

2020-2021 Fall  
MAT123 Midterm  
(30/11/2020)

1) Evaluate  $\lim_{x \rightarrow 0^+} (\sqrt{x})^{\ln(x+1)}$ .

2) Show that the function  $f(x)$  defined by

$$f(x) = \begin{cases} x \arctan \frac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ \frac{x - \cos x}{x^2}, & \text{if } x < 0 \end{cases}$$

is not continuous at the point  $x = 0$ .

3) Find an equation of the line which is tangent to the curve

$$\cos y^2 + xy + 1 = 0$$

at the point  $\left(\sqrt{\frac{2}{\pi}}, -\sqrt{\frac{\pi}{2}}\right)$ . Note that  $y = f(x)$ .

4) A block of ice in the shape of a cube originally having volume  $3000 \text{ cm}^3$ . When it is melting, the surface area is decreasing at the rate of  $36 \text{ cm}^2/\text{h}$ . At what rate does the length of each of its edges decrease at the time its volume is  $216 \text{ cm}^3$ ? Assume that during melting, the block of ice maintains its cubical shape.

5) a) Using the Intermediate Value Theorem and Rolle's theorem, show that the equation  $e^x + x = 0$  has only one root (Note that if this root  $c_1$ , then  $c_1 \in (-1, 0)$ ).

b) Determine the interval of increase, decrease, and concavity of the function  $f(x) = e^x + x$ . By constructing a table, sketch the graph.

6) Determine (but do not evaluate) the integral corresponding to the area of the region bounded by the curves  $y = -x^2 + 1$  and  $y = |x| - 1$

7) Evaluate  $\int x \ln x \, dx$ .

1) Let  $L$  be the value of the limit. Since the expression is continuous for  $x > 0$ , we can apply the logarithm function to each side of the equation. Then, we can swap the logarithm and the limit. Use the property of logarithms afterwards.

$$L = \lim_{x \rightarrow 0^+} (\sqrt{x})^{\ln(x+1)}$$

$$\ln(L) = \ln \left[ \lim_{x \rightarrow 0^+} (\sqrt{x})^{\ln(x+1)} \right] = \lim_{x \rightarrow 0^+} \ln \left[ (\sqrt{x})^{\ln(x+1)} \right] = \lim_{x \rightarrow 0^+} \ln \left[ (\sqrt{x})^{\ln(x+1)} \right]$$

$$\ln(L) = \lim_{x \rightarrow 0^+} [\ln(x+1) \cdot \ln(\sqrt{x})] \quad [0 \cdot \infty]$$

Make it so the limit is in the form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  in order to apply the L'Hôpital's rule.

$$\ln(L) = \lim_{x \rightarrow 0^+} \left[ \frac{\ln(\sqrt{x})}{\frac{1}{\ln(x+1)}} \right] \quad \left[ \frac{\infty}{\infty} \right]$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 0^+} \left[ \frac{\frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}{-\frac{1}{\ln^2(x+1)} \cdot \frac{1}{x+1}} \right] = \lim_{x \rightarrow 0^+} \left[ -\frac{\ln^2(x+1) \cdot (x+1)}{2x} \right]$$

$$= \lim_{x \rightarrow 0^+} \left[ -\frac{\ln^2(x+1)}{2x} \right] \cdot \lim_{x \rightarrow 0^+} (x+1) = \lim_{x \rightarrow 0^+} \left[ -\frac{\ln^2(x+1)}{2x} \right] \quad \left[ \frac{0}{0} \right]$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 0^+} \left[ -\frac{2 \ln(x+1) \cdot \frac{1}{x+1}}{2} \right] = \lim_{x \rightarrow 0^+} \left[ \frac{\ln(x+1)}{x+1} \right] = \frac{\ln 1}{1} = 0$$

$\ln(L) = 0$ , so  $\boxed{L = 1}$ .

2)  $\arctan \frac{1}{x}$  takes its values on  $-\frac{\pi}{2} \leq \arctan \frac{1}{x} \leq \frac{\pi}{2}$ . Multiply each side by  $x$ , then we get  $-\frac{x\pi}{2} \leq x \arctan \frac{1}{x} \leq \frac{x\pi}{2}$ . Take the limits of each side. By the squeeze theorem, we see that the limit of  $x \arctan \frac{1}{x}$  at the point  $x = 0$  is 0. This means that for  $f(x)$ , the limit from the right side also equals 0.

$$\begin{aligned} \lim_{x \rightarrow 0} \left( -\frac{x\pi}{2} \right) &\leq \lim_{x \rightarrow 0} \left( x \arctan \frac{1}{x} \right) \leq \lim_{x \rightarrow 0} \left( \frac{x\pi}{2} \right) \\ 0 &\leq \lim_{x \rightarrow 0} \left( x \arctan \frac{1}{x} \right) \leq 0 \\ \therefore \lim_{x \rightarrow 0} \left( x \arctan \frac{1}{x} \right) &= 0 \end{aligned}$$

From the left side, the limit is equal to as follows.

$$\lim_{x \rightarrow 0^-} \frac{x - \cos x}{x^2} = \lim_{x \rightarrow 0^-} (x - \cos x) \cdot \lim_{x \rightarrow 0^-} \frac{1}{x^2} = -\infty$$

Continuity requires the equality of one-sided limits and the value of the function at that point. However, the one-sided limits are not equal;  $0 \neq -\infty$ . Therefore,  $f(x)$  is discontinuous at  $x = 0$ .

3) Differentiate both sides implicitly.

$$\begin{aligned} \frac{d}{dx} (\cos y^2 + xy + 1) &= \frac{d}{dx} 0 \\ -\sin y^2 \cdot 2y \frac{dy}{dx} + y + x \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} (-\sin y^2 \cdot 2y + x) &= -y \\ \frac{dy}{dx} &= \frac{y}{\sin y^2 \cdot 2y - x} \end{aligned} \quad (1)$$

Evaluate  $\frac{dy}{dx}$  at the point.

$$\left. \frac{dy}{dx} \right|_{(\sqrt{\frac{2}{\pi}}, -\sqrt{\frac{\pi}{2}})} = \frac{y}{\sin y^2 \cdot 2y - x} = \frac{-\sqrt{\frac{\pi}{2}}}{\sin \left( (-\sqrt{\frac{\pi}{2}})^2 \right) \cdot 2 \left( -\sqrt{\frac{\pi}{2}} \right) - \sqrt{\frac{2}{\pi}}} = \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{2\pi} + \sqrt{\frac{2}{\pi}}} \quad (2)$$

Recall:  $y - y_0 = m(x - x_0)$ , where  $m$  is the slope. Substitute  $m$  with (2) and find the tangent line.

$$\boxed{y + \sqrt{\frac{\pi}{2}} = \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{2\pi} + \sqrt{\frac{2}{\pi}}} \left( x - \sqrt{\frac{2}{\pi}} \right)}$$

4) Let  $S(t)$ ,  $V(t)$ ,  $a(t)$  represent the surface area, the volume, and the length of one side of the object, respectively, as a function of time. We may write the following.

$$S(t) = 6a^2(t), \quad V(t) = a^3(t)$$

Given that at  $t = t_0$ ,  $V(t_0) = 216$ ,  $S'(t_0) = -36$ . Using the relationship with the sides,

$$\begin{aligned} V(t_0) &= a^3(t_0) = 216 \rightarrow a(t_0) = 6 \\ S'(t_0) &= 12a(t_0)a'(t_0) = -36 \\ \therefore 12 \cdot 6 \cdot a'(t_0) &= -36 \rightarrow a'(t_0) = -\frac{1}{2} \end{aligned}$$

$$\boxed{a'(t_0) = -\frac{1}{2} \text{ cm/h}}$$

5)

a) Let  $f(x) = e^x + x$ .  $f$  is continuous and differentiable for all  $x \in \mathbb{R}$ .

$$f(-1) = e^{-1} - 1 = \frac{1}{e} - 1, \quad f(0) = e - 0 = e$$

Since  $f(-1) < 0$  and  $f(0) > 0$  and  $f$  is continuous on the interval  $[-1, 0]$ , by IVT, there is at least one point  $x_1$  that satisfies  $f(x_1) = 0$ . Assume that there is another distinct root  $x_2$ . Rolle's theorem states that if  $f$  is continuous on a particular interval with endpoints having the same function value, there exists a point  $c$  on that interval such that  $f'(c) = 0$  there.

$$f'(c) = e^c + 1 \geq 1 \quad [e^c > 0]$$

This yields a contradiction. Therefore, there is *only* one root.

b) The expression is defined  $\forall x \in \mathbb{R}$ . Let us find the limit at infinity and the limit at negative infinity.

$$\lim_{x \rightarrow \infty} (e^x + x) = \infty \quad \lim_{x \rightarrow -\infty} (e^x + x) = -\infty$$

There are no vertical or horizontal asymptotes. However, there is a slant asymptote. Attempt a long polynomial division and we will find that the slant asymptote is  $y = x$ . Verify with the following limit:

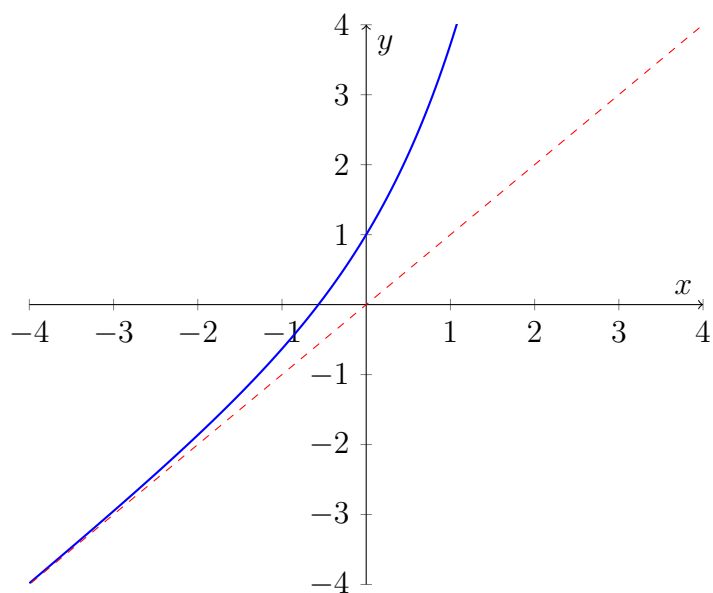
$$\lim_{x \rightarrow -\infty} [(e^x + x) - x] = \lim_{x \rightarrow -\infty} e^x = 0$$

Take the first and second derivatives.

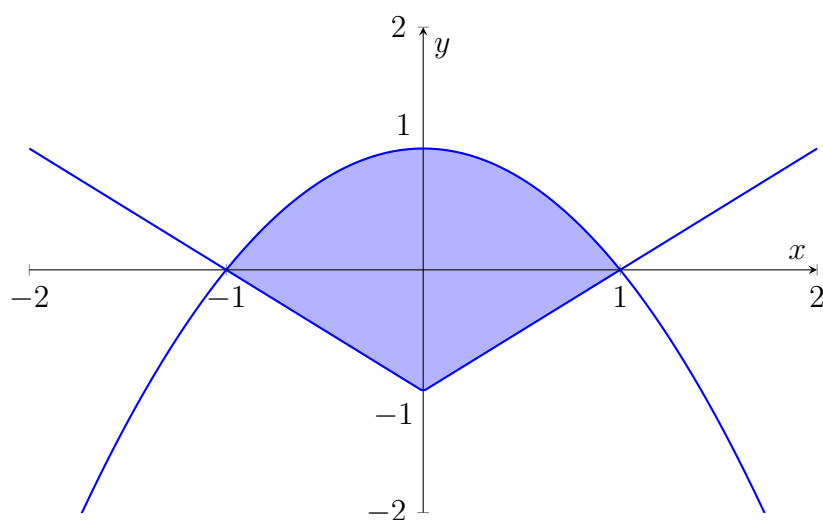
$$y' = e^x + 1, \quad y'' = e^x$$

We see that there are no critical or inflection points either. Now, set up a table and see what the graph looks like.

$x$	$(-\infty, \infty)$
$y$	$(-\infty, \infty)$
$y'$ sign	+
$y''$ sign	+



6)



The area of the region is as follows.

$$I = \int_{-1}^1 [(-x^2 + 1) - (|x| - 1)] dx = \int_{-1}^0 (-x^2 + x + 2) dx + \int_0^1 (-x^2 - x + 2) dx$$

7) We'll use integration by parts.

$$\left. \begin{array}{l} \ln x = u \rightarrow \frac{1}{x} dx = du \\ x dx = dv \rightarrow \frac{x^2}{2} = v \end{array} \right\} \quad \begin{aligned} I &= \int x \ln x dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\ &= \frac{x^2}{2} \cdot \ln x - \int \frac{x}{2} dx = \boxed{\frac{x^2}{2} \cdot \ln x - \frac{x^2}{4} + c, c \in \mathbb{R}} \end{aligned}$$