2017-2018 Summer
MAT123-02 Final
(13/08/2018)

1. Let us consider the area A of the region bounded by the curves $y = \mathrm{e}^x$, $y = x^2 - 1$ and the straight lines $x = -1$, $x = 1$. Write an integral (but do not evaluate) corresponding to the area A with respect to the y-axis.

2. Sketch the graph of

$$f(x) = \frac{4x}{x^2 + 1}.$$

3. Evaluate the following integrals.

(a) $\displaystyle\int \frac{\tan x}{\sec^4 x}\, dx$

(b) $\displaystyle\int \frac{\ln x}{x^3}\, dx$

(c) $\displaystyle\int \frac{1}{1 + \sin x}\, dx$

(d) $\displaystyle\int \frac{x + 7}{x^2\,(x + 2)}\, dx$

4. Use the Shell Method to determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$, $y = x$ about the line $y = 4$.

5. Find the area of the surface generated by rotating the curve $y = \mathrm{e}^x$, $0 \leq x \leq 1$ about the x-axis.

6. Use the Integral Test to investigate the convergence of the series $\displaystyle\sum_{k=0}^{\infty} k\mathrm{e}^{-k}$.

7. Find the Maclaurin series of the function $f(x) = \ln(1 + x)$.

8. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 5^n}$.

2017-2018 Summer Final (13/08/2018) Solutions
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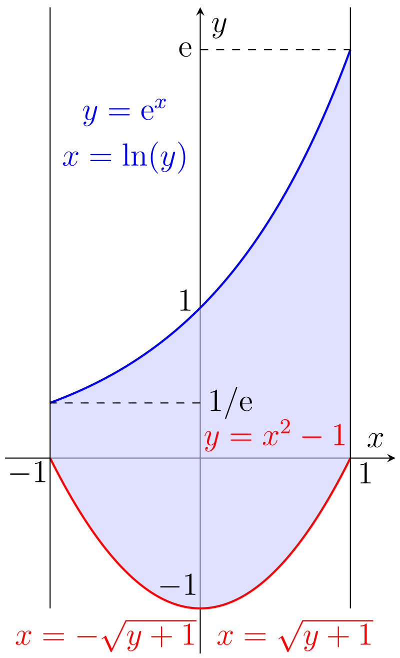
1.



$$\boxed{\begin{aligned} A = &\int_{-1}^{0} \left[\left(\sqrt{y+1}\right) - \left(-\sqrt{y+1}\right)\right] \, dy \\ &+ \int_{0}^{1/e} \left[(1) - (-1)\right] \, dy + \int_{1/e}^{e} \left[(1) - (\ln y)\right] \, dy \end{aligned}}$$

2. The function f is defined where the denominator is not equal to zero. However, the denominator is always positive. Therefore, the domain of f is $\mathbb{R}$.

Let us find the limit at infinity and negative infinity.

$$\lim_{x \to \infty} \frac{4x}{x^2+1} = \lim_{x \to \infty} \frac{4}{x + \frac{1}{x}} = 0$$

Similarly, at negative infinity, the limit is zero. Therefore, the x-axis is a horizontal asymptote. There is no vertical asymptote.

Take the first derivative and find the critical points. Apply the quotient rule.

$$y' = \frac{d}{dx}\left(\frac{4x}{x^2+1}\right) = \frac{4 \cdot (x^2+1) - 4x \cdot (2x)}{(x^2+1)^2} = \frac{-4x^2+4}{(x^2+1)^2}$$

The critical points occur at $x = \pm 1$. At these points, the first derivative is 0.

Take the second derivative. Apply the quotient rule again.

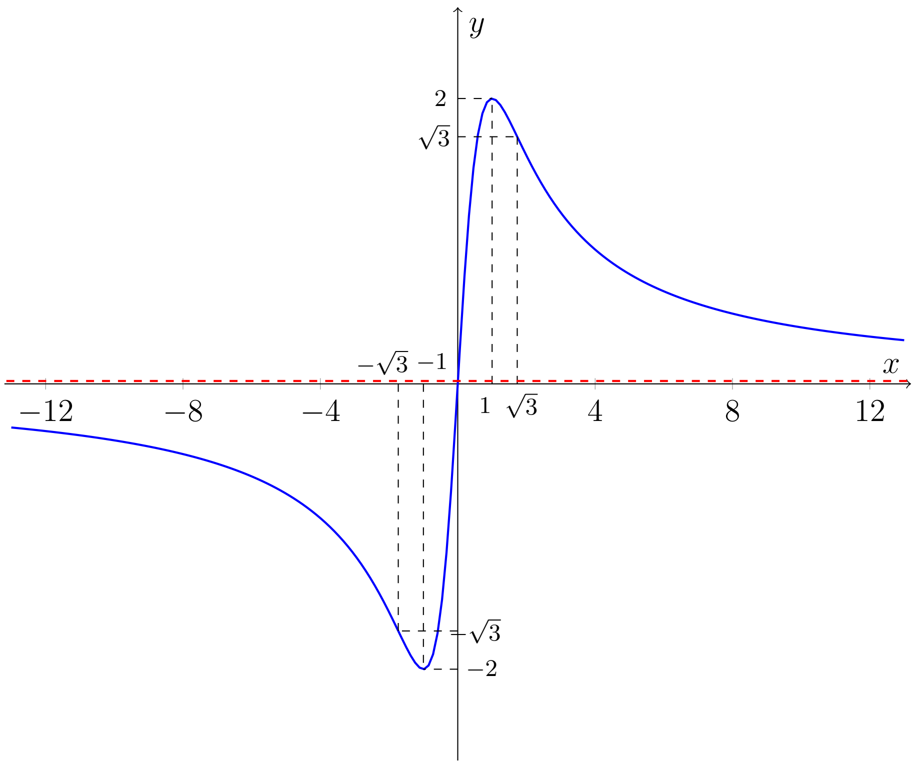
$$y'' = \frac{d}{dx}\left(\frac{-4x^2+4}{(x^2+1)^2}\right) = \frac{(-8x) \cdot (x^2+1)^2 - (-4x^2+4) \cdot 2(x^2+1) \cdot (2x)}{(x^2+1)^4} = \frac{8x(x^2-3)}{(x^2+1)^3}$$

The inflection points occur at $x = 0$ and $x = \pm\sqrt{3}$. At these points, the direction of the curvature changes.

Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f\left(-\sqrt{3}\right) = -\sqrt{3}, \quad f(-1) = -2, \quad f(0) = 0, \quad f(1) = 2, \quad f\left(\sqrt{3}\right) = \sqrt{3}$$

x	$\left(-\infty, -\sqrt{3}\right)$	$\left(-\sqrt{3}, -1\right)$	$(-1, 0)$	$(0, 1)$	$\left(1, \sqrt{3}\right)$	$\left(\sqrt{3}, \infty\right)$
y	$\left(-\sqrt{3}, 0\right)$	$\left(-2, -4\sqrt{3}\right)$	$(-2, 0)$	$(0, 2)$	$\left(\sqrt{3}, 2\right)$	$\left(0, \sqrt{3}\right)$
y' sign	-	-	$+$	$+$	-	-
y'' sign	-	$+$	$+$	-	-	$+$



3.

(a) Try to rewrite in the form of $\sin x$ and $\cos x$.

$$\int \frac{\tan x}{\sec^4 x}\,dx = \int \frac{\sin x}{\cos x} \cdot \cos^4 x\,dx = \int \sin x \cdot \cos^3 x\,dx = \int \sin x \cdot \left(1 - \sin^2 x\right) \cdot \cos x\,dx$$

Let $u = \sin x$, then $du = \cos x\,dx$.

$$\int \sin x \cdot \left(1 - \sin^2 x\right) \cdot \cos x\,dx = \int u\left(1 - u^2\right)\,du = \int \left(u - u^3\right)\,du = \frac{u^2}{2} - \frac{u^4}{4} + c$$

$$= \boxed{\frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + c, \quad c \in \mathbb{R}}$$

(b) Use the method of integration by parts.

$$\left.\begin{array}{l} u = \ln x \implies du = \dfrac{1}{x}\,dx \\[2ex] dv = \dfrac{1}{x^3}\,dx \implies v = -\dfrac{1}{2x^2} \end{array}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int \frac{\ln x}{x^3}\,dx = -\frac{\ln x}{2x^2} - \int -\frac{1}{2x^3}\,dx = \boxed{-\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c, \quad c \in \mathbb{R}}$$

(c) Expand the expression by multiplying and dividing by the conjugate of the denominator.

$$\int \frac{1}{1+\sin x}\,dx = \int \frac{1-\sin x}{(1+\sin x)(1-\sin x)}\,dx = \int \frac{1-\sin x}{1-\sin^2 x}\,dx = \int \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \int \left(\sec^2 x - \tan x \sec x\right)\,dx = \boxed{\tan x - \sec x + c, \quad c \in \mathbb{R}}$$

(d) Use the method of partial fraction decomposition.

$$\int \frac{x+7}{x^2\,(x+2)}\,dx = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}\right)\,dx$$

$$\begin{aligned} A(x)(x+2) + B(x+2) + C\left(x^2\right) &= x+7 \\ x^2(A+C) + x(2A+B) + 2B &= x+7 \end{aligned}$$

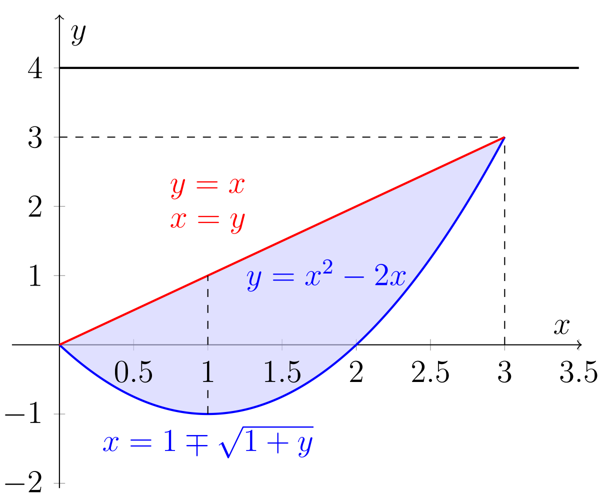
Equate the coefficients of like terms.

$$\left.\begin{array}{r} A+C=0 \\ 2A+B=1 \\ 2B=7 \end{array}\right\} \implies A = -\frac{5}{4}, \quad B = \frac{7}{2}, \quad C = \frac{5}{4}$$

Rewrite the integral by substituting the values into the unknowns.

$$\int \left(-\frac{5}{4x} + \frac{7}{2x^2} + \frac{5}{4(x+2)}\right)\,dx = \boxed{-\frac{5}{4}\ln|x| - \frac{7}{2x} + \frac{5}{4}\ln|x+2| + c, \quad c \in \mathbb{R}}$$

4.



$$V = \int_D 2\pi \cdot h(y) \cdot r(y)\,dy$$

$$V = \int_{-1}^{0} 2\pi\,(4-y)\left[\left(1+\sqrt{1+y}\right)-\left(1-\sqrt{1+y}\right)\right]\,dy$$

$$+\int_{0}^{3} 2\pi(4-y)\left[\left(1+\sqrt{1+y}\right)-(y)\right]\,dy$$

$$= 2\pi\int_{-1}^{0}(4-y)\left(2\sqrt{1+y}\right)\,dy + 2\pi\int_{0}^{3}(4-y)\left(1+\sqrt{1+y}-y\right)\,dy \tag{1}$$

Evaluate the first integral in (1). Let $u = 1+y$, then $du = dy$.

$$y=-1 \implies u=0, \qquad y=0 \implies u=1$$

$$\mathrm{I}_1 = \int_{-1}^{0}(4-y)\left(2\sqrt{1+y}\right)\,dy = 2\int_{0}^{1}(5-u)\sqrt{u}\,du = 10\int_{0}^{1}\sqrt{u}\,du - 2\int_{0}^{1}u^{3/2}\,du$$

$$= \frac{20}{3}u^{3/2}-\frac{4}{5}u^{5/2}\bigg|_{0}^{1} = \frac{20}{3}-\frac{4}{5}-0 = \frac{88}{15}$$

Evaluate the second integral in (1). Let $u = 1+y$, then $du = dy$.

$$y=0 \implies u=1, \qquad y=3 \implies u=4$$

$$\mathrm{I}_2 = \int_{0}^{3}(4-y)\left(1+\sqrt{1+y}-y\right)\,dy = \int_{1}^{4}(5-u)\left(-u+2+\sqrt{u}\right)\,du$$

$$= \int_{1}^{4}\left(-7u+10+u^2+5\sqrt{u}-u^{3/2}\right)\,du = \left[-\frac{7u^2}{2}+10u+\frac{u^3}{3}+\frac{10}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right]_{1}^{4}$$

$$= \left[\left(-56+40+\frac{64}{3}+\frac{80}{3}-\frac{64}{5}\right)-\left(-\frac{7}{2}+10+\frac{1}{3}+\frac{10}{3}-\frac{2}{5}\right)\right] = \frac{283}{30}$$

Therefore, the result is

$$2\pi\left(\frac{88}{15}+\frac{283}{30}\right) = \boxed{\frac{153\pi}{5}}$$

5. If the function $y = f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_{a}^{b} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$\dfrac{dy}{dx} = \mathrm{e}^{\mathrm{x}}$. Set $a = 0,\ b = 1,$ and then evaluate the integral.

$$S=\int_0^1 2\pi \mathrm{e}^x\sqrt{1+(\mathrm{e}^x)^2}\,dx$$

Let $u=\mathrm{e}^x$, then $du=\mathrm{e}^x\,dx$.

$$x=0 \implies u=\mathrm{e}^0=1, \qquad x=1 \implies u=\mathrm{e}^1=\mathrm{e}$$

$$S=\int_0^1 2\pi \mathrm{e}^x\sqrt{1+(\mathrm{e}^x)^2}\,dx=2\pi\int_1^{\mathrm{e}}\sqrt{1+u^2}\,du$$

We will now use a trigonometric substitution. Let $u=\tan t$ for $0<t<\dfrac{\pi}{2}$, then $du=\sec^2 t\,dt$.

$$S=2\pi\int_1^{\mathrm{e}}\sqrt{1+u^2}\,du=2\pi\int\sqrt{1+\tan^2 t}\cdot\sec^2 t\,dt=2\pi\int\sqrt{\sec^2 t}\cdot\sec^2 t\,dt$$

$$=2\pi\int|\sec t|\sec^2 t\,dt=2\pi\int\sec^3 t\,dt \qquad [\sec t>0]$$

Find the antiderivative of $\sec^3 t$ with the help of integration by parts.

$$\begin{aligned} w=\sec t \;&\rightarrow\; dw=\sec t\tan t\,dt \\ dz=\sec^2 t\,dt \;&\rightarrow\; z=\tan t \end{aligned}$$

$$\int\sec^3 t\,du=\tan t\cdot\sec t-\int\tan^2 t\sec t\,dt=\tan t\cdot\sec t-\int\frac{1-\cos^2 t}{\cos^3 t}\,dt$$

$$=\tan t\cdot\sec t-\int\sec^3 t\,dt+\int\sec t\,dt$$

Notice that the integral appears on the right side of the equation. Therefore,

$$\int\sec^3 t\,dt=\frac{1}{2}\cdot\tan t\cdot\sec t+\frac{1}{2}\cdot\int\sec t\,dt$$

The integral of $\sec t$ with respect to t is

$$\int\sec t\,dt=\ln|\tan t+\sec t|+c_1, \quad c_1\in\mathbb{R}$$

Recall $u=\tan t$.

$$u=\tan t \implies u^2=\tan^2 t=\sec^2 t-1 \implies \sec t=\sqrt{u^2+1}$$

$$S=2\pi\cdot\frac{1}{2}\left(\tan t\cdot\sec t+\ln|\tan t+\sec t|\right)+c=\pi\left[u\cdot\sqrt{u^2+1}+\ln\left|t+\sqrt{u^2+1}\right|\right]_1^{\mathrm{e}}$$

$$S = \pi \left[\left(\mathrm{e} \cdot \sqrt{\mathrm{e}^2 + 1} + \ln \left| \mathrm{e} + \sqrt{\mathrm{e}^2 + 1} \right| \right) - \left(\sqrt{2} + \ln \left| 1 + \sqrt{2} \right| \right) \right]$$

$$= \boxed{\pi \left[\mathrm{e} \cdot \sqrt{\mathrm{e}^2 + 1} - \sqrt{2} + \ln \left(\frac{\mathrm{e} + \sqrt{\mathrm{e}^2 + 1}}{1 + \sqrt{2}} \right) \right]}$$

6. Let the corresponding function be $f(x) = x\mathrm{e}^{-x}$. f is continuous for $x \geq 0$. f is positive for $x \geq 0$ because $x \geq 0$ and e^{-x} is positive everywhere. The function is also decreasing for $x \geq 1$. Verify this behavior by taking the first derivative of f. Apply the product rule.

$$f'(x) = 1 \cdot \mathrm{e}^{-x} - x\mathrm{e}^{-x} = (1 - x)\mathrm{e}^{-x}$$

$f'(x) > 0$ for $x \geq 1$. The Integral Test states that all the conditions must be satisfied for and after a specific value, for instance $x = 1$. Therefore, set the lower bound $x = 1$ and evaluate the integral. We will exclusively evaluate the first term of the sequence thereafter.

$$\int_1^\infty x\mathrm{e}^{-x}\,dx$$

Apply integration by parts and then evaluate the improper integrals by taking the limit.

$$\left.\begin{array}{r} u = x \implies du = dv \\ dv = \mathrm{e}^{-x}\,dx \implies v = -\mathrm{e}^{-x} \end{array}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int_1^\infty x\mathrm{e}^{-x}\,dx = \lim_{R\to\infty} \left[-x\mathrm{e}^{-x} \right]_1^R - \lim_{P\to\infty} \int_1^P -\mathrm{e}^{-x}\,dx = \lim_{R\to\infty} \left[-x\mathrm{e}^{-x} \right]_1^R - \lim_{P\to\infty} \mathrm{e}^{-x} \Big|_1^P$$

$$= \lim_{R\to\infty} \left(-R\mathrm{e}^{-R} + \mathrm{e}^{-1} \right) - \lim_{P\to\infty} \left(\mathrm{e}^{-P} - \mathrm{e}^{-1} \right) = \lim_{R\to\infty} \left(-R\mathrm{e}^{-R} \right) + 2\mathrm{e}^{-1}$$

To evaluate the limit, we assume that $-R\mathrm{e}^{-R}$ is a function of R. After that, put the expression in a form that we can apply L'Hôpital's rule in order to eliminate the form ∞/∞.

$$\lim_{R\to\infty} \left(-R\mathrm{e}^{-R} \right) = \lim_{R\to\infty} \frac{-R}{\frac{1}{\mathrm{e}^{-R}}} \overset{\text{L'H.}}{=} \lim_{R\to\infty} \frac{1}{\frac{1}{\mathrm{e}^{-2R}} \cdot \left(-\mathrm{e}^{-R} \right)} = \lim_{R\to\infty} \left(-\mathrm{e}^{-R} \right) = 0$$

Since the integral converges to $2\mathrm{e}^{-1}$, the series $\displaystyle\sum_{k=1}^{\infty} k\mathrm{e}^{-k}$ also converges. The first term of the series in the original question is $0 \cdot \mathrm{e}^0 = 0$. The sum of a convergent series and a finite number is still finite. Therefore, the sum $\displaystyle\sum_{k=0}^{\infty} k\mathrm{e}^{-k}$ converges.

7. The Maclaurin series of f is as follows.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Find $f(0),\ f'(0),\ f''(0),\ f'''(0), f^{(4)}(0)$ to look for the pattern.

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3} \quad f^{(4)}(x) = -\frac{6}{(1+x)^4}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2, \quad f^{(4)}(0) = -6$$

This is an alternating sequence where the coefficient of each term is the factorial of the subsequent number starting from 0 except for $k = 0$, that is, the first term of the series. At $k = 0$, the first term is 0. So,

$$f^k(0) = \begin{cases} (-1)^{k-1} \cdot (k-1)!, & \text{if } k > 0 \\ 0, & \text{if } k = 0 \end{cases}$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 + \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot (k-1)!}{k \cdot (k-1)!} x^k$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots}$$

8. Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1) \cdot (5)^{n+1}} \cdot \frac{n \cdot 5^n}{(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-2) \cdot n}{(n+1) \cdot 5} \right| = \frac{|x-2|}{5} \cdot \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = \frac{|x-2|}{5}$$

$$\frac{|x-2|}{5} < 1 \implies |x-2| < 5 \implies -5 < x-2 < 5 \implies -3 < x < 7 \quad \text{(convergent)}$$

Now, take a look at the endpoints.

$$x = -3 \implies \sum_{n=1}^{\infty} \frac{(-5)^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{n}$, is nonincreasing for $n \geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges. Try $x = 7$.

$$x = 7 \implies \sum_{n=1}^{\infty} \frac{(5)^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This is a p-series with $p = 1$, for which the series diverges by the p-series Test.

The convergence set for the power series is $\boxed{[-3, 7)}$.