

2014-2015 Summer  
MAT123 Final  
(23/08/2015)

1. Sketch the graph of  $f(x) = \ln(x^2 + 1)$ .
2. Determine the area of the region bounded by  $y = 4x + 3$  and  $y = 6 - x - 2x^2$ .
3. Use the Washer Method to find the volume of the solid obtained by revolving the region bounded by  $y = 2\sqrt{x-1}$ ,  $y = x - 1$  about the line  $x = -1$ , respectively.
4. Use the Cylindrical Shell Method to find the volume of the solid obtained by revolving the region bounded by  $x = y^2 - 4$  and  $x = 6 - 3y$  about  $y = -8$ .
5. Evaluate the following integrals.

(a)  $\int 4 \left( \frac{1}{x} - e^{-x} \right) \cos(e^{-x} + \ln x) \, dx$

(b)  $\int \frac{x^3 + 10x^2 + 3x + 36}{(x-1)^2 (x^2 + 4)^2} \, dx$

(c)  $\int \frac{\sqrt{25x^2 - 4}}{x} \, dx$

(d)  $\int x^2 \cos(4x) \, dx$

(e)  $\int \frac{dx}{2x^2 - 3x + 2}$

6. Investigate the convergence of the improper integral  $\int_{-\infty}^0 (1 + 2x) e^{-x} \, dx$ .
7. Evaluate the arc length  $x = \frac{2}{3}(y-1)^{3/2}$  for  $1 \leq y \leq 2$ .

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1.  $\ln x$  is defined for  $x > 0$ . Therefore,  $\ln(x^2 + 1)$  is defined on  $\mathbb{R}$  because  $x^2 + 1 \geq 1 > 0$ .

Let us find the limit at infinity and negative infinity.

$$\lim_{x \rightarrow \infty} \ln(x^2 + 1) = \lim_{x \rightarrow -\infty} \ln(x^2 + 1) = \infty$$

No asymptotes occur.

Take the first derivative and find the critical points. Apply the chain rule.

$$y' = \frac{d}{dx} \ln(x^2 + 1) = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1}$$

A critical point occurs at  $x = 0$ . At this point, the first derivative is 0.

Take the second derivative. Apply the quotient rule.

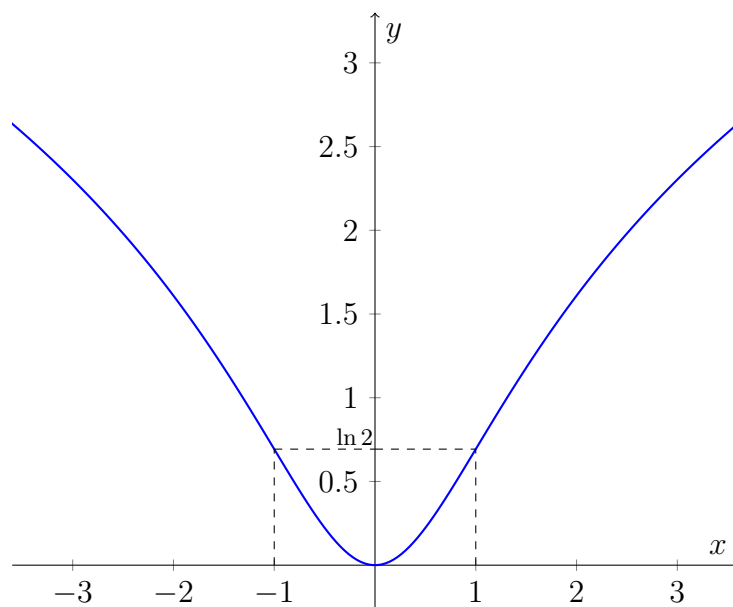
$$y'' = \frac{d}{dx} \left( \frac{2x}{x^2 + 1} \right) = \frac{2 \cdot (x^2 + 1) - 2x \cdot (2x)}{(x^2 + 1)^2} = \frac{2 - 2x^2}{(x^2 + 1)^2}$$

The inflection points occur at  $x = \pm 1$ . At these points, the direction of the curvature changes.

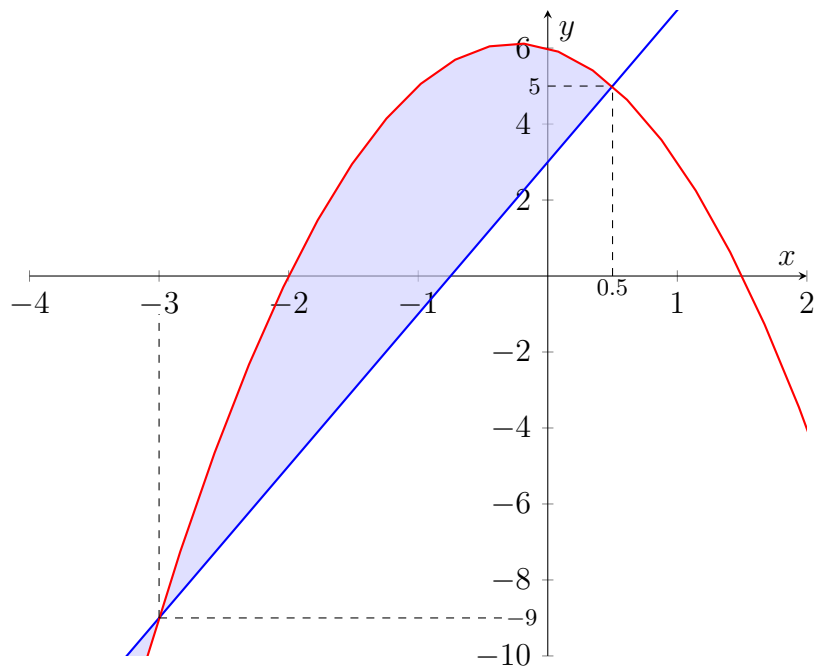
Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(-1) = f(1) = \ln 2, \quad f(0) = 0$$

$x$	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$y$	$(\ln 2, \infty)$	$(0, \ln 2)$	$(0, \ln 2)$	$(\ln 2, \infty)$
$y'$ sign	-	-	+	+
$y''$ sign	-	+	+	-



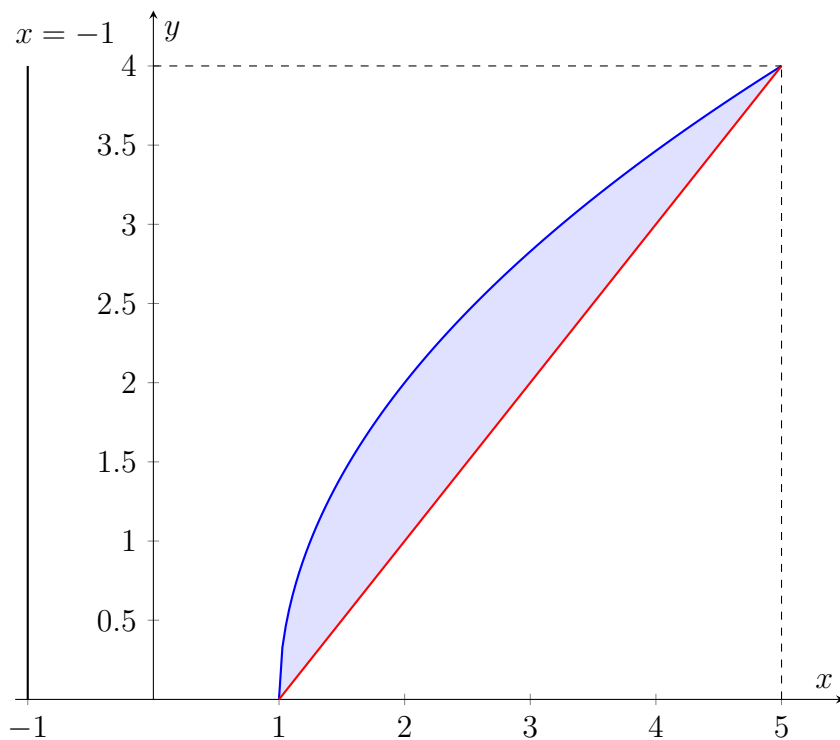
2.



$$\text{Area} = \int_{-3}^{1/2} [6 - x - 2x^2 - (4x + 3)] \, dx = \int_{-3}^{1/2} (3 - 5x - 2x^2) \, dx$$

$$= 3x - \frac{5x^2}{2} - \frac{2x^3}{3} \Big|_{-3}^{1/2} = \left( \frac{3}{2} - \frac{5}{8} - \frac{1}{12} \right) - \left( -9 - \frac{45}{2} + 18 \right) = \boxed{\frac{343}{24}}$$

3.



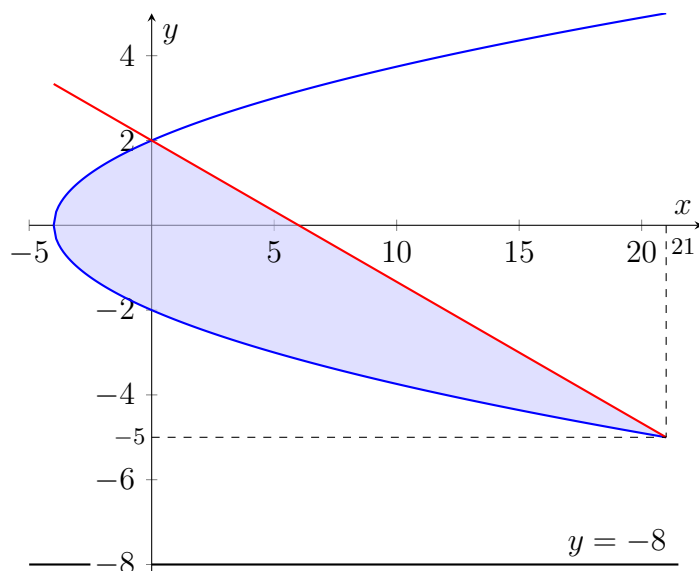
Rewrite the equations of the curves and solve for  $x$ . Let  $V$  be the volume of the solid.

$$y = 2\sqrt{x-1} \implies y^2 = 4x - 4 \implies x = \frac{y^2 + 4}{4}$$

$$y = x - 1 \implies x = y + 1$$

$$\begin{aligned} V &= \int_D \pi [R_2^2(y) - R_1^2(y)] dy = \int_0^4 \pi \left[ ((y+1)+1)^2 - \left( \left( \frac{y^2+4}{4} \right) + 1 \right)^2 \right] dy \\ &= \pi \int_0^4 \left[ (y+2)^2 - \left( \frac{y^2}{4} + 2 \right)^2 \right] dy = \pi \int_0^4 \left( y^2 + 4y + 4 - \frac{y^4}{16} - y^2 - 4 \right) dy \\ &= \pi \int_0^4 \left( 4y - \frac{y^4}{16} \right) dy = \pi \left[ 2y^2 - \frac{y^5}{80} \right]_0^4 = \pi \left[ \left( 32 - \frac{64}{5} \right) - (0) \right] = \boxed{\frac{96\pi}{5}} \end{aligned}$$

4.



Let  $V$  be the volume of the solid.

$$\begin{aligned} V &= \int_D 2\pi \cdot h(y) \cdot r(y) dy = 2\pi \int_{-5}^2 (y+8) \cdot [(6-3y) - (y^2-4)] dy \\ &= 2\pi \int_{-5}^2 (y+8)(-y^2-3y+10) dy = 2\pi \int_{-5}^2 (-y^3-3y^2+10y-8y^2-24y+80) dy \\ &= 2\pi \int_{-5}^2 (-y^3-11y^2-14y+80) dy = 2\pi \left[ -\frac{y^4}{4} - \frac{11y^3}{3} - 7y^2 + 80y \right]_{-5}^2 \\ &= 2\pi \left[ \left( -4 - \frac{88}{3} - 28 + 160 \right) - \left( \frac{625}{4} + \frac{1375}{3} - 175 - 400 \right) \right] = \boxed{\frac{4459\pi}{6}} \end{aligned}$$

5.

(a) Use the  $u$ -substitution method. Let  $u = e^{-x} + \ln x$ . Then  $du = \left(-e^{-x} + \frac{1}{x}\right) dx$ .

$$\begin{aligned} \int 4 \left( \frac{1}{x} - e^{-x} \right) \cos(e^{-x} + \ln x) dx &= \int 4 \cos u du = 4 \sin u + c \\ &= \boxed{4 \sin(e^{-x} + \ln x) + c, c \in \mathbb{R}} \end{aligned}$$

(b) Decompose the fraction into multiple partial fractions. Let  $A, B, C, D, E, F \in \mathbb{R}$ .

$$I = \int \frac{x^3 + 10x^2 + 3x + 36}{(x-1)^2 (x^2+4)^2} dx = \int \left( \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2} \right) dx$$

Let  $N = x^3 + 10x^2 + 3x + 36$ .

$$\begin{aligned} N &= A(x^2+4)^2(x-1) + B(x^2+4)^2 + (Cx+D)(x^2+4)(x-1)^2 + (Ex+F)(x-1)^2 \\ &= (x^2+4)^2[A(x-1) + B] + (x-1)^2[(Cx+D)(x^2+4) + Ex+F] \\ &= (x^4 + 8x^2 + 16)(Ax - A + B) + (x^2 - 2x + 1) \\ &\quad \cdot (Cx^3 + 4Cx + Dx^2 + 4D + Ex + F) \\ &= Ax^5 - Ax^4 + Bx^4 + 8Ax^3 - 8Ax^2 + 8Bx^2 + 16Ax - 16A + 16B \\ &\quad + Cx^5 + 4Cx^3 + Dx^4 + 4Dx^2 + Ex^3 + Fx^2 - 2Cx^4 - 8Cx^2 - 2Dx^3 - 8Dx \\ &\quad - 2Ex^2 - 2Fx + Cx^3 + 4Cx + Dx^2 + 4D + Ex + F \\ &= x^5(A+C) + x^4(-A+B+D-2C) + x^3(8A+4C+E-2D+C) \\ &\quad + x^2(-8A+8B+4D+F-8C-2E+D) + x(16A-8D-2F+4C+E) \\ &\quad - 16A + 16B + 4D + F \end{aligned}$$

Equate the coefficients of like terms.

$$\begin{aligned} A + C &= 0 \\ -A + B + D - 2C &= 0 \\ 8A + 5C + E - 2D &= 1 \\ -8A + 8B + 5D - 8C + F - 2E &= 10 \\ 16A - 8D - 2F + 4C + E &= 3 \\ -16A + 16B + 4D + F &= 36 \end{aligned} \tag{1}$$

From (1),  $A = -C$ . Rewrite  $C$  in terms of  $A$  and rearrange the equations.

$$A + B + D = 0 \quad (2)$$

$$3A + E - 2D = 1$$

$$8B + 5D + F - 2E = 10$$

$$12A - 8D - 2F + E = 3$$

$$-16A + 16B + 4D + F = 36$$

From (2),  $A + B = -D$ . Rewrite  $D$  in terms of  $A$  and  $B$  and rearrange the equations.

$$5A + 2B + E = 1 \quad (3)$$

$$-5A + 3B + F - 2E = 10 \quad (4)$$

$$20A + 8B - 2F + E = 3 \quad (5)$$

$$-20A + 12B + F = 36 \quad (6)$$

By using the couples (3) & (4) and (4) & (5), eliminate  $E$ .

$$\left. \begin{array}{rcl} 5A + 7B + F & = & 12 \\ 35A + 19B - 3F & = & 16 \\ -20A + 12B + F & = & 36 \end{array} \right\} \implies \left. \begin{array}{rcl} 50A + 40B & = & 52 \\ -25A + 55B & = & 124 \end{array} \right\} \implies A = -\frac{14}{25}, B = 2$$

Therefore,  $C = \frac{14}{25}$ ,  $D = -\frac{36}{25}$ . From (3),  $E = -\frac{1}{5}$ , and from (6),  $F = \frac{4}{5}$ .

Substitute the values into  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$ .

$$I = \int \left( -\frac{14}{25} \cdot \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{\frac{14}{25}x - \frac{36}{25}}{x^2 + 4} + \frac{-\frac{1}{5}x + \frac{4}{5}}{(x^2 + 4)^2} \right) dx \quad (7)$$

From now on, integrate term by term. Integrate the first term in (7).

$$\int -\frac{14}{25} \cdot \frac{1}{x-1} dx = -\frac{14}{25} \int \frac{1}{x-1} dx = -\frac{14}{25} \ln |x-1| + c \quad (8)$$

Integrate the second term in (7).

$$\int \frac{2}{(x-1)^2} dx = -\frac{2}{x-1} + c \quad (9)$$

Integrate the third term in (7).

$$\begin{aligned} \int \frac{\frac{14}{25}x - \frac{36}{25}}{x^2 + 4} dx &= \frac{1}{25} \int \frac{14x - 36}{x^2 + 4} dx = \frac{7}{25} \int \frac{2x}{x^2 + 4} dx - \frac{36}{25} \int \frac{1}{x^2 + 4} dx \\ &= \frac{7}{25} \ln |x^2 + 4| - \frac{36}{100} \int \frac{1}{\left(\frac{x}{2}\right)^2 + 1} dx \\ &= \frac{7}{25} \ln (x^2 + 4) - \frac{36}{50} \arctan \left( \frac{x}{2} \right) + c \end{aligned} \quad (10)$$

Integrate the last term in (7).

$$\int \frac{-\frac{1}{5}x + \frac{4}{5}}{(x^2 + 4)^2} dx = \frac{1}{5} \int \frac{4 - x}{(x^2 + 4)^2} dx = \frac{4}{5} \int \frac{1}{(x^2 + 4)^2} dx - \frac{1}{5} \int \frac{x}{(x^2 + 4)^2} dx \quad (11)$$

First, solve the integral on the left in (11). Let  $x = 2 \tan u$ , then  $dx = 2 \sec^2 u du$ .

$$\begin{aligned} \int \frac{1}{(x^2 + 4)^2} dx &= \int \frac{2 \sec^2 u}{(4 \tan^2 u + 4)^2} du = \int \frac{2 \sec^2 u}{16 \sec^4 u} du = \frac{1}{8} \int \frac{1}{\sec^2 u} du \\ &= \frac{1}{8} \int \cos^2 u du = \frac{1}{8} \int \left( \frac{1 + \cos 2u}{2} \right) du = \frac{1}{8} \left( \frac{u}{2} + \frac{\sin 2u}{4} \right) + c \\ &= \frac{u}{16} + \frac{\sin u \cos u}{16} + c \end{aligned}$$

Since  $x = 2 \tan u$ ,  $\tan u = \frac{x}{2}$

$$u = \arctan \frac{x}{2}, \quad \sin u = \frac{x}{\sqrt{x^2 + 4}}, \quad \cos u = \frac{2}{\sqrt{x^2 + 4}}$$

Rewrite the integral.

$$\int \frac{1}{(x^2 + 4)^2} dx = \frac{1}{16} \left( \arctan \frac{x}{2} + \frac{2x}{x^2 + 4} \right) + c \quad (12)$$

Now, solve the integral on the right in (11). Let  $u = x^2 + 4$ , then  $du = 2x dx$ .

$$\int \frac{x}{(x^2 + 4)^2} dx = \int \frac{du}{2u^2} = -\frac{1}{2u} + c = -\frac{1}{2(x^2 + 4)} + c \quad (13)$$

Rewrite the integral in (11) using (12) and (13).

$$\frac{4}{5} \int \frac{1}{(x^2 + 4)^2} dx - \frac{1}{5} \int \frac{x}{(x^2 + 4)^2} dx = \frac{1}{20} \left( \arctan \frac{x}{2} + \frac{2x + 2}{x^2 + 4} \right) + c \quad (14)$$

Eventually, using (8), (9), (10) and (14), rewrite (7).

$$I = \left[ -\frac{14}{25} \ln |x - 1| - \frac{2}{x - 1} + \frac{7}{25} \ln (x^2 + 4) - \frac{67}{100} \arctan \frac{x}{2} + \frac{x + 1}{10(x^2 + 4)} + c, \quad c \in \mathbb{R} \right]$$

(c) Let  $x = \frac{2}{5} \sec u$  for  $0 \leq u < \frac{\pi}{2}$ , then  $dx = \frac{2}{5} \sec u \tan u du$ .

$$I = \int \frac{\sqrt{25x^2 - 4}}{x} dx = \int \frac{\sqrt{4 \sec^2 u - 4}}{\frac{2}{5} \sec u} \cdot \frac{2}{5} \sec u \tan u du \quad [\tan^2 u + 1 = \sec^2 u]$$

$$I = 2 \int |\tan u| \tan u \, du \quad [\tan u > 0]$$

$$= 2 \int \tan^2 u \, du = 2 \int \sec^2 u \, du - 2 \int du = 2 \tan u - 2u + c$$

Recall that  $x = \frac{2}{5} \sec u$ .

$$\sec u = \frac{5x}{2} \implies \sec^2 u = \frac{25x^2}{4} \implies \tan u = \frac{\sqrt{25x^2 - 4}}{2} \implies u = \arctan\left(\frac{\sqrt{25x^2 - 4}}{2}\right)$$

$$\int \frac{\sqrt{25x^2 - 4}}{x} dx = \boxed{\sqrt{25x^2 - 4} - 2 \arctan\left(\frac{\sqrt{25x^2 - 4}}{2}\right) + c, \quad c \in \mathbb{R}}$$

(d) Use the method of integration by parts.

$$\left. \begin{aligned} u = x^2 &\implies du = 2x \, dx \\ dv = \cos(4x) \, dx &\implies v = \frac{1}{4} \sin(4x) \end{aligned} \right\} \rightarrow \int u \, dv = uv - \int v \, du$$

$$I = x^2 \cdot \frac{1}{4} \sin(4x) - \int \frac{1}{4} \sin(4x) \cdot 2x \, dx = \frac{x^2}{4} \sin(4x) - \frac{1}{2} \int x \sin(4x) \, dx$$

Apply the same procedure.

$$\left. \begin{aligned} w = x &\implies dw = dx \\ dz = \sin(4x) \, dx &\implies z = -\frac{1}{4} \cos(4x) \end{aligned} \right\} \rightarrow \int w \, dz = wz - \int z \, dw$$

$$I = \frac{x^2}{4} \sin(4x) - \frac{1}{2} \left[ \frac{-x}{4} \cos(4x) - \int -\frac{1}{4} \cos(4x) \, dx \right]$$

$$= \boxed{\frac{x^2}{4} \sin(4x) + \frac{x}{8} \cos(4x) - \frac{1}{32} \sin(4x) + c, \quad c \in \mathbb{R}}$$

(e)

$$\int \frac{dx}{2x^2 - 3x + 2} = \int \frac{dx}{2x^2 - 3x + \frac{9}{8} + \frac{7}{8}} = \int \frac{dx}{\left(\sqrt{2x} - \frac{3\sqrt{2}}{4}\right)^2 + \frac{7}{8}}$$

$$= \frac{8}{7} \int \frac{dx}{\frac{8 \left(\sqrt{2x} - \frac{3\sqrt{2}}{4}\right)^2}{7} + 1}$$



Let  $u = \frac{2\sqrt{2}}{\sqrt{7}} \left( \sqrt{2}x - \frac{3\sqrt{2}}{4} \right)$ , then  $du = \frac{4}{\sqrt{7}} dx$ .

$$\begin{aligned} \frac{8}{7} \int \frac{dx}{\frac{8 \left( \sqrt{2}x - \frac{3\sqrt{2}}{4} \right)^2}{7} + 1^2} &= \frac{8}{7} \int \frac{1}{u^2 + 1} \cdot \frac{\sqrt{7}}{4} du = \frac{2}{\sqrt{7}} \int \frac{1}{u^2 + 1} du = \frac{2}{\sqrt{7}} \arctan u + c \\ &= \frac{2}{\sqrt{7}} \arctan \left[ \frac{2\sqrt{2}}{\sqrt{7}} \left( \sqrt{2}x - \frac{3\sqrt{2}}{4} \right) \right] + c, \quad c \in \mathbb{R} \\ &= \boxed{\frac{2}{\sqrt{7}} \arctan \left( \frac{4x - 3}{\sqrt{7}} \right) + c, \quad c \in \mathbb{R}} \end{aligned}$$

6. Use the method of integration by parts.

$$\begin{aligned} \left. \begin{aligned} u &= 1 + 2x \implies du = 2 dx \\ dv &= e^{-x} dx \implies v = -e^{-x} \end{aligned} \right\} \rightarrow \int u dv = uv - \int v du \\ \int_{-\infty}^0 (1 + 2x) e^{-x} dx &= \lim_{R \rightarrow -\infty} -e^{-x}(1 + 2x) \Big|_R^0 - \int_{-\infty}^0 -e^{-x} \cdot 2 dx \\ &= \lim_{R \rightarrow -\infty} (-e^0 \cdot 1 + e^{-R}(1 + 2 \cdot R)) + \lim_{P \rightarrow -\infty} 2e^{-x} \Big|_P^0 \\ &= -\infty + 2 \lim_{P \rightarrow -\infty} (e^0 - e^{-P}) = -\infty - \infty = \boxed{-\infty} \end{aligned}$$

The integral diverges to negative infinity.

7. The length of a curve defined by  $x = f(y)$  whose derivative is continuous on the interval  $a \leq y \leq b$  can be evaluated using the integral

$$S = \int_a^b \sqrt{1 + \left( \frac{dx}{dy} \right)^2} dy.$$

Find  $\frac{dx}{dy}$ .

$$\frac{dx}{dy} = \frac{2}{3} \cdot \frac{3}{2} (y - 1)^{1/2} = \sqrt{y - 1}$$

Set  $a = 1$ ,  $b = 2$  and find the length.

$$S = \int_1^2 \sqrt{1 + \left( \sqrt{y - 1} \right)^2} dy = \int_1^2 \sqrt{y} dy = \frac{2}{3} y^{3/2} \Big|_1^2 = \boxed{\frac{2}{3} (2\sqrt{2} - 1)}$$