

2011-2012 Spring
MAT123-[Instructor] Midterm II
(08/05/2012)
Time: 15:00 - 16:45
Duration: 105 minutes

1. Consider the region bounded by the curve $y = x^2$, the x -axis, and the line $x = 2$, where $x \geq 0$.

(a) Find the volume of the solid generated by revolving the region about the x -axis by the disk method and sketch the solid.

(b) Find the volume of the solid generated by revolving the region about the y -axis by the shell method and sketch the solid.

2. Evaluate the following integrals.

$$\begin{array}{lll} \text{(a)} \int_0^3 |x^2 - 1| \, dx & \text{(b)} \int \frac{1}{x^2 + 2x + 1} \, dx & \text{(c)} \int \frac{1}{x^2 + 2x + 2} \, dx \\ \text{(d)} \int \frac{1}{x^2 + 3x + 2} \, dx & \text{(e)} \int_{-\pi/6}^0 \sqrt{1 - \cos(6x)} \, dx & \end{array}$$

3. Determine whether the improper integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$ is convergent or divergent. Evaluate if the integral is convergent.

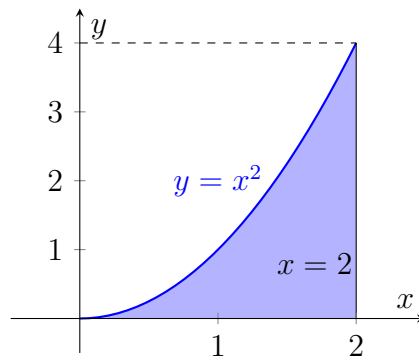
4. Evaluate the following limits. Explain all your work and write clearly.

$$\begin{array}{ll} \text{(a)} \lim_{x \rightarrow 0} \frac{3^x - 1}{5^x - 1} & \text{(b)} \lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sin t \, dt}{x} \end{array}$$

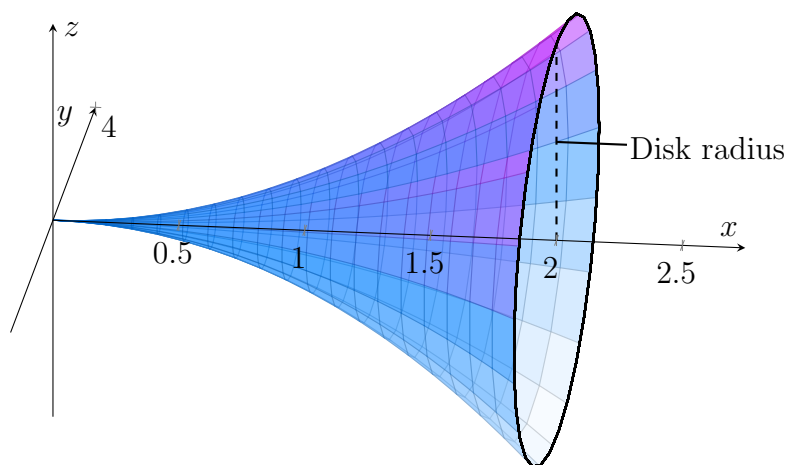
5. Determine whether each sequence converges or diverges. Evaluate the limit of each convergent sequence. Explain all your work and write clearly.

$$\begin{array}{lll} \text{(a)} a_n = \frac{2n + (-1)^n}{n} & \text{(b)} a_n = \arctan\left(\frac{n+1}{n}\right) & \text{(c)} a_n = \frac{n+1}{1-\sqrt{n}} \end{array}$$

1.

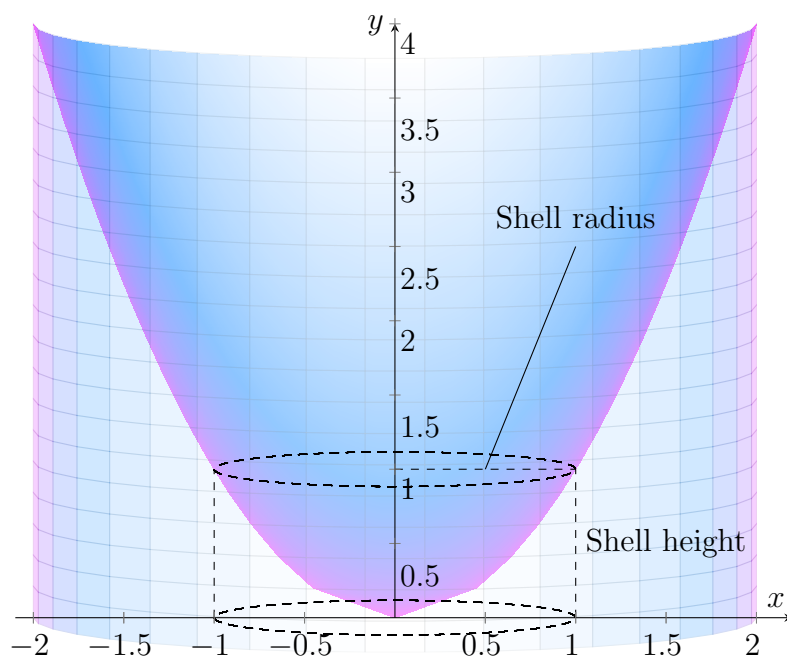


(a)



$$\text{Volume} = \int_{\mathcal{D}} \pi \cdot (r(x))^2 dx = \int_0^2 \pi (x^2)^2 dx = \pi \int_0^2 x^4 dx = \pi \left[\frac{x^5}{5} \right]_0^2 = \boxed{\frac{32\pi}{5}}$$

(b)



$$\begin{aligned}\text{Volume} &= \int_{\mathcal{D}} 2\pi \cdot r(x) \cdot h(x) dx = 2\pi \int_0^2 x \cdot x^2 dx = 2\pi \int_0^2 x^3 dx \\ &= 2\pi \left[\frac{x^4}{4} \right]_0^2 = 2\pi \left(\frac{2^4}{4} - 0 \right) = \boxed{8\pi}\end{aligned}$$

2.

(a) The expression $|x^2 - 1|$ is the same as $x^2 - 1$ for $x > 1$ and $1 - x^2$ for $x < 1$. We can write the equivalent expression below.

$$\begin{aligned}\int_0^3 |x^2 - 1| dx &= \int_0^1 (1 - x^2) dx + \int_1^3 (x^2 - 1) dx = \left[x - \frac{x^3}{3} \right]_0^1 + \left[\frac{x^3}{3} - x \right]_1^3 \\ &= \left[\left(1 - \frac{1^3}{3} \right) - 0 \right] + \left[\left(\frac{3^3}{3} - 3 \right) - \left(\frac{1^3}{3} - 1 \right) \right] = \boxed{\frac{22}{3}}\end{aligned}$$

(b)

$$\int \frac{1}{x^2 + 2x + 1} dx = \int \frac{1}{(x+1)^2} dx = \boxed{-\frac{1}{x+1} + c, \quad c \in \mathbb{R}}$$

(c)

$$\begin{aligned}\int \frac{1}{x^2 + 2x + 2} dx &= \int \frac{1}{(x+1)^2 + 1} dx \quad [u = x+1 \implies du = dx] \\ &= \int \frac{1}{u^2 + 1} du = \arctan(u) + c = \boxed{\arctan(x+1) + c, \quad c \in \mathbb{R}}\end{aligned}$$

(d) We can use the method of partial fraction decomposition.

$$\int \frac{1}{x^2 + 3x + 2} dx = \int \frac{1}{(x+1)(x+2)} dx = \int \left(\frac{A}{x+1} + \frac{B}{x+2} \right) dx \quad (1)$$

$$\begin{aligned}A(x+2) + B(x+1) &= 1 \\ x(A+B) + 2A + B &= 1 \\ \therefore A + B = 0 \quad [\text{eliminate } x] &\rightarrow 2A + B = 1\end{aligned}$$

$$\left. \begin{aligned} A + B &= 0 \\ 2A + B &= 1 \end{aligned} \right\} \quad A = 1, \quad B = -1$$

Plug the values of A and B into (1).

$$\int \left(\frac{A}{x+1} + \frac{B}{x+2} \right) dx = \int \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx = \boxed{\ln|x+1| - \ln|x+2| + c, \quad c \in \mathbb{R}}$$

(e) Use the trigonometric identity $\cos(2x) = 1 - 2\sin^2 x$.

$$\int_{-\pi/6}^0 \sqrt{1 - \cos(6x)} dx = \int_{-\pi/6}^0 \sqrt{2\sin^2(3x)} dx = \sqrt{2} \int_{-\pi/6}^0 |\sin(3x)| dx$$

$\sin x < 0$ for $-\pi < x < \pi$. Therefore, the integral can be rewritten as follows.

$$\begin{aligned} \sqrt{2} \int_{-\pi/6}^0 |\sin(3x)| dx &= \sqrt{2} \int_{-\pi/6}^0 -\sin(3x) dx = \sqrt{2} \left[\frac{1}{3} \cos(3x) \right]_{-\pi/6}^0 \\ &= \frac{\sqrt{2}}{3} \left[\cos 0 - \cos \left(-\frac{\pi}{2} \right) \right] = \boxed{\frac{\sqrt{2}}{3}} \end{aligned}$$

3. We have an improper integral where the limits are $\pm\infty$. Use limits to handle improper integrals accurately.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \lim_{A \rightarrow \infty} \int_{-A}^A \frac{1}{1+x^2} dx = \lim_{A \rightarrow \infty} \arctan(x) \Big|_{-A}^A \\ &= \lim_{A \rightarrow \infty} (\arctan A - \arctan(-A)) = \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \boxed{\pi} \end{aligned}$$

The value of the improper integral is finite. Therefore, this integral is convergent.

4.

(a) The limit is in the indeterminate form $0/0$. We can apply L'Hôpital's rule to eliminate the indeterminate form.

$$\lim_{x \rightarrow 0} \frac{3^x - 1}{5^x - 1} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 0} \frac{3^x \cdot \ln 3}{5^x \cdot \ln 5} = \log_5 3 \cdot \lim_{x \rightarrow 0} \left(\frac{3}{5} \right)^x = \boxed{\log_5 3}$$

(b) The first method is to evaluate the integral in the limit.

$$\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sin t dt}{x} = \lim_{x \rightarrow 0} \frac{-\cos t \Big|_{t=0}^{t=x^2}}{x} = \lim_{x \rightarrow 0} \frac{-\cos x^2 - (-\cos 0)}{x} = \lim_{x \rightarrow 0} \frac{1 - \cos x^2}{x}$$

Multiply by the expression inside the limit $\frac{x}{x}$. Notice that we obtain a standard form.

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1 - \cos x^2}{x} \cdot \frac{x}{x} \right) &= \lim_{x \rightarrow 0} \frac{1 - \cos x^2}{x^2} \cdot \lim_{x \rightarrow 0} x \quad \left[\lim_{u \rightarrow 0} \frac{1 - \cos u}{u} = 0 \right] \\ &= 1 \cdot 0 = \boxed{0} \end{aligned}$$

The second method is to use L'Hôpital's rule because of the 0/0 form.

$$\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sin t \, dt}{x} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \int_0^{x^2} \sin t \, dt}{1}$$

Let $u = x^2$, then $du = 2x \, dx$. By the Fundamental Theorem of Calculus, the limit can be rewritten as follows.

$$\lim_{x \rightarrow 0} \frac{d}{dx} \int_0^{x^2} \sin t \, dt = \lim_{x \rightarrow 0} \frac{d}{du} \left(\int_0^u \sin t \, dt \right) \frac{du}{dx} = \lim_{x \rightarrow 0} (\sin u \cdot 2x) = \lim_{x \rightarrow 0} [2x \sin(x^2)] = \boxed{0}$$

5.

(a)

$$a_n = \frac{2n + (-1)^n}{n} = 2 + \frac{(-1)^n}{n}$$

The sequence converges to $\boxed{2}$ because $\frac{(-1)^n}{n} \rightarrow 0$ as $n \rightarrow \infty$.

(b) arctan is continuous everywhere. Therefore, we can take the limit inside the expression.

$$\lim_{n \rightarrow \infty} \arctan \left(\frac{n+1}{n} \right) = \arctan \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = \arctan 1 = \frac{\pi}{4}$$

The sequence converges to $\boxed{\frac{\pi}{4}}$.

(c)

$$\lim_{n \rightarrow \infty} \frac{n+1}{1-\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n} \left(\sqrt{n} + \frac{1}{\sqrt{n}} \right)}{\sqrt{n} \left(\frac{1}{\sqrt{n}} - 1 \right)} = \lim_{n \rightarrow \infty} \frac{\sqrt{n} + \frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}} - 1} = -\infty$$

Notice that the denominator approaches -1 as $n \rightarrow \infty$ and the numerator approaches ∞ as $n \rightarrow \infty$. The sequence diverges to $\boxed{-\infty}$.