2023-2024 Fall
MAT124 Final
(11/01/2024)

1.

(a) Find the critical points of the function

$$f(x,y) = \mathrm{e}^y \left(y^2 - x^2\right)$$

and classify them.

(b) Find the maximum and minimum values of the function

$$f(x,y,z) = xyz^2$$

by using Lagrange multipliers on the region $D: x+y+z=1,\ x \geq 0,\ y \geq 0,\ z \geq 0.$

2.

(a) Sketch the domain of integration, and rewrite the integral by changing the order of integration.

$$\int_0^1 \int_{\sqrt{1-y}}^1 f(x,y)\,dx\,dy + \int_1^3 \int_{\frac{y-1}{2}}^1 f(x,y)\,dx\,dy$$

(b) Evaluate the integral

$$\iint_R \sqrt{x^2+y^2}\,dA$$

where R is the region bounded by the line $x=1$ on the left and by the circle $x^2+y^2=2$ on the right.

3. Convert

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3\,dz\,r\,dr\,d\theta, \qquad r \geq 0$$

(a) to rectangular coordinates with the order of integration $dz\,dx\,dy$.

(b) to spherical coordinates.

4.

(a) Is $\mathbf{F}(x,y) = 2xy\cos\left(x^2y\right)\mathbf{i} - x^2\cos\left(x^2y\right)\mathbf{j}$ conservative? Why?

(b) Show that

$$\mathbf{F}(x,y,z) = \left(y^2\cos\left(xy^2\right) + yz^2\right)\mathbf{i} + \left(2xy\cos\left(xy^2\right) + xz^2 + z\mathrm{e}^{yz}\right)\mathbf{j} + \left(2xyz + y\mathrm{e}^{yz} + 2z\right)\mathbf{k}$$

is conservative.

(c) Find its potential function.

5. $\mathbf{F}(x, y, z) = (y^2 \cos(xy^2) + yz^2)\,\mathbf{i} + (2xy\cos(xy^2) + xz^2 + ze^{yz})\,\mathbf{j} + (2xyz + ye^{yz} + 2z)\,\mathbf{k}$

(a) Let C be the curve of intersection of the cone $z^2 = 4x^2+9y^2$ and the plane $z = 1+x+2y$ and D be the part of the curve C that lies in the first octant $x \geq 0,\ y \geq 0,\ z \geq 0$ from $(1, 0, 2)$ to $(0, 1, 3)$. Evaluate $\int_D \mathbf{F} \cdot d\mathbf{r}$.

(b) Let C be the curve of intersection of $x^2 + y^2 = 1$ and $z = 40$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

6. Evaluate

$$\oint_C \frac{y^2}{x}\,dx + y \ln x\, dy$$

where C is the counterclockwise boundary of the region in the first quadrant bounded by the curves $y = x$, $y = \dfrac{x}{2}$, $y = \dfrac{1}{x}$ and $\dfrac{2}{x}$.

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1.

(a) To find the critical points of f, determine where both $f_x = f_y = 0$ or one of the partial derivatives does not exist. Apply the product rule appropriately.

$$f_x = \mathrm{e}^y(-2x), \quad f_y = \mathrm{e}^y\left(y^2 - x^2\right) + \mathrm{e}^y(2y) = \mathrm{e}^y\left(y^2 + 2y - x^2\right)$$

$$f_x = 0 \implies \mathrm{e}^y(-2x) = 0, \quad f_y = 0 \implies \mathrm{e}^y\left(y^2 + 2y - x^2\right) = 0$$

$$\mathrm{e}^y \neq 0 \implies x = 0, \quad y^2 + 2y - x^2 = 0 \implies y(y+2) - 0^2 = 0 \implies y = 0, \quad y = -2$$

The critical points occur at $(0,0)$ and $(0,-2)$. To classify these points, apply the second derivative test.

$$f_{xx} = -2\mathrm{e}^y, \quad f_{xy} = f_{yx} = \mathrm{e}^y(-2x), \quad f_{yy} = \mathrm{e}^y(y^2 + 2y - x^2) + \mathrm{e}^y\,(2y+2)$$

Calculate the Hessian determinant at these points.

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

$$(0,0) \quad \rightarrow \quad \begin{aligned} &f_{xx} = -2, \quad f_{xy} = f_{yx} = 0, \quad f_{yy} = 2 \\ &f_{xx}f_{yy} - f_{xy}^2 = -2 \cdot 2 - 0^2 = -4 < 0 \end{aligned}$$

$$(0,-2) \quad \rightarrow \quad \begin{aligned} &f_{xx} = -2\mathrm{e}^{-2}, \quad f_{xy} = f_{yx} = 0, \quad f_{yy} = -2\mathrm{e}^{-2} \\ &f_{xx}f_{yy} - f_{xy}^2 = -2\mathrm{e}^{-2} \cdot 2\mathrm{e}^{-2} - 0^2 = 4\mathrm{e}^{-4} > 0, \quad f_{xx} = -2\mathrm{e}^{-2} < 0 \end{aligned}$$

$$\boxed{\text{A saddle point occurs at } (0,0) \text{ and a local maximum occurs at } (0,-2).}$$

(b) Let $g(x,y,z) = x + y + z - 1$ be the constraint. Then solve the system of equations below.

$$\left.\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x,y,z) &= 0 \end{aligned}\right\} \quad \begin{aligned} &\nabla f = \langle yz^2, xz^2, 2xyz \rangle = \lambda\,\langle 1,1,1 \rangle = \lambda \nabla g \\ &\therefore yz^2 = xz^2 = 2xyz \end{aligned}$$

$$\begin{aligned} yz^2 = xz^2 &\implies y = x \\ yz^2 = 2xyz &\implies z = 2x \end{aligned}$$

Use the constraint and write y, z in terms of x.

$$x+y+z-1=0 \implies x+x+2x=1 \implies 4x=1 \implies x=\frac{1}{4}$$

$$x=\frac{1}{4} \implies y=\frac{1}{4}, \quad z=\frac{1}{2}$$

Consider the boundary as well. Set $x=0$, $y=0$, $z=0$ one by one. Notice that the value of the function becomes 0 on the boundary of the domain.

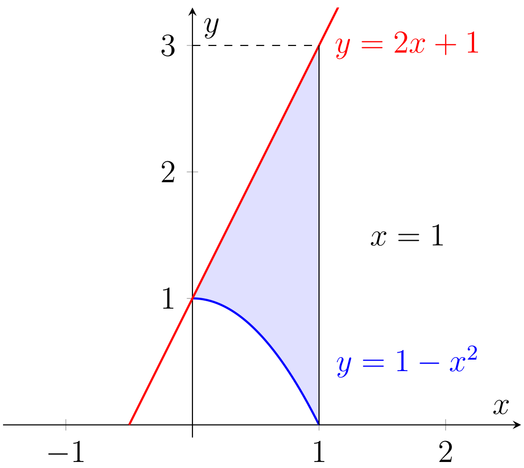
Compare all the values.

$$f(0,y,z)=f(x,0,z)=f(x,y,0)=0, \quad f\left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)=\frac{1}{4}\cdot\frac{1}{4}\cdot\left(\frac{1}{2}\right)^2=\frac{1}{64}$$

$$\boxed{\text{The maximum value is } \frac{1}{64}, \text{ the minimum value is } 0.}$$

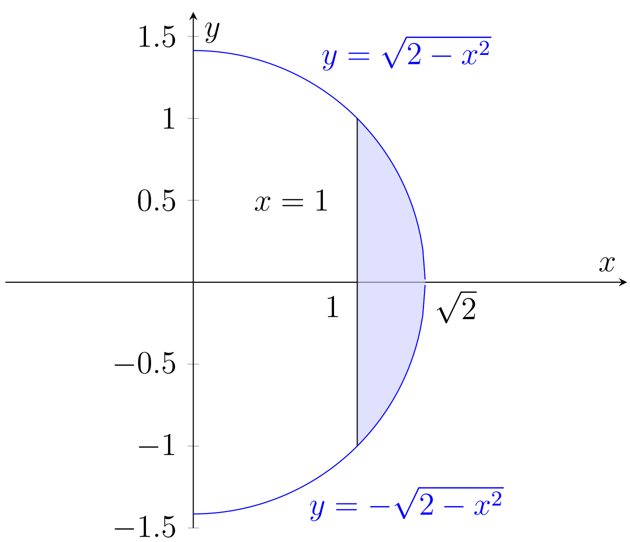
2.

(a)



$$\boxed{\int_0^1 \int_{1-x^2}^{2x+1} f(x,y)\,dy\,dx}$$

(b) Sketch the region.



$$\mathrm{I} = \int_{1}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

We may switch to polar coordinates to easily evaluate the integral.

$$\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \\ \theta = \tan^{-1}\dfrac{y}{x} \\ x^2+y^2 = r^2 \\ dA = dy\,dx = r\,dr\,d\theta \end{matrix} \quad \rightarrow \quad \begin{matrix} x = \sqrt{2}\cos\theta, \quad y = \sqrt{2}\sin\theta, \quad r_{\text{upper}} = \sqrt{2} \\ x = 1 \implies r\cos\theta = 1 \implies r_{\text{lower}} = \sec\theta \\ 1 = \sqrt{2}\cos\theta \implies \cos\theta = \dfrac{1}{\sqrt{2}} \implies \theta = \pm\dfrac{\pi}{4} \\ 0 \le \theta \le \pi/2 \end{matrix}$$

$$\mathrm{I} = \int_{-\pi/4}^{\pi/4} \int_{\sec\theta}^{\sqrt{2}} r \cdot r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \left[\frac{r^3}{3}\right]_{r=\sec\theta}^{r=\sqrt{2}} d\theta = \frac{1}{3}\int_{-\pi/4}^{\pi/4} \left(2\sqrt{2} - \sec^3\theta\right) d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{2\sqrt{2}}{3}\, d\theta - \frac{1}{3}\int_{-\pi/4}^{\pi/4} \sec^3\theta\, d\theta \tag{1}$$

Evaluate the left-hand side integral in (1).

$$\int_{-\pi/4}^{\pi/4} \frac{2\sqrt{2}}{3}\, d\theta = \frac{2\sqrt{2}}{3} \cdot \theta \Big|_{-\pi/4}^{\pi/4} = \frac{\pi\sqrt{2}}{3}$$

Evaluate the right-hand side integral in (1) with the help of integration by parts.

$$\begin{aligned} u = \sec\theta \ &\rightarrow \ du = \sec\theta\tan\theta\, d\theta \\ dv = \sec^2\theta\, d\theta \ &\rightarrow \ v = \tan\theta \end{aligned}$$

$$\int \sec^3\theta\, d\theta = \tan\theta \cdot \sec\theta - \int \tan^2\theta \sec\theta\, d\theta = \tan\theta \cdot \sec\theta - \int \frac{1-\cos^2\theta}{\cos^3\theta}\, d\theta$$

$$= \tan\theta \cdot \sec\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

Notice that the integral of $\sec^3\theta$ with respect to θ appears on the right side of the equation. Perform some algebra and then find the result of the integral.

$$\int \sec^3\theta\, d\theta = \frac{1}{2} \cdot \tan\theta \cdot \sec\theta + \frac{1}{2} \cdot \int \sec\theta\, d\theta$$

The integral of $\sec\theta$ with respect to θ is as follows.

$$\int \sec\theta\, d\theta = \ln|\tan\theta + \sec\theta| + c_0, \quad c_0 \in \mathbb{R}$$

Rewrite (1).

$$\mathrm{I} = \frac{\pi\sqrt{2}}{3} - \frac{1}{3}\cdot\frac{1}{2}\left(\tan\theta\cdot\sec\theta + \ln|\tan\theta+\sec\theta|\right)\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{\pi\sqrt{2}}{3} - \frac{1}{6}\left[\left(1\cdot\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right) - \left(-1\cdot\sqrt{2}+\ln\left(-1+\sqrt{2}\right)\right)\right]$$

$$= \boxed{\frac{(\pi-1)\sqrt{2}}{3} + \frac{1}{6}\cdot\ln\left(\frac{1+\sqrt{2}}{-1+\sqrt{2}}\right)}$$

3. From the bounds of the integral given in cylindrical coordinates, we infer that this is the region bounded above by the sphere $x^2+y^2+z^2=4$ and bounded below by the cone $z^2=x^2+y^2$

(a)

$$\begin{gathered} z = z \\ x = r\cos\theta \\ y = r\sin\theta \\ r^2 = x^2+y^2 \\ dV = r\,dz\,dr\,d\theta = dz\,dy\,dx \end{gathered} \quad\rightarrow\quad \begin{gathered} z=\sqrt{4-r^2} \implies z=\sqrt{4-x^2-y^2} \\ z=r \implies z=\sqrt{x^2+y^2} \\ 0\le\theta\le 2\pi,\quad 0\le r\le\sqrt{2} \\ \downarrow \\ -\sqrt{2-y^2}\le y\le\sqrt{2-y^2},\quad -\sqrt{2}\le y\le\sqrt{2} \end{gathered}$$

$$\boxed{\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\,dz\,dx\,dy}$$

(b) For spherical coordinates, we have

$$\begin{gathered} z=\rho\cos\phi \\ r=\rho\sin\phi \\ x^2+y^2+z^2=\rho^2 \\ dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{gathered} \quad\rightarrow\quad \begin{gathered} z=\sqrt{4-r^2} \implies \rho\cos\phi=\sqrt{4-\rho^2\sin^2\phi} \implies \rho_{\text{upper}}=2 \\ z=r \implies \rho\cos\phi=\rho\sin\phi \implies \rho_{\text{lower}}=0 \\ \cos\phi=\sin\phi \implies \tan\phi=1 \implies \phi_{\text{upper}}=\frac{\pi}{4} \\ \phi_{\text{lower}}=0,\quad 0\le\theta\le 2\pi \end{gathered}$$

$$\boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

4.

(a) For $\mathbf{F}$ to be conservative, it must be the gradient of some potential function ϕ. We may apply the component test to determine whether mixed partial derivatives are equal.

$$\left.\begin{aligned}\frac{\partial \mathrm{F}_1}{\partial y} &= 2x\cos\left(x^2y\right) - 2xy\sin\left(x^2y\right)\cdot x^2\\ \frac{\partial \mathrm{F}_2}{\partial x} &= -2x\cos\left(x^2y\right) + x^2\sin\left(x^2y\right)\cdot 2xy\end{aligned}\right\} \implies \frac{\partial \mathrm{F}_1}{\partial y} \neq \frac{\partial \mathrm{F}_2}{\partial x}$$

The mixed partial derivatives are not equal. Therefore, the force is not conservative.

(b) Like what we did above, determine the mixed partial derivatives.

$$\frac{\partial \mathrm{F}_1}{\partial y} = 2y\cos\left(xy^2\right) - y^2\sin\left(xy^2\right)\cdot 2xy + z^2) = \frac{\partial \mathrm{F}_2}{\partial x}$$

$$\frac{\partial \mathrm{F}_1}{\partial z} = 2yz = \frac{\partial \mathrm{F}_3}{\partial x}$$

$$\frac{\partial \mathrm{F}_2}{\partial z} = 2xz + \mathrm{e}^{yz} + zy\mathrm{e}^{yz} = \frac{\partial \mathrm{F}_3}{\partial y}$$

(c) Since $\mathbf{F}$ is conservative on $\mathbb{R}^3$, there exists a potential function f such that $\nabla f = \mathbf{F}$.

$$\frac{\partial f}{\partial x} = y^2\cos\left(xy^2\right) + yz^2, \quad \frac{\partial f}{\partial y} = 2xy\cos\left(xy^2\right) + xz^2 + z\mathrm{e}^{yz}, \quad \frac{\partial f}{\partial z} = 2xyz + y\mathrm{e}^{yz} + 2z$$

$$\int \frac{\partial f}{\partial x}\,dx = \int \left(y^2\cos\left(xy^2\right) + yz^2\right)\,dx = \sin\left(xy^2\right) + xyz^2 + g(y,z) = f(x,y,z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\sin\left(xy^2\right) + xyz^2 + g(y,z)\right) = 2xy\cos\left(xy^2\right) + xz^2 + g_y(y,z)$$

$$= 2xy\cos\left(xy^2\right) + xz^2 + z\mathrm{e}^{yz} \implies g_y(y,z) = z\mathrm{e}^{yz}$$

$$\int \frac{\partial f}{\partial y}\,dy = \int \left(2xy\cos\left(xy^2\right) + xz^2 + z\mathrm{e}^{yz}\right)\,dy = \sin\left(xy^2\right) + xyz^2 + \mathrm{e}^{yz} + h(z) = f(x,y,z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\sin\left(xy^2\right) + xyz^2 + \mathrm{e}^{yz} + h(z)\right) = 2xyz + y\mathrm{e}^{yz} + h_z(z)$$

$$= 2xyz + y\mathrm{e}^{yz} + 2z \implies h_z(z) = 2z$$

$$\int \frac{\partial f}{\partial z}\,dz = \int \left(2xyz + y\mathrm{e}^{yz} + 2z\right)\,dz = \sin\left(xy^2\right) + xyz^2 + \mathrm{e}^{yz} + z^2 + c = f(x,y,z)$$

The potential function for $\mathbf{F}$ is

$$\boxed{f(x,y,z) = \sin\left(xy^2\right) + xyz^2 + \mathrm{e}^{yz} + z^2 + c, \quad c \in \mathbb{R}}$$

5.

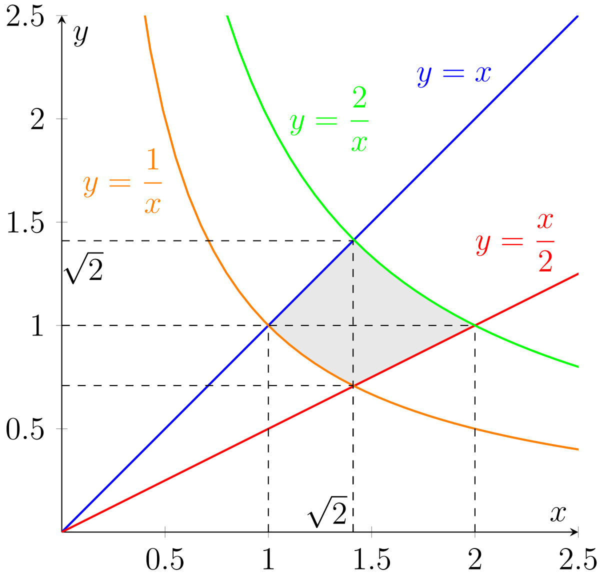
(a) We showed that $\mathbf{F}$ is conservative in 4(b). Using the Fundamental Theorem of Line Integrals, evaluate $f(0,1,3) - f(1,0,2)$.

$$\int_D \mathbf{F} \cdot d\mathbf{r} = f(0,1,3) - f(1,0,2) = \sin(0) + 0 + \mathrm{e}^3 + 3^2 + c - \left(\sin(1) + 0 + \mathrm{e}^0 + 2^2 + c\right)$$

$$= \boxed{\mathrm{e}^3 + 4}$$

(b) The curve of intersection is a circle, which is a closed curve. Since $\mathbf{F}$ is conservative, the value of the line integral is $\boxed{0}$.

6.



F_1 and F_2 have continuous partial derivatives. C is a closed curve with positive orientation. We may use the tangential form of Green's Theorem to evaluate the line integral.

$$\mathrm{I} = \oint_C \frac{y^2}{x}\,dx + y\ln x\,dy = \iint_R \left(\frac{\partial \mathrm{F}_2}{\partial x} - \frac{\partial \mathrm{F}_1}{\partial y}\right)\,dA = \iint_R -\frac{y}{x}\,dA$$

$$= \int_1^{\sqrt{2}} \int_{1/x}^{x} -\frac{y}{x}\,dy\,dx + \int_{\sqrt{2}}^{2} \int_{x/2}^{2/x} -\frac{y}{x}\,dy\,dx = \int_1^{\sqrt{2}} -\frac{y^2}{2x}\bigg|_{y=1/x}^{y=x}\,dx + \int_{\sqrt{2}}^{2} -\frac{y^2}{2x}\bigg|_{y=x/2}^{y=2/x}\,dx$$

$$= \int_1^{\sqrt{2}} \left(-\frac{x}{2} + \frac{1}{2x^3}\right)\,dx + \int_{\sqrt{2}}^{2} \left(-\frac{2}{x^3} + \frac{x}{8}\right)\,dx = \left[-\frac{x^2}{4} - \frac{1}{4x^2}\right]_1^{\sqrt{2}} + \left[\frac{1}{x^2} + \frac{x^2}{16}\right]_{\sqrt{2}}^{2}$$

$$= \left[-\frac{1}{2} - \frac{1}{8} - \left(-\frac{1}{4} - \frac{1}{4}\right)\right] + \left[\frac{1}{4} + \frac{1}{4} - \left(\frac{1}{2} + \frac{1}{8}\right)\right] = \boxed{-\frac{1}{4}}$$