

2024-2025 Fall
MAT123-02,05 Makeup
(31/01/2025)

1. (a) Find $\int \frac{\sin^3 x}{\sqrt{\cos x}} dx$.

(b) Find $\int \frac{dx}{x^3 - 4x^2 + 3x}$.

(c) Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\int_{\pi/2}^x \ln(\sin t) dt}{\sin x - 1}$.

(d) Evaluate the improper integral $\int_0^2 \frac{dx}{(x-1)^{2/3}}$.

2. Consider the region R bounded by the curves $y = \arctan x$, $y = \ln x$ and the lines $x = \frac{1}{\sqrt{3}}$ and $x = 1$.

(a) Sketch the region and find the area of the R .

(b) Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region R about the y -axis.

(c) Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of a solid obtained by rotating the region R about the line $y = 2$.

3. Determine whether each series is convergent or divergent. Explain your answer.

(a) $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

4. Find the Taylor series of the function $f(x) = \ln x$ at $c = 1$ and determine the interval of convergence.

2024-2025 Makeup (31/01/2025) Solutions
(Last update: 17/08/2025 20:56)

1. (a)

$$\int \frac{\sin^3 x}{\sqrt{\cos x}} dx = \int \frac{(1 - \cos^2 x) \cdot \sin x}{\sqrt{\cos x}} dx$$

Let $u = \cos x$, then $du = -\sin x dx$.

$$\begin{aligned} \int \frac{(1 - \cos^2 x) \cdot \sin x}{\sqrt{\cos x}} dx &= \int -\frac{(1 - u^2)}{\sqrt{u}} du = \int \left(u^{3/2} - \frac{1}{\sqrt{u}} \right) = \frac{2}{5} u^{5/2} - 2\sqrt{u} + c \\ &= \boxed{\frac{2}{5} (\cos x)^{5/2} - 2\sqrt{\cos x} + c, \quad c \in \mathbb{R}} \end{aligned}$$

(b) Use the method of partial fraction decomposition.

$$\int \frac{dx}{x^3 - 4x^2 + 3x} = \int \frac{dx}{x(x-3)(x-1)} = \int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1} \right) dx$$

$$\begin{aligned} A(x-3)(x-1) + Bx(x-1) + Cx(x-3) &= 1 \\ x^2(A+B+C) + x(-4A-B-3C) + 3A &= 1 \end{aligned}$$

Equate the coefficients of like terms.

$$\left. \begin{aligned} x^2(A+B+C) &= 0 \\ x(-4A-B-3C) &= 0 \\ 3A &= 1 \end{aligned} \right\} \rightarrow A = \frac{1}{3}, \quad \left. \begin{aligned} B+C &= -\frac{1}{3} \\ B+3C &= -\frac{4}{3} \end{aligned} \right\} \rightarrow B = \frac{1}{6}, \quad C = -\frac{1}{2}$$

Rewrite the integral.

$$\begin{aligned} \int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1} \right) dx &= \int \left(\frac{1}{3x} + \frac{1}{6(x-3)} - \frac{1}{2(x-1)} \right) dx \\ &= \boxed{\frac{1}{3} \ln |x| + \frac{1}{6} \ln |x-3| - \frac{1}{2} \ln |x-1| + c, \quad c \in \mathbb{R}} \end{aligned}$$

(c) The limit is in the indeterminate form $0/0$. Apply L'Hôpital's rule to eliminate the indeterminate form.

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\int_{\pi/2}^x \ln(\sin t) dt}{\sin x - 1} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{d}{dx} \int_{\pi/2}^x \ln(\sin t) dt}{\cos x}$$

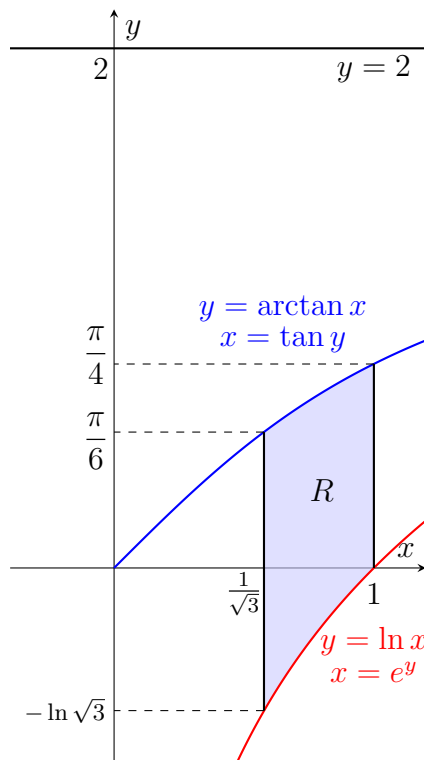
By the Fundamental Theorem of Calculus, we may rewrite the limit as follows.

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{d}{dx} \int_{\pi/2}^x \ln(\sin t) dt}{\cos x} &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln(\sin x)}{\cos x} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{1}{\sin x} \cdot \cos x}{-\sin x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{-\sin^2 x} = -\frac{\cos \frac{\pi}{2}}{\sin^2 \frac{\pi}{2}} \\ &= \boxed{0}\end{aligned}$$

(d) Take the limit as this is an improper integral.

$$\begin{aligned}\int_0^2 \frac{dx}{(x-1)^{2/3}} &= \lim_{R \rightarrow 1^-} \int_0^R \frac{dx}{(x-1)^{2/3}} + \lim_{P \rightarrow 1^+} \int_P^2 \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{R \rightarrow 1^-} 3(x-1)^{1/3} \Big|_0^R + \lim_{P \rightarrow 1^+} 3(x-1)^{1/3} \Big|_P^2 \\ &= 3 \lim_{R \rightarrow 1^-} ((R-1)^{1/3} - (-1)) + 3 \lim_{P \rightarrow 1^+} (1 - (P-1)^{1/3}) = \boxed{6}\end{aligned}$$

2. (a)



$$A = \int_{1/\sqrt{3}}^1 (\arctan x - \ln x) dx = \int_{1/\sqrt{3}}^1 \arctan x dx - \int_{1/\sqrt{3}}^1 \ln x dx \quad (1)$$

Calculate the first integral in (1) by integration by parts.

$$\left. \begin{array}{l} u = \arctan x \implies du = \frac{1}{x^2 + 1} dx \\ dv = dx \implies v = x \end{array} \right\} \rightarrow \int u dv = uv - \int v du$$

$$\int_{1/\sqrt{3}}^1 \arctan x dx = x \arctan x \Big|_{1/\sqrt{3}}^1 - \int_{1/\sqrt{3}}^1 \frac{x}{x^2 + 1} dx = \left(x \arctan x - \frac{1}{2} \ln |x^2 + 1| \right) \Big|_{1/\sqrt{3}}^1$$

$$= \left(\frac{\pi}{4} - \frac{\ln 2}{2} \right) - \left(\frac{\pi\sqrt{3}}{18} - \frac{1}{2} \cdot \ln \left(\frac{4}{3} \right) \right) = \frac{\pi(9 - 2\sqrt{3})}{36} + \frac{1}{2} \cdot \ln \frac{2}{3}$$

Calculate the second integral in (1) by integration by parts.

$$\left. \begin{array}{l} u = \ln x \implies du = \frac{1}{x} dx \\ dv = dx \implies v = x \end{array} \right\} \rightarrow \int u dv = uv - \int v du$$

$$\int_{1/\sqrt{3}}^1 \ln x dx = x \ln x \Big|_{1/\sqrt{3}}^1 - \int_{1/\sqrt{3}}^1 dx = (x \ln x - x) \Big|_{1/\sqrt{3}}^1 = (0 - 1) - \left(-\frac{\ln \sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right)$$

$$= \frac{\sqrt{3} \ln \sqrt{3} + \sqrt{3} - 3}{3}$$

The result is then

$$A = \boxed{\frac{\pi(9 - 2\sqrt{3})}{36} + \frac{1}{2} \cdot \ln \frac{2}{3} - \frac{\sqrt{3} \ln \sqrt{3} + \sqrt{3} - 3}{3}}$$

(b)

$$\boxed{\begin{aligned} & \int_{-\ln \sqrt{3}}^0 \pi \left[(e^y)^2 - \left(\frac{1}{\sqrt{3}} \right)^2 \right] dy + \int_0^{\pi/6} \pi \left[(1)^2 - \left(\frac{1}{\sqrt{3}} \right)^2 \right] dy \\ & + \int_{\pi/6}^{\pi/4} \pi \left[(1^2) - (\tan y)^2 \right] dy \end{aligned}}$$

(c)

$$\boxed{\begin{aligned} & \int_{-\ln \sqrt{3}}^0 2\pi (2 - y) \left(e^y - \frac{1}{\sqrt{3}} \right) dy + \int_0^{\pi/6} 2\pi (2 - y) \left(1 - \frac{1}{\sqrt{3}} \right) dy \\ & + \int_{\pi/6}^{\pi/4} 2\pi (2 - y) (1 - \tan y) dy \end{aligned}}$$

3. (a)

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}} = \sum_{n=0}^{\infty} \frac{-3 \cdot (-3)^n}{8^n} = -3 \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n$$

This is a geometric series where $r = -\frac{3}{8}$. $|r| = \frac{3}{8} < 1$. Therefore, the series $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$ converges.

(b) Take $f(x) = \frac{1}{x(\ln x)^2}$. f is positive and decreasing for $x \geq 2$ because x and $(\ln x)^2$ are positive and increasing for $x \geq 2$. x is a polynomial which is defined everywhere and $(\ln x)^2$ is continuous for $x \geq 2$. Since we took into account every criterion, we may apply the Integral Test. Handle the improper integrals by taking the limit.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{R \rightarrow \infty} \int_2^R \frac{dx}{x(\ln x)^2} = \lim_{R \rightarrow \infty} \left[-\frac{1}{\ln x} \right]_2^R = \lim_{R \rightarrow \infty} \left[-\frac{1}{\ln R} + \frac{1}{\ln 2} \right] = \frac{1}{\ln 2}$$

Since the integral converges, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ also converges.

4. The Taylor series of f at $c = 1$ is as follows.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} (x-1)^k$$

Find $f(1)$, $f'(1)$, $f''(1)$, $f'''(1)$, $f^{(4)}(1)$ to look for the pattern.

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f'''(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = -\frac{6}{x^4}$$

$$f(1) = 0, \quad f'(1) = 1, \quad f''(1) = -1, \quad f'''(1) = 2, \quad f^{(4)}(1) = -6$$

This is an alternating sequence where the coefficient of each term is the factorial of the subsequent number starting from 0 except for $k = 0$, that is, the first term of the series. At $k = 0$, the first term is 0. So,

$$f^k(1) = \begin{cases} (-1)^{k-1} \cdot (k-1)!, & \text{if } k > 0 \\ 0, & \text{if } k = 0 \end{cases}$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} x^k = 0 + \sum_{k=1}^{\infty} \frac{f^{(k)}(1)}{k!} (x-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot (k-1)!}{k \cdot (k-1)!} (x-1)^k$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot (x-1)^k}{k} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots}$$

Now, determine the interval of convergence. Apply the Ratio Test.

$$\begin{aligned}\lim_{k \rightarrow \infty} \left| \frac{(-1)^k (x-1)^{k+1}}{k+1} \cdot \frac{k}{(-1)^{k-1} (x-1)^k} \right| &= \lim_{k \rightarrow \infty} \left| \frac{(x-1) \cdot k}{(k+1) \cdot (-1)} \right| = |x-1| \lim_{k \rightarrow \infty} \left| \frac{k}{k+1} \right| \\ &= |x-1|\end{aligned}$$

$$|x-1| < 1 \implies -1 < x-1 < 1 \implies 0 < x < 2 \quad (\text{convergent})$$

Investigate the convergence at the endpoints.

$$x = 0 \rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cdot (-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{2k} \cdot (-1)}{k} = - \sum_{k=1}^{\infty} \frac{1}{k}$$

This is a p -series with $p = 1$, for which the series diverges by the p -series Test. Try $x = 2$.

$$x = 2 \rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{k}$, is nonincreasing for $k \geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges.

The convergence set for the power series is $\boxed{(0, 2]}$.