1. (a) Find $\displaystyle\int \frac{dx}{x^3 - 4x^2 + 3x}$.

(b) Evaluate $\displaystyle\lim_{x \to \frac{\pi}{2}} \frac{\displaystyle\int_{\pi/2}^{x} \ln(\sin t)\, dt}{\sin x - 1}$.

(c) Evaluate the improper integral $\displaystyle\int_{1}^{2} \frac{dx}{(x-1)^{2/3}}$.

2. Consider the region R bounded by the curves $y = \arctan x$, $y = \ln x$ and the lines $x = \dfrac{1}{\sqrt{3}}$ and $x = 1$.

(a) Find the area of the region R.

(b) Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region R about the y-axis.

(c) Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of a solid obtained by rotating the region R about the line $y = 2$.

3. Determine whether each series is convergent or divergent. Explain your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1))$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{n^2}{1+n^4}\right)$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

1. **(a)** Use the method of partial fraction decomposition.

$$\int \frac{dx}{x^3 - 4x^2 + 3x} = \int \frac{dx}{x(x-3)(x-1)} = \int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1} \right) dx$$

$$A(x-3)(x-1) + Bx(x-1) + Cx(x-3) = 1$$
$$x^2(A+B+C) + x(-4A-B-3C) + 3A = 1$$

Equate the coefficients of like terms.

$$\left. \begin{array}{r} x^2(A+B+C) = 0 \\ x(-4A-B-3C) = 0 \\ 3A = 1 \end{array} \right\} \to A = \frac{1}{3}, \qquad \left. \begin{array}{r} B+C = -\dfrac{1}{3} \\[2mm] B+3C = -\dfrac{4}{3} \end{array} \right\} \to B = \frac{1}{6}, \quad C = -\frac{1}{2}$$

Rewrite the integral.

$$\int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1} \right) dx = \int \left(\frac{1}{3x} + \frac{1}{6(x-3)} - \frac{1}{2(x-1)} \right) dx$$

$$= \boxed{ \frac{1}{3} \ln |x| + \frac{1}{6} \ln |x-3| - \frac{1}{2} \ln |x-1| + c, \quad c \in \mathbb{R} }$$

(b) The limit is in the indeterminate form $0/0$. Apply L'Hôpital's rule to eliminate the indeterminate form.

$$\lim_{x \to \frac{\pi}{2}} \frac{\displaystyle\int_{\pi/2}^{x} \ln(\sin t)\, dt}{\sin x - 1} \overset{\text{L'H.}}{=} \lim_{x \to \frac{\pi}{2}} \frac{\dfrac{d}{dx} \displaystyle\int_{\pi/2}^{x} \ln(\sin t)\, dt}{\cos x}$$
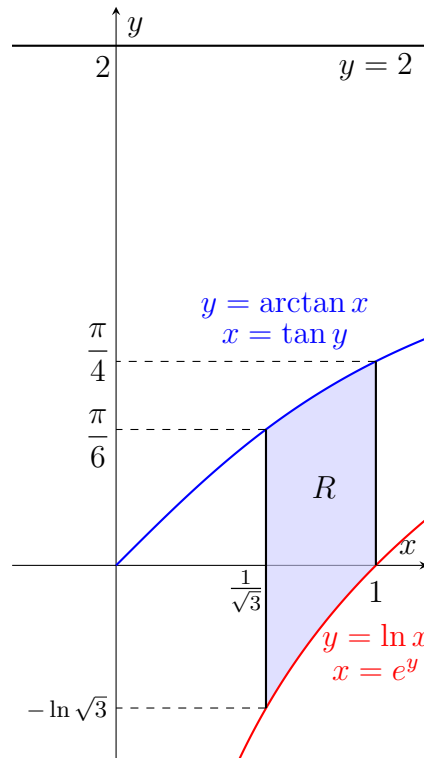
By the Fundamental Theorem of Calculus, we may rewrite the limit as follows.

$$\lim_{x \to \frac{\pi}{2}} \frac{\dfrac{d}{dx} \displaystyle\int_{\pi/2}^{x} \ln(\sin t)\, dt}{\cos x} = \lim_{x \to \frac{\pi}{2}} \frac{\ln(\sin x)}{\cos x} \overset{\text{L'H.}}{=} \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\sin x} \cdot \cos x}{-\sin x} = \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{-\sin^2 x} = -\frac{\cos \frac{\pi}{2}}{\sin^2 \frac{\pi}{2}}$$

$$= \boxed{0}$$

(c) Take the limit as this is an improper integral.

$$\int_{1}^{2} \frac{dx}{(x-1)^{2/3}} = \lim_{R \to 1^+} \int_{R}^{2} \frac{dx}{(x-1)^{2/3}} = \lim_{R \to 1^+} 3(x-1)^{1/3} \Big|_{R}^{2} = 3 \lim_{R \to 1^+} \left(1 - (R-1)^{1/3} \right) = \boxed{3}$$

2.

(a)

$$A = \int_{1/\sqrt{3}}^{1} (\arctan x - \ln x) \; dx = \int_{1/\sqrt{3}}^{1} \arctan x \; dx - \int_{1/\sqrt{3}}^{1} \ln x \; dx \qquad (1)$$

Calculate the first integral in (1) by integration by parts.

$$\left. \begin{array}{c} u = \arctan x \implies du = \dfrac{1}{x^2+1}\,dx \\[2mm] dv = dx \implies v = x \end{array} \right\} \to \int u\,dv = uv - \int v\,du$$

$$\int_{1/\sqrt{3}}^{1} \arctan x \; dx = x \arctan x \Big|_{1/\sqrt{3}}^{1} - \int_{1/\sqrt{3}}^{1} \frac{x}{x^2+1}\,dx = \left(x \arctan x - \frac{1}{2}\ln\left|x^2+1\right| \right)\Bigg|_{1/\sqrt{3}}^{1}$$

$$= \left(\frac{\pi}{4} - \frac{\ln 2}{2} \right) - \left(\frac{\pi\sqrt{3}}{18} - \frac{1}{2}\cdot\ln\left(\frac{4}{3}\right) \right) = \frac{\pi\left(9 - 2\sqrt{3}\right)}{36} + \frac{1}{2}\cdot\ln\frac{2}{3}$$

Calculate the second integral in (1) by integration by parts.

$$\left. \begin{array}{c} u = \ln x \implies du = \dfrac{1}{x}\,dx \\[2mm] dv = dx \implies v = x \end{array} \right\} \to \int u\,dv = uv - \int v\,du$$

$$\int_{1/\sqrt{3}}^{1} \ln x \, dx = x \ln x \Big|_{1/\sqrt{3}}^{1} - \int_{1/\sqrt{3}}^{1} dx = (x \ln x - x) \Big|_{1/\sqrt{3}}^{1} = (0 - 1) - \left(-\frac{\ln \sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right)$$

$$= \frac{\sqrt{3} \ln \sqrt{3} + \sqrt{3} - 3}{3}$$

The result is then

$$A = \boxed{\frac{\pi \left(9 - 2\sqrt{3}\right)}{36} + \frac{1}{2} \cdot \ln \frac{2}{3} - \frac{\sqrt{3} \ln \sqrt{3} + \sqrt{3} - 3}{3}}$$

(b)

$$\boxed{\begin{aligned} &\int_{-\ln \sqrt{3}}^{0} \pi \left[(e^y)^2 - \left(\frac{1}{\sqrt{3}} \right)^2 \right] dy + \int_{0}^{\pi/6} \pi \left[(1)^2 - \left(\frac{1}{\sqrt{3}} \right)^2 \right] dy \\ &+ \int_{\pi/6}^{\pi/4} \pi \left[(1^2) - (\tan y)^2 \right] dy \end{aligned}}$$

(c)

$$\boxed{\begin{aligned} &\int_{-\ln \sqrt{3}}^{0} 2\pi \left(2 - y\right) \left(e^y - \frac{1}{\sqrt{3}} \right) dy + \int_{0}^{\pi/6} 2\pi \left(2 - y\right) \left(1 - \frac{1}{\sqrt{3}} \right) dy \\ &+ \int_{\pi/6}^{\pi/4} 2\pi \left(2 - y\right) \left(1 - \tan y\right) dy \end{aligned}}$$

3. (a) Determine the nth partial sum.

$$\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1)) = (\cancel{\arctan 1} - \arctan 0) + (\cancel{\arctan 2} - \cancel{\arctan 1})$$

$$+ (\cancel{\arctan 3} - \cancel{\arctan 2}) + (\cancel{\arctan 4} - \cancel{\arctan 3}) + \dots$$

$$+ (\arctan n - \cancel{\arctan(n-1)}) = \arctan n - \arctan 0$$

$$= \arctan n$$

$$\lim_{n \to \infty} \arctan n = \frac{\pi}{2} \quad \text{(convergent)}$$

Therefore, the series $\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1))$ converges.

(b)

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}} = \sum_{n=0}^{\infty} \frac{-3 \cdot (-3)^n}{8^n} = -3 \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n$$

This is a geometric series where $r = -\frac{3}{8}$. $|r| = \frac{3}{8} < 1$. Therefore, the series $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$ converges.

(c) Apply the nth Term Test for divergence. We may take the limit inside the trigonometric function because $\cos x$ is continuous everywhere. Since $1 + n^4$ grows faster than n^2, the value of the expression $\frac{n^2}{1 + n^4}$ tends to zero.

$$\lim_{n \to \infty} \cos\left(\frac{n^2}{1 + n^4}\right) = \cos\left[\lim_{n \to \infty} \left(\frac{n^2}{1 + n^4}\right)\right] = \cos 0 = 1 \neq 0$$

By the nth Term Test for divergence, the series $\sum_{n=1}^{\infty} \cos\left(\frac{n^2}{1 + n^4}\right)$ diverges.

(d) Take $f(x) = \frac{1}{x(\ln x)^2}$. f is positive and decreasing for $x \geq 2$ because x and $(\ln x)^2$ are positive and increasing for $x \geq 2$. x is a polynomial which is defined everywhere and $(\ln x)^2$ is continuous for $x \geq 2$. Since we took into account every criterion, we may apply the Integral Test. Handle the improper integrals by taking the limit.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{R \to \infty} \int_2^R \frac{dx}{x(\ln x)^2} = \lim_{R \to \infty} \left[-\frac{1}{\ln x}\right]_2^R = \lim_{R \to \infty} \left[-\frac{1}{\ln R} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

Since the integral converges, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ also converges.