

2011-2012 Spring
MAT123-[Instructor] Midterm I
(06/04/2012)
Time: 15:00 - 16:45
Duration: 105 minutes

1. Find the equation of the tangent line to the curve $x^2 + 2xy + y^2 = 4$ at the point $(3, -1)$.
2. Find the point on the parabola $y = \sqrt{x}$ which is the closest to the point $(2, 0)$.
3. Evaluate the limit, if it exists, and explain your answer. Do not use L'Hôpital's rule.

(a) $\lim_{x \rightarrow 0} \sqrt{x^2 + 2x^3} \sin\left(\frac{1}{x}\right)$ (b) $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - x - 6}$ (c) $\lim_{x \rightarrow -\infty} \sqrt{x^2 - 4x} + x$
(d) $\lim_{h \rightarrow 0} \frac{(1+h)^{123} - 1}{h}$

4. Find the derivatives of the following functions.

(a) $f(x) = x^{\cos(x^3)}$ (b) $f(x) = \tan(e^{2x} \sin(3x))$

(c) Find the second derivative $f''(x)$ of $f(x) = \ln\left(\frac{x^2}{x^2 + 4}\right)$.

5. For the function $f(x) = \frac{1}{x^2 - 4}$,

- (a) Find the vertical and horizontal asymptotes.
- (b) Find the intervals of increase or decrease.
- (c) Find the local maximum and minimum values, if any.
- (d) Find the intervals of concavity and the inflection points, if any.
- (e) Sketch the graph of f .

2011-2012 Spring Midterm I (06/04/2012) Solutions
(Last update: 8/4/25 (4th of August) 3:11 PM)

1. y is implicitly defined as a function of x . Differentiate each side.

$$\begin{aligned}\frac{d}{dx}(x^2 + 2xy + y^2) &= \frac{d}{dx}(4) \\ 2x + 2y \cdot 1 + 2x \cdot \frac{dy}{dx} + 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx}(2x + 2y) &= -2x - 2y \\ \frac{dy}{dx} &= -1\end{aligned}$$

Recall the equation of a straight line: $y - y_0 = m(x - x_0)$, where m is simply $\frac{dy}{dx}$. Therefore, the tangent line at $(3, -1)$ is as follows.

$$\boxed{y + 1 = -1(x - 3)}$$

Furthermore, since $\frac{dy}{dx} = -1$, the tangent line consists of every point of the upper line. In other words, the tangent line is the upper line itself. The equation $x^2 + 2xy + y^2 = 4$ forms two parallel straight lines in the xy -coordinate system.

2. Let (x, \sqrt{x}) be a point on this parabola. The distance between the points can be expressed using the Pythagorean theorem as follows.

$$f(x) = L^2 = (2 - x)^2 + (\sqrt{x} - 0)^2$$

Take the derivative of both sides and set $f'(x) = \frac{dL}{dx} = 0$ to find the critical points.

$$f'(x) = 2L \frac{dL}{dx} = 2(2 - x) \cdot (-1) + 1 = 2x - 3 = 0 \implies x = \frac{3}{2}$$

Now, verify whether this is a local minimum by taking the second derivative.

$$f''(x) = (2x - 3)' = 2 > 0$$

Since this is a local minimum of f , the distance is closest at $x = \frac{3}{2}$. The point we're looking for is

$$\boxed{\left(\frac{3}{2}, \sqrt{\frac{3}{2}}\right)}$$

3.

(a) The inequality $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ holds for all $x \in \mathbb{R}$ except $x = 0$. For small x , we can multiply each side of the inequality by $\sqrt{x^2 + 2x^3}$. Using the squeeze theorem, the limit is equal to 0.

$$\begin{aligned} -1 &\leq \sin\left(\frac{1}{x}\right) \leq 1 \\ -\sqrt{x^2 + 2x^3} &\leq \sqrt{x^2 + 2x^3} \sin\left(\frac{1}{x}\right) \leq \sqrt{x^2 + 2x^3} \\ \lim_{x \rightarrow 0} -\sqrt{x^2 + 2x^3} &= \lim_{x \rightarrow 0} \sqrt{x^2 + 2x^3} = 0 \implies \lim_{x \rightarrow 0} \sqrt{x^2 + 2x^3} \sin\left(\frac{1}{x}\right) = \boxed{0} \end{aligned}$$

(b) Factorize each side of the fraction and eliminate like terms.

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - x - 6} = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)(x+2)} = \lim_{x \rightarrow 3} \frac{x+3}{x+2} = \boxed{\frac{6}{5}}$$

(c) The expression is in the form $\infty - \infty$. Expand the expression by multiplying by its conjugate to eliminate the indetermination.

$$\begin{aligned} \lim_{x \rightarrow -\infty} \sqrt{x^2 - 4x} + x &= \lim_{x \rightarrow -\infty} \left[\left(\sqrt{x^2 - 4x} + x \right) \cdot \frac{\sqrt{x^2 - 4x} - x}{\sqrt{x^2 - 4x} - x} \right] = \lim_{x \rightarrow -\infty} \frac{x^2 - 4x - x^2}{\sqrt{x^2 - 4x} - x} \\ &= \lim_{x \rightarrow -\infty} \frac{-4x}{\sqrt{x^2 - 4x} - x} = \lim_{x \rightarrow -\infty} \frac{-4x}{|x| \sqrt{1 - \frac{4}{x}} - x} \\ &= \lim_{x \rightarrow -\infty} \frac{-4}{-\sqrt{1 - \frac{4}{x}} - 1} = \lim_{x \rightarrow -\infty} \frac{4}{2} = \boxed{2} \end{aligned}$$

(d) Recall the definition of the derivative of a function at a point. Let f be a differentiable function, then

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

If we set $f(x) = x^{123}$, then we can differentiate f at $x = 1$.

$$\lim_{h \rightarrow 0} \frac{(1+h)^{123} - 1}{h} = f'(1) = 123 \cdot (1)^{122} = \boxed{123}$$

4.

(a) Take the logarithm of each side to compute the derivative easily.

$$\begin{aligned}
 f(x) &= x^{\cos(x^3)} \\
 \ln(f(x)) &= \ln \left[x^{\cos(x^3)} \right] = \cos(x^3) \cdot \ln x \\
 \frac{d}{dx} [\ln(f(x))] &= \frac{d}{dx} [\cos(x^3) \cdot \ln x] \\
 \frac{1}{f(x)} \cdot f'(x) &= -\sin(x^3) \cdot 3x^2 \cdot \ln x + \cos(x^3) \cdot \frac{1}{x} \\
 \boxed{f'(x) &= x^{\cos(x^3)} \cdot \left[-\sin(x^3) \cdot 3x^2 \cdot \ln x + \cos(x^3) \cdot \frac{1}{x} \right]}
 \end{aligned}$$

(b) Apply the chain rule accordingly.

$$\begin{aligned}
 f(x) &= \tan(e^{2x} \sin(3x)) \\
 \boxed{f'(x) &= \sec^2(e^{2x} \sin(3x)) \cdot [e^{2x} \cdot 2 \cdot \sin(3x) + e^{2x} \cdot \cos(3x) \cdot 3]}
 \end{aligned}$$

(c) Compute the first and second derivatives, respectively, applying the chain rule and the quotient rule accordingly.

$$\begin{aligned}
 f(x) &= \ln \left(\frac{x^2}{x^2 + 4} \right) \\
 f'(x) &= \frac{x^2 + 4}{x^2} \cdot \frac{2x \cdot (x^2 + 4) - x^2 \cdot 2x}{(x^2 + 4)^2} = \frac{8}{x^3 + 4x} \\
 f''(x) &= -\frac{8}{(x^3 + 4x)^2} \cdot (3x^2 + 4) = \boxed{-\frac{24x^2 + 32}{(x^3 + 4x)^2}}
 \end{aligned}$$

5)

(a) Find the horizontal asymptotes.

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^2 - 4} = 0$$

Find the vertical asymptotes. The expression is undefined for $x = \pm 2$.

$$\begin{aligned}
 \lim_{x \rightarrow 2^+} \frac{1}{x^2 - 4} &= \lim_{x \rightarrow -2^-} \frac{1}{x^2 - 4} = \infty \\
 \lim_{x \rightarrow 2^-} \frac{1}{x^2 - 4} &= \lim_{x \rightarrow -2^+} \frac{1}{x^2 - 4} = -\infty
 \end{aligned}$$

The horizontal asymptote is $y = 0$. The vertical asymptotes are $x = \pm 2$.

(b) Compute the first derivative and set it to 0 to find the critical points.

$$f'(x) = -\frac{1}{(x^2 - 4)^2} \cdot 2x$$

f is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$.

f is decreasing for $x > 0$, increasing for $x < 0$.

(c) The *only* critical point occurs at $x = 0$. Compute the second derivative to check whether this is a local minimum or local maximum.

$$f''(x) = -\frac{2 \cdot (x^2 - 4)^2 - 2x \cdot 2 \cdot (x^2 - 4) \cdot (2x)}{(x^2 - 4)^4} = \frac{8 + 6x^2}{(x^2 - 4)^3}$$

$$f''(0) = \frac{8 + 6 \cdot 0^2}{(0^2 - 4)^3} = -\frac{1}{8} < 0. \text{ Therefore, } (0, f(0)) \text{ is a local maximum.}$$

No local minimums exist.

The *only* local maximum occurs at $x = 0$, which is $\left(0, -\frac{1}{4}\right)$.

(d) $8 + 6x^2 \geq 0$. Therefore, no inflection points. f is concave up if $f''(x) > 0$, concave down if $f''(x) < 0$.

No inflection points exist.

f is concave up for $x > 2$ and $x < -2$. f is concave down for $|x| < 2$.

(e)

