2014-2015 Summer
MAT123 Final
(23/08/2015)

1. Sketch the graph of $f(x) = \ln(x^2+1)$.

2. Determine the area of the region bounded by $y = 4x+3$ and $y = 6 - x - 2x^2$.

3. Use the Washer Method to find the volume of the solid obtained by revolving the region bounded by $y = 2\sqrt{x-1}$, $y = x-1$ about the line $x = -1$, respectively.

4. Use the Cylindrical Shell Method to find the volume of the solid obtained by revolving the region bounded by $x = y^2 - 4$ and $x = 6 - 3y$ about $y = -8$.

5. Evaluate the following integrals.

(a) $\displaystyle\int 4\left(\frac{1}{x} - \mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x} + \ln x\right)\, dx$

(b) $\displaystyle\int \frac{x^3 + 10x^2 + 3x + 36}{(x-1)^2\,(x^2+4)^2}\, dx$

(c) $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\, dx$

(d) $\displaystyle\int x^2\cos(4x)\, dx$

(e) $\displaystyle\int \frac{dx}{2x^2-3x+2}$

6. Investigate the convergence of the improper integral $\displaystyle\int_{-\infty}^{0} (1+2x)\,\mathrm{e}^{-x}\, dx$.

7. Evaluate the arc length $x = \dfrac{2}{3}(y-1)^{3/2}$ for $1 \le y \le 2$.

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(Last update: 8/12/25 (12th of August) 3:12 AM)

1. $\ln x$ is defined for $x > 0$. Therefore, $\ln(x^2+1)$ is defined on $\mathbb{R}$ because $x^2 + 1 \geq 1 > 0$.

Let us find the limit at infinity and negative infinity.

$$\lim_{x\to\infty} \ln\left(x^2+1\right) = \lim_{x\to-\infty} \ln\left(x^2+1\right) = \infty$$

No asymptotes occur.

Take the first derivative and find the critical points. Apply the chain rule.

$$y' = \frac{d}{dx} \ln\left(x^2+1\right) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

A critical point occurs at $x = 0$. At this point, the first derivative is 0.

Take the second derivative. Apply the quotient rule.

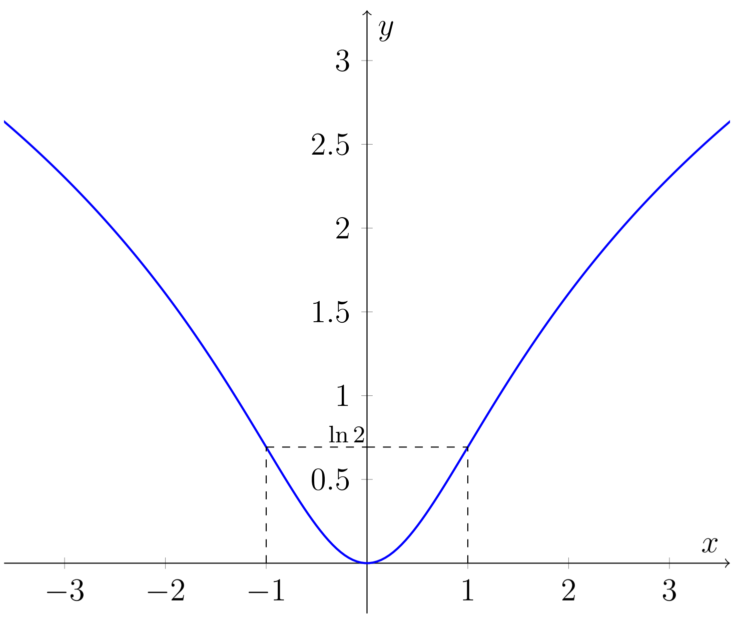
$$y'' = \frac{d}{dx}\left(\frac{2x}{x^2+1}\right) = \frac{2\cdot(x^2+1) - 2x\cdot(2x)}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$$

The inflection points occur at $x = \pm 1$. At these points, the direction of the curvature changes.

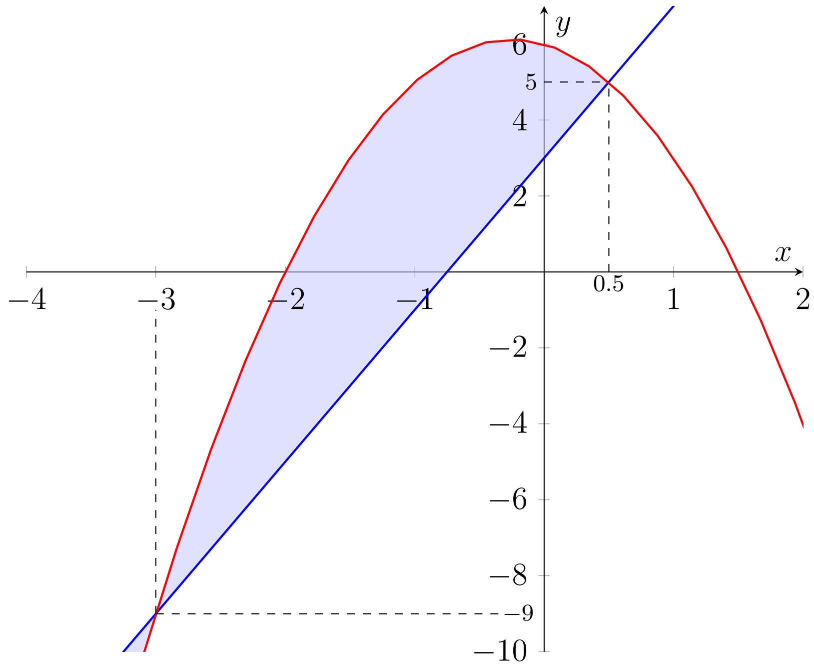
Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(-1) = f(1) = \ln 2, \quad f(0) = 0$$

x	$(-\infty, -1]$	$(-1, 0]$	$[0, 1)$	$[1, \infty)$
y	$[\ln 2, \infty)$	$[0, \ln 2)$	$[0, \ln 2)$	$[\ln 2, \infty)$
y' sign	-	-	+	+
y'' sign	-	+	+	-



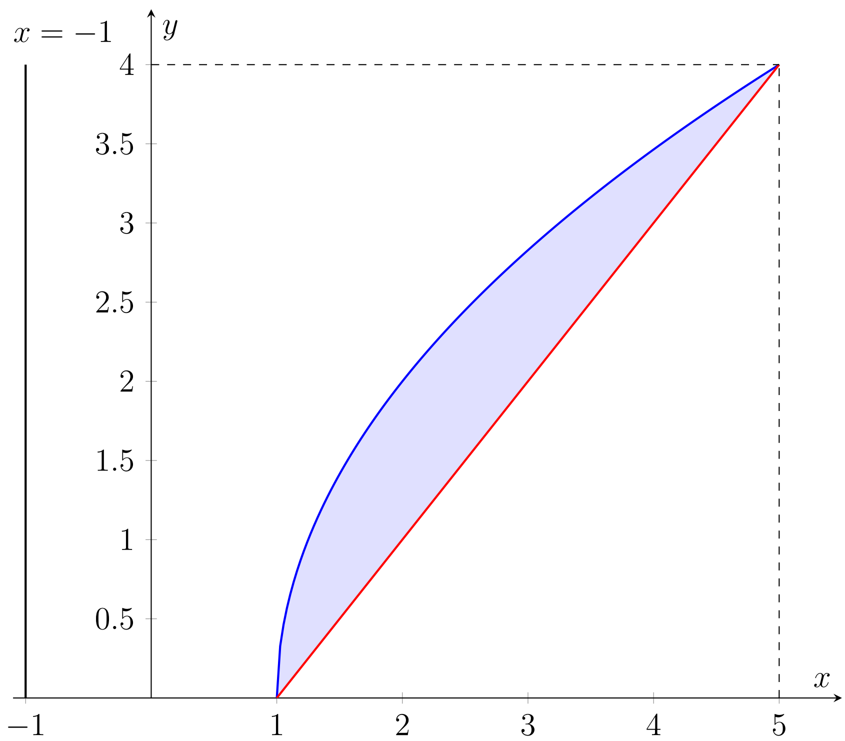
2.



$$\text{Area} = \int_{-3}^{1/2} \left[6 - x - 2x^2 - (4x+3)\right]\, dx = \int_{-3}^{1/2} \left(3 - 5x - 2x^2\right)\, dx$$

$$= 3x - \frac{5x^2}{2} - \frac{2x^3}{3}\bigg|_{-3}^{1/2} = \left(\frac{3}{2} - \frac{5}{8} - \frac{1}{12}\right) - \left(-9 - \frac{45}{2} + 18\right) = \boxed{\frac{343}{24}}$$

3.



Rewrite the equations of the curves and solve for x. Let V be the volume of the solid.

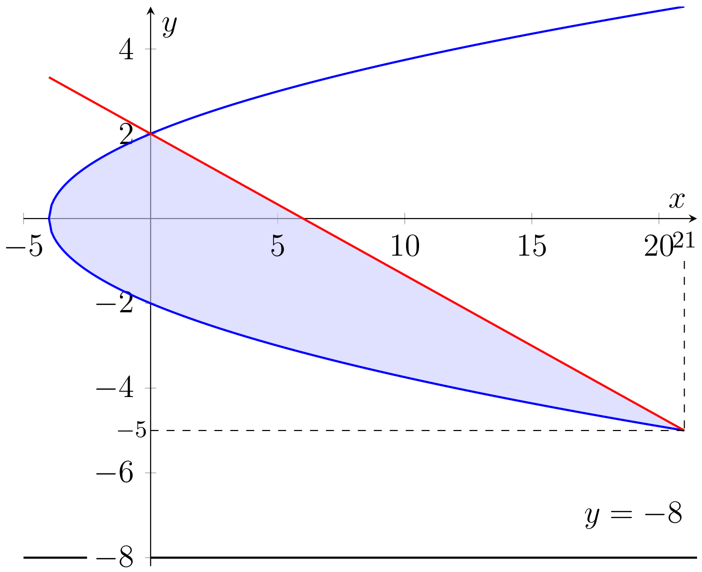
$$y = 2\sqrt{x-1} \implies y^2 = 4x - 4 \implies x = \frac{y^2+4}{4}$$
$$y = x - 1 \implies x = y + 1$$

$$V = \int_D \pi \left[R_2^2(y) - R_1^2(y)\right]\, dy = \int_0^4 \pi \left[((y+1)+1)^2 - \left(\left(\frac{y^2+4}{4}\right)+1\right)^2\right] dy$$

$$= \pi \int_0^4 \left[(y+2)^2 - \left(\frac{y^2}{4}+2\right)^2\right] dy = \pi \int_0^4 \left(y^2 + 4y + 4 - \frac{y^4}{16} - y^2 - 4\right) dy$$

$$= \pi \int_0^4 \left(4y - \frac{y^4}{16}\right) dy = \pi \left[2y^2 - \frac{y^5}{80}\right]_0^4 = \pi \left[\left(32 - \frac{64}{5}\right) - (0)\right] = \boxed{\frac{96\pi}{5}}$$

4.



Let V be the volume of the solid.

$$V = \int_D 2\pi \cdot h(y) \cdot r(y)\, dy = 2\pi \int_{-5}^{2} (y+8) \cdot \left[(6-3y) - (y^2-4)\right] dy$$

$$= 2\pi \int_{-5}^{2} (y+8)(-y^2 - 3y + 10)\, dy = 2\pi \int_{-5}^{2} \left(-y^3 - 3y^2 + 10y - 8y^2 - 24y + 80\right) dy$$

$$= 2\pi \int_{-5}^{2} \left(-y^3 - 11y^2 - 14y + 80\right) dy = 2\pi \left[-\frac{y^4}{4} - \frac{11y^3}{3} - 7y^2 + 80y\right]_{-5}^{2}$$

$$= 2\pi \left[\left(-4 - \frac{88}{3} - 28 + 160\right) - \left(\frac{625}{4} + \frac{1375}{3} - 175 - 400\right)\right] = \boxed{\frac{4459\pi}{6}}$$

5.

(a) Use the u-substitution method. Let $u = \mathrm{e}^{-x} + \ln x$. Then $du = \left(-\mathrm{e}^{-x} + \frac{1}{x}\right) dx$.

$$\int 4\left(\frac{1}{x} - \mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x} + \ln x\right) dx = \int 4\cos u \, du = 4\sin u + c$$

$$= \boxed{4\sin\left(\mathrm{e}^{-x} + \ln x\right) + c,\ c \in \mathbb{R}}$$

(b) Decompose the fraction into multiple partial fractions. Let $A,\ B,\ C,\ D,\ E,\ F \in \mathbb{R}$.

$$\mathrm{I} = \int \frac{x^3 + 10x^2 + 3x + 36}{(x-1)^2\ \left(x^2+4\right)^2} dx = \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2}\right) dx$$

Let $N = x^3 + 10x^2 + 3x + 36$.

$$N = A\left(x^2+4\right)^2(x-1) + B\left(x^2+4\right)^2 + (Cx+D)\left(x^2+4\right)(x-1)^2 + (Ex+F)(x-1)^2$$

$$= \left(x^2+4\right)^2[A(x-1)+B] + (x-1)^2\left[(Cx+D)\left(x^2+4\right) + Ex + F\right]$$

$$= \left(x^4 + 8x^2 + 16\right)(Ax - A + B) + \left(x^2 - 2x + 1\right)$$
$$\cdot \left(Cx^3 + 4Cx + Dx^2 + 4D + Ex + F\right)$$

$$= Ax^5 - Ax^4 + Bx^4 + 8Ax^3 - 8Ax^2 + 8Bx^2 + 16Ax - 16A + 16B$$
$$+ Cx^5 + 4Cx^3 + Dx^4 + 4Dx^2 + Ex^3 + Fx^2 - 2Cx^4 - 8Cx^2 - 2Dx^3 - 8Dx$$
$$- 2Ex^2 - 2Fx + Cx^3 + 4Cx + Dx^2 + 4D + Ex + F$$

$$= x^5(A+C) + x^4(-A+B+D-2C) + x^3(8A+4C+E-2D+C)$$
$$+ x^2(-8A+8B+4D+F-8C-2E+D) + x(16A-8D-2F+4C+E)$$
$$- 16A + 16B + 4D + F$$

Equate the coefficients of like terms.

$$A + C = 0 \tag{1}$$
$$-A + B + D - 2C = 0$$
$$8A + 5C + E - 2D = 1$$
$$-8A + 8B + 5D - 8C + F - 2E = 10$$
$$16A - 8D - 2F + 4C + E = 3$$
$$-16A + 16B + 4D + F = 36$$

From (1), $A = -C$. Rewrite C in terms of A and rearrange the equations.

$$\begin{aligned} A+B+D&=0 \quad (2)\\ 3A+E-2D&=1\\ 8B+5D+F-2E&=10\\ 12A-8D-2F+E&=3\\ -16A+16B+4D+F&=36 \end{aligned}$$

From (2), $A+B=-D$. Rewrite D in terms of A and B and rearrange the equations.

$$5A+2B+E=1 \tag{3}$$
$$-5A+3B+F-2E=10 \tag{4}$$
$$20A+8B-2F+E=3 \tag{5}$$
$$-20A+12B+F=36 \tag{6}$$

By using the couples (3) & (4) and (4) & (5), eliminate E.

$$\left.\begin{aligned} 5A+7B+F&=12\\ 35A+19B-3F&=16\\ -20A+12B+F&=36 \end{aligned}\right\} \implies \left.\begin{aligned} 50A+40B&=52\\ -25A+55B&=124 \end{aligned}\right\} \implies A=-\frac{14}{25},\ B=2$$

Therefore, $C=\frac{14}{25}$, $D=-\frac{36}{25}$. From (3), $E=-\frac{1}{5}$, and from (6), $F=\frac{4}{5}$.

Substitute the values into A, B, C, D, E, F.

$$\mathrm{I}=\int\left(-\frac{14}{25}\cdot\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{\frac{14}{25}x-\frac{36}{25}}{x^2+4}+\frac{-\frac{1}{5}x+\frac{4}{5}}{\left(x^2+4\right)^2}\right)dx \tag{7}$$

From now on, integrate term by term. Integrate the first term in (7).

$$\int-\frac{14}{25}\cdot\frac{1}{x-1}\,dx=-\frac{14}{25}\int\frac{1}{x-1}\,dx=-\frac{14}{25}\ln|x-1|+c \tag{8}$$

Integrate the second term in (7).

$$\int\frac{2}{(x-1)^2}\,dx=-\frac{2}{x-1}+c \tag{9}$$

Integrate the third term in (7).

$$\begin{aligned}\int\frac{\frac{14}{25}x-\frac{36}{25}}{x^2+4}\,dx&=\frac{1}{25}\int\frac{14x-36}{x^2+4}\,dx=\frac{7}{25}\int\frac{2x}{x^2+4}\,dx-\frac{36}{25}\int\frac{1}{x^2+4}\,dx\\ &=\frac{7}{25}\ln\left|x^2+4\right|-\frac{36}{100}\int\frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx\\ &=\frac{7}{25}\ln\left(x^2+4\right)-\frac{36}{50}\arctan\left(\frac{x}{2}\right)+c \qquad (10)\end{aligned}$$

Integrate the last term in (7).

$$\int \frac{-\frac{1}{5}x+\frac{4}{5}}{\left(x^2+4\right)^2}\,dx=\frac{1}{5}\int \frac{4-x}{\left(x^2+4\right)^2}\,dx=\frac{4}{5}\int \frac{1}{\left(x^2+4\right)^2}\,dx-\frac{1}{5}\int \frac{x}{\left(x^2+4\right)^2}\,dx \tag{11}$$

First, solve the integral on the left in (11). Let $x=2\tan u$, then $dx=2\sec^2 u\,du$.

$$\int \frac{1}{\left(x^2+4\right)^2}\,dx=\int \frac{2\sec^2 u}{\left(4\tan^2 u+4\right)^2}du=\int \frac{2\sec^2 u}{16\sec^4 u}\,du=\frac{1}{8}\int \frac{1}{\sec^2 u}\,du$$

$$=\frac{1}{8}\int \cos^2 u\,du=\frac{1}{8}\int \left(\frac{1-\cos 2u}{2}\right)\,du=\frac{1}{8}\left(\frac{u}{2}-\frac{\sin 2u}{4}\right)+c$$

$$=\frac{u}{16}-\frac{\sin u\cos u}{16}+c$$

Since $x=2\tan u$, $\tan u=\dfrac{x}{2}$

$$u=\arctan\frac{x}{2},\quad \sin u=\frac{x}{\sqrt{x^2+4}},\quad \cos u=\frac{2}{\sqrt{x^2+4}}$$

Rewrite the integral.

$$\int \frac{1}{\left(x^2+4\right)^2}\,dx=\frac{1}{16}\left(\arctan\frac{x}{2}-\frac{2x}{x^2+4}\right)+c \tag{12}$$

Now, solve the integral on the right in (11). Let $u=x^2+4$, then $du=2x\,dx$.

$$\int \frac{x}{\left(x^2+4\right)^2}\,dx=\int \frac{du}{2u^2}=-\frac{1}{2u}+c=-\frac{1}{2\left(x^2+4\right)}+c \tag{13}$$

Rewrite the integral in (11) using (12) and (13).

$$\frac{4}{5}\int \frac{1}{\left(x^2+4\right)^2}\,dx-\frac{1}{5}\int \frac{x}{\left(x^2+4\right)^2}\,dx=\frac{1}{20}\left(\arctan\frac{x}{2}+\frac{2x+2}{x^2+4}\right)+c \tag{14}$$

Eventually, using (8), (9), (10) and (14), rewrite (7).

$$\mathrm{I}=\boxed{-\frac{14}{25}\ln|x-1|-\frac{2}{x-1}+\frac{7}{25}\ln\left(x^2+4\right)-\frac{67}{100}\arctan\frac{x}{2}+\frac{x+1}{10\left(x^2+4\right)}+c,\ c\in\mathbb{R}}$$

(c) Let $x=\dfrac{2}{5}\sec u$ for $0\le u<\dfrac{\pi}{2}$, then $dx=\dfrac{2}{5}\sec u\tan u\,du$.

$$\mathrm{I}=\int \frac{\sqrt{25x^2-4}}{x}\,dx=\int \frac{\sqrt{4\sec^2 u-4}}{\frac{2}{5}\sec u}\cdot\frac{2}{5}\sec u\tan u\,du \qquad \left[\tan^2 u+1=\sec^2 u\right]$$

$$\mathrm{I} = 2\int |\tan u| \tan u \, du \qquad [|\tan u| > 0]$$

$$= 2\int \tan^2 u \, du = 2\int \sec^2 u \, du - 2\int du = 2\tan u - 2u + c$$

Recall that $x = \dfrac{2}{5}\sec u$.

$$\sec u = \frac{5x}{2} \implies \sec^2 u = \frac{25x^2}{4} \implies \tan u = \frac{\sqrt{25x^2-4}}{2} \implies u = \arctan\left(\frac{\sqrt{25x^2-4}}{2}\right)$$

$$\int \frac{\sqrt{25x^2-4}}{x}\, dx = \boxed{\sqrt{25x^2-4} - 2\arctan\left(\frac{\sqrt{25x^2-4}}{2}\right) + c, \quad c \in \mathbb{R}}$$

(d) Use the method of integration by parts.

$$\left.\begin{aligned} u = x^2 &\implies du = 2x\,dx \\ dv = \cos(4x)\,dx &\implies v = \frac{1}{4}\sin(4x) \end{aligned}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\mathrm{I} = x^2 \cdot \frac{1}{4}\sin(4x) - \int \frac{1}{4}\sin(4x)\cdot 2x\,dx = \frac{x^2}{4}\sin(4x) - \frac{1}{2}\int x\sin(4x)\,dx$$

Apply the same procedure.

$$\left.\begin{aligned} w = x &\implies dw = dx \\ dz = \sin(4x)\,dx &\implies z = -\frac{1}{4}\cos(4x) \end{aligned}\right\} \to \int w\,dz = wz - \int z\,dw$$

$$\mathrm{I} = \frac{x^2}{4}\sin(4x) - \frac{1}{2}\left[\frac{-x}{4}\cos(4x) - \int -\frac{1}{4}\cos(4x)\,dx\right]$$

$$= \boxed{\frac{x^2}{4}\sin(4x) + \frac{x}{8}\cos(4x) - \frac{1}{32}\sin(4x) + c, \quad c \in \mathbb{R}}$$

(e)

$$\int \frac{dx}{2x^2-3x+2}\,dx = \int \frac{dx}{2x^2-3x+\frac{9}{8}+\frac{7}{8}}\,dx = \int \frac{dx}{\left(\sqrt{2}x - \frac{3\sqrt{2}}{4}\right)^2 + \frac{7}{8}}$$

$$= \frac{8}{7}\int \frac{dx}{\frac{8\left(\sqrt{2}x - \frac{3\sqrt{2}}{4}\right)^2}{7} + 1}$$

Let $u = \dfrac{2\sqrt{2}}{\sqrt{7}}\left(\sqrt{2}x - \dfrac{3\sqrt{2}}{4}\right)$, then $du = \dfrac{4}{\sqrt{7}}\,dx$.

$$\frac{8}{7}\int \frac{dx}{\dfrac{8\left(\sqrt{2}x-\frac{3\sqrt{2}}{4}\right)^2}{7}+1^2} = \frac{8}{7}\int \frac{1}{u^2+1}\cdot\frac{\sqrt{7}}{4}\,du = \frac{2}{\sqrt{7}}\int\frac{1}{u^2+1}\,du = \frac{2}{\sqrt{7}}\arctan u + c$$

$$= \frac{2}{\sqrt{7}}\arctan\left[\frac{2\sqrt{2}}{\sqrt{7}}\left(\sqrt{2}x-\frac{3\sqrt{2}}{4}\right)\right] + c, \quad c\in\mathbb{R}$$

$$= \boxed{\frac{2}{\sqrt{7}}\arctan\left(\frac{4x-3}{\sqrt{7}}\right) + c, \quad c\in\mathbb{R}}$$

6. Use the method of integration by parts.

$$\left.\begin{array}{l} u = 1+2x \implies du = 2\,dx \\ dv = \mathrm{e}^{-x}\,dx \implies v = -\mathrm{e}^{-x} \end{array}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int_{-\infty}^{0}(1+2x)\,\mathrm{e}^{-x}\,dx = \lim_{R\to-\infty} -\mathrm{e}^{-x}(1+2x)\Big|_{R}^{0} - \int_{-\infty}^{0} -\mathrm{e}^{-x}\cdot 2\,dx$$

$$= \lim_{R\to-\infty}\left(-\mathrm{e}^{0}\cdot 1 + e^{-R}(1+2\cdot R)\right) + \lim_{P\to-\infty} 2\mathrm{e}^{-x}\Big|_{P}^{0}$$

$$= -\infty + 2\lim_{P\to-\infty}\left(\mathrm{e}^{0}-\mathrm{e}^{-P}\right) = -\infty-\infty = \boxed{-\infty}$$

The integral diverges to negative infinity.

7. The length of a curve defined by $x = f(y)$ whose derivative is continuous on the interval $a \le x \le b$ can be evaluated using the integral.

$$S = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

Find $\dfrac{dx}{dy}$.

$$\frac{dx}{dy} = \frac{2}{3}\cdot\frac{3}{2}(y-1)^{1/2} = \sqrt{y-1}$$

Set $a = 1,\ b = 2$ and find the length.

$$S = \int_1^2 \sqrt{1+\left(\sqrt{y-1}\right)^2}\,dy = \int_1^2 \sqrt{y}\,dy = \frac{2}{3}y^{3/2}\Big|_1^2 = \boxed{\frac{2}{3}(2\sqrt{2}-1)}$$