

2021-2022 Fall
MAT123-02,05 Final
(07/01/2022)

1.

(a) Find $\int \frac{dx}{x^3 - 4x^2 + 3x}$.

(b) Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\int_{\pi/2}^x \ln(\sin t) dt}{\sin x - 1}$.

(c) Evaluate the improper integral $\int_1^2 \frac{dx}{(x-1)^{2/3}}$.

2. Consider the region R bounded by the curves $y = \arctan x$, $y = \ln x$ and the lines $x = \frac{1}{\sqrt{3}}$ and $x = 1$.

(a) Find the area of the region R .

(b) Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region R about the y -axis.

(c) Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of a solid obtained by rotating the region R about the line $y = 2$.

3. Determine whether each series is convergent or divergent. Explain your answer.

(a) $\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1))$

(b) $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

(c) $\sum_{n=1}^{\infty} \cos\left(\frac{n^2}{1+n^4}\right)$

(d) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

1.

(a) Use the method of partial fraction decomposition.

$$\int \frac{dx}{x^3 - 4x^2 + 3x} = \int \frac{dx}{x(x-3)(x-1)} = \int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1} \right) dx$$

$$\begin{aligned} A(x-3)(x-1) + Bx(x-1) + Cx(x-3) &= 1 \\ x^2(A+B+C) + x(-4A-B-3C) + 3A &= 1 \end{aligned}$$

Equate the coefficients of like terms.

$$\left. \begin{aligned} x^2(A+B+C) &= 0 \\ x(-4A-B-3C) &= 0 \\ 3A &= 1 \end{aligned} \right\} \rightarrow A = \frac{1}{3}, \quad \left. \begin{aligned} B+C &= -\frac{1}{3} \\ B+3C &= -\frac{4}{3} \end{aligned} \right\} \rightarrow B = \frac{1}{6}, \quad C = -\frac{1}{2}$$

Rewrite the integral.

$$\int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1} \right) dx = \int \left(\frac{1}{3x} + \frac{1}{6(x-3)} - \frac{1}{2(x-1)} \right) dx$$

$$= \boxed{\frac{1}{3} \ln |x| + \frac{1}{6} \ln |x-3| - \frac{1}{2} \ln |x-1| + c, \quad c \in \mathbb{R}}$$

(b) The limit is in the indeterminate form $0/0$. Apply L'Hôpital's rule to eliminate the indeterminate form.

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\int_{\pi/2}^x \ln(\sin t) dt}{\sin x - 1} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{d}{dx} \int_{\pi/2}^x \ln(\sin t) dt}{\cos x}$$

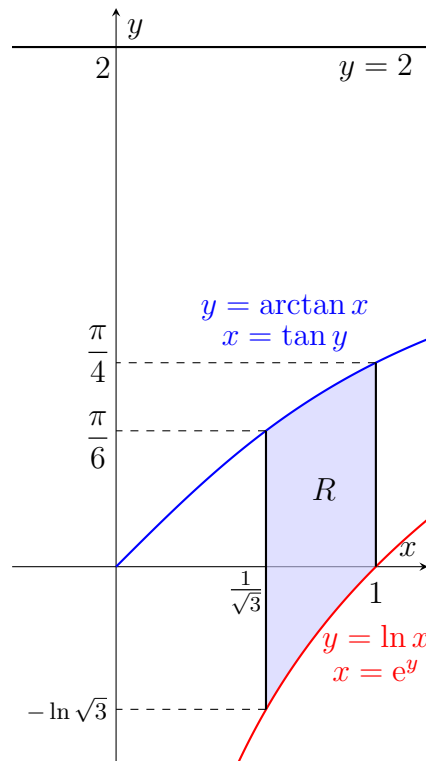
By the Fundamental Theorem of Calculus, we may rewrite the limit as follows.

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{d}{dx} \int_{\pi/2}^x \ln(\sin t) dt}{\cos x} &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln(\sin x)}{\cos x} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{1}{\sin x} \cdot \cos x}{-\sin x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{-\sin^2 x} = -\frac{\cos \frac{\pi}{2}}{\sin^2 \frac{\pi}{2}} \\ &= \boxed{0} \end{aligned}$$

(c) Take the limit as this is an improper integral.

$$\int_1^2 \frac{dx}{(x-1)^{2/3}} = \lim_{R \rightarrow 1^+} \int_R^2 \frac{dx}{(x-1)^{2/3}} = \lim_{R \rightarrow 1^+} 3(x-1)^{1/3} \Big|_R^2 = 3 \lim_{R \rightarrow 1^+} (1 - (R-1)^{1/3}) = \boxed{3}$$

2.



(a)

$$A = \int_{1/\sqrt{3}}^1 (\arctan x - \ln x) dx = \int_{1/\sqrt{3}}^1 \arctan x dx - \int_{1/\sqrt{3}}^1 \ln x dx \quad (1)$$

Calculate the first integral in (1) by integration by parts.

$$\left. \begin{aligned} u = \arctan x &\implies du = \frac{1}{x^2 + 1} dx \\ dv = dx &\implies v = x \end{aligned} \right\} \rightarrow \int u dv = uv - \int v du$$

$$\int_{1/\sqrt{3}}^1 \arctan x dx = x \arctan x \Big|_{1/\sqrt{3}}^1 - \int_{1/\sqrt{3}}^1 \frac{x}{x^2 + 1} dx = \left(x \arctan x - \frac{1}{2} \ln |x^2 + 1| \right) \Big|_{1/\sqrt{3}}^1$$

$$= \left(\frac{\pi}{4} - \frac{\ln 2}{2} \right) - \left(\frac{\pi\sqrt{3}}{18} - \frac{1}{2} \cdot \ln \left(\frac{4}{3} \right) \right) = \frac{\pi(9 - 2\sqrt{3})}{36} + \frac{1}{2} \cdot \ln \frac{2}{3}$$

Calculate the second integral in (1) by integration by parts.

$$\left. \begin{aligned} u = \ln x &\implies du = \frac{1}{x} dx \\ dv = dx &\implies v = x \end{aligned} \right\} \rightarrow \int u dv = uv - \int v du$$

$$\int_{1/\sqrt{3}}^1 \ln x \, dx = x \ln x \Big|_{1/\sqrt{3}}^1 - \int_{1/\sqrt{3}}^1 dx = (x \ln x - x) \Big|_{1/\sqrt{3}}^1 = (0 - 1) - \left(-\frac{\ln \sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right)$$

$$= \frac{\sqrt{3} \ln \sqrt{3} + \sqrt{3} - 3}{3}$$

The result is then

$$A = \boxed{\frac{\pi (9 - 2\sqrt{3})}{36} + \frac{1}{2} \cdot \ln \frac{2}{3} - \frac{\sqrt{3} \ln \sqrt{3} + \sqrt{3} - 3}{3}}$$

(b)

$$\boxed{\int_{-\ln \sqrt{3}}^0 \pi \left[(e^y)^2 - \left(\frac{1}{\sqrt{3}} \right)^2 \right] dy + \int_0^{\pi/6} \pi \left[(1)^2 - \left(\frac{1}{\sqrt{3}} \right)^2 \right] dy}$$

$$+ \int_{\pi/6}^{\pi/4} \pi [1^2 - (\tan y)^2] dy$$

(c)

$$\boxed{\int_{-\ln \sqrt{3}}^0 2\pi (2 - y) \left(e^y - \frac{1}{\sqrt{3}} \right) dy + \int_0^{\pi/6} 2\pi (2 - y) \left(1 - \frac{1}{\sqrt{3}} \right) dy}$$

$$+ \int_{\pi/6}^{\pi/4} 2\pi (2 - y) (1 - \tan y) dy$$

3.

(a) Determine the n th partial sum.

$$\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1)) = (\cancel{\arctan 1} - \arctan 0) + (\cancel{\arctan 2} - \cancel{\arctan 1})$$

$$+ (\cancel{\arctan 3} - \cancel{\arctan 2}) + (\cancel{\arctan 4} - \cancel{\arctan 3}) + \dots$$

$$+ (\arctan n - \cancel{\arctan(n-1)}) = \arctan n - \arctan 0$$

$$= \arctan n$$

$$\lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \quad (\text{convergent})$$

Therefore, the series $\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1))$ converges.

(b)

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}} = \sum_{n=0}^{\infty} \frac{-3 \cdot (-3)^n}{8^n} = -3 \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n$$

This is a geometric series where $r = -\frac{3}{8}$. $|r| = \frac{3}{8} < 1$. Therefore, the series $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$ converges.

(c) Apply the n th Term Test for divergence. We may take the limit inside the trigonometric function because $\cos x$ is continuous everywhere. Since $1 + n^4$ grows faster than n^2 , the value of the expression $\frac{n^2}{1 + n^4}$ tends to zero.

$$\lim_{n \rightarrow \infty} \cos \left(\frac{n^2}{1 + n^4} \right) = \cos \left[\lim_{n \rightarrow \infty} \left(\frac{n^2}{1 + n^4} \right) \right] = \cos 0 = 1 \neq 0$$

By the n th Term Test for divergence, the series $\sum_{n=1}^{\infty} \cos \left(\frac{n^2}{1 + n^4} \right)$ diverges.

(d) Take $f(x) = \frac{1}{x(\ln x)^2}$. f is positive and decreasing for $x \geq 2$ because x and $(\ln x)^2$ are positive and increasing for $x \geq 2$. x is a polynomial which is defined everywhere and $(\ln x)^2$ is continuous for $x \geq 2$. Since we took into account every criterion, we may apply the Integral Test. Handle the improper integrals by taking the limit.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{R \rightarrow \infty} \int_2^R \frac{dx}{x(\ln x)^2} = \lim_{R \rightarrow \infty} \left[-\frac{1}{\ln x} \right]_2^R = \lim_{R \rightarrow \infty} \left[-\frac{1}{\ln R} + \frac{1}{\ln 2} \right] = \frac{1}{\ln 2}$$

Since the integral converges, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ also converges.