

2023-2024 Spring
MAT124 Midterm
(26/04/2024)

1. Sketch the traces of the following surfaces with the coordinate planes $x = 0$, $y = 0$, and $z = 0$, and then sketch the graphs of them.

(a) $y = \ln z$ (b) $x^2 + 2y^2 - 3z^2 + 1 = 0$

2. State the $\epsilon - \delta$ definition of the limit of a function of two variables, and using the $\epsilon - \delta$ definition, show that

$$\lim_{(x,y) \rightarrow (0,0)} (2x^2 + y^2) = 0$$

3. For $t \in \mathbb{R}$, let l_1 and l_2 be two lines given by

$$\begin{array}{ll} x = 3t & x = 1 + 6\lambda \\ y = 4 - t & y = 2 - 2\lambda \\ z = 1 + 2t & z = 1 + 4\lambda \end{array} \quad \begin{array}{l} t \in \mathbb{R} \\ \lambda \in \mathbb{R} \end{array}$$

Determine an equation of the plane containing both l_1 and l_2 .

4. Let

$$f(x, y) = \begin{cases} \frac{x^2 \cdot e^{x^2+y}}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(i) Evaluate the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$.

(ii) Show that $f(x, y)$ is not differentiable at $(0, 0)$.

5. Let $u = u(x, y)$ and $v = v(x, y)$. The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Show that in the polar coordinate system ($x = r \cos \theta$, $y = r \sin \theta$),

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

6. Determine the direction at the point $(1, 1)$ in which the rate of change of

$$f(x, y) = \frac{2^{xy}}{x}$$

is the largest. Compute this rate of change.

7. Find all critical points of the function

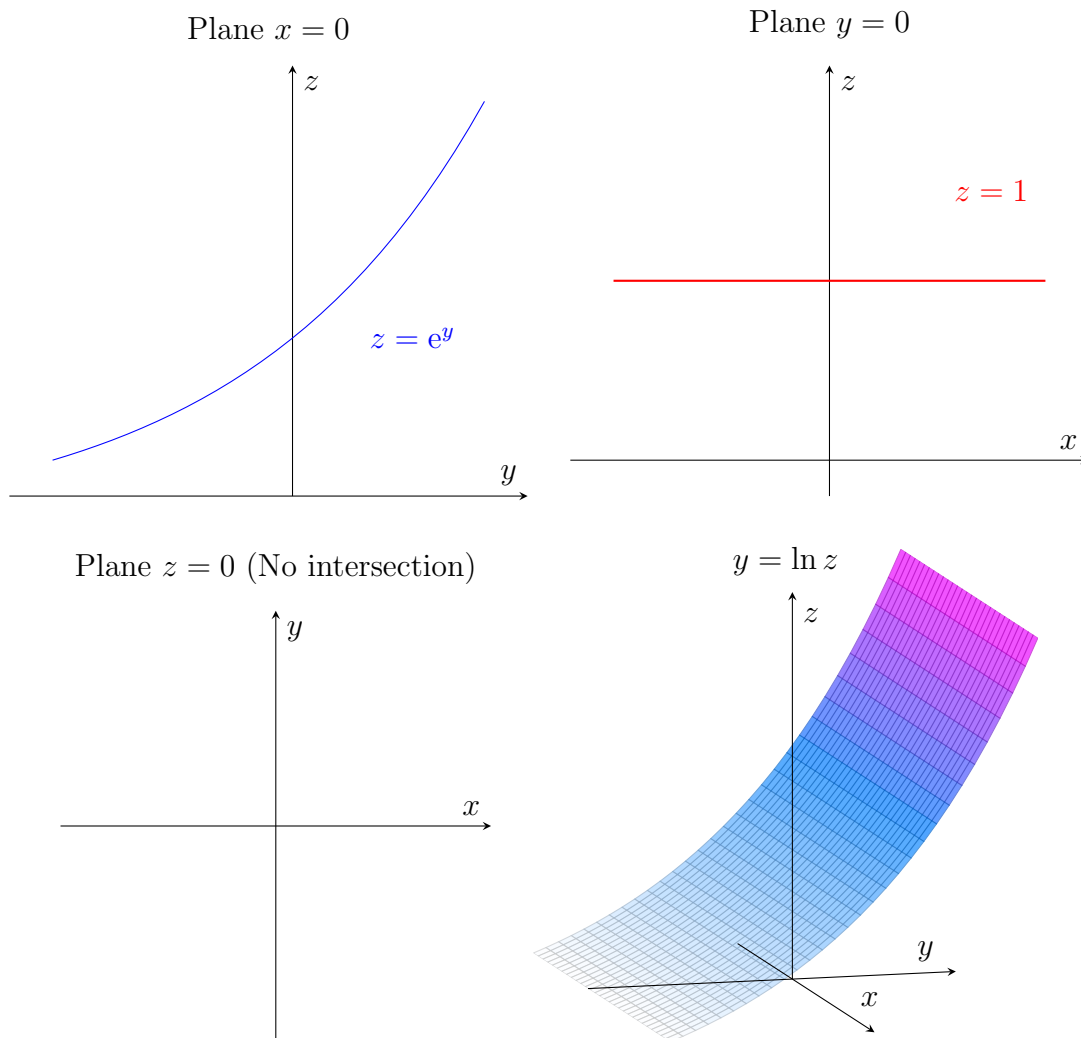
$$f(x, y) = (x - 1)(y + 1)(x - y + 3)$$

and then determine whether each critical point corresponds to a local maximum, a local minimum or a saddle point.

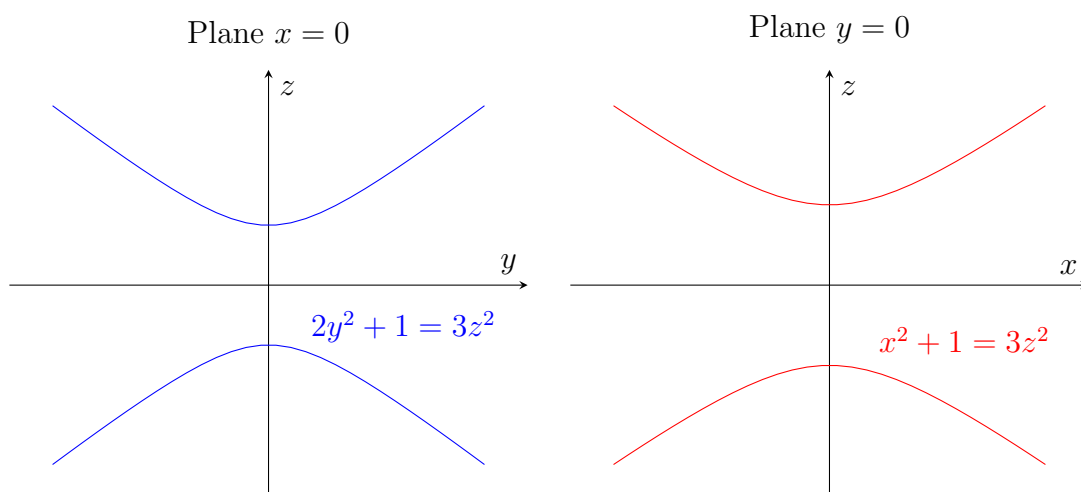
2023-2024 Spring Midterm (26/04/2024) Solutions
(Last update: 27/08/2025 23:24)

1.

(a)

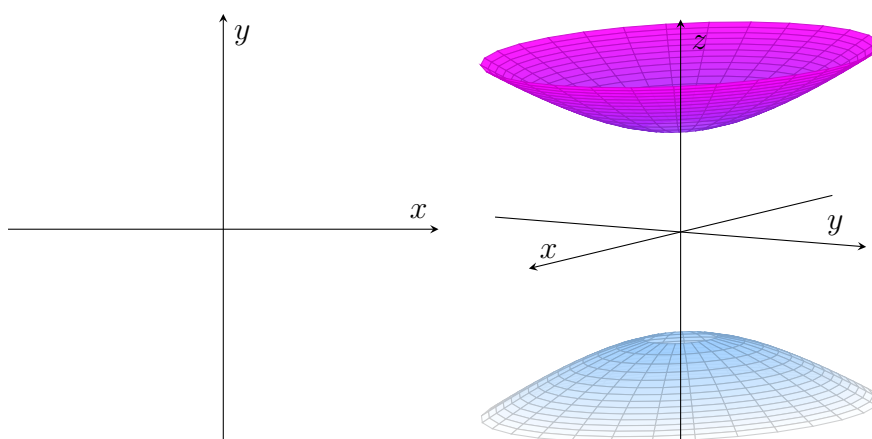


(b)



Plane $z = 0$ (No intersection)

$$x^2 + 2y^2 - 3z^2 + 1 = 0$$



2. The $\epsilon - \delta$ definition is the formal way of calculating the limit of a function at a point (x_0, y_0) . According to the definition, for every $\epsilon > 0$, there exists a δ such that

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \implies |f(x, y) - L| < \epsilon,$$

where L is the limit.

Then for every $\epsilon > 0$, there exists a δ such that

$$0 < \sqrt{x^2 + y^2} < \delta \implies |2x^2 + y^2| < \epsilon.$$

$$|2x^2 + y^2| \leq 2x^2 + y^2 \quad [x^2 \geq 0, \quad y^2 \geq 0]$$

$$2x^2 + y^2 \leq 2x^2 + 2y^2 = 2(x^2 + y^2) \leq 2\delta^2 \quad \left[\sqrt{x^2 + y^2} < \delta \implies x^2 + y^2 < \delta^2 \right]$$

$$\text{Let } \delta = \sqrt{\frac{\epsilon}{2}}.$$

$$|2x^2 + y^2| \leq 2(x^2 + y^2) < 2 \cdot \left(\sqrt{\frac{\epsilon}{2}} \right)^2 = 2 \cdot \frac{\epsilon}{2} = \epsilon$$

This shows that the limit is equal to 0.

3. The coefficients of the parametrization variables have the same ratio. That is, they are parallel. Choose $P_1(3, 3, 3)$ on l_1 and $P_2(1, 2, 1)$ on l_2 . The cross product of $\mathbf{u} = \langle 3, -1, 2 \rangle$, which is parallel to the lines, and the vector \mathbf{v} joining P_1 and P_2 , where $\mathbf{w} = \langle 3 - 1, 3 - 2, 3 - 1 \rangle = \langle 2, 1, 2 \rangle$, gives us the normal of the plane.

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= (-1 \cdot 2 - 1 \cdot 2)\mathbf{i} - (3 \cdot 2 - 2 \cdot 2)\mathbf{j} + (3 \cdot 1 - 2 \cdot (-1))\mathbf{k} = -4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

Using the definition $\mathbf{n} \cdot \overrightarrow{PP_1} = 0$, where P is a point on the plane, the equation of the plane is

$$\begin{aligned}\mathbf{n} \cdot \overrightarrow{PP_1} = 0 &\implies \langle -4, -2, 5 \rangle \cdot \langle x - 3, y - 3, z - 3 \rangle = 0 \\ &\implies -4(x - 3) - 2(y - 3) + 5(z - 3) = 0 \implies \boxed{-4x - 2y + 5z + 3 = 0}\end{aligned}$$

4.

(i) Use the definition of the partial derivative.

$$\begin{aligned}f_x|_{(0,0)} &= \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{(0+h)^2 \cdot e^{(0+h)^2+0}}{(0+h)^2 + 0^2}}{h} = \lim_{h \rightarrow 0} \frac{e^{h^2}}{h} \\ &= \infty \text{ (} f_x \text{ does not exist)}\end{aligned}$$

$$f_y|_{(0,0)} = \lim_{h \rightarrow 0} \frac{f(0, 0+h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0^2 \cdot e^{0^2+(0+h)}}{0^2 + (0+h)^2}}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

(ii) f_y is a finite number. However, since f_x does not exist, f is not differentiable at $(0, 0)$.

5. u and v are functions of x and y . x and y are functions of r and θ . Use the chain rule.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = u_x \cdot \cos \theta + u_y \cdot \sin \theta = v_y \cos \theta - v_x \sin \theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta} \implies \frac{1}{r} \frac{\partial v}{\partial \theta} = \frac{1}{r} (-v_x \cdot r \sin \theta + v_y \cdot r \cos \theta) = -v_x \sin \theta + v_y \cos \theta$$

$$\frac{\partial u}{\partial r} = -v_x \sin \theta + v_y \cos \theta = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r} = v_x \cdot \cos \theta + v_y \cdot \sin \theta = -u_y \cos \theta + u_x \sin \theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} \implies -\frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{1}{r} (-u_x r \sin \theta + u_y r \cos \theta) = u_x \sin \theta - u_y \cos \theta$$

$$\frac{\partial v}{\partial r} = u_x \sin \theta - u_y \cos \theta = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

6. The function f has the maximum rate of change if the gradient vector of f and the unit direction vector \mathbf{u} are in the same direction. Apply the quotient rule to compute the gradient of f .

$$\nabla f = \left\langle \frac{(2^{xy} \cdot y \cdot \ln 2) \cdot x - 2^{xy} \cdot (1)}{x^2}, 2^{xy} \cdot \ln 2 \right\rangle = \left\langle \frac{2^{xy} (yx \ln 2 - 1)}{x^2}, 2^{xy} \cdot \ln 2 \right\rangle$$

$$(\nabla f \cdot \mathbf{u})_{\max} = |\nabla f| |u| \cos 0 = |\nabla f|$$

$$\nabla f|_{(1,1)} = \langle 2 \ln 2 - 2, 2 \ln 2 \rangle \implies |\nabla f| = \sqrt{(2 \ln 2 - 2)^2 + (2 \ln 2)^2}$$

$$= 2\sqrt{2 \ln^2 2 - 2 \ln 2 + 1}$$

The unit direction vector u is

$$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{\langle 2 \ln 2 - 2, 2 \ln 2 \rangle}{2\sqrt{2 \ln^2 2 - 2 \ln 2 + 1}} = \left\langle \frac{\ln 2 - 1}{\sqrt{2 \ln^2 2 - 2 \ln 2 + 1}}, \frac{\ln 2}{\sqrt{2 \ln^2 2 - 2 \ln 2 + 1}} \right\rangle$$

The maximum rate of change: $2\sqrt{2 \ln^2 2 - 2 \ln 2 + 1}$

The direction vector: $\left\langle \frac{\ln 2 - 1}{\sqrt{2 \ln^2 2 - 2 \ln 2 + 1}}, \frac{\ln 2}{\sqrt{2 \ln^2 2 - 2 \ln 2 + 1}} \right\rangle$

7. Apply the chain rule.

$$f_x = (y + 1) [1 \cdot (x - y + 3) + (x - 1) \cdot 1] = (y + 1)(2x - y + 2)$$

$$= 2xy - y^2 + 2y + 2x - y + 2 = -y^2 + y + 2xy + 2x + 2$$

$$f_y = (x - 1) [1 \cdot (x - y + 3) + (y + 1) \cdot (-1)] = (x - 1)(x - 2y + 2)$$

$$= x^2 - 2xy + 2x - x + 2y - 2 = x^2 - 2xy + x + 2y - 2$$

The critical points occur where one of the partial derivatives does not exist or $f_x = f_y = 0$. f_x and f_y are continuous everywhere. Therefore, we may simply determine where $f_x = f_y = 0$.

$$f_x = f_y = 0 \implies \left. \begin{array}{l} 2xy - y^2 + 2y + 2x - y + 2 = 0 \\ x^2 - 2xy + x + 2y - 2 = 0 \end{array} \right\} \quad (1) \quad x^2 - y^2 + 3(x + y) = 0$$

$$\implies (x - y)(x + y) + 3(x + y) = 0 \implies (x - y + 3)(x + y) = 0$$

We have two cases: $x + 3 = y$ or $x = -y$.

$$\text{Case I : } x = -y \xrightarrow{(1)} -y^2 - 2y^2 + y - 2y + 2 = 0 \implies -3y^2 - y + 2 = 0$$

$$\implies (2 - 3y)(y + 1) = 0 \implies y_1 = \frac{2}{3}, y_2 = -1$$

$$y_1 = \frac{2}{3} \implies x_1 = -\frac{2}{3} \quad y_2 = -1 \implies x_2 = 1$$

$$\text{Case II : } x + 3 = y \xrightarrow{(1)} -(x + 3)^2 + 2x(x + 3) + x + 3 + 2x + 2 = 0$$

$$\implies -x^2 - 6x - 9 + 2x^2 + 6x + 3x + 5 = 0 \implies x^2 + 3x - 4 = 0$$

$$x_{3,4} = \frac{-3 \pm \sqrt{9 - 4 \cdot 1 \cdot (-4)}}{2 \cdot 1} = \frac{-3 \pm 5}{2} \implies x_3 = -4, x_4 = 1 \implies y_3 = -1, y_4 = 4$$

To classify these four critical points, apply the Second Derivative Test.

$$f_{xx} = 2y + 2, \quad f_{xy} = f_{yx} = 2x - 2y + 1, \quad f_{yy} = -2x + 2$$

$$\left(-\frac{2}{3}, \frac{2}{3}\right) \rightarrow \begin{cases} f_{xx} = \frac{10}{3}, & f_{xy} = -\frac{5}{3}, & f_{yy} = \frac{10}{3} \\ \left| \begin{array}{cc} \frac{10}{3} & -\frac{5}{3} \\ -\frac{5}{3} & \frac{10}{3} \end{array} \right| = \frac{10}{3} \cdot \frac{10}{3} - \left(-\frac{5}{3}\right) \cdot \left(-\frac{5}{3}\right) = \frac{25}{3}, & f_{xx} = \frac{10}{3} > 0 \end{cases}$$

$$(1, -1) \rightarrow \begin{cases} f_{xx} = 0, & f_{xy} = 5, & f_{yy} = 0 \\ \left| \begin{array}{cc} 0 & 5 \\ 5 & 0 \end{array} \right| = 0 \cdot 0 - 5 \cdot 5 = -25 < 0 \end{cases}$$

$$(-4, -1) \rightarrow \begin{cases} f_{xx} = 0, & f_{xy} = -5, & f_{yy} = 10 \\ \left| \begin{array}{cc} 0 & -5 \\ -5 & 10 \end{array} \right| = 0 \cdot 10 - (-5) \cdot (-5) = -25 < 0 \end{cases}$$

$$(1, 4) \rightarrow \begin{cases} f_{xx} = 10, & f_{xy} = 5, & f_{yy} = 0 \\ \left| \begin{array}{cc} 10 & 5 \\ 5 & 0 \end{array} \right| = 10 \cdot 0 - 5 \cdot 5 = -25 < 0 \end{cases}$$

A local minimum occurs at $\left(-\frac{2}{3}, \frac{2}{3}\right)$.
Saddle points occur at $(1, -1)$, $(-4, -1)$, and $(1, 4)$.