2022-2023 Spring
MAT124 Final
(12/06/2023)

1. Find the absolute extrema of $f(x, y) = 2x^2 - y^2$ on the closed, bounded set $x^2 + y^2 \leq 1$ in the plane.

2. Sketch the region corresponding to the double integral

$$\int_{-3}^{2} \int_{x^2}^{6-x} dy\, dx$$

and evaluate the integral by writing the equivalent integral with the order of integration reversed.

3. Using a double integral, find the area enclosed by the upper half of the cardioid $r = 1 + \sin\theta$.

4. Sketch the region R bounded above by the elliptic paraboloid $z = 2 - x^2 - y^2$ and below by the paraboloid $z = x^2 + y^2$. Using a double integral find the volume of R.

5. Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 4$ that is above the xy-plane and within the cylinder $x^2 + y^2 = 1$.

6.

(i) Sketch the graph of the region R bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by the plane $z = 4 - 2x$.

(ii) Evaluate the volume of R.

Solutions (Last update: 7/31/25 (31st of July) 7:34 PM)

1) Let $g(x, y, z) = x^2 + y^2 - 1$ and then solve the system of equations below using the method of Lagrange multipliers.

$$\left.\begin{array}{r} \nabla f = \lambda \nabla g \\ g(x, y) = 0 \end{array}\right\} \qquad \begin{array}{l} \nabla f = \langle 4x, -2y\rangle = \lambda \langle 2x, 2y\rangle = \lambda \nabla g \\ x^2 + y^2 - 1 = 0 \end{array}$$

$$\begin{aligned} 4x = \lambda(2x) &\implies 2x(2-\lambda) = 0 \implies \lambda = 2 \text{ or } x = 0 \\ -2y = \lambda(2y) &\implies -2y(1+\lambda) = 0 \implies \lambda = -1 \text{ or } y = 0 \end{aligned}$$

Now, use the constraint to find the coordinates.

$$\begin{aligned} \lambda = 2 &\implies y = 0 \implies x^2 + 0^2 - 1 = 0 \implies x = \pm 1 \\ \lambda = -1 &\implies x = 0 \implies 0^2 + y^2 - 1 = 0 \implies y = \pm 1 \end{aligned}$$

Evaluate f at these points: $(0, 1)$, $(0, -1)$, $(1, 0)$, or $(-1, 0)$.

$$\begin{aligned} f(0,1) = 2 \cdot 0^2 - 1^2 = -1, &\quad f(0,-1) = 2 \cdot 0^2 - (-1)^2 = -1, \\ f(1,0) = 2 \cdot 1^2 - 0^2 = 2, &\quad f(-1,0) = 2 \cdot (-1)^2 - 0^2 = 2 \end{aligned}$$

The *only* critical point occurs at $(0, 0)$.

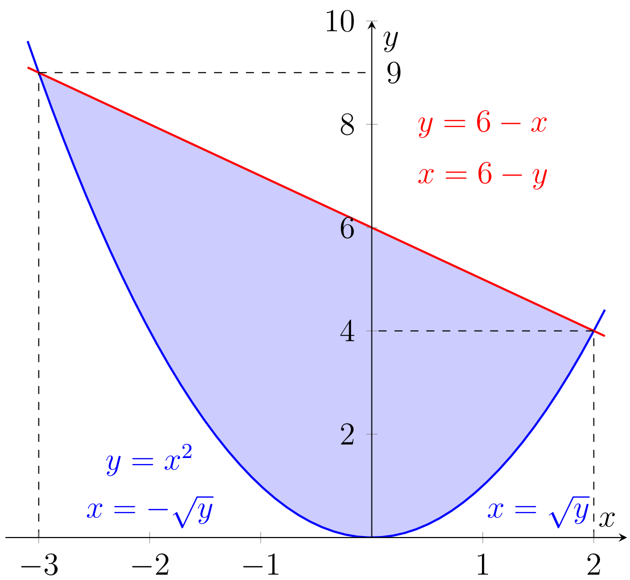
$$\frac{\partial f}{\partial x} = 4x = 0, \quad \frac{\partial f}{\partial y} = -2y = 0 \implies (x, y) = (0, 0) \quad \rightarrow \quad f(0,0) = 0$$

Compare all the values.

$$f(0,0) = 0, \quad f(0,1) = f(0,-1) = -1, \quad f(1,0) = f(-1,0) = 2$$

$$\boxed{f(0,1) = f(0,-1) = -1 \implies \text{abs. min.} \quad f(1,0) = f(-1,0) = 2 \implies \text{abs. max.}}$$

2)



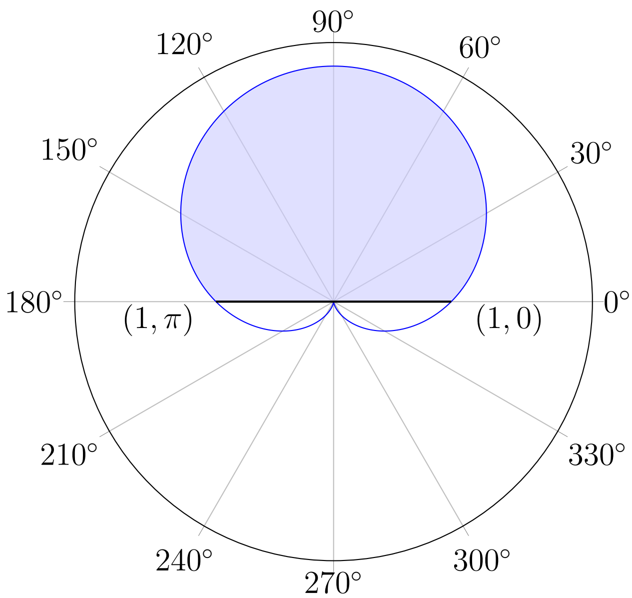
$$I = \int_{-3}^{2}\int_{x^2}^{6-x} dy\,dx = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_4^9\int_{-\sqrt{y}}^{6-y} dx\,dy$$

$$= \int_0^4 \left[\sqrt{y} - (-\sqrt{y})\right] dy + \int_4^9 \left[(6-y) - (-\sqrt{y})\right] dy$$

$$= 2\int_0^4 \sqrt{y}\,dy + \int_4^9 (6 - y + \sqrt{y})\,dy = 2\left[\frac{2}{3}y^{3/2}\right]_0^4 + \left[6y - \frac{y^2}{2} + \frac{2}{3}y^{3/2}\right]_4^9$$

$$= \frac{4}{3}\left[4^{3/2} - 0^{3/2}\right] + \left[\left(6\cdot 9 - \frac{9^2}{2} + \frac{2}{3}\cdot 9^{3/2}\right) - \left(6\cdot 4 - \frac{4^2}{2} + \frac{2}{3}\cdot 4^{3/2}\right)\right] = \boxed{\frac{125}{6}}$$

3) We are interested in the upper half of the cardioid. Therefore, $0 \le \theta \le \pi$.



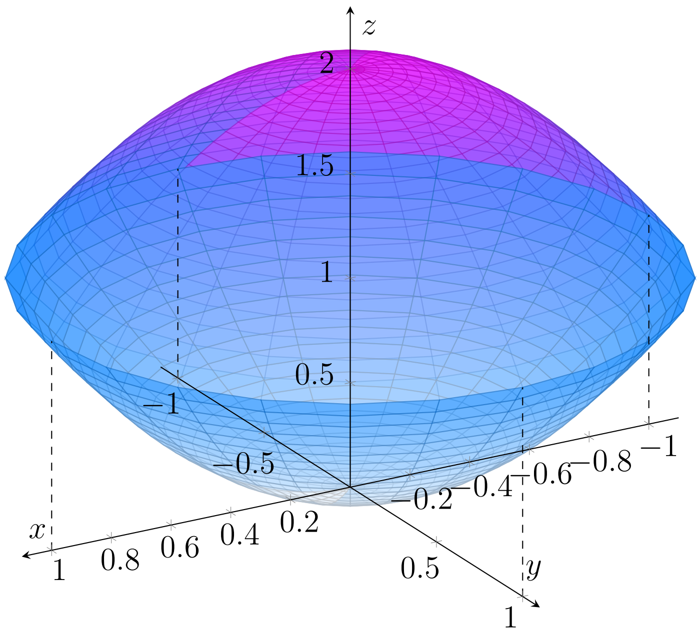
$$\text{Area} = \int_0^{\pi}\int_0^{1+\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi}\left[(1+\sin\theta)^2 - 0^2\right]\,d\theta = \frac{1}{2}\int_0^{\pi}\left(1 + 2\sin\theta + \sin^2\theta\right)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left(1 + 2\sin\theta + 1 - \cos^2\theta\right)\,d\theta = \frac{1}{2}\int_0^{\pi}\left(1 + 2\sin\theta + \frac{1}{2} - \frac{\cos(2\theta)}{2}\right)\,d\theta$$

$$= \frac{1}{2}\left[\theta - 2\cos\theta + \frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\left(\pi - 2\cos\pi + \frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right) - (0 - 2\cos 0 + 0 - \sin 0)\right] = \boxed{\frac{3\pi}{4} + 2}$$

4)



$$\mathrm{I} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[2 - x^2 - y^2 - \left(x^2 + y^2\right)\right] \, dy \, dx$$

This integral seems difficult. Use the transformation below to switch to polar coordinates.

$$\begin{array}{c} x^2 + y^2 = r^2 \\ dA = dy\,dx = r\,dr\,d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} r^2 \leq z \leq 2 - r^2 \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array}$$

$$\mathrm{I} = \int_0^{2\pi} \int_0^1 \left(2 - 2r^2\right) \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \left(2r - 2r^3\right) \, dr \, d\theta = \int_0^{2\pi} \left[r^2 - \frac{r^4}{2}\right]_{r=0}^{r=1} \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\theta = \boxed{\pi}$$

5) For the upper hemisphere, we have $z = \sqrt{4 - x^2 - y^2}$. The projection of the surface onto the xy-plane gives us the region $x^2 + y^2 \leq 1$. Find the surface area.

$$\begin{array}{c}\text{Surface}\\ \text{area}\end{array} = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + \left(\frac{x}{\sqrt{4 - x^2 - y^2}}\right)^2 + \left(\frac{y}{\sqrt{4 - x^2 - y^2}}\right)^2} \, dy \, dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + \frac{x^2 + y^2}{4 - x^2 - y^2}} \, dy \, dx = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\sqrt{4 - x^2 - y^2}} \, dy \, dx$$

From this point on, we can switch to polar coordinates using the transformation below.

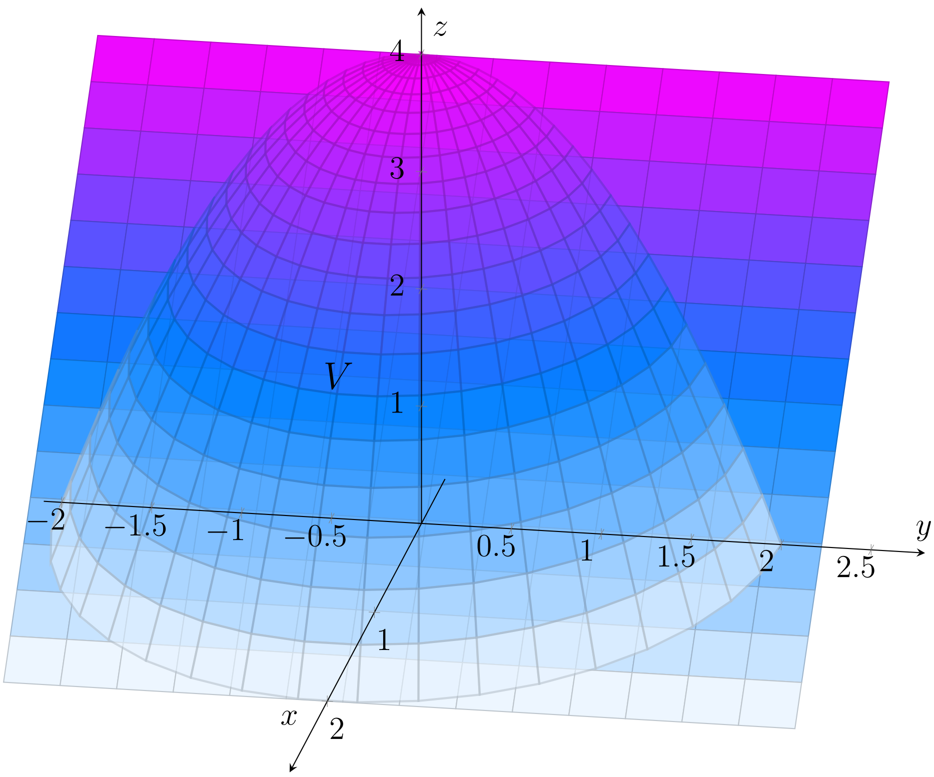
$$\begin{array}{l} r^2 = x^2 + y^2 \\ dA = r\,dr\,d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} \sqrt{4-x^2-y^2} = \sqrt{4-r^2} \\ x^2+y^2 \leq 1 \rightarrow r^2 \leq 1 \implies 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array}$$

$$\text{Surface area} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\sqrt{4-x^2-y^2}}\,dy\,dx = \int_0^{2\pi} \int_0^1 \frac{2}{\sqrt{4-r^2}} \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[-2\sqrt{4-r^2}\right]_{r=0}^{r=1} d\theta = \int_0^{2\pi} \left[-2\sqrt{3}-(-4)\right] d\theta = \boxed{4\pi\left(2-\sqrt{3}\right)}$$

6)

(i)



(ii) Using cylindrical coordinates, we may find the volume.

$$\begin{array}{c} z = z \\ x = r\cos\theta \\ y = r\sin\theta \\ r^2 = x^2+y^2 \\ dV = r\,dz\,dr\,d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} z = 4-x^2-y^2 \implies z = 4-r^2 \\ z = 4-2x \implies z = 4-2r\cos\theta \\ 4-2x = 4-x^2-y^2 \implies r = 2\cos\theta \\ \\ 4-2r\cos\theta \leq z \leq 4-r^2, \quad 0 \leq r \leq 2\cos\theta, \quad \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{array}$$

$$V = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_{4-2r\cos\theta}^{4-r^2} r\, dz\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \left[4 - r^2 - (4 - 2r\cos\theta)\right]\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (2r^2\cos\theta - r^3)\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{2r^3}{3}\cos\theta - \frac{r^4}{4}\right]_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\left(\frac{16}{3}\cos^4\theta - \frac{16\cos^4\theta}{4}\right) - 0\right] d\theta = \frac{4}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{4}{3}\int_{-\pi/2}^{\pi/2} \left(\frac{\cos(2\theta)+1}{2}\right)^2 d\theta = \frac{1}{3}\int_{-\pi/2}^{\pi/2} \left(\cos^2(2\theta) + 2\cos(2\theta) + 1\right)\, d\theta$$

$$= \frac{1}{3}\int_{-\pi/2}^{\pi/2} \left[\left(\frac{\cos(4\theta)+1}{2}\right) + 2\cos(2\theta) + 1\right] d\theta = \frac{1}{3}\left[\frac{\sin(4\theta)}{8} + \frac{\theta}{2} + \sin(2\theta) + \theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{3}\left[\left(\frac{\sin(2\pi)}{8} + \frac{\pi}{4} + \sin\pi + \frac{\pi}{2}\right) - \left(\frac{\sin(-2\pi)}{8} - \frac{\pi}{4} + \sin(-\pi) - \frac{\pi}{2}\right)\right] = \boxed{\frac{\pi}{2}}$$