2011-2012 Spring
MAT123-[Instructor] Midterm II
(08/05/2012)
Time: 15:00 - 16:45
Duration: 105 minutes

1. Consider the region bounded by the curve $y = x^2$, the x-axis, and the line $x = 2$, where $x \geq 0$.

(a) Find the volume of the solid generated by revolving the region about the x-axis by the disk method and sketch the solid.

(b) Find the volume of the solid generated by revolving the region about the y-axis by the shell method and sketch the solid.

2. Evaluate the following integrals.

(a) $\displaystyle\int_0^3 \left|x^2 - 1\right| \, dx$ (b) $\displaystyle\int \frac{1}{x^2 + 2x + 1} \, dx$ (c) $\displaystyle\int \frac{1}{x^2 + 2x + 2} \, dx$

(d) $\displaystyle\int \frac{1}{x^2 + 3x + 2} \, dx$ (e) $\displaystyle\int_{-\pi/6}^{0} \sqrt{1 - \cos(6x)} \, dx$

3. Determine whether the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx$ is convergent or divergent. Evaluate if the integral is convergent.

4. Evaluate the following limits. Explain all your work and write clearly.

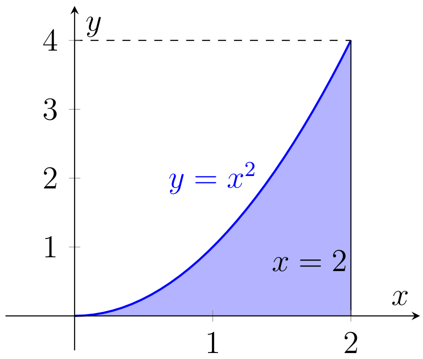
(a) $\displaystyle\lim_{x \to 0} \frac{3^x - 1}{5^x - 1}$ (b) $\displaystyle\lim_{x \to 0} \frac{\int_0^{x^2} \sin t \, dt}{x}$

5. Determine whether each sequence converges or diverges. Evaluate the limit of each convergent sequence. Explain all your work and write clearly.

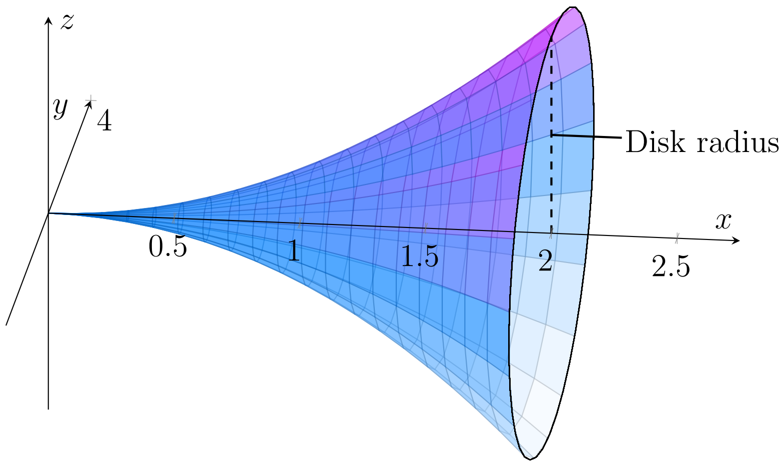
(a) $a_n = \dfrac{2n + (-1)^n}{n}$ (b) $a_n = \arctan\left(\dfrac{n + 1}{n}\right)$ (c) $a_n = \dfrac{n + 1}{1 - \sqrt{n}}$

2011-2012 Spring Midterm II (08/05/2012) Solutions
(Last update: 8/21/25 (21st of August) 3:13 PM)

1.

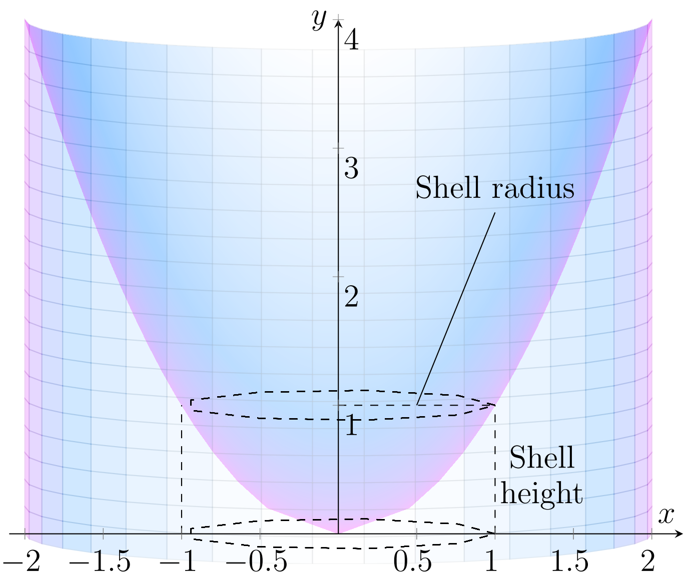


(a)



$$\text{Volume} = \int_{\mathcal{D}} \pi \cdot (r(x))^2\, dx = \int_0^2 \pi \left(x^2\right)^2\, dx = \pi \int_0^2 x^4\, dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \boxed{\frac{32\pi}{5}}$$

(b)



$$\text{Volume} = \int_{\mathcal{D}} 2\pi \cdot r(x) \cdot h(x)\, dx = 2\pi \int_0^2 x \cdot x^2\, dx = 2\pi \int_0^2 x^3\, dx$$

$$= 2\pi \left[\frac{x^4}{4}\right]_0^2 = 2\pi \left(\frac{2^4}{4} - 0\right) = \boxed{8\pi}$$

2.

(a) The expression $|x^2 - 1|$ is the same as $x^2 - 1$ for $x > 1$ and $1 - x^2$ for $x < 1$. We can write the equivalent expression below.

$$\int_0^3 |x^2 - 1| = dx \int_0^1 \left(1 - x^2\right)\, dx + \int_1^3 (x^2 - 1)\, dx = \left[x - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - x\right]_1^3$$

$$= \left[\left(1 - \frac{1^3}{3}\right) - 0\right] + \left[\left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right)\right] = \boxed{\frac{22}{3}}$$

(b)

$$\int \frac{1}{x^2 + 2x + 1}\, dx = \int \frac{1}{(x+1)^2}\, dx = \boxed{-\frac{1}{x+1} + c, \quad c \in \mathbb{R}}$$

(c)

$$\int \frac{1}{x^2 + 2x + 2}\, dx = \int \frac{1}{(x+1)^2 + 1}\, dx \quad [u = x + 1 \implies du = dx]$$

$$= \int \frac{1}{u^2 + 1}\, du = \arctan(u) + c = \boxed{\arctan(x+1) + c, \quad c \in \mathbb{R}}$$

(d) We can use the method of partial fraction decomposition.

$$\int \frac{1}{x^2 + 3x + 2}\, dx = \int \frac{1}{(x+1)(x+2)}\, dx = \int \left(\frac{A}{x+1} + \frac{B}{x+2}\right)\, dx \tag{1}$$

$$A(x+2) + B(x+1) = 1$$
$$x(A+B) + 2A + B = 1$$
$$\therefore A + B = 0 \quad [\text{eliminate } x] \to 2A + B = 1$$

$$\left.\begin{array}{r} A + B = 0 \\ 2A + B = 1 \end{array}\right\} \quad A = 1, \quad B = -1$$

Plug the values of A and B into (1).

$$\int \left(\frac{A}{x+1} + \frac{B}{x+2}\right)\, dx = \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right)\, dx = \boxed{\ln|x+1| - \ln|x+2| + c, \quad c \in \mathbb{R}}$$

(e) Use the trigonometric identity $\cos(2x) = 1 - 2\sin^2 x$.

$$\int_{-\pi/6}^{0} \sqrt{1-\cos(6x)}\,dx = \int_{-\pi/6}^{0} \sqrt{2\sin^2(3x)}\,dx = \sqrt{2}\int_{-\pi/6}^{0} |\sin(3x)|\,dx$$

$\sin x < 0$ for $-\pi < x < \pi$. Therefore, the integral can be rewritten as follows.

$$\sqrt{2}\int_{-\pi/6}^{0} |\sin(3x)|\,dx = \sqrt{2}\int_{-\pi/6}^{0} -\sin(3x)\,dx = \sqrt{2}\left[\frac{1}{3}\cos(3x)\right]_{-\pi/6}^{0}$$

$$= \frac{\sqrt{2}}{3}\left[\cos 0 - \cos\left(-\frac{\pi}{2}\right)\right] = \boxed{\frac{\sqrt{2}}{3}}$$

3. We have an improper integral where the limits are $\pm\infty$. Use limits to handle improper integrals accurately.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{A\to\infty}\int_{-A}^{A} \frac{1}{1+x^2}\,dx = \lim_{A\to\infty} \arctan(x)\Big|_{-A}^{A}$$

$$= \lim_{A\to\infty}(\arctan A - \arctan(-A)) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

The value of the improper integral is finite. Therefore, this integral is convergent.

4.

(a) The limit is in the indeterminate form $0/0$. We can apply L'Hôpital's rule to eliminate the indeterminate form.

$$\lim_{x\to 0}\frac{3^x - 1}{5^x - 1} \overset{\text{L'H.}}{=} \lim_{x\to 0}\frac{3^x\cdot\ln 3}{5^x\cdot\ln 5} = \log_5 3\cdot\lim_{x\to 0}\left(\frac{3}{5}\right)^x = \boxed{\log_5 3}$$

(b) The first method is to evaluate the integral in the limit.

$$\lim_{x\to 0}\frac{\int_0^{x^2}\sin t\,dt}{x} = \lim_{x\to 0}\frac{-\cos t\Big|_{t=0}^{t=x^2}}{x} = \lim_{x\to 0}\frac{-\cos x^2 - (-\cos 0)}{x} = \lim_{x\to 0}\frac{1-\cos x^2}{x}$$

Multiply by the expression inside the limit $\frac{x}{x}$. Notice that we obtain a standard form.

$$\lim_{x\to 0}\left(\frac{1-\cos x^2}{x}\cdot\frac{x}{x}\right) = \lim_{x\to 0}\frac{1-\cos x^2}{x^2}\cdot\lim_{x\to 0} x \qquad \left[\lim_{u\to 0}\frac{1-\cos u}{u} = 0\right]$$

$$= 1\cdot 0 = \boxed{0}$$

The second method is to use L'Hôpital's rule because of the 0/0 form.

$$\lim_{x\to 0} \frac{\displaystyle\int_0^{x^2} \sin t\, dt}{x} \overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{\displaystyle\frac{d}{dx}\int_0^{x^2} \sin t\, dt}{1}$$

Let $u = x^2$, then $du = 2x\, dx$. By the Fundamental Theorem of Calculus, the limit can be rewritten as follows.

$$\lim_{x\to 0} \frac{d}{dx}\int_0^{x^2} \sin t\, dt = \lim_{x\to 0} \frac{d}{du}\left(\int_0^{u} \sin t\, dt\right)\frac{du}{dx} = \lim_{x\to 0}(\sin u \cdot 2x) = \lim_{x\to 0}\left[2x \sin\left(x^2\right)\right] = \boxed{0}$$

5.

(a)

$$a_n = \frac{2n + (-1)^n}{n} = 2 + \frac{(-1)^n}{n}$$

The sequence converges to $\boxed{2}$ because $\dfrac{(-1)^n}{n} \to 0$ as $n \to \infty$.

(b) arctan is continuous everywhere. Therefore, we can take the limit inside the expression.

$$\lim_{n\to\infty} \arctan\left(\frac{n+1}{n}\right) = \arctan \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = \arctan 1 = \frac{\pi}{4}$$

The sequence converges to $\boxed{\dfrac{\pi}{4}}$.

(c)

$$\lim_{n\to\infty} \frac{n+1}{1-\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}\left(\sqrt{n} + \frac{1}{\sqrt{n}}\right)}{\sqrt{n}\left(\frac{1}{\sqrt{n}} - 1\right)} = \lim_{n\to\infty} \frac{\sqrt{n} + \frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}} - 1} = -\infty$$

Notice that the denominator approaches -1 as $n \to \infty$ and the numerator approaches ∞ as $n \to \infty$. The sequence diverges to $\boxed{-\infty}$.