MAT123 08.12.2025

QUESTIONS

Q1. Evaluate $\int 3y\sqrt{7-3y^2}\,dy$.

Q2. Evaluate $\int \tan^7 \frac{x}{2}\, \sec^2 \frac{x}{2}\,dx$.

Q3. Evaluate $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$.

Q4. Evaluate $\int \frac{(2x+3)e^{\sqrt{x^2+3x}}}{\sqrt{x^2+3x}}\,dx$.

Q5. Evaluate $\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$.

Q6. Evaluate $\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx$.

Q7. Evaluate $\int (\arcsin x)^2\,dx$.

Q8. Evaluate $\int x^2(\ln x)^2\,dx$.

Q9. Find $f'(1)$ if $\int_1^x f(t)\,dt = x + \ln(f(x))$ for all x.

Q10. Evaluate $\int \frac{(1+e^x)^2}{1+e^{2x}}\,dx$.

Q11. Evaluate $\int_0^{\ln 3} \frac{1}{e^x+2}\,dx$.

ANSWERS

Q1. Let $u = 7 - 3y^2$, so $du = -6y\,dy$ and $3y\,dy = -\frac{1}{2}du$.

$$\int 3y\sqrt{7-3y^2}\,dy = -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{3}u^{3/2} + C = \boxed{-\frac{1}{3}(7-3y^2)^{3/2} + C}$$

Q2. Let $u = \tan\frac{x}{2}$, so $du = \frac{1}{2}\sec^2\frac{x}{2}\,dx$ and $\sec^2\frac{x}{2}\,dx = 2\,du$.

$$\int \tan^7\frac{x}{2}\,\sec^2\frac{x}{2}\,dx = 2\int u^7\,du = \frac{u^8}{4} + C = \boxed{\frac{1}{4}\tan^8\frac{x}{2} + C}$$

Q3. Let $u = 1 + \sin^2(x-1)$, so $du = 2\sin(x-1)\cos(x-1)\,dx$.

$$\begin{aligned}\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx &= \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{3/2} + C\\ &= \boxed{\frac{1}{3}\big(1+\sin^2(x-1)\big)^{3/2} + C}\end{aligned}$$

Q4. Let $u = \sqrt{x^2+3x}$, so $u^2 = x^2 + 3x$ and $2u\,du = (2x+3)\,dx$.

$$\int \frac{(2x+3)e^{\sqrt{x^2+3x}}}{\sqrt{x^2+3x}}\,dx = \int \frac{2u\,e^u}{u}\,du = \int 2e^u\,du = 2e^u + C = \boxed{2e^{\sqrt{x^2+3x}} + C}$$

Q5. First let $x = \sin y$. Then

$$dx = \cos y\,dy, \qquad \sqrt{1-x^2} = \cos y, \qquad x^4 = \sin^4 y.$$

So the integral becomes

$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx = \int \frac{\cos y}{\sin^4 y}\,\cos y\,dy = \int \frac{\cos^2 y}{\sin^4 y}\,dy = \int \cot^2 y\,\csc^2 y\,dy.$$

Now set

$$u = \cot y, \qquad du = -\csc^2 y\,dy.$$

Then

$$\int \cot^2 y\,\csc^2 y\,dy = -\int u^2\,du.$$

Integrate:

$$-\int u^2\,du = -\frac{u^3}{3} + C.$$

Back-substitute $u = \cot y$. Since $x = \sin y$, we have

$$\cot y = \frac{\cos y}{\sin y} = \frac{\sqrt{1-x^2}}{x}.$$

Thus the final answer is

$$\boxed{-\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C}$$

Q6. Let

$$u = \ln x \implies du = \frac{dx}{x}.$$

Then the integral becomes

$$\int \frac{\sqrt{1-u^2}}{u}\,du.$$

Let

$$u = \sin y \implies du = \cos y\,dy, \quad \sqrt{1-u^2} = \cos y.$$

Then the integral becomes

$$\int \frac{\cos y}{\sin y} \cdot \cos y\,dy = \int \frac{\cos^2 y}{\sin y}\,dy.$$

$$\int \frac{\cos^2 y}{\sin y}\,dy = \int \frac{1-\sin^2 y}{\sin y}\,dy = \int \left(\frac{1}{\sin y} - \sin y\right)dy = \int (\csc y - \sin y)\,dy.$$

$$\int \csc y\,dy = \ln\left|\csc y - \cot y\right|, \quad \int \sin y\,dy = -\cos y.$$

So the integral becomes

$$\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx = \ln\left|\csc y - \cot y\right| + \cos y + C.$$

Since $u = \sin y = \ln x$, we have

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-(\ln x)^2},$$

and

$$\csc y - \cot y = \frac{1}{\sin y} - \frac{\cos y}{\sin y} = \frac{1-\sqrt{1-(\ln x)^2}}{\ln x}.$$

Thus, the final answer is

$$\boxed{\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx = \ln\left|\frac{1-\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C}.$$

Q7. Use integration by parts:

$$u = (\arcsin x)^2, \qquad dv = dx.$$

Then

$$du = 2\arcsin x \cdot \frac{1}{\sqrt{1-x^2}}\,dx, \qquad v = x.$$

Thus

$$\int (\arcsin x)^2\,dx = x(\arcsin x)^2 - 2\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx.$$

Let

$$I = \int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx.$$

Apply the second integration by parts.

$$u = \arcsin x, \qquad dv = \frac{x}{\sqrt{1-x^2}}\,dx.$$

Differentiate and integrate:

$$du = \frac{1}{\sqrt{1-x^2}}\,dx, \qquad v = -\sqrt{1-x^2}$$

Then

$$I = uv - \int v\,du = -\arcsin x\sqrt{1-x^2} - \int \left(-\sqrt{1-x^2}\right)\frac{1}{\sqrt{1-x^2}}\,dx$$

Simplify the integrand.

$$I = -\arcsin x\sqrt{1-x^2} + \int 1\,dx = -\arcsin x\sqrt{1-x^2} + x$$

Substitute back into the main integral.

$$\int (\arcsin x)^2\,dx = x(\arcsin x)^2 - 2\left[-\arcsin x\sqrt{1-x^2} + x\right] + C$$

Simplify.

$$\boxed{x(\arcsin x)^2 + 2\arcsin x\sqrt{1-x^2} - 2x + C}$$

Q8. Use integration by parts.

$$u = (\ln x)^2 \implies du = \frac{2\ln x}{x}\,dx, \qquad dv = x^2\,dx \implies v = \frac{x^3}{3}$$

Then

$$\int x^2(\ln x)^2\,dx = \frac{x^3}{3}(\ln x)^2 - \int \frac{x^3}{3}\cdot\frac{2\ln x}{x}\,dx = \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\int x^2\ln x\,dx.$$

Apply integration by parts again on $\int x^2 \ln x\,dx$ with

$$u = \ln x \implies du = \frac{1}{x}\,dx, \qquad dv = x^2\,dx \implies v = \frac{x^3}{3}.$$

Substitute directly.

$$\begin{aligned}
\int x^2(\ln x)^2\,dx &= \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\left(\frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx\right)\\
&= \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\left(\frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx\right)\\
&= \frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x + \frac{2}{3}\cdot\frac{x^3}{9} + C
\end{aligned}$$

$$\boxed{\int x^2(\ln x)^2\,dx = \frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x + \frac{2x^3}{27} + C}$$

Q9. Find $f'(1)$ if $\displaystyle\int_1^x f(t)\,dt = x + \ln(f(x))$.

Differentiate both sides. By the Fundamental Theorem of Calculus (Part I):

$$f(x) = 1 + \frac{f'(x)}{f(x)}$$

So

$$f'(x) = f(x)(f(x) - 1)$$

Evaluate original equation at $x = 1$:

$$0 = 1 + \ln(f(1)) \Rightarrow f(1) = e^{-1}$$

Thus

$$f'(1) = f(1)(f(1) - 1) = e^{-1}(e^{-1} - 1) = \boxed{\frac{1}{e^2} - \frac{1}{e}}.$$

Q10. Expand numerator.

$$\int \frac{1+2e^x+e^{2x}}{1+e^{2x}}\,dx = \int \left(\frac{1+e^{2x}}{1+e^{2x}} + \frac{2e^x}{1+e^{2x}}\right)dx = \int dx + \int \frac{2e^x}{1+e^{2x}}\,dx$$

Let $u = e^x$, $du = e^x\,dx$ for the right-hand integral.

$$\int \frac{2}{1+u^2}\,du = 2\arctan u + C = 2\arctan(e^x) + C$$

Thus

$$\int \frac{(1+e^x)^2}{1+e^{2x}}\,dx = \boxed{x + 2\arctan(e^x) + C}.$$

Q11. Add and subtract $\frac{1}{2}e^x$ in the numerator.

$$\frac{1}{e^x+2} = \frac{\frac{1}{2}e^x + 1 - \frac{1}{2}e^x}{e^x+2} = \frac{\frac{1}{2}e^x+1}{e^x+2} - \frac{\frac{1}{2}e^x}{e^x+2}$$

So the integral becomes

$$\int_0^{\ln 3} \frac{1}{e^x+2}\,dx = \int_0^{\ln 3} \frac{\frac{1}{2}e^x+1}{e^x+2}\,dx - \int_0^{\ln 3} \frac{\frac{1}{2}e^x}{e^x+2}\,dx.$$

Simplify each term. First integral:

$$\frac{\frac{1}{2}e^x+1}{e^x+2} = \frac{1}{2}\cdot\frac{e^x+2}{e^x+2} = \frac{1}{2}$$

$$\int_0^{\ln 3} \frac{\frac{1}{2}e^x+1}{e^x+2}\,dx = \int_0^{\ln 3} \frac{1}{2}\,dx = \frac{x}{2}\bigg|_0^{\ln 3} = \frac{\ln 3}{2}$$

Second integral:

$$\int_0^{\ln 3} \frac{\frac{1}{2}e^x}{e^x+2}\,dx = \frac{1}{2}\int_0^{\ln 3} \frac{e^x}{e^x+2}\,dx$$

Let $u = e^x + 2 \implies du = e^x\,dx$. Then

$$\frac{1}{2}\int_0^{\ln 3} \frac{e^x}{e^x+2}\,dx = \frac{1}{2}\int_3^5 \frac{du}{u} = \frac{1}{2}\ln\frac{5}{3}.$$

Combine results.

$$\int_0^{\ln 3} \frac{1}{e^x+2}\,dx = \frac{\ln 3}{2} - \frac{1}{2}\ln\frac{5}{3} = \boxed{\frac{1}{2}\ln\frac{9}{5} = \ln 3 - \frac{1}{2}\ln 5}$$