2015-2016 Fall
MAT123 Final
(12/01/2016)

1. Find the area of the region bounded by the circle $x^2 + y^2 = 1$ and the graph of the absolute value function $y = |x|$.

2. Use the Washer Method to find the volume of the solid obtained by revolving the region bounded by the parabola $y = 2x^2 - 3$ and the curves $y = -3$, $x = 2$ about the line $y = 7$.

3. Use the Cylindrical Shell Method to find the volume of the solid obtained by revolving the region bounded by $y = x^2$ and $y = -x + 1$ about the line $x = -1$.

4. Evaluate the following integrals.

(a) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

(b) $\displaystyle\int \frac{x^5 + x^4 - 8x^3 + 10x^2 + 12x}{x^2 - 3x + 2}\,dx$

(c) $\displaystyle\int \arccos x\,dx$

(d) $\displaystyle\int \frac{1}{x^2 + 3x + 1}\,dx$

(e) $\displaystyle\int \frac{\sin x}{1 + \sin x}\,dx$

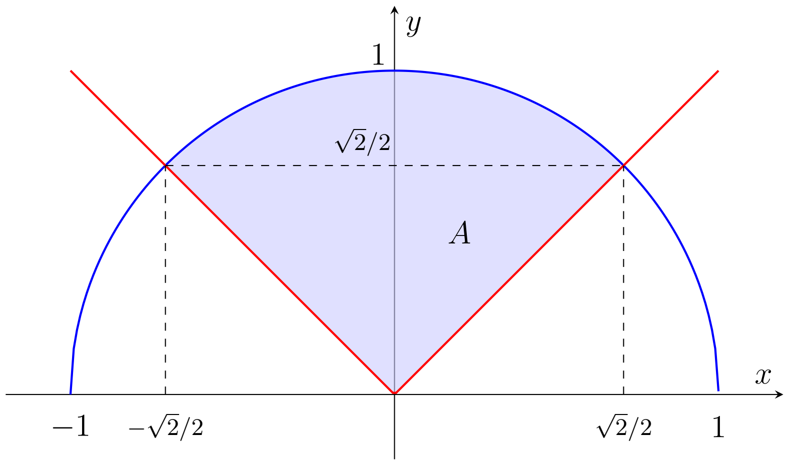
5. Find the surface area of the revolution by rotating the curve $y = \mathrm{e}^x$, $0 \le x \le 1$ about the x-axis.

6. Investigate the convergence of the improper integral $\displaystyle\int_0^{-\infty} \frac{\mathrm{e}^{-x}}{1 + \mathrm{e}^{-x}}\,dx$.

7. Using the Monotone Convergence Theorem, investigate the convergence of the sequence $\displaystyle\left(\frac{\ln n}{n}\right)_{n\in\mathbb{N}}$.

2015-2016 Final (12/01/2016) Solutions
(Last update: 8/17/25 (17th of August) 11:27 PM)

1.



$$A = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(\sqrt{1-x^2} - |x|\right)\, dx = \int_{-\sqrt{2}/2}^{0} \left(\sqrt{1-x^2} + x\right)\, dx + \int_{0}^{\sqrt{2}/2} \left(\sqrt{1-x^2} - x\right)\, dx$$

$$= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \sqrt{1-x^2}\, dx - \int_{0}^{\sqrt{2}/2} 2x\, dx \tag{1}$$

Calculate the right-hand integral (1).

$$\int_{0}^{\sqrt{2}/2} 2x\, dx = x^2\Big|_{0}^{\sqrt{2}/2} = \left(\frac{\sqrt{2}}{2}\right)^2 - (0)^2 = \frac{1}{2}$$

To calculate the left-hand integral in (1), we will use a trigonometric substitution. Let $x = \sin u$, then $dx = \cos u\, du$.

$$x = -\frac{\sqrt{2}}{2} \implies u = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}, \qquad x = \frac{\sqrt{2}}{2} \implies u = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

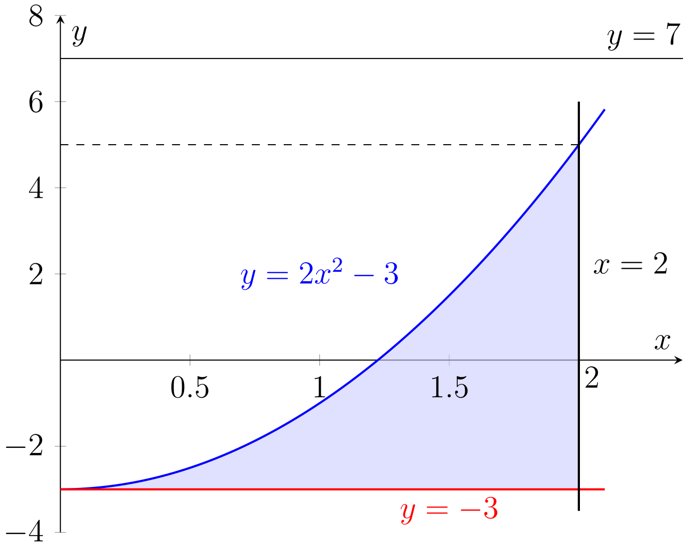
$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \sqrt{1-x^2}\, dx = \int_{-\pi/4}^{\pi/4} \sqrt{1-\sin^2 u}\cos u\, du = \int_{-\pi/4}^{\pi/4} |\cos u| \cos u\, du \quad [|\cos u| > 0]$$

$$= \int_{-\pi/4}^{\pi/4} \cos^2 u\, du = \int_{-\pi/4}^{\pi/4} \frac{1+\cos 2u}{2}\, du = \left[\frac{u}{2} + \frac{\sin 2u}{4}\right]_{-\pi/4}^{\pi/4}$$

$$= \left(\frac{\pi}{8} + \frac{1}{4}\right) - \left(-\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$

Evaluate (1). The area is $A = \left(\frac{\pi}{4} + \frac{1}{2}\right) - \left(\frac{1}{2}\right) = \boxed{\frac{\pi}{4}}$.

2.

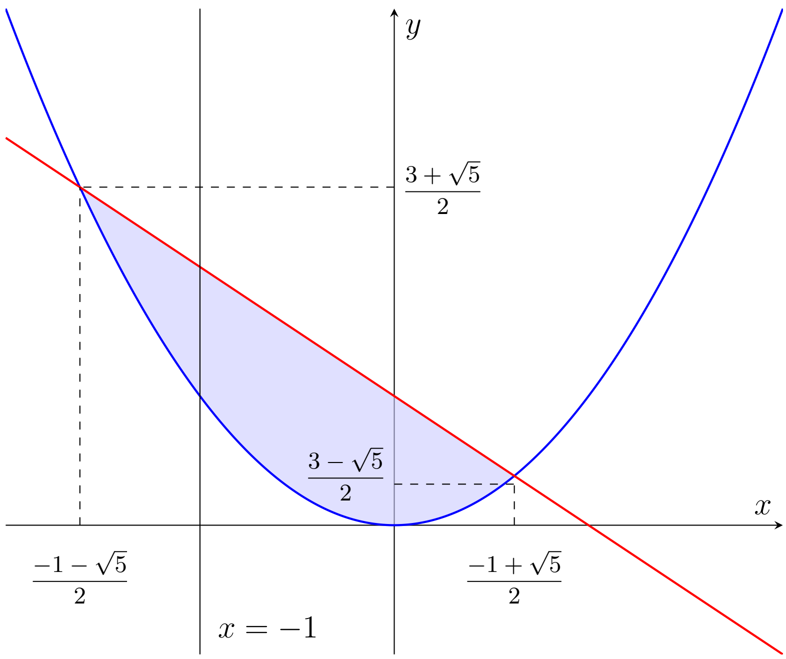


$$V=\int_D \pi\left[R_2^2(x)-R_1^2(x)\right]\,dx=\int_0^2 \pi\left[(7-(-3))^2-\left(7-\left(2x^2-3\right)\right)^2\right]\,dx$$

$$=\pi\int_0^2\left[(10)^2-\left(10-2x^2\right)^2\right]\,dx=\pi\int_0^2\left(100-100+40x^2-4x^4\right)\,dx$$

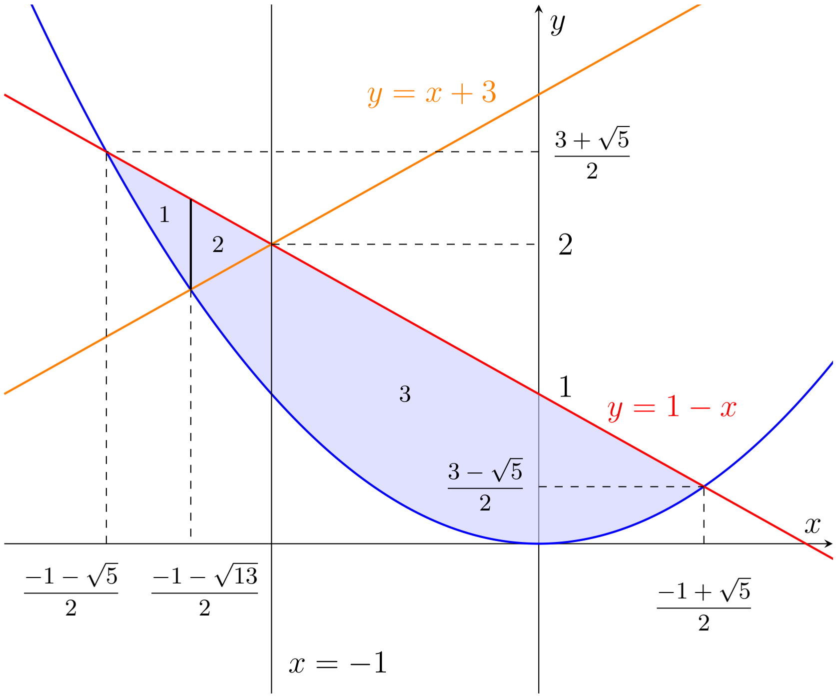
$$=\pi\int_0^2\left(40x^2-4x^4\right)\,dx=\pi\left[\frac{40x^3}{3}-\frac{4x^5}{5}\right]_0^2=\pi\left[\left(\frac{320}{3}-\frac{128}{5}\right)-(0)\right]=\boxed{\frac{1216\pi}{15}}$$

3.



Notice that the rotation axis passes through the region. Consider the right-hand region. If we rotate it around the axis, a piece of the region on the left will be inside the revolution. That is, the region bounded by the line that is symmetric to the line $y=-x+1$ around

$x = -1$ and the curve $y = x^2$. We do not need to rotate that region since it would lead to double revolution. The upper part of the left-hand region may be rotated around $x = -1$ independent of the right-hand region. Divide the left-hand region into three subregions. We get three different subregions to integrate over.



Let V be the volume of the solid.

$$V = \int_D 2\pi \cdot r(x) \cdot h(x)\, dx$$

$$= \int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{13}}{2}} 2\pi(-1-x)\left[(1-x) - x^2\right]\, dx + \int_{\frac{1-\sqrt{13}}{2}}^{-1} 2\pi(-1-x)\left[(1-x) - (x+3)\right]\, dx$$

$$+ \int_{-1}^{\frac{-1+\sqrt{5}}{2}} 2\pi\,(x+1)\left[(1-x) - x^2\right]\, dx$$

$$= 2\pi \int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{13}}{2}} \left(-1 + 2x^2 + x^3\right)\, dx + 2\pi \int_{\frac{1-\sqrt{13}}{2}}^{-1} \left(2x^2 + 4x + 2\right)\, dx$$

$$+ 2\pi \int_{-1}^{\frac{-1+\sqrt{5}}{2}} \left(1 - 2x^2 - x^3\right)\, dx$$

$$= 2\pi \left\{ \left[-x + \frac{2x^3}{3} + \frac{x^4}{4}\right]_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{13}}{2}} + \left[\frac{2x^3}{3} + 2x^2 + 2x\right]_{\frac{1-\sqrt{13}}{2}}^{-1} + \left[x - \frac{2x^3}{3} - \frac{x^4}{4}\right]_{-1}^{\frac{-1+\sqrt{5}}{2}} \right\}$$

$$V = 2\pi\left[\frac{\sqrt{13}-1}{2}+\frac{\left(1-\sqrt{13}\right)^3}{12}+\frac{\left(1-\sqrt{13}\right)^4}{64}-\frac{1+\sqrt{5}}{2}+\frac{\left(1+\sqrt{5}\right)^3}{12}-\frac{\left(1+\sqrt{5}\right)^4}{64}\right]$$

$$+\,2\pi\left[-\frac{2}{3}+2-2-\left(\frac{\left(1-\sqrt{13}\right)^3}{12}+\frac{\left(1-\sqrt{13}\right)^2}{2}+1-\sqrt{13}\right)\right]$$

$$+\,2\pi\left[\frac{-1+\sqrt{5}}{2}-\frac{\left(-1+\sqrt{5}\right)^3}{12}-\frac{\left(-1+\sqrt{5}\right)^4}{64}-\left(-1+\frac{2}{3}-\frac{1}{4}\right)\right]$$

After some mathematical operations, the volume is found to be

$$V = \frac{13\pi}{6}-\frac{\pi}{4}\left(47-13\sqrt{13}\right) = \boxed{\frac{\pi\left(39\sqrt{13}-115\right)}{12}}$$

The volume can also be evaluated by integrating over the domain and subtracting the region that is revolved twice. That is, the region beneath the region 2 mapped on the graph.

4.

(a) Let $u=\sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u \cdot 2\,du = -2\cos u + c = \boxed{-2\cos\sqrt{x}+c, \quad c\in\mathbb{R}}$$

(b) Perform a long polynomial division and rewrite the integral in two expressions.

$$\mathrm{I} = \int \frac{x^5+x^4-8x^3+10x^2+12x}{x^2-3x+2}\,dx = \int\left(x^3-4x^2-22x-48+\frac{-88x+96}{x^2-3x+2}\right)dx$$

$$= \int\left(x^3+4x^2+2x+8\right)dx + \int \frac{32x-16}{(x-2)(x-1)}\,dx \tag{2}$$

Calculate the integral on the left in (2).

$$\int\left(x^3+4x^2+2x+8\right)dx = \frac{x^4}{4}+\frac{4x^3}{3}+x^2+8x+c_1$$

Calculate the integral on the right in (2). Decompose the expression into partial fractions.

$$\int \frac{32x-16}{(x-2)(x-1)}\,dx = \int\left(\frac{A}{x-2}+\frac{B}{x-1}\right)dx$$

$$32x-16 = A(x-1)+B(x-2) = x(A+B)-A-2B$$

Equate the coefficients of like terms.

$$\left.\begin{array}{r} A+B=32 \\ -A-2B=-16 \end{array}\right\} \rightarrow A=48, \quad B=-16$$

Substitute the values into A and B.

$$\int\left(\frac{A}{x-2}+\frac{B}{x-1}\right) dx=\int\left(\frac{48}{x-2}-\frac{16}{x-1}\right) dx=48 \ln |x-2|-16 \ln |x-1|+c_2$$

Rewrite (2).

$$\mathrm{I}=\boxed{\frac{x^4}{4}+\frac{4x^3}{3}+x^2+8x+48 \ln |x-2|-16 \ln |x-1|+c, \quad c \in \mathbb{R}}$$

(c) Apply integration by parts.

$$\left.\begin{array}{r} u=\arccos x \rightarrow du=-\dfrac{1}{\sqrt{1-x^2}} dx \\ dv=dx \rightarrow v=x \end{array}\right\} \rightarrow \int v\, du=uv-\int v\, du$$

$$\int \arccos x\, dx=x \arccos x-\int \frac{-x}{\sqrt{1-x^2}} dx$$

Let us use a u-substitution for the integral on the right. Let $u=1-x^2$, then $du=-2x\, dx$

$$\int \frac{-x}{\sqrt{1-x^2}} dx=\int \frac{du}{2\sqrt{u}}=\sqrt{u}+c=\sqrt{1-x^2}+c$$

Therefore,

$$\int \arccos x\, dx=\boxed{x \arccos x-\sqrt{1-x^2}+c, \quad c \in \mathbb{R}}$$

(d) Use the method of partial fraction decomposition.

$$\mathrm{I}=\int \frac{dx}{x^2+3x+1}=\int \frac{dx}{x^2+3x+\frac{9}{4}-\frac{5}{4}}=\int \frac{dx}{\left(x+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}$$

$$=\int \frac{dx}{\left(x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\left(x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)}=\int\left(\frac{A}{x+\frac{3}{2}-\frac{\sqrt{5}}{2}}+\frac{B}{x+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right) dx$$

$$A\left(x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)+B\left(x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)=1$$

$$x(A+B)+A\left(\frac{3+\sqrt{5}}{2}\right)+B\left(\frac{3-\sqrt{5}}{2}\right)=1$$

Equate the coefficients of like terms.

$$\left.\begin{aligned} A+B&=0\\ A\left(\tfrac{3+\sqrt{5}}{2}\right)+B\left(\tfrac{3-\sqrt{5}}{2}\right)&=1 \end{aligned}\right\} \implies A=\frac{1}{\sqrt{5}},\quad B=-\frac{1}{\sqrt{5}}$$

Substitute the values into A and B.

$$\mathrm{I}=\int\left(\frac{A}{x+\frac{3}{2}-\frac{\sqrt{5}}{2}}+\frac{B}{x+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right)\,dx$$

$$=\int\left(\frac{1}{\sqrt{5}\left(x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)}-\frac{1}{\sqrt{5}\left(x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)}\right)\,dx$$

$$=\boxed{\frac{1}{\sqrt{5}}\left(\ln\left|x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right|-\ln\left|x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right|\right)+c,\quad c\in\mathbb{R}}$$

(e)

$$\mathrm{I}=\int\frac{\sin x}{1+\sin x}\,dx=\int\frac{1+\sin x-1}{1+\sin x}\,dx=\int dx-\int\frac{dx}{1+\sin x}$$

The integral on the left evaluates to $x+c_1$. Evaluate the other integral.

$$\int\frac{dx}{1+\sin x}=\int\frac{1-\sin x}{(1+\sin x)(1-\sin x)}\,dx=\int\frac{1-\sin x}{1-\sin^2 x}\,dx=\int\frac{1-\sin x}{\cos^2 x}\,dx$$

$$=\int\left(\sec^2 x-\tan x\sec x\right)\,dx=\tan x-\sec x+c_2$$

So, the result is

$$\mathrm{I}=\boxed{x-\tan x+\sec x+c,\quad c\in\mathbb{R}}$$

5. If the function $y=f(x)\geq 0$ is continuously differentiable on $[a,b]$, the area of the surface generated by revolving the graph of $y=f(x)$ about the x-axis is

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}=\mathrm{e}^{\mathrm{x}}$$

Set $a=0$ and $b=1$ and then evaluate the integral.

$$S=\int_0^1 2\pi \mathrm{e}^x \sqrt{1+(\mathrm{e}^x)^2}\, dx$$

Let $u=\mathrm{e}^x$, then $du=\mathrm{e}^x\, dx$.

$$x=0 \implies u=\mathrm{e}^0=1, \qquad x=1 \implies u=\mathrm{e}^1=\mathrm{e}$$

$$S=\int_0^1 2\pi \mathrm{e}^x \sqrt{1+(\mathrm{e}^x)^2}\, dx = 2\pi \int_1^{\mathrm{e}} \sqrt{1+u^2}\, du$$

We will now use a trigonometric substitution. Let $u=\tan t$ for $0<t<\frac{\pi}{2}$, then $du=\sec^2 t\, dt$.

$$S=2\pi \int_1^{\mathrm{e}} \sqrt{1+u^2}\, du = 2\pi \int \sqrt{1+\tan^2 t}\cdot \sec^2 t\, dt = 2\pi \int \sqrt{\sec^2 t}\cdot \sec^2 t\, dt$$

$$= 2\pi \int |\sec t| \sec^2 t\, dt = 2\pi \int \sec^3 t\, dt \qquad [\sec t>0]$$

Find the antiderivative of $\sec^3 t$ with the help of integration by parts.

$$w=\sec t \ \rightarrow\ dw=\sec t\tan t\, dt$$
$$dz=\sec^2 t\, dt \ \rightarrow\ z=\tan t$$

$$\int \sec^3 t\, du = \tan t\cdot \sec t - \int \tan^2 t \sec t\, dt = \tan t\cdot\sec t - \int \frac{1-\cos^2 t}{\cos^3 t}\, dt$$

$$= \tan t\cdot \sec t - \int \sec^3 t\, dt + \int \sec t\, dt$$

Notice that the integral appears on the right side of the equation. Therefore,

$$\int \sec^3 t\, dt = \frac{1}{2}\cdot \tan t\cdot \sec t + \frac{1}{2}\cdot \int \sec t\, dt$$

The integral of $\sec t$ with respect to t is

$$\int \sec t\, dt = \ln|\tan t+\sec t| + c_1, \quad c_1\in\mathbb{R}$$

Recall $u=\tan t$.

$$u=\tan t \implies u^2=\tan^2 t=\sec^2 t-1 \implies \sec t=\sqrt{u^2+1}$$

$$S = 2\pi \cdot \frac{1}{2}\left(\tan t \cdot \sec t + \ln\left|\tan t + \sec t\right|\right) + c = \pi\left[u \cdot \sqrt{u^2+1} + \ln\left|t + \sqrt{u^2+1}\right|\right]_1^{\mathrm{e}}$$

$$= \pi\left[\left(\mathrm{e} \cdot \sqrt{\mathrm{e}^2+1} + \ln\left|\mathrm{e} + \sqrt{\mathrm{e}^2+1}\right|\right) - \left(\sqrt{2} + \ln\left|1+\sqrt{2}\right|\right)\right]$$

$$= \boxed{\pi\left[\mathrm{e} \cdot \sqrt{\mathrm{e}^2+1} - \sqrt{2} + \ln\left(\frac{\mathrm{e}+\sqrt{\mathrm{e}^2+1}}{1+\sqrt{2}}\right)\right]}$$

6. Let $u = 1 + \mathrm{e}^{-x}$, then $du = -\mathrm{e}^{-x}\,dx$. Handle the improper integral by taking the limit.

$$x = 0 \implies u = 1 + \mathrm{e}^0 = 2, \qquad x \to -\infty \implies u \to \infty$$

$$\int_0^{-\infty} \frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}\,dx = \int_2^{\infty} -\frac{du}{u} = \lim_{R\to\infty} \int_2^{R} -\frac{du}{u} = \lim_{R\to\infty} \left(-\ln u\right)\Big|_2^R$$

$$= \lim_{R\to\infty} \left(-\ln R + \ln 2\right) = \boxed{-\infty}$$

The integral diverges to negative infinity.

7. According to the Monotone Convergence Theorem, if a sequence is both bounded and monotonic, the sequence converges. Take the corresponding function $f(x) = \dfrac{\ln x}{x}$. Apply the first derivative test and find the extrema.

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$
$$f'(x) = 0 \implies 1 - \ln x = 0 \implies \ln x = 1 \implies x = \mathrm{e}$$

A critical point occurs at $x = \mathrm{e}$. Apply the second derivative test and determine whether this is a local minimum or a local maximum.

$$f''(x) = \frac{-\frac{1}{x} \cdot x^2 - (1 - \ln x) \cdot 2x}{x^4} = \frac{2\ln x - 3}{x^3}$$
$$f''(\mathrm{e}) = \frac{-1}{\mathrm{e}^3} < 0$$

Therefore, this is a local maximum.

The first term of the sequence is $\dfrac{\ln 1}{1} = 0$. Take the limit at infinity. We may apply L'Hôpital's rule because we have an indeterminate form, which is ∞/∞.

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Since $\dfrac{\ln x}{x}$ is decreasing and bounded above by $f(\mathrm{e}) = \dfrac{1}{\mathrm{e}}$ and below by 0 for $x \geq \mathrm{e}$, by the Monotone Convergence Theorem, the sequence converges.