2023-2024 Spring
MAT124 Make-up
(27/06/2024)

1. Find the maximum and minimum values of $f(x, y, z) = x - y + z$ on the sphere $x^2 + y^2 + z^2 = 100$.

2. Sketch the region and reverse the order of the double integral

$$\int_0^{\cancel{4}\,3} \int_{y/3}^{\sqrt{4-y}} dx\, dy$$

3. Using a polar double integral, find the volume of the sphere with radius 4.

4. Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies in the cylinder $x^2 + y^2 = 1$.

5. Using the change of variables, evaluate the area of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$.

6. Let R be the solid region bounded below by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 9$. Let

$$\mathrm{I} = \iiint_R (x^2 + y^2)\, dV.$$

(i) Express (but do not evaluate) I as a triple integral in spherical coordinates.

(ii) Express (but do not evaluate) I as a triple integral in cylindrical coordinates.

2023-2024 Spring Make-up (27/06/2024) Solutions
(Last update: 8/4/25 (4th of August) 2:20 PM)

1. Let $g(x, y, z) = x^2 + y^2 + z^2 - 100$ for the constraint. Solve the system of equations below.

$$\left.\begin{array}{l}\nabla f = \lambda \nabla g \\ g(x,y,z) = 0\end{array}\right\} \qquad \nabla f = \langle 1, -1, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle = \lambda \nabla g$$

$$1 - 2\lambda x = 0 \implies x = \frac{1}{2\lambda}$$
$$-1 - 2\lambda y = 0 \implies y = -\frac{1}{2\lambda}$$
$$1 - 2\lambda z = 0 \implies z = \frac{1}{2\lambda}$$

Use the constraint to find the values of x, y, z.

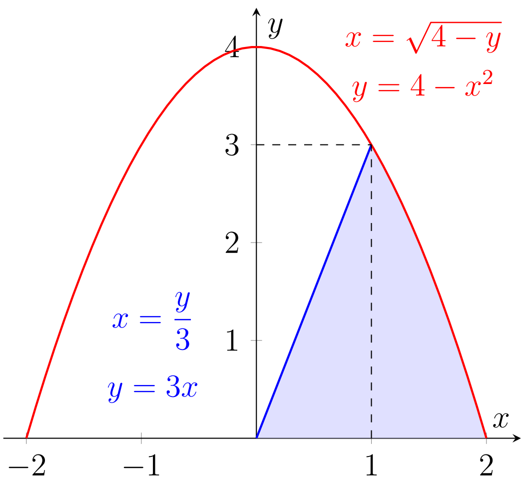
$$x^2 + y^2 + z^2 - 100 = 0 \implies \left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = \frac{3}{4\lambda^2} = 100 \implies \lambda = \pm\frac{\sqrt{3}}{20}$$

$$x = \frac{1}{2\lambda} = \pm\frac{10\sqrt{3}}{3}, \quad y = -\frac{1}{2\lambda} = \pm\frac{10\sqrt{3}}{3}, \quad z = \frac{1}{2\lambda} = \pm\frac{10\sqrt{3}}{3}$$

To find the maximum and minimum values of f, use the points $\left(\frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}\right)$ and $\left(-\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}\right)$, respectively.

$$\boxed{\begin{aligned} f_{\text{max}} &= \frac{10\sqrt{3}}{3} - \left(-\frac{10\sqrt{3}}{3}\right) + \frac{10\sqrt{3}}{3} = 10\sqrt{3} \\ f_{\text{min}} &= -\frac{10\sqrt{3}}{3} - \frac{10\sqrt{3}}{3} - \frac{10\sqrt{3}}{3} = -10\sqrt{3} \end{aligned}}$$

2. **Remark**: It is counterintuitive that the area of the region is negative for $y > 3$. Therefore, let's assume that the upper bound for y is 3 rather than the value stated in the original question. The lecturer might have made a typo.



$$\boxed{\int_0^1 \int_0^{3x} dy\,dx + \int_1^2 \int_0^{4-x^2} dy\,dx}$$

3. The equation of this sphere is $x^2 + y^2 + z^2 = 16$. Solve for z to find the bounds of z.

$$z_{\text{lower}} = -\sqrt{16 - x^2 - y^2}, \quad z_{\text{upper}} = \sqrt{16 - x^2 - y^2}$$

If we project the sphere onto the xy-plane, we will notice that the domain is $x^2 + y^2 \leq 16$. Use the transformation for polar coordinates.

$$\begin{array}{c} x = r\cos\theta \\ y = r\sin\theta \\ x^2 + y^2 = r^2 \\ dA = r\,dr\,d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} z = \sqrt{16 - x^2 - y^2} \implies z = \sqrt{16 - r^2} \\ z = -\sqrt{16 - x^2 - y^2} \implies z = -\sqrt{16 - r^2} \\ x^2 + y^2 \leq 16 \implies r^2 \leq 16 \implies 0 \leq r \leq 4, \quad 0 \leq \theta \leq 2\pi \end{array}$$

$$\mathrm{I} = \int_0^{2\pi} \int_0^4 \left[\sqrt{16 - r^2} - \left(\sqrt{16 - r^2}\right)\right] r\,dr\,d\theta = 2\int_0^{2\pi} \int_0^4 r\sqrt{16 - r^2}\,dr\,d\theta$$

$$= 2\int_0^{2\pi} \left[-\frac{1}{3}\left(16 - r^2\right)^{3/2}\right]_{r=0}^{r=4} d\theta = \frac{2}{3}\int_0^{2\pi} [0 - (-64)]\,d\theta = \frac{128}{3}\int_0^{2\pi} d\theta = \boxed{\frac{256\pi}{3}}$$

4. The domain is $x^2 + y^2 \leq 1$. Using the double integral below, we find the surface area.

$$\text{Surface area} = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + (2x)^2 + (2y)^2}\,dy\,dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx$$

If we switch to polar coordinates, we can easily evaluate the integral.

$$\text{Surface area} = \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\,r\,dr\,d\theta = \frac{1}{12}\int_0^{2\pi} \left[\left(1 + 4r^2\right)^{3/2}\right]_{r=0}^{r=1} d\theta$$

$$= \frac{1}{12}\int_0^{2\pi} \left(5\sqrt{5} - 1\right) d\theta = \boxed{\frac{\pi}{6}\left(5\sqrt{5} - 1\right)}$$

5. Let $x = 4r\cos\theta,\ y = 5r\sin\theta$.

$$\frac{x^2}{16} + \frac{y^2}{25} = 1 \implies \frac{(4r\cos\theta)^2}{16} + \frac{(5r\sin\theta)^2}{25} = 1 \implies r^2\left(\sin^2\theta + \cos^2\theta\right) = 1$$

$$r^2 = 1 \implies r = 1 \qquad 0 \leq \theta \leq 2\pi$$

Calculate the Jacobian determinant.

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 4\cos\theta & -4r\sin\theta \\ 5\sin\theta & 5r\cos\theta \end{vmatrix} = 20r\cos^2\theta - \left(-20r\sin^2\theta\right) = 20r$$

Then we have the integral

$$\int_0^{2\pi}\int_0^1 \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \int_0^{2\pi}\int_0^1 20r\,dr\,d\theta = 20\int_0^{2\pi}\left[\frac{1}{2}r^2\right]_{r=0}^{r=1} d\theta = 10\int_0^{2\pi} d\theta = \boxed{20\pi}$$

6.

(i) For spherical coordinates, we have

$$\begin{gathered} z = \rho\cos\theta \\ r = \rho\sin\theta \\ x^2+y^2 = r^2 \\ x^2+y^2+z^2 = \rho^2 \\ dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{gathered} \quad \rightarrow \quad \begin{gathered} z = \sqrt{3x^2+3y^2} \implies \rho\cos\theta = \sqrt{3}\rho\sin\theta \implies \theta = \frac{\pi}{6} \\ x^2+y^2 = r^2 = \rho^2\sin^2\theta \\ x^2+y^2+z^2 = 9 \implies \rho^2 = 9 \implies \rho = 3 \\ 0 \le \theta \le 2\pi \end{gathered}$$

The integral in spherical coordinates can be expressed as follows.

$$\boxed{\mathrm{I} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho^2\sin^2\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta}$$

(ii) For cylindrical coordinates, we have

$$\begin{gathered} z = z \\ r^2 = x^2+y^2 \\ dV = r\,dz\,dr\,d\theta \end{gathered} \quad \rightarrow \quad \begin{gathered} z = \sqrt{3x^2+3y^2} \implies z = r\sqrt{3} \\ x^2+y^2 = r^2 \\ x^2+y^2+z^2 = 9 \implies z = \sqrt{9-r^2} \\ 0 \le \theta \le 2\pi \end{gathered}$$

Find where the curves intersect to find the upper limit of r.

$$r\sqrt{3} = \sqrt{9-r^2} \implies 3r^2 = 9 - r^2 \implies r^2 = \frac{9}{4} \implies r = \frac{3}{2}$$

The integral in cylindrical coordinates can be expressed as follows.

$$\boxed{\mathrm{I} = \int_0^{2\pi}\int_0^{3/2}\int_{r\sqrt{3}}^{\sqrt{9-r^2}} r^2\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{3/2}\int_{r\sqrt{3}}^{\sqrt{9-r^2}} r^3\,dz\,dr\,d\theta}$$