

2019-2020 Spring
MAT124-02,05 Final
(01/07/2020)

1. Find the maximum and minimum values of the function $f(x, y, z) = x - y + z$ on the sphere $x^2 + y^2 + z^2 = 100$.

2. A cylindrical tank is 4 ft high and has an outer diameter of 2 ft. The walls of the tank are 0.2 in. thick. Approximate the volume of the interior of the tank assuming the tank has a top and bottom that are both also 0.2 in. thick.

3. Let $z = f(x, y)$ be a differentiable function of x and y , and let $x = r \cos \theta$ and $y = r \sin \theta$ for $r > 0$ and $0 < \theta < 2\pi$. Show that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

4. Reverse the order of integration in

$$\int_1^2 \int_x^{x^3} f(x, y) dy dx + \int_2^8 \int_x^8 f(x, y) dy dx.$$

5. Evaluate the double integral

$$\int_1^2 \int_{y^2}^{y^5} e^{x/y^2} dx dy.$$

6. Use a double integral to find the area of the region that lies inside the circle $r = \cos \theta$ and outside the cardioid $r = 1 - \cos \theta$.

7. Evaluate the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$, by using a triple integral in cylindrical coordinates.

8. Let T be the solid in the first octant bounded above by the sphere $x^2 + y^2 + z^2 = 7$ and below by the paraboloid $z = x^2 + y^2$. Express (do not evaluate) the integral

$$\iiint_T \sin \left(\sqrt{x^2 + y^2 + z^2} \right) dV$$

in spherical coordinates.

2019-2020 Spring Final (01/07/2020) Solutions
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1. Let $g(x, y, z) = x^2 + y^2 + z^2 - 100$ and then, solve the system of equations below using the method of Lagrange multipliers.

$$\left. \begin{array}{l} \nabla f = \lambda \nabla g \\ g(x, y, z) = 0 \end{array} \right\} \quad \nabla f = \langle 1, -1, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle = \lambda \nabla g$$

$$\therefore x = \frac{1}{2\lambda}, \quad y = -\frac{1}{2\lambda}, \quad z = \frac{1}{2\lambda}$$

Use the constraint.

$$g(x, y, z) = 0 \implies \left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 100 \implies \frac{3}{4\lambda^2} = 100 \implies \lambda = \pm \frac{\sqrt{3}}{20}$$

$$\lambda = \pm \frac{\sqrt{3}}{20\lambda} \implies x = \pm \frac{10\sqrt{3}}{3}, \quad y = \mp \frac{10\sqrt{3}}{3}, \quad z = \pm \frac{10\sqrt{3}}{3}$$

The absolute extrema occur at $\left(\frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}\right)$ and $\left(-\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}\right)$.

$$f\left(\frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}\right) = 10\sqrt{3}, \quad f\left(-\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}\right) = -10\sqrt{3}$$

The minimum value is $-10\sqrt{3}$ and the maximum value is $10\sqrt{3}$.

2. Recall that 12 inches is 1 foot. The volume of a right circular cylinder is

$$V(r, h) = \pi r^2 h$$

The total differential is

$$dV = V_r dr + V_h dh = \pi (2r \cdot h) dr + \pi (r^2 \cdot 1) dh$$

Set $r = 1$, $h = 4$, $dr = -0.2/12 = -1/60$, $dh = -0.4/12 = -1/30$.

$$dV = \pi (2 \cdot 1 \cdot 4) \cdot \left(-\frac{1}{60}\right) + \pi (1^2) \left(-\frac{1}{30}\right) = -\frac{\pi}{6}$$

Calculate the volume of the outer cylinder.

$$V(1, 4) = \pi \cdot 1^2 \cdot 4 = 4\pi$$

Take $\Delta V \approx dV = -\frac{\pi}{6}$. Therefore, the volume of the interior can be approximated as follows.

$$V \approx 4\pi - \frac{\pi}{6} = \frac{23\pi}{6}$$

3. We have $x = r \cos \theta$ and $y = r \sin \theta$. Compute the first-order partial derivatives.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}, \quad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$\frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta,$$

Rewrite $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \cos \theta + \frac{\partial z}{\partial y} \cdot \sin \theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot (-r \sin \theta) + \frac{\partial z}{\partial y} \cdot (r \cos \theta) \implies \frac{1}{r} \cdot \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot (-\sin \theta) + \frac{\partial z}{\partial y} \cdot \cos \theta$$

Take the squares of both sides of the equations and add up side by side.

$$\left(\frac{\partial z}{\partial r} \right)^2 = \left(\frac{\partial z}{\partial x} \cdot \cos \theta + \frac{\partial z}{\partial y} \cdot \sin \theta \right)^2, \quad \left(\frac{1}{r} \cdot \frac{\partial z}{\partial \theta} \right)^2 = \left(\frac{\partial z}{\partial x} \cdot (-\sin \theta) + \frac{\partial z}{\partial y} \cdot \cos \theta \right)^2$$

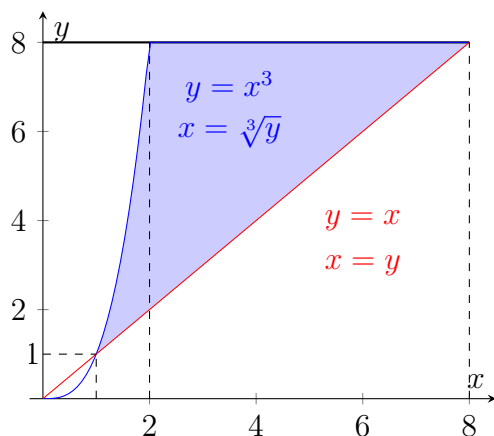
$$\begin{aligned} \left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2 &= \left(\frac{\partial z}{\partial x} \right)^2 \cos^2 \theta + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \cos \theta \sin \theta + \left(\frac{\partial z}{\partial y} \right)^2 \sin^2 \theta \\ &\quad + \left(\frac{\partial z}{\partial x} \right)^2 \sin^2 \theta - \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \cos \theta \sin \theta + \left(\frac{\partial z}{\partial y} \right)^2 \cos^2 \theta \end{aligned}$$

The terms with $\sin \theta \cos \theta$ cancel each other. Recall the equation $\sin^2 x + \cos^2 x = 1$. The equation then becomes

$$\left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2 = \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2,$$

which we set out to demonstrate.

4.

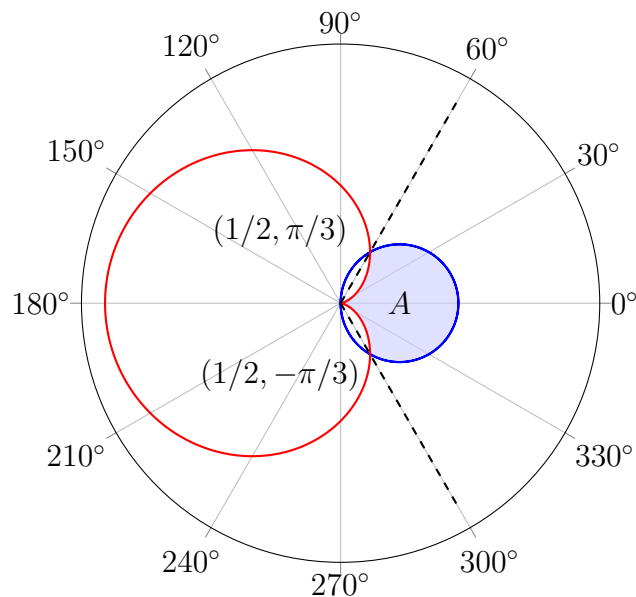


$$\int_1^8 \int_{\sqrt[3]{y}}^y f(x, y) dx dy$$

5.

$$\begin{aligned}\int_1^2 \int_{y^2}^{y^5} e^{x/y^2} dx dy &= \int_1^2 \left[y^2 \cdot e^{x/y^2} \right]_{x=y^2}^{x=y^5} dy = \int_1^2 \left(y^2 \cdot e^{y^3} - y^2 \cdot e \right) dy = \left[\frac{1}{3} e^{y^3} - \frac{1}{3} e y^3 \right]_1^2 \\ &= \frac{1}{3} e^8 - \frac{8e}{3} - \left(\frac{1}{3} e^1 - \frac{1}{3} e \right) = \boxed{\frac{e}{3} (e^7 - 8)}\end{aligned}$$

6.



$$\begin{aligned}A &= \int_{-\pi/3}^{\pi/3} \int_{1-\cos\theta}^{\cos\theta} r dr d\theta = \frac{1}{2} \int_{-\pi/3}^{\pi/3} [\cos^2\theta - (1 - \cos\theta)^2] d\theta = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (2\cos\theta - 1) d\theta \\ &= \frac{1}{2} \left[2\sin\theta - \theta \right]_{-\pi/3}^{\pi/3} = \frac{1}{2} \left[\left(2\sin\frac{\pi}{3} - \frac{\pi}{3} \right) - \left(2\sin\left(-\frac{\pi}{3}\right) + \frac{\pi}{3} \right) \right] = \boxed{\sqrt{3} - \frac{\pi}{3}}\end{aligned}$$

7. For cylindrical coordinates, we have

$$\begin{aligned}z &= \sqrt{x^2 + y^2} \implies z = \sqrt{r^2} \implies z_{\text{lower}} = r \\ \left. \begin{aligned} z &= z \\ r^2 &= x^2 + y^2 \\ dV &= r dz dr d\theta \end{aligned} \right\} \rightarrow \begin{aligned} z &= 2 - x^2 - y^2 \implies z_{\text{upper}} = 2 - r^2 \\ 0 &\leq \theta \leq 2\pi \end{aligned}\end{aligned}$$

Find where the surfaces $z = r$ and $z = 2 - r^2$ intersect to determine the upper bound of r .

$$\left. \begin{aligned} z &= r \\ z &= 2 - r^2 \end{aligned} \right\} \quad r^2 + r - 2 = 0 \implies (r+2)(r-1) = 0 \implies r_{\text{upper}} = 1$$

$$\begin{aligned}
I &= \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \left[z \right]_{z=r}^{z=2-r^2} r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (2 - r^2 - r) \, r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 (2r - r^3 - r^2) \, dr \, d\theta = \int_0^{2\pi} \left[r^2 - \frac{r^4}{4} - \frac{r^3}{3} \right]_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{5}{12} d\theta = \frac{5}{12} \cdot \theta \Big|_0^{2\pi} \\
&= \boxed{\frac{5\pi}{6}}
\end{aligned}$$

8. For spherical coordinates, we have

$$\begin{aligned}
x^2 + y^2 + z^2 = 7 &\implies \rho^2 = 7 \implies \rho_{\text{upper},1} = \sqrt{7} \\
z = x^2 + y^2 &\implies \rho \cos \phi = \rho^2 \sin^2 \phi \implies \rho_{\text{upper},2} = \cot \phi \csc \phi \\
z = \rho \cos \phi & \\
r = \rho \sin \phi & \\
x^2 + y^2 + z^2 = \rho^2 & \\
dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta & \\
\sin(\sqrt{x^2 + y^2 + z^2}) = \sin(\sqrt{\rho^2}) = \sin \rho & \\
0 \leq \theta \leq \frac{\pi}{2} &
\end{aligned}$$

Find where the surfaces $z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 7$ intersect to find the bounds of ϕ .

$$\begin{aligned}
z^2 + z - 7 &= 0 \implies z_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-7)}}{2} \\
z > 0 &\implies z = \rho \cos \phi = \sqrt{7} \cos \phi = \frac{-1 + \sqrt{29}}{2} \\
\cos \phi &= \frac{-1 + \sqrt{29}}{2\sqrt{7}} \implies \phi = \arccos\left(\frac{-1 + \sqrt{29}}{2\sqrt{7}}\right)
\end{aligned}$$

For $\phi < \arccos\left(\frac{-1 + \sqrt{29}}{2\sqrt{7}}\right)$, the upper bound for ρ is $\sqrt{7}$. For $\phi > \arccos\left(\frac{-1 + \sqrt{29}}{2\sqrt{7}}\right)$, the lower bound is $\cot \phi \csc \phi$.

$$\begin{aligned}
&\int_0^{\pi/2} \int_0^{\arccos\left(\frac{-1+\sqrt{29}}{2\sqrt{7}}\right)} \int_0^{\sqrt{7}} \sin \rho \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
&+ \int_0^{\pi/2} \int_{\arccos\left(\frac{-1+\sqrt{29}}{2\sqrt{7}}\right)}^{\pi/2} \int_0^{\cot \phi \csc \phi} \sin \rho \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\end{aligned}$$

Since we choose the minimum of the upper bounds of ρ , we can write the equivalent expression.

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\min(\sqrt{7}, \cot \phi \csc \phi)} \sin \rho \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$