2011-2012 Fall
MAT123-[Instructor02]-02, [Instructor05]-05 Midterm I
(15/11/2012)
Time: 13:00 - 15:00
Duration: 120 minutes

1. Evaluate the limits, if they exist, and explain your answer. Don't use L'Hôpital's rule.

(a) $\lim_{x\to 1} \frac{x^2+x-2}{\sqrt{x}-1}$ (b) $\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3}$ (c) $\lim_{x\to -\infty} \left(\sqrt{x^2-x+1}-\sqrt{x^2-2x}\right)$

2. Find the derivatives of the following functions.

(a) $f(x) = \tan^3\left(4\sin^2(3x)\right)$ (b) $f(x) = \left(\cos x^2\right)^x$

(c) Find $f'(0)$ of $f(x) = \ln\left(\frac{3^x}{3^x+1}\right)$

3. Evaluate $\lim_{x\to 0} (\mathrm{e}^x - x)^{\frac{1}{x}}$.

4.

(a) Let $F(x)$ be a one-to-one function with inverse F^{-1}. Define a new function

$$p(x) = 1 - 2F\left(\frac{x}{3}\right)$$

Find a formula for p^{-1} in terms of F^{-1}.

(b) Find the derivative of the inverse of the function $f(x) = \arctan x + \mathrm{e}^{123x}$ at $x = 1$. That is, find $\left(f^{-1}\right)'(1)$.

5. Find the equation of the tangent line to the curve $x^2y^2 - 36x = 37$ at $(-1, 1)$.

6. The length of a hypotenuse of a right triangle is constant at 5 cm, and the length of one of its sides is decreasing at a rate of 2 cm/sec. Find the rate of change of the area of the triangle when this side is 3 cm long.

2011-2012 Fall Midterm I (15/11/2012) Solutions
(Last update: 7/22/25 (22nd of July) 8:27 PM)

1.

(a) Multiply each side by the conjugate of the denominator to eliminate the indetermination.

$$\lim_{x\to 1}\frac{x^2+x-2}{\sqrt{x}-1}=\lim_{x\to 1}\left[\frac{(x+2)(x-1)}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}\right]=\lim_{x\to 1}\frac{(x+2)(x-1)\left(\sqrt{x}+1\right)}{x-1}$$

$$=\lim_{x\to 1}\left[(x+2)\left(\sqrt{x}+1\right)\right]=3\cdot 2=\boxed{6}$$

(b)

$$\lim_{x\to 3}\frac{\sin(x^2-9)}{x-3}=\lim_{x\to 3}\left[\frac{\sin(x^2-9)}{x-3}\cdot\frac{x+3}{x+3}\right]=\lim_{x\to 3}\left[\frac{\sin(x^2-9)}{x^2-9}\right]\cdot\lim_{x\to 3}(x+3)$$

The value $\lim\limits_{x\to 0}\dfrac{\sin u}{u}$ can be evaluated by using the squeeze theorem, and it could be expected that we knew the value of this limit prior to the examination. Set $u=x^2-9$. So, the left-hand limit is 1.

$$=\lim_{x\to 3}\left[\frac{\sin(x^2-9)}{x^2-9}\right]\cdot\lim_{x\to 3}(x+3)=1\cdot 6=\boxed{6}$$

(c) This is an indeterminate $(\infty-\infty)$ form. We expand the expression by using its conjugate and divide each side of the fraction by x to eliminate the determination.

$$\lim_{x\to -\infty}\left(\sqrt{x^2-x+1}-\sqrt{x^2-2x}\right)$$

$$=\lim_{x\to -\infty}\left[\sqrt{x^2-x+1}-\sqrt{x^2-2x}\cdot\frac{\sqrt{x^2-x+1}+\sqrt{x^2-2x}}{\sqrt{x^2-x+1}+\sqrt{x^2-2x}}\right]$$

$$=\lim_{x\to -\infty}\frac{x^2-x+1-(x^2-2x)}{\sqrt{x^2-x+1}+\sqrt{x^2-2x}}=\lim_{x\to -\infty}\left(\frac{x+1}{\sqrt{x^2-x+1}+\sqrt{x^2-2x}}\cdot\frac{x}{x}\right)$$

$$=\lim_{x\to -\infty}\frac{\dfrac{x+1}{x}}{\dfrac{\sqrt{x^2-x+1}+\sqrt{x^2-2x}}{x}}=\lim_{x\to -\infty}\frac{1+\dfrac{1}{x}}{\sqrt{1-\dfrac{1}{x}+\dfrac{1}{x^2}}+\sqrt{1-\dfrac{2}{x}}}$$

$$=\frac{1+0}{\sqrt{1-0+0}+\sqrt{1-0}}=\boxed{\frac{1}{2}}$$

2.

(a) Apply the chain rule accordingly.

$$\boxed{f'(x) = 3\tan^2\left(4\sin^2(3x)\right) \cdot \sec^2\left(4\sin^2(3x)\right) \cdot 8\sin(3x) \cdot \cos(3x) \cdot 3}$$

(b) Take the logarithm of both sides to differentiate easily.

$$\ln(f(x)) = \ln\left[\left(\cos x^2\right)^x\right] = x\ln\left(\cos x^2\right)$$

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}\left[x\ln\left(\cos x^2\right)\right]$$

$$\frac{1}{f(x)} \cdot f'(x) = 1 \cdot \left[\ln\left(\cos x^2\right)\right] + x \cdot \frac{1}{\cos x^2} \cdot \left(-\sin x^2\right) \cdot 2x$$

$$f'(x) = f(x)\left[\ln\left(\cos x^2\right) - 2x^2 \cdot \tan x^2\right]$$

$$\boxed{f'(x) = \left(\cos x^2\right)^x \cdot \left[\ln\left(\cos x^2\right) - 2x^2 \cdot \tan x^2\right]}$$

(c) Rewrite the right-hand side using the property of logarithms. Take the first derivative afterwards.

$$f(x) = \ln\left(\frac{3^x}{3^x+1}\right) = \ln(3^x) - \ln(3^x+1)$$

$$f'(x) = \frac{1}{3^x} \cdot 3^x \cdot \ln(3) - \frac{1}{3^x+1} \cdot 3^x \cdot \ln(3) = \ln(3) \cdot \left[1 - \frac{3^x}{3^x+1}\right]$$

$$f'(0) = \ln(3) \cdot \left[1 - \frac{3^0}{3^0+1}\right] = \boxed{\frac{\ln 3}{2}}$$

3. Let L be the value of the limit.

$$L = \lim_{x\to 0}\left(\mathrm{e}^x - x\right)^{\frac{1}{x}}$$

$$\ln(L) = \ln\left[\lim_{x\to 0}\left(\mathrm{e}^x - x\right)^{\frac{1}{x}}\right]$$

The expression is defined for $x \neq 0$. Therefore, we can take the logarithm function inside the limit. After that, apply L'Hôpital's rule to eliminate the indeterminate form.

$$\ln(L) = \lim_{x\to 0}\ln\left[\left(\mathrm{e}^x - x\right)^{\frac{1}{x}}\right] = \lim_{x\to 0}\frac{\ln\left(\mathrm{e}^x - x\right)}{x} \quad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 0}\frac{\frac{1}{\mathrm{e}^x - x} \cdot \left(\mathrm{e}^x - 1\right)}{1} = \lim_{x\to 0}\frac{\mathrm{e}^x - 1}{\mathrm{e}^x - x} = \frac{\mathrm{e}^0 - 1}{\mathrm{e}^0 - 0} = 0$$

If $\ln(L) = 0$, then $\boxed{L = 1}$.

4.

(a)

$$\begin{aligned}
p(x) &= 1 - 2F\left(\frac{x}{3}\right) \\
p(x) - 1 &= -2F\left(\frac{x}{3}\right) \\
\frac{1-p(x)}{2} &= F\left(\frac{x}{3}\right) \\
F^{-1}\left(\frac{1-p(x)}{2}\right) &= \frac{x}{3} \\
3F^{-1}\left(\frac{1-p(x)}{2}\right) &= x \\
3F^{-1}\left(\frac{1-p\left(p^{-1}(x)\right)}{2}\right) &= p^{-1}(x)
\end{aligned}$$

$$\boxed{p^{-1}(x) = 3F^{-1}\left(\frac{1-x}{2}\right)}$$

(b) Find a root so that $f(x_0) = \arctan(x_0) + \mathrm{e}^{123 \cdot x_0} = 1$. We can intuitively say that the root is small because both $\arctan x$ and e^x are strictly increasing everywhere. Try $x = 0$.

$$f(0) = \arctan 0 + \mathrm{e}^{123 \cdot 0} = 0 + 1 = 1$$

Therefore, $f^{-1}(1) = 0$. The derivative of an inverse function at a given point is

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(f^{-1}(1))}$$

Calculate $f'(x)$ and then, $(f^{-1})(1)$.

$$f'(x) = \frac{1}{1+x^2} + 123\mathrm{e}^{122x}$$

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(0)} = \left(\frac{1}{1+0^2} + 123\mathrm{e}^{122 \cdot 0}\right)^{-1} = \boxed{\frac{1}{124}}$$

5) Consider $y = f(x)$. Differentiate both sides implicitly.

$$\frac{d}{dx}\left(x^2y^2 - 36x\right) = \frac{d}{dx}37$$

$$2xy^2 + x^2 \cdot 2y \cdot \frac{dy}{dx} - 36 = 0$$

$$x^2 \cdot 2y \cdot \frac{dy}{dx} = 36 - 2xy^2$$

$$\frac{dy}{dx} = \frac{36 - 2xy^2}{x^2 \cdot 2y}$$

Calculate $\frac{dy}{dx}$ at $(-1, 1)$.

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{36 - 2(-1) \cdot 1^2}{(-1)^2 \cdot 2 \cdot 1} = 19 \tag{1}$$

Using the straight line formula, we find the tangent line. Recall: $y - y_0 = m(x - x_0)$, where m can be substituted with (1).

$$\boxed{y - 1 = 19(x + 1)}$$

6. Let $x(t)$, $y(t)$, $l(t)$ represent the lengths of the sides as functions of time. We can set up the following equation for the area of the right triangle.

$$A(t) = \frac{x(t) \cdot y(t)}{2}$$

Take the derivative of both sides.

$$A'(t) = \frac{1}{2}\left(x'(t) \cdot y(t) + x(t) \cdot y'(t)\right) \tag{2}$$

We also know that, by the Pythagorean theorem,

$$l^2(t) = x^2(t) + y^2(t)$$

Take the derivative of both sides.

$$2l(t)l'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

Since the length of the hypotenuse is constant, $l'(t) = 0$. Therefore,

$$x(t)x'(t) = -y(t)y'(t) \tag{3}$$

At $t = t_0$, we have $l(t_0) = 5$, $x(t_0) = 3$, $x'(t_0) = -2$, and by the Pythagorean theorem, $y(t_0) = \sqrt{5^2 - 3^2} = 4$. Calculate $y'(t_0)$ from (3).

$$y'(t_0) = -\frac{3 \cdot (-2)}{4} = \frac{3}{2}$$

Plug the necessary values into (2) to find the rate of change of the area.

$$A'(t_0) = \frac{1}{2}\left((-2) \cdot 4 + 3 \cdot \frac{3}{2}\right) = \boxed{-\frac{7}{4}\,\mathrm{cm}^2/\mathrm{s}}$$