

2020-2021 Fall  
MAT123 Final  
(18/01/2021)

1. Consider the region  $R$  bounded by the curve  $y = x^3$ , and the straight lines  $y = -x$  and  $y = x + 6$ .

(i) Write down the integral corresponding to the area of  $R$  with respect to  $x$ .

(ii) Write down the integral corresponding to the area of  $R$  with respect to  $y$ .

2. Consider the solid  $S$  obtained by revolving the region in the first quadrant bounded by  $x = -y^2 + 1$ ,  $y^2 = x$  and  $y = 1/2$  about  $x = -1$ .

(i) Using the Shell Method, write down the integral corresponding to the volume of the solid  $S$ .

(ii) Using the Washer Method, write down the integral corresponding to the volume of the solid  $S$ .

3. Evaluate the following integrals.

(a)  $\int \frac{dx}{x^{2/3}(\sqrt[3]{x} + 4)}$

(b)  $\int (\ln x)^2 dx$

(c)  $\int \frac{dx}{\sqrt{3+x^2}}$

(d)  $\int \frac{dx}{2 + \sin x}$

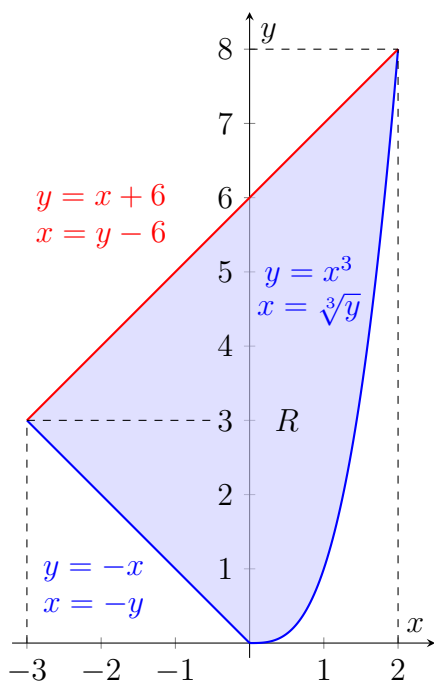
4. Using the Monotone Convergence Theorem, show that the sequence  $\left(\frac{n^2 + 1}{n^3}\right)_{n \in \mathbb{N}}$  is convergent.

5. Use the Integral Test to determine whether the series  $\sum_{k=1}^{\infty} \frac{1}{k^4 + k^2}$  converges or diverges.

6. Find the convergence set for the power series  $\sum_{k=1}^{\infty} \frac{(x+1)^k}{2k}$ .

7. Using a Maclaurin series, show that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

1.



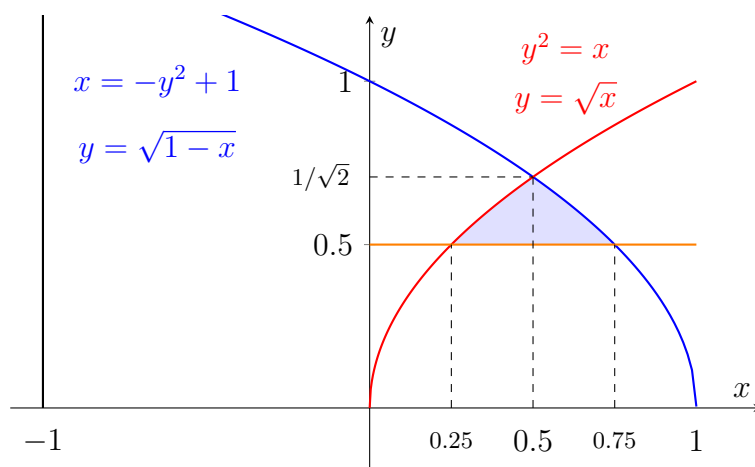
(i)

$$\int_{-3}^0 [(x+6) - (-x)] dx + \int_0^2 [(x+6) - (x^3)] dx$$

(ii)

$$\int_0^3 [(\sqrt[3]{y}) - (-y)] dy + \int_3^8 [(\sqrt[3]{y}) - (y-6)] dy$$

2.



(i)

$$\int_{1/4}^{1/2} 2\pi(x+1) \left[ (\sqrt{x}) - \left(\frac{1}{2}\right) \right] dx + \int_{1/2}^{3/4} 2\pi(x+1) \left[ (\sqrt{1-x}) - \left(\frac{1}{2}\right) \right] dx$$

(ii)

$$\int_{1/2}^{1/\sqrt{2}} \pi \left[ (-y^2 + 1 + 1)^2 - (y^2 + 1)^2 \right] dy$$

3.

(a) Let  $u = \sqrt[3]{x} + 4$ , then  $du = \frac{1}{3x^{2/3}} dx$ .

$$\int \frac{dx}{x^{2/3}(\sqrt[3]{x} + 4)} = \int \frac{3du}{u} = 3 \ln |u| + c = \boxed{3 \ln |\sqrt[3]{x} + 4| + c, \quad c \in \mathbb{R}}$$

(b) Apply integration by parts.

$$\left. \begin{aligned} u = (\ln x)^2 &\implies du = 2 \ln x \cdot \frac{1}{x} dx \\ dv = dx &\implies v = x \end{aligned} \right\} \rightarrow \int u dv = uv - \int v du$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int 2 \ln x dx$$

Apply integration by parts once again.

$$\left. \begin{aligned} u = \ln x &\implies du = \frac{1}{x} dx \\ dv = dx &\implies v = x \end{aligned} \right\} \rightarrow \int u dv = uv - \int v du$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int 2 \ln x dx$$

$$x(\ln x)^2 - \int 2 \ln x dx = x(\ln x)^2 - 2 \left[ x \ln x - \int dx \right] = \boxed{x(\ln x)^2 - 2x \ln x + 2x + c, \quad c \in \mathbb{R}}$$

(c) Let  $x = \sqrt{3} \tan u$  for  $0 < u < \frac{\pi}{2}$ , then  $dx = \sqrt{3} \sec^2 u du$ .

$$\begin{aligned} \int \frac{dx}{\sqrt{3+x^2}} &= \int \frac{\sqrt{3} \sec^2 u}{\sqrt{3+3 \tan^2 u}} du = \int \frac{\sec^2 u}{\sqrt{1+\tan^2 u}} du = \int \frac{\sec^2 u}{|\sec u|} du \\ &= \int \sec u du \quad [\sec u > 0] \\ &= \ln |\tan u + \sec u| + c, \quad c \in \mathbb{R} \end{aligned}$$

Recall  $x = \sqrt{3} \tan u$ .

$$x^2 = 3 \tan^2 u = 3 \sec^2 u - 3 \implies \sec^2 u = \frac{x^2 + 3}{3} \implies \sec u = \frac{\sqrt{x^2 + 3}}{\sqrt{3}}$$

$$\ln |\tan u + \sec u| + c = \ln \left| \frac{x}{\sqrt{3}} + \frac{\sqrt{x^2 + 3}}{\sqrt{3}} \right| + c, \quad c \in \mathbb{R}$$

We can omit the constant part.

$$\boxed{\ln \left( x + \sqrt{x^2 + 3} \right) + c, \quad c \in \mathbb{R}}$$

(d) We may utilize the tangent half-angle substitution, which is also called the Weierstrass substitution. Let  $t = \tan\left(\frac{x}{2}\right)$ . After some mathematical operations, we get the following.

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2} dt$$

$$\begin{aligned} \int \frac{dx}{2+\sin x} &= \int \frac{2}{1+t^2} \cdot \frac{1}{2+\frac{2t}{1+t^2}} dt = \int \frac{dt}{t^2+t+1} = \int \frac{dt}{t^2+t+\frac{1}{4}+\frac{3}{4}} \\ &= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{4}{3} \int \frac{dt}{\frac{4}{3}\left(t+\frac{1}{2}\right)^2+1} = \frac{4}{3} \int \frac{dt}{\left(\frac{2}{\sqrt{3}}\right)^2\left(t+\frac{1}{2}\right)^2+1} \end{aligned}$$

Let  $u = \frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)$ , then  $du = \frac{2}{\sqrt{3}} dt$ .

$$\frac{2\sqrt{3}}{3} \int \frac{du}{u^2+1} = \frac{2\sqrt{3}}{3} \arctan u + c = \frac{2\sqrt{3}}{3} \arctan \left( \frac{2}{\sqrt{3}} \left( t + \frac{1}{2} \right) \right) + c$$

$$= \boxed{\frac{2\sqrt{3}}{3} \arctan \left( \frac{2}{\sqrt{3}} \left( \tan\left(\frac{x}{2}\right) + \frac{1}{2} \right) \right) + c, \quad c \in \mathbb{R}}$$

4. Take  $f(x) = \frac{x^2+1}{x^3}$ . We have  $f'(x) = -\frac{x^2+3}{x^4} < 0$  for all  $x \geq 1$ . That means  $f$  is decreasing for  $x \geq 1$ . We also have

$$f(1) = 2, \quad \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left( \frac{1}{x} + \frac{1}{x^3} \right) \implies 0 < f(x) \leq 2, \quad x \geq 1$$

Since the sequence is bounded and monotonic, by the Monotone Convergence Theorem, the sequence converges.

5. Take the corresponding function  $f(x) = \frac{1}{x^4+x^2}$ .  $f$  is positive for  $x \geq 1$  because  $x^4 > 0$  and  $x^2 > 0$ .  $f$  is also continuous for  $x \geq 1$  because the denominator is a polynomial whose *only* root is zero, which is out of the boundary of the integral. Investigate the monotonicity of  $f$  by taking the first derivative.

$$f'(x) = -\frac{4x^3+2x}{(x^4+x^2)^2} \implies f'(x) \leq 0 \text{ for } x \geq 1$$

We may now apply the Integral Test since the criteria have been satisfied.

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^4 + x^2} &= \int_1^\infty \frac{dx}{x^2(x^2 + 1)} = \int_1^\infty \left( \frac{1}{x^2} - \frac{1}{x^2 + 1} \right) dx \\
&= \lim_{R \rightarrow \infty} \int_1^R \left( \frac{1}{x^2} - \frac{1}{x^2 + 1} \right) dx = \lim_{R \rightarrow \infty} \left[ -\frac{1}{x} - \arctan x \right]_1^R \\
&= \lim_{R \rightarrow \infty} \left[ \left( -\frac{1}{R} - \arctan R \right) - \left( -1 - \arctan 1 \right) \right] = 1 - \frac{\pi}{4} \quad (\text{convergent})
\end{aligned}$$

By the Integral Test, the series  $\sum_{k=1}^{\infty} \frac{1}{k^4 + k^2}$  also converges.

6. Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left| \frac{(x+1)^{k+1}}{2(k+1)} \cdot \frac{2k}{(x+1)^k} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x+1) \cdot k}{k+1} \right| = |x+1| \cdot \lim_{n \rightarrow \infty} \left| \frac{k}{k+1} \right| = |x+1| \\
|x+1| < 1 &\implies -1 < x+1 < 1 \implies -2 < x < 0 \quad (\text{convergent})
\end{aligned}$$

Now, take a look at the endpoints.

$$x = -2 \rightarrow \sum_{k=1}^{\infty} \frac{(-1)^k}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

This is an alternating series. The non-alternating part, which is  $\frac{1}{k}$ , is nonincreasing for  $k \geq 1$  and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges. Try  $x = 0$ .

$$x = 0 \rightarrow \sum_{k=1}^{\infty} \frac{1^k}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k}$$

This is a  $p$ -series with  $p = 1$ , for which the series diverges by the  $p$ -series Test.

The convergence set for the power series is

$$\boxed{[-2, 0)}$$

7. The Maclaurin series of  $\sin x$  is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Rewrite the limit using this expansion.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \left[ \frac{1}{x} \cdot \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \right] = \lim_{x \rightarrow 0} \left( 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \right) = \boxed{1}$$