2021-2022 Spring
MAT124 Final
(23/05/2022)

1. Find the largest and smallest values of $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 1$ with $x \geq 0$ and $y \geq 0$.

2. Sketch the region corresponding to the double integral

$$\int_0^{\frac{-1+\sqrt{5}}{2}} \int_{\sqrt{y}}^{\sqrt{1-y^2}} dx\, dy$$

and reverse the order of integration. Do not evaluate the integral!

3. Sketch the region inside the circle $r = 1$ and outside the cardioid $r = 1 + \sin\theta$ and then, using a double integral, find the area of the region.

4. Using a double integral, find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the circular region $x^2 + y^2 \leq 2$ in the xy-plane.

5. Let us consider the frustum of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.

(i) Sketch the graph of the frustum.

(ii) Find the surface area of the frustum.

6. Let S be the region in the cylinder $x^2 + y^2 = 1$ bounded above by the plane $z = 6$ and below by the paraboloid $z = 1 - x^2 - y^2$.

(i) Using the spherical coordinates, set up (but do not evaluate!) an integral for the volume of the solid S.

(ii) Using the cylindrical coordinates, find the volume of the solid S.

Solutions (Last update: 7/27/25 (27th of July) 6:34 PM)

1) Let $g(x, y) = x + y - 1$ and then, solve the system of equations below using the method of Lagrange multipliers.

$$\left.\begin{array}{l}\nabla f = \lambda \nabla g \\ g(x,y) = 0\end{array}\right\} \qquad \begin{array}{l}\nabla f = \langle 2x, 2y\rangle = \lambda\langle 1, 1\rangle = \lambda\nabla g \\ \therefore \lambda = 2x \quad \text{and} \quad \lambda = 2y \quad \text{and} \quad x + y - 1 = 0\end{array}$$

$$\lambda = 2x = 2y \implies x = y$$

$$x + y - 1 = 0 \to 2x - 1 = 0 \to x = \frac{1}{2} = y$$

Evaluate $f\left(\frac{1}{2}, \frac{1}{2}\right)$ to find the minimum value.

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

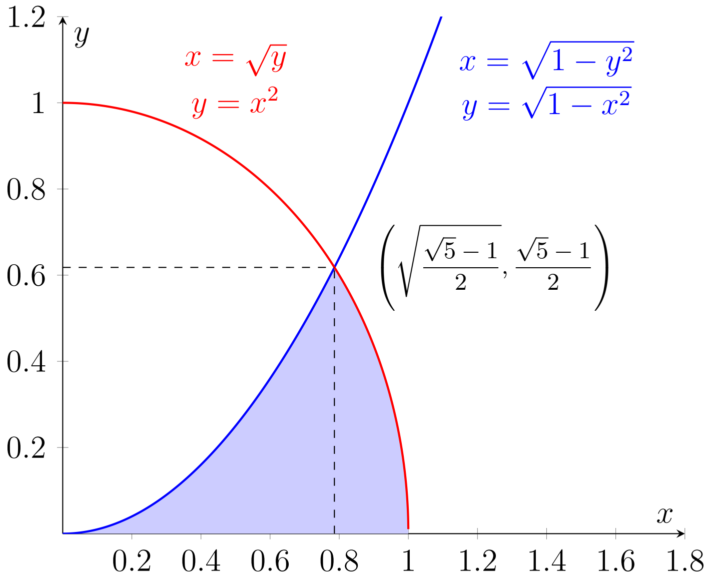
$f(x, y)$ is defined for $x \geq 0$ and $y \geq 0$. This implies that $0 \leq y \leq 1$ and $0 \leq x \leq 1$ by the constraint. The domain of f is closed and bounded, and f is continuous. By the extreme value theorem, the absolute minima and maxima must exist. Using the method of Lagrange multipliers, we could find only one point. This means that the other value exists on the boundary of f.

$$x = 0 \to 0 + y - 1 = 0 \to y = 1, \quad y = 0 \to x + 0 - 1 = 0 \to x = 1$$
$$f(0, 1) = 0^2 + 1^2 = 1, \quad f(1, 0) = 1^2 + 0^2 = 1$$

Eventually, compare the values we obtain.

$$\boxed{\frac{1}{2}\text{ is the absolute minimum, } 1 \text{ is the absolute maximum.}}$$

2)

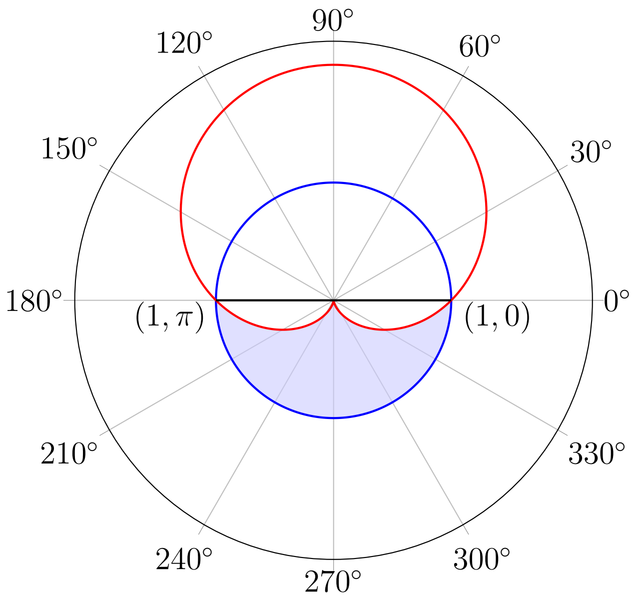


The integral with reverse order is as follows.

$$\boxed{\int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} \int_0^{x^2} dy\,dx + \int_{\sqrt{\frac{\sqrt{5}-1}{2}}}^{1} \int_0^{\sqrt{1-x^2}} dy\,dx}$$

3) Find where these two curves intersect and then find the area.

$$1 = 1 + \sin\theta \implies \sin\theta = 0 \implies \theta = k\pi, \quad k \in \mathbb{Z}$$



$$\text{Area} = \int_\pi^{2\pi} \int_{1+\sin\theta}^{1} r\,dr\,d\theta = \frac{1}{2}\int_\pi^{2\pi} \left[1^2 - (1+\sin\theta)^2\right] d\theta$$

$$= \frac{1}{2}\int_\pi^{2\pi} \left[-2\sin\theta - \sin^2\theta\right] d\theta \quad \left[\sin^2\theta + \cos^2\theta = 1\right]$$

$$= -\int_\pi^{2\pi} \sin\theta\,d\theta - \frac{1}{2}\int_\pi^{2\pi} (1-\cos^2\theta)\,d\theta \quad [2\cos^2\theta - 1 = \cos(2\theta)]$$

$$= -\int_\pi^{2\pi} \sin\theta\,d\theta + \int_\pi^{2\pi} \frac{\cos(2\theta)-1}{4}\,d\theta = \int_\pi^{2\pi} \left[-\frac{1}{4} + \frac{1}{4}\cos(2\theta) - \sin\theta\right] d\theta$$

$$= \left[-\frac{\theta}{4} + \frac{1}{8}\sin(2\theta) + \cos\theta\right]_\pi^{2\pi} = \left[\left(-\frac{\pi}{2}+0+1\right) - \left(-\frac{\pi}{4}+0-1\right)\right] = \boxed{2 - \frac{\pi}{4}}$$

4) We have the sphere $x^2+y^2+z^2 = 16$. So, the upper bound is $z = \sqrt{16-x^2-y^2}$, while the lower bound is $z = 0$. If we project the domain onto the xy-plane, we see that the upper and lower bounds for y are $\sqrt{2-x^2}$ and $-\sqrt{2-x^2}$, respectively. For x, the integration starts from $-\sqrt{2}$ and ends at $\sqrt{2}$. The volume of the object can be evaluated using the following integral.

$$\mathrm{I} = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \left[\sqrt{16 - x^2 - y^2} - 0\right] \, dy \, dx$$

This integral seems a little bit hard. We can switch to polar coordinates for ease. Use the transformation below.

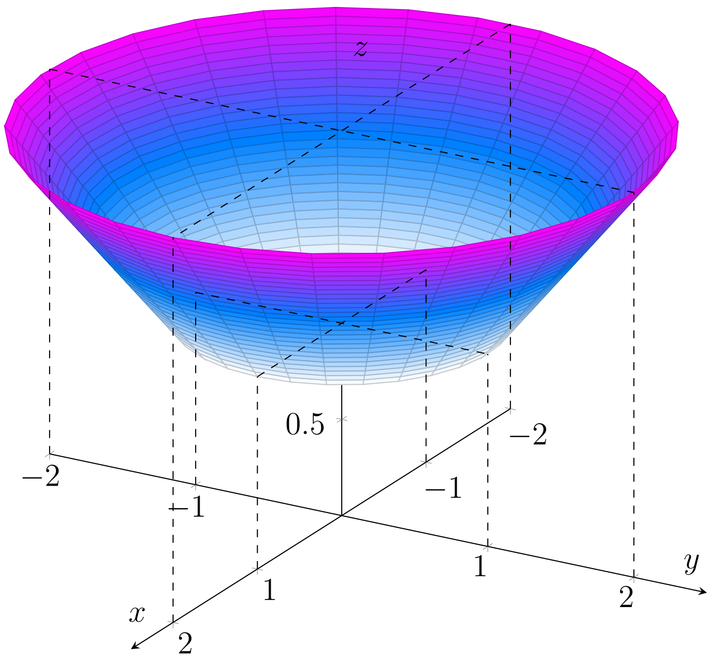
$$\begin{matrix} x^2 + y^2 = r^2 \\ dA = dy\,dx = r\,dr\,d\theta \end{matrix} \quad \rightarrow \quad \begin{matrix} 0 \leq z \leq \sqrt{16 - r^2} \\ 0 \leq r \leq \sqrt{2} \\ 0 \leq \theta \leq 2\pi \end{matrix}$$

$$\mathrm{I} = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{16 - r^2} \, r \, dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}\left(16 - r^2\right)^{3/2}\right]_{r=0}^{r=\sqrt{2}} d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \left[-14^{3/2} - \left(-16^{3/2}\right)\right] \, d\theta = \boxed{\frac{2\pi}{3}\left(16^{3/2} - 14^{3/2}\right)}$$

5)

(i)



(ii) Using the double integral below, we find the lateral surface area.

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\, dA$$

$$= \iint_D \sqrt{1 + \left(\frac{x^2+y^2}{x^2+y^2}\right)}\, dA = \iint_D \sqrt{1+1}\, dA = \sqrt{2} \iint_D dA$$

If we switch to polar coordinates, we can easily evaluate the integral.

$$A = \sqrt{2} \int_0^{2\pi} \int_1^2 r\, dr\, d\theta = \sqrt{2} \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r=1}^{r=2} d\theta = \sqrt{2} \int_0^{2\pi} \frac{3}{2}\, d\theta = \boxed{3\pi\sqrt{2}}$$

6)

(i) For spherical coordinates, we have

$$\begin{array}{c} z = \rho\cos\theta \\ r = \rho\sin\theta \\ x^2+y^2+z^2 = \rho^2 \\ dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} x^2+y^2 = 1 \;\rightarrow\; \rho^2\sin^2\phi = 1 \\ z = 6 \;\rightarrow\; \rho\cos\phi = 6 \\ z = 1 - x^2 - y^2 \;\rightarrow\; \rho\cos\phi = 1 - \rho^2\sin^2\phi \end{array}$$

We have the lower bound for ρ, which is the solution of $\rho\cos\phi = 1 - \rho^2\sin^2\phi$

$$\rho\cos\phi = 1 - \rho^2\sin^2\phi$$

$$\rho^2\sin^2\phi + \rho\cos\phi - 1 = 0$$

$$\rho_{1,2} = \frac{-\cos\phi \pm \sqrt{\cos^2\phi - 4\cdot\sin^2\phi\cdot(-1)}}{2\sin^2\phi} \qquad \left[x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\right]$$

$$\rho > 0 \implies \rho_{\text{lower}} = \frac{-\cos\phi + \sqrt{\cos^2\phi + 4\sin^2\phi}}{2\sin^2\phi}$$

However, we have two distinct upper bounds for ρ. We need to find the value of ϕ where the surfaces $\rho\cos\phi = 6$ and $\rho^2\sin^2\phi = 1$ intersect.

$$\rho^2\sin^2\phi = 1 \;\rightarrow\; \rho\sin\phi = 1$$

$$\left.\begin{array}{c} \rho\sin\phi = 1 \\ \rho\cos\phi = 6 \end{array}\right\} \quad \tan\phi = \frac{1}{6} \implies \phi = \arctan\frac{1}{6}$$

For $\phi < \arctan\frac{1}{6}$, the upper bound is $\frac{6}{\cos\phi}$. Meanwhile, for $\phi > \arctan\frac{1}{6}$, it is $\frac{1}{\sin\phi}$. The region in the $xy-$plane is circular, therefore $0 \le \theta \le 2\pi$. As for ϕ, we have $0 \le \phi \le \frac{\pi}{2}$. The volume of the object in spherical coordinates can be expressed as follows.

$$\boxed{\begin{aligned} V = & \int_0^{2\pi}\int_0^{\arctan\frac{1}{6}}\int_{\frac{-\cos\phi+\sqrt{\cos^2\phi+4\sin^2\phi}}{2\sin^2\phi}}^{\frac{6}{\cos\phi}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ & +\int_0^{2\pi}\int_{\arctan\frac{1}{6}}^{\pi/2}\int_{\frac{-\cos\phi+\sqrt{\cos^2\phi+4\sin^2\phi}}{2\sin^2\phi}}^{\frac{1}{\sin\phi}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}}$$

In fact, we are looking for the minimum value of the upper bound for ρ between $\dfrac{1}{\sin\phi}$ and $\dfrac{6}{\cos\phi}$. Hence, we can write the equivalent expression as follows.

$$\boxed{V = \int_0^{2\pi}\int_0^{\pi/2}\int_{\frac{-\cos\phi+\sqrt{\cos^2\phi+4\sin^2\phi}}{2\sin^2\phi}}^{\min\left(\frac{6}{\cos\phi},\,\frac{1}{\sin\phi}\right)} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

(ii) For cylindrical coordinates, we have

$$\begin{matrix} z = z \\ r^2 = x^2+y^2 \\ dV = r\,dz\,dr\,d\theta \end{matrix} \quad \rightarrow \quad \begin{matrix} x^2+y^2=1 \rightarrow r^2=1 \rightarrow r=1 \\ z=6 \\ z=1-x^2-y^2 \rightarrow z=1-r^2 \end{matrix}$$

The volume can be expressed as follows.

$$V = \int_0^{2\pi}\int_0^1\int_{1-r^2}^6 r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \big[z\big]_{z=1-r^2}^{z=6} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r^3+5r\right)\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}+\frac{5r^2}{2}\right]_{r=0}^{r=1} d\theta$$

$$= \big[\theta\big]_0^{2\pi}\cdot\left[\left(\frac{1^4}{4}+\frac{5\cdot 1^2}{2}\right)-(0)\right] = \boxed{\frac{11\pi}{2}}$$