2024-2025 Fall
MAT123-02,05 Makeup
(31/01/2025)

1.

(a) Find $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$.

(b) Find $\displaystyle\int \frac{dx}{x^3 - 4x^2 + 3x}$.

(c) Evaluate $\displaystyle\lim_{x \to \frac{\pi}{2}} \frac{\int_{\pi/2}^{x} \ln(\sin t)\,dt}{\sin x - 1}$.

(d) Evaluate the improper integral $\displaystyle\int_0^2 \frac{dx}{(x-1)^{2/3}}$.

2. Consider the region R bounded by the curves $y = \arctan x$, $y = \ln x$ and the lines $x = \dfrac{1}{\sqrt{3}}$ and $x = 1$.

(a) Sketch the region and find the area of the R.

(b) Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region R about the y-axis.

(c) Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of a solid obtained by rotating the region R about the line $y = 2$.

3. Determine whether each series is convergent or divergent. Explain your answer.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

4. Find the Taylor series of the function $f(x) = \ln x$ at $c = 1$ and determine the interval of convergence.

2024-2025 Makeup (31/01/2025) Solutions
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1.

(a)

$$\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx = \int \frac{(1-\cos^2 x)\cdot \sin x}{\sqrt{\cos x}}\,dx$$

Let $u = \cos x$, then $du = -\sin x\,dx$.

$$\int \frac{(1-\cos^2 x)\cdot \sin x}{\sqrt{\cos x}}\,dx = \int -\frac{(1-u^2)}{\sqrt{u}}\,du = \int \left(u^{3/2} - \frac{1}{\sqrt{u}}\right) = \frac{2}{5}u^{5/2} - 2\sqrt{u} + c$$

$$= \boxed{\frac{2}{5}(\cos x)^{5/2} - 2\sqrt{\cos x} + c, \quad c \in \mathbb{R}}$$

(b) Use the method of partial fraction decomposition.

$$\int \frac{dx}{x^3 - 4x^2 + 3x} = \int \frac{dx}{x(x-3)(x-1)} = \int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1}\right)dx$$

$$\begin{aligned} A(x-3)(x-1) + Bx(x-1) + Cx(x-3) &= 1 \\ x^2(A+B+C) + x(-4A-B-3C) + 3A &= 1 \end{aligned}$$

Equate the coefficients of like terms.

$$\left.\begin{aligned} x^2(A+B+C) &= 0 \\ x(-4A-B-3C) &= 0 \\ 3A &= 1 \end{aligned}\right\} \to A = \frac{1}{3}, \qquad \left.\begin{aligned} B + C &= -\frac{1}{3} \\ B + 3C &= -\frac{4}{3} \end{aligned}\right\} \to B = \frac{1}{6}, \quad C = -\frac{1}{2}$$

Rewrite the integral.

$$\int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1}\right)dx = \int \left(\frac{1}{3x} + \frac{1}{6(x-3)} - \frac{1}{2(x-1)}\right)dx$$

$$= \boxed{\frac{1}{3}\ln|x| + \frac{1}{6}\ln|x-3| - \frac{1}{2}\ln|x-1| + c, \quad c \in \mathbb{R}}$$

(c) The limit is in the indeterminate form 0/0. Apply L'Hôpital's rule to eliminate the indeterminate form.

$$\lim_{x \to \frac{\pi}{2}} \frac{\displaystyle\int_{\pi/2}^{x} \ln(\sin t)\,dt}{\sin x - 1} \overset{\text{L'H.}}{=} \lim_{x \to \frac{\pi}{2}} \frac{\displaystyle\frac{d}{dx}\int_{\pi/2}^{x} \ln(\sin t)\,dt}{\cos x}$$

By the Fundamental Theorem of Calculus, we may rewrite the limit as follows.

$$\lim_{x\to\frac{\pi}{2}} \frac{\dfrac{d}{dx}\displaystyle\int_{\pi/2}^{x} \ln(\sin t)\,dt}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{\ln(\sin x)}{\cos x} \overset{\text{L'H.}}{=} \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin x}\cdot\cos x}{-\sin x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{-\sin^2 x} = -\frac{\cos\frac{\pi}{2}}{\sin^2\frac{\pi}{2}}$$

$$= \boxed{0}$$

(d) Take the limit as this is an improper integral.

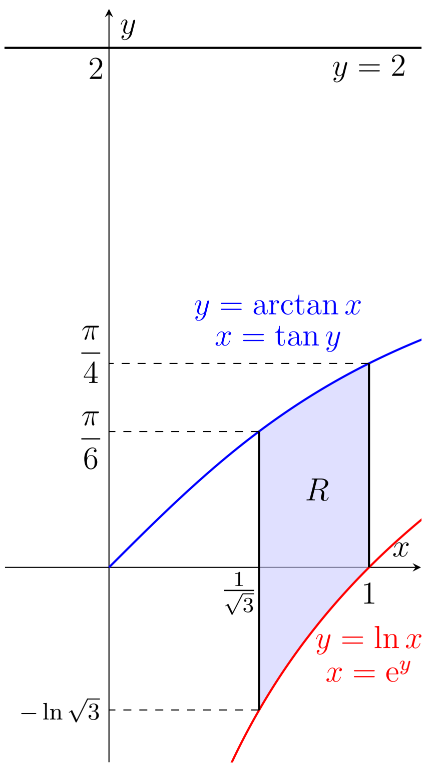
$$\int_0^2 \frac{dx}{(x-1)^{2/3}} = \lim_{R\to 1^-} \int_0^R \frac{dx}{(x-1)^{2/3}} + \lim_{P\to 1^+} \int_P^2 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{R\to 1^-} 3(x-1)^{1/3}\Big|_0^R + \lim_{P\to 1^+} 3(x-1)^{1/3}\Big|_P^2$$

$$= 3\lim_{R\to 1^-} \left((R-1)^{1/3} - (-1)\right) + 3\lim_{P\to 1^+} \left(1-(P-1)^{1/3}\right) = \boxed{6}$$

2.

(a)



$$A = \int_{1/\sqrt{3}}^{1} (\arctan x - \ln x)\ dx = \int_{1/\sqrt{3}}^{1} \arctan x\,dx - \int_{1/\sqrt{3}}^{1} \ln x\,dx \tag{1}$$

Calculate the first integral in (1) by integration by parts.

$$\left.\begin{aligned} u = \arctan x &\implies du = \frac{1}{x^2+1}\,dx \\ dv = dx &\implies v = x \end{aligned}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int_{1/\sqrt{3}}^{1} \arctan x\,dx = x\arctan x\Big|_{1/\sqrt{3}}^{1} - \int_{1/\sqrt{3}}^{1} \frac{x}{x^2+1}\,dx = \left(x\arctan x - \frac{1}{2}\ln\left|x^2+1\right|\right)\Big|_{1/\sqrt{3}}^{1}$$

$$= \left(\frac{\pi}{4} - \frac{\ln 2}{2}\right) - \left(\frac{\pi\sqrt{3}}{18} - \frac{1}{2}\cdot\ln\left(\frac{4}{3}\right)\right) = \frac{\pi\left(9-2\sqrt{3}\right)}{36} + \frac{1}{2}\cdot\ln\frac{2}{3}$$

Calculate the second integral in (1) by integration by parts.

$$\left.\begin{aligned} u = \ln x &\implies du = \frac{1}{x}\,dx \\ dv = dx &\implies v = x \end{aligned}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int_{1/\sqrt{3}}^{1} \ln x\,dx = x\ln x\Big|_{1/\sqrt{3}}^{1} - \int_{1/\sqrt{3}}^{1} dx = (x\ln x - x)\Big|_{1/\sqrt{3}}^{1} = (0-1) - \left(-\frac{\ln\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)$$

$$= \frac{\sqrt{3}\ln\sqrt{3} + \sqrt{3} - 3}{3}$$

The result is then

$$A = \boxed{\frac{\pi\left(9-2\sqrt{3}\right)}{36} + \frac{1}{2}\cdot\ln\frac{2}{3} - \frac{\sqrt{3}\ln\sqrt{3}+\sqrt{3}-3}{3}}$$

(b)

$$\boxed{\begin{aligned} &\int_{-\ln\sqrt{3}}^{0} \pi\left[\left(e^y\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right] dy + \int_{0}^{\pi/6} \pi\left[(1)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right] dy \\ &+ \int_{\pi/6}^{\pi/4} \pi\left[\left(1^2\right) - (\tan y)^2\right] dy \end{aligned}}$$

(c)

$$\boxed{\begin{aligned} &\int_{-\ln\sqrt{3}}^{0} 2\pi\,(2-y)\left(e^y - \frac{1}{\sqrt{3}}\right) dy + \int_{0}^{\pi/6} 2\pi\,(2-y)\left(1 - \frac{1}{\sqrt{3}}\right) dy \\ &+ \int_{\pi/6}^{\pi/4} 2\pi\,(2-y)\,(1-\tan y)\,dy \end{aligned}}$$

3.

(a)

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}} = \sum_{n=0}^{\infty} \frac{-3 \cdot (-3)^n}{8^n} = -3 \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n$$

This is a geometric series where $r = -\dfrac{3}{8}$. $|r| = \dfrac{3}{8} < 1$. Therefore, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$ converges.

(b) Take $f(x) = \dfrac{1}{x(\ln x)^2}$. f is positive and decreasing for $x \geq 2$ because x and $(\ln x)^2$ are positive and increasing for $x \geq 2$. x is a polynomial which is defined everywhere and $(\ln x)^2$ is continuous for $x \geq 2$. Since we took into account every criterion, we may apply the Integral Test. Handle the improper integrals by taking the limit.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{R\to\infty} \int_2^{R} \frac{dx}{x(\ln x)^2} = \lim_{R\to\infty} \left[-\frac{1}{\ln x}\right]_2^R = \lim_{R\to\infty} \left[-\frac{1}{\ln R} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

Since the integral converges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ also converges.

4. The Taylor series of f at $c = 1$ is as follows.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

Find $f(1)$, $f'(1)$, $f''(1)$, $f'''(1)$ $f^{(4)}(1)$ to look for the pattern.

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f'''(x) = \frac{2}{x^3} \quad f^{(4)}(x) = -\frac{6}{x^4}$$

$$f(1) = 0, \quad f'(1) = 1, \quad f''(1) = -1, \quad f'''(1) = 2, \quad f^{(4)}(1) = -6$$

This is an alternating sequence where the coefficient of each term is the factorial of the subsequent number starting from 0 except for $k = 0$, that is, the first term of the series. At $k = 0$, the first term is 0. So,

$$f^k(1) = \begin{cases} (-1)^{k-1} \cdot (k-1)!, & \text{if } k > 0 \\ 0, & \text{if } k = 0 \end{cases}$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}x^k = 0 + \sum_{k=1}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot (k-1)!}{k \cdot (k-1)!}(x-1)^k$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot (x-1)^k}{k} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + ...}$$

Now, determine the interval of convergence. Apply the Ratio Test.

$$\lim_{k\to\infty}\left|\frac{(-1)^k(x-1)^{k+1}}{k+1}\cdot\frac{k}{(-1)^{k-1}(x-1)^k}\right| = \lim_{k\to\infty}\left|\frac{(x-1)\cdot k}{(k+1)\cdot(-1)}\right| = |x-1|\lim_{k\to\infty}\left|\frac{k}{k+1}\right| = |x-1|$$

$$|x-1|<1 \implies -1<x-1<1 \implies 0<x<2 \quad \text{(convergent)}$$

Investigate the convergence at the endpoints.

$$x=0 \to \sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cdot(-1)^k}{k} = \sum_{k=1}^{\infty}\frac{(-1)^{2k}\cdot(-1)}{k} = -\sum_{k=1}^{\infty}\frac{1}{k}$$

This is a p-series with $p=1$, for which the series diverges by the p-series Test. Try $x=2$.

$$x=2 \to \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{k}$, is nonincreasing for $k \geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges.

The convergence set for the power series is $\boxed{(0,2]}$.