

2023-2024 Fall  
MAT124 Final  
(11/01/2024)

1.

(a) Find the critical points of the function

$$f(x, y) = e^y (y^2 - x^2)$$

and classify them.

(b) Find the maximum and minimum values of the function

$$f(x, y, z) = xyz^2$$

by using Lagrange multipliers on the region  $D : x + y + z = 1, x \geq 0, y \geq 0, z \geq 0$ .

2.

(a) Sketch the domain of integration, and rewrite the integral by changing the order of integration.

$$\int_0^1 \int_{\sqrt{1-y}}^1 f(x, y) dx dy + \int_1^3 \int_{\frac{y-1}{2}}^1 f(x, y) dx dy$$

(b) Evaluate the integral

$$\iint_R \sqrt{x^2 + y^2} dA$$

where  $R$  is the region bounded by the line  $x = 1$  on the left and by the circle  $x^2 + y^2 = 2$  on the right.

3. Convert

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3 dz r dr d\theta, \quad r \geq 0$$

(a) to rectangular coordinates with the order of integration  $dz dx dy$ .

(b) to spherical coordinates.

4.

(a) Is  $\mathbf{F}(x, y) = 2xy \cos(x^2y) \mathbf{i} - x^2 \cos(x^2y) \mathbf{j}$  conservative? Why?

(b) Show that

$$\mathbf{F}(x, y, z) = (y^2 \cos(xy^2) + yz^2) \mathbf{i} + (2xy \cos(xy^2) + xz^2 + ze^{yz}) \mathbf{j} + (2xyz + ye^{yz} + 2z) \mathbf{k}$$

is conservative.

(c) Find its potential function.

5.  $\mathbf{F}(x, y, z) = (y^2 \cos(xy^2) + yz^2) \mathbf{i} + (2xy \cos(xy^2) + xz^2 + ze^{yz}) \mathbf{j} + (2xyz + ye^{yz} + 2z) \mathbf{k}$

(a) Let  $C$  be the curve of intersection of the cone  $z^2 = 4x^2 + 9y^2$  and the plane  $z = 1 + x + 2y$  and  $D$  be the part of the curve  $C$  that lies in the first octant  $x \geq 0, y \geq 0, z \geq 0$  from  $(1, 0, 2)$  to  $(0, 1, 3)$ . Evaluate  $\int_D \mathbf{F} \cdot d\mathbf{r}$ .

(b) Let  $C$  be the curve of intersection of  $x^2 + y^2 = 1$  and  $z = 40$ . Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ .

6. Evaluate

$$\oint_C \frac{y^2}{x} dx + y \ln x dy$$

where  $C$  is the counterclockwise boundary of the region in the first quadrant bounded by the curves  $y = x$ ,  $y = \frac{x}{2}$ ,  $y = \frac{1}{x}$  and  $\frac{2}{x}$ .

2023-2024 Fall Final (11/01/2024) Solutions  
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1.

(a) To find the critical points of  $f$ , determine where both  $f_x = f_y = 0$  or one of the partial derivatives does not exist. Apply the product rule appropriately.

$$f_x = e^y(-2x), \quad f_y = e^y(y^2 - x^2) + e^y(2y) = e^y(y^2 + 2y - x^2)$$

$$f_x = 0 \implies e^y(-2x) = 0, \quad f_y = 0 \implies e^y(y^2 + 2y - x^2) = 0$$

$$e^y \neq 0 \implies x = 0, \quad y^2 + 2y - x^2 = 0 \implies y(y + 2) - 0^2 = 0 \implies y = 0, \quad y = -2$$

The critical points occur at  $(0, 0)$  and  $(0, -2)$ . To classify these points, apply the second derivative test.

$$f_{xx} = -2e^y, \quad f_{xy} = f_{yx} = e^y(-2x), \quad f_{yy} = e^y(y^2 + 2y - x^2) + e^y(2y + 2)$$

Calculate the Hessian determinant at these points.

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

$$(0, 0) \rightarrow \begin{aligned} f_{xx} &= -2, \quad f_{xy} = f_{yx} = 0, \quad f_{yy} = 2 \\ f_{xx}f_{yy} - f_{xy}^2 &= -2 \cdot 2 - 0^2 = -4 < 0 \end{aligned}$$

$$(0, -2) \rightarrow \begin{aligned} f_{xx} &= -2e^{-2}, \quad f_{xy} = f_{yx} = 0, \quad f_{yy} = -2e^{-2} \\ f_{xx}f_{yy} - f_{xy}^2 &= -2e^{-2} \cdot 2e^{-2} - 0^2 = 4e^{-4} > 0, \quad f_{xx} = -2e^{-2} < 0 \end{aligned}$$

A saddle point occurs at  $(0, 0)$  and a local maximum occurs at  $(0, -2)$ .

(b) Let  $g(x, y, z) = x + y + z - 1$  be the constraint. Then solve the system of equations below.

$$\left. \begin{aligned} \nabla f &= \lambda \nabla g \\ g(x, y, z) &= 0 \end{aligned} \right\} \quad \begin{aligned} \nabla f &= \langle yz^2, xz^2, 2xyz \rangle = \lambda \langle 1, 1, 1 \rangle = \lambda \nabla g \\ \therefore yz^2 &= xz^2 = 2xyz \end{aligned}$$

$$yz^2 = xz^2 \implies y = x$$

$$yz^2 = 2xyz \implies z = 2x$$

Use the constraint and write  $y, z$  in terms of  $x$ .

$$x + y + z - 1 = 0 \implies x + x + 2x = 1 \implies 4x = 1 \implies x = \frac{1}{4}$$

$$x = \frac{1}{4} \implies y = \frac{1}{4}, \quad z = \frac{1}{2}$$

Consider the boundary as well. Set  $x = 0$ ,  $y = 0$ ,  $z = 0$  one by one. Notice that the value of the function becomes 0 on the boundary of the domain.

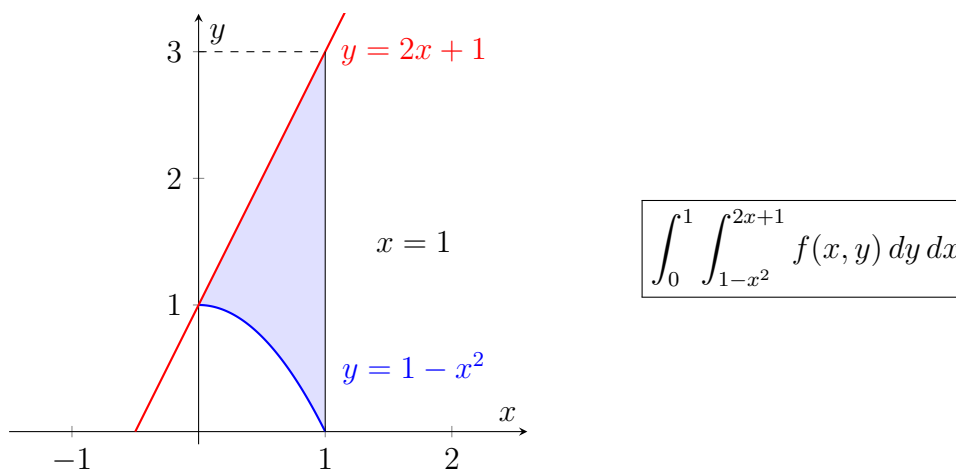
Compare all the values.

$$f(0, y, z) = f(x, 0, z) = f(x, y, 0) = 0, \quad f\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} \cdot \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{64}$$

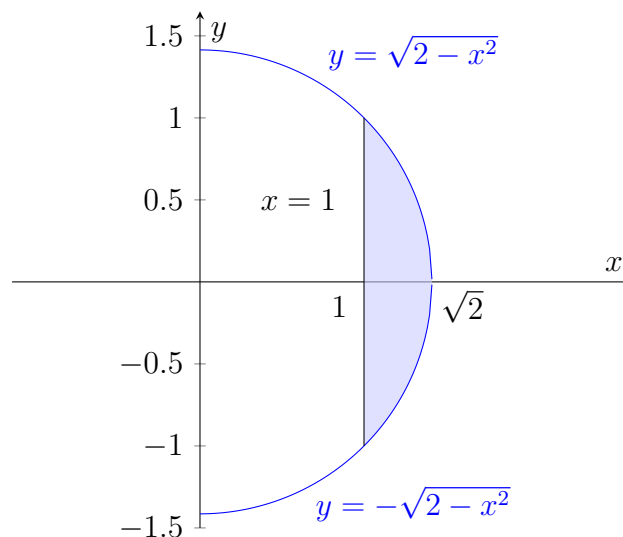
The maximum value is  $\frac{1}{64}$ , the minimum value is 0.

2.

(a)



(b) Sketch the region.



$$I = \int_1^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \sqrt{x^2 + y^2} dy dx$$

We may switch to polar coordinates to easily evaluate the integral.

$$\begin{array}{ll} x = r \cos \theta & x = \sqrt{2} \cos \theta, \quad y = \sqrt{2} \sin \theta, \quad r_{\text{upper}} = \sqrt{2} \\ y = r \sin \theta & \\ \theta = \tan^{-1} \frac{y}{x} & \rightarrow x = 1 \implies r \cos \theta = 1 \implies r_{\text{lower}} = \sec \theta \\ x^2 + y^2 = r^2 & 1 = \sqrt{2} \cos \theta \implies \cos \theta = \frac{1}{\sqrt{2}} \implies \theta = \pm \frac{\pi}{4} \\ dA = dy dx = r dr d\theta & 0 \leq \theta \leq \pi/2 \end{array}$$

$$\begin{aligned} I &= \int_{-\pi/4}^{\pi/4} \int_{\sec \theta}^{\sqrt{2}} r \cdot r dr d\theta = \int_{-\pi/4}^{\pi/4} \left[ \frac{r^3}{3} \right]_{r=\sec \theta}^{r=\sqrt{2}} d\theta = \frac{1}{3} \int_{-\pi/4}^{\pi/4} (2\sqrt{2} - \sec^3 \theta) d\theta \\ &= \int_{-\pi/4}^{\pi/4} \frac{2\sqrt{2}}{3} d\theta - \frac{1}{3} \int_{-\pi/4}^{\pi/4} \sec^3 \theta d\theta \end{aligned} \quad (1)$$

Evaluate the left-hand side integral in (1).

$$\int_{-\pi/4}^{\pi/4} \frac{2\sqrt{2}}{3} d\theta = \frac{2\sqrt{2}}{3} \cdot \theta \Big|_{-\pi/4}^{\pi/4} = \frac{\pi\sqrt{2}}{3}$$

Evaluate the right-hand side integral in (1) with the help of integration by parts.

$$\begin{aligned} u &= \sec \theta \rightarrow du = \sec \theta \tan \theta d\theta \\ dv &= \sec^2 \theta d\theta \rightarrow v = \tan \theta \end{aligned}$$

$$\begin{aligned} \int \sec^3 \theta d\theta &= \tan \theta \cdot \sec \theta - \int \tan^2 \theta \sec \theta d\theta = \tan \theta \cdot \sec \theta - \int \frac{1 - \cos^2 \theta}{\cos^3 \theta} d\theta \\ &= \tan \theta \cdot \sec \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta \end{aligned}$$

Notice that the integral of  $\sec^3 \theta$  with respect to  $\theta$  appears on the right side of the equation. Perform some algebra and then find the result of the integral.

$$\int \sec^3 \theta d\theta = \frac{1}{2} \cdot \tan \theta \cdot \sec \theta + \frac{1}{2} \cdot \int \sec \theta d\theta$$

The integral of  $\sec \theta$  with respect to  $\theta$  is as follows.

$$\int \sec \theta d\theta = \ln |\tan \theta + \sec \theta| + c_0, \quad c_0 \in \mathbb{R}$$

Rewrite (1).

$$\begin{aligned}
I &= \frac{\pi\sqrt{2}}{3} - \frac{1}{3} \cdot \frac{1}{2} (\tan \theta \cdot \sec \theta + \ln |\tan \theta + \sec \theta|) \Big|_{-\pi/4}^{\pi/4} \\
&= \frac{\pi\sqrt{2}}{3} - \frac{1}{6} \left[ \left( 1 \cdot \sqrt{2} + \ln \left( 1 + \sqrt{2} \right) \right) - \left( -1 \cdot \sqrt{2} + \ln \left( -1 + \sqrt{2} \right) \right) \right] \\
&= \boxed{\frac{(\pi-1)\sqrt{2}}{3} + \frac{1}{6} \cdot \ln \left( \frac{1+\sqrt{2}}{-1+\sqrt{2}} \right)}
\end{aligned}$$

3. From the bounds of the integral given in cylindrical coordinates, we infer that this is the region bounded above by the sphere  $x^2 + y^2 + z^2 = 4$  and bounded below by the cone  $z^2 = x^2 + y^2$

(a)

$$\begin{array}{ll}
\begin{array}{l} z = z \\ x = r \cos \theta \\ y = r \sin \theta \\ r^2 = x^2 + y^2 \\ dV = r \, dz \, dr \, d\theta = dz \, dy \, dx \end{array} & \rightarrow \begin{array}{l} z = \sqrt{4-r^2} \implies z = \sqrt{4-x^2-y^2} \\ z = r \implies z = \sqrt{x^2+y^2} \\ 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq \sqrt{2} \\ \downarrow \\ -\sqrt{2-y^2} \leq y \leq \sqrt{2-y^2}, \quad -\sqrt{2} \leq y \leq \sqrt{2} \end{array} \\
\end{array}$$

$$\boxed{\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3 \, dz \, dx \, dy}$$

(b) For spherical coordinates, we have

$$\begin{array}{ll}
\begin{array}{l} z = \rho \cos \phi \\ r = \rho \sin \phi \\ x^2 + y^2 + z^2 = \rho^2 \\ dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{array} & \rightarrow \begin{array}{l} z = \sqrt{4-r^2} \implies \rho \cos \phi = \sqrt{4-\rho^2 \sin^2 \phi} \implies \rho_{\text{upper}} = 2 \\ z = r \implies \rho \cos \phi = \rho \sin \phi \implies \rho_{\text{lower}} = 0 \\ \cos \phi = \sin \phi \implies \tan \phi = 1 \implies \phi_{\text{upper}} = \frac{\pi}{4} \\ \phi_{\text{lower}} = 0, \quad 0 \leq \theta \leq 2\pi \end{array} \\
\end{array}$$

$$\boxed{\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 3\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta}$$

4.

(a) For  $\mathbf{F}$  to be conservative, it must be the gradient of some potential function  $\phi$ . We may apply the component test to determine whether mixed partial derivatives are equal.

$$\left. \begin{aligned} \frac{\partial F_1}{\partial y} &= 2x \cos(x^2 y) - 2xy \sin(x^2 y) \cdot x^2 \\ \frac{\partial F_2}{\partial x} &= -2x \cos(x^2 y) + x^2 \sin(x^2 y) \cdot 2xy \end{aligned} \right\} \implies \frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$$

The mixed partial derivatives are not equal. Therefore, the force is not conservative.

(b) Like what we did above, determine the mixed partial derivatives.

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= 2y \cos(xy^2) - y^2 \sin(xy^2) \cdot 2xy + z^2 = \frac{\partial F_2}{\partial x} \\ \frac{\partial F_1}{\partial z} &= 2yz = \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial z} &= 2xz + e^{yz} + zye^{yz} = \frac{\partial F_3}{\partial y} \end{aligned}$$

(c) Since  $\mathbf{F}$  is conservative on  $\mathbb{R}^3$ , there exists a potential function  $f$  such that  $\nabla f = \mathbf{F}$ .

$$\frac{\partial f}{\partial x} = y^2 \cos(xy^2) + yz^2, \quad \frac{\partial f}{\partial y} = 2xy \cos(xy^2) + xz^2 + ze^{yz}, \quad \frac{\partial f}{\partial z} = 2xyz + ye^{yz} + 2z$$

$$\int \frac{\partial f}{\partial x} dx = \int (y^2 \cos(xy^2) + yz^2) dx = \sin(xy^2) + xyz^2 + g(y, z) = f(x, y, z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (\sin(xy^2) + xyz^2 + g(y, z)) = 2xy \cos(xy^2) + xz^2 + g_y(y, z)$$

$$= 2xy \cos(xy^2) + xz^2 + ze^{yz} \implies g_y(y, z) = ze^{yz}$$

$$\int \frac{\partial f}{\partial y} dy = \int (2xy \cos(xy^2) + xz^2 + ze^{yz}) dy = \sin(xy^2) + xyz^2 + e^{yz} + h(z) = f(x, y, z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} (\sin(xy^2) + xyz^2 + e^{yz} + h(z)) = 2xyz + ye^{yz} + h_z(z)$$

$$= 2xyz + ye^{yz} + 2z \implies h_z(z) = 2z$$

$$\int \frac{\partial f}{\partial z} dz = \int (2xyz + ye^{yz} + 2z) dz = \sin(xy^2) + xyz^2 + e^{yz} + z^2 + c = f(x, y, z)$$

The potential function for  $\mathbf{F}$  is

$$\boxed{f(x, y, z) = \sin(xy^2) + xyz^2 + e^{yz} + z^2 + c, \quad c \in \mathbb{R}}$$

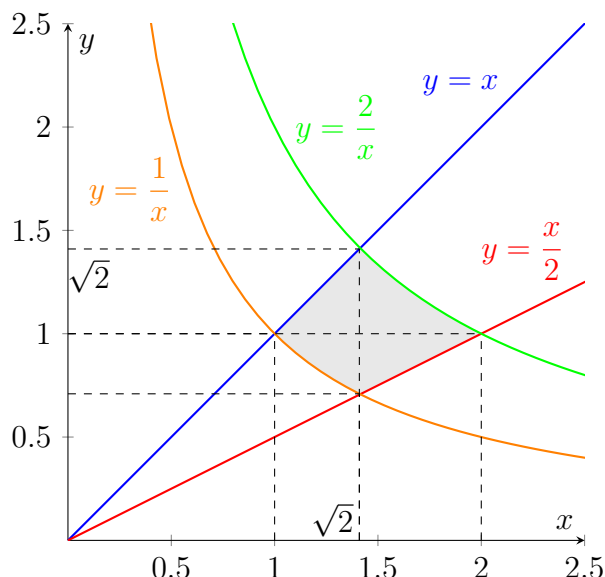
5.

(a) We showed that  $\mathbf{F}$  is conservative in 4(b). Using the Fundamental Theorem of Line Integrals, evaluate  $f(0, 1, 3) - f(1, 0, 2)$ .

$$\begin{aligned}\int_D \mathbf{F} \cdot d\mathbf{r} &= f(0, 1, 3) - f(1, 0, 2) = \sin(0) + 0 + e^3 + 3^2 + c - (\sin(1) + 0 + e^0 + 2^2 + c) \\ &= \boxed{e^3 + 4}\end{aligned}$$

(b) The curve of intersection is a circle, which is a closed curve. Since  $\mathbf{F}$  is conservative, the value of the line integral is  $\boxed{0}$ .

6.



$F_1$  and  $F_2$  have continuous partial derivatives.  $C$  is a closed curve with positive orientation. We may use the tangential form of Green's Theorem to evaluate the line integral.

$$\begin{aligned}I &= \oint_C \frac{y^2}{x} dx + y \ln x dy = \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \iint_R -\frac{y}{x} dA \\ &= \int_1^{\sqrt{2}} \int_{1/x}^x -\frac{y}{x} dy dx + \int_{\sqrt{2}}^2 \int_{x/2}^{2/x} -\frac{y}{x} dy dx = \int_1^{\sqrt{2}} -\frac{y^2}{2x} \Big|_{y=1/x}^{y=x} dx + \int_{\sqrt{2}}^2 -\frac{y^2}{2x} \Big|_{y=x/2}^{y=2/x} dx \\ &= \int_1^{\sqrt{2}} \left( -\frac{x}{2} + \frac{1}{2x^3} \right) dx + \int_{\sqrt{2}}^2 \left( -\frac{2}{x^3} + \frac{x}{8} \right) dx = \left[ -\frac{x^2}{4} - \frac{1}{4x^2} \right]_1^{\sqrt{2}} + \left[ \frac{1}{x^2} + \frac{x^2}{16} \right]_{\sqrt{2}}^2 \\ &= \left[ -\frac{1}{2} - \frac{1}{8} - \left( -\frac{1}{4} - \frac{1}{4} \right) \right] + \left[ \frac{1}{4} + \frac{1}{4} - \left( \frac{1}{2} + \frac{1}{8} \right) \right] = \boxed{-\frac{1}{4}}\end{aligned}$$