**2015-2016 Fall**
**MAT123 Final**
**(12/01/2016)**

**1.** Find the area of the region bounded by the circle $x^2 + y^2 = 1$ and the graph of the absolute value function $y = |x|$.

**2.** Use the Washer Method to find the volume of the solid obtained by revolving the region bounded by the parabola $y = 2x^2 - 3$ and the curves $y = -3$, $x = 2$ about the line $y = 7$.

**3.** Use the Cylindrical Shell Method to find the volume of the solid obtained by revolving the region bounded by $y = x^2$ and $y = -x + 1$ about the line $x = -1$.

**4.** Evaluate the following integrals.

**(a)** $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

**(b)** $\displaystyle\int \frac{x^5 + x^4 - 8x^3 + 10x^2 + 12x}{x^2 - 3x + 2}\,dx$

**(c)** $\displaystyle\int \arccos x\,dx$

**(d)** $\displaystyle\int \frac{1}{x^2 + 3x + 1}\,dx$

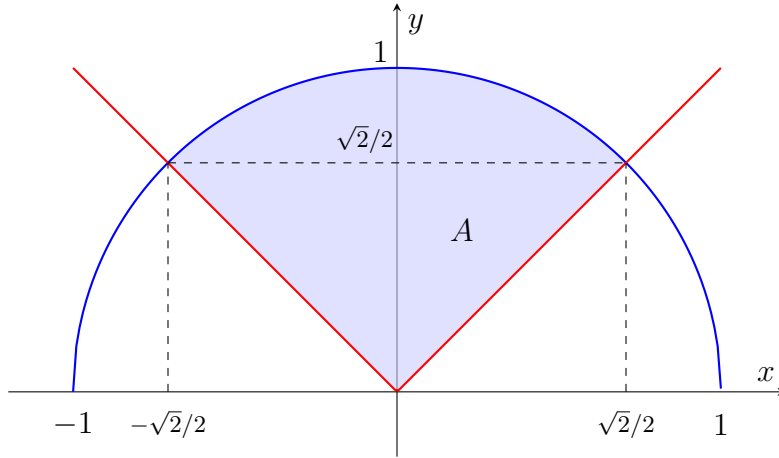**(e)** $\displaystyle\int \frac{\sin x}{1 + \sin x}\,dx$

**5.** Find the surface area of the revolution by rotating the curve $y = e^x$, $0 \leq x \leq 1$ about the $x$-axis.

**6.** Investigate the convergence of the improper integral $\displaystyle\int_0^{-\infty} \frac{e^{-x}}{1 + e^{-x}}\,dx$.

**7.** Using the Monotone Convergence Theorem, investigate the convergence of the sequence $\left(\dfrac{\ln n}{n}\right)_{n \in \mathbb{N}}$.

**1.**



$$A = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left( \sqrt{1-x^2} - |x| \right) dx = \int_{-\sqrt{2}/2}^{0} \left( \sqrt{1-x^2} + x \right) dx + \int_{0}^{\sqrt{2}/2} \left( \sqrt{1-x^2} - x \right) dx$$

$$= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \sqrt{1-x^2} \, dx - \int_{0}^{\sqrt{2}/2} 2x \, dx \tag{1}$$

Calculate the right-hand integral (1).

$$\int_{0}^{\sqrt{2}/2} 2x \, dx = x^2 \Big|_{0}^{\sqrt{2}/2} = \left( \frac{\sqrt{2}}{2} \right)^2 - (0)^2 = \frac{1}{2}$$

To calculate the left-hand integral in (1), we will use a trigonometric substitution. Let $x = \sin u$, then $dx = \cos u \, du$.

$$x = -\frac{\sqrt{2}}{2} \implies u = \arcsin \left( -\frac{\sqrt{2}}{2} \right) = -\frac{\pi}{4}, \qquad x = \frac{\sqrt{2}}{2} \implies u = \arcsin \left( \frac{\sqrt{2}}{2} \right) = \frac{\pi}{4}$$
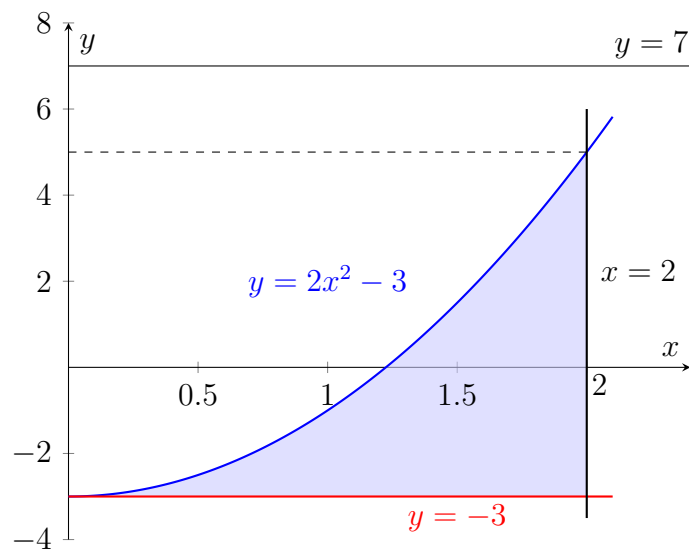
$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \sqrt{1-x^2} \, dx = \int_{-\pi/4}^{\pi/4} \sqrt{1-\sin^2 u} \cos u \, du = \int_{-\pi/4}^{\pi/4} |\cos u| \cos u \, du \quad [|\cos u| > 0]$$

$$= \int_{-\pi/4}^{\pi/4} \cos^2 u \, du = \int_{-\pi/4}^{\pi/4} \frac{1+\cos 2u}{2} \, du = \left[ \frac{u}{2} + \frac{\sin 2u}{4} \right]_{-\pi/4}^{\pi/4}$$

$$= \left( \frac{\pi}{8} + \frac{1}{4} \right) - \left( -\frac{\pi}{8} - \frac{1}{4} \right) = \frac{\pi}{4} - \frac{1}{2}$$

Evaluate (1). The area is $A = \left( \frac{\pi}{4} + \frac{1}{2} \right) - \left( \frac{1}{2} \right) = \boxed{\frac{\pi}{4}}$.
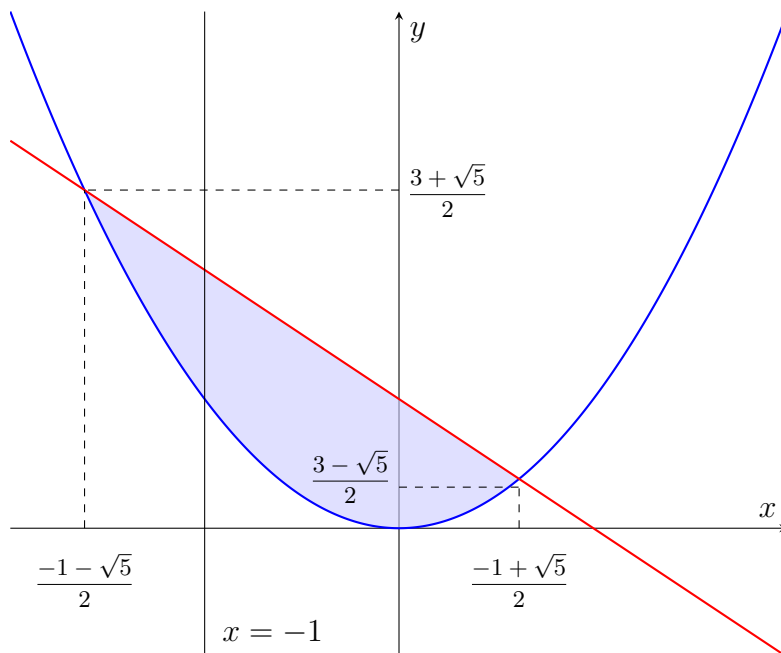
**2.**



$$V = \int_D \pi \left[ R_2^2(x) - R_1^2(x) \right] \, dx = \int_0^2 \pi \left[ (7 - (-3))^2 - \left( 7 - \left( 2x^2 - 3 \right) \right)^2 \right] \, dx$$

$$= \pi \int_0^2 \left[ (10)^2 - \left( 10 - 2x^2 \right)^2 \right] \, dx = \pi \int_0^2 \left( 100 - 100 + 40x^2 - 4x^4 \right) \, dx$$
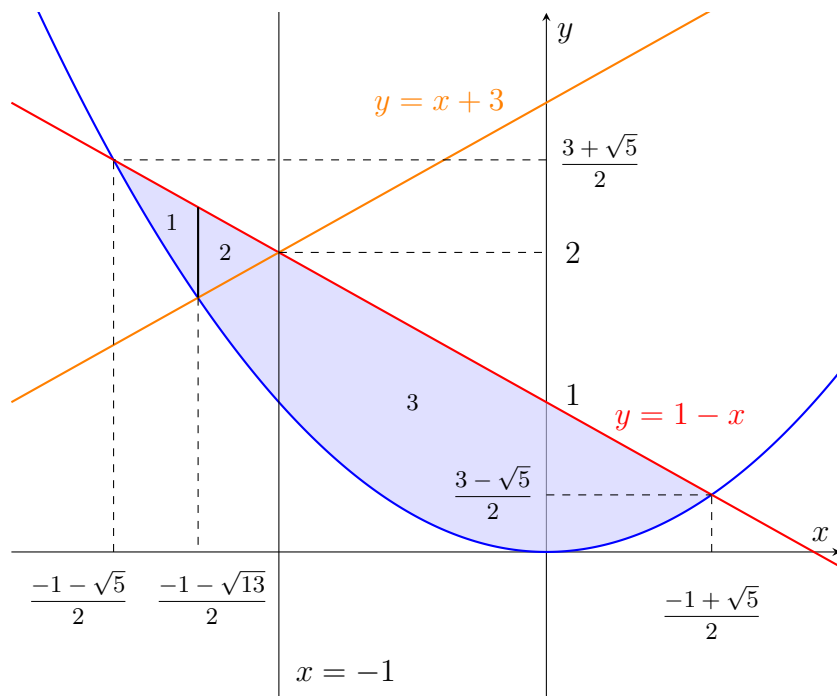
$$= \pi \int_0^2 \left( 40x^2 - 4x^4 \right) \, dx = \pi \left[ \frac{40x^3}{3} - \frac{4x^5}{5} \right]_0^2 = \pi \left[ \left( \frac{320}{3} - \frac{128}{5} \right) - (0) \right] = \boxed{\frac{1216\pi}{15}}$$

**3.**



Notice that the rotation axis passes through the region. Consider the right-hand region. If we rotate it around the axis, a piece of the region on the left will be inside the revolution.

That is, the region bounded by the line that is symmetric to the line $y = -x + 1$ around $x = -1$ and the curve $y = x^2$. We do not need to rotate that region since it would lead to double revolution. The upper part of the left-hand region may be rotated around $x = -1$ independent of the right-hand region. Divide the left-hand region into three subregions. We get three different subregions to integrate over.



Let $V$ be the volume of the solid.

$$V = \int_D 2\pi \cdot r(x) \cdot h(x)\, dx$$

$$= \int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{13}}{2}} 2\pi(-1-x)\left[(1-x)-x^2\right] dx + \int_{\frac{1-\sqrt{13}}{2}}^{-1} 2\pi(-1-x)\left[(1-x)-(x+3)\right] dx$$

$$+ \int_{-1}^{\frac{-1+\sqrt{5}}{2}} 2\pi\,(x+1)\left[(1-x)-x^2\right] dx$$

$$= 2\pi \int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{13}}{2}} \left(-1+2x^2+x^3\right) dx + 2\pi \int_{\frac{1-\sqrt{13}}{2}}^{-1} \left(2x^2+4x+2\right) dx$$

$$+ 2\pi \int_{-1}^{\frac{-1+\sqrt{5}}{2}} \left(1-2x^2-x^3\right) dx$$

$$= 2\pi \left\{ \left[-x+\frac{2x^3}{3}+\frac{x^4}{4}\right]_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{13}}{2}} + \left[\frac{2x^3}{3}+2x^2+2x\right]_{\frac{1-\sqrt{13}}{2}}^{-1} + \left[x-\frac{2x^3}{3}-\frac{x^4}{4}\right]_{-1}^{\frac{-1+\sqrt{5}}{2}} \right\}$$

$$V = 2\pi \left[ \frac{\sqrt{13}-1}{2} + \frac{\left(1-\sqrt{13}\right)^3}{12} + \frac{\left(1-\sqrt{13}\right)^4}{64} - \frac{1+\sqrt{5}}{2} + \frac{\left(1+\sqrt{5}\right)^3}{12} - \frac{\left(1+\sqrt{5}\right)^4}{64} \right]$$

$$+ 2\pi \left[ -\frac{2}{3} + 2 - 2 - \left( \frac{\left(1-\sqrt{13}\right)^3}{12} + \frac{\left(1-\sqrt{13}\right)^2}{2} + 1 - \sqrt{13} \right) \right]$$

$$+ 2\pi \left[ \frac{-1+\sqrt{5}}{2} - \frac{\left(-1+\sqrt{5}\right)^3}{12} - \frac{\left(-1+\sqrt{5}\right)^4}{64} - \left( -1 + \frac{2}{3} - \frac{1}{4} \right) \right]$$

After some mathematical operations, the volume is found to be

$$V = \frac{13\pi}{6} - \frac{\pi}{4}\left( 47 - 13\sqrt{13} \right) = \boxed{\frac{\pi\left( 39\sqrt{13} - 115 \right)}{12}}$$

The volume can also be evaluated by integrating over the domain and subtracting the region that is revolved twice. That is, the region beneath the region 2 mapped on the graph.

**4. (a)** Let $u = \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u \cdot 2\,du = -2\cos u + c = \boxed{-2\cos\sqrt{x} + c, \quad c \in \mathbb{R}}$$

**(b)** Perform a long polynomial division and rewrite the integral in two expressions.

$$I = \int \frac{x^5 + x^4 - 8x^3 + 10x^2 + 12x}{x^2 - 3x + 2}\,dx = \int \left( x^3 - 4x^2 - 22x - 48 + \frac{-88x + 96}{x^2 - 3x + 2} \right) dx$$

$$= \int \left( x^3 + 4x^2 + 2x + 8 \right) dx + \int \frac{32x - 16}{(x-2)(x-1)}\,dx \tag{2}$$

Calculate the integral on the left in (2).

$$\int \left( x^3 + 4x^2 + 2x + 8 \right) dx = \frac{x^4}{4} + \frac{4x^3}{3} + x^2 + 8x + c_1$$

Calculate the integral on the right in (2). Decompose the expression into partial fractions.

$$\int \frac{32x - 16}{(x-2)(x-1)}\,dx = \int \left( \frac{A}{x-2} + \frac{B}{x-1} \right) dx$$

$$32x - 16 = A(x-1) + B(x-2) = x(A+B) - A - 2B$$

Equate the coefficients of like terms.

$$\left.\begin{array}{r} A + B = 32 \\ -A - 2B = -16 \end{array}\right\} \rightarrow A = 48, \quad B = -16$$

Substitute the values into $A$ and $B$.

$$\int \left(\frac{A}{x-2} + \frac{B}{x-1}\right) dx = \int \left(\frac{48}{x-2} - \frac{16}{x-1}\right) dx = 48 \ln|x-2| - 16 \ln|x-1| + c_2$$

Rewrite (2).

$$\boxed{I = \frac{x^4}{4} + \frac{4x^3}{3} + x^2 + 8x + 48 \ln|x-2| - 16 \ln|x-1| + c, \quad c \in \mathbb{R}}$$

**(c)** Apply integration by parts.

$$\left.\begin{array}{l} u = \arccos x \rightarrow du = -\dfrac{1}{\sqrt{1-x^2}} dx \\ dv = dx \rightarrow v = x \end{array}\right\} \rightarrow \int v\, du = uv - \int v\, du$$

$$\int \arccos x\, dx = x \arccos x - \int \frac{-x}{\sqrt{1-x^2}} dx$$

Let us use a $u$-substitution for the integral on the right. Let $u = 1 - x^2$, then $du = -2x\, dx$

$$\int \frac{-x}{\sqrt{1-x^2}} dx = \int \frac{du}{2\sqrt{u}} = \sqrt{u} + c = \sqrt{1-x^2} + c$$

Therefore,

$$\int \arccos x\, dx = \boxed{x \arccos x - \sqrt{1-x^2} + c, \quad c \in \mathbb{R}}$$

**(d)** Use the method of partial fraction decomposition.

$$I = \int \frac{dx}{x^2 + 3x + 1} = \int \frac{dx}{x^2 + 3x + \frac{9}{4} - \frac{5}{4}} = \int \frac{dx}{\left(x + \frac{3}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= \int \frac{dx}{\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)} = \int \left(\frac{A}{x + \frac{3}{2} - \frac{\sqrt{5}}{2}} + \frac{B}{x + \frac{3}{2} + \frac{\sqrt{5}}{2}}\right) dx$$

$$A\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right) + B\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right) = 1$$

$$x(A + B) + A\left(\frac{3 + \sqrt{5}}{2}\right) + B\left(\frac{3 - \sqrt{5}}{2}\right) = 1$$

Equate the coefficients of like terms.

$$\left.\begin{array}{c} A + B = 0 \\ A\left(\frac{3+\sqrt{5}}{2}\right) + B\left(\frac{3-\sqrt{5}}{2}\right) = 1 \end{array}\right\} \implies A = \frac{1}{\sqrt{5}}, \quad B = -\frac{1}{\sqrt{5}}$$

Substitute the values into $A$ and $B$.

$$I = \int \left(\frac{A}{x + \frac{3}{2} - \frac{\sqrt{5}}{2}} + \frac{B}{x + \frac{3}{2} + \frac{\sqrt{5}}{2}}\right) dx$$

$$= \int \left(\frac{1}{\sqrt{5}\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)} - \frac{1}{\sqrt{5}\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)}\right) dx$$

$$= \boxed{\frac{1}{\sqrt{5}}\left(\ln\left|x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right| - \ln\left|x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right|\right) + c, \quad c \in \mathbb{R}}$$

**(e)**

$$I = \int \frac{\sin x}{1 + \sin x} dx = \int \frac{1 + \sin x - 1}{1 + \sin x} dx = \int dx - \int \frac{dx}{1 + \sin x}$$

The integral on the left evaluates to $x + c_1$. Evaluate the other integral.

$$\int \frac{dx}{1 + \sin x} = \int \frac{1 - \sin x}{(1 + \sin x)(1 - \sin x)} dx = \int \frac{1 - \sin x}{1 - \sin^2 x} dx = \int \frac{1 - \sin x}{\cos^2 x} dx$$

$$= \int \left(\sec^2 x - \tan x \sec x\right) dx = \tan x - \sec x + c_2$$

So, the result is

$$I = \boxed{x - \tan x + \sec x + c, \quad c \in \mathbb{R}}$$

**5.** If the function $y = f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the $x$-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = e^x$$

Set $a = 0$ and $b = 1$ and then evaluate the integral.

$$S = \int_0^1 2\pi e^x \sqrt{1 + (e^x)^2}\, dx$$

Let $u = e^x$, then $du = e^x\, dx$.

$$x = 0 \implies u = e^0 = 1, \qquad x = 1 \implies u = e^1 = e$$

$$S = \int_0^1 2\pi e^x \sqrt{1 + (e^x)^2}\, dx = 2\pi \int_1^e \sqrt{1 + u^2}\, du$$

We will now use a trigonometric substitution. Let $u = \tan t$ for $0 < t < \dfrac{\pi}{2}$, then $du = \sec^2 t\, dt$.

$$S = 2\pi \int_1^e \sqrt{1 + u^2}\, du = 2\pi \int \sqrt{1 + \tan^2 t} \cdot \sec^2 t\, dt = 2\pi \int \sqrt{\sec^2 t} \cdot \sec^2 t\, dt$$

$$= 2\pi \int |\sec t| \sec^2 t\, dt = 2\pi \int \sec^3 t\, dt \qquad [\sec t > 0]$$

Find the antiderivative of $\sec^3 t$ with the help of integration by parts.

$$w = \sec t \;\to\; dw = \sec t \tan t\, dt$$
$$dz = \sec^2 t\, dt \;\to\; z = \tan t$$

$$\int \sec^3 t\, du = \tan t \cdot \sec t - \int \tan^2 t \sec t\, dt = \tan t \cdot \sec t - \int \frac{1 - \cos^2 t}{\cos^3 t}\, dt$$

$$= \tan t \cdot \sec t - \int \sec^3 t\, dt + \int \sec t\, dt$$

Notice that the integral appears on the right side of the equation. Therefore,

$$\int \sec^3 t\, dt = \frac{1}{2} \cdot \tan t \cdot \sec t + \frac{1}{2} \cdot \int \sec t\, dt$$

The integral of $\sec t$ with respect to $t$ is

$$\int \sec t\, dt = \ln |\tan t + \sec t| + c_1, \quad c_1 \in \mathbb{R}$$

Recall $u = \tan t$.

$$u = \tan t \implies u^2 = \tan^2 t = \sec^2 t - 1 \implies \sec t = \sqrt{u^2 + 1}$$

$$S = 2\pi \cdot \frac{1}{2} \left( \tan t \cdot \sec t + \ln |\tan t + \sec t| \right) + c = \pi \left[ u \cdot \sqrt{u^2 + 1} + \ln \left| t + \sqrt{u^2 + 1} \right| \right]_1^e$$

$$= \pi \left[ \left( e \cdot \sqrt{e^2 + 1} + \ln \left| e + \sqrt{e^2 + 1} \right| \right) - \left( \sqrt{2} + \ln \left| 1 + \sqrt{2} \right| \right) \right]$$

$$\boxed{= \pi \left[ e \cdot \sqrt{e^2 + 1} - \sqrt{2} + \ln \left( \frac{e + \sqrt{e^2 + 1}}{1 + \sqrt{2}} \right) \right]}$$

**6.** Let $u = 1 + e^{-x}$, then $du = -e^{-x}\,dx$. Handle the improper integral by taking the limit.

$$x = 0 \implies u = 1 + e^0 = 2, \qquad x \to -\infty \implies u \to \infty$$

$$\int_0^{-\infty} \frac{e^{-x}}{1 + e^{-x}}\,dx = \int_2^\infty -\frac{du}{u} = \lim_{R \to \infty} \int_2^R -\frac{du}{u} = \lim_{R \to \infty} \left( -\ln u \right) \Big|_2^R$$

$$= \lim_{R \to \infty} \left( -\ln R + \ln 2 \right) = \boxed{-\infty}$$

The integral diverges to negative infinity.

**7.** According to the Monotone Convergence Theorem, if a sequence is both bounded and monotonic, the sequence converges. Take the corresponding function $f(x) = \dfrac{\ln x}{x}$. Apply the first derivative test and find the extrema.

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f'(x) = 0 \implies 1 - \ln x = 0 \implies \ln x = 1 \implies x = e$$

A critical point occurs at $x = e$. Apply the second derivative test and determine whether this is a local minimum or a local maximum.

$$f''(x) = \frac{-\frac{1}{x} \cdot x^2 - (1 - \ln x) \cdot 2x}{x^4} = \frac{2 \ln x - 3}{x^3}$$

$$f''(e) = \frac{-1}{e^3} < 0$$

Therefore, this is a local maximum.

The first term of the sequence is $\dfrac{\ln 1}{1} = 0$. Take the limit at infinity. We may apply L'Hôpital's rule because we have an indeterminate form, which is $\infty/\infty$.

$$\lim_{x \to \infty} \frac{\ln x}{x} \stackrel{\text{L'H.}}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$$

Since $\dfrac{\ln x}{x}$ is decreasing and bounded above by $f(e) = \dfrac{1}{e}$ and below by 0 for $x \geq e$, by the Monotone Convergence Theorem, the sequence converges.