

1.

- (a) Find the critical points of the function

$$f(x, y) = 4x^3 - 6xy + y^2 + 2y$$

and classify them.

- (b) Find the maximum and minimum values of the function

$$f(x, y, z) = x + y + z$$

by using Lagrange multipliers on the ellipsoid $x^2 + 4y^2 + 9z^2 = 1764$.

2.

- (a) Sketch the domain of integration and rewrite the integral by changing the order of integration.

$$\int_0^1 \int_0^{x\sqrt{3}} e^{-x^2-y^2} dy dx + \int_1^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2} dy dx$$

- (b) Evaluate the integral

$$\iint_R \frac{xy}{1+x^4} dA$$

where R is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.

3. The following integral gives the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and between the planes $z = 0$ and $z = 1$.

$$V = 4 \int_0^{\pi/2} \int_0^{\sqrt{3}} \int_0^1 r dz dr d\theta + 4 \int_0^{\pi/2} \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-r^2}} r dz dr d\theta$$

- (a) Write the integral in rectangular coordinates with the order of integration $dz dy dx$.

- (b) Write the integral in spherical coordinates.

4.

- (a) Is $\mathbf{F}(x, y) = 2xy \sin(x^2y) \mathbf{i} - x^2 \sin(x^2y) \mathbf{j}$ conservative? Why?

- (b) Show that

$$\mathbf{F}(x, y, z) = (2x + y^2 + z \cos x) \mathbf{i} + (2xy + e^z) \mathbf{j} + (1 + ye^z + \sin x) \mathbf{k}$$

is conservative.

- (c) Find its potential function.

5. $\mathbf{F}(x, y, z) = (2x + y^2 + z \cos x) \mathbf{i} + (2xy + e^z) \mathbf{j} + (1 + ye^z + \sin x) \mathbf{k}$

(a) Let C be the curve of intersection of the cone $z^2 = 4x^2 + 9y^2$ and the plane $z = 1+x+2y$, and let D be the part of the curve C that lies in the first octant $x \geq 0, y \geq 0, z \geq 0$ from $(1, 0, 2)$ to $(0, 1, 3)$. Evaluate $\int_D \mathbf{F} \cdot d\mathbf{r}$.

(b) Let C be the curve of intersection of $x^2 + y^2 = 1$ and $z = 40$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

6. Evaluate

$$\oint_C \left(x^3 \sin(\sqrt{x^2 + 4}) - xe^{x+2y} \right) dx + \left(\cos(y^3 + y) - 4ye^{x+2y} \right) dy$$

where C is the counterclockwise boundary of the parallelogram with vertices $(2, 0)$, $(0, -1)$, $(-2, 0)$, and $(0, 1)$.

2023-2024 Fall Resist (29/01/2024) Solutions
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1.

(a) To find the critical points of f , determine where both $f_x = f_y = 0$ or one of the partial derivatives does not exist.

$$f_x = 12x^2 - 6y, \quad f_y = -6x + 2y + 2$$

$$f_x = 0 \implies y = 2x^2, \quad f_y = 0 \implies y = 3x - 1$$

$$f_x = f_y = 0 \implies 2x^2 - 3x + 1 = 0 \implies x_{1,2} = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{3 \pm 1}{4}$$

$$x = \frac{1}{2} \implies y = 2 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{2}, \quad x = 1 \implies y = 2 \cdot (1)^2 = 2$$

The critical points occur at $(1/2, 1/2)$ and $(1, 2)$. To classify these points, apply the second derivative test.

$$f_{xx} = 24x, \quad f_{xy} = f_{yx} = -6, \quad f_{yy} = 2$$

Calculate the Hessian determinant at these points.

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

$$(1/2, 1/2) \rightarrow \begin{aligned} f_{xx} &= 12, & f_{xy} = f_{yx} &= -6, & f_{yy} &= 2 \\ f_{xx}f_{yy} - f_{xy}^2 &= 12 \cdot 2 - (-6)^2 = -12 < 0 \end{aligned}$$

$$(1, 2) \rightarrow \begin{aligned} f_{xx} &= 24, & f_{xy} = f_{yx} &= -6, & f_{yy} &= 2 \\ f_{xx}f_{yy} - f_{xy}^2 &= 24 \cdot 2 - (-6)^2 = 12 > 0, & f_{xx} &> 0 \end{aligned}$$

A local min occurs at $(1, 2)$ and a saddle point occurs at $(1/2, 1/2)$.

(b) Let $g(x, y, z) = x^2 + 4y^2 + 9z^2 - 1764$ be the constraint. Then solve the system of equations below.

$$\left. \begin{array}{l} \nabla f = \lambda \nabla g \\ g(x, y, z) = 0 \end{array} \right\} \quad \begin{aligned} \nabla f &= \langle 1, 1, 1 \rangle = \lambda \langle 2x, 8y, 18z \rangle = \lambda \nabla g \\ \therefore x &= \frac{1}{2\lambda}, \quad y = \frac{1}{8\lambda}, \quad z = \frac{1}{18\lambda} \end{aligned}$$

Use the constraint.

$$x^2 + 4y^2 + 9z^2 - 1764 = 0 \implies \left(\frac{1}{2\lambda}\right)^2 + 4\left(\frac{1}{8\lambda}\right)^2 + 9\left(\frac{1}{18\lambda}\right)^2 = 1764$$

$$\implies \frac{49}{144\lambda^2} = 42^2 \implies \lambda = \pm \frac{1}{72}$$

$$\lambda = \pm \frac{1}{72} \implies x = \pm 36, \quad y = \pm 9, \quad z = \pm 4$$

Find the minimum and maximum values using the points $(-36, -9, -4)$ and $(36, 9, 4)$, respectively.

$$f_{\min} = -36 - 9 - 4 = -49, \quad f_{\max} = 36 + 9 + 4 = 49$$

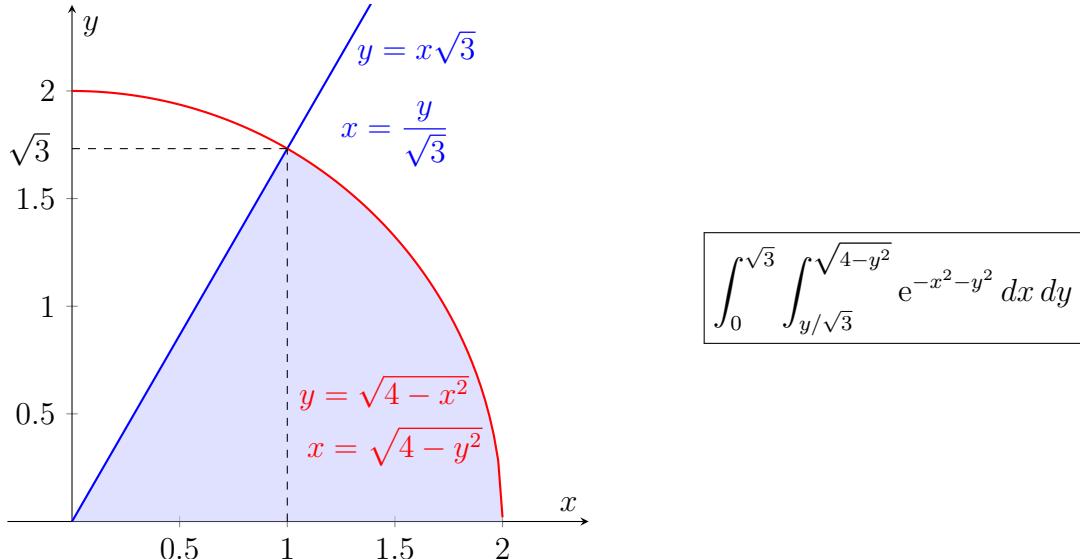
Compare all the values.

$$f(0, y, z) = f(x, 0, z) = f(x, y, 0) = 0, \quad f\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} \cdot \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{64}$$

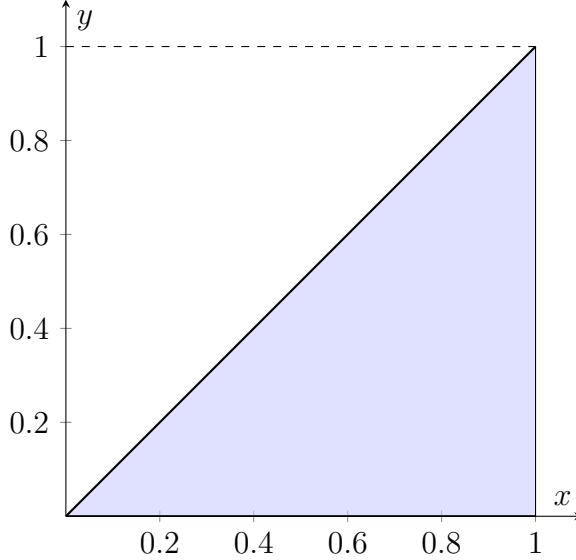
The maximum value is $\frac{1}{64}$, the minimum value is 0.

2.

(a)



(b) Sketch the region.



$$\begin{aligned}
\iint_R \frac{xy}{1+x^4} dA &= \int_0^1 \int_0^x \frac{xy}{1+x^4} dy dx = \int_0^1 \frac{x}{1+x^4} \left[\frac{y^2}{2} \right]_{y=0}^{y=x} dx = \frac{1}{2} \int_0^1 \frac{x^3}{1+x^4} dx \\
&= \frac{1}{2} \cdot \left[\frac{1}{4} \ln |1+x^4| \right]_0^1 = \frac{1}{2} \cdot \frac{1}{4} (\ln 2 - \ln 1) = \boxed{\frac{1}{8} \ln 2}
\end{aligned}$$

3.

(a)

$$\begin{aligned}
z &= z \\
x &= r \cos \theta \\
y &= r \sin \theta \\
r^2 &= x^2 + y^2 \\
dV &= r dz dr d\theta = dz dy dx
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
z &= \sqrt{4-r^2} \implies z = \sqrt{4-x^2-y^2} \\
z &= 1
\end{aligned}$$

Notice that we have two distinct upper bounds for z , which are $z = \sqrt{4-x^2-y^2}$ and $z = 1$. The lower bound for z is $z = 0$. For the upper bounds of z , we choose the minimum of the bounds. If we project the shape onto the xy -plane, we get $x^2 + y^2 = 4$. Rewrite the integral in rectangular coordinates.

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\min(1, \sqrt{4-x^2-y^2})} dz dy dx$$

Another method is to calculate the volume of the corresponding hemisphere and then extract the upper part of the hemisphere. The volume of a hemisphere is given by the formula

$$V_{\text{hemisphere}} = \frac{2}{3} \pi r^3,$$

where r is the radius. Now, focus on the upper part of the hemisphere. The upper bound for z is the sphere $x^2 + y^2 + z^2 = 4$, and the lower bound is $z = 1$. The solid lies above $x^2 + y^2 = 3$. The equivalent form of the answer above is as follows.

$$V = \frac{16\pi}{3} - \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dz dy dx$$

(b) For spherical coordinates, we have

$$\begin{aligned} z &= \rho \cos \phi & z &= \sqrt{4 - r^2} \implies \rho \cos \phi = \sqrt{4 - \rho^2 \sin^2 \phi} \implies \rho = 2 \\ r &= \rho \sin \phi & & \\ x^2 + y^2 + z^2 &= \rho^2 & \rightarrow & \\ dV &= \rho^2 \sin \phi d\rho d\phi d\theta & z = 1 & \implies \rho \cos \phi = 1 \implies \rho = \frac{1}{\cos \phi} \end{aligned}$$

For ρ , we have two different upper bounds. We choose the minimum of these bounds.

$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\min(2, \frac{1}{\cos \phi})} \rho^2 \sin \phi d\rho d\phi d\theta$$

Alternatively, we may find the angle of intersection of the plane $z = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

$$\frac{1}{\cos \phi} = 2 \implies \cos \phi = \frac{1}{2} \implies \phi = \frac{\pi}{3}$$

From $\phi = 0$ to $\phi = \frac{\pi}{3}$, the upper bound for ρ is $\rho = \frac{1}{\cos \phi}$. From $\phi = \frac{\pi}{3}$ to $\phi = \frac{\pi}{2}$, it is $\rho = 2$. The equivalent integral is as follows.

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{1/\cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta + \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^2 \sin \phi d\rho d\phi d\theta$$

4.

(a) For \mathbf{F} to be conservative, it must be the gradient of some potential function ϕ . We may apply the component test to determine whether the mixed partial derivatives are equal.

$$\left. \begin{aligned} \frac{\partial F_1}{\partial y} &= 2x \sin(x^2y) + 2xy \cos(x^2y) \cdot x^2 \\ \frac{\partial F_2}{\partial x} &= -2x \sin(x^2y) - x^2 \cos(x^2y) \cdot 2xy \end{aligned} \right\} \implies \frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$$

The mixed partial derivatives are not equal. Therefore, the force is not conservative.

(b) Like what we did above, determine the mixed partial derivatives.

$$\frac{\partial F_1}{\partial y} = 2y = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \cos x = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = e^z = \frac{\partial F_3}{\partial y}$$

(c) Since \mathbf{F} is conservative on \mathbb{R}^3 , there exists a potential function f such that $\nabla f = \mathbf{F}$.

$$\frac{\partial f}{\partial x} = 2x + y^2 + z \cos x, \quad \frac{\partial f}{\partial y} = 2xy + e^z, \quad \frac{\partial f}{\partial z} = 1 + ye^z + \sin x$$

$$\int \frac{\partial f}{\partial x} dx = \int (2x + y^2 + z \cos x) dx = x^2 + xy^2 + z \sin x + g(y, z) = f(x, y, z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2 + xy^2 + z \sin x + g(y, z)) = 2xy + g_y(y, z) = 2xy + e^z \implies g_y(y, z) = e^z$$

$$\int \frac{\partial f}{\partial y} dy = \int (2xy + e^z) dy = x^2 + z \sin x + xy^2 + ye^z + h(z) = f(x, y, z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} (x^2 + z \sin x + xy^2 + e^z + h(z)) = \sin x + ye^z + h_z(z)$$

$$= 1 + ye^z + \sin x \implies h_z(z) = 1$$

$$\int \frac{\partial f}{\partial z} dz = \int (1 + ye^z + \sin x) dz = x^2 + z \sin x + xy^2 + ye^z + z + c = f(x, y, z)$$

The potential function for \mathbf{F} is

$$f(x, y, z) = x^2 + z \sin x + xy^2 + ye^z + z + c, \quad c \in \mathbb{R}$$

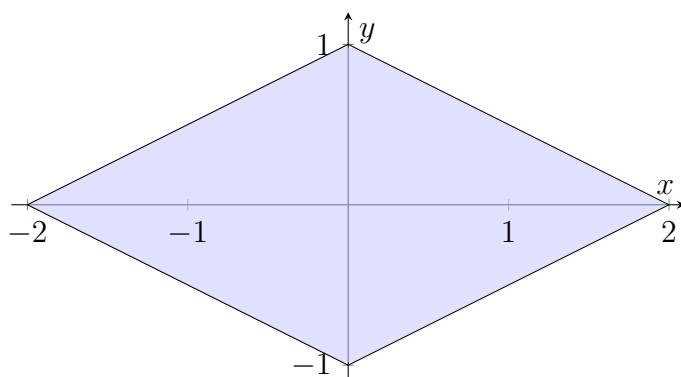
5.

(a) We showed that \mathbf{F} is conservative in 4(b). Using the Fundamental Theorem of Line Integrals, evaluate $f(0, 1, 3) - f(1, 0, 2)$.

$$\int_D \mathbf{F} \cdot d\mathbf{r} = f(0, 1, 3) - f(1, 0, 2) = e^3 + 3 + c - (1 + 2 \sin 1 + 2 + c) = [e^3 - 2 \sin 1]$$

(b) The curve of intersection is a circle, which is a closed curve. Since \mathbf{F} is conservative, the value of the line integral is $[0]$.

6.



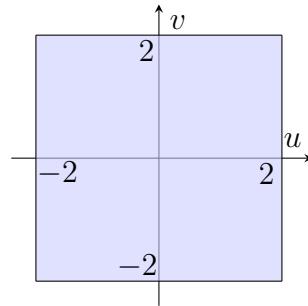
F_1 and F_2 have continuous partial derivatives. C is a closed curve with positive orientation. We may use the tangential form of Green's Theorem to evaluate the line integral.

$$\begin{aligned}
I &= \oint_C \left(x^3 \sin(\sqrt{x^2 + 4}) - xe^{x+2y} \right) dx + \left(\cos(y^3 + y) - 4ye^{x+2y} \right) dy \\
&= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \iint_R \left(-4ye^{x+2y} \cdot 1 - (-xe^{x+2y} \cdot 2) \right) dA \\
&= \iint_R (2x - 4y) \cdot e^{x+2y} dA
\end{aligned} \tag{1}$$

From the edges of the parallelogram, we have $x = 2y + 2, x = 2 - 2y, x = -2 - 2y, x = 2y - 2$. Now, use the method of change of variables. Let $u = x - 2y, v = x + 2y$. Then $x = \frac{u+v}{2}, y = \frac{v-u}{4}$.

$$\begin{aligned}
x = 2y + 2 &\implies \frac{u+v}{2} = 2 \left(\frac{v-u}{4} \right) + 2 \implies u = 2 \\
x = 2 - 2y &\implies \frac{u+v}{2} = 2 - 2 \left(\frac{v-u}{4} \right) \implies v = 2 \\
x = -2 - 2y &\implies \frac{u+v}{2} = -2 - 2 \left(\frac{v-u}{4} \right) \implies v = -2 \\
x = 2y - 2 &\implies \frac{u+v}{2} = 2 \left(\frac{v-u}{4} \right) - 2 \implies u = -2
\end{aligned}$$

The region in uv -coordinates becomes as follows. Calculate the Jacobian determinant and rewrite the integral in (1).



$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{vmatrix} = \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4}$$

$$\begin{aligned}
I &= \iint_R (2x - 4y) \cdot e^{x+2y} dA = \int_{-2}^2 \int_{-2}^2 2ue^v \cdot \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \int_{-2}^2 \int_{-2}^2 2ue^v \cdot \left| \frac{1}{4} \right| du dv \\
&= \int_{-2}^2 e^v \left[\frac{u^2}{4} \right]_{u=-2}^{u=2} dv = \int_{-2}^2 e^v \cdot 0 dv = \boxed{0}
\end{aligned}$$