

2011-2012 Fall
MAT123-[Instructor02]-02, [Instructor05]-05 Midterm I
(15/11/2012)
Time: 13:00 - 15:00
Duration: 120 minutes

1. Evaluate the limits, if they exist, and explain your answer. Don't use L'Hôpital's rule.

(a) $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{\sqrt{x} - 1}$ (b) $\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3}$ (c) $\lim_{x \rightarrow -\infty} \left(\sqrt{x^2 - x + 1} - \sqrt{x^2 - 2x} \right)$

2. Find the derivatives of the following functions.

(a) $f(x) = \tan^3(4 \sin^2(3x))$ (b) $f(x) = (\cos x^2)^x$

(c) Find $f'(0)$ of $f(x) = \ln \left(\frac{3^x}{3^x + 1} \right)$

3. Evaluate $\lim_{x \rightarrow 0} (e^x - x)^{\frac{1}{x}}$.

4.

- (a) Let $F(x)$ be a one-to-one function with inverse F^{-1} . Define a new function

$$p(x) = 1 - 2F\left(\frac{x}{3}\right)$$

Find a formula for p^{-1} in terms of F^{-1} .

- (b) Find the derivative of the inverse of the function $f(x) = \arctan x + e^{123x}$ at $x = 1$. That is, find $(f^{-1})'(1)$.

5. Find the equation of the tangent line to the curve $x^2y^2 - 36x = 37$ at $(-1, 1)$.

6. The length of a hypotenuse of a right triangle is constant at 5 cm, and the length of one of its sides is decreasing at a rate of 2 cm/sec. Find the rate of change of the area of the triangle when this side is 3 cm long.

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(Last update: 16/08/2025 16:19)

1.

(a) Multiply each side by the conjugate of the denominator to eliminate the indetermination.

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{\sqrt{x} - 1} &= \lim_{x \rightarrow 1} \left[\frac{(x+2)(x-1)}{\sqrt{x} - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \right] = \lim_{x \rightarrow 1} \frac{(x+2)(x-1)(\sqrt{x} + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} [(x+2)(\sqrt{x} + 1)] = 3 \cdot 2 = \boxed{6}\end{aligned}$$

(b)

$$\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} = \lim_{x \rightarrow 3} \left[\frac{\sin(x^2 - 9)}{x - 3} \cdot \frac{x + 3}{x + 3} \right] = \lim_{x \rightarrow 3} \left[\frac{\sin(x^2 - 9)}{x^2 - 9} \right] \cdot \lim_{x \rightarrow 3} (x + 3)$$

The value $\lim_{x \rightarrow 0} \frac{\sin u}{u}$ can be evaluated by using the squeeze theorem, and it could be expected that we knew the value of this limit prior to the examination. Set $u = x^2 - 9$. So, the left-hand limit is 1.

$$= \lim_{x \rightarrow 3} \left[\frac{\sin(x^2 - 9)}{x^2 - 9} \right] \cdot \lim_{x \rightarrow 3} (x + 3) = 1 \cdot 6 = \boxed{6}$$

(c) This is an indeterminate ($\infty - \infty$) form. We expand the expression by using its conjugate and dividing each side of the fraction by x to eliminate the indeterminate form.

$$\begin{aligned}\lim_{x \rightarrow -\infty} (\sqrt{x^2 - x + 1} - \sqrt{x^2 - 2x}) \\ &= \lim_{x \rightarrow -\infty} \left[\sqrt{x^2 - x + 1} - \sqrt{x^2 - 2x} \cdot \frac{\sqrt{x^2 - x + 1} + \sqrt{x^2 - 2x}}{\sqrt{x^2 - x + 1} + \sqrt{x^2 - 2x}} \right] \\ &= \lim_{x \rightarrow -\infty} \frac{x^2 - x + 1 - (x^2 - 2x)}{\sqrt{x^2 - x + 1} + \sqrt{x^2 - 2x}} = \lim_{x \rightarrow -\infty} \left(\frac{x + 1}{\sqrt{x^2 - x + 1} + \sqrt{x^2 - 2x}} \cdot \frac{x}{x} \right) \\ &= \lim_{x \rightarrow -\infty} \frac{\frac{x + 1}{x}}{\frac{\sqrt{x^2 - x + 1} + \sqrt{x^2 - 2x}}{x}} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + \sqrt{1 - \frac{2}{x}}} \\ &= \frac{1 + 0}{\sqrt{1 - 0 + 0} + \sqrt{1 - 0}} = \boxed{\frac{1}{2}}\end{aligned}$$

2.

(a) Apply the chain rule accordingly.

$$\boxed{f'(x) = 3 \tan^2(4 \sin^2(3x)) \cdot \sec^2(4 \sin^2(3x)) \cdot 8 \sin(3x) \cdot \cos(3x) \cdot 3}$$

(b) Take the logarithm of both sides to differentiate easily.

$$\ln(f(x)) = \ln[(\cos x^2)^x] = x \ln(\cos x^2)$$

$$\frac{d}{dx} \ln(f(x)) = \frac{d}{dx} [x \ln(\cos x^2)]$$

$$\frac{1}{f(x)} \cdot f'(x) = 1 \cdot [\ln(\cos x^2)] + x \cdot \frac{1}{\cos x^2} \cdot (-\sin x^2) \cdot 2x$$

$$f'(x) = f(x) [\ln(\cos x^2) - 2x^2 \cdot \tan x^2]$$

$$\boxed{f'(x) = (\cos x^2)^x \cdot [\ln(\cos x^2) - 2x^2 \cdot \tan x^2]}$$

(c) Rewrite the right-hand side using the property of logarithms. Take the first derivative afterwards.

$$f(x) = \ln\left(\frac{3^x}{3^x + 1}\right) = \ln(3^x) - \ln(3^x + 1)$$

$$f'(x) = \frac{1}{3^x} \cdot 3^x \cdot \ln(3) - \frac{1}{3^x + 1} \cdot 3^x \cdot \ln(3) = \ln(3) \cdot \left[1 - \frac{3^x}{3^x + 1}\right]$$

$$f'(0) = \ln(3) \cdot \left[1 - \frac{3^0}{3^0 + 1}\right] = \boxed{\frac{\ln 3}{2}}$$

3. Let L be the value of the limit.

$$L = \lim_{x \rightarrow 0} (e^x - x)^{\frac{1}{x}}$$

$$\ln(L) = \ln \left[\lim_{x \rightarrow 0} (e^x - x)^{\frac{1}{x}} \right]$$

The expression is defined for $x \neq 0$. Therefore, we can take the logarithm function inside the limit. After that, apply L'Hôpital's rule to eliminate the indeterminate form.

$$\ln(L) = \lim_{x \rightarrow 0} \ln \left[(e^x - x)^{\frac{1}{x}} \right] = \lim_{x \rightarrow 0} \frac{\ln(e^x - x)}{x} \quad \left[\frac{0}{0} \right]$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{e^x - x} \cdot (e^x - 1)}{1} = \lim_{x \rightarrow 0} \frac{e^x - 1}{e^x - x} = \frac{e^0 - 1}{e^0 - 0} = 0$$

If $\ln(L) = 0$, then $\boxed{L = 1}$.

4.

(a)

$$\begin{aligned}
 p(x) &= 1 - 2F\left(\frac{x}{3}\right) \\
 p(x) - 1 &= -2F\left(\frac{x}{3}\right) \\
 \frac{1 - p(x)}{2} &= F\left(\frac{x}{3}\right) \\
 F^{-1}\left(\frac{1 - p(x)}{2}\right) &= \frac{x}{3} \\
 3F^{-1}\left(\frac{1 - p(x)}{2}\right) &= x \\
 3F^{-1}\left(\frac{1 - p(p^{-1}(x))}{2}\right) &= p^{-1}(x) \\
 \boxed{p^{-1}(x) = 3F^{-1}\left(\frac{1 - x}{2}\right)}
 \end{aligned}$$

(b) Find a root so that $f(x_0) = \arctan(x_0) + e^{123 \cdot x_0} = 1$. We can intuitively say that the root is small because both $\arctan x$ and e^x are strictly increasing everywhere. Try $x = 0$.

$$f(0) = \arctan 0 + e^{123 \cdot 0} = 0 + 1 = 1$$

Therefore, $f^{-1}(1) = 0$. The derivative of an inverse function at a given point is

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

Calculate $f'(x)$ and $(f^{-1})(1)$.

$$\begin{aligned}
 f'(x) &= \frac{1}{1 + x^2} + 123e^{122x} \\
 (f^{-1})'(1) &= \frac{1}{f'(0)} = \left(\frac{1}{1 + 0^2} + 123e^{122 \cdot 0}\right)^{-1} = \boxed{\frac{1}{124}}
 \end{aligned}$$

5. Consider $y = f(x)$. Differentiate both sides implicitly.

$$\begin{aligned}
 \frac{d}{dx}(x^2y^2 - 36x) &= \frac{d}{dx} 37 \\
 2xy^2 + x^2 \cdot 2y \cdot \frac{dy}{dx} - 36 &= 0 \\
 x^2 \cdot 2y \cdot \frac{dy}{dx} &= 36 - 2xy^2 \\
 \frac{dy}{dx} &= \frac{36 - 2xy^2}{x^2 \cdot 2y}
 \end{aligned}$$

Calculate $\frac{dy}{dx}$ at $(-1, 1)$.

$$\left. \frac{dy}{dx} \right|_{(-1,1)} = \frac{36 - 2(-1) \cdot 1^2}{(-1)^2 \cdot 2 \cdot 1} = 19 \quad (1)$$

Using the straight line formula, we find the tangent line. Recall: $y - y_0 = m(x - x_0)$, where m can be substituted with (1).

$$\boxed{y - 1 = 19(x + 1)}$$

6. Let $x(t)$, $y(t)$, $l(t)$ represent the lengths of the sides as functions of time. We can set up the following equation for the area of the right triangle.

$$A(t) = \frac{x(t) \cdot y(t)}{2}$$

Take the derivative of both sides.

$$A'(t) = \frac{1}{2} (x'(t) \cdot y(t) + x(t) \cdot y'(t)) \quad (2)$$

We also know that, by the Pythagorean theorem,

$$l^2(t) = x^2(t) + y^2(t)$$

Take the derivative of both sides.

$$2l(t)l'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

Since the length of the hypotenuse is constant, $l'(t) = 0$. Therefore,

$$x(t)x'(t) = -y(t)y'(t) \quad (3)$$

At $t = t_0$, we have $l(t_0) = 5$, $x(t_0) = 3$, $x'(t_0) = -2$, and by the Pythagorean theorem, $y(t_0) = \sqrt{5^2 - 3^2} = 4$. Calculate $y'(t_0)$ from (3).

$$y'(t_0) = -\frac{3 \cdot (-2)}{4} = \frac{3}{2}$$

Plug the necessary values into (2) to find the rate of change of the area.

$$A'(t_0) = \frac{1}{2} \left((-2) \cdot 4 + 3 \cdot \frac{3}{2} \right) = \boxed{-\frac{7}{4} \text{ cm}^2/\text{s}}$$