2022-2023 Spring
MAT124 Midterm
(08/05/2023)

1. Sketch the traces of the following surfaces with the coordinate planes $x = 0$, $y = 0$, and $z = 0$, and then sketch the graphs of them.

(a) $\dfrac{x^2}{16} + \dfrac{y^2}{4} - z^2 = 1$

(b) $z = \dfrac{x^2}{4} + \dfrac{y^2}{4} - 6$

2. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{y^3\sqrt{x}}{2\left(x^2+y^4\right)}$$

does not exist.

3. Find the equation of the plane passing through the point $P_0(0,1,2)$ and which is perpendicular to the line that is tangent to the curve of intersection of the surfaces

$$xz^2 - 2xy + y^2 = 2 \qquad \text{and} \qquad xz - x^2y + z^2 = 1$$

at the point $P_1\left(0, \sqrt{2}, 1\right)$.

4. Find $\dfrac{\partial w}{\partial s}$ where $w = xy \ln\left(1 + \sqrt{x^2+y^2}\right) + xz$ and $x = t + s$, $y = \mathrm{e}^s$, $z = \ln\left(s^2 + t\right)$.

5. Use incremental approximation to estimate the value $\tan\left((0.97) \cdot (2.05)^2\right)$.

6. Find the direction vector in which the function

$$f(x,y,z) = \sqrt{x + yz}$$

has the minimum rate of change at the point $(1,1,3)$. Also, find this rate of change.

7. Find the absolute extrema of

$$f(x,y) = x^2 + y - xy + 4$$

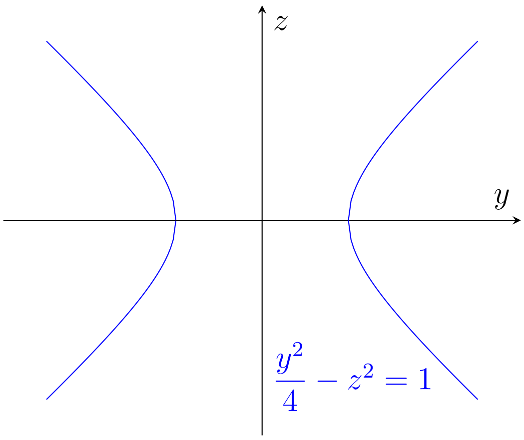
on the triangular region with vertices $(0,0)$, $(4,0)$, $(0,4)$.

2022-2023 Spring Midterm (08/05/2023) Solutions
(Last update: 8/25/25 (25th of August) 11:50 PM)

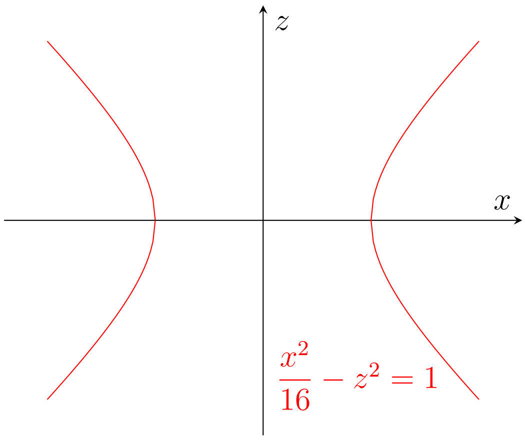
1.

(a)

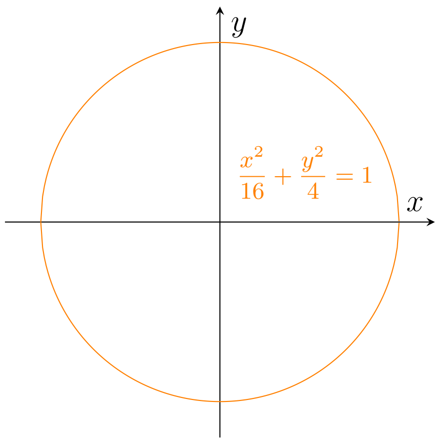
Plane $x = 0$



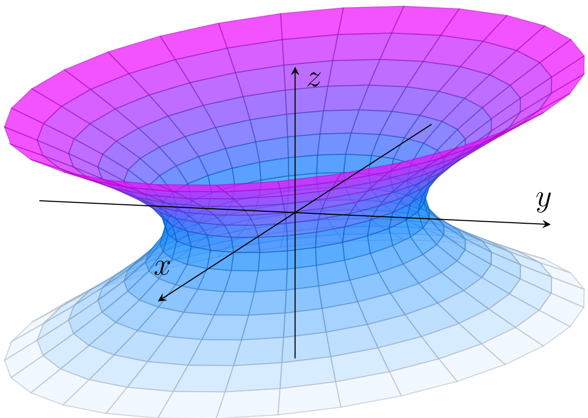

Plane $y = 0$




Plane $z = 0$

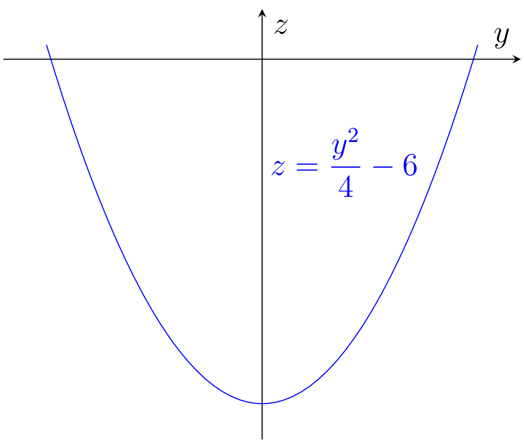



$$\frac{x^2}{16} + \frac{y^2}{4} - z^2 = 1$$

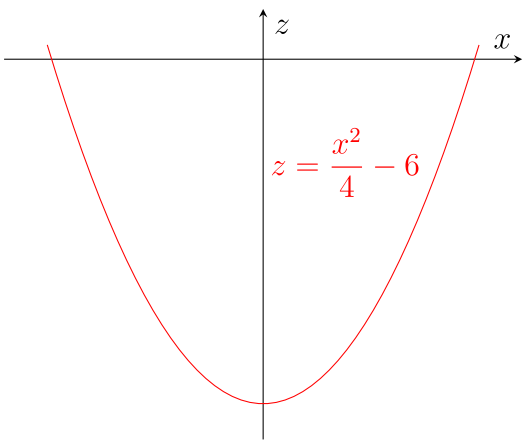


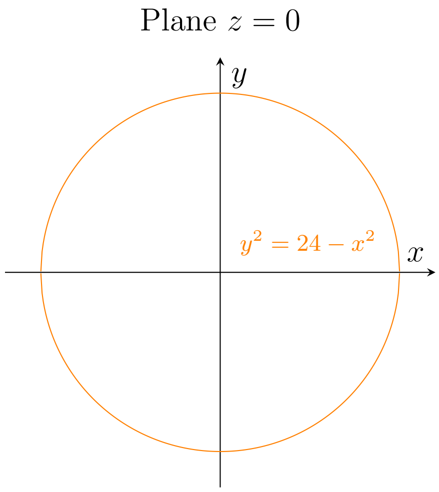

(b)

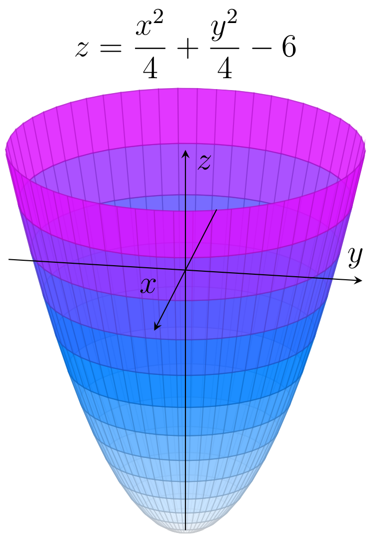
Plane $x = 0$




Plane $y = 0$






2. Apply the Two-Path Test.

$$x = y \implies \lim_{(x,y)\to(0,0)} \frac{y^3\sqrt{x}}{2\left(x^2+y^4\right)} = \lim_{x\to 0} \frac{x^{7/2}}{2\left(x^2+x^4\right)} = \lim_{x\to 0} \frac{x^{3/2}}{2\left(1+x^2\right)} = \frac{0}{2} = 0$$

$$x = y^2 \implies \lim_{(x,y)\to(0,0)} \frac{y^3\sqrt{x}}{2\left(x^2+y^4\right)} = \lim_{y\to 0} \frac{y^4}{4y^4} = \lim_{y\to 0} \frac{1}{4} = \frac{1}{4}$$

Since $0 \neq \frac{1}{4}$, by the Two-Path Test, the limit does not exist.

3. Let $f(x,y,z) = xz^2 - 2xy + y^2 = 2$ and $g(x,y,z) = xz - x^2y + z^2 = 1$ be the level surfaces. The cross product of the gradient of these functions give us the vector that is parallel to the line of intersection. Compute the gradients.

$$\nabla f = \left\langle z^2 - 2y,\ -2x + 2y,\ 2xz \right\rangle, \quad \nabla g = \left\langle z - 2xy,\ -x^2,\ x + 2z \right\rangle$$

$$\nabla f\,\big|_{(0,\sqrt{2},1)} = \left\langle 1 - 2\sqrt{2},\ 2\sqrt{2},\ 0 \right\rangle, \quad \nabla g\,\big|_{(0,\sqrt{2},1)} = \langle 1,\ 0,\ 2\rangle$$

Find the cross product.

$$\mathbf{n} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 - 2\sqrt{2} & 2\sqrt{2} & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 2\sqrt{2} & 0 \\ 0 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 - 2\sqrt{2} & 0 \\ 1 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 - 2\sqrt{2} & 2\sqrt{2} \\ 1 & 0 \end{vmatrix}$$

$$= \left(2\sqrt{2}\cdot 2 - 0\cdot 0\right)\mathbf{i} - \left[\left(1 - 2\sqrt{2}\right)\cdot 2 - 0\cdot 1\right]\mathbf{j} + \left[\left(1 - 2\sqrt{2}\right)\cdot 0 - 2\sqrt{2}\cdot 1\right)\mathbf{k}$$

$$= 4\sqrt{2}\,\mathbf{i} + \left(4\sqrt{2} - 2\right)\mathbf{j} + -2\sqrt{2}\,\mathbf{k}$$

The line of intersection is the normal line of the plane. Therefore, **n** is the normal vector of the plane. The equation of the plane with the normal vector **n** and containing the point $P(x_0, y_0, z_0)$ is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Therefore, the equation for this plane is

$$4\sqrt{2}(x - 0) + \left(4\sqrt{2} - 2\right)(y - 1) - 2\sqrt{2}(z - 2) = 0$$

$$\boxed{2x\sqrt{2} + y\left(2\sqrt{2} - 1\right) - z\sqrt{2} + 1 = 0}$$

4. Apply the chain rule.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

Compute the partial derivatives.

$$\frac{\partial w}{\partial x} = y \ln\left(1 + \sqrt{x^2 + y^2}\right) + xy \cdot \frac{1}{1 + \sqrt{x^2 + y^2}} \cdot \left(\frac{x}{\sqrt{x^2 + y^2}}\right) + z$$

$$\frac{\partial w}{\partial y} = x \ln\left(1 + \sqrt{x^2 + y^2}\right) + xy \cdot \frac{1}{1 + \sqrt{x^2 + y^2}} \cdot \left(\frac{y}{\sqrt{x^2 + y^2}}\right), \quad \frac{\partial w}{\partial z} = x$$

$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial s} = \mathrm{e}^s, \quad \frac{\partial z}{\partial s} = \frac{2s}{s^2 + t}$$

$$\begin{aligned} \frac{\partial w}{\partial s} &= \left[y \ln\left(1 + \sqrt{x^2 + y^2}\right) + xy \cdot \frac{1}{1 + \sqrt{x^2 + y^2}} \cdot \left(\frac{x}{\sqrt{x^2 + y^2}}\right) + z\right] \cdot 1 \\ &\quad + \left[x \ln\left(1 + \sqrt{x^2 + y^2}\right) + xy \cdot \frac{1}{1 + \sqrt{x^2 + y^2}} \cdot \left(\frac{y}{\sqrt{x^2 + y^2}}\right)\right] \cdot \mathrm{e}^s + \frac{2xs}{s^2 + t} \end{aligned}$$

Write in terms of $s$ and $t$ rigorously.

$$\boxed{\begin{aligned} \frac{\partial w}{\partial s} = \; & \mathrm{e}^s \ln\left(1 + \sqrt{(t+s)^2 + \mathrm{e}^{2s}}\right) + \frac{(t+s)^2 \cdot \mathrm{e}^s}{\sqrt{(t+s)^2 + \mathrm{e}^{2s}} + (t+s)^2 + \mathrm{e}^{2s}} + \ln\left(s^2 + t\right) \\ & +(t+s)\mathrm{e}^s \ln\left(1 + \sqrt{(t+s)^2 + \mathrm{e}^{2s}}\right) + \frac{(t+s) \cdot \mathrm{e}^{3s}}{\sqrt{(t+s)^2 + \mathrm{e}^{2s}} + (t+s)^2 + \mathrm{e}^{2s}} \\ & + \frac{2s(t+s)}{s^2 + t} \end{aligned}}$$

5. Let $f(x,y) = \tan(xy^2)$. The total differential of $f$ is

$$df = f_x\,dx + f_y\,dy = \sec^2\left(xy^2\right)\cdot y^2\,dx + \sec^2\left(xy^2\right)\cdot 2xy\,dy$$

Since $x = 0.97 \approx 1$ and $y = 2.05 \approx 2$, we may approximate the value of $\tan(0.97\cdot 2.05^2)$ near $\tan(1\cdot 2^2) = \tan 4$. Take $x = 1$, $y = 2$, $dx = 0.97 - 1 = -0.03$, $dy = 2.05 - 2 = 0.05$.

$$df = \sec^2 4\cdot 4\cdot(-0.03) + \sec^2 4\cdot 4\cdot(0.05) = 0.08\sec^2 4$$

Since $f(x+\Delta x,\ y+\Delta y) \approx f(x,y) + df$, the value of $\tan(0.97\cdot 2.05^2)$ is approximately

$$\boxed{\tan 4 + 0.08\sec^2 4}$$

6. The function $f$ has the minimum rate of change if the gradient vector of $f$ and the unit direction vector $\mathbf{u}$ are antiparallel. That is, they have opposite directions.

$$\nabla f = \left\langle \frac{1}{2\sqrt{x+yz}}, \frac{z}{2\sqrt{x+yz}}, \frac{y}{2\sqrt{x+yz}} \right\rangle$$

$$(\nabla f\cdot\mathbf{u})_{\min} = |\nabla f|\,|u|\cos\pi = -|\nabla f|$$

$$\nabla f|_{(1,1,3)} = \left\langle \frac{1}{4}, \frac{3}{4}, \frac{1}{4} \right\rangle \implies -|\nabla f| = -\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = -\frac{\sqrt{11}}{4}$$

Since $\mathbf{u}$ has the opposite direction to that of the gradient, we may also find the components of $\mathbf{u}$.

$$\mathbf{u} = -\frac{\nabla f}{|\nabla f|} = -\frac{\left\langle \frac{1}{4}, \frac{3}{4}, \frac{1}{4} \right\rangle}{\frac{\sqrt{11}}{4}} = \left\langle -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}, -\frac{1}{\sqrt{11}} \right\rangle$$

$$\boxed{\text{The minimum rate of change: } -\frac{\sqrt{11}}{4},\ \text{ the direction vector: } \left\langle -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}, -\frac{1}{\sqrt{11}} \right\rangle}$$

7. $f$ is continuous on a bounded and closed set on $\mathbb{R}$. By the Extreme Value Theorem, the extrema must exist in the region or on the boundary.

First, determine where $f_x = f_y = 0$ to find the critical points.

$$f_x = 2x - y, \quad f_y = 1 - x$$

$$f_x = f_y = 0 \implies 2x - y = 0, \quad 1 - x = 0 \implies x = 1, \quad y = 2$$

$$f(1,2) = 5$$

Take a look at the boundary. From $(0,0)$ to $(4,0)$, we have $y = 0$.

$$y = 0 \implies f(x,0) = x^2 + 4 \quad \to \quad \frac{d}{dx}\left(x^2 - 4\right) = 2x = 0 \implies x = 0$$

$$f(0,0) = 4$$

From $(0,0)$ to $(0,4)$, we have $x = 0$.

$$x = 0 \implies f(0,y) = y + 4 \quad \rightarrow \quad \frac{d}{dy}(y+4) = 1 \neq 0$$

From $(4,0)$ to $(0,4)$, we have $x = y$.

$$y = 4 - x \implies f(x, 4-x) = x^2 + (4-x) - x(4-x) + 4 = 2x^2 - 5x + 8$$

$$\frac{d}{dx}\left(2x^2 - 5x + 8\right) = 4x - 5 = 0 \implies x = \frac{5}{4}, \quad y = \frac{11}{4}$$

$$f\left(\frac{5}{4}, \frac{11}{4}\right) = \frac{39}{8}$$

We also have $f(0,0) = 5$, $f(4,0) = 20$, $f(0,4) = 8$ from the vertices of the triangular region.

Compare all the values $f(0,0)$, $f(1,2)$, $f(4,0)$, $f(0,4)$, $f\left(\frac{5}{4}, \frac{11}{4}\right)$.

$$\boxed{\text{Absolute minimum: } f(0,0) = 4, \quad \text{absolute maximum: } f(4,0) = 20.}$$