

2023-2024 Spring
MAT124 Makeup
(27/06/2024)

1. Find the maximum and minimum values of $f(x, y, z) = x - y + z$ on the sphere $x^2 + y^2 + z^2 = 100$.

2. Sketch the region and reverse the order of the double integral

$$\int_0^3 \int_{y/3}^{\sqrt{4-y}} dx dy$$

3. Using a polar double integral, find the volume of the sphere with radius 4.

4. Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies in the cylinder $x^2 + y^2 = 1$.

5. Using the change of variables, evaluate the area of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$.

6. Let R be the solid region bounded below by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 9$. Let

$$I = \iiint_R (x^2 + y^2) dV.$$

(i) Express (but do not evaluate) I as a triple integral in spherical coordinates.

(ii) Express (but do not evaluate) I as a triple integral in cylindrical coordinates.

1. Let $g(x, y, z) = x^2 + y^2 + z^2 - 100$ for the constraint. Solve the system of equations below.

$$\left. \begin{array}{l} \nabla f = \lambda \nabla g \\ g(x, y, z) = 0 \end{array} \right\} \quad \nabla f = \langle 1, -1, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle = \lambda \nabla g$$

$$1 - 2\lambda x = 0 \implies x = \frac{1}{2\lambda}$$

$$-1 - 2\lambda y = 0 \implies y = -\frac{1}{2\lambda}$$

$$1 - 2\lambda z = 0 \implies z = \frac{1}{2\lambda}$$

Use the constraint to find the values of x, y, z .

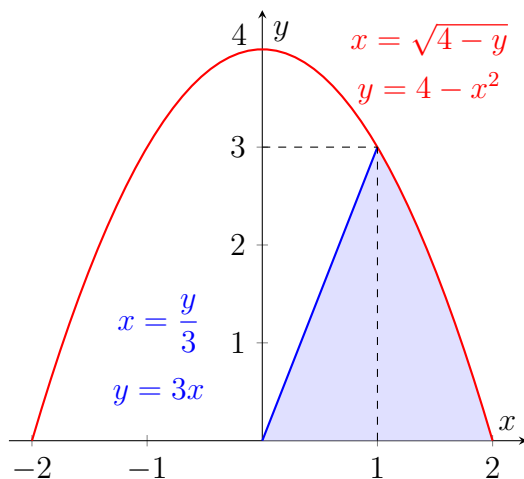
$$x^2 + y^2 + z^2 - 100 = 0 \implies \left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = \frac{3}{4\lambda^2} = 100 \implies \lambda = \pm \frac{\sqrt{3}}{20}$$

$$x = \frac{1}{2\lambda} = \pm \frac{10\sqrt{3}}{3}, \quad y = -\frac{1}{2\lambda} = \pm \frac{10\sqrt{3}}{3}, \quad z = \frac{1}{2\lambda} = \pm \frac{10\sqrt{3}}{3}$$

To find the maximum and minimum values of f , use the points $\left(\frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}\right)$ and $\left(-\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}\right)$, respectively.

$$\begin{aligned} f_{\max} &= \frac{10\sqrt{3}}{3} - \left(-\frac{10\sqrt{3}}{3}\right) + \frac{10\sqrt{3}}{3} = 10\sqrt{3} \\ f_{\min} &= -\frac{10\sqrt{3}}{3} - \frac{10\sqrt{3}}{3} - \frac{10\sqrt{3}}{3} = -10\sqrt{3} \end{aligned}$$

2. **Remark:** It is counterintuitive that the area of the region is negative for $y > 3$. Therefore, let's assume that the upper bound for y is 3 rather than the value stated in the original question. The lecturer might have made a typo.



$$\int_0^1 \int_0^{3x} dy \, dx + \int_1^2 \int_0^{4-x^2} dy \, dx$$

3. The equation of this sphere is $x^2 + y^2 + z^2 = 16$. Solve for z to find the bounds of z .

$$z_{\text{lower}} = -\sqrt{16 - x^2 - y^2}, \quad z_{\text{upper}} = \sqrt{16 - x^2 - y^2}$$

If we project the sphere onto the xy -plane, we will notice that the domain is $x^2 + y^2 \leq 16$. Use the transformation for polar coordinates.

$$\begin{array}{ll} \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \\ x^2 + y^2 = r^2 \\ dA = r \, dr \, d\theta \end{array} & \rightarrow \begin{array}{l} z = \sqrt{16 - x^2 - y^2} \implies z = \sqrt{16 - r^2} \\ z = -\sqrt{16 - x^2 - y^2} \implies z = -\sqrt{16 - r^2} \\ x^2 + y^2 \leq 16 \implies r^2 \leq 16 \implies 0 \leq r \leq 4, \quad 0 \leq \theta \leq 2\pi \end{array} \end{array}$$

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^4 \left[\sqrt{16 - r^2} - \left(-\sqrt{16 - r^2} \right) \right] r \, dr \, d\theta = 2 \int_0^{2\pi} \int_0^4 r \sqrt{16 - r^2} \, dr \, d\theta \\ &= 2 \int_0^{2\pi} \left[-\frac{1}{3} (16 - r^2)^{3/2} \right]_{r=0}^{r=4} d\theta = \frac{2}{3} \int_0^{2\pi} [0 - (-64)] \, d\theta = \frac{128}{3} \int_0^{2\pi} d\theta = \boxed{\frac{256\pi}{3}} \end{aligned}$$

4. The domain is $x^2 + y^2 \leq 1$. Using the double integral below, we find the surface area.

$$\begin{aligned} \text{Surface area} &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} \, dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + (2x)^2 + (2y)^2} \, dy \, dx \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + 4x^2 + 4y^2} \, dy \, dx \end{aligned}$$

If we switch to polar coordinates, we can easily evaluate the integral.

$$\begin{aligned} \text{Surface area} &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \, r \, dr \, d\theta = \frac{1}{12} \int_0^{2\pi} \left[(1 + 4r^2)^{3/2} \right]_{r=0}^{r=1} d\theta \\ &= \frac{1}{12} \int_0^{2\pi} (5\sqrt{5} - 1) \, d\theta = \boxed{\frac{\pi}{6} (5\sqrt{5} - 1)} \end{aligned}$$

5. Let $x = 4r \cos \theta$, $y = 5r \sin \theta$.

$$\begin{aligned} \frac{x^2}{16} + \frac{y^2}{25} = 1 &\implies \frac{(4r \cos \theta)^2}{16} + \frac{(5r \sin \theta)^2}{25} = 1 \implies r^2 (\sin^2 \theta + \cos^2 \theta) = 1 \\ r^2 &= 1 \implies r = 1 \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

Calculate the Jacobian determinant.

$$\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 4 \cos \theta & -4r \sin \theta \\ 5 \sin \theta & 5r \cos \theta \end{vmatrix} = 20r \cos^2 \theta - (-20r \sin^2 \theta) = 20r$$

Then we have the integral

$$\int_0^{2\pi} \int_0^1 \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta = \int_0^{2\pi} \int_0^1 20r dr d\theta = 20 \int_0^{2\pi} \left[\frac{1}{2} r^2 \right]_{r=0}^{r=1} d\theta = 10 \int_0^{2\pi} d\theta = \boxed{20\pi}$$

6.

(i) For spherical coordinates, we have

$$\begin{array}{ll} \begin{array}{l} z = \rho \cos \phi \\ r = \rho \sin \phi \\ x^2 + y^2 = r^2 \\ x^2 + y^2 + z^2 = \rho^2 \\ dV = \rho^2 \sin \phi d\rho d\phi d\theta \end{array} & \rightarrow \begin{array}{l} z = \sqrt{3x^2 + 3y^2} \implies \rho \cos \phi = \sqrt{3}\rho \sin \phi \implies \phi = \frac{\pi}{6} \\ x^2 + y^2 = r^2 = \rho^2 \sin^2 \phi \\ x^2 + y^2 + z^2 = 9 \implies \rho^2 = 9 \implies \rho = 3 \\ 0 \leq \theta \leq 2\pi \end{array} \end{array}$$

The integral in spherical coordinates can be expressed as follows.

$$\boxed{I = \int_0^{2\pi} \int_0^{\pi/6} \int_0^3 \rho^2 \sin^2 \phi \cdot \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/6} \int_0^3 \rho^4 \sin^3 \phi d\rho d\phi d\theta}$$

(ii) For cylindrical coordinates, we have

$$\begin{array}{ll} \begin{array}{l} z = z \\ r^2 = x^2 + y^2 \\ dV = r dz dr d\theta \end{array} & \rightarrow \begin{array}{l} z = \sqrt{3x^2 + 3y^2} \implies z = r\sqrt{3} \\ x^2 + y^2 = r^2 \\ x^2 + y^2 + z^2 = 9 \implies z = \sqrt{9 - r^2} \\ 0 \leq \theta \leq 2\pi \end{array} \end{array}$$

Find where the curves intersect to find the upper limit of r .

$$r\sqrt{3} = \sqrt{9 - r^2} \implies 3r^2 = 9 - r^2 \implies r^2 = \frac{9}{4} \implies r = \frac{3}{2}$$

The integral in cylindrical coordinates can be expressed as follows.

$$\boxed{I = \int_0^{2\pi} \int_0^{3/2} \int_{r\sqrt{3}}^{\sqrt{9-r^2}} r^2 \cdot r dz dr d\theta = \int_0^{2\pi} \int_0^{3/2} \int_{r\sqrt{3}}^{\sqrt{9-r^2}} r^3 dz dr d\theta}$$