2023-2024 Fall
MAT124 Resit
(29/01/2024)

1.

(a) Find the critical points of the function

$$f(x,y) = 4x^3 - 6xy + y^2 + 2y$$

and classify them.

(b) Find the maximum and minimum values of the function

$$f(x,y,z) = x + y + z$$

by using Lagrange multipliers on the ellipsoid $x^2 + 4y^2 + 9z^2 = 1764$.

2.

(a) Sketch the domain of integration and rewrite the integral by changing the order of integration.

$$\int_0^1 \int_0^{x\sqrt{3}} \mathrm{e}^{-x^2-y^2}\, dy\, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} \mathrm{e}^{-x^2-y^2}\, dy\, dx$$

(b) Evaluate the integral

$$\iint_R \frac{xy}{1+x^4}\, dA$$

where R is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

3. The following integral gives the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and between the planes $z = 0$ and $z = 1$.

$$V = 4\int_0^{\pi/2} \int_0^{\sqrt{3}} \int_0^1 r\, dz\, dr\, d\theta + 4\int_0^{\pi/2} \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$$

(a) Write the integral in rectangular coordinates with the order of integration $dz\, dy\, dx$.

(b) Write the integral in spherical coordinates.

4.

(a) Is $\mathbf{F}(x,y) = 2xy\sin(x^2y)\,\mathbf{i} - x^2\sin(x^2y)\,\mathbf{j}$ conservative? Why?

(b) Show that

$$\mathbf{F}(x,y,z) = \big(2x + y^2 + z\cos x\big)\,\mathbf{i} + (2xy + \mathrm{e}^z)\,\mathbf{j} + (1 + y\mathrm{e}^z + \sin x)\,\mathbf{k}$$

is conservative.

(c) Find its potential function.

5. $\mathbf{F}(x,y,z) = (2x + y^2 + z\cos x)\,\mathbf{i} + (2xy + \mathrm{e}^z)\,\mathbf{j} + (1 + y\mathrm{e}^z + \sin x)\,\mathbf{k}$

(a) Let C be the curve of intersection of the cone $z^2 = 4x^2+9y^2$ and the plane $z = 1+x+2y$, and let D be the part of the curve C that lies in the first octant $x \geq 0,\ y \geq 0,\ z \geq 0$ from $(1,0,2)$ to $(0,1,3)$. Evaluate $\int_D \mathbf{F} \cdot d\mathbf{r}$.

(b) Let C be the curve of intersection of $x^2 + y^2 = 1$ and $z = 40$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

6. Evaluate

$$\oint_C \left(x^3 \sin\left(\sqrt{x^2+4}\right) - x\mathrm{e}^{x+2y}\right)\,dx + \left(\cos\left(y^3+y\right) - 4y\mathrm{e}^{x+2y}\right)\,dy$$

where C is the counterclockwise boundary of the parallelogram with vertices $(2,0)$, $(0,-1)$, $(-2,0)$, and $(0,1)$.

2023-2024 Fall Resit (29/01/2024) Solutions
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1.

(a) To find the critical points of f, determine where both $f_x = f_y = 0$ or one of the partial derivatives does not exist.

$$f_x = 12x^2 - 6y, \quad f_y = -6x + 2y + 2$$

$$f_x = 0 \implies y = 2x^2, \quad f_y = 0 \implies y = 3x - 1$$

$$f_x = f_y = 0 \implies 2x^2 - 3x + 1 = 0 \implies x_{1,2} = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{3 \pm 1}{4}$$

$$x = \frac{1}{2} \implies y = 2 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{2}, \quad x = 1 \implies y = 2 \cdot (1)^2 = 2$$

The critical points occur at $(1/2, 1/2)$ and $(1, 2)$. To classify these points, apply the second derivative test.

$$f_{xx} = 24x, \quad f_{xy} = f_{yx} = -6, \quad f_{yy} = 2$$

Calculate the Hessian determinant at these points.

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

$$(1/2, 1/2) \quad \to \quad \begin{array}{l} f_{xx} = 12, \quad f_{xy} = f_{yx} = -6, \quad f_{yy} = 2 \\ f_{xx} f_{yy} - f_{xy}^2 = 12 \cdot 2 - (-6)^2 = -12 < 0 \end{array}$$

$$(1, 2) \quad \to \quad \begin{array}{l} f_{xx} = 24, \quad f_{xy} = f_{yx} = -6, \quad f_{yy} = 2 \\ f_{xx} f_{yy} - f_{xy}^2 = 24 \cdot 2 - (-6)^2 = 12 > 0, \quad f_{xx} > 0 \end{array}$$

$$\boxed{\text{A local min occurs at } (1,2) \text{ and a saddle point occurs at } (1/2, 1/2).}$$

(b) Let $g(x, y, z) = x^2 + 4y^2 + 9z^2 - 1764$ be the constraint. Then solve the system of equations below.

$$\left.\begin{array}{l} \nabla f = \lambda \nabla g \\ g(x, y, z) = 0 \end{array}\right\} \quad \begin{array}{l} \nabla f = \langle 1, 1, 1 \rangle = \lambda \langle 2x, 8y, 18z \rangle = \lambda \nabla g \\ \therefore x = \dfrac{1}{2\lambda}, \quad y = \dfrac{1}{8\lambda}, \quad z = \dfrac{1}{18\lambda} \end{array}$$

Use the constraint.

$$x^2+4y^2+9z^2-1764=0 \implies \left(\frac{1}{2\lambda}\right)^2+4\left(\frac{1}{8\lambda}\right)^2+9\left(\frac{1}{18\lambda}\right)^2=1764$$

$$\implies \frac{49}{144\lambda^2}=42^2 \implies \lambda=\pm\frac{1}{72}$$

$$\lambda=\pm\frac{1}{72} \implies x=\pm 36, \quad y=\pm 9, \quad z=\pm 4$$

To find the minimum and maximum values, consider the points $(-36,-9,-4)$ and $(36,9,4)$, respectively.

$$\boxed{f_{\text{min}}=-36-9-4=-49, \quad f_{\text{max}}=36+9+4=49}$$

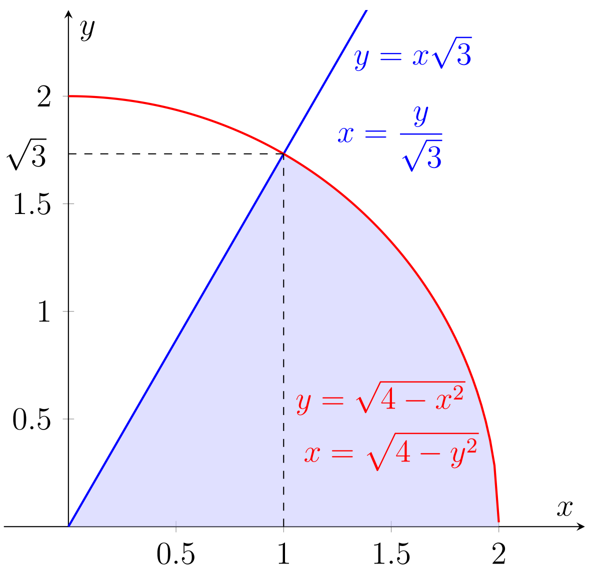
Compare all the values.

$$f(0,y,z)=f(x,0,z)=f(x,y,0)=0, \quad f\left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)=\frac{1}{4}\cdot\frac{1}{4}\cdot\left(\frac{1}{2}\right)^2=\frac{1}{64}$$

$$\boxed{\text{The maximum value is } \frac{1}{64}, \text{ the minimum value is } 0.}$$

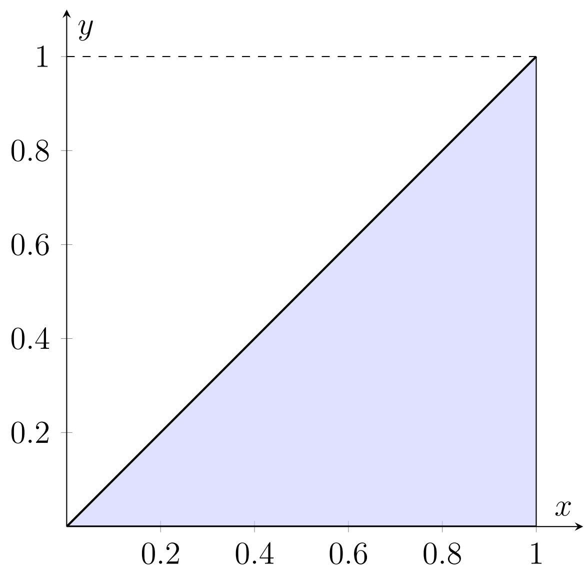
2.

(a)



$$\boxed{\int_0^{\sqrt{3}} \int_{y/\sqrt{3}}^{\sqrt{4-y^2}} e^{-x^2-y^2}\, dx\, dy}$$

(b) Sketch the region.



$$\iint_R \frac{xy}{1+x^4}\,dA = \int_0^1 \int_0^x \frac{xy}{1+x^4}\,dy\,dx = \int_0^1 \frac{x}{1+x^4}\left[\frac{y^2}{2}\right]_{y=0}^{y=x} dx = \frac{1}{2}\int_0^1 \frac{x^3}{1+x^4}\,dx$$

$$= \frac{1}{2}\cdot\left[\frac{1}{4}\ln\left|1+x^4\right|\right]_0^1 = \frac{1}{2}\cdot\frac{1}{4}(\ln 2 - \ln 1) = \boxed{\frac{1}{8}\ln 2}$$

3.

(a)

$$\begin{aligned} z &= z \\ x &= r\cos\theta \\ y &= r\sin\theta \\ r^2 &= x^2+y^2 \\ dV &= r\,dz\,dr\,d\theta = dz\,dy\,dx \end{aligned} \quad \rightarrow \quad \begin{gathered} z = \sqrt{4-r^2} \implies z = \sqrt{4-x^2-y^2} \\ z = 1 \end{gathered}$$

Notice that we have two distinct upper bounds for z, which are $z = \sqrt{4-x^2-y^2}$ and $z = 1$. The lower bound for z is $z = 0$. For the upper bounds of z, we choose the minimum of the bounds. If we project the shape onto the xy-plane, we get $x^2+y^2 = 4$. Rewrite the integral in rectangular coordinates.

$$\boxed{V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\min\left(1,\sqrt{4-x^2-y^2}\right)} dz\,dy\,dx}$$

Another method is to calculate the volume of the corresponding hemisphere and then extract the upper part of the hemisphere. The volume of a hemisphere is given by the formula

$$V_{\text{hemisphere}} = \frac{2}{3}\pi r^3,$$

where r is the radius. Now, focus on the upper part of the hemisphere. The upper bound for z is the sphere $x^2+y^2+z^2=4$, and the lower bound is $z=1$. The solid lies above $x^2+y^2=3$. The equivalent form of the answer above is as follows.

$$\boxed{V=\frac{16\pi}{3}-\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{1}^{\sqrt{4-x^2-y^2}}dz\,dy\,dx}$$

(b) For spherical coordinates, we have

$$\begin{aligned} z &= \rho\cos\phi \\ r &= \rho\sin\phi \\ x^2+y^2+z^2 &= \rho^2 \\ dV &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned} \quad\to\quad \begin{aligned} z=\sqrt{4-r^2} \implies \rho\cos\phi=\sqrt{4-\rho^2\sin^2\phi} \implies \rho=2 \\ z=1 \implies \rho\cos\phi=1 \implies \rho=\frac{1}{\cos\phi} \end{aligned}$$

For ρ, we have two different upper bounds. We choose the minimum of these bounds.

$$\boxed{V=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\min\left(2,\,\frac{1}{\cos\phi}\right)}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

Alternatively, we may find the angle of intersection of the plane $z=1$ and the sphere $x^2+y^2+z^2=4$.

$$\frac{1}{\cos\phi}=2 \implies \cos\phi=\frac{1}{2} \implies \phi=\frac{\pi}{3}$$

From $\phi=0$ to $\phi=\dfrac{\pi}{3}$, the upper bound for ρ is $\rho=\dfrac{1}{\cos\phi}$. From $\phi=\dfrac{\pi}{3}$ to $\phi=\dfrac{\pi}{2}$, it is $\rho=2$. The equivalent integral is as follows.

$$\boxed{V=\int_0^{2\pi}\int_0^{\pi/3}\int_0^{1/\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta+\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

4.

(a) For $\mathbf{F}$ to be conservative, it must be the gradient of some potential function ϕ. We may apply the component test to determine whether the mixed partial derivatives are equal.

$$\left.\begin{aligned} \frac{\partial \mathrm{F}_1}{\partial y} &= 2x\sin\left(x^2y\right)+2xy\cos\left(x^2y\right)\cdot x^2 \\ \frac{\partial \mathrm{F}_2}{\partial x} &= -2x\sin\left(x^2y\right)-x^2\cos\left(x^2y\right)\cdot 2xy \end{aligned}\right\} \implies \frac{\partial \mathrm{F}_1}{\partial y}\neq\frac{\partial \mathrm{F}_2}{\partial x}$$

The mixed partial derivatives are not equal. Therefore, the force is not conservative.

(b) Like what we did above, determine the mixed partial derivatives.

$$\frac{\partial \mathrm{F}_1}{\partial y}=2y=\frac{\partial \mathrm{F}_2}{\partial x},\quad \frac{\partial \mathrm{F}_1}{\partial z}=\cos x=\frac{\partial \mathrm{F}_3}{\partial x},\quad \frac{\partial \mathrm{F}_2}{\partial z}=\mathrm{e}^z=\frac{\partial \mathrm{F}_3}{\partial y}$$

(c) Since $\mathbf{F}$ is conservative on $\mathbb{R}^3$, there exists a potential function f such that $\nabla f = \mathbf{F}$.

$$\frac{\partial f}{\partial x} = 2x + y^2 + z\cos x, \quad \frac{\partial f}{\partial y} = 2xy + \mathrm{e}^z, \quad \frac{\partial f}{\partial z} = 1 + y\mathrm{e}^z + \sin x$$

$$\int \frac{\partial f}{\partial x}\, dx = \int \left(2x + y^2 + z\cos x\right)\, dx = x^2 + xy^2 + z\sin x + g(y,z) = f(x,y,z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + xy^2 + z\sin x + g(y,z)\right) = 2xy + g_y(y,z) = 2xy + \mathrm{e}^z \implies g_y(y,z) = \mathrm{e}^z$$

$$\int \frac{\partial f}{\partial y}\, dy = \int \left(2xy + \mathrm{e}^z\right)\, dy = x^2 + z\sin x + xy^2 + y\mathrm{e}^z + h(z) = f(x,y,z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2 + z\sin x + xy^2 + \mathrm{e}^z + h(z)\right) = \sin x + y\mathrm{e}^z + h_z(z)$$

$$= 1 + y\mathrm{e}^z + \sin x \implies h_z(z) = 1$$

$$\int \frac{\partial f}{\partial z}\, dz = \int \left(1 + y\mathrm{e}^z + \sin x\right)\, dz = x^2 + z\sin x + xy^2 + y\mathrm{e}^z + z + c = f(x,y,z)$$

The potential function for $\mathbf{F}$ is

$$\boxed{f(x,y,z) = x^2 + z\sin x + xy^2 + y\mathrm{e}^z + z + c, \quad c \in \mathbb{R}}$$

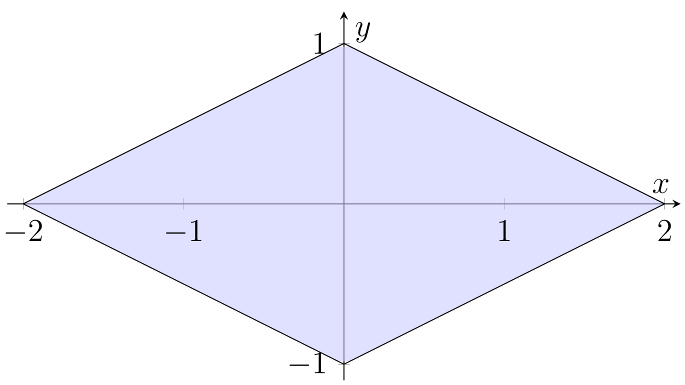
5.

(a) We showed that $\mathbf{F}$ is conservative in $4(b)$. Using the Fundamental Theorem of Line Integrals, evaluate $f(0,1,3) - f(1,0,2)$.

$$\int_D \mathbf{F}\cdot d\mathbf{r} = f(0,1,3) - f(1,0,2) = \mathrm{e}^3 + 3 + c - (1 + 2\sin 1 + 2 + c) = \boxed{\mathrm{e}^3 - 2\sin 1}$$

(b) The curve of intersection is a circle, which is a closed curve. Since $\mathbf{F}$ is conservative, the value of the line integral is $\boxed{0}$.

6.



F_1 and F_2 have continuous partial derivatives. C is a closed curve with positive orientation. We may use the tangential form of Green's Theorem to evaluate the line integral.

$$\mathrm{I} = \oint_C \left(x^3 \sin\left(\sqrt{x^2+4}\right) - x\mathrm{e}^{x+2y}\right)\, dx + \left(\cos\left(y^3+y\right) - 4y\mathrm{e}^{x+2y}\right)\, dy$$

$$= \iint_R \left(\frac{\partial \mathrm{F}_2}{\partial x} - \frac{\partial \mathrm{F}_1}{\partial y}\right) dA = \iint_R \left(-4y\mathrm{e}^{x+2y} \cdot 1 - \left(-x\mathrm{e}^{x+2y} \cdot 2\right)\right) dA$$

$$= \iint_R (2x-4y) \cdot \mathrm{e}^{x+2y}\, dA \tag{1}$$

From the edges of the parallelogram, we have $x = 2y+2, x = 2-2y, x = -2-2y, x = 2y-2$. Now, use the method of change of variables. Let $u = x-2y,\ v = x+2y$. Then $x = \dfrac{u+v}{2},\quad y = \dfrac{v-u}{4}$.

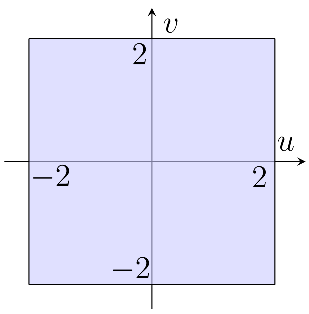
$$x = 2y+2 \implies \frac{u+v}{2} = 2\left(\frac{v-u}{4}\right) + 2 \implies u = 2$$

$$x = 2-2y \implies \frac{u+v}{2} = 2-2\left(\frac{v-u}{4}\right) \implies v = 2$$

$$x = -2-2y \implies \frac{u+v}{2} = -2-2\left(\frac{v-u}{4}\right) \implies v = -2$$

$$x = 2y-2 \implies \frac{u+v}{2} = 2\left(\frac{v-u}{4}\right) - 2 \implies u = -2$$

The region in uv-coordinates becomes as follows. Calculate the Jacobian determinant and rewrite the integral in (1).



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{-1}{4} & \dfrac{1}{4} \end{vmatrix} = \frac{1}{2}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{4}$$

$$\mathrm{I} = \iint_R (2x-4y)\cdot \mathrm{e}^{x+2y}\, dA = \int_{-2}^{2}\int_{-2}^{2} 2u\mathrm{e}^{v} \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv = \int_{-2}^{2}\int_{-2}^{2} 2u\mathrm{e}^{v} \cdot \left|\frac{1}{4}\right| du\, dv$$

$$= \int_{-2}^{2} \mathrm{e}^{v} \left[\frac{u^2}{4}\right]_{u=-2}^{u=2} dv = \int_{-2}^{2} \mathrm{e}^{v} \cdot 0\, dv = \boxed{0}$$