MAT123 26.12.2025

QUESTIONS

Q1. Determine if the following sequences converge or diverge.

a. $a_n = \dfrac{1}{\sqrt{n^2-1} - \sqrt{n^2+n}}$ **b.** $b_n = \dfrac{\sin^2 n}{2^n}$ **c.** $c_n = \dfrac{n^2}{2n-1} \sin \dfrac{1}{n}$

d. $a_n = \displaystyle\int_1^n \frac{1}{x^p}\, dx, \quad p > 1$ **e.** $b_n = \dfrac{n!}{n^n}$

Q2. Determine whether each of the following series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{n+1} + 3^n}{4^n}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$ **c.** $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n}$

d. $\displaystyle\sum_{n=1}^{\infty} \ln \sqrt{\frac{n+1}{n}}$ **e.** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \, \ln(\ln n)}$ **f.** $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$

Q3. Find the radius, center, and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$.

Q4. Find the Maclaurin series for the function $\dfrac{x^2}{1+x}$.

ANSWERS

Q1. To determine whether the sequence converges or diverges, we compute its limit as $n \to \infty$.

a. Rationalize the denominator; i.e., multiply and divide by the conjugate of the denominator.

$$a_n = \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}} \cdot \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{\sqrt{n^2-1}+\sqrt{n^2+n}} = \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{(n^2-1)-(n^2+n)}$$

Simplify the denominator.

$$(n^2-1)-(n^2+n) = -n-1 \implies a_n = \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{-n-1}$$

Factor out n from the square roots.

$$\sqrt{n^2-1} = |n|\sqrt{1-\frac{1}{n^2}}, \quad \sqrt{n^2+n} = |n|\sqrt{1+\frac{1}{n}}$$

$|n|$ simplifies to n because $n > 0$. Hence,

$$a_n = \frac{n\left(\sqrt{1-\frac{1}{n^2}}+\sqrt{1+\frac{1}{n}}\right)}{-n-1}$$

Divide numerator and denominator by n:

$$a_n = \frac{\sqrt{1-\frac{1}{n^2}}+\sqrt{1+\frac{1}{n}}}{-\left(1+\frac{1}{n}\right)}$$

Take the limit.

$$\lim_{n\to\infty} a_n = \frac{1+1}{-1} = -2 \qquad \left[\text{because} \lim_{n\to\infty}\frac{1}{n^2} = \lim_{n\to\infty}\frac{1}{n} = 0\right]$$

Therefore, $\boxed{\lim_{n\to\infty} a_n = -2}$

b. Use bounds on the numerator. For all real numbers n,

$$-1 \le \sin n \le 1 \implies 0 \le \sin^2 n \le 1.$$

Thus,

$$0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}.$$

Since $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{2^n} = 0$, by the Squeeze Theorem, the sequence $\frac{\sin^2 n}{2^n}$ also converges to

$$\boxed{0}$$

c. Rewrite the expression to isolate known limits.

$$c_n = \left(\frac{n^2}{2n-1}\right)\sin\frac{1}{n}$$

Rewrite the rational factor:

$$\frac{n^2}{2n-1} = \frac{n}{2-\frac{1}{n}} \implies c_n = \frac{n}{2-\frac{1}{n}}\sin\frac{1}{n}$$

Use a standard trigonometric limit. Recall that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Let $x = \frac{1}{n}$. Then

$$\sin\frac{1}{n} = \frac{1}{n}\cdot\frac{\sin\frac{1}{n}}{\frac{1}{n}} \implies c_n = \frac{n}{2-\frac{1}{n}}\cdot\frac{1}{n}\cdot\frac{\sin\frac{1}{n}}{\frac{1}{n}} = \frac{1}{2-\frac{1}{n}}\cdot\frac{\sin\frac{1}{n}}{\frac{1}{n}}$$

Take the limit.

$$\lim_{n\to\infty}\left(\frac{1}{2-\frac{1}{n}}\right) = \frac{1}{2}, \qquad \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \implies \lim_{n\to\infty}\left[\left(2-\frac{1}{n}\right)\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right] = \frac{1}{2}\cdot 1 = \boxed{\frac{1}{2}}$$

d. Evaluate the integral.

$$\lim_{n\to\infty}\int_1^n \frac{1}{x^p}\,dx = \lim_{n\to\infty}\left.\frac{x^{1-p}}{1-p}\right|_1^n = \lim_{n\to\infty}\left(\underbrace{\frac{n^{1-p}}{1-p}}_{0} - \frac{1^{1-p}}{1-p}\right) = \lim_{n\to\infty}\frac{1}{p-1}$$

$$\boxed{\text{The limit of the sequence is } \frac{1}{p-1}.}$$

e. Write the sequence as a product.

$$b_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n}$$

Estimate the product. For $k \leq n$, we have $\frac{k}{n} \leq 1$. In particular, for $k = 1, 2, \ldots, n$,

$$\frac{k}{n} \leq \frac{n}{n} = 1.$$

Hence,

$$b_n = \frac{1 \cdot 2 \cdot 3 \ \ldots \ n}{n \cdot n \cdot n \ \ldots \ n} \leq \frac{1}{n} \cdot 1 \cdot 1 \ \ldots \ 1 = \frac{1}{n}. \implies b_n \leq \frac{1}{n}$$

$n!$ and n^n are both positive for $n > 0$. Therefore,

$$0 \leq b_n \leq \frac{1}{n}.$$

Since $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, by the Squeeze Theorem, the sequence $\dfrac{n!}{n^n}$ also converges to

$$\boxed{0}$$

Q2. a. Split the series.

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+1} + 3^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{4^n} + \sum_{n=1}^{\infty} \frac{3^n}{4^n}.$$

Simplify each term. For the first series,

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{4^n} = \sum_{n=1}^{\infty} \frac{(-2)^n \cdot (-2)}{4^n} = -2 \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$$

For the second series,

$$\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n.$$

Both series are geometric with common ratios

$$r_1 = -\frac{1}{2}, \qquad r_2 = \frac{3}{4}$$

and satisfy $|r_1| < 1$ and $|r_2| < 1$. Remember the geometric series formula.

$$\sum_{n=1}^{\infty} ar^n = \frac{r}{1-r}$$

First series:

$$-2\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = -2\left(\frac{-\frac{1}{2}}{1-\left(-\frac{1}{2}\right)}\right) = \frac{2}{3}.$$

Second series:

$$\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = \frac{\frac{3}{4}}{1-\frac{3}{4}} = 3.$$

Add the results.

$$\sum_{n=1}^{\infty}\frac{(-2)^{n+1}+3^n}{4^n} = \frac{2}{3}+3 = \frac{11}{3}.$$

$$\boxed{\text{The series converges and its sum is } \frac{11}{3}.}$$

b. Divide the numerator and denominator by n^2:

$$a_n = \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}.$$

As $n \to \infty$, $\dfrac{2}{n}, \dfrac{3}{n} \to 0$. So

$$\lim_{n\to\infty} a_n = \frac{1}{(1)(1)} = 1.$$

Apply the Divergence Test. Since $\lim\limits_{n\to\infty} a_n = 1 \neq 0$, the Divergence Test implies that the series

$$\sum_{n=1}^{\infty}\frac{n(n+1)}{(n+2)(n+3)}$$

$\boxed{\text{diverges}}$.

c. Let $a_n = \dfrac{\cos(n\pi)}{5^n}$. Apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cos((n+1)\pi)}{5^{n+1}}\cdot\frac{5^n}{\cos(n\pi)}\right| = \lim_{n\to\infty}\frac{|\cos((n+1)\pi)|}{|\cos(n\pi)|}\cdot\frac{1}{5}$$

Since $\cos(n\pi) = (-1)^n$, we have $|\cos(n\pi)| = 1$ for all n. Therefore,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} < 1.$$

By the Ratio Test, the series converges absolutely. Therefore, the series $\boxed{\text{converges}}$.

d. First rewrite the general term:

$$\ln\sqrt{\frac{n+1}{n}} = \frac{1}{2}\ln\left(\frac{n+1}{n}\right) = \frac{1}{2}\big(\ln(n+1) - \ln n\big).$$

Examine the partial sums. Let

$$S_N = \sum_{n=1}^{N}\ln\sqrt{\frac{n+1}{n}} = \frac{1}{2}\sum_{n=1}^{N}\big(\ln(n+1) - \ln n\big).$$

Take the limit.

$$\lim_{N\to\infty} S_N = \frac{1}{2}\Big[(\cancel{\ln 2} - \ln 1) + \cancel{\ln 3} - \cancel{\ln 2}) + (\cancel{\ln 4} - \cancel{\ln 3})$$
$$+ \ldots + (\cancel{\ln N} - \cancel{\ln(N-1)}) + (\ln(N+1) - \cancel{\ln N})\Big] = \frac{1}{2}(\ln(N+1) - \ln 1)$$

$$\lim_{N\to\infty} S_N = \frac{1}{2}\ln(N+1) = \infty$$

Since the partial sums diverge to infinity, the series

$$\sum_{n=1}^{\infty}\ln\sqrt{\frac{n+1}{n}}$$

$\boxed{\text{diverges.}}$

e. We determine whether the series converges or diverges using the Integral Test. Verify the conditions of the Integral Test.

For $x \geq 3$, the function $f(x) = \dfrac{1}{x\,\ln x\,\ln(\ln x)}$ is positive, continuous, and decreasing. Thus, the Integral Test applies.

Evaluate the corresponding improper integral. Consider

$$\int_3^{\infty}\frac{1}{x\,\ln x\,\ln(\ln x)}\,dx.$$

Use the substitution

$$u = \ln(\ln x), \quad du = \frac{1}{x \ln x}\, dx.$$

For $x = 3,\ u = \ln\ln 3$. For $x \to \infty,\ u \to \infty$.

Then the integral becomes

$$\int_3^\infty \frac{1}{x\ \ln x\ \ln(\ln x)}\, dx = \int_{\ln\ln 3}^\infty \frac{1}{u}\, du.$$

This is an improper, where we need to take the limit.

$$\int_{\ln\ln 3}^\infty \frac{1}{u}\, du = \lim_{R\to\infty} \int_{\ln\ln 3}^R \frac{1}{u}\, du = \lim_{R\to\infty} \ln|u| \Big|_{\ln\ln 3}^R = \lim_{R\to\infty} (\ln|R| - \ln\ln\ln 3) = \infty$$

Since the corresponding improper integral diverges, the series

$$\sum_{n=3}^\infty \frac{1}{n\ \ln n\ \ln(\ln n)}$$

also $\boxed{\text{diverges}}$ by the Integral Test.

f. We determine whether the series converges or diverges using the Limit Comparison Test. Consider the series

$$\sum_{n=1}^\infty \frac{1}{\sqrt{n}},$$

which is a p-series with $p = \frac{1}{2} < 1$ and therefore diverges. Let

$$a_n = \sqrt{\frac{n+1}{n^2+2}}, \qquad b_n = \frac{1}{\sqrt{n}}.$$

Then

$$\frac{a_n}{b_n} = \sqrt{\frac{n+1}{n^2+2}} \cdot \sqrt{n} = \sqrt{\frac{n(n+1)}{n^2+2}}.$$

Divide numerator and denominator inside the square root by n^2:

$$\frac{a_n}{b_n} = \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n^2}}}.$$

Take the limit.

$$\lim_{n\to\infty} \frac{1}{n} = 0, \qquad \lim_{n\to\infty} \frac{1}{n^2} = 0 \qquad \Longrightarrow \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = \sqrt{\frac{1}{1}} = 1$$

Since $0 < \lim\limits_{n\to\infty} \frac{a_n}{b_n} < \infty$ and $\sum b_n$ diverges, the given series also $\boxed{\text{diverges}}$ by the Limit Comparison Test.

Q3. The series is of the form

$$\sum_{n=1}^{\infty} a_n x^n, \qquad a_n = \left(\frac{n}{n+1}\right)^{n^2}.$$

Hence, the center of the power series is 0.

Use the Root Test to find the radius of convergence. Consider

$$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}\,|x| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n} |x|.$$

Calculate $\lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^{n}$.

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{\underbrace{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n}}_{\text{standard limit}}} = \frac{1}{e}$$

We obtain

$$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = \frac{|x|}{e}.$$

By the Root Test, the series converges if

$$\frac{|x|}{e} < 1 \implies |x| < e.$$

Thus, the radius of convergence is $R = e$. Test the endpoints.

Case 1: $x = e$. The series becomes

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} e^n = \sum_{n=1}^{\infty} \left[e\left(\frac{n}{n+1}\right)^{n}\right]^{n}.$$

Now, we need to compute $\lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^{n^2} e^n$ for the Divergence Test. Relate this series to the corresponding function and calculate

$$L = \lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{x^2} e^x.$$

First rewrite the expression using exponentials:

$$\left(\frac{x}{x+1}\right)^{x^2} e^x = e^{\left(x^2 \ln\left(\frac{x}{x+1}\right) + x\right)}.$$

Thus it suffices to compute

$$\lim_{x\to\infty}\left[x^2\ln\left(\frac{x}{x+1}\right)+x\right].$$

Rewrite.

$$\lim_{x\to\infty}\frac{\frac{1}{x}+\ln\left(\frac{x}{1+x}\right)}{1/x^2}.$$

This limit is of the indeterminate form $\frac{0}{0}$, so we apply L'Hôpital's Rule. Differentiate the numerator and denominator.

Numerator:

$$\frac{d}{dx}\left(\frac{1}{x}+\ln\left(\frac{x}{x+1}\right)\right)=-\frac{1}{x^2}+\left(\frac{x+1}{x}\cdot\frac{1\cdot(x+1)-x\cdot(1)}{(x+1)^2}\right).$$

Denominator:

$$\frac{d}{dx}\left(\frac{1}{x^2}\right)=-\frac{2}{x^3}.$$

Thus the limit becomes

$$\lim_{x\to\infty}\frac{-\frac{1}{x^2}+\frac{x+1}{x}\cdot\frac{1}{(x+1)^2}}{-\frac{2}{x^3}}=\lim_{x\to\infty}\frac{-\frac{1}{x^2}+\frac{1}{x}\cdot\frac{1}{x+1}}{-\frac{2}{x^3}}$$

Multiply numerator and denominator by x^3:

$$\lim_{x\to\infty}\frac{-x+\frac{x^2}{x+1}}{-2}=\lim_{x\to\infty}\frac{-x+(x-\frac{x}{x+1})}{-2}=\lim_{x\to\infty}\frac{x}{2(x+1)}=\frac{1}{2}.$$

Hence,

$$\lim_{x\to\infty}\left[x^2\ln\left(\frac{x}{x+1}\right)+x\right]=\frac{1}{2}.$$

Finally,

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{x^2}e^x=\sqrt{e}.$$

Since the limit converges to $\sqrt{e}$, which is different than 0, by the Divergence Test, the series at this point diverges.

Case 2: $x=-e$. The series becomes

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{n+1}\right)^{n^2}e^n=\sum_{n=1}^{\infty}(-1)^n\left[e\left(\frac{n}{n+1}\right)^n\right]^n.$$

From earlier, we found out that $\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{x^2} e^x = \sqrt{e}$. The limit we want to evaluate $\lim_{x\to\infty}(-1)^n\left(\frac{x}{x+1}\right)^{x^2} e^x$ does not exist because the function oscillates between $-\sqrt{e}$ and $\sqrt{e}$ as $x \to \infty$. Therefore, the series at this point is also divergent.

The series converges for

$$|x| < e$$

and diverges at $x = \pm e$.

$$\boxed{\begin{array}{c}\text{Center: } 0,\\ \text{Radius of convergence: } R = e,\\ \text{Interval of convergence: } (-e, e).\end{array}}$$

Q4. We begin with the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots, \qquad |x| < 1.$$

Replace x by $-x$ to obtain

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n, \qquad |x| < 1.$$

Now multiply both sides by x^2:

$$\frac{x^2}{1+x} = x^2\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n x^{n+2}.$$

Therefore, the Maclaurin series for the function is

$$\boxed{\frac{x^2}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^{n+2}, \qquad |x| < 1.}$$