

2020-2021 Spring
MAT124 Final
(09/06/2021)

1. Sketch the region corresponding to the double integral

$$\int_{-1}^2 \int_{x^2}^{x+2} dy \, dx$$

and reverse the order of integration.

2. Find the volume of the solid bounded above by the cone $z = \sqrt{x^2 + y^2}$ and below by the circular region $x^2 + y^2 \leq 4$ in the xy -plane.

3. Let R be the region lying outside $r = 1$ and inside $r = 1 + \cos \theta$.

(i) Sketch the graph of the region R .

(ii) Set up (but do not evaluate) a double integral in polar coordinates for the area of the region R .

4. Let S be the portion of the surface $z = y^2$ that lies over the rectangular region in the xy -plane with the vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$ and $(1, 0, 0)$.

(i) Sketch the graph of S .

(ii) Evaluate the surface area.

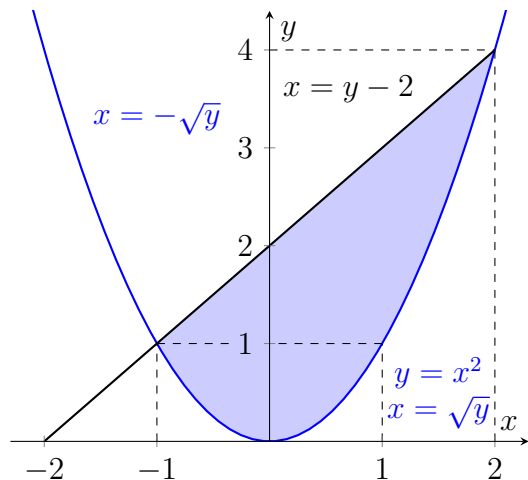
5. Evaluate $\iiint_E z \, dV$, where E is the region bounded below by $x^2 + y^2 + z^2 = 4$ and above by $x^2 + y^2 + z^2 = 9$.

6. Let S be the region bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the xy -plane.

(i) Using the spherical coordinates, set up (but do not evaluate) an integral for the volume of the solid S .

(ii) Using the cylindrical coordinates, set up (but do not evaluate) an integral for the volume of the solid S .

1.



$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx dy$$

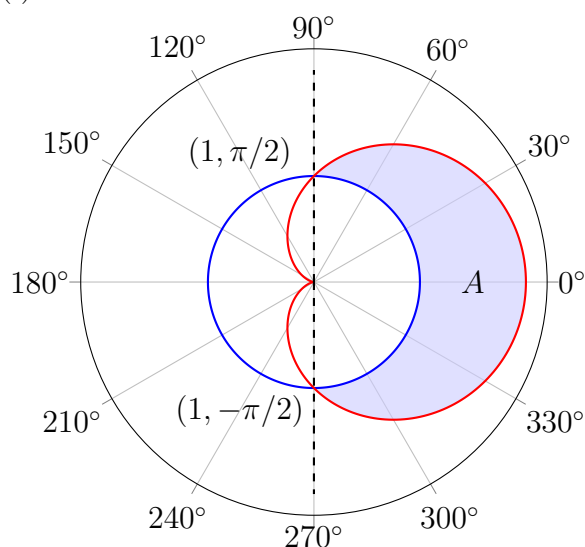
2. Using polar coordinates, we can find the volume with a double integral.

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \implies z = \sqrt{r^2} \implies z = r \\ r^2 &= x^2 + y^2 \quad \rightarrow \quad x^2 + y^2 \leq 4 \rightarrow r^2 \leq 4 \implies 0 \leq r \leq 2 \\ dA &= r dr d\theta \quad \rightarrow \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

$$\begin{aligned} \text{Volume} &= \int_0^{2\pi} \int_0^2 [r - 0] \cdot r dr d\theta = \int_0^{2\pi} \int_0^2 r^2 dr d\theta = \int_0^{2\pi} \left[\frac{r^3}{3} \right]_{r=0}^{r=2} d\theta \\ &= \frac{8}{3} \int_0^{2\pi} d\theta = \boxed{\frac{16\pi}{3}} \end{aligned}$$

3.

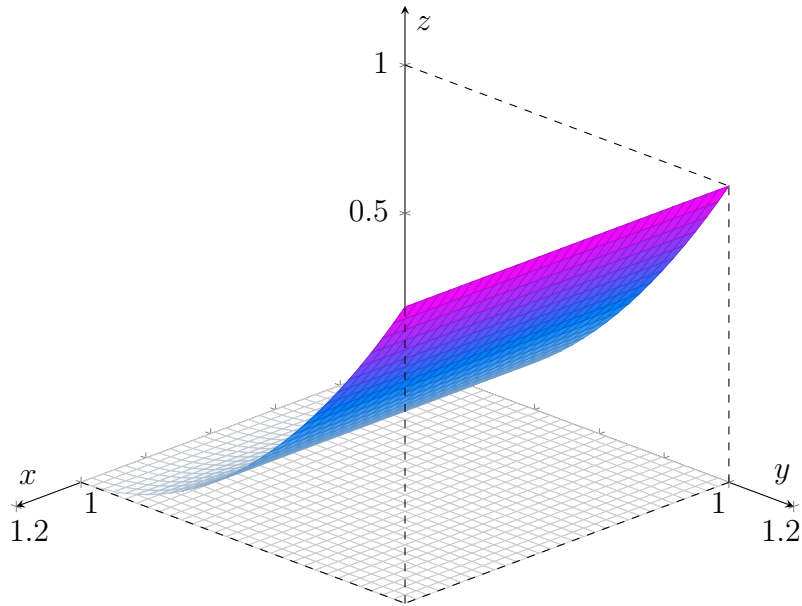
(i)



$$A = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} r dr d\theta$$

4.

(i)



(ii) Calculate the partial derivatives to find the surface area.

$$z = y^2 \implies \frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 2y$$

$$\begin{aligned} \text{Surface area} &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA = \int_0^1 \int_0^1 \sqrt{1 + (0)^2 + (2y)^2} dx dy \\ &= \int_0^1 \int_0^1 \sqrt{4y^2 + 1} dx dy = \int_0^1 \left[x \sqrt{4y^2 + 1} \right]_{x=0}^{x=1} dy \\ &= \int_0^1 \sqrt{4y^2 + 1} dy \left[y = \frac{1}{2} \tan u \implies dy = \frac{1}{2} \sec^2 u du, \quad \begin{matrix} u_{\text{upper}} = \arctan 2 \\ u_{\text{lower}} = 0 \end{matrix} \right] \\ &= \frac{1}{2} \int_0^{\arctan 2} \sqrt{\tan^2 u + 1} \cdot \sec^2 u du = \frac{1}{2} \int_0^{\arctan 2} \sec^3 u du \end{aligned}$$

To evaluate the last integral, we will use integration by parts.

$$\begin{aligned} w &= \sec u \rightarrow dw = \sec u \tan u du \\ dz &= \sec^2 u du \rightarrow z = \tan u \end{aligned}$$

$$\begin{aligned}
\int_0^{\arctan 2} \sec^3 u \, du &= \tan u \cdot \sec u \Big|_0^{\arctan 2} - \int_0^{\arctan 2} \tan^2 u \sec u \, du \\
&= \tan u \cdot \sec u \Big|_0^{\arctan 2} - \int_0^{\arctan 2} \frac{1 - \cos^2 u}{\cos^3 u} \, du \\
&= \tan u \cdot \sec u \Big|_0^{\arctan 2} - \int_0^{\arctan 2} \sec^3 u \, du + \int_0^{\arctan 2} \sec u \, du
\end{aligned}$$

Notice that the integral we want to evaluate appears on the right side. After a little algebra, we can evaluate the integral.

$$\int_0^{\arctan 2} \sec^3 u \, du = \frac{1}{2} \cdot \tan u \cdot \sec u \Big|_0^{\arctan 2} + \frac{1}{2} \cdot \int_0^{\arctan 2} \sec u \, du$$

The integral of $\sec u$ with respect to u is as follows. One can derive it with particular methods.

$$\int_0^{\arctan 2} \sec u \, du = \ln |\tan u + \sec u| \Big|_0^{\arctan 2}$$

So, the surface area becomes as follows.

$$\begin{aligned}
\text{Surface area} &= \frac{1}{2} \int_0^{\arctan 2} \sec^3 u \, du = \frac{1}{4} (\tan u \cdot \sec u + \ln |\tan u + \sec u|) \Big|_0^{\arctan 2} \\
&= \frac{1}{4} [2 \sec(\arctan 2) + \ln(2 + \sec(\arctan 2)) - 0] = \boxed{\frac{1}{4} [2\sqrt{5} + \ln(2 + \sqrt{5})]}
\end{aligned}$$

5) By means of spherical coordinates, we can easily evaluate the integral. For spherical coordinates, we have

$$\begin{aligned}
z &= \rho \cos \theta \\
r &= \rho \sin \theta \\
x^2 + y^2 + z^2 &= \rho^2 \\
dV &= \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
x^2 + y^2 + z^2 &= 4 \rightarrow \rho^2 = 4 \implies \rho_{\min} = 2 \\
x^2 + y^2 + z^2 &= 9 \rightarrow \rho^2 = 9 \implies \rho_{\max} = 3 \\
z &\equiv \rho \cos \phi \\
0 &\leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi
\end{aligned}$$

$$\begin{aligned}
\iiint_E z \, dV &= \int_0^{2\pi} \int_0^\pi \int_2^3 \rho \cos \phi \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^\pi \int_2^3 \rho^3 \sin(2\phi) \, d\rho \, d\phi \, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^\pi \left[\frac{\rho^4}{4} \right]_{\rho=2}^{\rho=3} \sin(2\phi) \, d\phi \, d\theta = \frac{65}{8} \int_0^{2\pi} \int_0^\pi \sin(2\phi) \, d\phi \, d\theta \\
&= \frac{65}{8} \int_0^{2\pi} \left[-\frac{1}{2} \cos(2\phi) \right]_{\phi=0}^{\phi=\pi} d\theta = \frac{65}{8} \int_0^{2\pi} 0 \, d\theta = \boxed{0}
\end{aligned}$$

6)

(i) For spherical coordinates, we have

$$\begin{array}{lcl} \begin{array}{l} z = \rho \cos \theta \\ r = \rho \sin \theta \\ x^2 + y^2 + z^2 = \rho^2 \\ dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{array} & \rightarrow & \begin{array}{l} z = 1 - x^2 - y^2 \implies \rho \cos \phi = 1 - \rho^2 \sin^2 \phi \quad (1) \\ 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \rho \leq \rho_{\text{upper}} \end{array} \end{array}$$

Find ρ_{upper} with equation (1).

$$\rho \cos \phi = 1 - \rho^2 \sin^2 \phi \implies \rho^2 \sin^2 \phi + \rho \cos \phi - 1 = 0$$

$$\rho_{1,2} = \frac{-\cos \phi \pm \sqrt{\cos^2 \phi - 4 \cdot \sin^2 \phi \cdot (-1)}}{2 \sin^2 \phi} \quad \left[x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right]$$

$$\rho > 0 \implies \rho_{\text{upper}} = \frac{-\cos \phi + \sqrt{\cos^2 \phi + 4 \sin^2 \phi}}{2 \sin^2 \phi}$$

$$\text{Volume} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\frac{-\cos \phi + \sqrt{\cos^2 \phi + 4 \sin^2 \phi}}{2 \sin^2 \phi}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

(ii) For cylindrical coordinates, we have

$$\begin{array}{lcl} \begin{array}{l} z = z \\ r^2 = x^2 + y^2 \\ dV = r \, dz \, dr \, d\theta \end{array} & \rightarrow & \begin{array}{l} z = 1 - x^2 - y^2 \implies z = 1 - r^2 \\ 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 1 \end{array} \end{array}$$

$$\text{Volume} = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r \, dz \, dr \, d\theta$$