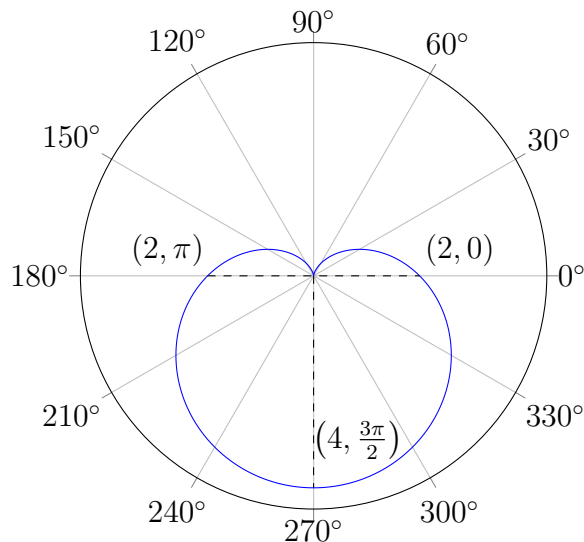


2011-2012 Spring
MAT124 Midterm I
(28/03/2012)
Time: 13:00 - 15:00
Duration: 120 minutes

1. Consider the curve with the polar equations $r = 2 - 2 \sin \theta$.
 - (a) Sketch this curve.
 - (b) Find the area of the region enclosed by this curve.
 - (c) Find the tangent vector(s) to this curve at $\theta = \frac{\pi}{4}$.
2. Consider the points $A(0, -1, 1)$, $B(-1, 0, 1)$, $C(1, 0, 2)$, $D(0, 0, 1)$ in \mathbb{R}^3 .
 - (a) Find the equation of the plane passing through A, B and C .
 - (b) Find the distance from the point D to the plane passing through A, B and C .
 - (c) Find the equation of the line passing through B and D .
3. Consider the space curve whose vector equation is given by
$$\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} + e^t \mathbf{k} \quad \text{for } 0 \leq t \leq 1.$$
 - (a) Sketch the curve.
 - (a) Find the length of the curve.
4. Evaluate the limit, if it exists, and explain your answer.
 - (a) $\lim_{(x,y) \rightarrow (-1,-2)} \frac{y+2}{x^2y - xy + 2x^2 - 2x}$
 - (b) $\lim_{(x,y) \rightarrow (0,0)} \frac{4x^2y}{x^4 + y^2}$
5. Answer each independent question below.
 - (a) Let $f(x, y) = \ln(x^2 + y^2)$. Evaluate $f_{xx} + f_{yy}$.
 - (b) Sketch the surface given by $z = 2x^2 + y^2 + 4y + 6$.

1.

(a)



(b)

$$\begin{aligned}
 A &= \frac{1}{2} \int_a^b r^2 d\theta = \frac{1}{2} \int_0^{2\pi} (2 - 2 \sin \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (4 - 8 \sin \theta + 4 \sin^2 \theta) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (4 - 8 \sin \theta + 4 \sin^2 \theta) d\theta = \int_0^{2\pi} (2 - 4 \sin \theta + 1 - \cos 2\theta) d\theta \\
 &= \left[2\theta + 4 \cos \theta + \theta - \frac{1}{2} \sin 2\theta \right]_0^{2\pi} = [(4\pi + 4 + 2\pi) - (4)] = \boxed{6\pi}
 \end{aligned}$$

(c) The slope of the tangent line in terms of r and θ is

$$\left. \frac{dy}{dx} \right|_{(r,\theta)} = \frac{f'(\theta) \cdot \sin \theta + f(\theta) \cdot \cos \theta}{f'(\theta) \cdot \cos \theta - f(\theta) \cdot \sin \theta}$$

$$f(\theta) = 2 - 2 \sin \theta, \quad f'(\theta) = -2 \cos \theta$$

$$\frac{dy}{dx} = \frac{-2 \cos \theta \cdot \sin \theta + (2 - 2 \sin \theta) \cdot \cos \theta}{-2 \cos \theta \cdot \cos \theta - (2 - 2 \sin \theta) \cdot \sin \theta} = \frac{-2 \sin 2\theta + 2 \cos \theta}{-\sin 2\theta - 2 \sin \theta + 2 \sin^2 \theta}$$

$$\left. \frac{dy}{dx} \right|_{(2-\sqrt{2}, \pi/4)} = \frac{-2 + \sqrt{2}}{-\sqrt{2}} = \sqrt{2} - 1$$

The tangent vectors to the curve at $\theta = \frac{\pi}{4}$ are

$$\boxed{\left\langle k, k(\sqrt{2} - 1) \right\rangle, \quad k \in \mathbb{R}}$$

2.

(a) Choose three arbitrary points and determine the parallel vector of each of the two line segments that connect the points.

$$\begin{aligned}\overrightarrow{AB} &= \langle -1 - 0, 0 - (-1), 1 - 1 \rangle = \langle -1, 1, 0 \rangle \\ \overrightarrow{AC} &= \langle 1 - 0, 0 - (-1), 2 - 1 \rangle = \langle 1, 1, 1 \rangle\end{aligned}$$

The cross product of these vectors gives us the normal vector of the plane.

$$\begin{aligned}\mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \\ &= (1 \cdot 1 - 0 \cdot 1)\mathbf{i} - (-1 \cdot 1 - 0 \cdot 1)\mathbf{j} + (-1 \cdot 1 - 1 \cdot 1)\mathbf{k} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}\end{aligned}$$

The plane has the equation $\mathbf{n} \cdot \overrightarrow{PP_0} = 0$. Since $C(1, 0, 2)$ is on the plane, the equation of the plane is

$$\boxed{1(x - 1) + 1(y - 0) - 2(z - 2) = 0 \implies x + y - 2z + 3 = 0}$$

(b) Let P be a point on a plane, then the distance from any point R to the plane is the length of the vector projection of \overrightarrow{PR} onto \mathbf{n} . That is,

$$d = \left| \overrightarrow{PR} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$$

The distance from D to the plane passing through A, B, C can be calculated using

$$d = \left| \overrightarrow{AD} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \langle 0 - 0, 0 - (-1), 1 - 1 \rangle \cdot \frac{\langle 1, 1, -2 \rangle}{\sqrt{1^2 + 1^2 + (-2)^2}} \right| = \boxed{\frac{1}{\sqrt{6}}}$$

(c) We have

$$\mathbf{v} = \overrightarrow{BD} = \langle 0 - (-1), 0 - 0, 1 - 1 \rangle = \langle 1, 0, 0 \rangle$$

The parametric equations for a line that passes through the point $P_0(x_0, y_0, z_0)$ is given by

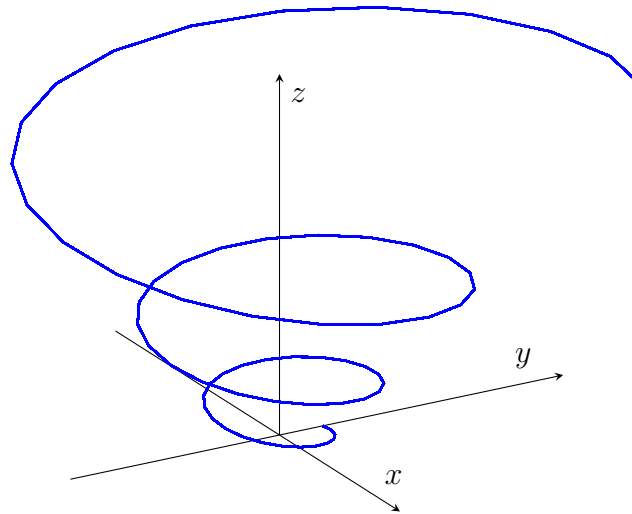
$$\left. \begin{aligned} x &= x_0 + v_1 t \\ y &= y_0 + v_2 t \\ z &= z_0 + v_3 t \end{aligned} \right\} \quad t \in \mathbb{R}$$

Therefore, the parametric equations for the line passing through B and D is

$$\boxed{\left. \begin{aligned} x &= -1 + t \\ y &= 0 \\ z &= 1 \end{aligned} \right\} \quad t \in \mathbb{R}}$$

3.

(a)



(b) The length of the parametrized curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a < t < b$ can be evaluated using the integral

$$L = \int_a^b \left| \frac{d\mathbf{r}}{dt} \right| dt$$

The length of the curve is then

$$\begin{aligned} L &= \int_0^1 \left| \frac{d\mathbf{r}}{dt} \right| dt = \int_0^1 |\langle e^t \sin t + e^t \cdot \cos t, e^t \cos t - e^t \sin t, e^t \rangle| dt \\ &= \int_0^1 \sqrt{(e^t \sin t + e^t \cdot \cos t)^2 + (e^t \cos t - e^t \sin t)^2 + (e^t)^2} dt \\ &= \int_0^1 \sqrt{e^{2t} (\sin^2 t + 2 \sin t \cos t + \cos^2 t) + e^t (\cos^2 t - 2 \sin t \cos t + \sin^2 t) + e^{2t}} dt \\ &= \int_0^1 \sqrt{3e^{2t}} dt = \int_0^1 \sqrt{3}e^t dt = \sqrt{3}e^t \Big|_0^1 = \boxed{\sqrt{3}(e - 1)} \end{aligned}$$

4.

(a) Factor the denominator.

$$\begin{aligned}\lim_{(x,y) \rightarrow (-1,-2)} \frac{y+2}{x^2y - xy + 2x^2 - 2x} &= \lim_{(x,y) \rightarrow (-1,-2)} \frac{y+2}{x^2(y+2) - x(y+2)} \\ &= \lim_{(x,y) \rightarrow (-1,-2)} \frac{1}{x^2 - x} = \frac{1}{1 - (-1)} = \boxed{\frac{1}{2}}\end{aligned}$$

(b) Apply the Two-Path Test.

$$\begin{aligned}y = x &\implies \lim_{(x,y) \rightarrow (0,0)} \frac{4x^3}{x^4 + x^2} = \lim_{x \rightarrow 0} \frac{4x}{x^2 + 1} = \frac{0}{1} = 0 \\ y = x^2 &\implies \lim_{(x,y) \rightarrow (0,0)} \frac{4x^4}{2x^4} = \lim_{x \rightarrow 0} \frac{4}{2} = 2\end{aligned}$$

Since $0 \neq 2$, by the Two-Path Test, the limit does not exist.

5.

(a) Compute the first partial derivatives.

$$f_x = \frac{1}{x^2 + y^2} \cdot 2x = \frac{2x}{x^2 + y^2}, \quad f_y = \frac{1}{x^2 + y^2} \cdot 2y = \frac{2y}{x^2 + y^2}$$

Compute the second partial derivatives.

$$\begin{aligned}f_{xx} &= \frac{2(x^2 + y^2) - (2x) \cdot (2x)}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}, \quad f_{yy} = \frac{2(x^2 + y^2) - (2y) \cdot (2y)}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} \\ f_{xx} + f_{yy} &= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} = \boxed{0}\end{aligned}$$

(b)

$$z = 2x^2 + y^2 + 4y + 6$$

