1. Evaluate the limits, if they exist, and explain your answer. Don't use L'Hôpital's rule.

(a) $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{\sqrt{x} - 1}$　　**(b)** $\lim\limits_{x \to 3} \dfrac{\sin(x^2 - 9)}{x - 3}$　　**(c)** $\lim\limits_{x \to -\infty} \left(\sqrt{x^2 - x + 1} - \sqrt{x^2 - 2x} \right)$

2. Find the derivatives of the following functions.

(a) $f(x) = \tan^3 \left(4\sin^2(3x) \right)$　　**(b)** $f(x) = \left(\cos x^2 \right)^x$

(c) Find $f'(0)$ of $f(x) = \ln \left(\dfrac{3^x}{3^x + 1} \right)$

3. Evaluate $\lim\limits_{x \to 0} \left(e^x - x \right)^{\frac{1}{x}}$.

4. (a) Let $F(x)$ be a one-to-one function with inverse F^{-1}. Define a new function

$$p(x) = 1 - 2F \left(\frac{x}{3} \right)$$

Find a formula for p^{-1} in terms of F^{-1}.

(b) Find the derivative of the inverse of the function $f(x) = \arctan x + e^{123x}$ at $x = 1$. That is, find $\left(f^{-1} \right)' (1)$.

5. Find the equation of the tangent line to the curve $x^2 y^2 - 36x = 37$ at $(-1, 1)$.

6. The length of a hypotenuse of a right triangle is constant at 5 cm, and the length of one of its sides is decreasing at a rate of 2 cm/sec. Find the rate of change of the area of the triangle when this side is 3 cm long.

1. (a) Multiply each side by the conjugate of the denominator to eliminate the indetermination.

$$\lim_{x \to 1} \frac{x^2 + x - 2}{\sqrt{x} - 1} = \lim_{x \to 1} \left[\frac{(x+2)(x-1)}{\sqrt{x} - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \right] = \lim_{x \to 1} \frac{(x+2)(x-1)\left(\sqrt{x} + 1\right)}{x - 1}$$

$$= \lim_{x \to 1} \left[(x+2)\left(\sqrt{x} + 1\right) \right] = 3 \cdot 2 = \boxed{6}$$

(b)

$$\lim_{x \to 3} \frac{\sin(x^2 - 9)}{x - 3} = \lim_{x \to 3} \left[\frac{\sin(x^2 - 9)}{x - 3} \cdot \frac{x + 3}{x + 3} \right] = \lim_{x \to 3} \left[\frac{\sin(x^2 - 9)}{x^2 - 9} \right] \cdot \lim_{x \to 3}(x + 3)$$

The value $\lim_{x \to 0} \dfrac{\sin u}{u}$ can be evaluated by using the squeeze theorem, and it could be expected that we knew the value of this limit prior to the examination. Set $u = x^2 - 9$. So, the left-hand limit is 1.

$$= \lim_{x \to 3} \left[\frac{\sin(x^2 - 9)}{x^2 - 9} \right] \cdot \lim_{x \to 3}(x + 3) = 1 \cdot 6 = \boxed{6}$$

(c) This is an indeterminate $(\infty - \infty)$ form. We expand the expression by using its conjugate and dividing each side of the fraction by x to eliminate the indeterminate form.

$$\lim_{x \to -\infty} \left(\sqrt{x^2 - x + 1} - \sqrt{x^2 - 2x} \right)$$

$$= \lim_{x \to -\infty} \left[\sqrt{x^2 - x + 1} - \sqrt{x^2 - 2x} \cdot \frac{\sqrt{x^2 - x + 1} + \sqrt{x^2 - 2x}}{\sqrt{x^2 - x + 1} + \sqrt{x^2 - 2x}} \right]$$

$$= \lim_{x \to -\infty} \frac{x^2 - x + 1 - (x^2 - 2x)}{\sqrt{x^2 - x + 1} + \sqrt{x^2 - 2x}} = \lim_{x \to -\infty} \left(\frac{x + 1}{\sqrt{x^2 - x + 1} + \sqrt{x^2 - 2x}} \cdot \frac{x}{x} \right)$$

$$= \lim_{x \to -\infty} \frac{\dfrac{x + 1}{x}}{\dfrac{|x|\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + |x|\sqrt{1 - \frac{1}{x}}}{x}} = \lim_{x \to -\infty} -\frac{1 + \dfrac{1}{x}}{\sqrt{1 - \dfrac{1}{x} + \dfrac{1}{x^2}} + \sqrt{1 - \dfrac{2}{x}}} \quad \left[|x| = -x \right]$$

$$= -\frac{1 - 0}{\sqrt{1 + 0 - 0} + \sqrt{1 + 0}} = \boxed{-\frac{1}{2}}$$

2. (a) Apply the chain rule accordingly.

$$f'(x) = 3\tan^2\left(4\sin^2(3x)\right) \cdot \sec^2\left(4\sin^2(3x)\right) \cdot 8\sin(3x) \cdot \cos(3x) \cdot 3$$

(b) Take the logarithm of both sides to differentiate easily.

$$\ln\left(f(x)\right) = \ln\left[\left(\cos x^2\right)^x\right] = x\ln\left(\cos x^2\right)$$

$$\frac{d}{dx}\ln\left(f(x)\right) = \frac{d}{dx}\left[x\ln\left(\cos x^2\right)\right]$$

$$\frac{1}{f(x)} \cdot f'(x) = 1 \cdot \left[\ln\left(\cos x^2\right)\right] + x \cdot \frac{1}{\cos x^2} \cdot \left(-\sin x^2\right) \cdot 2x$$

$$f'(x) = f(x)\left[\ln\left(\cos x^2\right) - 2x^2 \cdot \tan x^2\right]$$

$$f'(x) = \left(\cos x^2\right)^x \cdot \left[\ln\left(\cos x^2\right) - 2x^2 \cdot \tan x^2\right]$$

(c) Rewrite the right-hand side using the property of logarithms. Take the first derivative afterwards.

$$f(x) = \ln\left(\frac{3^x}{3^x + 1}\right) = \ln\left(3^x\right) - \ln\left(3^x + 1\right)$$

$$f'(x) = \frac{1}{3^x} \cdot 3^x \cdot \ln(3) - \frac{1}{3^x + 1} \cdot 3^x \cdot \ln(3) = \ln(3) \cdot \left[1 - \frac{3^x}{3^x + 1}\right]$$

$$f'(0) = \ln(3) \cdot \left[1 - \frac{3^0}{3^0 + 1}\right] = \boxed{\frac{\ln 3}{2}}$$

3. Let L be the value of the limit.

$$L = \lim_{x \to 0}\left(e^x - x\right)^{\frac{1}{x}}$$

$$\ln(L) = \ln\left[\lim_{x \to 0}\left(e^x - x\right)^{\frac{1}{x}}\right]$$

The expression is defined for $x \neq 0$. Therefore, we can take the logarithm function inside the limit. After that, apply L'Hôpital's rule to eliminate the indeterminate form.

$$\ln(L) = \lim_{x \to 0}\ln\left[\left(e^x - x\right)^{\frac{1}{x}}\right] = \lim_{x \to 0}\frac{\ln\left(e^x - x\right)}{x} \qquad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x \to 0}\frac{\frac{1}{e^x - x} \cdot \left(e^x - 1\right)}{1} = \lim_{x \to 0}\frac{e^x - 1}{e^x - x} = \frac{e^0 - 1}{e^0 - 0} = 0$$

If $\ln(L) = 0$, then $\boxed{L = 1}$.

4. (a)

$$p(x) = 1 - 2F\left(\frac{x}{3}\right)$$

$$p(x) - 1 = -2F\left(\frac{x}{3}\right)$$

$$\frac{1 - p(x)}{2} = F\left(\frac{x}{3}\right)$$

$$F^{-1}\left(\frac{1 - p(x)}{2}\right) = \frac{x}{3}$$

$$3F^{-1}\left(\frac{1 - p(x)}{2}\right) = x$$

$$3F^{-1}\left(\frac{1 - p\left(p^{-1}(x)\right)}{2}\right) = p^{-1}(x)$$

$$\boxed{p^{-1}(x) = 3F^{-1}\left(\frac{1 - x}{2}\right)}$$

(b) Find a root so that $f(x_0) = \arctan(x_0) + e^{123 \cdot x_0} = 1$. We can intuitively say that the root is small because both $\arctan x$ and e^x are strictly increasing everywhere. Try $x = 0$.

$$f(0) = \arctan 0 + e^{123 \cdot 0} = 0 + 1 = 1$$

Therefore, $f^{-1}(1) = 0$. The derivative of an inverse function at a given point is

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(f^{-1}(1))}$$

Calculate $f'(x)$ and $(f^{-1})'(1)$.

$$f'(x) = \frac{1}{1 + x^2} + 123e^{122x}$$

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(0)} = \left(\frac{1}{1 + 0^2} + 123e^{122 \cdot 0}\right)^{-1} = \boxed{\frac{1}{124}}$$

5. Consider $y = f(x)$. Differentiate both sides implicitly.

$$\frac{d}{dx}\left(x^2 y^2 - 36x\right) = \frac{d}{dx} 37$$

$$2xy^2 + x^2 \cdot 2y \cdot \frac{dy}{dx} - 36 = 0$$

$$x^2 \cdot 2y \cdot \frac{dy}{dx} = 36 - 2xy^2$$

$$\frac{dy}{dx} = \frac{36 - 2xy^2}{x^2 \cdot 2y}$$

Calculate $\frac{dy}{dx}$ at $(-1, 1)$.

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{36 - 2(-1) \cdot 1^2}{(-1)^2 \cdot 2 \cdot 1} = 19 \tag{1}$$

Using the straight line formula, we find the tangent line. Recall: $y - y_0 = m(x - x_0)$, where m can be substituted with (1).

$$\boxed{y - 1 = 19(x + 1)}$$

6. Let $x(t)$, $y(t)$, $l(t)$ represent the lengths of the sides as functions of time. We can set up the following equation for the area of the right triangle.

$$A(t) = \frac{x(t) \cdot y(t)}{2}$$

Take the derivative of both sides.

$$A'(t) = \frac{1}{2}\left(x'(t) \cdot y(t) + x(t) \cdot y'(t)\right) \tag{2}$$

We also know that, by the Pythagorean theorem,

$$l^2(t) = x^2(t) + y^2(t)$$

Take the derivative of both sides.

$$2l(t)l'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

Since the length of the hypotenuse is constant, $l'(t) = 0$. Therefore,

$$x(t)x'(t) = -y(t)y'(t) \tag{3}$$

At $t = t_0$, we have $l(t_0) = 5$, $x(t_0) = 3$, $x'(t_0) = -2$, and by the Pythagorean theorem, $y(t_0) = \sqrt{5^2 - 3^2} = 4$. Calculate $y'(t_0)$ from (3).

$$y'(t_0) = -\frac{3 \cdot (-2)}{4} = \frac{3}{2}$$

Plug the necessary values into (2) to find the rate of change of the area.

$$A'(t_0) = \frac{1}{2}\left((-2) \cdot 4 + 3 \cdot \frac{3}{2}\right) = \boxed{-\frac{7}{4} \, \text{cm}^2/\text{s}}$$