

$$\begin{aligned} I &= \int \tan x \cdot \sec^6 x \, dx \quad [\tan^2 x + 1 = \sec^2 x] \\ &= \int \tan x \cdot \sec^2 x \cdot (1 + \tan^2 x)^2 \, dx = \frac{1}{2} \int u^2 \, du = \frac{u^3}{6} + c \end{aligned}$$

$$\boxed{I = \frac{(\tan^2 x + 1)^3}{6} + c, \, c \in \mathbb{R}}$$

(c) This is an improper integral; we need to make use of the limit concept. The expression is undefined for $x = 4$.

$$\begin{aligned} I &= \int_4^8 \frac{1}{(x-4)^3} \, dx = \lim_{R \rightarrow 4^+} \int_R^8 \frac{1}{(x-4)^3} \, dx = \lim_{R \rightarrow 4^+} \left[-\frac{1}{2(x-4)^2} \right]_R^8 \\ &= \lim_{R \rightarrow 4^+} \left[-\frac{1}{32} + \frac{1}{(R-4)^2} \right] = \boxed{\infty} \end{aligned}$$

(d) We'll use integration by parts.

$$\begin{aligned} u &= e^x \rightarrow du = e^x \, dx \\ dv &= e^x \sin e^x \, dx \rightarrow v = -\cos e^x \end{aligned}$$

$$\begin{aligned} \int e^{2x} \sin e^x \, dx &= (-\cos e^x) \cdot e^x - \int e^x \cdot (-\cos e^x) \, dx \\ &= \boxed{-e^x \cos e^x + \sin e^x + c, \, c \in \mathbb{R}} \end{aligned}$$

(e) We may utilize the tangent half-angle substitution, which is sometimes called the Weierstrass substitution. Let $t = \tan\left(\frac{x}{2}\right)$. After some mathematical operations, we get the following. One can later derive the formulas.

$$\sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \quad dx = \frac{2}{1+t^2} \, dt$$

Rewrite the integral and apply partial fraction decomposition.

$$\begin{aligned} I &= \int \frac{dx}{\sin x - \cos x} = \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} \, dt = \int \frac{2}{t^2 + 2t - 1} \, dt = 2 \int \frac{1}{(t+1)^2 - (\sqrt{2})^2} \, dt \\ &= 2 \int \frac{1}{(t+1-\sqrt{2})(t+1+\sqrt{2})} \, dt = 2 \int \left(\frac{A}{t+1+\sqrt{2}} + \frac{B}{t+1-\sqrt{2}} \right) \, dt \quad (4) \end{aligned}$$

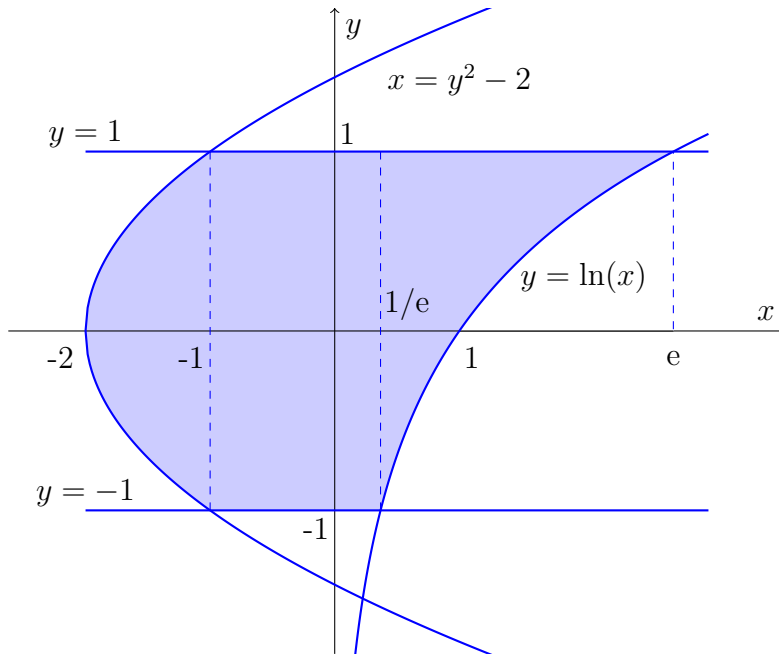
$$\begin{aligned}
A(t+1+\sqrt{2}) + B(t+1-\sqrt{2}) &= 1 \\
t(A+B) + A+B + \sqrt{2}(A-B) &= 1 \\
\therefore A+B &= 0 \quad [\text{eliminate } t] \rightarrow A-B = \frac{1}{\sqrt{2}} \\
\left. \begin{aligned} A+B &= 0 \\ A-B &= \frac{1}{\sqrt{2}} \end{aligned} \right\} & \quad A = \frac{1}{2\sqrt{2}}, \quad B = -\frac{1}{2\sqrt{2}}
\end{aligned}$$

Plug the values of A and B into (4).

$$I = \frac{\sqrt{2}}{2} \int \left(\frac{1}{t+1+\sqrt{2}} - \frac{1}{t+1-\sqrt{2}} \right) dt = \frac{\sqrt{2}}{2} \ln \left(\frac{|t+1+\sqrt{2}|}{|t+1-\sqrt{2}|} \right) + c, \quad c \in \mathbb{R}$$

$$I = \frac{\sqrt{2}}{2} \ln \left(\frac{\left| \tan\left(\frac{x}{2}\right) + 1 + \sqrt{2} \right|}{\left| \tan\left(\frac{x}{2}\right) + 1 - \sqrt{2} \right|} \right) + c, \quad c \in \mathbb{R}$$

7)



(i) The variable is y . Hence, the limits are $-1, 1$, respectively.

$$A = \int_{-1}^1 [e^y - (y^2 - 2)] dy$$

(ii) We have three different regions. This leads us to take three different integrals.

$$A = \int_{-2}^{-1} [\sqrt{x+2} - (-\sqrt{x+2})] dx + \int_{-1}^{1/e} [1 - (-1)] dx + \int_{1/e}^e (1 - \ln x) dx$$