2020-2021 Fall
MAT123 Midterm
(30/11/2020)

1. Evaluate $\lim\limits_{x\to 0^+} (\sqrt{x})^{\ln(x+1)}$.

2. Show that the function $f(x)$ defined by

$$f(x) = \begin{cases} x\arctan\dfrac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ \dfrac{x-\cos x}{x^2}, & \text{if } x < 0 \end{cases}$$

is not continuous at the point $x = 0$.

3. Find an equation of the line which is tangent to the curve

$$\cos y^2 + xy + 1 = 0$$

at the point $\left(\sqrt{\dfrac{2}{\pi}}, -\sqrt{\dfrac{\pi}{2}}\right)$. Note that $y = f(x)$.

4. A block of ice in the shape of a cube originally having volume of 3000 cm^3. When it is melting, the surface area is decreasing at the rate of 36 cm^2/h. At what rate does the length of each of its edges decrease at the time its volume is 216 cm^3? Assume that during melting, the block of ice maintains its cubical shape.

5. **(a)** Using the Intermediate Value Theorem and Rolle's theorem, show that the equation $e^x + x = 0$ has only one root (Note that if this root c_1, then $c_1 \in (-1, 0)$).

(b) Determine the interval of increase, decrease, and concavity of the function $f(x) = \mathrm{e}^x + x$. By constructing a table, sketch the graph.

6. Determine (but do not evaluate) the integral corresponding to the area of the region bounded by the curves $y = -x^2 + 1$ and $y = |x| - 1$.

7. Evaluate $\displaystyle\int x \ln x \, dx$.

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1. Let L be the value of the limit. Since the expression is continuous for $x > 0$, we can apply the logarithm function to each side of the equation. Then, we can swap the logarithm and the limit. Use the property of logarithms afterwards.

$$L = \lim_{x\to 0^+} \left(\sqrt{x}\right)^{\ln(x+1)}$$

$$\ln(L) = \ln\left[\lim_{x\to 0^+} \left(\sqrt{x}\right)^{\ln(x+1)}\right] = \lim_{x\to 0^+} \ln\left[\left(\sqrt{x}\right)^{\ln(x+1)}\right] = \lim_{x\to 0^+} \ln\left[\left(\sqrt{x}\right)^{\ln(x+1)}\right]$$

$$\ln(L) = \lim_{x\to 0^+} \left[\ln(x+1)\cdot\ln\left(\sqrt{x}\right)\right] \quad [0\cdot\infty]$$

Make it so the limit is in the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ in order to apply the L'Hôpital's rule.

$$\ln(L) = \lim_{x\to 0^+}\left[\frac{\ln\left(\sqrt{x}\right)}{\frac{1}{\ln(x+1)}}\right] \quad \left[\frac{\infty}{\infty}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 0^+}\left[\frac{\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}{-\frac{1}{\ln^2(x+1)}\cdot\frac{1}{x+1}}\right] = \lim_{x\to 0^+}\left[-\frac{\ln^2(x+1)\cdot(x+1)}{2x}\right]$$

$$= \lim_{x\to 0^+}\left[-\frac{\ln^2(x+1)}{2x}\right]\cdot\lim_{x\to 0^+}(x+1) = \lim_{x\to 0^+}\left[-\frac{\ln^2(x+1)}{2x}\right] \quad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 0^+}\left[-\frac{2\ln(x+1)\cdot\frac{1}{x+1}}{2}\right] = \lim_{x\to 0^+}\left[\frac{\ln(x+1)}{x+1}\right] = \frac{\ln 1}{1} = 0$$

$\ln(L) = 0$, so $\boxed{L = 1}$.

2. $\arctan\dfrac{1}{x}$ takes it values on $-\dfrac{\pi}{2} \le \arctan\dfrac{1}{x} \le \dfrac{\pi}{2}$. Multiply each side by x, then we get $-\dfrac{x\pi}{2} \le x\arctan\dfrac{1}{x} \le \dfrac{x\pi}{2}$. Take the limits of each side. By the squeeze theorem, we see that the limit of $x\arctan\dfrac{1}{x}$ at the point $x = 0$ is 0. This means that for $f(x)$, the limit from the right side also equals 0.

$$\lim_{x\to 0}\left(-\frac{x\pi}{2}\right) \le \lim_{x\to 0}\left(x\arctan\frac{1}{x}\right) \le \lim_{x\to 0}\left(\frac{x\pi}{2}\right)$$

$$0 \le \lim_{x\to 0}\left(x\arctan\frac{1}{x}\right) \le 0$$

$$\therefore \lim_{x\to 0}\left(x\arctan\frac{1}{x}\right) = 0$$

From the left side, the limit is equal to as follows.

$$\lim_{x\to 0^-}\frac{x-\cos x}{x^2} = \lim_{x\to 0^-}(x-\cos x)\cdot\lim_{x\to 0^-}\frac{1}{x^2} = -\infty$$

Continuity requires the equality of one-sided limits and the value of the function at that point. However, the one-sided limits are not equal; $0 \neq -\infty$. Therefore, $f(x)$ is discontinuous at $x = 0$.

3. Differentiate both sides implicitly.

$$\begin{aligned}\frac{d}{dx}\left(\cos y^2 + xy + 1\right) &= \frac{d}{dx}(0)\\ -\sin y^2\cdot 2y\frac{dy}{dx} + y + x\frac{dy}{dx} &= 0\\ \frac{dy}{dx}\left(-\sin y^2\cdot 2y + x\right) &= -y\end{aligned}$$

$$\frac{dy}{dx} = \frac{y}{\sin y^2\cdot 2y - x} \tag{1}$$

Evaluate $\dfrac{dy}{dx}$ at the point.

$$\left.\frac{dy}{dx}\right|_{\left(\sqrt{\frac{2}{\pi}},-\sqrt{\frac{\pi}{2}}\right)} = \frac{y}{\sin y^2\cdot 2y - x} = \frac{-\sqrt{\frac{\pi}{2}}}{\sin\left(\left(-\sqrt{\frac{\pi}{2}}\right)^2\right)\cdot 2\left(-\sqrt{\frac{\pi}{2}}\right) - \sqrt{\frac{2}{\pi}}} = \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{2\pi}+\sqrt{\frac{2}{\pi}}} \tag{2}$$

Recall: $y - y_0 = m(x - x_0)$, where m is the slope. Substitute m with (2) and find the tangent line.

$$\boxed{y + \sqrt{\frac{\pi}{2}} = \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{2\pi}+\sqrt{\frac{2}{\pi}}}\left(x - \sqrt{\frac{2}{\pi}}\right)}$$

4. Let $S(t)$, $V(t)$, $a(t)$ represent the surface area, the volume, and the length of one side of the object, respectively, as a function of time. We may write the following.

$$S(t) = 6a^2(t), \quad V(t) = a^3(t)$$

Given that at $t = t_0$, $V(t_0) = 216$, $S'(t_0) = -36$. Using the relationship with the sides,

$$\begin{aligned}&V(t_0) = a^3(t_0) = 216 \rightarrow a(t_0) = 6\\ &S'(t_0) = 12a(t_0)a'(t_0) = -36\\ &\therefore 12\cdot 6\cdot a'(t_0) = -36 \rightarrow a'(t_0) = -\frac{1}{2}\end{aligned}$$

$$\boxed{a'(t_0) = -\frac{1}{2}\ \text{cm/h}}$$

5. **(a)** Let $f(x) = e^x + x$. f is continuous and differentiable for all $x \in \mathbb{R}$.

$$f(-1) = e^{-1} - 1 = \frac{1}{e} - 1, \quad f(0) = e - 0 = e$$

Since $f(-1) < 0$ and $f(0) > 0$ and f is continuous on the interval $[-1, 0]$, by IVT, there is at least one point x_1 that satisfies $f(x_1) = 0$. Assume that there is another distinct root x_2. Rolle's theorem states that if f is continuous on a particular interval with endpoints having the same function value, there exists a point c on that interval such that $f'(c) = 0$ there.

$$f'(c) = e^c + 1 \geq 1 \qquad [e^c > 0]$$

This yields a contradiction. Therefore, there is *only* one root.

(b) The expression is defined $\forall x \in \mathbb{R}$. Let us find the limit at infinity and the limit at negative infinity.

$$\lim_{x \to \infty} (e^x + x) = \infty \qquad \lim_{x \to -\infty} (e^x + x) = -\infty$$

There are no vertical or horizontal asymptotes. However, there is a slant asymptote. Attempt a long polynomial division and we will find that the slant asymptote is $y = x$. Verify with the following limit:

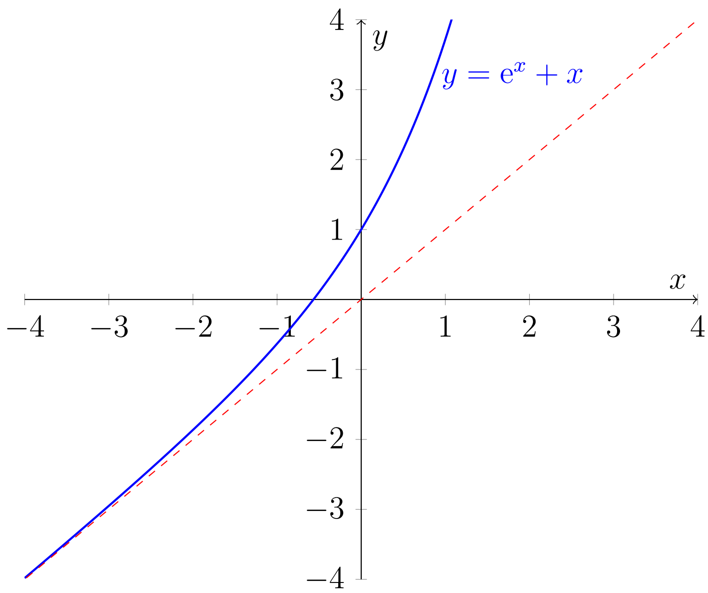
$$\lim_{x \to -\infty} [(e^x + x) - x] = \lim_{x \to -\infty} e^x = 0$$

Take the first and second derivatives.

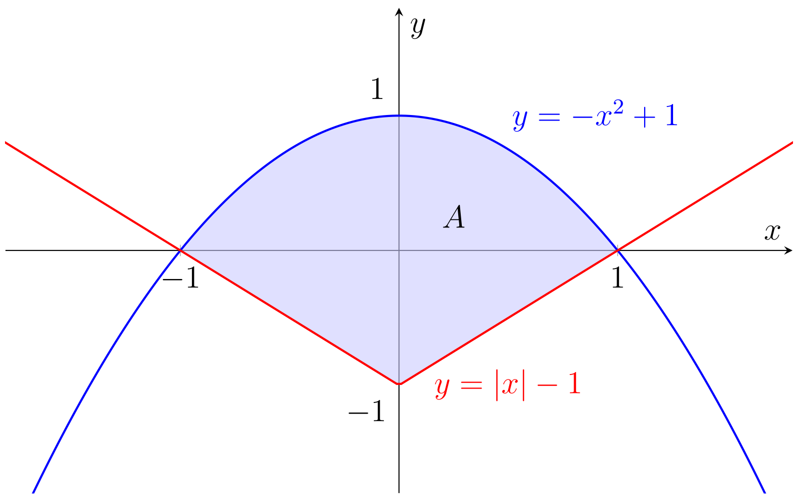
$$y' = e^x + 1, \quad y'' = e^x$$

We see that there are no critical or inflection points either. Now, set up a table and see what the graph looks like.

x	$(-\infty, \infty)$
y	$(-\infty, \infty)$
y' sign	$+$
y'' sign	$+$



6.



The area of the region is as follows.

$$A = \int_{-1}^{1} \left[(-x^2+1) - (|x|-1)\right]\, dx = \int_{-1}^{0} (-x^2+x+2)\, dx + \int_{0}^{1} (-x^2-x+2)\, dx$$

7. We'll use integration by parts.

$$\left.\begin{array}{l} \ln x = u \;\rightarrow\; \dfrac{1}{x}\, dx = du \\ x\, dx = dv \;\rightarrow\; \dfrac{x^2}{2} = v \end{array}\right\} \qquad \begin{aligned} I &= \int x \ln x\, dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx \\ &= \frac{x^2}{2} \cdot \ln x - \int \frac{x}{2}\, dx = \boxed{\frac{x^2}{2} \cdot \ln x - \frac{x^2}{4} + c,\; c \in \mathbb{R}} \end{aligned}$$