2015-2016 Fall Semester
MAT123-07 Midterm
(10/12/2015)

1) Find all local extrema and inflection points of the function $f(x) = \frac{1}{x} + \frac{1}{x^2}$. On which intervals is the function increasing, decreasing, concave upward, or concave downward? Find all asymptotes. Graph the function.

2) Use Rolle's theorem to show that $3\tan x + x^3 = 2$ has exactly one solution on the interval $[0, \pi/4]$.

3) Find the tangent line to the graph of the equation $x\sin(xy - y^2) = x^2 - 1$, at $(1, 1)$.

4) Evaluate the limits

a. $\lim\limits_{x\to 0^+} (1 + \sin x)^{\frac{1}{x}}$

b. $\lim\limits_{x\to 3} \dfrac{x^2 - \lfloor x^2 \rfloor}{x - 3}$

5) A point P is moving in the xy-plane. When P is at $(4, 3)$, its distance to the origin is increasing at a rate of $\sqrt{2}$ cm/s, and its distance to the point $(7, 0)$ is decreasing at a rate of 3 cm/s. Determine the rate of change of the x-coordinate of P at that moment.

6) Sketch the region bounded by $y = 2|x|$ and $y = 8 - x^2$. Find the area of the region.

Solutions (Last update: 7/12/25 (12th of July) 5:35 PM)

1) Take the first derivative and set to 0.

$$f'(x) = -\frac{1}{x^2} - \frac{2}{x^3}$$

$$f'(x) = 0 \rightarrow \frac{2}{x^3} = -\frac{1}{x^2} \rightarrow x = -2 \,(\text{candidate for a critical point})$$

Take the second derivative and set to 0.

$$f''(x) = \frac{2}{x^3} + \frac{6}{x^4}$$

$$f''(x) = 0 \rightarrow \frac{1}{x^3} = -\frac{3}{x^4} \rightarrow x = -3 \,(\text{candidate for an inflection point})$$

$\{-2, -3\} \in D$. Therefore, $f(-2)$ gives rise to a local extremum. The sign of the first derivative changes from minus to plus, meaning $f(-2)$ is a local minimum. $x = -3$ gives rise to an inflection point because the sign of the second derivative also changes.

Find the asymptotes.

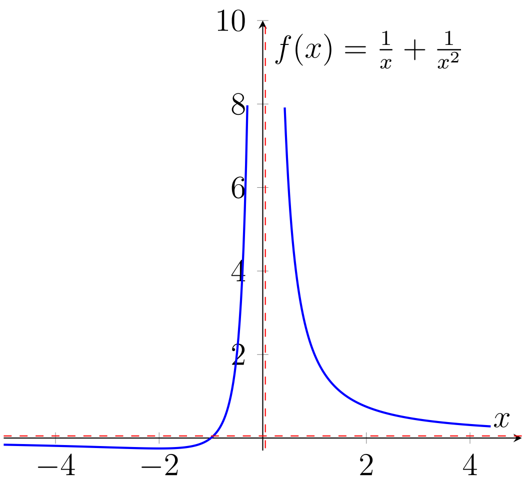
$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$$

$y = 0$ is the horizontal asymptote and $x = 0$ is the vertical asymptote.

Let us find monotonicity and concavity. If the sign of $f'(x)$ is minus, the function is decreasing on the corresponding interval; otherwise, increasing. If the sign of $f''(x)$ is minus, the graph of the function is concave downward; otherwise, concave upward.

x	$(-\infty, -3)$	$(-3, -2)$	$(-2, 0)$	$(0, \infty)$
f' sign	-	-	$+$	-
f'' sign	-	$+$	$+$	$+$

Eventually, sketch the graph.



2) Let $f(x) = 3\tan x + x^3 - 2$. f is continuous on $[0, \pi/4]$ and differentiable on $(0, \pi/4)$. By IVT (Intermediate Value Theorem), there exists at least one point where $f(x) = 0$ because $f(0) = -2$ and $f(\pi/4) = 1 + (\pi/4)^3$. Assume that we have two roots on the interval, so at some point c, $f'(c) = 0$.

$$f'(x) = 3\sec^2 x + 3x^2 \to f'(c) = 3\sec^2 c + 3c^2 = 0 \to \sec^2 c = -c^2$$

Since $-c^2 \leq 0$ and $\sec^2 c > 0$, there is no c that satisfies the equation. This contradicts our assumption that we have two roots on the interval. By Rolle's theorem, there is only one root on the interval $[0, \pi/4]$.

3) Implicitly differentiate both sides.

$$\frac{d}{dx}[x\sin(xy - y^2)] = \frac{d}{dx}(x^2 - 1)$$

$$1 \cdot \sin(xy - y^2) + x \cdot \cos(xy - y^2) \cdot \left[\left(1 \cdot y + x\frac{dy}{dx}\right) - 2y\frac{dy}{dx}\right] = 2x$$

Rearrange the equation to solve for $\frac{dy}{dx}$ through a careful and rigorous attempt.

$$x \cdot \cos(xy - y^2) \cdot \left[\left(1 \cdot y + x\frac{dy}{dx}\right) - 2y\frac{dy}{dx}\right] = 2x - \sin(xy - y^2)$$

$$y + \frac{dy}{dx}(x - 2y) = \frac{2x - \sin(xy - y^2)}{x \cdot \cos(xy - y^2)}$$

$$\frac{dy}{dx}(x - 2y) = \frac{2x - \sin(xy - y^2)}{x \cdot \cos(xy - y^2)} - y$$

$$\frac{dy}{dx} = \frac{\frac{2x - \sin(xy - y^2)}{x \cdot \cos(xy - y^2)} - y}{(x - 2y)} \tag{1}$$

Calculate $\left.\frac{dy}{dx}\right|_{(1,1)}$ from (1). This will give us the slope of the tangent line.

$$\left.\frac{dy}{dx}\right|_{(1,1)} = -1$$

Recall: $y - y_0 = m(x - x_0)$. m is $\frac{dy}{dx}$ at $x = 1$. So, the tangent line is:

$$y - 1 = -(x - 1) \to \boxed{y = 2 - x}$$

4)

a) Let L be the value of the limit. Then, take the logarithm of both sides.

$$L = \lim_{x \to 0^+} (1 + \sin x)^{\frac{1}{x}}$$

$$\ln(L) = \ln\left[\lim_{x\to 0^+}(1+\sin x)^{\frac{1}{x}}\right]$$

The expression on the right is continuous for $x > 0$. Therefore, we can take the logarithm inside the limit.

$$\ln(L) = \lim_{x\to 0^+}\ln\left[(1+\sin x)^{\frac{1}{x}}\right] = \lim_{x\to 0^+}\left[\frac{\ln(1+\sin x)}{x}\right]$$

If we substitute $x = 0$, the limit is in the form $0/0$. L'Hôpital's rule states that we may take the derivatives of both sides of the fraction if there's a $0/0$ indeterminate form. Apply the chain rule accordingly.

$$\lim_{x\to 0^+}\left[\frac{\ln(1+\sin x)}{x}\right] \overset{\text{L'H.}}{=} \lim_{x\to 0^+}\left[\frac{\frac{1}{1+\sin x}\cdot\cos x}{1}\right] = \lim_{x\to 0^+}\left[\frac{\cos x}{1+\sin x}\right]$$

The limit can now be evaluated by substituting $x = 0$.

$$\lim_{x\to 0^+}\left[\frac{\cos x}{1+\sin x}\right] = \frac{\cos 0}{1+\sin 0} = 1$$

Now, $\ln(L) = 1$. Simply, take L out of the logarithm.

$$\boxed{L = \mathrm{e}}$$

b) Look at the one-sided limits. Let us first evaluate the limit from the right side. Above and near $x = 3$, the floor function will return 9.

$$\lim_{x\to 3^+}\frac{x^2-\lfloor x^2\rfloor}{x-3} = \lim_{x\to 3^+}\frac{x^2-9}{x-3} = \lim_{x\to 3^+}\frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3^+}(x+3) = 6$$

From the left side, the output is the largest integer less than 9. Therefore, $\lfloor x^2\rfloor = 8$.

$$\lim_{x\to 3^-}\frac{x^2-\lfloor x^2\rfloor}{x-3} = \lim_{x\to 3^-}\frac{x^2-8}{x-3} = -\infty$$

The one-sided limits are not equal to each other. Therefore, the limit does not exist.

5) $x = x(t)$ and $y = y(t)$. The distance between the point P and the origin, and the distance between the point P and the point (7,0) are, respectively, given by:

$$f(t) = \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}$$

$$g(t) = \sqrt{(x-7)^2+(y-0)^2} = \sqrt{x^2-14x+49+y^2}$$

Take the first derivative with respect to time.

$$f'(t) = \frac{1}{2\sqrt{x^2+y^2}}\cdot\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right) \tag{2}$$

$$g'(t) = \frac{1}{2\sqrt{x^2 - 14x + 49 + y^2}} \cdot \left((2x - 14)\frac{dx}{dt} + 2y\frac{dy}{dt}\right) \tag{3}$$

For $t = t_0$, it is given $x(t_0) = 4$, $y(t_0) = 3$ and $f(t_0) = \sqrt{4^2 + 3^2} = 5$, $g(t_0) = \sqrt{3^2 + 3^2} = 3\sqrt{2}$. We then obtain a system of two equations by substituting values in (2) and (3):

$$f'(t_0) = \frac{1}{10} \cdot \left(8\frac{dx}{dt} + 6\frac{dy}{dt}\right) = \sqrt{2}$$

$$g'(t_0) = \frac{1}{6\sqrt{2}} \cdot \left((-6)\frac{dx}{dt} + 6\frac{dy}{dt}\right) = -3$$

Let us simplify the equations.

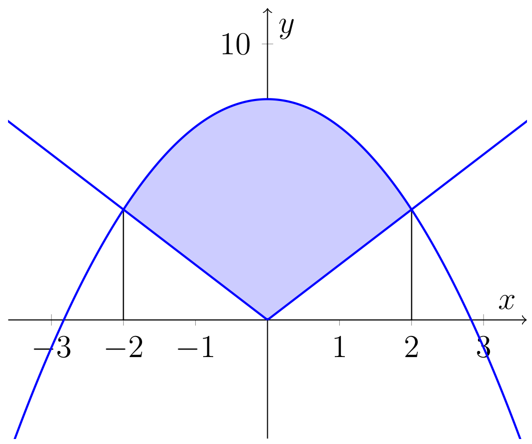
$$4x'(t_0) + 3y'(t_0) = 5\sqrt{2}$$
$$-3x'(t_0) + 3y'(t_0) = -9\sqrt{2}$$

The question asks us to find the change in the x-coordinate of P. Therefore, negate the latter equation and solve for $x'(t_0)$.

$$\boxed{x'(t_0) = 2\sqrt{2}}$$

The change in the y-coordinate is left as a practice for the reader.

6)



The area can be found by integrating the difference in y with respect to x. We split the integral into two because the absolute value function changes sign.

$$\text{I} = \int_{-2}^{2} (8 - x^2 - 2|x|)\, dx = \int_{-2}^{0} (8 - x^2 + 2x)\, dx + \int_{0}^{2} (8 - x^2 - 2x)\, dx$$

$$\text{I} = \left[8x - \frac{x^3}{3} + x^2\right]_{-2}^{0} + \left[8x - \frac{x^3}{3} - x^2\right]_{0}^{2}$$

$$\text{I} = 0 - \left(-16 + \frac{8}{3} + 4\right) + \left(16 - \frac{8}{3} - 4\right) - 0 = \boxed{\frac{56}{3}}$$