

2015-2016 Fall Semester
MAT123-07 Midterm
(10/12/2015)

- 1) Find all local extrema and inflection points of the function $f(x) = \frac{1}{x} + \frac{1}{x^2}$. On which intervals is the function increasing, decreasing, concave upward, or concave downward? Find all asymptotes. Graph the function.
- 2) Use Rolle's theorem to show that $3 \tan x + x^3 = 2$ has exactly one solution on the interval $[0, \pi/4]$.
- 3) Find the tangent line to the graph of the equation $x \sin(xy - y^2) = x^2 - 1$, at $(1, 1)$.
- 4) Evaluate the limits
 - a. $\lim_{x \rightarrow 0^+} (1 + \sin x)^{\frac{1}{x}}$
 - b. $\lim_{x \rightarrow 3} \frac{x^2 - \lfloor x^2 \rfloor}{x - 3}$
- 5) A point P is moving in the xy -plane. When P is at $(4, 3)$, its distance to the origin is increasing at a rate of $\sqrt{2}$ cm/s, and its distance to the point $(7, 0)$ is decreasing at a rate of 3 cm/s. Determine the rate of change of the x -coordinate of P at that moment.
- 6) Sketch the region bounded by $y = 2|x|$ and $y = 8 - x^2$. Find the area of the region.

1) Take the first derivative and set to 0.

$$f'(x) = -\frac{1}{x^2} - \frac{2}{x^3}$$

$$f'(x) = 0 \rightarrow \frac{2}{x^3} = -\frac{1}{x^2} \rightarrow x = -2 \text{ (candidate for a critical point)}$$

Take the second derivative and set to 0.

$$f''(x) = \frac{2}{x^3} + \frac{6}{x^4}$$

$$f''(x) = 0 \rightarrow \frac{1}{x^3} = -\frac{3}{x^4} \rightarrow x = -3 \text{ (candidate for an inflection point)}$$

$\{-2, -3\} \in D$. Therefore, $f(-2)$ gives rise to a local extremum. The sign of the first derivative changes from minus to plus, meaning $f(-2)$ is a local minimum. $x = -3$ gives rise to an inflection point because the sign of the second derivative also changes.

Find the asymptotes.

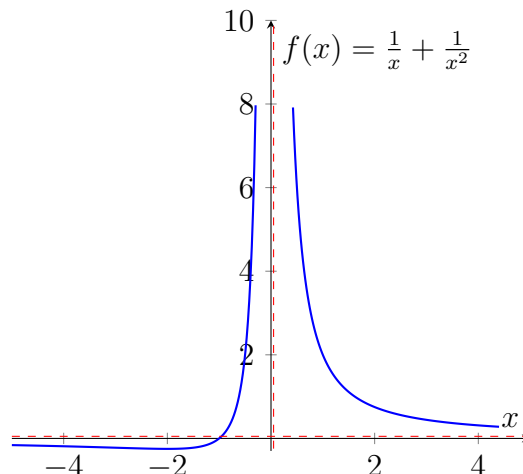
$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$$

$y = 0$ is the horizontal asymptote and $x = 0$ is the vertical asymptote.

Let us find the concavity and monotonicity. If the sign of $f'(x)$ is minus, the function is decreasing on the corresponding interval; otherwise, increasing. If the sign of $f''(x)$ is minus, the graph of the function is concave downward; otherwise, concave upward.

x	$(-\infty, -3)$	$(-3, -2)$	$(-2, 0)$	$(0, \infty)$
f' sign	-	-	+	-
f'' sign	-	+	+	+

Eventually, sketch the graph.



2) Let $f(x) = 3 \tan x + x^3 - 2$. f is continuous on $[0, \pi/4]$ and differentiable on $(0, \pi/4)$. By IVT (Intermediate Value Theorem), there exists at least one point where $f(x) = 0$ because $f(0) = -2$ and $f(\pi/4) = 1 + (\pi/4)^3$. Assume that we have two roots on the interval, so at some point c , $f'(c) = 0$.

$$f'(x) = 3 \sec^2 x + 3x^2 \rightarrow f'(c) = 3 \sec^2 c + 3c^2 = 0 \rightarrow \sec^2 c = -c^2$$

Since $-c^2 \leq 0$ and $\sec^2 c > 0$, there is no c that satisfies the equation. This contradicts our assumption that we have two roots on the interval. By Rolle's theorem, there is only one root on the interval $[0, \pi/4]$.

3) Implicitly differentiate both sides.

$$\frac{d}{dx}[x \sin(xy - y^2)] = \frac{d}{dx}(x^2 - 1)$$

$$1 \cdot \sin(xy - y^2) + x \cdot \cos(xy - y^2) \cdot \left[\left(1 \cdot y + x \frac{dy}{dx} \right) - 2y \frac{dy}{dx} \right] = 2x$$

Rearrange the equation to solve for $\frac{dy}{dx}$ through a careful and rigorous attempt.

$$x \cdot \cos(xy - y^2) \cdot \left[\left(1 \cdot y + x \frac{dy}{dx} \right) - 2y \frac{dy}{dx} \right] = 2x - \sin(xy - y^2)$$

$$y + \frac{dy}{dx}(x - 2y) = \frac{2x - \sin(xy - y^2)}{x \cdot \cos(xy - y^2)}$$

$$\frac{dy}{dx}(x - 2y) = \frac{2x - \sin(xy - y^2)}{x \cdot \cos(xy - y^2)} - y$$

$$\frac{dy}{dx} = \frac{\frac{2x - \sin(xy - y^2)}{x \cdot \cos(xy - y^2)} - y}{(x - 2y)} \quad (1)$$

Calculate $\frac{dy}{dx} \Big|_{(1,1)}$ from (1). This will give us the slope of the tangent line.

$$\frac{dy}{dx} \Big|_{(1,1)} = -1$$

Recall: $y - y_0 = m(x - x_0)$. m is $\frac{dy}{dx}$ at $x = 1$. So, the tangent line is:

$$y - 1 = -(x - 1) \rightarrow \boxed{y = 2 - x}$$

4)

a) Let L be the value of the limit. Then, take the logarithm of both sides.

$$L = \lim_{x \rightarrow 0^+} (1 + \sin x)^{\frac{1}{x}}$$

$$\ln(L) = \ln \left[\lim_{x \rightarrow 0^+} (1 + \sin x)^{\frac{1}{x}} \right]$$

The expression on the right is continuous for $x > 0$. Therefore, we can take the logarithm inside the limit.

$$\ln(L) = \lim_{x \rightarrow 0^+} \ln \left[(1 + \sin x)^{\frac{1}{x}} \right] = \lim_{x \rightarrow 0^+} \left[\frac{\ln(1 + \sin x)}{x} \right]$$

If we substitute $x = 0$, the limit is in the form $0/0$. L'Hôpital's rule states that we may take the derivatives of both sides of the fraction if there's a $0/0$ indeterminate form. Apply the chain rule accordingly.

$$\lim_{x \rightarrow 0^+} \left[\frac{\ln(1 + \sin x)}{x} \right] \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 0^+} \left[\frac{\frac{1}{1 + \sin x} \cdot \cos x}{1} \right] = \lim_{x \rightarrow 0^+} \left[\frac{\cos x}{1 + \sin x} \right]$$

The limit can now be evaluated by substituting $x = 0$.

$$\lim_{x \rightarrow 0^+} \left[\frac{\cos x}{1 + \sin x} \right] = \frac{\cos 0}{1 + \sin 0} = 1$$

Now, $\ln(L) = 1$. Simply, take L out of the logarithm.

$$\boxed{L = e}$$

b) Look at the one-sided limits. Let us first evaluate the limit from the right side. Above and near $x = 3$, the floor function will return 9.

$$\lim_{x \rightarrow 3^+} \frac{x^2 - \lfloor x^2 \rfloor}{x - 3} = \lim_{x \rightarrow 3^+} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3^+} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3^+} (x + 3) = 6$$

From the left side, the output is the largest integer less than 9. Therefore, $\lfloor x^2 \rfloor = 8$.

$$\lim_{x \rightarrow 3^-} \frac{x^2 - \lfloor x^2 \rfloor}{x - 3} = \lim_{x \rightarrow 3^-} \frac{x^2 - 8}{x - 3} = -\infty$$

The one-sided limits are not equal to each other. Therefore, the limit does not exist.

5) $x = x(t)$ and $y = y(t)$. The distance between the point P and the origin, and the distance between the point P and the point (7,0) are, respectively, given by:

$$f(t) = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$$

$$g(t) = \sqrt{(x - 7)^2 + (y - 0)^2} = \sqrt{x^2 - 14x + 49 + y^2}$$

Take the first derivative with respect to time.

$$f'(t) = \frac{1}{2(x^2 + y^2)} \cdot \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right) \quad (2)$$

$$g'(t) = \frac{1}{2(x^2 - 14x + 49 + y^2)} \cdot \left((2x - 14) \frac{dx}{dt} + 2y \frac{dy}{dt} \right) \quad (3)$$

For $t = t_0$, it is given $x(t_0) = 4$, $y(t_0) = 3$ and $f(t_0) = \sqrt{4^2 + 3^2} = 5$, $g(t_0) = \sqrt{3^2 + 3^2} = 3\sqrt{2}$. We then obtain a system of two equations by substituting values in (2) and (3):

$$\begin{aligned} f'(t_0) &= \frac{1}{50} \cdot \left(8 \frac{dx}{dt} + 6 \frac{dy}{dt} \right) = \sqrt{2} \\ g'(t_0) &= \frac{1}{36} \cdot \left((-6) \frac{dx}{dt} + 6 \frac{dy}{dt} \right) = 3 \end{aligned}$$

Let us simplify the equations for readiness.

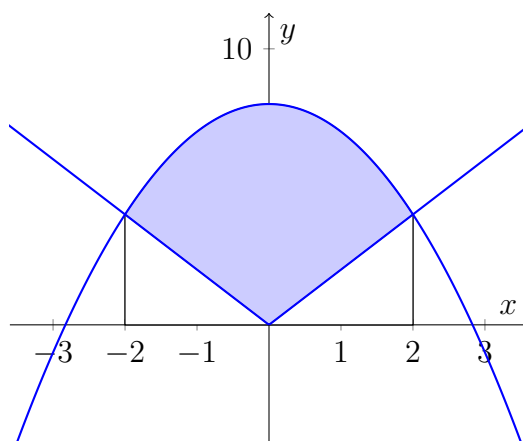
$$\begin{aligned} 8x'(t_0) + 6y'(t_0) &= 50\sqrt{2} \\ -6x'(t_0) + 6y'(t_0) &= 108 \end{aligned}$$

The question wants us to find the change in the x -coordinate of P. Therefore, negate the latter equation and solve for $x'(t_0)$.

$$\boxed{x'(t_0) = \frac{25\sqrt{2} - 54}{7}}$$

The change in the y -coordinate is left as a practice for the reader.

6)



The area can be found by integrating the difference in y with respect to x . We split the integral into two because the absolute value function changes sign.

$$\begin{aligned} I &= \int_{-2}^2 (8 - x^2 - 2|x|) dx = \int_{-2}^0 (8 - x^2 + 2x) dx + \int_0^2 (8 - x^2 - 2x) dx \\ I &= \left[8x - \frac{x^3}{3} + x^2 \right]_{-2}^0 + \left[8x - \frac{x^3}{3} - x^2 \right]_0^2 \\ I &= 0 - \left(-16 + \frac{8}{3} + 4 \right) + \left(16 - \frac{8}{3} - 4 \right) - 0 = \boxed{\frac{56}{3}} \end{aligned}$$