2024-2025 Fall

MAT123-02,05 Final

(09/01/2025)

1. Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{2\cos x + 3}$

(b) $\displaystyle\int \frac{\sqrt{5-x^2}}{x}\, dx$

(c) $\displaystyle\int \frac{dx}{1-x^4}$

(d) $\displaystyle\int \ln\left(1+x^2\right)\, dx$

2. Consider the region R bounded by the curves $y = x$, $y = x^2$ and $y = x^2/2$.

(a) Find the area of the region R.

(b) Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region R about the y-axis.

(c) Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of the solid obtained by rotating the region R about the line $y = 2$.

3. Use the Integral Test to determine the existence of the sum of the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$.

4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)^{1/n}$ is convergent or divergent.

5. Find the Maclaurin series of $f(x) = \sqrt{e^x}$ and determine the interval of convergence of this series.

2024-2025 Final (09/01/2025) Solutions
(Last update: 8/21/25 (21st of August) 12:12 PM)

1.

(a) Use the tangent half-angle substitution, which is also called the Weierstrass substitution. Let $t = \tan\left(\frac{x}{2}\right)$. Then

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2}\,dt$$

$$\int \frac{dx}{2\cos x + 3} = \int \frac{1}{2\left(\frac{1-t^2}{1+t^2}\right)+3} \cdot \frac{2}{1+t^2}\,dt = \int \frac{2}{5+t^2}\,dt = \frac{2}{5}\int \frac{1}{\left(1+\left(\frac{t}{\sqrt{5}}\right)^2\right)}\,dt$$

Let $u = \frac{t}{\sqrt{5}}$, then $\sqrt{5}\,du = dt$.

$$\frac{2}{5}\int \frac{1}{\left(1+\left(\frac{t}{\sqrt{5}}\right)^2\right)}\,dt = \frac{2\sqrt{5}}{5}\int \frac{1}{1+u^2}\,du = \frac{2\sqrt{5}}{5}\arctan u + c = \frac{2\sqrt{5}}{5}\arctan\frac{t}{\sqrt{5}} + c$$

$$= \boxed{\frac{2\sqrt{5}}{5}\arctan\left(\frac{1}{\sqrt{5}}\cdot\tan\left(\frac{x}{2}\right)\right) + c, \quad c \in \mathbb{R}}$$

(b) Let $x = \sqrt{5}\sin u$ for $-\frac{\pi}{2} \le u < \frac{\pi}{2}$, then $dx = \sqrt{5}\cos u\,du$.

$$\mathrm{I} = \int \frac{\sqrt{5-x^2}}{x}\,dx = \int \frac{\sqrt{5-5\sin^2 u}}{\sqrt{5}\sin u}\cdot\sqrt{5}\cos u\,du \qquad \left[\sin^2 u + \cos^2 u = 1\right]$$

$$= \sqrt{5}\int \sqrt{\cos^2 u}\cot u\,du \qquad [\cos u > 0]$$

$$= \sqrt{5}\int \cos u \cdot \cot u\,du = \sqrt{5}\int \frac{\cos^2 u}{\sin u}\,du = \sqrt{5}\int \frac{1-\sin^2 u}{\sin u}\,du$$

$$= \sqrt{5}\int (\csc u - \sin u)\,du = \sqrt{5}\left(-\ln|\cot u + \csc u| + \cos u\right) + c$$

Recall that $x = \sqrt{5}\sin u$.

$$\sin u = \frac{x}{\sqrt{5}} \implies \sin^2 u = \frac{x^2}{5} \implies \cos^2 u = \frac{5-x^2}{5} \implies \cos u = \frac{\sqrt{5-x^2}}{\sqrt{5}}$$

$$\cot u = \frac{\cos u}{\sin u} = \frac{\sqrt{5-x^2}}{x}, \qquad \csc u = \frac{1}{\sin u} = \frac{\sqrt{5}}{x}$$

Therefore,

$$\mathrm{I} = \boxed{-\sqrt{5}\ln\left|\frac{\sqrt{5-x^2}}{x} + \frac{\sqrt{5}}{x}\right| + \sqrt{5-x^2} + c, \quad c \in \mathbb{R}}$$

(c) Use partial fractions to compute the integral.

$$\mathrm{I} = \int \frac{1}{1-x^4}\,dx = \int \frac{1}{(1-x^2)(1+x^2)}\,dx = \int \frac{1}{(1-x)(1+x)(1+x^2)}\,dx$$

$$= \int \left(\frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}\right) dx$$

$$A(1+x)(1+x^2) + B(1-x)(1+x^2) + (Cx+D)(1+x)(1-x) = 1$$
$$x^3(A-B-C) + x^2(A+B-D) + x(A-B+C) + A+B+D = 1$$

Equate the coefficients of like terms.

$$\left.\begin{array}{lr} A-B-C=0 & (1) \\ A+B-D=0 & (2) \\ A-B+C=0 & (3) \\ A+B+D=1 & (4) \end{array}\right\}$$

$$(1)\ \&\ (3) \to 2A-2B=0$$
$$(2)\ \&\ (4) \to 2A+2B=1$$
$$\therefore A = \frac{1}{4}, \quad B = \frac{1}{4}$$
$$(1) \to C=0, \quad (2) \to D = \frac{1}{2}$$

Rewrite the integral.

$$\mathrm{I} = \int \left(\frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}\right) dx = \int \left(\frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}\right) dx$$

$$= \boxed{-\frac{1}{4}\ln|1-x| + \frac{1}{4}\ln|1+x| + \frac{1}{2}\arctan(x) + c,\ c \in \mathbb{R}}$$

(d) Use the method of integration by parts.

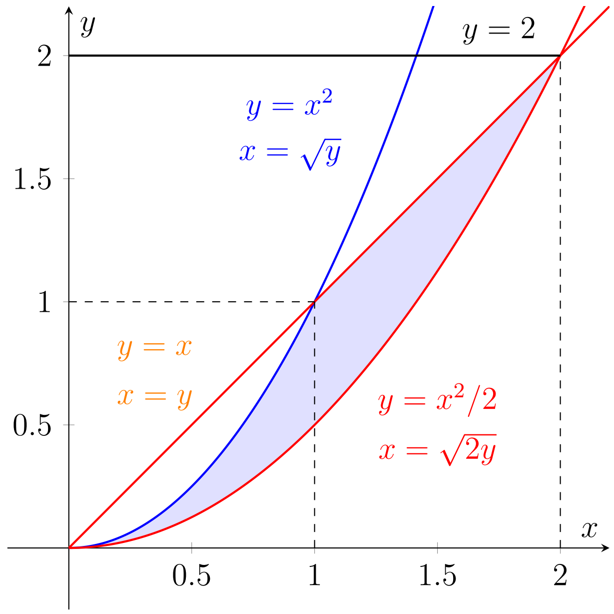
$$\left.\begin{array}{l} u = \ln(1+x^2) \implies du = \dfrac{2x}{1+x^2}\,dx \\ dv = dx \implies v = x \end{array}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int \ln\left(1+x^2\right) dx = x\ln\left(1+x^2\right) - \int \frac{2x^2}{1+x^2}\,dx = x\ln\left(1+x^2\right) - \int \frac{2x^2+2-2}{1+x^2}\,dx$$

$$= x\ln\left(1+x^2\right) - 2\int dx + 2\int \frac{dx}{1+x^2}$$

$$= \boxed{x\ln\left(1+x^2\right) - 2x + 2\arctan x + c, \quad c \in \mathbb{R}}$$

2.



(a)

$$A=\int_0^1\left(x^2-\frac{x^2}{2}\right)\,dx+\int_1^2\left(x-\frac{x^2}{2}\right)\,dx=\left[\frac{x^3}{3}-\frac{x^3}{6}\right]_0^1+\left[\frac{x^2}{2}-\frac{x^3}{6}\right]_1^2$$

$$=\left[\left(\frac{1}{3}-\frac{1}{6}\right)-0\right]+\left[\left(2-\frac{8}{6}\right)-\left(\frac{1}{2}-\frac{1}{6}\right)\right]=\boxed{\frac{1}{2}}$$

(b)

$$V=\int_D\pi\left[r_2^2(y)-r_1^2(y)\right]\,dy=\boxed{\int_0^1\pi\left[\left(\sqrt{2y}\right)^2-(\sqrt{y})^2\right]\,dy+\int_1^2\pi\left[\left(\sqrt{2y}\right)^2-y^2\right]\,dy}$$

(c)

$$V=\int_D 2\pi\cdot h(y)\cdot r(y)\,dy=\boxed{\int_0^1 2\pi(2-y)\left(\sqrt{2y}-\sqrt{y}\right)dy+\int_1^2 2\pi(2-y)\left(\sqrt{2y}-y\right)dy}$$

3. Take $f(x)=x\mathrm{e}^{-x^2}$. f is continuous because the product of a polynomial and an exponential expression is still continuous. f is positive and decreasing for $x\geq 1$. Verify the monotonicity of f by taking the first derivative.

$$\frac{df}{dx}=1\cdot\mathrm{e}^{-x^2}+x\mathrm{e}^{-x^2}\cdot(-2x)=\mathrm{e}^{-x^2}\left(1-2x^2\right)$$

$$f'(x)<0\quad\text{for}\quad x>\frac{\sqrt{2}}{2}\implies f'(x)<0\quad\text{for}\quad x\geq 1$$

We may now apply the Integral Test. Take the limit for the improper integral.

$$\int_1^\infty x\mathrm{e}^{-x^2}\,dx=\lim_{R\to\infty}\int_1^R x\mathrm{e}^{-x^2}\,dx=\lim_{R\to\infty}-\frac{1}{2}\mathrm{e}^{-x^2}\Big|_1^R=\lim_{R\to\infty}-\frac{1}{2}\left(\mathrm{e}^{-R^2}-\mathrm{e}^{-1}\right)=\frac{1}{2\mathrm{e}}$$

The integral converges. Then the series $\sum_{n=1}^{\infty} ne^{-n^2}$ also converges.

4. Apply the nth Term Test for the non-alternating part. Let L be the value of the limit.

$$L = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} \implies \ln(L) = \ln\left[\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n}\right]$$

We can also take the limit inside the function because ln is continuous on its domain.

$$\ln(L) = \lim_{n\to\infty} \ln\left[\left(\frac{1}{n}\right)^{1/n}\right] = \lim_{n\to\infty} \frac{\ln\left(\frac{1}{n}\right)}{n}$$

To be able to use L'Hôpital's rule, take the corresponding function $f(x) = \dfrac{\ln\left(\frac{1}{x}\right)}{x}$.

$$\lim_{x\to\infty} \frac{\ln\left(\frac{1}{x}\right)}{x} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{x\cdot\left(-\frac{1}{x^2}\right)}{1} = \lim_{x\to\infty} -\frac{1}{x} = 0 \implies \ln(L) = 0 \implies L = 1$$

The limit $\lim_{n\to\infty}(-1)^{n+1}$ does not exist because of the oscillation. So, $\lim_{n\to\infty}(-1)^{n+1}\left(\frac{1}{n}\right)^{1/n}$ does not exist as well. Therefore, by the nth Term Test for divergence, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)^{1/n}$ is divergent.

5. The Maclaurin series of f is given by $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$.

Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ to look for the pattern.

$$f'(x) = \frac{1}{2\sqrt{e^x}}\cdot e^x = \frac{1}{2}\sqrt{e^x}, \quad f''(x) = \frac{1}{4\sqrt{e^x}}\cdot e^x = \frac{1}{4}\sqrt{e^x}, \quad f'''(x) = \frac{1}{8\sqrt{e^x}}\cdot e^x = \frac{1}{8}\sqrt{e^x}$$

$$f(0) = 1, \quad f'(0) = \frac{1}{2}, \quad f''(0) = \frac{1}{4}, \quad f'''(0) = \frac{1}{8}$$

Therefore, $f^{(k)}(0) = \left(\frac{1}{2}\right)^k$. Rewrite the summation formula.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^k \cdot x^k}{k!} = \boxed{\sum_{k=0}^{\infty} \frac{x^k}{2^k\cdot k!} = 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \ldots}$$

Find the interval of convergence by applying the Ratio Test.

$$\lim_{k\to\infty}\left|\frac{x^{k+1}}{2^{k+1}\cdot(k+1)!}\cdot\frac{2^k\cdot k!}{x^k}\right| = \lim_{k\to\infty}\frac{x}{2k} = 0 < 1$$

$\boxed{\text{The series is convergent on } \mathbb{R}.}$