MAT123 12.12.2025

QUESTIONS

Q1. Evaluate $\displaystyle\int \frac{2x^2+3}{x^2(4x^2+9)}\,dx$.

Q2. Calculate $\displaystyle\int_0^{\infty} \frac{x^2+x}{(x^2+1)^2}\,dx$.

Q3. Calculate $\displaystyle\int_0^{-2\pi} \sin(\theta)\cdot(\theta+\pi)^{\frac{1}{3}}\cdot\arctan(\theta+\pi)\,d\theta$.

Q4. Evaluate $\displaystyle\int \frac{1}{3-2\cos x+\sin x}\,dx$.

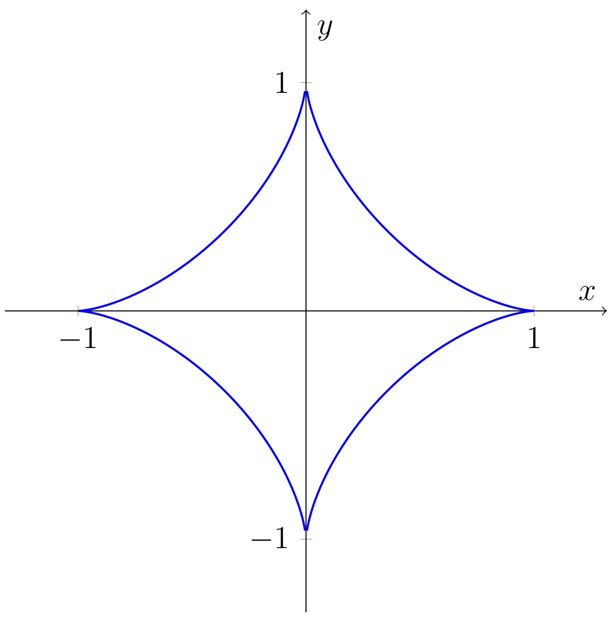
Q5. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin(2x)}{(a\sin^2 x+b\cos^2 x)^2}\,dx$, where $a, b > 0$.

Q6. Find the area of the region bounded between $x = 6 - y^2$, $x = y$, and $y \geq 1$.

Q7. Find the length of the graph of the function $y = \ln(1-x^2)$, $0 \leq x \leq 1/2$.

Q8. What values of p have the following property: The area of the region between the curve $y = x^{-p}, 1 \leq x \leq \infty$, and the x-axis is infinite but the volume of the solid generated by revolving the region about the x-axis is finite.

Q9. The graph of the equation $x^{2/3} + y^{2/3} = 1$ is an astroid (*see figure below*). Find the area of the surface generated by revolving the curve about the x-axis.



ANSWERS

Q1. Use partial fraction decomposition.

Since the denominator contains a repeated linear factor x^2 and an irreducible quadratic factor $4x^2+9$, we write

$$\frac{2x^2+3}{x^2(4x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{4x^2+9}.$$

Equate the denominator of each fraction. After we equate, we obtain these numerators:

$$2x^2+3 = Ax(4x^2+9) + B(4x^2+9) + (Cx+D)x^2.$$

Compare coefficients. Equating coefficients of like powers of x, we get

$$A=0, \quad B=\frac{1}{3}, \quad C=0, \quad D=\frac{2}{3}.$$

Rewrite the integrand. Substituting these values gives

$$\frac{2x^2+3}{x^2(4x^2+9)} = \frac{1}{3x^2} + \frac{2}{3(4x^2+9)}.$$

Integrate term by term. The first integral is easy.

$$\int \frac{1}{3x^2}\,dx = -\frac{1}{3x} + c_1, \tag{1}$$

To evaluate the second integral, we need to put it in a form that we can evaluate using standard integrals.

$$\int \frac{2}{3(4x^2+9)}\,dx = \frac{2}{3}\int \frac{1}{4x^2+9}\,dx = \frac{2}{3}\cdot\frac{1}{4}\int \frac{1}{x^2+\left(\frac{3}{2}\right)^2}\,dx = \frac{1}{6}\int \frac{1}{x^2+\left(\frac{3}{2}\right)^2}\,dx$$

$$\text{Let } x = \frac{3u}{2} \quad \Rightarrow \quad dx = \frac{3}{2}\,du$$

$$= \frac{1}{6}\int \frac{1}{\left(\frac{3}{2}\right)^2(u^2+1)}\cdot\frac{3}{2}\,du = \frac{1}{9}\int \frac{1}{u^2+1}\,du = \frac{1}{9}\tan^{-1}(u) = \frac{1}{9}\tan^{-1}\left(\frac{2x}{3}\right) + c_2 \tag{2}$$

Combining (1) and (2), we get the following.

$$\boxed{\int \frac{2x^2+3}{x^2(4x^2+9)}\,dx = -\frac{1}{3x} + \frac{1}{9}\tan^{-1}\left(\frac{2x}{3}\right) + C}$$

Q2. Use partial fraction decomposition.

$$\frac{x^2+x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Equate the denominator of each fraction. After we equate, we obtain these numerators:, we get

$$x^2+x = (Ax+B)(x^2+1) + (Cx+D).$$

Expanding,

$$x^2+x = Ax^3 + Ax + Bx^2 + B + Cx + D.$$

Comparing coefficients yields

$$A=0, \quad B=1, \quad C=1, \quad D=-1.$$

Hence,

$$\int \frac{x^2+x}{(x^2+1)^2}\,dx = \int \frac{1}{x^2+1}\,dx + \int \frac{x-1}{(x^2+1)^2}\,dx.$$

Now, integrate term by term. The first integral appears to be an improper integral of Type I. Take the limit and then evaluate the integral.

$$\begin{aligned}\int_0^\infty \frac{1}{x^2+1}\,dx &= \lim_{R\to\infty}\int_0^R \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \arctan x\Big|_0^R = \lim_{R\to\infty}(\arctan R - \arctan 0)\\ &= \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned} \tag{3}$$

Evaluate the second integral by using a trigonometric substitution. Now, we can ignore the limits of integration. After we return to the variable x, we must consider the upper and lower limits.

$$\text{Let } x = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta\,d\theta$$

$$\begin{aligned}\int \frac{x-1}{(x^2+1)^2}\,dx &= \int \frac{\tan\theta - 1}{\sec^4\theta}\sec^2\theta\,d\theta = \int (\tan\theta - 1)\cos^2\theta\,d\theta\\ &= \int \left(\sin\theta\cos\theta - \cos^2\theta\right)d\theta = \int \frac{1}{2}\sin 2\theta\,d\theta - \int \frac{1+\cos 2\theta}{2}\,d\theta\\ &= -\frac{1}{4}\cos 2\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C\end{aligned}$$

Recall the substitution $x = \tan\theta$, then $\arctan x = \theta$.

$$-\frac{1}{4}\cos(2\arctan x) - \frac{\arctan x}{2} - \frac{\sin(2\arctan x)}{4} + C$$

We evaluated the indefinite integral. However, the original integral we want to evaluate is an improper integral of Type I. Therefore, we take the limit.

$$\int_0^\infty \frac{x-1}{(x^2+1)^2}\,dx = \lim_{R\to\infty}\left[-\frac{1}{4}\cos(2\arctan x) - \frac{\arctan x}{2} - \frac{\sin(2\arctan x)}{4}\right]_0^R$$
$$= \left(-\frac{1}{4}\cos(\pi) - \frac{\pi}{4} - \frac{\sin\pi}{4}\right) - \left(-\frac{1}{4}\cos 0 - 0 - \frac{\sin 0}{2}\right)$$
$$= \left(\frac{1}{4} - \frac{\pi}{4} - 0\right) - \left(-\frac{1}{4} - 0 - 0\right) = \frac{1}{2} - \frac{\pi}{4} \tag{4}$$

Combine (3) and (4).

$$\boxed{\frac{1}{2}+\frac{\pi}{4}}$$

Q3. Let $u = \theta + \pi \implies du = d\theta$. Then $\sin(\theta) = \sin(u-\pi) = -\sin(u)$.
When $\theta = 0$, $u = \pi$, and when $\theta = -2\pi$, $u = -\pi$. So the integral becomes

$$\int_0^{-2\pi} \sin(\theta)\cdot(\theta+\pi)^{1/3}\cdot\arctan(\theta+\pi)\,d\theta = \int_\pi^{-\pi} -\sin(u)\cdot u^{1/3}\cdot\arctan(u)\,du$$
$$= \int_{-\pi}^{\pi} \sin(u)\cdot u^{1/3}\cdot\arctan(u)\,du.$$

The integrand $f(u) = \sin(u)\cdot u^{1/3}\cdot\arctan(u)$ is an odd function because each factor is odd. Therefore, by the property of integrals of odd functions over symmetric limits,

$$\int_{-\pi}^{\pi} f(u)\,du = 0.$$

$$\boxed{0}$$

Q4. Use the Weierstrass substitution (tangent half-angle substitution):

$$t = \tan\frac{x}{2} \implies dx = \frac{2\,dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}.$$

Then

$$3 - 2\cos x + \sin x = \frac{5t^2+2t+1}{1+t^2}, \quad dx = \frac{2\,dt}{1+t^2}.$$

So the integral becomes

$$\int \frac{dx}{3-2\cos x+\sin x} = \int \frac{2\,dt}{5t^2+2t+1}.$$

Next, we complete the square in the denominator:

$$5t^2+2t+1 = 5\left(t^2+\frac{2}{5}t+\frac{1}{5}\right) = 5\left(\left(t+\frac{1}{5}\right)^2+\frac{4}{25}\right).$$

Factor out $\frac{4}{25}$ to get a 1 inside the parentheses:

$$5\left(\left(t+\frac{1}{5}\right)^2+\frac{4}{25}\right)=5\cdot\frac{4}{25}\left(\frac{25}{4}\left(t+\frac{1}{5}\right)^2+1\right)=\frac{4}{5}\left(\left(\frac{5}{2}t+\frac{1}{2}\right)^2+1\right).$$

So the integral becomes

$$\int\frac{2\,dt}{5t^2+2t+1}=\int\frac{2\,dt}{\frac{4}{5}\left(\left(\frac{5}{2}t+\frac{1}{2}\right)^2+1\right)}=\frac{5}{2}\int\frac{dt}{\left(\frac{5}{2}t+\frac{1}{2}\right)^2+1}.$$

Finally, let

$$u=\frac{5}{2}t+\frac{1}{2}\implies du=\frac{5}{2}dt\implies dt=\frac{2}{5}du,$$

so the integral becomes

$$\frac{5}{2}\int\frac{dt}{\left(\frac{5}{2}t+\frac{1}{2}\right)^2+1}=\frac{5}{2}\cdot\frac{2}{5}\int\frac{du}{u^2+1}=\int\frac{du}{u^2+1}=\arctan(u)+C.$$

Substitute $u=\frac{5}{2}t+\frac{1}{2}$ and $t=\tan\frac{x}{2}$ back.

$$\boxed{\arctan\left(\frac{5}{2}\tan\frac{x}{2}+\frac{1}{2}\right)+C}$$

Q5. Let $u=\sin x\implies du=\cos x\,dx$. Then

$$\sin 2x=2\sin x\cos x=2u\sqrt{1-u^2},\quad dx=\frac{du}{\sqrt{1-u^2}}.$$

$$x=\frac{\pi}{2}\implies u=1,\qquad x=0\implies u=0$$

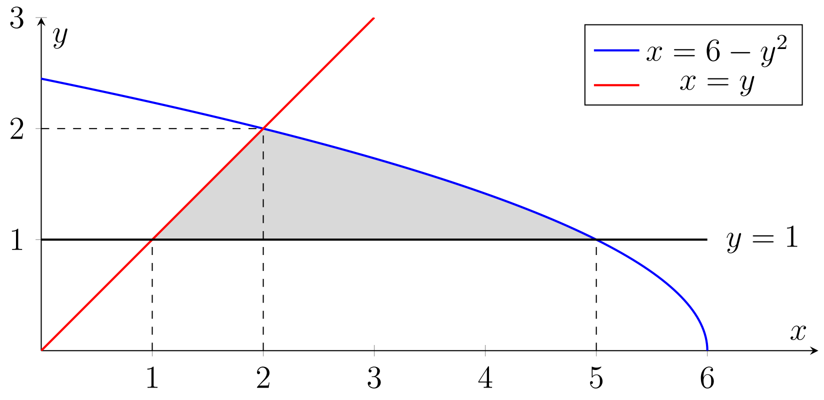
So the integral becomes

$$I=\int_0^1\frac{2u\sqrt{1-u^2}}{(au^2+b(1-u^2))^2}\cdot\frac{du}{\sqrt{1-u^2}}=\int_0^1\frac{2u}{((a-b)u^2+b)^2}\,du.$$

Let $v=(a-b)u^2+b\implies dv=2u(a-b)\,du$, and $u=1\implies w=b$, $u=0\implies w=a$. So

$$I=\frac{1}{a-b}\int_b^a\frac{dv}{v^2}=\frac{1}{a-b}\left[-\frac{1}{v}\right]_b^a=\frac{1}{a-b}\left(\frac{1}{b}-\frac{1}{a}\right)=\boxed{\frac{1}{ab}}.$$

Q6.



Method 1: Integrate with respect to y.

Horizontal distance: $x_{\text{right}} - x_{\text{left}} = (6 - y^2) - y$.

$$A = \int_1^2 (6 - y^2 - y)\,dy = \int_1^2 6\,dy - \int_1^2 y^2\,dy - \int_1^2 y\,dy = 6 - \frac{7}{3} - \frac{3}{2} = \boxed{\frac{13}{6}}$$

Method 2: Integrate with respect to x.

Express curves as functions of x: $y = \sqrt{6 - x}$ and $y = x$.

For $1 \le x \le 2$, the upper function is $y = x$. Meanwhile, the lower function is $y = 1$. For $2 \le x \le 5$, these are $y = \sqrt{6 - x}$ and $y = 1$, respectively. Therefore, we take two integrals.

$$\begin{aligned}\int_1^2 (x - 1)\,dx + \int_2^5 (\sqrt{6 - x} - 1)\,dx &= \left[\frac{x^2}{2} - x\right]_1^2 + \left[-\frac{2}{3}(6 - x)^{3/2} - x\right]_2^5 \\ &= \left(0 - \left(\frac{1}{2} - 1\right)\right) + \left(\left(-\frac{2}{3} - 5\right) - \left(-\frac{16}{3} - 2\right)\right) \\ &= \frac{1}{2} + \frac{5}{3} = \boxed{\frac{13}{6}}\end{aligned}$$

Q7. The arc length formula is

$$L = \int_0^{1/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

$$y = \ln(1 - x^2) \implies \frac{dy}{dx} = \frac{-2x}{1 - x^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4x^2}{(1 - x^2)^2} = \frac{(1 - x^2)^2 + 4x^2}{(1 - x^2)^2} = \frac{(1 + x^2)^2}{(1 - x^2)^2} = \left(\frac{1 + x^2}{1 - x^2}\right)^2$$

Hence,

$$L = \int_0^{1/2} \frac{1 + x^2}{1 - x^2}\,dx.$$

Split the fraction.

$$\frac{1+x^2}{1-x^2} = -1 + \frac{2}{1-x^2}$$

Thus, the integral becomes

$$L = \int_0^{1/2} \left(-1 + \frac{2}{1-x^2}\right) dx = \int_0^{1/2} (-1)\, dx + \int_0^{1/2} \frac{2}{1-x^2}\, dx.$$

The first integral is simple.

$$\int_0^{1/2} (-1)\, dx = -\frac{1}{2} \tag{5}$$

For the second integral, factor using partial fractions.

$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

$$\int_0^{1/2} \frac{2}{1-x^2}\, dx = \int_0^{1/2} \frac{1}{1-x}\, dx + \int_0^{1/2} \frac{1}{1+x}\, dx$$

Integrate each term.

$$\int_0^{1/2} \frac{1}{1-x}\, dx = -\ln|1-x|\Big|_0^{1/2} = \ln 1 - \ln\frac{1}{2} = \ln 2 \tag{6}$$

$$\int_0^{1/2} \frac{1}{1+x}\, dx = \ln|1+x|\Big|_0^{1/2} = \ln\frac{3}{2} - \ln 1 = \ln\frac{3}{2} \tag{7}$$

Combine (5), (6), and (7).

$$L = -\frac{1}{2} + \ln 2 + \ln\frac{3}{2} = -\frac{1}{2} + \ln 3 - \ln 2 + \ln 2 = \ln 3 - \frac{1}{2}$$

$$\boxed{L = \ln 3 - \frac{1}{2}}$$

Q8. The area is

$$A = \int_1^\infty x^{-p}\, dx.$$

For $p \neq 1$, we have

$$\int_1^\infty x^{-p}\, dx = \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^b = \lim_{b\to\infty} \frac{b^{1-p}-1}{1-p}.$$

- If $p < 1$, then $1-p > 0$ and $b^{1-p} \to \infty$, so the integral diverges.
- If $p > 1$, then $1-p < 0$ and $b^{1-p} \to 0$, so the integral converges.
- If $p = 1$, $\int_1^\infty x^{-1} dx = \int_1^\infty \frac{dx}{x} = \lim_{R\to\infty} (\ln R - \ln 1) = \infty$.

Condition for infinite area: $p \leq 1$.

The volume generated by revolving around the x-axis is

$$V = \pi \int_1^\infty (x^{-p})^2 \, dx = \pi \int_1^\infty x^{-2p} \, dx.$$

For $2p \neq 1$,

$$\pi \int_1^\infty x^{-2p} \, dx = \pi \lim_{b \to \infty} \frac{x^{1-2p}}{1-2p} \bigg|_1^b = \pi \lim_{b \to \infty} \frac{b^{1-2p} - 1}{1-2p}.$$

- If $2p > 1 \implies p > \frac{1}{2}$, then $1 - 2p < 0$ and $b^{1-2p} \to 0$, so the integral converges.
- If $2p < 1 \implies p < \frac{1}{2}$, the integral diverges.
- If $2p = 1 \implies p = \frac{1}{2}$, the integral diverges because $\displaystyle\int_1^\infty \frac{1}{x} \, dx = \infty$.

Condition for finite volume: $p > \frac{1}{2}$.

Combine conditions.

$$p \leq 1 \quad \text{(area is infinite)}, \qquad p > \frac{1}{2} \quad \text{(volume is finite)}$$

$$\boxed{\frac{1}{2} < p \leq 1}$$

Q9. We may consider the curves in Quadrant I and II. Then it is redundant to use the lower part since the rotated curves are the same. Solve for y.

$$y = (1 - x^{2/3})^{3/2}, \quad x \in [-1, 1]$$

Surface area formula:

$$S = 2\pi \int_{-1}^1 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Compute dy/dx.

$$\frac{dy}{dx} = \frac{3}{2}(1 - x^{2/3})^{1/2} \cdot \left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}(1 - x^{2/3})^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = x^{-2/3}(1 - x^{2/3})$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + x^{-2/3}(1 - x^{2/3})} = \sqrt{x^{-2/3}} = \left|x^{-1/3}\right|$$

Set up the integral.

$$S = 2\pi \int_{-1}^1 (1 - x^{2/3})^{3/2} \cdot \left|x^{-1/3}\right| \, dx$$

The expression $\left|x^{-2/3}\right|$ is positive for $x > 0$ and negative for $x < 0$. We may rewrite this integral as follows.

$$\begin{aligned}
S &= 2\pi \int_{-1}^{1} (1 - x^{2/3})^{3/2} \cdot \left|x^{-1/3}\right| \, dx \\
&= 2\pi \int_{-1}^{0} (1 - x^{2/3})^{3/2} \left(-x^{-1/3}\right) \, dx + 2\pi \int_{0}^{1} (1 - x^{2/3})^{3/2} \left(x^{-1/3}\right) \, dx \\
&= 4\pi \int_{0}^{1} (1 - x^{2/3})^{3/2} x^{-1/3} \, dx \qquad \text{[symmetry property]}
\end{aligned}$$

Let $u = x^{2/3} \implies du = \frac{2}{3}x^{-1/3}\, dx$. Then

$$x^{-1/3}dx = \frac{3}{2}du, \quad (1 - x^{2/3})^{3/2} = (1 - u)^{3/2}.$$

$$S = 4\pi \int_{0}^{1} (1 - u)^{3/2} \cdot \frac{3}{2} \, du = 6\pi \int_{0}^{1} (1 - u)^{3/2} \, du$$

Integrate.

$$S = 6\pi \int_{0}^{1} (1 - u)^{3/2} \, du = 6\pi \cdot \left[-\frac{2}{5} (1 - u)^{5/2} \right]_{0}^{1} = 6\pi \left[0 - \left(-\frac{2}{5} \right) \right] = \boxed{\frac{12\pi}{5}}$$