2020-2021 Fall
MAT123 Final
(18/01/2021)

1. Consider the region R bounded by the curve $y = x^3$, and the straight lines $y = -x$ and $y = x + 6$.

(i) Write down the integral corresponding to the area of R with respect to x.

(ii) Write down the integral corresponding to the area of R with respect to y.

2. Consider the solid S obtained by revolving the region in the first quadrant bounded by $x = -y^2 + 1$, $y^2 = x$ and $y = 1/2$ about $x = -1$.

(i) Using the Shell Method, write down the integral corresponding to the volume of the solid S.

(ii) Using the Washer Method, write down the integral corresponding to the volume of the solid S.

3. Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{x^{2/3}\left(\sqrt[3]{x}+4\right)}$

(b) $\displaystyle\int (\ln x)^2\, dx$

(c) $\displaystyle\int \frac{dx}{\sqrt{3+x^2}}$

(d) $\displaystyle\int \frac{dx}{2+\sin x}$

4. Using the Monotone Convergence Theorem, show that the sequence $\displaystyle\left(\frac{n^2+1}{n^3}\right)_{n\in\mathbb{N}}$ is convergent.

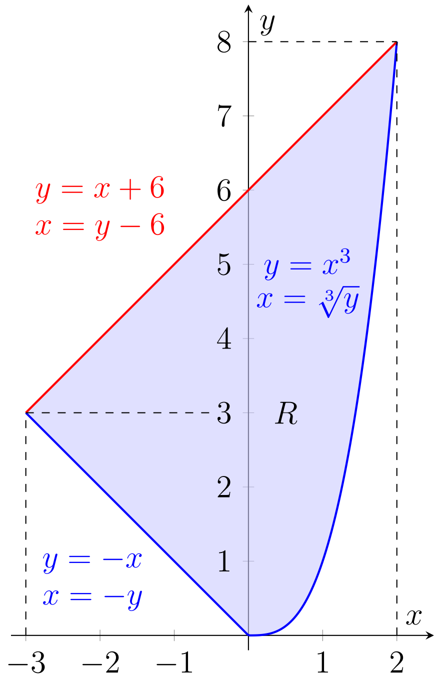
5. Use the Integral Test to determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4+k^2}$ converges or diverges.

6. Find the convergence set for the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(x+1)^k}{2k}$.

7. Using a Maclaurin series, show that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$.

2020-2021 Fall Final (18/01/2021) Solutions
(Last update: 16/08/2025 15:00)

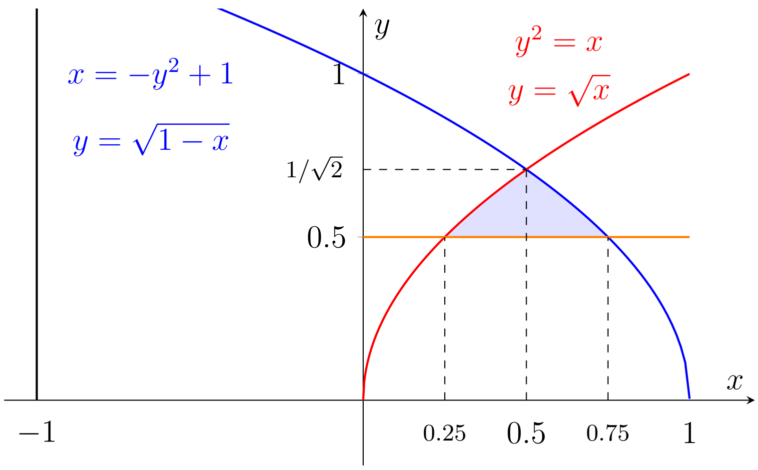
1.



(i)
$$\int_{-3}^{0} [(x+6) - (-x)]\, dx + \int_{0}^{2} \left[(x+6) - \left(x^3\right)\right]\, dx$$

(ii)
$$\int_{0}^{3} [(\sqrt[3]{y}) - (-y)]\, dy + \int_{3}^{8} [(\sqrt[3]{y}) - (y-6)]\, dy$$

2.



(i)

$$\int_{1/4}^{1/2} 2\pi(x+1)\left[\left(\sqrt{x}\right) - \left(\frac{1}{2}\right)\right] dx + \int_{1/2}^{3/4} 2\pi(x+1)\left[\left(\sqrt{1-x}\right) - \left(\frac{1}{2}\right)\right] dx$$

(ii)

$$\int_{1/2}^{1/\sqrt{2}} \pi\left[\left(-y^2+1+1\right)^2 - \left(y^2+1\right)^2\right] dy$$

3. **(a)** Let $u = \sqrt[3]{x} + 4$, then $du = \dfrac{1}{3x^{2/3}}\, dx$.

$$\int \frac{dx}{x^{2/3}\left(\sqrt[3]{x}+4\right)} = \int \frac{3du}{u} = 3\ln|u| + c = \boxed{3\ln\left|\sqrt[3]{x}+4\right| + c, \quad c \in \mathbb{R}}$$

(b) Apply integration by parts.

$$\left.\begin{array}{r} u = (\ln x)^2 \implies du = 2\ln x \cdot \dfrac{1}{x}\, dx \\ dv = dx \implies v = x \end{array}\right\} \to \int u\, dv = uv - \int v\, du$$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - \int 2\ln x\, dx$$

Apply integration by parts once again.

$$\left.\begin{array}{r} u = \ln x \implies du = \dfrac{1}{x}\, dx \\ dv = dx \implies v = x \end{array}\right\} \to \int u\, dv = uv - \int v\, du$$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - \int 2\ln x\, dx$$

$$x(\ln x)^2 - \int 2\ln x\, dx = x(\ln x)^2 - 2\left[x\ln x - \int dx\right] = \boxed{x(\ln x)^2 - 2x\ln x + 2x + c,\ c \in \mathbb{R}}$$

(c) Let $x = \sqrt{3}\tan u$ for $0 < u < \dfrac{\pi}{2}$, then $dx = \sqrt{3}\sec^2 u\, du$.

$$\int \frac{dx}{\sqrt{3+x^2}} = \int \frac{\sqrt{3}\sec^2 u}{\sqrt{3+3\tan^2 u}}\, du = \int \frac{\sec^2 u}{\sqrt{1+\tan^2 u}}\, du = \int \frac{\sec^2 u}{|\sec u|}\, du$$

$$= \int \sec u\, du \qquad [\sec u > 0]$$

$$= \ln|\tan u + \sec u| + c, \quad c \in \mathbb{R}$$

Recall $x = \sqrt{3}\tan u$.

$$x^2 = 3\tan^2 u = 3\sec^2 u - 3 \implies \sec^2 u = \frac{x^2+3}{3} \implies \sec u = \frac{\sqrt{x^2+3}}{\sqrt{3}}$$

$$\ln|\tan u + \sec u| + c = \boxed{\ln\left|\frac{x}{\sqrt{3}} + \frac{\sqrt{x^2+3}}{\sqrt{3}}\right| + c, \quad c \in \mathbb{R}}$$

We can omit the constant part.

$$\boxed{\ln\left(x+\sqrt{x^2+3}\right)+c, \quad c\in\mathbb{R}}$$

(d) We may utilize the tangent half-angle substitution, which is also called the Weierstrass substitution. Let $t = \tan\left(\dfrac{x}{2}\right)$. After some mathematical operations, we get the following.

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2}{1+t^2}\,dt$$

$$\int \frac{dx}{2+\sin x} = \int \frac{2}{1+t^2}\cdot\frac{1}{2+\frac{2t}{1+t^2}}\,dt = \int \frac{dt}{t^2+t+1} = \int \frac{dt}{t^2+t+\frac{1}{4}+\frac{3}{4}}$$

$$= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{4}{3}\int \frac{dt}{\frac{4}{3}\left(t+\frac{1}{2}\right)^2+1} = \frac{4}{3}\int \frac{dt}{\left(\frac{2}{\sqrt{3}}\right)^2\left(t+\frac{1}{2}\right)^2+1}$$

Let $u = \dfrac{2}{\sqrt{3}}\left(t+\dfrac{1}{2}\right)$, then $du = \dfrac{2}{\sqrt{3}}\,dt$.

$$\frac{2\sqrt{3}}{3}\int \frac{du}{u^2+1} = \frac{2\sqrt{3}}{3}\arctan u + c = \frac{2\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right)+c$$

$$= \boxed{\frac{2\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\left(\tan\left(\frac{x}{2}\right)+\frac{1}{2}\right)\right)+c, \quad c\in\mathbb{R}}$$

4. Take $f(x) = \dfrac{x^2+1}{x^3}$. We have $f'(x) = -\dfrac{x^2+3}{x^4} < 0$ for all $x \geq 1$. That means f is decreasing for $x \geq 1$. We also have

$$f(1) = 2, \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{1}{x}+\frac{1}{x^3}\right) \qquad \Longrightarrow 0 < f(x) \leq 2, \quad x \geq 1$$

Since the sequence is bounded and monotonic, by the Monotone Convergence Theorem, the sequence converges.

5. Take the corresponding function $f(x) = \dfrac{1}{x^4+x^2}$. f is positive for $x \geq 1$ because $x^4 > 0$ and $x^2 > 0$. f is also continuous for $x \geq 1$ because the denominator is a polynomial whose *only* root is zero, which is out of the boundary of the integral. Investigate the monotonicity of f by taking the first derivative.

$$f'(x) = -\frac{4x^3+2x}{\left(x^4+x^2\right)^2} \implies f'(x) \leq 0 \text{ for } x \geq 1$$

We may now apply the Integral Test since the criteria have been satisfied.

$$\int_1^\infty \frac{dx}{x^4+x^2} = \int_1^\infty \frac{dx}{x^2\left(x^2+1\right)} = \int_1^\infty \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) dx$$

$$= \lim_{R\to\infty} \int_1^R \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) dx = \lim_{R\to\infty} \left[-\frac{1}{x} - \arctan x\right]_1^R$$

$$= \lim_{R\to\infty} \left[\left(-\frac{1}{R} - \arctan R\right) - (-1 - \arctan 1)\right] = 1 - \frac{\pi}{4} \quad \text{(convergent)}$$

By the Integral Test, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4+k^2}$ also converges.

6. Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\lim_{n\to\infty} \left|\frac{(x+1)^{k+1}}{2(k+1)} \cdot \frac{2k}{(x+1)^k}\right| = \lim_{n\to\infty} \left|\frac{(x+1)\cdot k}{k+1}\right| = |x+1| \cdot \lim_{n\to\infty} \left|\frac{k}{k+1}\right| = |x+1|$$

$$|x+1| < 1 \implies -1 < x+1 < 1 \implies -2 < x < 0 \quad \text{(convergent)}$$

Now, take a look at the endpoints.

$$x = -2 \to \sum_{k=1}^{\infty} \frac{(-1)^k}{2k} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{k}$, is nonincreasing for $k \geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges. Try $x = 0$.

$$x = 0 \to \sum_{k=1}^{\infty} \frac{1^k}{2k} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k}$$

This is a p-series with $p = 1$, for which the series diverges by the p-series Test.

The convergence set for the power series is

$$\boxed{[-2, 0)}$$

7. The Maclaurin series of $\sin x$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

Rewrite the limit using this expansion.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \left[\frac{1}{x} \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)\right] = \lim_{x\to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...\right) = \boxed{1}$$