2020-2021 Spring
MAT124 Final
(09/06/2021)

1. Sketch the region corresponding to the double integral

$$\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$$

and reverse the order of integration.

2. Find the volume of the solid bounded above by the cone $z=\sqrt{x^2+y^2}$ and below by the circular region $x^2+y^2\leq 4$ in the xy-plane.

3. Let R be the region lying outside $r=1$ and inside $r=1+\cos\theta$.

(i) Sketch the graph of the region R.

(ii) Set up (but do not evaluate) a double integral in polar coordinates for the area of the region R.

4. Let S be the portion of the surface $z=y^2$ that lies over the rectangular region in the xy-plane with the vertices $(0,0,0),(0,1,0),(1,1,0)$ and $(1,0,0)$.

(i) Sketch the graph of S.

(ii) Evaluate the surface area.

5. Evaluate $\iiint_E z\,dV$, where E is the region bounded below by $x^2+y^2+z^2=4$ and above by $x^2+y^2+z^2=9$.

6. Let S be the region bounded above by the paraboloid $z=1-x^2-y^2$ and below by the xy-plane.

(i) Using the spherical coordinates, set up (but do not evaluate) an integral for the volume of the solid S.

(ii) Using the cylindrical coordinates, set up (but do not evaluate) an integral for the volume of the solid S.

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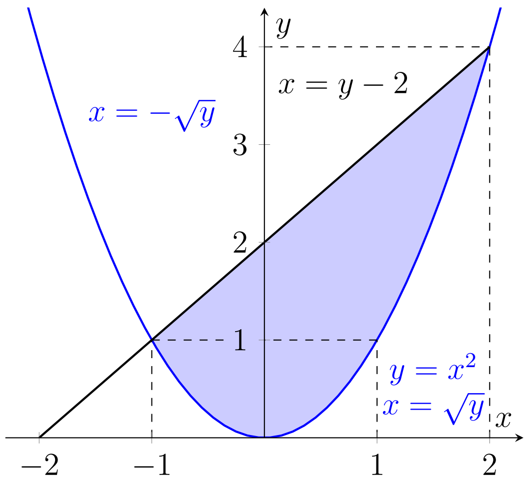
1.



$$\boxed{\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\,dy}$$

2. Using polar coordinates, we can find the volume with a double integral.

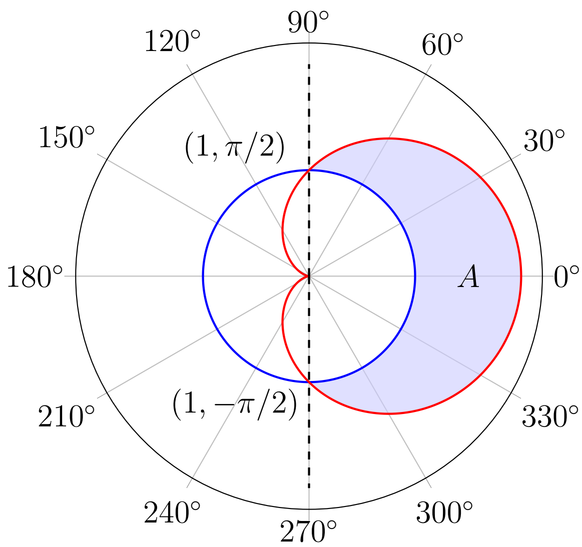
$$\begin{matrix} r^2 = x^2 + y^2 \\ dA = r\,dr\,d\theta \end{matrix} \quad \rightarrow \quad \begin{matrix} z = \sqrt{x^2+y^2} \implies z = \sqrt{r^2} \implies z = r \\ x^2 + y^2 \le 4 \rightarrow r^2 \le 4 \implies 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$\text{Volume} = \int_0^{2\pi} \int_0^2 [r - 0] \cdot r\,dr\,d\theta = \int_0^{2\pi} \int_0^2 r^2\,dr\,d\theta = \int_0^{2\pi} \left[\frac{r^3}{3}\right]_{r=0}^{r=2} d\theta$$

$$= \frac{8}{3} \int_0^{2\pi} d\theta = \boxed{\frac{16\pi}{3}}$$

3.

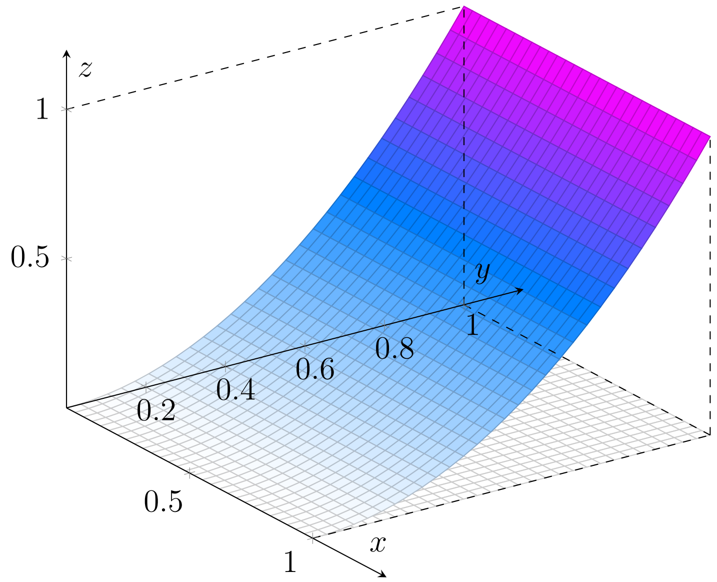
(i)



(ii) $\boxed{A = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} r\,dr\,d\theta}$

4.

(i)



(ii) Calculate the partial derivatives to find the surface area.

$$z = y^2 \implies \quad \frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 2y$$

$$\text{Surface area} = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int_0^1 \int_0^1 \sqrt{1 + (0)^2 + (2y)^2}\, dx\, dy$$

$$= \int_0^1 \int_0^1 \sqrt{4y^2+1}\, dx\, dy = \int_0^1 \left[x\sqrt{4y^2+1}\right]_{x=0}^{x=1} dy$$

$$= \int_0^1 \sqrt{4y^2+1}\, dy \quad \left[y = \frac{1}{2}\tan u \implies dy = \frac{1}{2}\sec^2 u\, du, \quad \begin{matrix} u_{\text{upper}} = \arctan 2 \\ u_{\text{lower}} = 0 \end{matrix}\right]$$

$$= \frac{1}{2}\int_0^{\arctan 2} \sqrt{\tan^2 u + 1} \cdot \sec^2 u\, du = \frac{1}{2}\int_0^{\arctan 2} \sec^3 u\, du$$

To evaluate the last integral, we will use integration by parts.

$$\begin{aligned} w = \sec u \;&\rightarrow\; dw = \sec u \tan u\, du \\ dz = \sec^2 u\, du \;&\rightarrow\; z = \tan u \end{aligned}$$

$$\int_0^{\arctan 2} \sec^3 u \, du = \tan u \cdot \sec u \Big|_0^{\arctan 2} - \int_0^{\arctan 2} \tan^2 u \sec u \, du$$

$$= \tan u \cdot \sec u \Big|_0^{\arctan 2} - \int_0^{\arctan 2} \frac{1 - \cos^2 u}{\cos^3 u} \, du$$

$$= \tan u \cdot \sec u \Big|_0^{\arctan 2} - \int_0^{\arctan 2} \sec^3 u \, du + \int_0^{\arctan 2} \sec u \, du$$

Notice that the integral we want to evaluate appears on the right side. After a little algebra, we can evaluate the integral.

$$\int_0^{\arctan 2} \sec^3 u \, du = \frac{1}{2} \cdot \tan u \cdot \sec u \Big|_0^{\arctan 2} + \frac{1}{2} \cdot \int_0^{\arctan 2} \sec u \, du$$

The integral of $\sec u$ with respect to u is as follows. One can derive it with particular methods.

$$\int_0^{\arctan 2} \sec u \, du = \ln |\tan u + \sec u| \Big|_0^{\arctan 2}$$

So, the surface area becomes as follows.

$$\text{Surface area} = \frac{1}{2} \int_0^{\arctan 2} \sec^3 u \, du = \frac{1}{4} \left(\tan u \cdot \sec u + \ln |\tan u + \sec u| \right) \Big|_0^{\arctan 2}$$

$$= \frac{1}{4} \left[2 \sec(\arctan 2) + \ln(2 + \sec(\arctan 2)) - 0 \right] = \boxed{\frac{1}{4} \left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right) \right]}$$

5. By means of spherical coordinates, we can easily evaluate the integral. For spherical coordinates, we have

$$\begin{array}{c} z = \rho \cos \theta \\ r = \rho \sin \theta \\ x^2 + y^2 + z^2 = \rho^2 \\ dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} x^2 + y^2 + z^2 = 4 \rightarrow \rho^2 = 4 \implies \rho_{\min} = 2 \\ x^2 + y^2 + z^2 = 9 \rightarrow \rho^2 = 9 \implies \rho_{\max} = 3 \\ z \equiv \rho \cos \phi \\ 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi \end{array}$$

$$\iiint_E z \, dV = \int_0^{2\pi} \int_0^{\pi} \int_2^3 \rho \cos \phi \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} \int_2^3 \rho^3 \sin(2\phi) \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^4}{4} \right]_{\rho=2}^{\rho=3} \sin(2\phi) \, d\phi \, d\theta = \frac{65}{8} \int_0^{2\pi} \int_0^{\pi} \sin(2\phi) \, d\phi \, d\theta$$

$$= \frac{65}{8} \int_0^{2\pi} \left[-\frac{1}{2} \cos(2\phi) \right]_{\phi=0}^{\phi=\pi} d\theta = \frac{65}{8} \int_0^{2\pi} 0 \, d\theta = \boxed{0}$$

6.

(i) For spherical coordinates, we have

$$\begin{array}{c} z = \rho\cos\phi \\ r = \rho\sin\phi \\ x^2+y^2+z^2=\rho^2 \\ dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} z = 1-x^2-y^2 \implies \rho\cos\phi = 1-\rho^2\sin^2\phi \quad (1) \\ 0 \le \phi \le \dfrac{\pi}{2}, \quad 0 \le \theta \le 2\pi, \quad 0 \le \rho \le \rho_{\text{upper}} \end{array}$$

Find ρ_{upper} with equation (1).

$$\rho\cos\phi = 1-\rho^2\sin^2\phi \implies \rho^2\sin^2\phi + \rho\cos\phi - 1 = 0$$

$$\rho_{1,2} = \frac{-\cos\phi \pm \sqrt{\cos^2\phi - 4\cdot\sin^2\phi\cdot(-1)}}{2\sin^2\phi} \qquad \left[x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$$

$$\rho > 0 \implies \rho_{\text{upper}} = \frac{-\cos\phi + \sqrt{\cos^2\phi + 4\sin^2\phi}}{2\sin^2\phi}$$

$$\boxed{\text{Volume} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\frac{-\cos\phi+\sqrt{\cos^2\phi+4\sin^2\phi}}{2\sin^2\phi}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

(ii) For cylindrical coordinates, we have

$$\begin{array}{c} z = z \\ r^2 = x^2+y^2 \\ dV = r\, dz\, dr\, d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} z = 1-x^2-y^2 \implies z = 1-r^2 \\ 0 \le \theta \le 2\pi, \quad 0 \le r \le 1 \end{array}$$

$$\boxed{\text{Volume} = \int_0^{2\pi}\int_0^{1}\int_0^{1-r^2} r\, dz\, dr\, d\theta}$$