

2022-2023 Fall
MAT123-02,05 Final
(11/01/2023)

1. The radius R of a spherical ball is measured as 14 in.

(a) Use differentials to estimate the maximum propagated error in computing the volume V if R is measured with a maximum error of $1/8$ inches.

(b) With what accuracy must the radius R be measured to guarantee an error of at most 2 in^3 in the calculated volume?

2. Evaluate the following integrals.

(a) $\int \frac{dx}{\sqrt{x}(\sqrt{x}+2)}$

(b) ~~$\int \frac{\sin x}{x^2+1} dx$~~

(c) $\int \frac{dx}{\sqrt{3-x^2}}$

(d) $\int \frac{dx}{2+\cos x}$

3. Use the Shell Method and then the Washer Method to set up an integral (but do not evaluate) the volume of the solid generated by revolving the region R about the y -axis, where R is bounded by the curve $y = x^2$ and the line $y = -x + 1$.

4. Find the area of the surface obtained by rotating the arc of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $[1/2, 1]$ about the x -axis.

5. Using the Integral Test, determine whether the series

$$\sum_{n=1}^{\infty} \frac{2}{3n+5}$$

converges or diverges.

6. Find the Maclaurin series for $f(x) = \frac{1}{x^2 - 5x + 6}$.

2022-2023 Final (11/01/2023) Solutions
(Last update: 18/08/2025 01:51)

1.

(a) The volume of a sphere with radius r is

$$V = \frac{4}{3}\pi r^3$$

The differential of V is

$$dV = 4\pi r^2 dr$$

The maximum error is known to be $1/8$ inches. So, $|dr| \leq 1/8$. The maximum propagated error is then

$$dV = 4\pi \cdot 14^2 \cdot \frac{1}{8} = \boxed{98\pi \text{ in}^3}.$$

(b) $|dV| = 2$ at most. Solve the differential form for dr .

$$dr = \frac{dV}{4\pi r^2} = \frac{2}{4\pi \cdot 14^2} = \boxed{\frac{1}{392\pi} \text{ inches}}$$

2.

(a) Let $x = u^2$, then $u = \sqrt{x}$ and $dx = 2u du$.

$$\begin{aligned} \int \frac{dx}{\sqrt{x}(\sqrt{x}+2)} &= \int \frac{2u}{u(u+2)} du = 2 \int \frac{du}{u+2} = 2 \ln |u+2| + c \\ &= \boxed{2 \ln |\sqrt{x}+2| + c, \quad c \in \mathbb{R}} \end{aligned}$$

(b) This question is beyond the scope of the curriculum, and students are not expected to solve it using the knowledge they have acquired in this course.

(c) Let $x = \sqrt{3} \sin u$ for $-\frac{\pi}{2} < u < \frac{\pi}{2}$, then $dx = \sqrt{3} \cos u du$.

$$\begin{aligned} \int \frac{dx}{\sqrt{3-x^2}} &= \int \frac{\sqrt{3} \cos u}{\sqrt{3-3\sin^2 u}} du = \int \frac{\cos u}{\sqrt{\cos^2 u}} du = \int \frac{\cos u}{|\cos u|} du \\ &= \int \frac{\cos u}{\cos u} du \quad [\cos u > 0] \\ &= \int du = u + c \end{aligned}$$

If $x = \sqrt{3} \sin u$, then $\sin u = \frac{x}{\sqrt{3}} \implies u = \arcsin\left(\frac{x}{\sqrt{3}}\right)$. The answer is then

$$\arcsin\left(\frac{x}{\sqrt{3}}\right) + c, \quad c \in \mathbb{R}$$

(d) We may utilize the tangent half-angle substitution, which is sometimes called the Weierstrass substitution. Let $t = \tan\left(\frac{x}{2}\right)$. Then

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2} dt$$

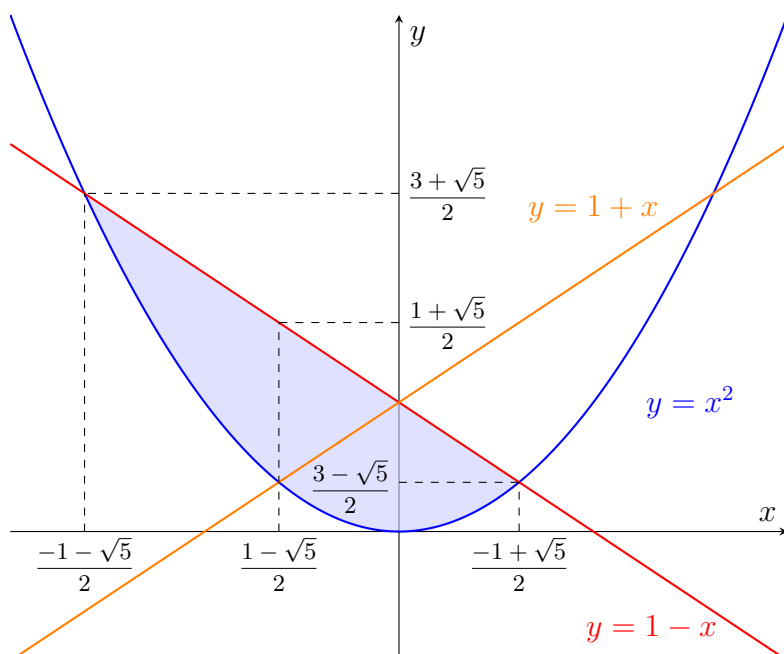
Rewrite the integral.

$$\int \frac{dx}{2 + \cos x} = \int \frac{\frac{2}{1+t^2}}{2 + \frac{1-t^2}{1+t^2}} dt = \int \frac{2}{3+t^2} dt = \int \frac{2}{3\left(1 + \frac{t^2}{3}\right)} dt = \frac{2}{3} \int \frac{1}{1 + \left(\frac{t}{\sqrt{3}}\right)^2} dt$$

Let $u = \frac{t}{\sqrt{3}}$, then $\sqrt{3} du = dt$.

$$\begin{aligned} \frac{2}{3} \int \frac{dt}{1 + \left(\frac{t}{\sqrt{3}}\right)^2} &= \frac{2\sqrt{3}}{3} \int \frac{du}{1+u^2} = \frac{2\sqrt{3}}{3} \arctan u + c = \frac{2\sqrt{3}}{3} \arctan \frac{t}{\sqrt{3}} + c \\ &= \frac{2\sqrt{3}}{3} \arctan\left(\frac{1}{\sqrt{3}} \tan\left(\frac{x}{2}\right)\right) + c, \quad c \in \mathbb{R} \end{aligned}$$

3.



We have the symmetry of the region that is bounded to the right of the y -axis. Therefore, it is not necessary to apply the method to the symmetrical region on the left. The volume of this solid is

$$\boxed{\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{5}}{2}} 2\pi(-x) [(1-x) - (x^2)] dx + \int_{\frac{1-\sqrt{5}}{2}}^0 2\pi(-x) [(1-x) - (1+x)] dx + \int_0^{\frac{-1+\sqrt{5}}{2}} 2\pi(x) [(1-x) - (x^2)] dx}$$

4. If the function $y = f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the x -axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$. Set $a = 1/2$, $b = 1$ and then evaluate the integral.

$$\begin{aligned} S &= \int_{1/2}^1 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} dx \\ &= \int_{1/2}^1 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} dx = \int_{1/2}^1 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} dx \\ &= \int_{1/2}^1 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} dx = \int_{1/2}^1 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right) \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx \\ &= \int_{1/2}^1 2\pi \left(\frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3}\right) dx = 2\pi \left[\frac{x^6}{72} + \frac{x^2}{24} + \frac{x^2}{8} - \frac{1}{8x^2}\right]_{1/2}^1 \\ &= 2\pi \left[\left(\frac{1}{72} + \frac{1}{24} + \frac{1}{8} - \frac{1}{8}\right) - \left(\frac{1}{4608} + \frac{1}{96} + \frac{1}{32} - \frac{1}{2}\right)\right] = \boxed{\frac{2367\pi}{2304}} \end{aligned}$$

5. Take the corresponding function $f(x) = \frac{2}{3x+5}$. The function is continuous for $x \geq 1$ because the denominator is a first-degree polynomial whose root is $x_0 = -\frac{5}{3} < 1$. f is also positive and increasing for $x \geq 1$. Since the criteria hold, we may apply the Integral Test. Handle the improper integral by taking the limit.

$$\begin{aligned} \int_1^\infty \frac{2}{3x+5} dx &= \lim_{R \rightarrow \infty} \int_1^R \frac{2}{3x+5} dx = \lim_{R \rightarrow \infty} \frac{2}{3} \ln |3x+5| \Big|_1^R = \frac{2}{3} \lim_{R \rightarrow \infty} (\ln |3R+5| - \ln 8) \\ &= \infty \end{aligned}$$

Since the integral diverges, by the Integral Test, the series $\sum_{n=1}^{\infty} \frac{2}{3n+5}$ also diverges.

6. Recall the equality $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$.

$$f(x) = \frac{1}{x^2 - 5x + 6} = \frac{1}{(x-3)(x-2)} = \frac{1}{x-3} - \frac{1}{x-2}$$

The Maclaurin series of f is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, $f^{(4)}(0)$ to look for the pattern.

$$\begin{aligned} f'(x) &= -\frac{1}{(x-3)^2} + \frac{1}{(x-2)^2}, & f''(x) &= \frac{2}{(x-3)^3} - \frac{2}{(x-2)^3} \\ f'''(x) &= -\frac{6}{(x-3)^4} + \frac{6}{(x-2)^4}, & f^{(4)}(x) &= \frac{24}{(x-3)^5} - \frac{24}{(x-2)^5} \end{aligned}$$

$$\begin{aligned} f(0) &= -\frac{1}{3} + \frac{1}{2}, & f'(0) &= -\frac{1}{9} + \frac{1}{4}, & f''(0) &= -\frac{2}{27} + \frac{2}{8} \\ f'''(0) &= -\frac{6}{81} + \frac{6}{16}, & f^{(4)}(0) &= -\frac{24}{243} + \frac{24}{32} \end{aligned}$$

This is a sequence where each term is defined by the following.

$$f^{(k)}(0) = (k!) \cdot \left(-\frac{1}{3^{k+1}} + \frac{1}{2^{k+1}} \right)$$

Rewrite the sum.

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k &= \sum_{k=0}^{\infty} \frac{(k!) \cdot x^k}{k!} \cdot \left(-\frac{1}{3^{k+1}} + \frac{1}{2^{k+1}} \right) \\ &= \boxed{\sum_{k=0}^{\infty} x^k \left(\frac{1}{2^{k+1}} - \frac{1}{3^{k+1}} \right)} \end{aligned}$$