

2011-2012 Spring
MAT123-[Instructor] Final
(29/05/2012)
Time: 15:00 - 16:50
Duration: 110 minutes

1. Find the length of the curve $y^2 = 4(x + 1)^3$ for $0 \leq x \leq 1$, $y > 0$.

2. Given $f(x) = x + 2x^2 + x^3$, find $(f^{-1})'(4)$.

3. Evaluate the following integrals.

(a) $\int \cos^3 x \sin^2 x \, dx$ (b) $\int \frac{x^2}{\sqrt{16 - x^2}} \, dx$ (c) $\int \frac{x^3 - 1}{x^3 - x} \, dx$ (d) $\int x^{123} \ln x \, dx$

4. Evaluate $\lim_{x \rightarrow \infty} (\ln x)^{\frac{1}{x}}$.

5.

(a) Determine the series $\sum_{n=1}^{\infty} \frac{5}{3^n}$ converges or diverges. Give reasons for your answer.

(b) Determine the series $\sum_{n=1}^{\infty} \cos\left(\frac{1}{5^n}\right)$ converges or diverges. Give reasons for your answer.

(c) Use the Ratio Test to determine if the series $\sum_{n=1}^{\infty} \frac{n!}{e^{2n}}$ converges or diverges.

(d) Use the Integral Test to determine if the series $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ converges or diverges. Be sure to check that the conditions of the integral test are satisfied.

(Bonus)

(a) Geometrically, what does $f'(x)$ mean?

(b) If $f(t)$ describes the displacement of an object in time t , what is $f'(t)$?

2011-2012 Spring Final (29/05/2012) Solutions
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1. y is implicitly defined as a function of x . Differentiate each side and solve for $\frac{dy}{dx}$.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4(x+1)^3) \implies 2y \frac{dy}{dx} = 12(x+1)^2 \implies \frac{dy}{dx} = \frac{6(x+1)^2}{y}$$

Since we're interested in the upper part of the curve (i.e., $y > 0$), $y = 2(x+1)^{3/2}$.

$$\frac{dy}{dx} = \frac{6(x+1)^2}{2(x+1)^{3/2}} = 3\sqrt{x+1}$$

The length of a curve defined by $y = f(x)$ whose derivative is continuous on the interval $a \leq x \leq b$ can be evaluated using the integral

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Set $a = 0$, $b = 1$, $\frac{dy}{dx} = 3\sqrt{x+1}$ and find the length.

$$S = \int_0^1 \sqrt{1 + (3\sqrt{x+1})^2} dx = \int_0^1 \sqrt{9x+10} dx$$

Let $u = 9x + 10$, then $du = 9 dx$.

$$S = \int_0^1 \sqrt{9x+10} dx = \int \frac{1}{9} \sqrt{u} du = \frac{1}{9} \cdot \frac{2}{3} u^{3/2} + c = \frac{2}{27} (9x+10)^{3/2} \Big|_0^1$$

$$= \boxed{\frac{2}{27} (19^{3/2} - 10^{3/2})}$$

2. The derivative of f^{-1} at a point can be calculated using the rule

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Find the point where $f(x) = 4$. We could intuitively say $f(1) = 4$ because $f(1) = 1 + 2 \cdot 1^3 + 1^3 = 4$. Therefore,

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(1)}$$

Calculate the derivative of f at the point $x = 1$.

$$f'(x) = 1 + 4x + 3x^2 \implies f'(1) = 1 + 4 \cdot 1 + 3 \cdot 1^2 = 8$$

So,

$$(f^{-1})'(4) = \boxed{\frac{1}{8}}$$

3.

(a)

$$\begin{aligned} I &= \int \cos^3 x \sin^2 x \, dx \quad [\sin^2 + \cos^2 x = 1] \\ &= \int \cos x \cdot (1 - \sin^2 x) \cdot \sin^2 x \, dx \end{aligned}$$

Let $u = \sin x$, then $du = \cos x \, dx$.

$$\begin{aligned} I &= \int \cos x \cdot (1 - \sin^2 x) \cdot \sin^2 x \, dx = \int (1 - u^2) u^2 \, du = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + c \\ &= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c} \end{aligned}$$

(b) Let $x = 4 \sin u$, then $dx = 4 \cos u \, du$ for $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

$$\begin{aligned} I &= \int \frac{x^2}{\sqrt{16 - x^2}} \, dx = \int \frac{16 \sin^2 u}{\sqrt{16 - 16 \sin^2 u}} \cdot 4 \cos u \, du = \int \frac{16 \sin^2 u \cos u}{|\cos u|} \, du \left[|\cos u| > 0 \right] \\ &= \int 16 \sin^2 u \, du = 16 \int (1 - \cos^2 u) \, du = 16 \int \frac{1 - \cos 2u}{2} \, du = 8 \left(u - \frac{\sin 2u}{2} \right) + c \\ &= 8u - 8 \sin u \cos u + c \end{aligned}$$

Recall: $x = 4 \sin u$. Then

$$\begin{aligned} x^2 = 16 \sin^2 u &\implies x^2 = 16 - 16 \cos^2 u \implies \cos^2 u = \frac{16 - x^2}{16} \implies \cos u = \sqrt{1 - \frac{x^2}{16}} \\ \sin u &= \frac{x}{4} \implies u = \arcsin \frac{x}{4} \end{aligned}$$

Rewrite the integral.

$$I = \boxed{8 \arcsin \frac{x}{4} - 2x \sqrt{1 - \frac{x^2}{16}} + c, \quad c \in \mathbb{R}}$$

(c) Use the method of partial fraction decomposition.

$$\begin{aligned} I &= \int \frac{x^3 - 1}{x^3 - x} dx = \int \frac{(x-1)(x^2 + x + 1)}{x(x-1)(x+1)} dx = \int \frac{x^2 + x + 1}{x^2 + x} dx = \int \left(1 + \frac{1}{x^2 + x}\right) dx \\ &= \int dx + \int \frac{1}{x(x+1)} dx = x + \int \left(\frac{A}{x} + \frac{B}{x+1}\right) dx \end{aligned}$$

$$\begin{aligned} A(x+1) + B(x) &= 1 \\ x(A+B) + A &= 1 \\ \therefore A+B &= 0 \quad [\text{eliminate } x] \rightarrow A=1 \implies B=-1 \end{aligned}$$

$$I = x + \int \left(\frac{A}{x} + \frac{B}{x+1}\right) dx = x + \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \boxed{x + \ln|x| - \ln|x+1| + c, \quad c \in \mathbb{R}}$$

(d) Use the method of integration by parts.

$$\left. \begin{aligned} u = \ln x &\implies du = \frac{1}{x} dx \\ dv = x^{123} dx &\implies v = \frac{x^{124}}{124} \end{aligned} \right\} \rightarrow \int u dv = uv - \int v du$$

$$I = \ln x \cdot \frac{x^{124}}{124} - \int \frac{x^{124}}{124} \cdot \frac{1}{x} dx = \boxed{\frac{\ln x \cdot x^{124}}{124} + \frac{x^{124}}{124^2} + c, \quad c \in \mathbb{R}}$$

4. Let L be the value of the limit.

$$L = \lim_{x \rightarrow \infty} (\ln x)^{\frac{1}{x}}$$

Take the logarithm of both sides. We can take the logarithm inside the limit because the expression is continuous for $x > 0$. After that, apply L'Hôpital's rule where $0/0$ or ∞/∞ forms occur.

$$\ln(L) = \ln \left(\lim_{x \rightarrow \infty} (\ln x)^{\frac{1}{x}} \right) = \lim_{x \rightarrow \infty} \ln \left[(\ln x)^{\frac{1}{x}} \right] = \lim_{x \rightarrow \infty} \frac{\ln(\ln x)}{x} \quad \left[\frac{\infty}{\infty} \right]$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x \ln x} = 0$$

Since $\ln(L) = 0$, $\boxed{L = 1}$.

5.

(a) Since the numerator is constant, we can take it out of the summation.

$$\sum_{n=1}^{\infty} \frac{5}{3^n} = 5 \cdot \sum_{n=1}^{\infty} \frac{1}{3^n} = 5 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

This is a geometric series with $r = \frac{1}{3} < 1$. Therefore, the series converges.

(b) Take the limit of the sequence at infinity. We can take the limit inside the trigonometric function because it is continuous everywhere.

$$\lim_{n \rightarrow \infty} \cos\left(\frac{1}{5^n}\right) = \cos\left(\lim_{n \rightarrow \infty} \frac{1}{5^n}\right) = \cos(0) = 1 \neq 0$$

By the n th Term Test for divergence, the series diverges.

(c) Let $a_n = \frac{n!}{e^{2n}}$. Then,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{e^{2(n+1)}} \cdot \frac{e^{2n}}{n!} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{e^2} \right| = \infty > 1$$

By the Ratio Test, the series diverges.

(d) Let $a_n = f(n)$, where $n \in \mathbb{N}$. The function $f(x) = \frac{x}{e^{x^2}}$ is positive, continuous and decreasing for $x > 1$.

$$\left. \begin{array}{l} x > 0 \\ e^{x^2} > 0 \end{array} \right\} \text{ for } x > 1 \implies \frac{x}{e^{x^2}} > 0$$

e^{x^2} grows at a higher rate than x . Therefore, f is decreasing. The expressions are continuous for $x > 1$. We may now apply the Integral Test. Handle the improper integral with the limit.

$$\int_1^{\infty} \frac{x}{e^{x^2}} dx = \lim_{R \rightarrow \infty} \left. -\frac{1}{2} e^{-x^2} \right|_1^R = \lim_{R \rightarrow \infty} -\frac{1}{2} (e^{-R^2} - e^{-1}) = \frac{1}{2e} \quad (\text{converges})$$

By the Integral Test, the series converges.

(Bonus)

(a) Let $y = f(x)$ be a continuous function on a bounded interval, and let f be differentiable on the same interval except possibly at the endpoints. Then $f'(x)$ gives the first derivative. $f'(x)$ gives the instantaneous rate of change of the function at a certain point and it gives the slope of the line that is tangent to the graph of the function at that point.

(b) Given $f(t)$ describes the displacement, $f'(t)$ corresponds to the instantaneous velocity of the object.