

2012-2013 Fall
MAT123-[Instructor02]-02, [Instructor05]-05 Midterm I
(19/12/2012)
Time: 09:00 - 11:00
Duration: 120 minutes

1. A 10 m long wire is cut into two pieces. One piece is bent into an equilateral triangle and the other piece is bent into a circle. If the sum of the areas enclosed by each part is minimum, what is the length of each piece?

2. Integrate the following functions, and write each step in detail.

$$\begin{array}{lll} \text{(a)} \int \frac{dy}{\sqrt{y}(1+\sqrt{y})^2} & \text{(b)} \int_{\pi/6}^{\pi/4} \frac{\cot x}{\ln(\sin x)} dx & \text{(c)} \int \frac{x^3 - 4x - 10}{x^2 - x - 6} dx \\ \text{(d)} \int \tan x \sec^{123} x dx & \text{(e)} \int \sqrt{t - t^2} dt & \text{(f)} \int \sqrt{1 + \sqrt{x}} dx \end{array}$$

3. Integrate the following functions, and write each step in detail.

$$\text{(a)} \int_0^2 \frac{dx}{\sqrt{|x-1|}} \quad \text{(b)} \int_{-\infty}^{\infty} x^2 e^{-x^3} dx$$

4. By using the Washer Method, find the volume of the solid generated by revolving the region bounded by the curves $x = y^3$ and $y + x^2 = 0$ about the x -axis.

1. Let x be the length of the wire that is used to form the equilateral triangle. So, $10 - x$ is the length of the other piece. The area of the triangle is

$$A_1 = \frac{x^2\sqrt{3}}{4}$$

The perimeter of the circle is $10 - x$. Therefore, $2\pi r = 10 - x$, where r is the radius of the circle. The area of the circle can be extracted from the formula $A_2 = \pi r^2$. Solve the former equation for r , and express the area in terms of x .

$$r = \frac{10 - x}{2\pi} \rightarrow A_2 = \pi \left(\frac{10 - x}{2\pi} \right)^2 = \frac{100 - 20x + x^2}{4\pi}$$

Let $A(x)$ be the function of length representing the sum of the areas.

$$A(x) = \frac{x^2\sqrt{3}}{4} + \frac{100 - 20x + x^2}{4\pi}$$

Minimize $A(x)$ by taking the first derivative and setting it to 0.

$$A'(x) = \frac{x\sqrt{3}}{2} + \frac{x - 10}{2\pi} = 0 \rightarrow 10 = x + x \cdot \pi\sqrt{3} \rightarrow x = \frac{10}{1 + \pi\sqrt{3}}$$

The length of the piece used to form the triangle is $\frac{10}{1 + \pi\sqrt{3}}$, the length of the other piece is $\frac{10\pi\sqrt{3}}{1 + \pi\sqrt{3}}$.

2.

(a) Let $u = 1 + \sqrt{y}$, then $du = \frac{dy}{2\sqrt{y}}$.

$$\int \frac{dy}{\sqrt{y}(1 + \sqrt{y})^2} = \int \frac{1}{u^2} \cdot 2 du = -\frac{2}{u} + c = \boxed{-\frac{2}{1 + \sqrt{y}} + c, c \in \mathbb{R}}$$

(b) Let $u = \ln(\sin x)$, then $du = \frac{1}{\sin x} \cdot \cos x dx$.

$$\int_{\pi/6}^{\pi/4} \frac{\cot x}{\ln(\sin x)} dx = \int \frac{1}{u} du = \ln |u| + c = \left[\ln |\ln(\sin x)| \right]_{\pi/6}^{\pi/4}$$

$$= \ln \left| \ln \frac{\sqrt{2}}{2} \right| - \ln \left| \ln \frac{1}{2} \right| = \ln \frac{\left| \ln \frac{\sqrt{2}}{2} \right|}{\left| \ln \frac{1}{2} \right|}$$

$$= \ln \left(\frac{\ln 2 - \ln \sqrt{2}}{\ln 2 - \ln 1} \right) = \ln \frac{1}{2} = \boxed{-\ln 2}$$

(c) Attempt a long polynomial division and split into two integrals.

$$\begin{aligned} I &= \int \frac{x^3 - 4x - 10}{x^2 - x - 6} dx = \int (x + 1) dx + \int \frac{3x - 4}{(x - 3)(x + 2)} dx \\ &= \frac{x^2}{2} + x + \int \left(\frac{A}{x - 3} + \frac{B}{x + 2} \right) dx \end{aligned}$$

$$\begin{aligned} A(x + 2) + B(x - 3) &= 3x - 4 \\ x(A + B) + 2A - 3B &= 3x - 4 \end{aligned}$$

$$\left. \begin{aligned} A + B &= 3 \\ 2A - 3B &= -4 \end{aligned} \right\} \quad A = 1, \quad B = 2$$

$$I = \frac{x^2}{2} + x + \int \left(\frac{1}{x - 3} + \frac{2}{x + 2} \right) dx = \boxed{\frac{x^2}{2} + x + \ln |x - 3| + 2 \ln |x + 2| + c, \quad c \in \mathbb{R}}$$

(d) Let $u = \sec x$, then $du = \tan x \sec x dx$.

$$\int \tan x \sec^{123} x dx = \int u^{122} du = \frac{u^{123}}{123} + c = \boxed{\frac{\sec^{123} x}{123} + c, \quad c \in \mathbb{R}}$$

(e) Let $t = \sin^2 x$, then $dt = 2 \sin x \cos x dx$.

$$\begin{aligned} \int \sqrt{t - t^2} dt &= \int \sqrt{\sin^2 x - \sin^4 x} \cdot 2 \sin x \cos x dx = \int \sqrt{\sin^2 x \cdot (1 - \sin^2 x)} \cdot \sin(2x) dx \\ &= \int \sin x \cos x \cdot \sin(2x) dx = \frac{1}{2} \int \sin^2(2x) dx = \frac{1}{2} \int (1 - \cos^2(2x)) dx \\ &= \frac{1}{2} \int \frac{1 - \cos(4x)}{2} dx = \frac{1}{4} \left[x - \frac{1}{4} \sin(4x) \right] + c, \quad c \in \mathbb{R} \end{aligned}$$

We will try to rewrite the result in terms of t .

$$\begin{aligned} t - t^2 \geq 0 &\implies 0 \leq t \leq 1 \\ t = \sin^2 x &\implies \sqrt{t} = \sin x \implies \arcsin(\sqrt{t}) = x \\ t = \sin^2 x = 1 - \cos^2 x &\implies \cos x = \sqrt{1 - t} \\ \sin(4x) = 2 \sin(2x) \cos(2x) &= 4 \sin x \cos x (2 \cos^2 x - 1) = 4\sqrt{t} \cdot \sqrt{1 - t} \cdot (1 - 2t) \end{aligned}$$

$$\int \sqrt{t - t^2} dt = \boxed{\frac{1}{4} \left[\arcsin(\sqrt{t}) - \sqrt{t(1 - t)} \cdot (1 - 2t) \right] + c, \quad c \in \mathbb{R}}$$

(f) Let $u = \sqrt{1 + \sqrt{x}}$.

$$u^2 = 1 + \sqrt{x} \implies u^2 - 1 = \sqrt{x} \implies (u^2 - 1)^2 = x \implies 2(u^2 - 1) \cdot 2u \, du = dx$$

$$\int \sqrt{1 + \sqrt{x}} \, dx = \int u \cdot (2u^2 - 2) \cdot 2u \, du = \int (4u^4 - 4u^2) \, du = \frac{4u^5}{5} - \frac{4u^3}{3} + c$$

$$= \boxed{\frac{4 \left(\sqrt{1 + \sqrt{x}} \right)^5}{5} - \frac{4 \left(\sqrt{1 + \sqrt{x}} \right)^3}{3} + c, c \in \mathbb{R}}$$

3.

(a)

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{S \rightarrow 1^-} \int_0^S \frac{dx}{\sqrt{1-x}} + \lim_{P \rightarrow 1^+} \int_P^2 \frac{dx}{\sqrt{x-1}}$$

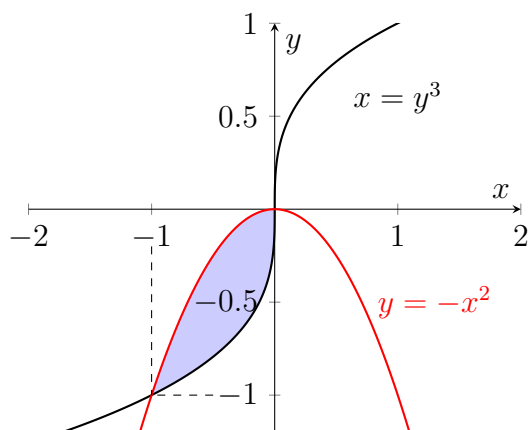
$$= \lim_{S \rightarrow 1^-} [-2\sqrt{1-x}]_0^S + \lim_{P \rightarrow 1^+} [2\sqrt{x-1}]_P^2$$

$$= \lim_{S \rightarrow 1^-} [-2\sqrt{1-S}] + 2\sqrt{1-0} + 2\sqrt{2-1} - \lim_{P \rightarrow 1^+} [2\sqrt{P-1}] = \boxed{4}$$

(b)

$$\int_{-\infty}^{\infty} x^2 e^{-x^3} \, dx = \lim_{a \rightarrow \infty} \int_{-a}^a x^2 e^{-x^3} \, dx = \lim_{a \rightarrow \infty} -\frac{1}{3} [e^{-x^3}]_{-a}^a = \lim_{a \rightarrow \infty} -\frac{1}{3} [e^{-a^3} - e^{a^3}] = \boxed{\infty}$$

4)



$$V = \int_{-1}^0 \pi \left[(x^{1/3})^2 - (x^2)^2 \right] \, dx$$

$$= \pi \int_{-1}^0 (x^{2/3} - x^4) \, dx$$

$$= \pi \left[\frac{3}{5} x^{5/3} - \frac{x^5}{5} \right]_{-1}^0$$

$$= \pi \cdot \left[0 - \left(-\frac{3}{5} + \frac{1}{5} \right) \right] = \boxed{\frac{2\pi}{5}}$$