2011-2012 Spring
MAT124 Midterm I
(28/03/2012)
Time: 13:00 - 15:00
Duration: 120 minutes

1. Consider the curve with the polar equations $r = 2 - 2\sin\theta$.

   (a) Sketch this curve.

   (b) Find the area of the region enclosed by this curve.

   (c) Find the tangent vector(s) to this curve at $\theta = \dfrac{\pi}{4}$.

2. Consider the points $A(0,-1,1)$, $B(-1,0,1)$, $C(1,0,2)$, $D(0,0,1)$ in $\mathbb{R}^3$.

   (a) Find the equation of the plane passing through A, B and C.

   (b) Find the distance from the point D to the plane passing through A, B and C.

   (c) Find the equation of the line passing through B and D.

3. Consider the space curve whose vector equation is given by

$$r(t) = \mathrm{e}^t \sin t\,\mathbf{i} + \mathrm{e}^t \cos t\,\mathbf{j} + \mathrm{e}^t\,\mathbf{k} \quad \text{for} \quad 0 \le t \le 1\,.$$

   (a) Sketch the curve.

   (a) Find the length of the curve.

4. Evaluate the limit, if it exists, and explain your answer.

   (a) $\displaystyle\lim_{(x,y)\to(-1,-2)} \frac{y+2}{x^2y - xy + 2x^2 - 2x}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^4+y^2}$

5. Answer each independent question below.

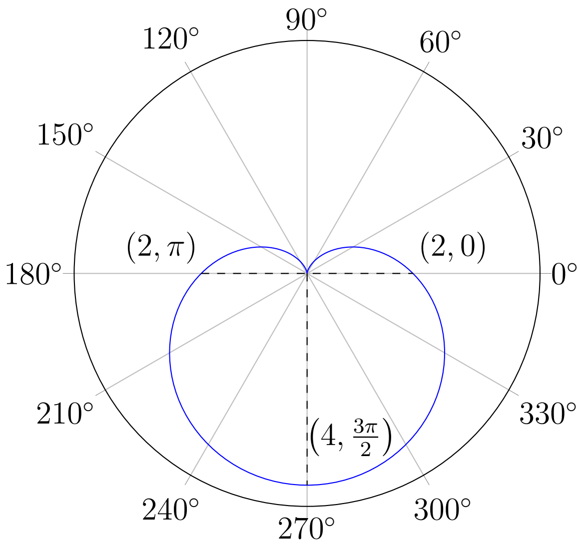
   (a) Let $f(x,y) = \ln(x^2+y^2)$. Evaluate $f_{xx} + f_{yy}$.

   (b) Sketch the surface given by $z = 2x^2 + y^2 + 4y + 6$.

2011-2012 Spring Midterm I (28/03/2012) Solutions
(Last update: 8/26/25 (28th of August) 2:01 PM)

1.

(a)



(b)

$$A = \frac{1}{2}\int_a^b r^2\,d\theta = \frac{1}{2}\int_0^{2\pi} (2-2\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} \left(4 - 8\sin\theta + 4\sin^2\theta\right)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(4 - 8\sin\theta + 4\sin^2\theta\right)\,d\theta = \int_0^{2\pi} (2 - 4\sin\theta + 1 - \cos 2\theta)\,d\theta$$

$$= \left[2\theta + 4\cos\theta + \theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = [(4\pi + 4 + 2\pi) - (4)] = \boxed{6\pi}$$

(c) The slope of the tangent line in terms of r and θ is

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\cdot\sin\theta + f(\theta)\cdot\cos\theta}{f'(\theta)\cdot\cos\theta - f(\theta)\cdot\sin\theta}$$

$$f(\theta) = 2 - 2\sin\theta, \quad f'(\theta) = -2\cos\theta$$

$$\frac{dy}{dx} = \frac{-2\cos\theta\cdot\sin\theta + (2-2\sin\theta)\cdot\cos\theta}{-2\cos\theta\cdot\cos\theta - (2-2\sin\theta)\cdot\sin\theta} = \frac{-2\sin 2\theta + 2\cos\theta}{-\sin 2\theta - 2\sin\theta + 2\sin^2\theta}$$

$$\left.\frac{dy}{dx}\right|_{\left(2-\sqrt{2},\,\pi/4\right)} = \frac{-2+\sqrt{2}}{-\sqrt{2}} = \sqrt{2} - 1$$

The tangent vectors to the curve at $\theta = \dfrac{\pi}{4}$ are

$$\boxed{\left\langle k,\; k\left(\sqrt{2}-1\right)\right\rangle, \quad k \in \mathbb{R}}$$

2.

(a) Choose three arbitrary points and determine the parallel vector of each of the two line segments that connect the points.

$$\overrightarrow{AB} = \langle -1-0,\ 0-(-1),\ 1-1\rangle = \langle -1, 1, 0\rangle$$
$$\overrightarrow{AC} = \langle 1-0,\ 0-(-1),\ 2-1\rangle = \langle 1, 1, 1\rangle$$

The cross product of these vectors gives us the normal vector of the plane.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= (1\cdot 1 - 0\cdot 1)\mathbf{i} - (-1\cdot 1 - 0\cdot 1)\mathbf{j} + (-1\cdot 1 - 1\cdot 1)\mathbf{k} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

The plane has the equation $\mathbf{n}\cdot\overrightarrow{PP_0} = 0$. Since $C(1, 0, 2)$ is on the plane, the equation of the plane is

$$\boxed{1(x-1) + 1(y-0) - 2(z-2) = 0 \implies x + y - 2z + 3 = 0}$$

(b) Let P be a point on a plane, then the distance from any point R to the plane is the length of the vector projection of $\overrightarrow{PR}$ onto $\mathbf{n}$. That is,

$$d = \left|\overrightarrow{PR}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

The distance from D to the plane passing through A, B, C can be calculated using

$$d = \left|\overrightarrow{AD}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\langle 0-0, 0-(-1), 1-1\rangle\cdot\frac{\langle 1, 1, -2\rangle}{\sqrt{1^2+1^2+(-2)^2}}\right| = \boxed{\frac{1}{\sqrt{6}}}$$

(c) We have

$$\mathbf{v} = \overrightarrow{BD} = \langle 0-(-1), 0-0, 1-1\rangle = \langle 1, 0, 0\rangle$$

The parametric equations for a line that passes through the point $P_0(x_0, y_0, z_0)$ is given by

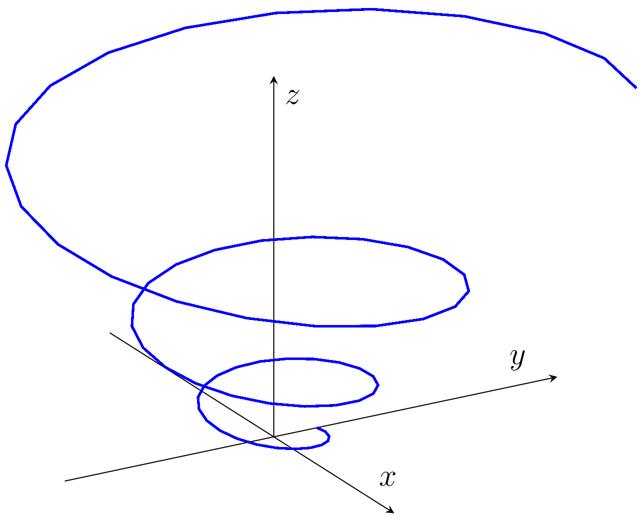
$$\left.\begin{array}{l} x = x_0 + v_1 t \\ y = y_0 + v_2 t \\ z = z_0 + v_3 t \end{array}\right\} \quad t \in \mathbb{R}$$

Therefore, the parametric equations for the line passing through B and D is

$$\boxed{\left.\begin{array}{l} x = -1 + t \\ y = 0 \\ z = 1 \end{array}\right\} \quad t \in \mathbb{R}}$$

3.

(a)



(b) The length of the parametrized curve $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$ for $a < t < b$ can be evaluated using the integral

$$L = \int_a^b \left|\frac{d\mathbf{r}}{dt}\right| \, dt$$

The length of the curve is then

$$L = \int_0^1 \left|\frac{d\mathbf{r}}{dt}\right| \, dt = \int_0^1 \left|\left\langle \mathrm{e}^t \sin t + \mathrm{e}^t \cdot \cos t,\ \mathrm{e}^t \cos t - \mathrm{e}^t \sin t,\ \mathrm{e}^t \right\rangle\right| \, dt$$

$$= \int_0^1 \sqrt{\left(\mathrm{e}^t \sin t + \mathrm{e}^t \cdot \cos t\right)^2 + \left(\mathrm{e}^t \cos t - \mathrm{e}^t \sin t\right)^2 + \left(\mathrm{e}^t\right)^2} \, dt$$

$$= \int_0^1 \sqrt{\mathrm{e}^{2t}\left(\sin^2 t + 2\sin t \cos t + \cos^2 t\right) + \mathrm{e}^t\left(\cos^2 t - 2\sin t \cos t + \sin^2 t\right) + \mathrm{e}^{2t}} \, dt$$

$$= \int_0^1 \sqrt{3\mathrm{e}^{2t}} \, dt = \int_0^1 \sqrt{3}\mathrm{e}^t \, dt = \sqrt{3}\mathrm{e}^t\Big|_0^1 = \boxed{\sqrt{3}(\mathrm{e} - 1)}$$

4.

(a) Factor the denominator.

$$\lim_{(x,y)\to(-1,-2)} \frac{y+2}{x^2y - xy + 2x^2 - 2x} = \lim_{(x,y)\to(-1,-2)} \frac{y+2}{x^2(y+2) - x(y+2)}$$

$$= \lim_{(x,y)\to(-1,-2)} \frac{1}{x^2 - x} = \frac{1}{1-(-1)} = \boxed{\frac{1}{2}}$$

(b) Apply the Two-Path Test.

$$y = x \implies \lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{4x^3}{x^4+x^2} = \lim_{x\to 0} \frac{4x}{x^2+1} = \frac{0}{1} = 0$$

$$y = x^2 \implies \lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{4x^4}{2x^4} = \lim_{x\to 0} \frac{4}{2} = 2$$

Since $0 \neq 2$, by the Two-Path Test, the limit does not exist.

5.

(a) Compute the first partial derivatives.

$$f_x = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}, \qquad f_y = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$$

Compute the second partial derivatives.

$$f_{xx} = \frac{2\left(x^2+y^2\right) - (2x)\cdot(2x)}{\left(x^2+y^2\right)^2} = \frac{2y^2-2x^2}{\left(x^2+y^2\right)^2}, \; f_{yy} = \frac{2\left(x^2+y^2\right) - (2y)\cdot(2y)}{\left(x^2+y^2\right)^2} = \frac{2x^2-2y^2}{\left(x^2+y^2\right)^2}$$

$$f_{xx} + f_{yy} = \frac{2y^2-2x^2}{\left(x^2+y^2\right)^2} + \frac{2x^2-2y^2}{\left(x^2+y^2\right)^2} = \boxed{0}$$

(b)

$$z = 2x^2 + y^2 + 4y + 6$$