2019-2020 Spring
MAT124 Resit
(01/07/2020)

1. The temperature $T$ at the point $(x, y, z)$ in a region of space is given by the formula $T = 100 - xy - xz - yz$. Find the lowest temperature on the plane $x + y + z = 10$.

2. Show that if $z = f(r, \theta)$, where $r$ and $\theta$ are defined as functions of $x$ and $y$ by the equations $x = r\cos\theta$ and $y = r\sin\theta$, then the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ becomes

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = 0\,.$$

3. Evaluate the integral $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1 - x^3}\, dx\, dy$.

4. Evaluate

$$\int_2^4 \int_2^y \, dx\, dy + \int_4^8 \int_2^{16/y} \, dx\, dy$$

by reversing the order of integration.

5. Use a double integral to find the area inside the circle $r = \cos\theta$ and outside the cardioid $r = 1 - \cos\theta$.

6. Use polar coordinates to evaluate the double integral

$$\iint_D \sin\left(x^2 + y^2\right)\, dA,$$

where $D$ is the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and the lines $y = 0$, $x = \sqrt{3}y$.

7. Using cylindrical coordinates, evaluate

$$\iiint_D \frac{dV}{x^2 + y^2 + z^2},$$

where $D$ is the solid region bounded below by the paraboloid $2z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 8$.

8. Using spherical coordinates, evaluate the triple integral

$$\iiint_D \sqrt{x^2 + y^2 + z^2}\, dV,$$

where $D$ is the portion of the solid sphere $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.

2019-2020 Spring Resit (01/07/2020) Solutions
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1) Let $g(x, y, z) = x + y + z - 10$ and then, solve the system of equations below using the method of Lagrange multipliers.

$$\left.\begin{aligned} \nabla T &= \lambda \nabla g \\ g(x, y, z) &= 0 \end{aligned}\right\} \quad \nabla T = \langle -y - z, -x - z, -x - y \rangle = \lambda \langle 1, 1, 1 \rangle = \lambda \nabla g$$

$$\begin{aligned} T_x + T_y + T_z &= (-y - z) + (-x - z) + (-x - y) = -2x - 2y - 2z \\ &= \lambda + \lambda + \lambda = 3\lambda \implies x + y + z = \frac{-3\lambda}{2} \end{aligned}$$

Use the constraint to find the value of $\lambda$.

$$g(x, y, z) = 0 \implies \frac{-3\lambda}{2} - 10 = 0 \implies \lambda = -\frac{20}{3}$$

So far, we have the equations below. Solve the system of equations and find the values of $x, y, z$ one by one.

$$\left.\begin{aligned} -y - z &= -\frac{20}{3} \quad (1) \\ -x - z &= -\frac{20}{3} \quad (2) \\ -x - y &= -\frac{20}{3} \quad (3) \end{aligned}\right\} \quad \begin{aligned} &(1)\ \&\ (2) \to x - y = 0 \quad (4) \\ &(3)\ \&\ (4) \to y = \frac{10}{3} \qquad (5) \\ &\therefore z = \frac{10}{3}, \quad x = \frac{10}{3} \end{aligned}$$

We now have all the values. Substitute in $f(x, y, z)$ to find the minimum value of the temperature.

$$f_{\min} = f\left(\frac{10}{3}, \frac{10}{3}, \frac{10}{3}\right) = 100 - \left(\frac{10}{3}\right)^2 - \left(\frac{10}{3}\right)^2 - \left(\frac{10}{3}\right)^2 = \boxed{\frac{200}{3}}$$

2) We have $x = r\cos\theta$ and $y = r\sin\theta$.

$$x^2 = r^2\cos^2\theta, \quad y^2 = r^2\sin^2\theta \implies x^2 + y^2 = r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2, \quad \therefore r = \sqrt{x^2 + y^2}$$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \implies \theta = \tan^{-1}\frac{y}{x}$$

Compute the first-order partial derivatives.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \sin\theta$$

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)=\frac{-y}{x^2+y^2}=\frac{-\sin\theta}{r},\qquad \frac{\partial\theta}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}=\frac{\cos\theta}{r}$$

Rewrite $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cdot\frac{x}{\sqrt{x^2+y^2}}+\frac{\partial z}{\partial\theta}\cdot\frac{-y}{x^2+y^2},\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial r}\cdot\frac{y}{\sqrt{x^2+y^2}}+\frac{\partial z}{\partial\theta}\cdot\frac{x}{x^2+y^2}$$

Compute the second-order partial derivatives.

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial r}\cdot\frac{x}{\sqrt{x^2+y^2}}+\frac{\partial z}{\partial\theta}\cdot\frac{-y}{x^2+y^2}\right)$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}=&\left[\left(\frac{\partial^2 z}{\partial r^2}\cdot\frac{\partial r}{\partial x}+\frac{\partial^2 z}{\partial\theta\,\partial r}\cdot\frac{\partial\theta}{\partial x}\right)\cdot\frac{x}{\sqrt{x^2+y^2}}+\frac{\partial z}{\partial r}\cdot\frac{1\cdot\sqrt{x^2+y^2}-x\cdot\frac{x}{\sqrt{x^2+y^2}}}{\left(\sqrt{x^2+y^2}\right)^2}\right]\\&+\left[\left(\frac{\partial^2 z}{\partial\theta^2}\cdot\frac{\partial\theta}{\partial x}+\frac{\partial^2 z}{\partial r\,\partial\theta}\cdot\frac{\partial r}{\partial x}\right)\cdot\frac{-y}{x^2+y^2}+\frac{\partial z}{\partial\theta}\cdot\frac{y}{(x^2+y^2)^2}\cdot 2x\right]\end{aligned}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial r}\cdot\frac{y}{\sqrt{x^2+y^2}}+\frac{\partial z}{\partial\theta}\cdot\frac{x}{x^2+y^2}\right)$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2}=&\left[\left(\frac{\partial^2 z}{\partial r^2}\cdot\frac{\partial r}{\partial y}+\frac{\partial^2 z}{\partial\theta\,\partial r}\cdot\frac{\partial\theta}{\partial y}\right)\cdot\frac{y}{\sqrt{x^2+y^2}}+\frac{\partial z}{\partial r}\cdot\frac{1\cdot\sqrt{x^2+y^2}-y\cdot\frac{y}{\sqrt{x^2+y^2}}}{\left(\sqrt{x^2+y^2}\right)^2}\right]\\&+\left[\left(\frac{\partial^2 z}{\partial\theta^2}\cdot\frac{\partial\theta}{\partial y}+\frac{\partial^2 z}{\partial r\,\partial\theta}\cdot\frac{\partial r}{\partial y}\right)\cdot\frac{x}{x^2+y^2}+\frac{\partial z}{\partial\theta}\cdot\frac{-x}{(x^2+y^2)^2}\cdot 2y\right]\end{aligned}$$

Add the second-order partial derivatives and set to 0. The last terms eliminate each other. Write $x$ and $y$ in terms of $r$ and $\theta$.

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=&\left[\left(\frac{\partial^2 z}{\partial r^2}\cdot\cos\theta+\frac{\partial^2 z}{\partial\theta\,\partial r}\cdot\frac{-\sin\theta}{r}\right)\cdot\cos\theta+\frac{\partial z}{\partial r}\cdot\frac{\sin^2\theta}{r}\right]\\&+\left[\left(\frac{\partial^2 z}{\partial\theta^2}\cdot\frac{-\sin\theta}{r}+\frac{\partial^2 z}{\partial r\,\partial\theta}\cdot\cos\theta\right)\cdot\frac{-\sin\theta}{r}\right]\\&+\left[\left(\frac{\partial^2 z}{\partial r^2}\cdot\sin\theta+\frac{\partial^2 z}{\partial\theta\,\partial r}\cdot\frac{\cos\theta}{r}\right)\cdot\sin\theta+\frac{\partial z}{\partial r}\cdot\frac{\cos^2\theta}{r}\right]\\&+\left[\left(\frac{\partial^2 z}{\partial\theta^2}\cdot\frac{\cos\theta}{r}+\frac{\partial^2 z}{\partial r\,\partial\theta}\cdot\sin\theta\right)\cdot\frac{\cos\theta}{r}\right]=0\end{aligned}$$

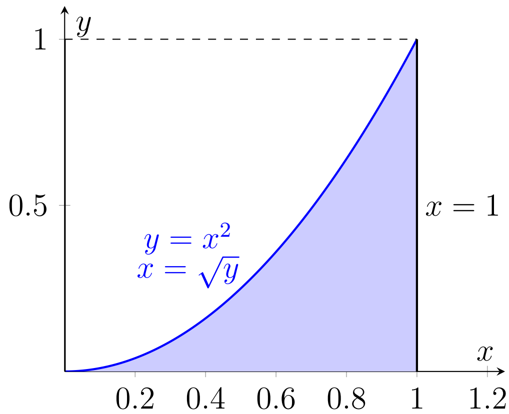
Inspect the terms that add up to 0. Recall $\sin^2\theta + \cos^2\theta = 1$, then the equation reduces to

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} \cdot \left(\cos^2\theta + \sin^2\theta\right) + \frac{\partial z}{\partial r} \cdot \frac{\sin^2\theta + \cos^2\theta}{r} + \frac{\partial^2 z}{\partial \theta^2} \cdot \left(\sin^2\theta + \cos^2\theta\right)$$

$$= \frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial z}{\partial r} + \frac{\partial^2 z}{\partial \theta^2} = 0,$$

which we set out to demonstrate.

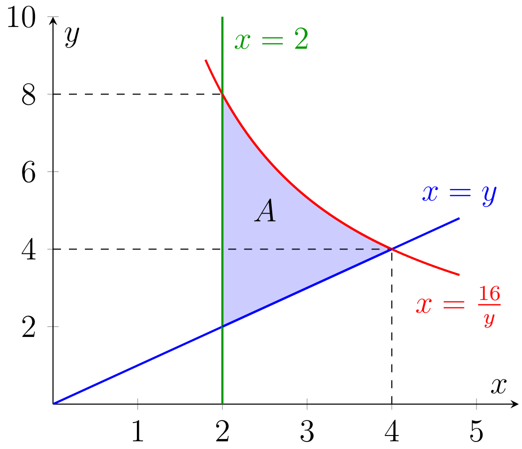
3) Change the order of integration using the graph below and then evaluate the integral.




$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1-x^3}\, dx\, dy = \int_0^1 \int_0^{x^2} \sqrt{1-x^3}\, dy\, dx = \int_0^1 x^2\sqrt{1-x^3}\, dx \quad \left[\begin{array}{c} u = 1 - x^3 \\ du = -3x^2\, dx \end{array}\right]$$

$$= \int \frac{\sqrt{u}}{-3}\, du = -\frac{2}{9}u^{3/2} + c = -\frac{2}{9}\left(1-x^3\right)^{3/2}\Bigg|_0^1 = 0 - \left[-\frac{2}{9}\right] = \boxed{\frac{2}{9}}$$

4)




$$A = \int_2^4 \int_2^y dx\, dy + \int_4^8 \int_2^{16/y} dx\, dy$$

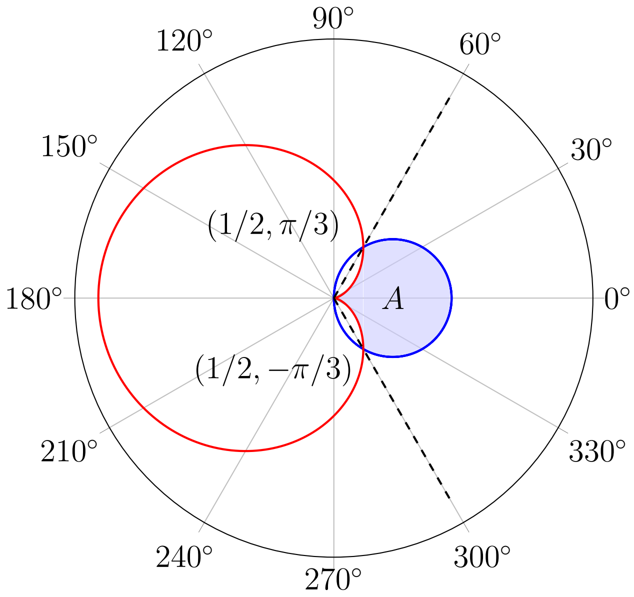
$$= \int_2^4 \int_x^{\frac{16}{x}} dy\, dx = \int_2^4 \left(\frac{16}{x} - x\right) dx$$

$$= \left[16\ln|x| - \frac{x^2}{2}\right]_2^4$$

$$= \left[\left(16\ln 4 - \frac{4^2}{2}\right) - \left(16\ln 2 - \frac{2^2}{2}\right)\right]$$

$$= \boxed{16\ln 2 - 6}$$

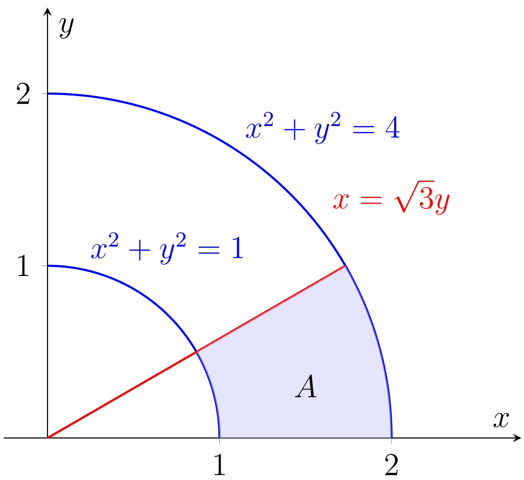
5)




$$A=\int_{-\pi/3}^{\pi/3}\int_{1-\cos\theta}^{\cos\theta} r\,dr\,d\theta=\frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[\cos^2\theta-(1-\cos\theta)^2\right]\,d\theta=\frac{1}{2}\int_{-\pi/3}^{\pi/3}(2\cos\theta-1)\,d\theta$$

$$=\frac{1}{2}\Big[2\sin\theta-\theta\Big]_{-\pi/3}^{\pi/3}=\frac{1}{2}\left[\left(2\sin\frac{\pi}{3}-\frac{\pi}{3}\right)-\left(2\sin\left(-\frac{\pi}{3}\right)+\frac{\pi}{3}\right)\right]=\boxed{\sqrt{3}-\frac{\pi}{3}}$$

6)




Use the following transformation to switch to polar coordinates.

$$\begin{array}{c} x=r\cos\theta \\ y=r\sin\theta \\ x^2+y^2=r^2 \\ \theta=\tan^{-1}\dfrac{y}{x} \\ dA=r\,dr\,d\theta \end{array}\quad\rightarrow\quad\begin{array}{c} x^2+y^2=1 \implies r^2=1 \implies r=1 \\ x^2+y^2=4 \implies r^2=4 \implies r=2 \\ y=0 \implies \theta=0 \\ x=\sqrt{3}y \implies \tan^{-1}\dfrac{y}{x}=\tan^{-1}\dfrac{1}{\sqrt{3}} \implies \theta=\dfrac{\pi}{6} \\ \\ \therefore 1\le r\le 2,\quad 0\le\theta\le\dfrac{\pi}{6} \end{array}$$

$$\int_0^{\pi/6}\int_1^2 \sin(r^2)\, r\, dr\, d\theta = \int_0^{\pi/6}\left[-\frac{1}{2}\cos r^2\right]_1^2 d\theta = -\frac{1}{2}\int_0^{\pi/6}(\cos 4 - \cos 1)\; d\theta$$

$$= \frac{1}{2}(\cos 1 - \cos 4)\cdot \theta\Big|_0^{\pi/6} = \boxed{\frac{\pi}{12}(\cos 1 - \cos 4)}$$

7) For cylindrical coordinates, we have

$$\begin{array}{c} z = z \\ r^2 = x^2 + y^2 \\ dV = r\, dz\, dr\, d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} 2z = x^2 + y^2 \;\rightarrow\; z = \dfrac{r^2}{2} \\ x^2 + y^2 + z^2 = 8 \;\rightarrow\; z = \sqrt{8 - r^2} \end{array}$$

Find where the surfaces $2z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 8$ intersect to determine the limits of $r$.

$$\begin{aligned} x^2 + y^2 + z^2 = 8 &\implies 2z + z^2 = 8 \implies (z+4)(z-2) = 0 \implies z = 2 \\ &\implies 4 = x^2 + y^2 = r^2 \implies r = 2 \end{aligned}$$

The lower limit of $r$ is apparently 0. The region in the $xy$-plane is circular if we project the domain. Therefore, $0 \leq \theta \leq 2\pi$. Now, set up the triple integral in polar coordinates.

$$\mathrm{I} = \iiint_D \frac{dV}{x^2 + y^2 + z^2} = \int_0^{2\pi}\int_0^2\int_{r^2/2}^{\sqrt{8-r^2}} \frac{r}{r^2 + z^2}\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2/2}^{\sqrt{8-r^2}} \frac{1}{1 + \left(\frac{z}{r}\right)^2}\cdot\frac{1}{r}\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^2 \left[\arctan\left(\frac{z}{r}\right)\right]_{r^2/2}^{z=\sqrt{8-r^2}} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \arctan\left(\frac{\sqrt{8-r^2}}{r}\right) dr\, d\theta - \int_0^{2\pi}\int_0^2 \arctan\left(\frac{r}{2}\right) dr\, d\theta$$

$\theta$ is independent of $r$. Therefore, we can write the following.

$$\mathrm{I} = 2\pi\int_0^2 \arctan\left(\frac{\sqrt{8-r^2}}{r}\right) dr - 2\pi\int_0^2 \arctan\left(\frac{r}{2}\right) dr \tag{1}$$

Now, use integration by parts for the left-hand integral in (1). Apply the chain rule and the quotient rule rigorously.

$$u = \arctan\left(\frac{\sqrt{8-r^2}}{r}\right) \rightarrow du = \frac{1}{1 + \left(\frac{\sqrt{8-r^2}}{r}\right)^2}\cdot\frac{\frac{1}{2\sqrt{8-r^2}}\cdot(-2r)\cdot r - \sqrt{8-r^2}\cdot 1}{r^2}\, dr$$

$$dv = dr \rightarrow v = r$$

Notice that we have an improper integral, where we need to use limits.

$$\int_0^2 \arctan\left(\frac{\sqrt{8-r^2}}{r}\right)\,dr = \lim_{T\to 0^+}\left[ r\cdot\arctan\left(\frac{\sqrt{8-r^2}}{r}\right)\Bigg|_T^2 - \int_T^2 r\cdot\frac{-1}{\sqrt{8-r^2}}\,dr\right]$$

$$= \lim_{T\to 0^+}\left[ r\cdot\arctan\left(\frac{\sqrt{8-r^2}}{r}\right) - \sqrt{8-r^2}\right]_T^2 \tag{2}$$

Compute the other integral in (1) using integration by parts.

$$u = \arctan\left(\frac{r}{2}\right) \to du = \frac{1}{1+\left(\frac{r}{2}\right)^2}\cdot\frac{1}{2}\,dr$$
$$dv = dr \to v = r$$

$$\int_0^2 \arctan\left(\frac{r}{2}\right)\,dr = r\cdot\arctan\left(\frac{r}{2}\right)\Bigg|_0^2 - \int_0^2 r\cdot\frac{2}{4+r^2}\,dr$$

$$= \left[r\cdot\arctan\left(\frac{r}{2}\right) - \ln\left|4+r^2\right|\right]_0^2 \tag{3}$$

Rewrite (1) using (2) and (3).

$$\mathrm{I} = 2\pi\lim_{T\to 0^+}\left[ r\cdot\arctan\left(\frac{\sqrt{8-r^2}}{r}\right) - \sqrt{8-r^2}\right]_T^2 - 2\pi\left[r\cdot\arctan\left(\frac{r}{2}\right) - \ln\left|4+r^2\right|\right]_0^2$$

$$\mathrm{I} = 2\pi\left(2\cdot\arctan 1 - 2\right) - 2\pi\lim_{T\to 0^+}\left[T\cdot\arctan\frac{\sqrt{8-T^2}}{T} - \sqrt{8-T^2}\right]$$

$$- 2\pi\left[\left(2\cdot\arctan 1 - \ln 8\right) - \left(0 - \ln 4\right)\right]$$

$$\mathrm{I} = \pi\left(\ln 4 - 4 + 2\lim_{T\to 0^+}\left(\sqrt{8-T^2}\right)\right) - 2\pi\lim_{T\to 0^+}\left(T\cdot\arctan\frac{\sqrt{8-T^2}}{T}\right) \tag{4}$$

We need to evaluate the limit on the right side in (4) using the squeeze theorem.

$$-\frac{\pi}{2} \le \arctan\left(\frac{\sqrt{8-T^2}}{T}\right) \le \frac{\pi}{2}$$

$$-\frac{T\cdot\pi}{2} \le T\cdot\arctan\left(\frac{\sqrt{8-T^2}}{T}\right) \le \frac{T\cdot\pi}{2}$$

$$\lim_{T\to 0^+}\frac{-T\cdot\pi}{2} = \lim_{T\to 0^+}\frac{T\cdot\pi}{2} = 0 \implies \lim_{T\to 0^+}\left(T\cdot\arctan\frac{\sqrt{8-T^2}}{T}\right) = 0$$

The limit on the left side in (4) is simply equal to $2\sqrt{2}$. The value of the integral is then

$$\boxed{\mathrm{I} = \pi\left(\ln 4 - 4 + 4\sqrt{2}\right)}$$

8) For spherical coordinates, we have

$$\begin{array}{c} z = \rho\cos\phi \\ r = \rho\sin\phi \\ x^2+y^2+z^2 = \rho^2 \\ dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} x^2+y^2+z^2 \le 1 \implies \rho^2 \le 1 \implies 0 \le \rho \le 1 \\ \sqrt{x^2+y^2+z^2} = 1 \rightarrow \sqrt{\rho^2} = 1 \implies \rho = 1 \\ \\ \therefore 0 \le \theta \le \dfrac{\pi}{2}, \quad 0 \le \phi \le \dfrac{\pi}{2} \end{array}$$

Set up the integral and then evaluate.

$$\mathrm{I} = \iiint_D \sqrt{x^2+y^2+z^2}\, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1}\sin\phi\, d\phi\, d\theta = \frac{1}{4}\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\, d\phi\, d\theta = \frac{1}{4}\int_0^{\pi/2}\Big[-\cos\phi\Big]_0^{\pi/2} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} d\theta = \boxed{\frac{\pi}{8}}$$