2024-2025 Fall
MAT123 Midterm
(02/12/2024)

1. Let

$$f(x) = \begin{cases} \dfrac{\tan ax}{\tan bx}, & \text{if } x < 0 \\ 4, & \text{if } x = 0 \\ ax + b, & \text{if } x > 0 \end{cases}$$

Determine the values of a and b such that f is continuous at the point $x = 0$.

2. Use differential to approximate $3\sqrt[3]{66} + 2\sqrt{66}$.

3. **(a)** Without using L'Hôpital's rule, evaluate $\lim\limits_{x \to 0} \dfrac{5 - 6\cos x + \cos^2 x}{x \sin x}$.

(b) Prove that $\lim\limits_{x \to -3} \sqrt{-x-2} = 1$ by using the formal definition of limit.

(c) Evaluate $\lim\limits_{x \to 1^+} \left(\sqrt{x}\right)^{\ln(x-1)}$.

4. Coffee is draining out of a conical filter at a rate of 2.25 in.^{3}/min. If the cone is 5 in. tall and has a radius of 2 in., how fast is the coffee level dropping when the coffee is 3 in. deep?

5. Using the Mean Value Theorem, show that $\ln(x+1) < x$ for $x > 0$.

6. Let $f(x) = \dfrac{x^2 - 2}{(x-1)^2}$.

(a) Determine the interval of increase, decrease and concavity of f.

(b) Construct a table.

(c) Sketch the graph of f.

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1. To ensure continuity at $x = 0$, the one-sided limit values must be equal to the value of the function at that point.

$$\lim_{x\to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x\to 0^+} (ax + b) = f(0) = 4$$

The easy part is that we can calculate the limit from the right.

$$\lim_{x\to 0^+} (ax + b) = 0 + b = b$$

Hence, $b = 4$. To calculate from the left, we need another technique.

$$\lim_{x\to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x\to 0^-} \left(\frac{\sin ax}{\cos ax} \cdot \frac{\cos bx}{\sin bx} \cdot \frac{bx}{bx} \cdot \frac{ax}{ax}\right)$$

$$= \lim_{x\to 0^-} \left(\frac{\sin ax}{ax}\right) \cdot \lim_{x\to 0^-} \left(\frac{1}{\frac{\sin bx}{bx}}\right) \cdot \lim_{x\to 0^-} \left(\frac{\cos(bx) \cdot ax}{\cos(ax) \cdot bx}\right)$$

$$= 1 \cdot \frac{1}{\lim\limits_{x\to 0^-} \dfrac{\sin bx}{bx}} \cdot \lim_{x\to 0^-} \left(\frac{\cos(bx) \cdot a}{\cos(ax) \cdot b}\right) = 1 \cdot 1 \cdot \left(\frac{\cos(0) \cdot a}{\cos(0) \cdot b}\right)$$

$$= \frac{a}{b}$$

Now, set $\dfrac{a}{b} = b \implies a = 16.$ $\boxed{a = 16,\ b = 4}$

2. Let $f(x) = x^{1/3}$ and $g(x) = x^{1/2}$. Using the differential approximation, we get

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x = x^{1/3} + \frac{1}{3}x^{-2/3}\Delta x$$
$$g(x + \Delta x) \approx g(x) + g'(x)\Delta x = x^{1/2} + \frac{1}{2}x^{-1/2}\Delta x$$

Set $x = 64$ and $\Delta x = 2$.

$$3\sqrt[3]{66} + 2\sqrt{66} \approx 3\left(64^{1/3} + \frac{1}{3} \cdot 64^{-2/3} \cdot 2\right) + 2\left(64^{1/2} + \frac{1}{2} \cdot 64^{-1/2} \cdot 2\right)$$
$$= 3\left(4 + \frac{1}{24}\right) + 2\left(8 + \frac{1}{8}\right) = \boxed{28.375}$$

3. **(a)** Factor the numerator and use the conjugate of the expression $\cos x - 1$.

$$\lim_{x\to 0} \frac{5 - 6\cos x + \cos^2 x}{x \sin x} = \lim_{x\to 0} \frac{(\cos x - 1)(\cos x - 5)}{x \sin x}$$

$$= \lim_{x\to 0} \frac{(\cos x - 1)(\cos x - 5)(\cos x + 1)}{(x\sin x)(\cos x + 1)} = \lim_{x\to 0} \left(-\frac{\sin^2 x \cdot (\cos x - 5)}{x \sin x \cdot (\cos x + 1)} \right)$$

$$= \lim_{x\to 0} \left(-\frac{\sin x \cdot (\cos x - 5)}{x \cdot (\cos x + 1)} \right) = -\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\cos x - 5}{\cos x + 1} = -1 \cdot \frac{\cos 0 - 5}{\cos 0 + 1} = \boxed{2}$$

(b) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x+3| < \delta \implies |f(x) - 1| < \epsilon$.

$$|f(x) - 1| = \left|\sqrt{-x-2} - 1\right| = \left| \left(\sqrt{-x-2} - 1\right) \cdot \frac{\sqrt{-x-2}+1}{\sqrt{-x-2}+1} \right|$$

$$= \left| \frac{-x-3}{\sqrt{-x-2}+1} \right| \le \frac{|-x-3|}{1} = |x+3| \qquad \left[\sqrt{-x-2} + 1 \ge 0 + 1 = 1\right]$$

We need to ensure that for all $\epsilon > 0$, there exists such a δ satisfying the inequality. To control the expression $\sqrt{-x-2} + 1$ in the denominator, we can assume that $|x - 3| < 1$. Then, the inequality $\sqrt{-x-2} + 1 \ge 1$ holds. We need to guarantee $\delta = 1$ when $\epsilon > 1$ because of the restriction $|x - 3| < 1$. Therefore, let $\delta = \min(1, \epsilon)$.

$$\left|\sqrt{-x-2} - 1\right| \le |x - 3| < \delta \le \epsilon$$

(c) Let L be the value of the limit. Then, take the logarithm of both sides. Since the expression is continuous for $x > 1$, we can take the logarithm function inside the limit.

$$L = \lim_{x\to 1^+} \left(\sqrt{x}\right)^{\ln(x-1)} \implies \ln(L) = \ln \left[\lim_{x\to 1^+} \left(\sqrt{x}\right)^{\ln(x-1)} \right]$$

$$\ln(L) = \lim_{x\to 1^+} \ln\left[(\sqrt{x})^{\ln(x-1)} \right] = \lim_{x\to 1^+} \left[\ln(x-1) \cdot \ln(\sqrt{x}) \right] = \lim_{x\to 1^+} \frac{\ln(x-1)}{\frac{1}{\ln(\sqrt{x})}} \qquad \left[\frac{\infty}{\infty}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 1^+} \frac{\frac{1}{x-1}}{\frac{1}{-\ln^2(\sqrt{x})} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}} = \lim_{x\to 1^+} \frac{\ln^2(\sqrt{x}) \cdot 2x}{1 - x} \qquad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 1^+} \frac{2\ln\left(\sqrt{x}\right) \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \cdot 2x + \ln^2(\sqrt{x}) \cdot 2}{-1} = \lim_{x\to 1^+} \left[-2\ln\left(\sqrt{x}\right) - 2\ln^2\left(\sqrt{x}\right) \right]$$

$$= 2\ln\left(\sqrt{1}\right) + 2\ln^2\left(\sqrt{1}\right) = 0$$

$\ln(L) = 0$. Therefore, $\boxed{L = 1}$.

4. Let $f(x)$ represent the volume of coffee in the cone in cubic inches. The coffee in the cone will have a conical shape while draining. We may set up the equation below using the formula of the volume of a cone.

$$f(t) = \frac{1}{3} \cdot h(t) \cdot \pi r^2(t)$$

$h(t), r(t)$ represent the height and radius of the circular area that coffee forms, respectively, in inches. We can eliminate r to proceed with h. r and h are proportional.

$$\frac{r}{h} = \frac{2}{5} \implies r = \frac{2h}{5}$$

$$f(t) = \frac{4\pi h^3(t)}{75}$$

Take the derivative of both sides.

$$f'(t) = \frac{4\pi}{25} \cdot h^2(t) \cdot h'(t)$$

Given that at $t = t_0$, $f'(t_0) = -2.25$, $h(t_0) = 3$. We may now find $h'(t_0)$. Solve for $h'(t_0)$.

$$h'(t_0) = \frac{25f'(t_0)}{4\pi h^2(t_0)} = \frac{25 \cdot (-2.25)}{4\pi \cdot (3)^2} = \boxed{-\frac{1.5625}{\pi} \text{ inches/minute}}$$

5. Let $f(x) = \ln(1+x) - x$. We have $f(0) = \ln(1+0) - 0 = 0$. The mean value theorem (MVT) states that if a function $g(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point c such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$

f is continuous on $[0, x]$ and differentiable on $(0, x)$. By MVT, $\dfrac{f(x) - f(0)}{x - 0} = f'(c)$ provided for some point c such that $0 < c < x$.

$$f'(c) = \frac{1}{c+1} - 1 = \frac{\ln(x+1) - x}{x} = \frac{f(x) - f(0)}{x - 0}$$
$$\frac{1}{c+1} = \frac{\ln(x+1)}{x} \implies c + 1 = \frac{x}{\ln(x+1)}$$
$$c = \frac{x - \ln(x+1)}{\ln(x+1)}$$

From the inequality $0 < c < x$,

$$0 < \frac{x - \ln(x+1)}{\ln(x+1)}$$
$$0 < x - \ln(x+1)$$
$$\ln(x+1) < x$$

6. (a) First off, find the domain. The expression is undefined when the denominator is zero. Therefore, $(x-1)^2 \neq 0 \implies x \neq 1$. The only vertical asymptote occurs at $x = 1$.

$$\mathcal{D} = \mathbb{R} - \{1\}$$

Let us find the limit at infinity.

$$\lim_{x\to\infty} \frac{x^2-2}{(x-1)^2} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2x}{2(x-1)} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2}{2} = 1$$

Similarly,

$$\lim_{x\to-\infty} \frac{x^2-2}{(x-1)^2} = 1$$

The horizontal asymptote occurs only at $y = 0$.

Take the first derivative by applying the quotient rule.

$$y' = \frac{(2x)\cdot(x-1)^2 - (x^2-2)\cdot 2(x-1)}{(x-1)^4} = \frac{4-2x}{(x-1)^3}$$

y' is undefined for $x = 1$, and $y' = 0$ for $x = 2$. Since 1 is not in the domain, the *only* critical point is $x = 2$.

Take the second derivative.

$$y'' = \frac{(-2)\cdot(x-1)^3 - (4-2x)\cdot 3(x-1)^2}{(x-1)^6} = \frac{4x-10}{(x-1)^4}$$

The only inflection point occurs at $x = \frac{5}{2}$.

(b) Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f\left(-\sqrt{2}\right) = f\left(\sqrt{2}\right) = 0,\ f(0) = -2,\ f(2) = 2,\ f(5/2) = 17/9$$

x	$\left(-\infty,-\sqrt{2}\right)$	$\left(-\sqrt{2},0\right)$	$(0,1)$	$\left(1,\sqrt{2}\right)$	$\left(\sqrt{2},2\right)$	$\left(2,\frac{5}{2}\right)$	$\left(\frac{5}{2},\infty\right)$
y	$(1,0)$	$(-2,0)$	$(-\infty,-2)$	$(-\infty,0)$	$(0,2)$	$\left(2,\frac{17}{9}\right)$	$\left(\frac{17}{9},1\right)$
y' sign	-	-	-	+	+	-	-
y'' sign	-	-	-	-	-	-	+

(c)