

2024-2025 Fall
MAT123 Midterm
(02/12/2024)

1) Let

$$f(x) = \begin{cases} \frac{\tan ax}{\tan bx}, & \text{if } x < 0 \\ 4, & \text{if } x = 0 \\ ax + b, & \text{if } x > 0 \end{cases}$$

Determine the values of a and b such that f is continuous at the point $x = 0$.

2. Use differential to approximate $3\sqrt[3]{66} + 2\sqrt{66}$.

3. (a) Without using L'Hôpital's rule, evaluate $\lim_{x \rightarrow 0} \frac{5 - 6 \cos x + \cos^2 x}{x \sin x}$.

(b) Prove that $\lim_{x \rightarrow -3} \sqrt{-x - 2} = 1$ by using the formal definition of limit.

(c) Evaluate $\lim_{x \rightarrow 1^+} (\sqrt{x})^{\ln(x-1)}$.

4. Coffee is draining out of a conical filter at a rate of $2.25 \text{ in.}^3/\text{min}$. If the cone is 5 in. tall and has a radius of 2 in., how fast is the coffee level dropping when the coffee is 3 in. deep?

5. Using the mean value theorem, show that $\ln(x + 1) < x$ for $x > 0$.

6. Let $f(x) = \frac{x^2 - 2}{(x - 1)^2}$

(a) Determine the interval of increase, decrease and concavity of f .

(b) Construct a table.

(c) Sketch the graph of f .

1) To ensure continuity at $x = 0$, the one-sided limit values must be equal to the value of the function at that point.

$$\lim_{x \rightarrow 0^-} \frac{\tan ax}{\tan bx} = \lim_{x \rightarrow 0^+} (ax + b) = f(0) = 4$$

The easy part is that we can calculate the limit from the right.

$$\lim_{x \rightarrow 0^+} (ax + b) = 0 + b = b$$

Hence, $b = 4$. To calculate from the left, we need another technique.

$$\begin{aligned} \lim_{x \rightarrow 0^-} \frac{\tan ax}{\tan bx} &= \lim_{x \rightarrow 0^-} \left(\frac{\sin ax}{\cos ax} \cdot \frac{\cos bx}{\sin bx} \cdot \frac{bx}{bx} \cdot \frac{ax}{ax} \right) \\ &= \lim_{x \rightarrow 0^-} \left(\frac{\sin ax}{ax} \right) \cdot \lim_{x \rightarrow 0^-} \left(\frac{1}{\frac{\sin bx}{bx}} \right) \cdot \lim_{x \rightarrow 0^-} \left(\frac{\cos(bx) \cdot ax}{\cos(ax) \cdot bx} \right) \\ &= 1 \cdot \frac{1}{\lim_{x \rightarrow 0^-} \frac{\sin bx}{bx}} \cdot \lim_{x \rightarrow 0^-} \left(\frac{\cos(bx) \cdot a}{\cos(ax) \cdot b} \right) = 1 \cdot 1 \cdot \left(\frac{\cos(0) \cdot a}{\cos(0) \cdot b} \right) \\ &= \frac{a}{b} \end{aligned}$$

Now, set $\frac{a}{b} = b \rightarrow a = 16$. $a = 16, b = 4$

2) Let $f(x) = x^{1/3}$ and $g(x) = x^{1/2}$. Using the differential approximation, we get

$$\begin{aligned} f(x + \Delta x) &\approx f(x) + f'(x)\Delta x = x^{1/3} + \frac{1}{3}x^{-2/3}\Delta x \\ g(x + \Delta x) &\approx g(x) + g'(x)\Delta x = x^{1/2} + \frac{1}{2}x^{-1/2}\Delta x \end{aligned}$$

Set $x = 64$ and $\Delta x = 2$.

$$\begin{aligned} 3\sqrt[3]{66} + 2\sqrt{66} &\approx 3 \left(64^{1/3} + \frac{1}{3}64^{-2/3} \cdot 2 \right) + 2 \left(64^{1/2} + \frac{1}{2}64^{-1/2} \cdot 2 \right) \\ &= 3 \left(4 + \frac{1}{24} \right) + 2 \left(8 + \frac{1}{8} \right) = \boxed{28.375} \end{aligned}$$

3)

(a)

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{5 - 6 \cos x + \cos^2 x}{x \sin x} &= \lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - 5)}{x \sin x} \\
&= \lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - 5)(\cos x + 1)}{(x \sin x)(\cos x + 1)} = \lim_{x \rightarrow 0} \left(-\frac{\sin^2 x \cdot (\cos x - 5)}{x \sin x \cdot (\cos x + 1)} \right) \\
&= \lim_{x \rightarrow 0} \left(-\frac{\sin x \cdot (\cos x - 5)}{x \cdot (\cos x + 1)} \right) = -\lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\cos x - 5}{\cos x + 1} = -1 \cdot \frac{\cos 0 - 5}{\cos 0 + 1} = \boxed{2}
\end{aligned}$$

(b) For all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x + 3| < \delta \implies |f(x) - 1| < \epsilon$$

$$\begin{aligned}
|f(x) - 1| &= \left| \sqrt{-x - 2} - 1 \right| = \left| \sqrt{-x - 2} - 1 \cdot \frac{\sqrt{-x - 2} + 1}{\sqrt{-x - 2} + 1} \right| \\
&= \left| \frac{-x - 3}{\sqrt{-x - 2} + 1} \right| \leq |-x - 3| \quad [\sqrt{-x - 2} \geq 0] \\
&= |x + 3| < \delta = \epsilon
\end{aligned}$$

If we set $\delta = \epsilon$, the proof is complete.

(c) Let L be the value of the limit. Then, take the logarithm of both sides. Since the expression is continuous for $x > 1$, we can take the logarithm function inside the limit.

$$\begin{aligned}
L &= \lim_{x \rightarrow 1^+} (\sqrt{x})^{\ln(x-1)} \quad [1^{-\infty}] \\
\ln(L) &= \ln \left[\lim_{x \rightarrow 1^+} (\sqrt{x})^{\ln(x-1)} \right]
\end{aligned}$$

$$\begin{aligned}
\ln(L) &= \lim_{x \rightarrow 1^+} \ln [(\sqrt{x})^{\ln(x-1)}] = \lim_{x \rightarrow 1^+} [\ln(x-1) \cdot \ln(\sqrt{x})] \quad [\infty \cdot 0] \\
&= \lim_{x \rightarrow 1^+} \left[\frac{\ln(x-1)}{\frac{1}{\ln(\sqrt{x})}} \right] \quad \left[\frac{\infty}{\infty} \right] \\
&= \lim_{x \rightarrow 1^+} \left[\frac{\frac{1}{x-1}}{\frac{1}{-\ln^2(\sqrt{x})} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}} \right] = \lim_{x \rightarrow 1^+} \left[-\frac{\ln^2(\sqrt{x}) \cdot 2x}{x-1} \right] \quad \left[\frac{0}{0} \right] \\
&= \lim_{x \rightarrow 1^+} \left(-\frac{2 \ln(\sqrt{x}) \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \cdot 2x + \ln^2(\sqrt{x}) \cdot 2}{1} \right) = \lim_{x \rightarrow 1^+} [2 \ln(\sqrt{x}) + 2 \ln^2(\sqrt{x})] \\
&= [2 \ln(\sqrt{1}) + 2 \ln^2(\sqrt{1})] = 0
\end{aligned}$$

$\ln(L) = 0$. Therefore, $\boxed{L = 1}$.

4) Let $f(x)$ represent the volume of coffee in the cone in cubic inches. The coffee in the cone will have a conical shape while draining. We may set up the equation below using the formula of the volume of a cone.

$$f(t) = \frac{1}{3} \cdot h(t) \cdot \pi r^2(t)$$

$h(t), r(t)$ represent the height and radius of the circular area that coffee forms, respectively, in inches. We can eliminate r to proceed with h . r and h are proportional.

$$\frac{r}{h} = \frac{2}{5} \rightarrow r = \frac{2h}{5}$$

$$f(t) = \frac{4\pi h^3(t)}{75}$$

Take the derivative of both sides.

$$f'(t) = \frac{4\pi}{25} \cdot h^2(t) \cdot h'(t)$$

Given that at $t = t_0$, $f'(t_0) = -2.25$, $h(t_0) = 3$. We may now find $h'(t_0)$. Solve for $h'(t_0)$.

$$h'(t_0) = \frac{25f'(t_0)}{4\pi h^2(t_0)} = \frac{25 \cdot (-2.25)}{4\pi \cdot (3)^2} = \boxed{-\frac{1.5625}{\pi} \text{ inches/minute}}$$

5) Let $f(x) = \ln(1+x) - x$. We have $f(0) = \ln(1+0) - 0 = 0$. The mean value theorem (MVT) states that if a function $g(x)$ is continuous on $[a, b]$ and differentiable on (a, b) , then there exists a point c such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$

f is continuous on $[0, x]$ and differentiable on $(0, x)$. By MVT, $\frac{f(x) - f(0)}{x - 0} = f'(c)$ provided for some point c such that $0 < c < x$.

$$\begin{aligned} f'(c) &= \frac{1}{c+1} - 1 = \frac{\ln(x+1) - x}{x} = \frac{f(x) - f(0)}{x - 0} \\ \frac{1}{c+1} &= \frac{\ln(x+1)}{x} \rightarrow c+1 = \frac{x}{\ln(x+1)} \\ c &= \frac{x - \ln(x+1)}{\ln(x+1)} \end{aligned}$$

From the inequality $0 < c < x$,

$$\begin{aligned} 0 &< \frac{x - \ln(x+1)}{\ln(x+1)} \\ 0 &< x - \ln(x+1) \\ \ln(x+1) &< x \end{aligned}$$

6)

(a) First off, find the domain. The expression is undefined when the denominator is zero. Therefore, $(x-1)^2 \neq 0 \rightarrow x \neq 1$. The only vertical asymptote occurs at $x = 1$.

$$\mathcal{D} = \mathbb{R} - \{1\}$$

Let us find the limit at infinity.

$$\lim_{x \rightarrow \infty} \frac{x^2 - 2}{(x-1)^2} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{2x}{2(x-1)} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{2}{2} = 1$$

Similarly,

$$\lim_{x \rightarrow -\infty} \frac{x^2 - 2}{(x-1)^2} = 1$$

The horizontal asymptote occurs only at $y = 0$.

Take the first derivative by applying the quotient rule.

$$y' = \frac{(2x) \cdot (x-1)^2 - (x^2 - 2) \cdot 2(x-1)}{(x-1)^4} = \frac{4 - 2x}{(x-1)^3}$$

y' is undefined for $x = 1$, and $y' = 0$ for $x = 2$. Since 1 is not in the domain, the *only* critical point is $x = 2$.

Take the second derivative.

$$y'' = \frac{(-2) \cdot (x-1)^3 - (4-2x) \cdot 3(x-1)^2}{(x-1)^6} = \frac{4x-10}{(x-1)^4}$$

The only inflection point occurs at $x = \frac{5}{2}$.

(b) Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(-\sqrt{2}) = f(\sqrt{2}) = 0, f(0) = -2, f(2) = 2, f(5/2) = 17/9$$

x	$(-\infty, -\sqrt{2}]$	$(-\sqrt{2}, 0]$	$[0, 1)$	$(1, \sqrt{2}]$	$[\sqrt{2}, 2]$	$[2, \frac{5}{2}]$	$[\frac{5}{2}, \infty)$
y	$(1, 0]$	$[-2, 0)$	$(-\infty, -2]$	$(-\infty, 0]$	$[0, 2]$	$[2, \frac{17}{9}]$	$[\frac{17}{9}, 1]$
y' sign	-	-	-	+	+	-	-
y'' sign	-	-	-	-	-	-	+

(c)

