

2024-2025 Fall  
MAT123-02,05 Final  
(09/01/2025)

1. Evaluate the following integrals.

(a)  $\int \frac{dx}{2 \cos x + 3}$

(b)  $\int \frac{\sqrt{5-x^2}}{x} dx$

(c)  $\int \frac{dx}{1-x^4}$

(d)  $\int \ln(1+x^2) dx$

2. Consider the region  $R$  bounded by the curves  $y = x$ ,  $y = x^2$  and  $y = x^2/2$ .

(a) Find the area of the region  $R$ .

(b) Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region  $R$  about the  $y$ -axis.

(c) Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of the solid obtained by rotating the region  $R$  about the line  $y = 2$ .

3. Use the Integral Test to determine the existence of the sum of the series  $\sum_{n=1}^{\infty} ne^{-n^2}$ .

4. Determine whether the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n}\right)^{1/n}$  is convergent or divergent.

5. Find the Maclaurin series of  $f(x) = \sqrt{e^x}$  and determine the interval of convergence of this series.

1.

(a) Use the tangent half-angle substitution, which is also called the Weierstrass substitution. Let  $t = \tan\left(\frac{x}{2}\right)$ . Then

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2} dt$$

$$\int \frac{dx}{2\cos x + 3} = \int \frac{1}{2\left(\frac{1-t^2}{1+t^2}\right) + 3} \cdot \frac{2}{1+t^2} dt = \int \frac{2}{5+t^2} dt = \frac{2}{5} \int \frac{1}{\left(1 + \left(\frac{t}{\sqrt{5}}\right)^2\right)} dt$$

Let  $u = \frac{t}{\sqrt{5}}$ , then  $\sqrt{5} du = dt$ .

$$\begin{aligned} \frac{2}{5} \int \frac{1}{\left(1 + \left(\frac{t}{\sqrt{5}}\right)^2\right)} dt &= \frac{2\sqrt{5}}{5} \int \frac{1}{1+u^2} du = \frac{2\sqrt{5}}{5} \arctan u + c = \frac{2\sqrt{5}}{5} \arctan \frac{t}{\sqrt{5}} + c \\ &= \boxed{\frac{2\sqrt{5}}{5} \arctan \left( \frac{1}{\sqrt{5}} \cdot \tan \left( \frac{x}{2} \right) \right) + c, \quad c \in \mathbb{R}} \end{aligned}$$

(b) Let  $x = \sqrt{5} \sin u$  for  $-\frac{\pi}{2} \leq u < \frac{\pi}{2}$ , then  $dx = \sqrt{5} \cos u du$ .

$$\begin{aligned} I &= \int \frac{\sqrt{5-x^2}}{x} dx = \int \frac{\sqrt{5-5\sin^2 u}}{\sqrt{5} \sin u} \cdot \sqrt{5} \cos u du \quad [\sin^2 u + \cos^2 u = 1] \\ &= \sqrt{5} \int \sqrt{\cos^2 u} \cot u du \quad [\cos u > 0] \\ &= \sqrt{5} \int \cos u \cdot \cot u du = \sqrt{5} \int \frac{\cos^2 u}{\sin u} du = \sqrt{5} \int \frac{1 - \sin^2 u}{\sin u} du \\ &= \sqrt{5} \int (\csc u - \sin u) du = \sqrt{5} (-\ln |\cot u + \csc u| + \cos u) + c \end{aligned}$$

Recall that  $x = \sqrt{5} \sin u$ .

$$\begin{aligned} \sin u = \frac{x}{\sqrt{5}} &\implies \sin^2 u = \frac{x^2}{5} \implies \cos^2 u = \frac{5-x^2}{5} \implies \cos u = \frac{\sqrt{5-x^2}}{\sqrt{5}} \\ \cot u &= \frac{\cos u}{\sin u} = \frac{\sqrt{5-x^2}}{x}, \quad \csc u = \frac{1}{\sin u} = \frac{\sqrt{5}}{x} \end{aligned}$$

Therefore,

$$I = \boxed{-\sqrt{5} \ln \left| \frac{\sqrt{5-x^2}}{x} + \frac{\sqrt{5}}{x} \right| + \sqrt{5-x^2} + c, \quad c \in \mathbb{R}}$$

(c) Use partial fractions to compute the integral.

$$\begin{aligned} I &= \int \frac{1}{1-x^4} dx = \int \frac{1}{(1-x^2)(1+x^2)} dx = \int \frac{1}{(1-x)(1+x)(1+x^2)} dx \\ &= \int \left( \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2} \right) dx \end{aligned}$$

$$\begin{aligned} A(1+x)(1+x^2) + B(1-x)(1+x^2) + (Cx+D)(1+x)(1-x) &= 1 \\ x^3(A-B-C) + x^2(A+B-D) + x(A-B+C) + A+B+D &= 1 \end{aligned}$$

Equate the coefficients of like terms.

$$\left. \begin{array}{l} A-B-C=0 \quad (1) \\ A+B-D=0 \quad (2) \\ A-B+C=0 \quad (3) \\ A+B+D=1 \quad (4) \end{array} \right\} \begin{array}{l} (1) \& (3) \rightarrow 2A-2B=0 \\ (2) \& (4) \rightarrow 2A+2B=1 \\ \therefore A=\frac{1}{4}, \quad B=\frac{1}{4} \\ (1) \rightarrow C=0, \quad (2) \rightarrow D=\frac{1}{2} \end{array}$$

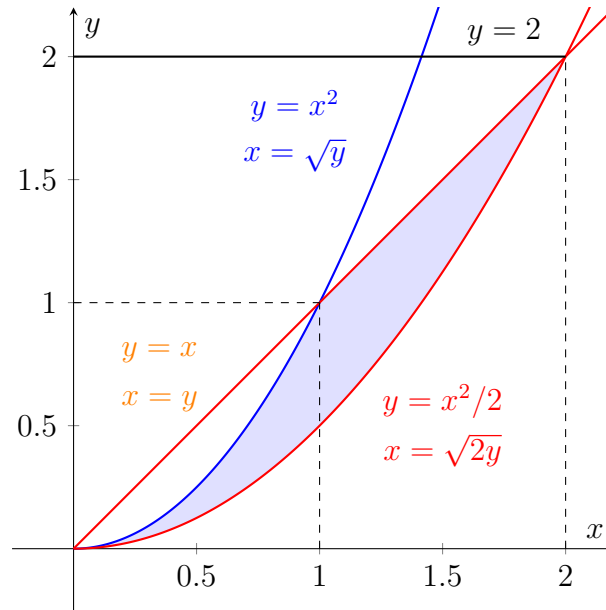
Rewrite the integral.

$$\begin{aligned} I &= \int \left( \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2} \right) dx = \int \left( \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)} \right) dx \\ &= \boxed{-\frac{1}{4} \ln |1-x| + \frac{1}{4} \ln |1+x| + \frac{1}{2} \arctan(x) + c, \quad c \in \mathbb{R}} \end{aligned}$$

(d) Use the method of integration by parts.

$$\begin{aligned} \left. \begin{array}{l} u = \ln(1+x^2) \implies du = \frac{2x}{1+x^2} dx \\ dv = dx \implies v = x \end{array} \right\} \rightarrow \int u dv = uv - \int v du \\ \int \ln(1+x^2) dx = x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx = x \ln(1+x^2) - \int \frac{2x^2+2-2}{1+x^2} dx \\ = x \ln(1+x^2) - 2 \int dx + 2 \int \frac{dx}{1+x^2} \\ = \boxed{x \ln(1+x^2) - 2x + 2 \arctan x + c, \quad c \in \mathbb{R}} \end{aligned}$$

2.



(a)

$$A = \int_0^1 \left( x^2 - \frac{x^2}{2} \right) dx + \int_1^2 \left( x - \frac{x^2}{2} \right) dx = \left[ \frac{x^3}{3} - \frac{x^3}{6} \right]_0^1 + \left[ \frac{x^2}{2} - \frac{x^3}{6} \right]_1^2$$

$$= \left[ \left( \frac{1}{3} - \frac{1}{6} \right) - 0 \right] + \left[ \left( 2 - \frac{8}{6} \right) - \left( \frac{1}{2} - \frac{1}{6} \right) \right] = \boxed{\frac{1}{2}}$$

(b)

$$V = \int_D \pi [r_2^2(y) - r_1^2(y)] dy = \boxed{\int_0^1 \pi \left[ (\sqrt{2y})^2 - (\sqrt{y})^2 \right] dy + \int_1^2 \pi \left[ (\sqrt{2y})^2 - y^2 \right] dy}$$

(c)

$$V = \int_D 2\pi \cdot h(y) \cdot r(y) dy = \boxed{\int_0^1 2\pi(2-y) (\sqrt{2y} - \sqrt{y}) dy + \int_1^2 2\pi(2-y) (\sqrt{2y} - y) dy}$$

3. Take  $f(x) = xe^{-x^2}$ .  $f$  is continuous because the product of a polynomial and an exponential expression is still continuous.  $f$  is positive and decreasing for  $x \geq 1$ . Verify the monotonicity of  $f$  by taking the first derivative.

$$\frac{df}{dx} = 1 \cdot e^{-x^2} + xe^{-x^2} \cdot (-2x) = e^{-x^2} (1 - 2x^2)$$

$$f'(x) < 0 \quad \text{for} \quad x > \frac{\sqrt{2}}{2} \implies f'(x) < 0 \quad \text{for} \quad x \geq 1$$

We may now apply the Integral Test. Take the limit for the improper integral.

$$\int_1^\infty xe^{-x^2} dx = \lim_{R \rightarrow \infty} \int_1^R xe^{-x^2} dx = \lim_{R \rightarrow \infty} \left. -\frac{1}{2}e^{-x^2} \right|_1^R = \lim_{R \rightarrow \infty} -\frac{1}{2} (e^{-R^2} - e^{-1}) = \frac{1}{2e}$$

The integral converges. Then the series  $\sum_{n=1}^{\infty} ne^{-n^2}$  also converges.

4. Apply the  $n$ th Term Test for the non-alternating part. Let  $L$  be the value of the limit.

$$L = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \right)^{1/n} \implies \ln(L) = \ln \left[ \lim_{n \rightarrow \infty} \left( \frac{1}{n} \right)^{1/n} \right]$$

We can also take the limit inside the function because  $\ln$  is continuous on its domain.

$$\ln(L) = \lim_{n \rightarrow \infty} \ln \left[ \left( \frac{1}{n} \right)^{1/n} \right] = \lim_{n \rightarrow \infty} \frac{\ln \left( \frac{1}{n} \right)}{n}$$

To be able to use L'Hôpital's rule, take the corresponding function  $f(x) = \frac{\ln \left( \frac{1}{x} \right)}{x}$ .

$$\lim_{x \rightarrow \infty} \frac{\ln \left( \frac{1}{x} \right)}{x} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{x \cdot \left( -\frac{1}{x^2} \right)}{1} = \lim_{x \rightarrow \infty} -\frac{1}{x} = 0 \implies \ln(L) = 0 \implies L = 1$$

The limit  $\lim_{n \rightarrow \infty} (-1)^{n+1}$  does not exist because of the oscillation. So,  $\lim_{n \rightarrow \infty} (-1)^{n+1} \left( \frac{1}{n} \right)^{1/n}$  does not exist as well. Therefore, by the  $n$ th Term Test for divergence, the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{n} \right)^{1/n}$  is divergent.

5. The Maclaurin series of  $f$  is given by  $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ .

Find  $f(0)$ ,  $f'(0)$ ,  $f''(0)$ ,  $f'''(0)$  to look for the pattern.

$$f'(x) = \frac{1}{2\sqrt{e^x}} \cdot e^x = \frac{1}{2}\sqrt{e^x}, \quad f''(x) = \frac{1}{4\sqrt{e^x}} \cdot e^x = \frac{1}{4}\sqrt{e^x}, \quad f'''(x) = \frac{1}{8\sqrt{e^x}} \cdot e^x = \frac{1}{8}\sqrt{e^x}$$

$$f(0) = 1, \quad f'(0) = \frac{1}{2}, \quad f''(0) = \frac{1}{4}, \quad f'''(0) = \frac{1}{8}$$

Therefore,  $f^{(k)}(0) = \left( \frac{1}{2} \right)^k$ . Rewrite the summation formula.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{\left( \frac{1}{2} \right)^k \cdot x^k}{k!} = \boxed{\sum_{k=0}^{\infty} \frac{x^k}{2^k \cdot k!} = 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \dots}$$

Find the interval of convergence by applying the Ratio Test.

$$\lim_{k \rightarrow \infty} \left| \frac{x^{k+1}}{2^{k+1} \cdot (k+1)!} \cdot \frac{2^k \cdot k!}{x^k} \right| = \lim_{k \rightarrow \infty} \frac{x}{2k} = 0 < 1$$

The series is convergent on  $\mathbb{R}$ .