MAT123 03.11.2025

QUESTIONS

Q1: Evaluate $\lim_{x\to 0^+}\left(\frac{2}{\pi}\arccos x\right)^{1/x}$.

Q2: Evaluate $\lim_{x\to 0^+}(e^x-1)^{1/\ln x}$.

Q3: Find the equation of the line that is tangent to the curve $(x^2+y^2)^3=(x-y)^3$ at $(1,-1)$.

Q4: Find the maximum possible total surface area of a cylinder inscribed in a hemisphere of radius 1.

Q5: Find the closest point(s) on the curve $y=x^2$ to the point $(0,1)$.

Q6: Evaluate $\lim_{x\to 0}\frac{\cos^2 x-\cos\left(x\sqrt{2}\right)}{x^4}$.

Q7: Determine the constants so that $\lim_{x\to\infty}x^3\left(a+\frac{b}{x}+\arctan x\right)=c$.

Q8: Find the absolute extreme values of $f(x)=|x^2-x-12|$ on $[-4,5]$.

Q9: Find the first derivative of $f(x)=\ln(\ln x)+\tan^3\left(\frac{x+1}{x-1}\right)+\pi^{\sin^3 x}$.

Q10: What value of a makes $f(x)=x^2+\frac{a}{x}$ have

a. a local minimum at $x=2$?

b. a point of inflection at $x=1$?

ANSWERS

Q1: $e^{-\frac{2}{\pi}}$ **Q2:** e **Q3:** $y=x-2$ **Q4:** $\pi\left(1+\sqrt{2}\right)$

Q5: $\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)$, $\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)$

Q6: $\frac{1}{6}$

Q7: $a=-\frac{\pi}{2},\ b=1,\ c=\frac{1}{3}$

Q8: Absolute minimum is 0 at $x=-3,-4$; absolute maximum is $\frac{49}{4}$ at $x=\frac{1}{2}$

Q9: $\frac{1}{x\ln x}-6\tan^2\left(\frac{x+1}{x-1}\right)\cdot\sec^2\left(\frac{x+1}{x-1}\right)\cdot\frac{1}{(x-1)^2}+\pi^{\sin^3 x}\left[3\sin^2 x\cdot\cos x\cdot\ln\pi\right]$

Q10: **a.** 16, **b.** -1

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(Q1) $\lim_{x\to 0^+}\left(\frac{2}{\pi}\arccos x\right)^{1/x} = L$

$$\Rightarrow \ln L = \ln\left[\lim_{x\to 0^+}\left(\frac{2}{\pi}\arccos x\right)^{1/x}\right] = \lim_{x\to 0^+}\ln\left(\frac{2}{\pi}\arccos x\right)^{1/x} = \lim_{x\to 0^+}\frac{\ln\left(\frac{2}{\pi}\arccos x\right)}{x} \quad \left[\frac{0}{0}\right]$$

↳ ln is continuous for $x>0$

$$\Rightarrow \ln L \overset{L'H.}{=} \lim_{x\to 0^+}\frac{\frac{1}{\frac{2}{\pi}\arccos x}\cdot\frac{2}{\pi}\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right)}{1} = -\lim_{x\to 0^+}\frac{1}{\arccos x\cdot\sqrt{1-x^2}} = \frac{1}{(\pi/2)\cdot\sqrt{1-0}} = \frac{2}{\pi}$$

Since $\ln L = \frac{2}{\pi}$, $\boxed{L = e^{-2/\pi}}$

(Q2) $\lim_{x\to 0^+}(e^x-1)^{1/\ln x} = L$

$$\Rightarrow \ln L = \ln\left[\lim_{x\to 0^+}(e^x-1)^{1/\ln x}\right] = \lim_{x\to 0^+}\ln\left[(e^x-1)^{1/\ln x}\right] = \lim_{x\to 0^+}\frac{\ln(e^x-1)}{\ln x} \quad \left[\frac{\infty}{\infty}\right]$$

↳ ln is continuous for $x>0$

$$\Rightarrow \ln L \overset{L'H.}{=} \lim_{x\to 0^+}\frac{\frac{1}{e^x-1}\cdot e^x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{xe^x}{e^x-1} \quad \left[\frac{0}{0}\right]$$

$$\Rightarrow \ln L \overset{L'H.}{=} \lim_{x\to 0^+}\frac{1\cdot e^x + xe^x}{e^x} = \lim_{x\to 0^+}\frac{(x+1)e^x}{e^x} = \lim_{x\to 0^+}(x+1) = 0+1 = 1$$

Since $\ln L = 1$, $\boxed{L = e}$

(Q3) Apply implicit differentiation and use the chain rule,

$$\frac{d}{dx}\left[(x^2+y^2)^3\right] = \frac{d}{dx}\left[(x-y)^3\right] \Rightarrow 3(x^2+y^2)^2\cdot\left(2x+2y\frac{dy}{dx}\right) = 3(x-y)^2\cdot\left(1-\frac{dy}{dx}\right)$$

Collect the terms with $\frac{dy}{dx}$,

$$6x(x^2+y^2)^2 + 6y(x^2+y^2)^2\frac{dy}{dx} = 3(x-y)^2 - 3(x-y)^2\frac{dy}{dx} \Rightarrow 6y(x^2+y^2)^2\frac{dy}{dx} + 3(x-y)^2\frac{dy}{dx} = 3(x-y)^2 - 6x(x^2+y^2)^2$$

$$\Rightarrow \frac{dy}{dx}\left[6y(x^2+y^2)^2 + 3(x-y)^2\right] = 3(x-y)^2 - 6x(x^2+y^2)^2 \Rightarrow \frac{dy}{dx} = \frac{3(x-y)^2 - 6x(x^2+y^2)^2}{6y(x^2+y^2)^2 + 3(x-y)^2}$$

We have the point $(1,-1)$.

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,-1)} = \frac{3(2)^2 - 6(2)^2}{-6(2)^2 + 3(2)^2} = \frac{12-24}{-24+12} = \frac{-12}{-12} = 1$$

Recall the straight line formula $y - y_0 = m(x - x_0)$, m is the derivative of this curve at $(1,-1)$. Therefore,

$$y-(-1) = 1(x-1) \Rightarrow \boxed{y = x-2}$$

Q4

$\Rightarrow h^2+r^2=1$

$\Rightarrow h=\sqrt{1-r^2}$

Let S be the total surface area of the cylinder.

$S=2\pi r^2+2\pi r h=2\pi\left(r^2+r\sqrt{1-r^2}\right)$

Set $\frac{dS}{dr}=0$ to find the extremum points.

$$\frac{dS}{dr}=2\pi\left(2r+\sqrt{1-r^2}+r\cdot\frac{1}{2}(1-r^2)^{-1/2}\cdot(-2r)\right)=0 \Rightarrow 2r+\sqrt{1-r^2}-r^2(1-r^2)^{-1/2}=0 \quad \text{(Multiply by } (1-r^2)^{1/2}\text{)}$$

$$\Rightarrow 2r(1-r^2)^{1/2}+1-r^2-r^2=0 \Rightarrow 2r(1-r^2)^{1/2}=2r^2-1 \quad \text{(Square both sides)} \Rightarrow 4r^2(1-r^2)=4r^4-4r^2+1$$

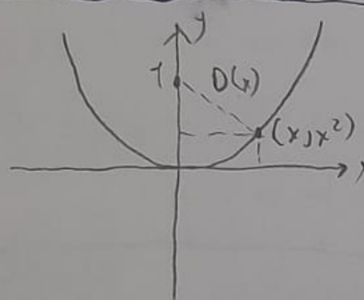
$$\Rightarrow 8r^4-8r^2+1=0 \quad \text{Let } r^2=u, \text{ then we have } 8u^2-8u+1=0$$

$$u_{1,2}=\frac{8\pm\sqrt{8^2-4\cdot8\cdot1}}{8\cdot2}=\frac{1}{2}\pm\frac{\sqrt2}{4} \Rightarrow u=\frac12+\frac{\sqrt2}{4} \Rightarrow r^2=\frac12+\frac{\sqrt2}{4} \Rightarrow r=\sqrt{\frac12+\frac{\sqrt2}{4}}$$

$$S_{max}=S\left(r=\sqrt{\tfrac12+\tfrac{\sqrt2}{4}}\right)=2\pi\left[\frac12+\frac{\sqrt2}{4}+\sqrt{\frac12+\frac{\sqrt2}{4}}\cdot\sqrt{1-\left(\frac12+\frac{\sqrt2}{4}\right)}\right]=2\pi\left[2+\frac{\sqrt2}{4}+\sqrt{\left(\frac12+\frac{\sqrt2}{4}\right)\cdot\left(\frac12-\frac{\sqrt2}{4}\right)}\right]$$

$$=2\pi\left[2+\frac{\sqrt2}{4}+\sqrt{\frac14-\frac18}\right]=\pi\left[1+\frac{\sqrt2}{2}+\frac{1}{\sqrt2}\right]=\boxed{\pi\left(1+\sqrt2\right)}$$

Q5



By the Pythagorean theorem, $(1-x^2)^2+x^2=D^2(x)$.

Take the derivative of each side with respect to x and find extremum points by setting $D'(x)=0$.

$$\frac{d}{dx}\left[(1-x^2)^2+x^2\right]=\frac{d}{dx}\left[D^2(x)\right] \Rightarrow 2(1-x^2)(-2x)+2x=2D(x)D'(x)=0$$

$$\Rightarrow 4x^3-2x=0 \Rightarrow 2x(2x^2-1)=0$$

$$\Rightarrow x_1=0,\ x_2=\sqrt2/2,\ x_3=-\sqrt2/2$$

Determine $D(x)$ at x_1, x_2, x_3 individually and compare,

$D^2(x_1)=(1-0)^2+0^2=1$, $D^2(x_2)=\left(1-\frac12\right)^2+\frac12=\frac34$, $D^2(x_3)=\left(1-\frac12\right)^2+\frac12=\frac34$

$D(x_1)=1$, $D(x_2)=\sqrt3/2$, $D(x_3)=\sqrt3/2$

Since $D(x_2)=D(x_3)>D(x_1)$, we have two points on the curve closest to $(0,1)$.

$x_2=\frac{\sqrt2}{2} \rightarrow y_2=\frac12$, $x_3=-\frac{\sqrt2}{2} \rightarrow y_3=\frac12$

$\boxed{\text{Points: } \left\{\left(\frac{\sqrt2}{2},\frac12\right),\left(-\frac{\sqrt2}{2},\frac12\right)\right\}}$

Q6

$$\lim_{x\to0}\frac{\cos^2x-\cos(x\sqrt2)}{x^4} \quad \left[\frac00\right]$$

$$\overset{L'H.}{=}\lim_{x\to0}\frac{\overbrace{-2\cos x\sin x}^{-\sin 2x}+\sin(x\sqrt2)\cdot\sqrt2}{4x^3} \quad \left[\frac00\right]$$

$$\overset{L'H.}{=}\lim_{x\to0}\frac{-2\cos2x+\cos(x\sqrt2)\cdot2}{12x^2} \quad \left[\frac00\right]$$

$$\overset{L'H.}{=}\lim_{x\to0}\frac{4\sin2x-\sin(x\sqrt2)\cdot2\sqrt2}{24x} \quad \left[\frac00\right]$$

$$\overset{L'H.}{=}\lim_{x\to0}\frac{8\cos2x-\cos(x\sqrt2)\cdot4}{24}=\frac{8\cdot1-1\cdot4}{24}=\boxed{\frac16}$$

Q7 $\lim_{x\to\infty} x^3\left(a+\frac{b}{x}+\arctan x\right)=C$ $\left[\infty\cdot\left(a+0+\frac{\pi}{2}\right)\right]$

Since we need this limit to approach a constant, there has to be an indeterminate form, such as $\infty \cdot 0$. If we require $a+\frac{\pi}{2}=0$, $\boxed{a=-\frac{\pi}{2}}$. Rewrite the limit.

$$\lim_{x\to\infty} x^3\left(-\frac{\pi}{2}+\frac{b}{x}+\arctan x\right)=C \Rightarrow \lim_{x\to\infty}\frac{-\frac{\pi}{2}+\frac{b}{x}+\arctan x}{1/x^3}=C \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H.}{\Rightarrow} \lim_{x\to\infty}\frac{\frac{-b}{x^2}+\frac{1}{1+x^2}}{-3/x^4}=\lim_{x\to\infty}\left(\frac{x^4b}{3x^2}-\frac{x^4}{3+3x^2}\right)=\lim_{x\to\infty}\left[\frac{x^4b(3+3x^2)-x^4\cdot 3x^2}{(3+3x^2)3x^2}\right]=\lim_{x\to\infty}\frac{3bx^6+3bx^4-3x^6}{3x^2(3+3x^2)}$$

$$=\lim_{x\to\infty}\frac{x^6(3b-3)+3bx^4}{3x^2(3+3x^2)}$$

[Notice that the numerator contains x^6, while the denominator does not. If we set $b=1$, the dominant term will disappear. Therefore, $\boxed{b=1}$]

$$=\lim_{x\to\infty}\frac{3\cdot 1x^4}{3x^2(3+3x^2)}=\lim_{x\to\infty}\frac{x^2}{3+3x^2}=\lim_{x\to\infty}\frac{1}{\frac{3}{x^2}+3}=\frac{1}{0+3}=\boxed{\frac{1}{3}=C}$$

Q8 $f(x)$ is continuous on $[-4,5]$. By the extreme value theorem, absolute extrema of f must occur. Let's redefine f by splitting the intervals into three. Evaluate f at the critical points and endpoints.

$$f(x)=\begin{cases} x^2-x-12, & x\leq -3 \\ 12+x-x^2, & -3<x<4 \\ x^2-x-12, & 4\leq x \end{cases}$$

Case I: $f(x)=x^2-x-12$, $x\leq-3$; $f'(x)=2x-1$, $x<-3$ → $f(-4)=16+4-12=8$, $f(-3)=9+3-12=0$

Case II: $f(x)=12+x-x^2$, $-3<x<4$; $f'(x)=1-2x$, $-3<x<4$ → $f'\left(\frac{1}{2}\right)=0 \Rightarrow 12+\frac{1}{2}-\frac{1}{4}=\frac{49}{4}$

Case III: $f(x)=x^2-x-12$, $4\geq x$; $f'(x)=2x-1$, $4>x$ → $f(4)=16-4-12=0$, $f(5)=25-5-12=8$

Compare all the values we obtained. The absolute maximum is $\frac{49}{4}$, the absolute minimum is 0.

Q9 Apply the chain rule.

$$f'(x)=\frac{1}{\ln x}\cdot\frac{1}{x}+3\tan^2\left(\frac{x+1}{x-1}\right)\sec^2\left(\frac{x+1}{x-1}\right)\cdot\frac{1\cdot(x-1)-1(x+1)}{(x-1)^2}+\pi^{\sin^3 x}\cdot\ln\pi\cdot 3\sin^2 x\cdot\cos x$$

Q10 a. At a point giving rise to a local minimum, the first derivative is zero.

$$f'(x)=2x-\frac{a}{x^2},\quad f'(2)=4-\frac{a}{4}=0 \Rightarrow \boxed{a=16}$$

b. At an inflection point, the second derivative of f is zero.

$$f''(x)=2+\frac{2a}{x^3},\quad f''(1)=2+\frac{2a}{1}=0 \Rightarrow \boxed{a=-1}$$