2023-2024 Spring
MAT124 Midterm
(26/04/2024)

1. Sketch the traces of the following surfaces with the coordinate planes $x = 0$, $y = 0$, and $z = 0$, and then sketch the graphs of them.

(a) $y = \ln z$ (b) $x^2 + 2y^2 - 3z^2 + 1 = 0$

2. State the $\epsilon - \delta$ definition of the limit of a function of two variables, and using the $\epsilon - \delta$ definition, show that

$$\lim_{(x,y)\to(0,0)} \left(2x^2 + y^2\right) = 0$$

3. For $t \in \mathbb{R}$, let l_1 and l_2 be two lines given by

$$\begin{aligned} x &= 3t \\ y &= 4 - t \quad t \in \mathbb{R} \\ z &= 1 + 2t \end{aligned} \qquad\qquad \begin{aligned} x &= 1 + 6\lambda \\ y &= 2 - 2\lambda \quad \lambda \in \mathbb{R} \\ z &= 1 + 4\lambda \end{aligned}$$

Determine an equation of the plane containing both l_1 and l_2.

4. Let

$$f(x,y) = \begin{cases} \dfrac{x^2 \cdot e^{x^2+y}}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(i) Evaluate the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

(ii) Show that $f(x,y)$ is not differentiable at $(0,0)$.

5. Let $u = u(x,y)$ and $v = v(x,y)$. The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Show that in the polar coordinate system ($x = r\cos\theta,\ y = r\sin\theta$),

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

6. Determine the direction at the point $(1,1)$ in which the rate of change of

$$f(x,y) = \frac{2^{xy}}{x}$$

is the largest. Compute this rate of change.

7. Find all critical points of the function

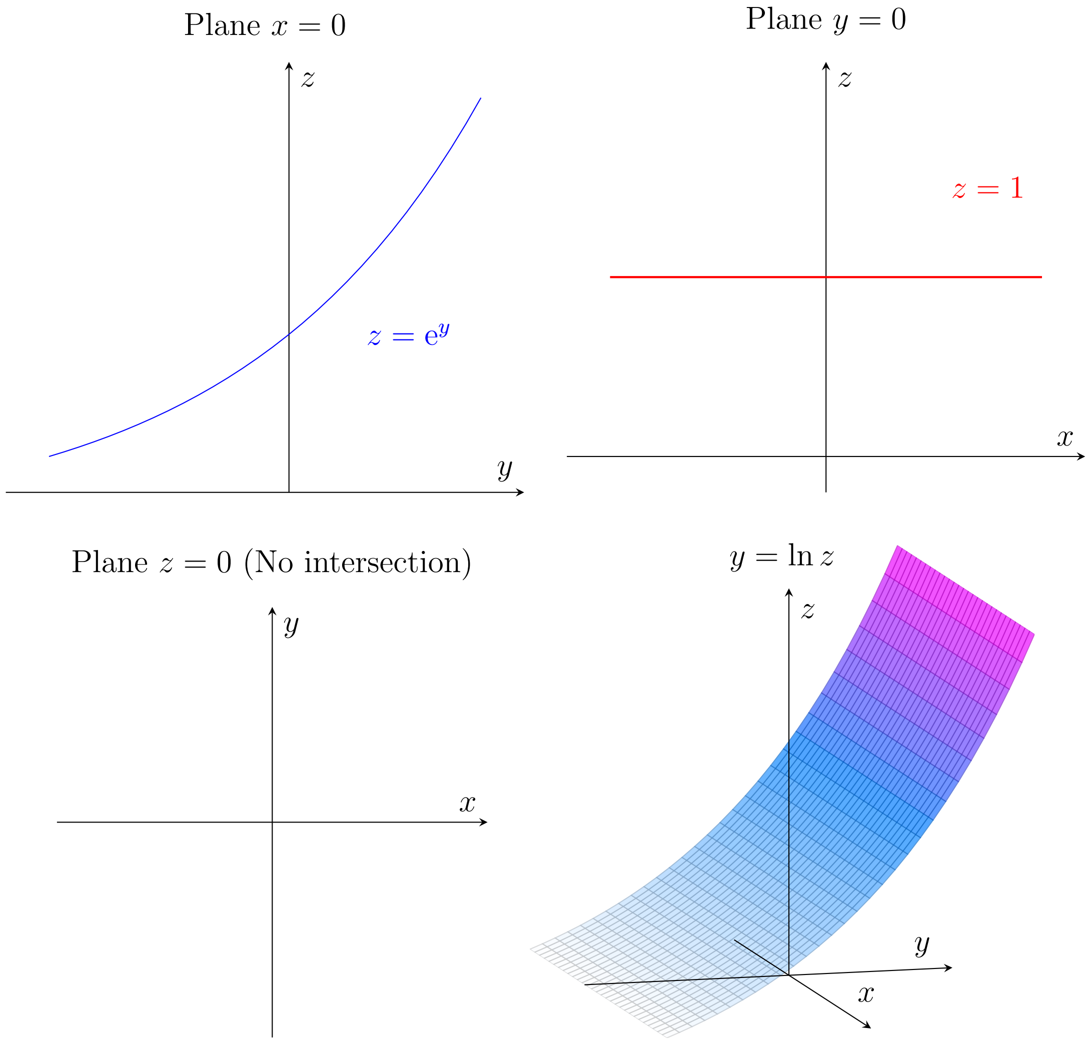
$$f(x,y) = (x-1)(y+1)(x-y+3)$$

and then determine whether each critical point corresponds to a local maximum, a local minimum or a saddle point.

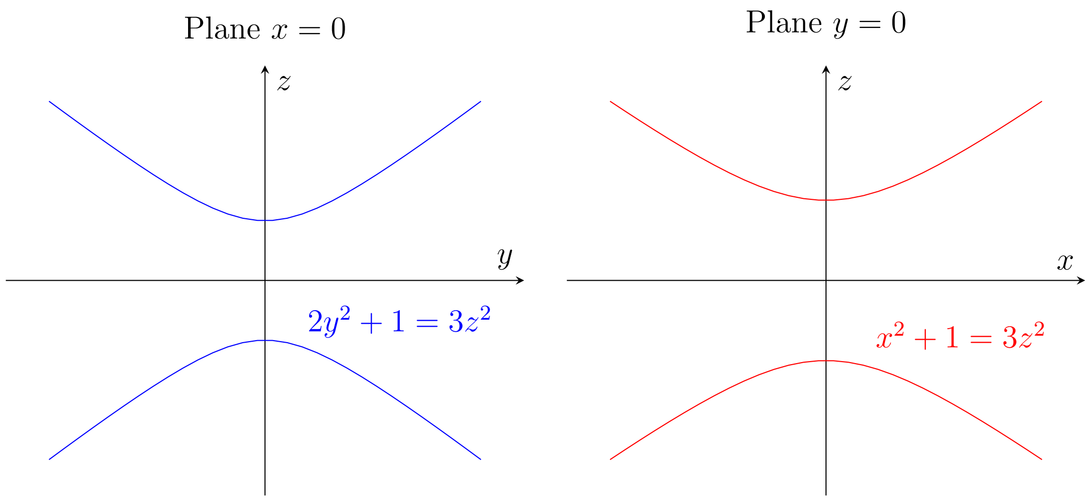
2023-2024 Spring Midterm (26/04/2024) Solutions
(Last update: 8/27/25 (27th of August) 11:24 PM)

1.

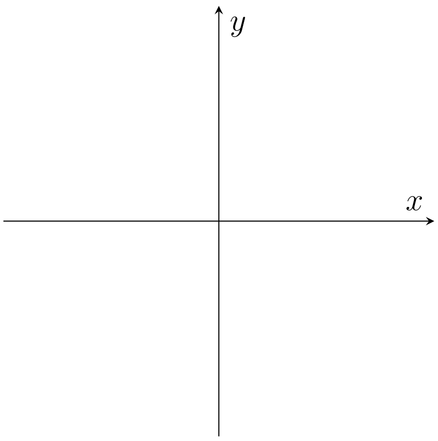
(a)



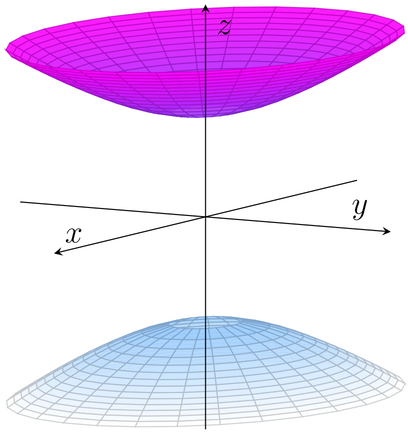
(b)



Plane $z = 0$ (No intersection)



$x^2 + 2y^2 - 3z^2 + 1 = 0$



2. The $\epsilon - \delta$ definition is the formal way of calculating the limit of a function at a point (x_0, y_0). According to the definition, for every $\epsilon > 0$, there exists a δ such that

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \implies |f(x, y) - L| < \epsilon,$$

where L is the limit.

Then for every $\epsilon > 0$, there exists a δ such that

$$0 < \sqrt{x^2 + y^2} < \delta \implies \left|2x^2 + y^2\right| < \epsilon.$$

$$\left|2x^2 + y^2\right| \leq 2x^2 + y^2 \qquad \left[x^2 \geq 0, \quad y^2 \geq 0\right]$$

$$2x^2 + y^2 \leq 2x^2 + 2y^2 = 2\left(x^2 + y^2\right) \leq 2\delta^2 \qquad \left[\sqrt{x^2 + y^2} < \delta \implies x^2 + y^2 < \delta^2\right]$$

Let $\delta = \sqrt{\dfrac{\epsilon}{2}}$.

$$\left|2x^2 + y^2\right| \leq 2\left(x^2 + y^2\right) < 2 \cdot \left(\sqrt{\frac{\epsilon}{2}}\right)^2 = 2 \cdot \frac{\epsilon}{2} = \epsilon$$

This shows that the limit is equal to 0.

3. The coefficients of the parametrization variables have the same ratio. That is, they are parallel. Choose $P_1(3, 3, 3)$ on l_1 and $P_2(1, 2, 1)$ on l_2. The cross product of $\mathbf{u} = \langle 3, -1, 2\rangle$, which is parallel to the lines, and the vector $\mathbf{v}$ joining P_1 and P_2, where $\mathbf{w} = \langle 3 - 1,\, 3 - 2,\, 3 - 1\rangle = \langle 2, 1, 2\rangle$, gives us the normal of the plane.

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= (-1 \cdot 2 - 1 \cdot 2)\mathbf{i} - (3 \cdot 2 - 2 \cdot 2)\mathbf{j} + (3 \cdot 1 - 2 \cdot (-1))\mathbf{k} = -4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

Using the definition $\mathbf{n} \cdot \overrightarrow{PP_1} = 0$, where P is a point on the plane, the equation of the plane is

$$\mathbf{n} \cdot \overrightarrow{PP_1} = 0 \implies \langle -4, -2, 5 \rangle \cdot \langle x-3,\ y-3,\ z-3 \rangle = 0$$

$$\implies -4(x-3) - 2(y-3) + 5(z-3) = 0 \implies \boxed{-4x - 2y + 5z + 3 = 0}$$

4.

(i) Use the definition of the partial derivative.

$$f_x|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\dfrac{(0+h)^2 \cdot \mathrm{e}^{(0+h)^2+0}}{(0+h)^2 + 0^2}}{h} = \lim_{h\to 0} \frac{\mathrm{e}^{h^2}}{h}$$

$$= \infty \ (f_x \text{ does not exist})$$

$$f_y|_{(0,0)} = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\dfrac{0^2 \cdot \mathrm{e}^{0^2+(0+h)}}{0^2 + (0+h)^2}}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

(ii) f_y is a finite number. However, since f_x does not exist, f is not differentiable at $(0,0)$.

5. u and v are functions of x and y. x and y are functions of r and θ. Use the chain rule.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = u_x \cdot \cos\theta + u_y \cdot \sin\theta = v_y \cos\theta - v_x \sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta} \implies \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}(-v_x \cdot r\sin\theta + v_y \cdot r\cos\theta) = -v_x \sin\theta + v_y \cos\theta$$

$$\frac{\partial u}{\partial r} = -v_x \sin\theta + v_y \cos\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r} = v_x \cdot \cos\theta + v_y \cdot \sin\theta = -u_y \cos\theta + u_x \sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} \implies -\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{1}{r}(-u_x r\sin\theta + u_y r\cos\theta) = u_x \sin\theta - u_y \cos\theta$$

$$\frac{\partial v}{\partial r} = u_x \sin\theta - u_y \cos\theta = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

6. The function f has the maximum rate of change if the gradient vector of f and the unit direction vector $\mathbf{u}$ are in the same direction. Apply the quotient rule to compute the gradient of f.

$$\nabla f = \left\langle \frac{(2^{xy} \cdot y \cdot \ln 2) \cdot x - 2^{xy} \cdot (1)}{x^2},\ 2^{xy} \cdot \ln 2 \right\rangle = \left\langle \frac{2^{xy}\,(yx \ln 2 - 1)}{x^2},\ 2^{xy} \cdot \ln 2 \right\rangle$$

$$(\nabla f \cdot \mathbf{u})_{\text{max}} = |\nabla f|\,|u| \cos 0 = |\nabla f|$$

$$\nabla f|_{(1,1)} = \langle 2\ln 2 - 2,\ 2\ln 2 \rangle \implies |\nabla f| = \sqrt{(2\ln 2 - 2)^2 + (2\ln 2)^2}$$

$$= 2\sqrt{2\ln^2 2 - 2\ln 2 + 1}$$

The unit direction vector u is

$$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{\langle 2\ln 2 - 2,\ 2\ln 2 \rangle}{2\sqrt{2\ln^2 2 - 2\ln 2 + 1}} = \left\langle \frac{\ln 2 - 1}{\sqrt{2\ln^2 2 - 2\ln 2 + 1}}, \frac{\ln 2}{\sqrt{2\ln^2 2 - 2\ln 2 + 1}} \right\rangle$$

$$\boxed{\begin{array}{c} \text{The maximum rate of change: } 2\sqrt{2\ln^2 2 - 2\ln 2 + 1} \\[1em] \text{The direction vector: } \left\langle \dfrac{\ln 2 - 1}{\sqrt{2\ln^2 2 - 2\ln 2 + 1}}, \dfrac{\ln 2}{\sqrt{2\ln^2 2 - 2\ln 2 + 1}} \right\rangle \end{array}}$$

7. Apply the chain rule.

$$f_x = (y+1)\,[1 \cdot (x - y + 3) + (x - 1) \cdot 1] = (y+1)(2x - y + 2)$$

$$= 2xy - y^2 + 2y + 2x - y + 2 = -y^2 + y + 2xy + 2x + 2$$

$$f_y = (x-1)\,[1 \cdot (x - y + 3) + (y + 1) \cdot (-1)] = (x-1)(x - 2y + 2)$$

$$= x^2 - 2xy + 2x - x + 2y - 2 = x^2 - 2xy + x + 2y - 2$$

The critical points occur where one of the partial derivatives does not exist or $f_x = f_y = 0$. f_x and f_y are continuous everywhere. Therefore, we may simply determine where $f_x = f_y = 0$.

$$f_x = f_y = 0 \implies \left.\begin{array}{r} 2xy - y^2 + 2y + 2x - y + 2 = 0 \quad (1) \\ x^2 - 2xy + x + 2y - 2 = 0 \end{array}\right\} \quad x^2 - y^2 + 3(x+y) = 0$$

$$\implies (x-y)(x+y) + 3(x+y) = 0 \implies (x - y + 3)(x + y) = 0$$

We have two cases: $x + 3 = y$ or $x = -y$.

$$\text{Case I}: x=-y \overset{(1)}{\Longrightarrow} -y^2-2y^2+y-2y+2=0 \Longrightarrow -3y^2-y+2=0$$

$$\Longrightarrow (2-3y)(y+1)=0 \Longrightarrow y_1=\frac{2}{3},\ y_2=-1$$

$$y_1=\frac{2}{3} \Longrightarrow x_1=-\frac{2}{3} \qquad y_2=-1 \Longrightarrow x_2=1$$

$$\text{Case II}: x+3=y \overset{(1)}{\Longrightarrow} -(x+3)^2+2x(x+3)+x+3+2x+2=0$$

$$\Longrightarrow -x^2-6x-9+2x^2+6x+3x+5=0 \Longrightarrow x^2+3x-4=0$$

$$x_{3,4}=\frac{-3\pm\sqrt{9-4\cdot1\cdot(-4)}}{2\cdot1}=\frac{-3\pm5}{2} \Longrightarrow x_3=-4,\ x_4=1 \Longrightarrow y_3=-1,\ y_4=4$$

To classify these four critical points, apply the Second Derivative Test.

$$f_{xx}=2y+2, \quad f_{xy}=f_{yx}=2x-2y+1, \quad f_{yy}=-2x+2$$

$$\left(-\frac{2}{3},\frac{2}{3}\right) \to \begin{cases} f_{xx}=\dfrac{10}{3}, \quad f_{xy}=-\dfrac{5}{3}, \quad f_{yy}=\dfrac{10}{3} \\ \begin{vmatrix} \dfrac{10}{3} & -\dfrac{10}{3} \\ -\dfrac{5}{3} & \dfrac{10}{3} \end{vmatrix} = \dfrac{10}{3}\cdot\dfrac{10}{3}-\left(-\dfrac{5}{3}\right)\cdot\left(-\dfrac{5}{3}\right)=\dfrac{25}{3}, \quad f_{xx}=\dfrac{10}{3}>0 \end{cases}$$

$$(1,-1) \to \begin{cases} f_{xx}=0, \quad f_{xy}=5, \quad f_{yy}=0 \\ \begin{vmatrix} 0 & 5 \\ 5 & 0 \end{vmatrix} = 0\cdot0-5\cdot5=-25<0 \end{cases}$$

$$(-4,-1) \to \begin{cases} f_{xx}=0, \quad f_{xy}=-5, \quad f_{yy}=10 \\ \begin{vmatrix} 0 & -5 \\ -5 & 10 \end{vmatrix} = 0\cdot10-(-5)\cdot(-5)=-25<0 \end{cases}$$

$$(1,4) \to \begin{cases} f_{xx}=10, \quad f_{xy}=5, \quad f_{yy}=0 \\ \begin{vmatrix} 10 & 5 \\ 5 & 0 \end{vmatrix} = 10\cdot0-5\cdot5=-25<0 \end{cases}$$

$$\boxed{\begin{array}{c} \text{A local minimum occurs at } \left(-\dfrac{2}{3},\dfrac{2}{3}\right). \\ \text{Saddle points occur at } (1,-1),(-4,-1), \text{ and } (1,4). \end{array}}$$