

2021-2022 Spring
MAT124 Midterm
(20/04/2022)

1. Find the convergence set of the power series $\sum_{k=1}^{\infty} \frac{(\ln k)x^k}{k}$.
2. Show that the vectors $\mathbf{u} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} + 3\mathbf{k}$ are coplanar (all in the same plane).
3. Find the point where the line

$$x - 1 = z, y = 3$$

intersects the plane $2y - z = 5$.

4. Sketch the graphs of the following surfaces.

(a) $z = \sin x$

(b) $y = z^2 - x^2$

5. Let f be the function $f(x, y) = \frac{xy^2}{x^2 + y^4}$ for $(x, y) \neq (0, 0)$. Show that f has no limit at $(0, 0)$.

6. Van der Waal's equation in physical chemistry states that a gas occupying volume V at temperature T (Kelvin) exerts pressure P , where

$$\left(P + \frac{A}{V^2}\right)(V - B) = kT$$

for physical constants A, B and k . Compute the following rates.

- (a) the rate of change of volume with respect to temperature.
- (b) the rate of change of pressure with respect to volume.

7. Consider the surface $z = x^2 + y^2 - xy$, a paraboloid, on which a particle moves with x and y coordinates given by $x = \cos t$ and $y = \sin t$. Using the chain rule, find $\frac{dz}{dt}$ when $t = 0$.

8. Find the parametric equations of the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $2x^2 + 2y^2 + z^2 = 8$ at $P = (-1, 1, 2)$.

2021-2022 Spring Midterm (20/04/2022) Solutions
(Last update: 26/08/2025 23:42)

1. Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\lim_{k \rightarrow \infty} \left| \frac{\ln(k+1) \cdot x^{k+1}}{k+1} \cdot \frac{k}{\ln(k) \cdot x^k} \right| = \lim_{k \rightarrow \infty} \left| x \cdot \frac{k}{k+1} \cdot \frac{\ln(k+1)}{\ln(k)} \right| = |x| \cdot 1 = |x|$$

$k/(k+1) \rightarrow 1$ and $\ln(k)/\ln(k+1) \rightarrow 1$ as $k \rightarrow \infty$. Therefore, the limit is $|x|$.

$$|x| < 1 \implies -1 < x < 1 \quad (\text{convergent})$$

Investigate the convergence at the endpoints.

$$x = 1 \implies \sum_{k=1}^{\infty} \frac{(\ln k) \cdot 1^k}{k} = \sum_{k=1}^{\infty} \frac{\ln k}{k}$$

Take the corresponding function $f(x) = \frac{\ln x}{x}$. The function is continuous and positive for $x > 1$. It is also decreasing for $x > e$ because x grows faster than $\ln x$. Confirm the behavior by taking the first derivative.

$$f'(x) = \frac{1 - \ln x}{x^2} < 0 \quad \text{for } x > e$$

We may now apply the Integral Test.

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{R \rightarrow \infty} \int_1^R \frac{\ln x}{x} dx = \lim_{R \rightarrow \infty} \frac{1}{2} (\ln x)^2 \Big|_1^R = \frac{1}{2} \lim_{R \rightarrow \infty} ((\ln R)^2 - (\ln 1)^2) = \infty$$

Since the integral diverges, by the Integral Test, the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ also diverges. Try $x = -1$.

$$x = -1 \implies \sum_{k=1}^{\infty} \frac{(\ln k) \cdot (-1)^k}{k}$$

This is an alternating series. The non-alternating part, which is $\frac{\ln k}{k}$, is nonincreasing for $k > e$ and it is positive. The previous cases are already confirmed for $x = 1$. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges.

Thus, the convergence set for the power series is $\boxed{[-1, 1)}$.

2. If $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, the vectors are coplanar. Because $\mathbf{u} \times \mathbf{v}$ is perpendicular to \mathbf{u} and \mathbf{v} , and the resulting vector is also perpendicular to \mathbf{w} . Compute the cross product.

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 2 & -1 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 3 & 1 \\ -1 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \\ &= (-1 \cdot 3 - 1 \cdot (-1))\mathbf{i} - (-1 \cdot 1 - 2 \cdot 1)\mathbf{j} + (-1 \cdot 1 - 3 \cdot 2)\mathbf{k} = -2\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}\end{aligned}$$

Compute the dot product.

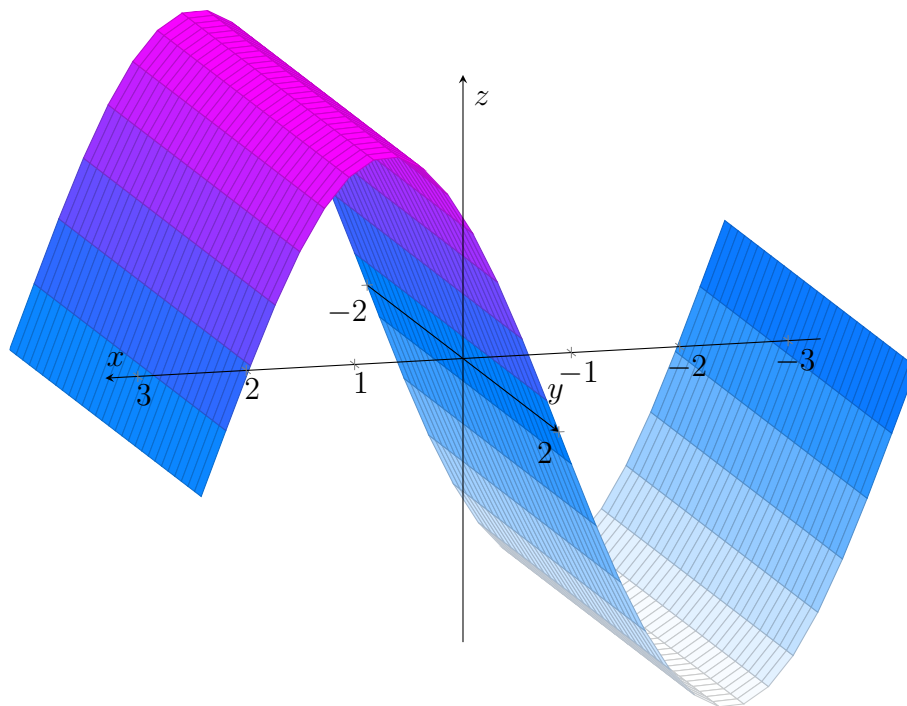
$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (7\mathbf{j} + 3\mathbf{k}) \cdot (-2\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}) = 0 \cdot 2 + 7 \cdot 3 + 3 \cdot (-7) = 0$$

Therefore, all the vectors are coplanar.

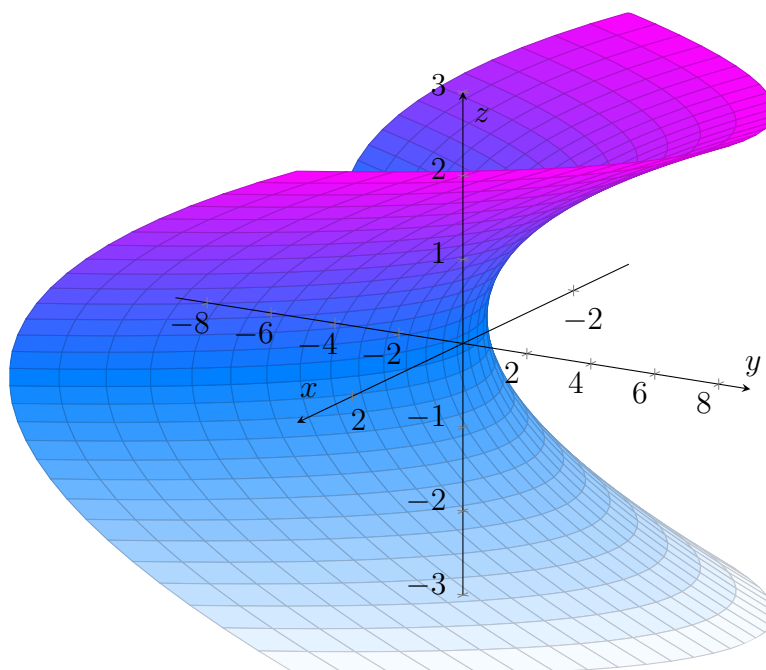
3. For $y = 3$, $2y - z = 5 \implies 6 - z = 5 \implies z = 1$. For $z = 1$, $x - 1 = z \implies x - 1 = 1 \implies x = 2$. Therefore, the point where the line and the plane intersect is $\boxed{(2, 3, 1)}$.

4.

(a)



(b)



5. Apply the Two-Path Test.

$$y = x \implies \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{x \rightarrow 0} \frac{x^3}{x^2 + x^4} = \lim_{x \rightarrow 0} \frac{x}{1 + x^2} = \frac{0}{1} = 0$$

$$x = y^2 \implies \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{y \rightarrow 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

Since $0 \neq \frac{1}{2}$, by the Two-Path Test, the limit does not exist.

6.

(a) Apply the product rule.

$$\frac{\partial}{\partial T} \left[\left(P + \frac{A}{V^2} \right) (V - B) \right] = \frac{\partial}{\partial T} (kT)$$

$$\left(\frac{\partial P}{\partial T} - \frac{2A}{V^3} \cdot \frac{\partial V}{\partial T} \right) (V - B) + \left(P + \frac{A}{V^2} \right) \left(\frac{\partial V}{\partial T} \right) = k$$

$$(V - B) \frac{\partial P}{\partial T} + \frac{\partial V}{\partial T} \left(-\frac{2A}{V^2} + \frac{2AB}{V^3} \right) + \frac{\partial V}{\partial T} \left(P + \frac{A}{V^2} \right) = k$$

$$\boxed{\frac{\partial V}{\partial T} = \frac{k + (B - V) \frac{\partial P}{\partial T}}{P + \frac{2AB}{V^3} - \frac{A}{V^2}}}$$

(b) Apply the product rule again.

$$\begin{aligned}
\frac{\partial}{\partial V} \left[\left(P + \frac{A}{V^2} \right) (V - B) \right] &= \frac{\partial}{\partial V} (kT) \\
\left(\frac{\partial P}{\partial V} - \frac{2A}{V^3} \right) (V - B) + \left(P + \frac{A}{V^2} \right) \cdot 1 &= k \frac{\partial T}{\partial V} \\
(V - B) \frac{\partial P}{\partial V} - \frac{2A}{V^2} + \frac{2AB}{V^3} &= k \frac{\partial T}{\partial V} - P - \frac{A}{V^2} \\
\boxed{\frac{\partial P}{\partial V} = \frac{k \frac{\partial T}{\partial V} - P + \frac{A}{V^2} - \frac{2AB}{V^3}}{V - B}}
\end{aligned}$$

7. $z = z(x, y)$, $x = x(t)$ and $y = y(t)$. Apply the chain rule.

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = (2x - y)(-\sin t) + (2y - x)(\cos t) \\
&= (2 \cos t - \sin t)(-\sin t) + (2 \sin t - \cos t)(\cos t) = \sin^2 t - \cos^2 t = -\cos 2t
\end{aligned}$$

The result is then

$$\left. \frac{dz}{dt} \right|_{t=0} = -\cos 0 = \boxed{-1}$$

8. Let $f(x, y, z) = x^2 + y^2 - z = 0$ and $g(x, y, z) = 2x^2 + 2y^2 + z^2 = 8$ be level surfaces. The tangent line is perpendicular to both ∇f and ∇g at P .

$$\begin{aligned}
\nabla f &= \langle 2x, 2y, -1 \rangle, & \nabla g &= \langle 4x, 4y, 2z \rangle \\
\nabla f(P) &= \langle -2, 2, -1 \rangle, & \nabla g(P) &= \langle -4, 4, 4 \rangle
\end{aligned}$$

$$\begin{aligned}
\mathbf{T} = \nabla f \times \nabla g &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & -1 \\ -4 & 4 & 4 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -1 \\ 4 & 4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & -1 \\ -4 & 4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 2 \\ -4 & 4 \end{vmatrix} \\
&= (2 \cdot 4 - 4 \cdot (-1))\mathbf{i} - (-2 \cdot 4 - (-4) \cdot (-1))\mathbf{j} + (-2 \cdot 4 - 2 \cdot (-4))\mathbf{k} = 12\mathbf{i} + 12\mathbf{j}
\end{aligned}$$

The parametric equations for the tangent line is

$$\boxed{\left. \begin{aligned} x &= -1 + 12t \\ y &= 1 + 12t \\ z &= 2 \end{aligned} \right\} \quad t \in \mathbb{R}}$$