MAT123 31.10.2025

QUESTIONS

Q1: Let f be the function defined by $f(x) = \begin{cases} (e-2)\,x^2, & x \leq 0 \\ \dfrac{e^x - x - 1}{x}, & x > 0 \end{cases}$.

- (a) Is f a continuous function?
- (b) Is f a differentiable function?

Q2: Evaluate $\lim\limits_{x \to 0} \dfrac{\sin\left(\dfrac{1}{x^2}\right)}{\dfrac{1}{x^2}}$.

Q3: Evaluate $\lim\limits_{x \to 0} \dfrac{x}{\sqrt{1 + \tan(3x)} - 1}$ without using L'Hôpital's rule.

Q4: Find the equations of the tangent lines to the curve $y = x^3 + x$, which pass through the point $(2, 2)$.

Q5: Find $\dfrac{dy}{dx}$ from the equation $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$.

Q6: If $x^3 + y^3 = 16$,

- (a) find $\dfrac{dy}{dx}$.
- (b) find the value of $\dfrac{d^2y}{dx^2}$ at the point $(2, 2)$.

Q7: The coordinates of a particle in the metric xy-plane are differentiable functions of time t with $\dfrac{dx}{dt} = -1$ m/s and $\dfrac{dy}{dt} = -5$ m/s. How fast is the particle's distance from the origin changing as it passes through the point $(5, 12)$?

Q8: A spherical iron ball 8 cm in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 cm^3/min, what is the change rate of the thickness of the ice when it is 2 cm thick? What is the rate of change of the outer surface?

Q9: Let $0 < a < b$. Show that $\dfrac{b-a}{1+b^2} < \arctan b - \arctan a < \dfrac{b-a}{1+a^2}$.

Q10: Evaluate $\lim\limits_{x \to 1} \dfrac{x^2 - 4x + 3}{\sqrt{x} - 1}$.

Q11: Evaluate $\lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x^3 - 82x + 9}$.

ANSWERS

Q1: (a) For f to be a continuous function, f must be continuous on its entire domain. For $x \leq 0$, $f(x) = (e-2)\,x^2$ becomes a polynomial expression, which is continuous for $x < 0$. For $x > 0$, $f(x) = \dfrac{e^x - x - 1}{x}$ becomes an expression comprising an exponential and a polynomial in the numerator and a polynomial in the denominator, which is continuous for $x > 0$. The *only* point at which we need to check the continuity is $x = 0$.

Check if $f(0)$ is defined.

$$f(0) = (e-2)0^2 = 0 \implies \text{It is defined at } x = 0.$$

Check the one-sided limits. Let's first check the limit from the left.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (e-2)x^2 = (e-2)0^2 = 0$$

Check the limit from the right.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} \frac{e^x - x - 1}{x}$$

Notice that if we put $x = 0$, the expression is in the form $\dfrac{0}{0}$. To eliminate this indeterminate form, we rearrange the limit with the following steps.

$$\lim_{x \to 0^+} \frac{e^x - x - 1}{x} = \lim_{x \to 0^+} \frac{e^x - 1}{x} - \underbrace{\lim_{x \to 0^+} \frac{x}{x}}_{1} = \lim_{x \to 0^+} \frac{e^x - 1}{x} - 1$$

Recall the definition of the derivative from the right side. Let $g(x) = e^x$, then we may rewrite the resulting limit as

$$\lim_{x \to 0^+} \frac{e^x - 1}{x} - 1 = \lim_{x \to 0^+} \frac{e^x - e^0}{x - 0} - 1 = \lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0} - 1 = g'_+(0) - 1$$

The first derivative of e^x from the right side is e^x. Therefore,

$$g'_+(0) - 1 = e^0 - 1 = 1 - 1 = 0$$

Since $\lim\limits_{x \to 0} f(x) = f(0)$, the criteria for continuity are satisfied. Therefore, at $x = 0$, f is continuous.

(b) f is differentiable for $x \neq 0$. If f is differentiable at $x = 0$, then f is differentiable on its entire domain. We check whether the following equality holds.

$$\lim_{x \to 0} f'(x) = f'(0)$$

Investigate the right- and left-hand derivatives.

$$f'_{-}(x) = (e-2)\cdot 2x \implies f'_{-}(0) = 0$$

$$f'_{+}(x) = \frac{(e^x-1)x - (e^x - x - 1)\cdot 1}{x^2} = \frac{xe^x - e^x + 1}{x^2}$$
$$f'_{+}(0) \overset{\text{L'H.}}{=} \lim_{x\to 0^+} \frac{e^x(x-1)+e^x}{2x} = \lim_{x\to 0^+} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$$

Since $f'_{+}(0) \neq f'_{-}(0)$, f is not differentiable at $x=0$.

Q2: Rewrite the limit.

$$\lim_{x\to 0} \frac{\sin\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}} = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right)$$

We cannot separate this limit into two different limits because the limit $\lim_{x\to 0}\sin\left(\frac{1}{x^2}\right)$ does not exist. The inside expression oscillates wildly as $x \to 0$. We will solve this limit by applying the squeeze theorem.

We have the inequality below for all $x \in \mathbb{R}\setminus\{0\}$

$$-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$$

Multiply each side by x^2. The inequality direction stays the same because $x^2 \geq 0$.

$$-x^2 \leq x^2\sin\left(\frac{1}{x^2}\right) \leq x^2$$

Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, by the squeeze theorem, the limit $\lim_{x\to 0} x^2\sin\left(\frac{1}{x^2}\right)$ is also $\boxed{0}$.

Q3: Multiply and divide by the conjugate of the denominator.

$$L = \lim_{x\to 0}\frac{x}{\sqrt{1+\tan(3x)}-1} = \lim_{x\to 0}\frac{x}{\sqrt{1+\tan(3x)}-1}\cdot\frac{\sqrt{1+\tan(3x)}+1}{\sqrt{1+\tan(3x)}+1}$$
$$L = \lim_{x\to 0}\frac{x\sqrt{1+\tan(3x)}+x}{\tan(3x)} = \lim_{x\to 0}\frac{x(\sqrt{1+\tan(3x)}+1)}{\tan(3x)}$$

Recall that $\lim_{u\to 0}\frac{\sin u}{u} = 1$. Apply the **quotient rule** and rearrange the limit.

$$L = \lim_{x\to 0}\frac{\sqrt{1+\tan(3x)}+1}{\frac{\tan(3x)}{x}} = \lim_{x\to 0}\frac{\sqrt{1+\tan(3x)}+1}{\frac{\sin(3x)}{x\cos(3x)}\cdot\frac{3}{3}} = \frac{\lim_{x\to 0}\left(\sqrt{1+\tan(3x)}+1\right)}{\underbrace{\lim_{x\to 0}\frac{\sin(3x)}{3x}}_{1}\cdot\lim_{x\to 0}\frac{3}{\cos(3x)}}$$

$$L = \frac{\frac{\sqrt{1+\tan 0}+1}{3}}{\cos(0)} = \boxed{\frac{2}{3}}$$

Q4: We have the point of tangency (a, a^3+a) for $x = a$ on the curve. The derivative at this point is

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = 3a^2 + 1.$$

Recall the straight line formula $y - y_0 = m(x - x_0)$, where m is the slope of the line. Since we have the derivative at $x = a$, m is just $3a^2 + 1$. We also have the point $(a, a^3 + a)$ on the line. So,

$$y - (a^3 + a) = (3a^2 + 1)(x - a)$$

To find the specific values for a, try the point $(2, 2)$.

$$2 - a^3 - a = (3a^2 + 1)(2 - a) \implies 2 - a^3 - a = 6a^2 - 3a^3 + 2 - a \implies 6a^2 - 2a^3 = 0$$
$$\implies 2a^2(3 - a) = 0 \implies a_1 = 0, \quad a_2 = 3$$

If $a = 0$, we have the tangent line $y = x$. If $a = 3$, we have $y - 30 = 28x - 84 \implies y = 28x - 54$.

$$\boxed{\text{Tangent lines:} \quad y = x, \quad y = 28x - 54}$$

Q5: Use **implicit differentiation**, where $y = f(x)$. Apply **the product rule** and **the chain rule**.

$$y \sin\left(\frac{1}{y}\right) = 1 - xy$$
$$\frac{d}{dx}\left[y \sin\left(\frac{1}{y}\right)\right] = \frac{d}{dx}(1 - xy)$$
$$\frac{dy}{dx} \cdot \sin\left(\frac{1}{y}\right) + y \cdot \cos\left(\frac{1}{y}\right) \cdot \left(-\frac{1}{y^2}\right) \cdot \frac{dy}{dx} = -1 \cdot y - x\frac{dy}{dx}$$

Regroup the terms to solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} \cdot \sin\left(\frac{1}{y}\right) + x\frac{dy}{dx} - \frac{1}{y} \cdot \cos\left(\frac{1}{y}\right) \cdot \frac{dy}{dx} = -y$$
$$\frac{dy}{dx}\left[\sin\left(\frac{1}{y}\right) + x - \frac{1}{y} \cdot \cos\left(\frac{1}{y}\right)\right] = -y$$
$$\boxed{\frac{dy}{dx} = \frac{-y}{\sin\left(\frac{1}{y}\right) + x - \frac{1}{y} \cdot \cos\left(\frac{1}{y}\right)}}$$

Q6: (a) Apply implicit differentiation, where $y = f(x)$.

$$x^3 + y^3 = 16$$
$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(16)$$
$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 0 \implies \boxed{\frac{dy}{dx} = -\frac{x^2}{y^2}}$$

(b) Take the second derivative using the result of (a).

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x^2}{y^2}\right)$$
$$\frac{d^2y}{dx^2} = -\frac{2x \cdot y^2 - x^2 \cdot 2y \cdot \frac{dy}{dx}}{y^4} = \frac{2x\left(-y + x\frac{dy}{dx}\right)}{y^3}$$

Substitute $-\frac{x^2}{y^2}$ into $\frac{dy}{dx}$.

$$\frac{d^2y}{dx^2} = \frac{2x\left(-y - x \cdot \frac{x^2}{y^2}\right)}{y^3} = -\frac{2xy + \frac{2x^4}{y^2}}{y^3}$$

Find $\left.\frac{d^2y}{dx^2}\right|_{(2,2)}$.

$$\left.\frac{d^2y}{dx^2}\right|_{(2,2)} = -\frac{2 \cdot 2 \cdot 2 + \frac{2 \cdot 2^4}{2^2}}{2^3} = \boxed{-2}$$

Q7: The distance $D(t)$ from the origin to the particle can be expressed using the Pythagorean theorem below:

$$D^2(t) = x^2(t) + y^2(t),$$

where $x(t)$ and $y(t)$ indicate the $x-$ and $y-$ position of the particle, respectively. Take the derivative of both sides.

$$\frac{d}{dt}\left(D^2(t)\right) = \frac{d}{dt}\left(x^2(t) + y^2(t)\right)$$
$$2D(t)\frac{dD}{dt} = 2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt}$$

We're asked to find $\frac{dD}{dt}$. Therefore, solve for $\frac{dD}{dt}$.

$$\frac{dD}{dt} = \frac{x(t)\frac{dx}{dt} + y(t)\frac{dy}{dt}}{D(t)}$$

Using the info given in the question, the answer is

$$\frac{dD}{dt} = \frac{5(-1)+12(-5)}{\sqrt{5^2+12^2}} = \boxed{-5 \text{ m/s}}$$

Q8: Let $r_1(t)$ be the thickness of the ice layer. Then the radius of the sphere is $r_1(t)+\frac{8}{2} = r_1(t)+4$ cm. The volume of the ice can be calculated using the formula

$$V = \frac{4}{3}\pi\left(r_1(t)+3\right)^3 - \frac{4}{3}\pi\cdot 8^3 = \frac{4}{3}\pi\left[(r_1(t)+4)^3 - 8^3\right]$$

Given that the volume decreases at the rate of 10 cm^3/ min, that is, $\dfrac{dV}{dt} = -10$,

$$\frac{dV}{dt} = \frac{d}{dt}\left\{\frac{4}{3}\pi\left[(r_1(t)+4)^3 - 8^3\right]\right\} = 4\pi\cdot[r_1(t)+4]^2\cdot\frac{dr_1}{dt}$$

Also given in the question that $r_1(t) = 2$ cm, calculate the rate of change of thickness, which is $\dfrac{dr_1}{dt}$. Solve the equation above for $\dfrac{dr_1}{dt}$

$$\frac{dr_1}{dt} = \frac{\frac{dV}{dt}}{4\pi\cdot[r_1(t)+4]^2} = \frac{-10}{4\pi\cdot 6^2} = \boxed{-\frac{5}{72\pi} \text{ cm/min}}$$

The formula for the outer surface is

$$S(t) = 4\pi\cdot(r_1(t)+4)^2$$

Take the derivative of both sides.

$$\frac{dS}{dt} = 8\pi\cdot(r_1(t)+4)\cdot\frac{dr_1}{dt}$$

We can calculate the rate of change easily.

$$\frac{dS}{dt} = 8\pi\cdot 6\cdot\left(-\frac{5}{72\pi}\right) = \boxed{-\frac{10}{3} \text{ cm}^2\text{/min}}$$

Q9: We will use the **Mean Value Theorem (MVT)**. We have $0 < a < b$, so we can investigate the interval $a < x < b$.

Let $f(x) = \arctan x$. f is continuous and differentiable on its entire domain. This implies that f is continuous on $[a, b]$ and differentiable on (a, b). Then by MVT, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Therefore, we have

$$f'(c) = \frac{\arctan b - \arctan a}{b-a}$$

We also have $f'(x) = \dfrac{1}{1+x^2} \implies f'(c) = \dfrac{1}{1+c^2} = \dfrac{\arctan b - \arctan a}{b-a}$. From the inequality $a < c < b$,

$$a^2 < c^2 < b^2 \qquad [0 < a < c < b]$$

$$a^2 + 1 < c^2 + 1 < b^2 + 1$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \quad \text{[The inequality direction is reversed]}$$

Recall $\dfrac{1}{1+c^2} = \dfrac{\arctan b - \arctan a}{b-a}$. Substitute it into the inequality.

$$\frac{1}{1+a^2} > \frac{\arctan b - \arctan a}{b-a} > \frac{1}{1+b^2}$$

$$\frac{b-a}{1+a^2} > \arctan b - \arctan a > \frac{b-a}{1+b^2}$$

$$\boxed{\frac{b-a}{1+b^2} < \arctan b - \arctan a < \frac{b-a}{1+a^2}}$$

Q10: Multiply and divide by the conjugate of the denominator.

$$\begin{aligned} L &= \lim_{x\to 1} \frac{x^2-4x+3}{\sqrt{x}-1} = \lim_{x\to 1} \frac{(x-3)(x-1)}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{(x-3)(x-1)(\sqrt{x}+1)}{x-1} \\ &= \lim_{x\to 1} (x-3)(\sqrt{x}+1) = -2\cdot 2 = \boxed{-4} \end{aligned}$$

Q11: Rearrange the limit.

$$\begin{aligned} L &= \lim_{x\to 9} \frac{\sqrt{x}-3}{x^3-82x+9} = \lim_{x\to 9} \frac{\sqrt{x}-3}{x^3-81x-x+9} = \lim_{x\to 9} \frac{\sqrt{x}-3}{x(x^2-81)-(x-9)} \\ &= \lim_{x\to 9} \frac{\sqrt{x}-3}{(x-9)\cdot x(x+9)-(x-9)\cdot 1} = \lim_{x\to 9} \frac{\sqrt{x}-3}{(x-9)\left[x(x+9)-1\right]} \\ &= \lim_{x\to 9} \frac{\sqrt{x}-3}{(x-9)(x^2+9x-1)} \end{aligned}$$

Multiply and divide by the conjugate of the numerator.

$$\begin{aligned} L &= \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(x^2+9x-1)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{x-9}{(x-9)(x^2+9x-1)(\sqrt{x}+3)} \\ &= \lim_{x\to 9} \frac{1}{(x^2+9x-1)(\sqrt{x}+3)} = \frac{1}{(9^2+9\cdot 9-1)(\sqrt{9}+3)} = \boxed{\frac{1}{966}} \end{aligned}$$

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