2011-2012 Spring
MAT123-[Instructor] Final
(29/05/2012)
Time: 15:00 - 16:50
Duration: 110 minutes

1. Find the length of the curve $y^2 = 4(x+1)^3$ for $0 \le x \le 1,\ y > 0$.

2. Given $f(x) = x + 2x^2 + x^3$, find $\left(f^{-1}\right)'(4)$.

3. Evaluate the following integrals.

   (a) $\int \cos^3 x \sin^2 x \, dx$ (b) $\int \frac{x^2}{\sqrt{16 - x^2}} \, dx$ (c) $\int \frac{x^3 - 1}{x^3 - x} \, dx$ (d) $\int x^{123} \ln x \, dx$

4. Evaluate $\lim\limits_{x \to \infty} (\ln x)^{\frac{1}{x}}$.

5.

(a) Determine the series $\sum\limits_{n=1}^{\infty} \frac{5}{3^n}$ converges or diverges. Give reasons for your answer.

(b) Determine the series $\sum\limits_{n=1}^{\infty} \cos\left(\frac{1}{5^n}\right)$ converges or diverges. Give reasons for your answer.

(c) Use the Ratio Test to determine if the series $\sum\limits_{n=1}^{\infty} \frac{n!}{e^{2n}}$ converges or diverges.

(d) Use the Integral Test to determine if the series $\sum\limits_{n=1}^{\infty} \frac{n}{e^{n^2}}$ converges or diverges. Be sure to check that the conditions of the integral test are satisfied.

(Bonus)

(a) Geometrically, what does $f'(x)$ mean?

(b) If $f(t)$ describes the displacement of an object in time t, what is $f'(t)$?

2011-2012 Spring Final (29/05/2012) Solutions
(Last update: 30/08/2025 00:25)

1. y is implicitly defined as a function of x. Differentiate each side and solve for $\dfrac{dy}{dx}$.

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(4(x+1)^3\right) \implies 2y\frac{dy}{dx} = 12(x+1)^2 \implies \frac{dy}{dx} = \frac{6(x+1)^2}{y}$$

Since we're interested in the upper part of the curve (i.e., $y > 0$), $y = 2(x+1)^{3/2}$.

$$\frac{dy}{dx} = \frac{6(x+1)^2}{2(x+1)^{3/2}} = 3\sqrt{x+1}$$

The length of a curve defined by $y = f(x)$ whose derivative is continuous on the interval $a \leq x \leq b$ can be evaluated using the integral

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Set $a = 0,\ b = 1,\ \dfrac{dy}{dx} = 3\sqrt{x+1}$ and find the length.

$$S = \int_0^1 \sqrt{1 + \left(3\sqrt{x+1}\right)^2}\, dx = \int_0^1 \sqrt{9x+10}\, dx$$

Let $u = 9x + 10$, then $du = 9\, dx$.

$$S = \int_0^1 \sqrt{9x+10}\, dx = \int \frac{1}{9}\sqrt{u}\, du = \frac{1}{9}\cdot\frac{2}{3}u^{3/2} + c = \frac{2}{27}\left(9x+10\right)^{3/2}\Big|_0^1$$

$$= \boxed{\frac{2}{27}\left(19^{3/2} - 10^{3/2}\right)}$$

2. The derivative of f^{-1} at a point can be calculated using the rule

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Find the point where $f(x) = 4$. We could intuitively say $f(1) = 4$ because $f(1) = 1 + 2\cdot 1^3 + 1^3 = 4$. Therefore,

$$\left(f^{-1}\right)'(4) = \frac{1}{f'\left(f^{-1}(4)\right)} = \frac{1}{f'(1)}$$

Calculate the derivative of f at the point $x = 1$.

$$f'(x) = 1 + 4x + 3x^2 \implies f'(1) = 1 + 4\cdot 1 + 3\cdot 1^2 = 8$$

So,

$$\left(f^{-1}\right)'(4) = \boxed{\frac{1}{8}}$$

3.

(a)

$$\mathrm{I} = \int \cos^3 x \sin^2 x \, dx \qquad \left[\sin^2 + \cos^2 x = 1\right]$$

$$= \int \cos x \cdot \left(1 - \sin^2 x\right) \cdot \sin^2 x \, dx$$

Let $u = \sin x$, then $du = \cos x \, dx$.

$$\mathrm{I} = \int \cos x \cdot \left(1 - \sin^2 x\right) \cdot \sin^2 x \, dx = \int \left(1 - u^2\right) u^2 \, du = \int \left(u^2 - u^4\right) \, du = \frac{u^3}{3} - \frac{u^5}{5} + c$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c}$$

(b) Let $x = 4 \sin u$, then $dx = 4 \cos u \, du$ for $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

$$\mathrm{I} = \int \frac{x^2}{\sqrt{16 - x^2}} \, dx = \int \frac{16 \sin^2 u}{\sqrt{16 - 16 \sin^2 u}} \cdot 4 \cos u \, du = \int \frac{16 \sin^2 u \cos u}{|\cos u|} \, du \ [\cos u > 0]$$

$$= \int 16 \sin^2 u \, du = 16 \int \left(1 - \cos^2 u\right) \, du = 16 \int \frac{1 - \cos 2u}{2} \, du = 8 \left(u - \frac{\sin 2u}{2}\right) + c$$

$$= 8u - 8 \sin u \cos u + c$$

Recall: $x = 4 \sin u$. Then

$$x^2 = 16 \sin^2 u \implies x^2 = 16 - 16 \cos^2 u \implies \cos^2 u = \frac{16 - x^2}{16} \implies \cos u = \sqrt{1 - \frac{x^2}{16}}$$

$$\sin u = \frac{x}{4} \implies u = \arcsin \frac{x}{4}$$

Rewrite the integral.

$$\mathrm{I} = \boxed{8 \arcsin \frac{x}{4} - 2x \sqrt{1 - \frac{x^2}{16}} + c, \quad c \in \mathbb{R}}$$

(c) Use the method of partial fraction decomposition.

$$\mathrm{I} = \int \frac{x^3-1}{x^3-x}\,dx = \int \frac{(x-1)\left(x^2+x+1\right)}{x(x-1)(x+1)}\,dx = \int \frac{x^2+x+1}{x^2+x}\,dx = \int \left(1+\frac{1}{x^2+x}\right)\,dx$$

$$= \int dx + \int \frac{1}{x(x+1)}\,dx = x + \int \left(\frac{A}{x}+\frac{B}{x+1}\right)\,dx$$

$$\begin{aligned} A(x+1)+B(x) &= 1 \\ x(A+B)+A &= 1 \\ \therefore A+B = 0 \quad [\text{eliminate}\, x] \;\rightarrow\; A &= 1 \implies B = -1 \end{aligned}$$

$$\mathrm{I} = x + \int \left(\frac{A}{x}+\frac{B}{x+1}\right)\,dx = x + \int \left(\frac{1}{x}-\frac{1}{x+1}\right)\,dx = \boxed{x+\ln|x|-\ln|x+1|+c,\; c\in\mathbb{R}}$$

(d) Use the method of integration by parts.

$$\left.\begin{aligned} u = \ln x &\implies du = \frac{1}{x}\,dx \\ dv = x^{123}\,dx &\implies v = \frac{x^{124}}{124} \end{aligned}\right\} \rightarrow \int u\,dv = uv - \int v\,du$$

$$\mathrm{I} = \ln x \cdot \frac{x^{124}}{124} - \int \frac{x^{124}}{124}\cdot\frac{1}{x}\,dx = \boxed{\frac{\ln x\cdot x^{124}}{124} + \frac{x^{124}}{124^2} + c, \quad c\in\mathbb{R}}$$

4. Let L be the value of the limit.

$$L = \lim_{x\to\infty} (\ln x)^{\frac{1}{x}}$$

Take the logarithm of both sides. We can take the logarithm inside the limit because the expression is continuous for $x > 0$. After that, apply L'Hôpital's rule where $0/0$ or ∞/∞ forms occur.

$$\ln(L) = \ln\left(\lim_{x\to\infty} (\ln x)^{\frac{1}{x}}\right) = \lim_{x\to\infty} \ln\left[\ln(x)^{\frac{1}{x}}\right] = \lim_{x\to\infty} \frac{\ln(\ln x)}{x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x\ln x} = 0$$

Since $\ln(L) = 0$, $\boxed{L = 1}$.

5.

(a) Since the numerator is constant, we can take it out of the summation.

$$\sum_{n=1}^{\infty} \frac{5}{3^n} = 5 \cdot \sum_{n=1}^{\infty} \frac{1}{3^n} = 5 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

This is a geometric series with $r = \frac{1}{3} < 1$. Therefore, the series converges.

(b) Take the limit of the sequence at infinity. We can take the limit inside the trigonometric function because it is continuous everywhere.

$$\lim_{n\to\infty} \cos\left(\frac{1}{5^n}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{5^n}\right) = \cos(0) = 1 \neq 0$$

By the nth Term Test for divergence, the series diverges.

(c) Let $a_n = \frac{n!}{\mathrm{e}^{2\mathrm{n}}}$. Then,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{\mathrm{e}^{2(n+1)}} \cdot \frac{\mathrm{e}^{2n}}{n!}\right| = \lim_{n\to\infty} \left|\frac{n+1}{\mathrm{e}^2}\right| = \infty > 1$$

By the Ratio Test, the series diverges.

(d) Let $a_n = f(n)$, where $n \in \mathbb{N}$. The function $f(x) = \frac{x}{\mathrm{e}^{x^2}}$ is positive, continuous and decreasing for $x > 1$.

$$\left.\begin{array}{r} x > 0 \\ \mathrm{e}^{x^2} > 0 \end{array}\right\} \text{ for } x > 1 \implies \frac{x}{\mathrm{e}^{x^2}} > 0$$

e^{x^2} grows at a higher rate than x. Therefore, f is decreasing. The expressions are continuous for $x > 1$. We may now apply the Integral Test. Handle the improper integral with the limit.

$$\int_1^{\infty} \frac{x}{\mathrm{e}^{x^2}}\,dx = \lim_{R\to\infty} -\frac{1}{2}\mathrm{e}^{-x^2}\Big|_1^R = \lim_{R\to\infty} -\frac{1}{2}\left(\mathrm{e}^{-R^2} - \mathrm{e}^{-1}\right) = \frac{1}{2\mathrm{e}} \quad \text{(converges)}$$

By the Integral Test, the series converges.

(Bonus)

(a) Let $y = f(x)$ be a continuous function on a bounded interval, and let f be differentiable on the same interval except possibly at the endpoints. Then $f'(x)$ gives the first derivative. $f'(x)$ gives the instantaneous rate of change of the function at a certain point and it gives the slope of the line that is tangent to the graph of the function at that point.

(b) Given $f(t)$ describes the displacement, $f'(t)$ corresponds to the instantaneous velocity of the object.