2021-2022 Fall
MAT123-02,05 Final
(07/01/2022)

1.

(a) Find $\displaystyle\int \frac{dx}{x^3 - 4x^2 + 3x}$.

(b) Evaluate $\displaystyle\lim_{x \to \frac{\pi}{2}} \frac{\int_{\pi/2}^{x} \ln(\sin t)\, dt}{\sin x - 1}$.

(c) Evaluate the improper integral $\displaystyle\int_1^2 \frac{dx}{(x-1)^{2/3}}$.

2. Consider the region R bounded by the curves $y = \arctan x$, $y = \ln x$ and the lines $x = \dfrac{1}{\sqrt{3}}$ and $x = 1$.

(a) Find the area of the region R.

(b) Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region R about the y-axis.

(c) Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of a solid obtained by rotating the region R about the line $y = 2$.

3. Determine whether each series is convergent or divergent. Explain your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1))$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{n^2}{1+n^4}\right)$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

2021-2022 Final (07/01/2022) Solutions
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1.

(a) Use the method of partial fraction decomposition.

$$\int \frac{dx}{x^3-4x^2+3x} = \int \frac{dx}{x(x-3)(x-1)} = \int \left(\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x-1}\right) dx$$

$$\begin{aligned} A(x-3)(x-1)+Bx(x-1)+Cx(x-3) &= 1 \\ x^2(A+B+C)+x(-4A-B-3C)+3A &= 1 \end{aligned}$$

Equate the coefficients of like terms.

$$\left.\begin{aligned} x^2(A+B+C) &= 0 \\ x(-4A-B-3C) &= 0 \\ 3A &= 1 \end{aligned}\right\} \to A = \frac{1}{3}, \qquad \left.\begin{aligned} B+C &= -\frac{1}{3} \\ B+3C &= -\frac{4}{3} \end{aligned}\right\} \to B = \frac{1}{6}, \quad C = -\frac{1}{2}$$

Rewrite the integral.

$$\int \left(\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x-1}\right) dx = \int \left(\frac{1}{3x}+\frac{1}{6(x-3)}-\frac{1}{2(x-1)}\right) dx$$

$$= \boxed{\frac{1}{3}\ln|x|+\frac{1}{6}\ln|x-3|-\frac{1}{2}\ln|x-1|+c, \quad c\in\mathbb{R}}$$

(b) The limit is in the indeterminate form 0/0. Apply L'Hôpital's rule to eliminate the indeterminate form.

$$\lim_{x\to\frac{\pi}{2}} \frac{\displaystyle\int_{\pi/2}^{x} \ln(\sin t)\,dt}{\sin x - 1} \overset{\text{L'H.}}{=} \lim_{x\to\frac{\pi}{2}} \frac{\displaystyle\frac{d}{dx}\int_{\pi/2}^{x} \ln(\sin t)\,dt}{\cos x}$$

By the Fundamental Theorem of Calculus, we may rewrite the limit as follows.

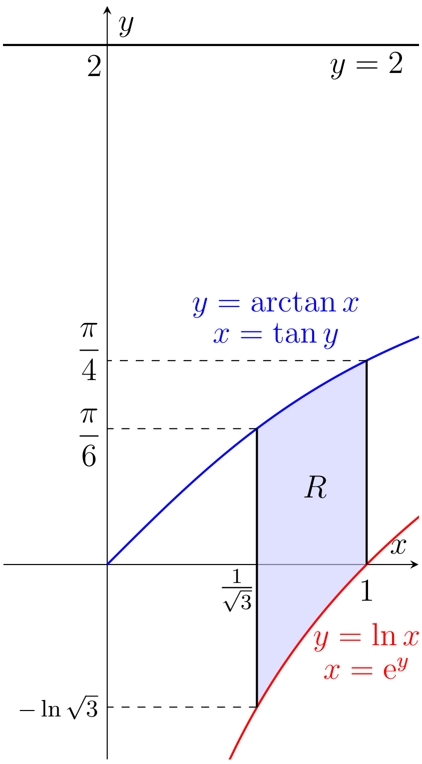
$$\lim_{x\to\frac{\pi}{2}} \frac{\displaystyle\frac{d}{dx}\int_{\pi/2}^{x} \ln(\sin t)\,dt}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{\ln(\sin x)}{\cos x} \overset{\text{L'H.}}{=} \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin x}\cdot\cos x}{-\sin x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{-\sin^2 x} = -\frac{\cos\frac{\pi}{2}}{\sin^2\frac{\pi}{2}}$$

$$= \boxed{0}$$

(c) Take the limit as this is an improper integral.

$$\int_1^2 \frac{dx}{(x-1)^{2/3}} = \lim_{R\to1^+}\int_R^2 \frac{dx}{(x-1)^{2/3}} = \lim_{R\to1^+} 3(x-1)^{1/3}\Big|_R^2 = 3\lim_{R\to1^+}\left(1-(R-1)^{1/3}\right) = \boxed{3}$$

2.



(a)

$$A = \int_{1/\sqrt{3}}^{1} (\arctan x - \ln x)\ dx = \int_{1/\sqrt{3}}^{1} \arctan x\, dx - \int_{1/\sqrt{3}}^{1} \ln x\, dx \tag{1}$$

Calculate the first integral in (1) by integration by parts.

$$\left.\begin{aligned} u = \arctan x \implies du &= \frac{1}{x^2+1}\, dx \\ dv = dx \implies v &= x \end{aligned}\right\} \to \int u\, dv = uv - \int v\, du$$

$$\int_{1/\sqrt{3}}^{1} \arctan x\, dx = x \arctan x \Big|_{1/\sqrt{3}}^{1} - \int_{1/\sqrt{3}}^{1} \frac{x}{x^2+1}\, dx = \left(x \arctan x - \frac{1}{2} \ln\left| x^2 + 1 \right| \right) \Bigg|_{1/\sqrt{3}}^{1}$$

$$= \left(\frac{\pi}{4} - \frac{\ln 2}{2} \right) - \left(\frac{\pi\sqrt{3}}{18} - \frac{1}{2} \cdot \ln\left(\frac{4}{3}\right) \right) = \frac{\pi\left(9 - 2\sqrt{3}\right)}{36} + \frac{1}{2} \cdot \ln\frac{2}{3}$$

Calculate the second integral in (1) by integration by parts.

$$\left.\begin{aligned} u = \ln x \implies du &= \frac{1}{x}\, dx \\ dv = dx \implies v &= x \end{aligned}\right\} \to \int u\, dv = uv - \int v\, du$$

$$\int_{1/\sqrt{3}}^{1} \ln x \, dx = x \ln x \Big|_{1/\sqrt{3}}^{1} - \int_{1/\sqrt{3}}^{1} dx = (x \ln x - x) \Big|_{1/\sqrt{3}}^{1} = (0-1) - \left(-\frac{\ln\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right)$$

$$= \frac{\sqrt{3}\ln\sqrt{3} + \sqrt{3} - 3}{3}$$

The result is then

$$\boxed{A = \frac{\pi\left(9 - 2\sqrt{3}\right)}{36} + \frac{1}{2} \cdot \ln\frac{2}{3} - \frac{\sqrt{3}\ln\sqrt{3} + \sqrt{3} - 3}{3}}$$

(b)

$$\boxed{\begin{aligned} &\int_{-\ln\sqrt{3}}^{0} \pi \left[(e^y)^2 - \left(\frac{1}{\sqrt{3}}\right)^2 \right] dy + \int_{0}^{\pi/6} \pi \left[(1)^2 - \left(\frac{1}{\sqrt{3}}\right)^2 \right] dy \\ &+ \int_{\pi/6}^{\pi/4} \pi \left[(1^2) - (\tan y)^2 \right] dy \end{aligned}}$$

(c)

$$\boxed{\begin{aligned} &\int_{-\ln\sqrt{3}}^{0} 2\pi\,(2-y)\left(e^y - \frac{1}{\sqrt{3}}\right) dy + \int_{0}^{\pi/6} 2\pi\,(2-y)\left(1 - \frac{1}{\sqrt{3}}\right) dy \\ &+ \int_{\pi/6}^{\pi/4} 2\pi\,(2-y)\,(1-\tan y)\, dy \end{aligned}}$$

3.

(a) Determine the nth partial sum.

$$\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1)) = (\cancel{\arctan 1} - \arctan 0) + (\cancel{\arctan 2} - \cancel{\arctan 1})$$

$$+ (\cancel{\arctan 3} - \cancel{\arctan 2}) + (\cancel{\arctan 4} - \cancel{\arctan 3}) + \ldots$$

$$+ (\arctan n - \cancel{\arctan(n-1)}) = \arctan n - \arctan 0$$

$$= \arctan n$$

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \quad \text{(convergent)}$$

Therefore, the series $\sum_{n=1}^{\infty}(\arctan n - \arctan(n-1))$ converges.

(b)

$$\sum_{n=0}^{\infty}\frac{(-3)^{n+1}}{2^{3n}} = \sum_{n=0}^{\infty}\frac{-3\cdot(-3)^n}{8^n} = -3\sum_{n=0}^{\infty}\left(-\frac{3}{8}\right)^n$$

This is a geometric series where $r = -\frac{3}{8}$. $|r| = \frac{3}{8} < 1$. Therefore, the series $\sum_{n=0}^{\infty}\frac{(-3)^{n+1}}{2^{3n}}$ converges.

(c) Apply the nth Term Test for divergence. We may take the limit inside the trigonometric function because $\cos x$ is continuous everywhere. Since $1+n^4$ grows faster than n^2, the value of the expression $\frac{n^2}{1+n^4}$ tends to zero.

$$\lim_{n\to\infty}\cos\left(\frac{n^2}{1+n^4}\right) = \cos\left[\lim_{n\to\infty}\left(\frac{n^2}{1+n^4}\right)\right] = \cos 0 = 1 \neq 0$$

By the nth Term Test for divergence, the series $\sum_{n=1}^{\infty}\cos\left(\frac{n^2}{1+n^4}\right)$ diverges.

(d) Take $f(x) = \frac{1}{x(\ln x)^2}$. f is positive and decreasing for $x \geq 2$ because x and $(\ln x)^2$ are positive and increasing for $x \geq 2$. x is a polynomial which is defined everywhere and $(\ln x)^2$ is continuous for $x \geq 2$. Since we took into account every criterion, we may apply the Integral Test. Handle the improper integrals by taking the limit.

$$\int_2^{\infty}\frac{dx}{x(\ln x)^2} = \lim_{R\to\infty}\int_2^{R}\frac{dx}{x(\ln x)^2} = \lim_{R\to\infty}\left[-\frac{1}{\ln x}\right]_2^R = \lim_{R\to\infty}\left[-\frac{1}{\ln R}+\frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

Since the integral converges, the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ also converges.