2019-2020 Summer
MAT124 Final
(28/08/2020)

1. Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 - 3y$ on the closed disk $x^2 + y^2 \leq 4$.

2. The velocity of a particle moving in space is

$$\mathbf{V}(t) = \ln t\,\mathbf{i} + \sin t\,\mathbf{j} + t^3\,\mathbf{k}$$

Find the particle's position as a function of t if the position at time $t = 1$ is $\mathbf{R}(1) = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

3. Evaluate the integral $\displaystyle\int_0^1 \int_{y^2}^1 y^3 \cos x^3 \, dx \, dy$.

4. Let R be the region lying inside $r = 1$ and outside $r = 1 + \cos\theta$.

(i) Sketch the graph of the region R.

(ii) Set up (but do not evaluate) a double integral in polar coordinates for the area of the region R.

5. Let S be the surface of the portion of the cone $z^2 = x^2 + y^2$ that is contained in the cylinder $x^2 + y^2 = 9$.

(i) Sketch the graph of S.

(ii) Evaluate the surface area using polar coordinates.

6. Let S be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.

(i) Using the spherical coordinates, set up (but do not evaluate) an integral for the volume of the solid S.

(ii) Using the cylindrical coordinates, set up (but do not evaluate) an integral for the volume of the solid S.

2019-2020 Summer Final (28/08/2020) Solutions
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1. Determine the critical points by setting $f_x = f_y = 0$.

$$f_x = 2x, \quad f_y = 2y - 3$$

$$f_x = f_y = 0 \implies x = 0, \quad y = \frac{3}{2}$$

The *only* critical point occurs at $(0, 3/2)$. The value of the function at this point is $f(0, 3/2) = 0^2 + (3/2)^2 - 3(3/2) = -9/4$.

Using Lagrange multipliers, find the corresponding points on the boundary. Let $g(x, y) = x^2 + y^2 - 4$ be the constraint. Solve the system of equations below.

$$\left.\begin{array}{l} \nabla f = \lambda \nabla g \\ g(x, y) = 0 \end{array}\right\} \quad \nabla f = \langle 2x, 2y - 3 \rangle = \lambda \langle 2x, 2y \rangle = \lambda \nabla g$$

$$2x(1 - \lambda) = 0 \implies x = 0 \quad \text{or} \quad \lambda = 1$$

$$2y(1 - \lambda) - 3 = 0 \implies 1 - \lambda = \frac{3}{2y} \implies \lambda = 1 - \frac{3}{2y}$$

$$x = 0 \implies g(0, y) = 0^2 + y^2 - 4 = 0 \implies y = \pm 2$$

$$\lambda = 1 \implies 2y - 3 = 2y \implies 0 \neq -3 \quad \therefore \lambda \neq -1$$

Evaluate f at the points $(0, 2)$ and $(0, -2)$.

$$f(0, 2) = 0^2 + 2^2 - 3 \cdot 2 = -2, \quad f(0, -2) = 0^2 + (-2)^2 - 3 \cdot (-2) = 10$$

Compare the values $f(0, 3/2)$, $f(0, 2)$, $f(0, -2)$.

$$\boxed{f_{\max} = f(0, -2) = 10, \quad f_{\min} = f(0, 3/2) = -9/4}$$

2. The velocity of a particle can be obtained by taking the first derivative of the position vector of the particle. Since we have the velocity of the particle, we can take the antiderivative. The velocity vector is defined for $t > 0$. It is continuous and integrable for $t > 0$.

$$\mathbf{V}(t) = \ln t \, \mathbf{i} + \sin t \, \mathbf{j} + t^3 \, \mathbf{k}$$

$$\int \mathbf{V}(t)\, dt = \left(\int \ln t \, dt \right) \mathbf{i} + \left(\int \sin t \, dt \right) \mathbf{j} + \left(\int t^3 \, dt \right) \mathbf{k}$$

The first integral on the right side can be evaluated with integration by parts. The second integral is related to the integration of trigonometric functions. The last integral is just ordinary integration. The position vector is as follows.

$$\mathbf{R}(t) = (t \ln t - t + c_1)\, \mathbf{i} + (-\cos t + c_2)\, \mathbf{j} + \left(\frac{t^4}{4} + c_3 \right) \mathbf{k}$$

Evaluate $\mathbf{R}(1)$ to find the constants.

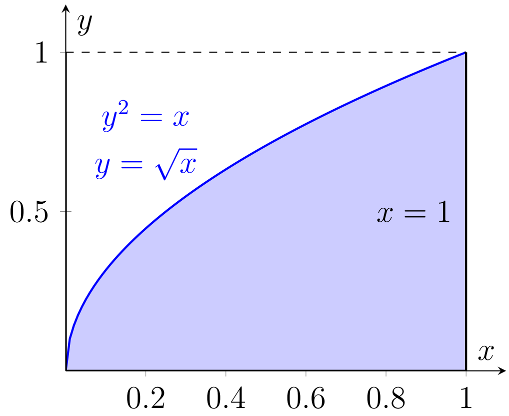
$$\mathbf{R}(1) = (-1 + c_1)\,\mathbf{i} + (-\cos 1 + c_2)\,\mathbf{j} + \left(\frac{1}{4} + c_3\right)\,\mathbf{k} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$c_1 = 3, \quad c_2 = 1 + \cos 1, \quad c_3 = -\frac{5}{4}$$

Rewrite the position vector by substituting the numbers into the constants.

$$\boxed{\mathbf{R}(t) = (t\ln t - t + 3)\,\mathbf{i} + (-\cos t + 1 + \cos 1)\,\mathbf{j} + \left(\frac{t^4}{4} - \frac{5}{4}\right)\,\mathbf{k}}$$

3. It is difficult to solve the integral with this order of integration. Sketch the region and change the order of integration.

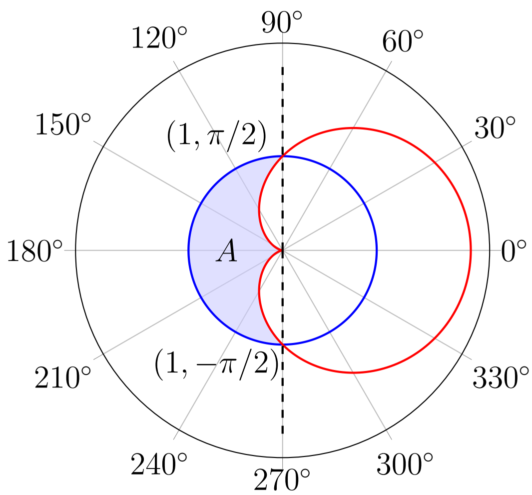


$$\int_0^1 \int_{y^2}^1 y^3 \cos\left(x^3\right)\,dx\,dy = \int_0^1 \int_0^{\sqrt{x}} y^3 \cos\left(x^3\right)\,dy\,dx = \int_0^1 \cos\left(x^3\right)\left[\frac{y^4}{4}\right]_{y=0}^{y=\sqrt{x}}\,dx$$

$$= \frac{1}{4}\int_0^1 x^2 \cos\left(x^3\right)\,dx = \frac{1}{12}\left[\sin\left(x^3\right)\right]_0^1 = \boxed{\frac{1}{12}\sin 1}$$

4.

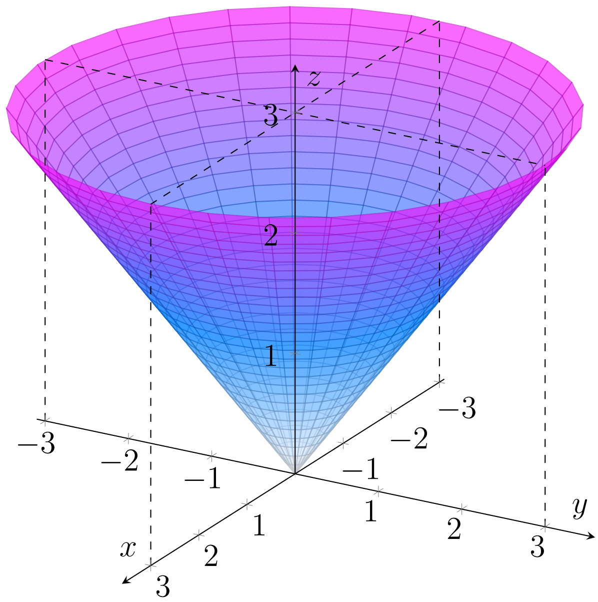
(i)



(ii) $\boxed{A = \int_{\pi/2}^{3\pi/2} \int_{1+\cos\theta}^{1} r\,dr\,d\theta}$

5.

(i)



$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\, dA$$

$$= \iint_D \sqrt{1 + \left(\frac{x^2+y^2}{x^2+y^2}\right)}\, dA = \iint_D \sqrt{1+1}\, dA = \sqrt{2} \iint_D dA$$

If we switch to polar coordinates, we can easily evaluate the integral.

$$A = \sqrt{2} \int_0^{2\pi} \int_0^3 r\, dr\, d\theta = \sqrt{2} \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=3} d\theta = \sqrt{2} \int_0^{2\pi} \frac{9}{2}\, d\theta = \boxed{9\pi\sqrt{2}}$$

6.

(i) For spherical coordinates, we have

$$\begin{gathered} z = \rho\cos\phi \\ r = \rho\sin\phi \\ x^2 + y^2 + z^2 = \rho^2 \\ dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \end{gathered} \quad \rightarrow \quad \begin{aligned} z = 8 - x^2 - y^2 &\implies \rho\cos\phi = 8 - \rho^2\sin^2\phi \quad (1) \\ z = x^2 + y^2 &\implies \rho\cos\phi = \rho^2\sin^2\phi \quad (2) \end{aligned}$$

Now, notice that we have two distinct upper bounds for ρ. From $\phi = 0$ to the angle of intersection, the integration is bounded above by $z = 8 - x^2 - y^2$, where from the angle of intersection to $\phi = \pi/2$, the integration is bounded above by $z = x^2 + y^2$.

Solve (1) for ρ to find the upper bound.

$$\rho\cos\phi = 8 - \rho^2\sin^2\phi \implies \rho^2\sin^2\phi + \rho\cos\phi - 8 = 0$$

$$\rho_{1,2} = \frac{-\cos\phi \pm \sqrt{\cos^2\phi - 4\sin^2\phi\cdot(-8)}}{2\sin^2\phi} \qquad \left[x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\right]$$

$$\rho > 0 \implies \rho_{\text{upper, 1}} = \frac{-\cos\phi + \sqrt{\cos^2\phi + 32\sin^2\phi}}{2\sin^2\phi}$$

Solve (2) for ρ to find the other upper bound.

$$\rho\cos\phi = \rho^2\sin^2\phi \implies \rho_{\text{upper, 2}} = \frac{\cos\phi}{\sin^2\phi} = \cot\phi\csc\phi$$

Find where two surfaces intersect by equating (1) and (2).

$$8 - \rho^2\sin^2\phi = \rho^2\sin^2\phi \implies \rho^2\sin^2\phi = 4 \implies \rho\sin\phi = 2 \implies r = 2$$

$$z = x^2 + y^2 = r^2 \implies z = 4 \implies \rho = \sqrt{x^2+y^2+z^2} = 2\sqrt{5} \implies \sin\phi = \frac{2}{\rho} = \frac{1}{\sqrt{5}}$$

$$\phi = \arcsin\frac{1}{\sqrt{5}}$$

We know $0 \le \theta \le 2\pi$. Therefore, the volume in spherical coordinates is as follows.

$$\boxed{\begin{aligned} V = &\int_0^{2\pi}\int_0^{\arcsin\frac{1}{\sqrt{5}}}\int_0^{\frac{-\cos\phi+\sqrt{\cos^2\phi+32\sin^2\phi}}{2\sin^2\phi}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &+\int_0^{2\pi}\int_{\arcsin\frac{1}{\sqrt{5}}}^{\pi/2}\int_0^{\cot\phi\csc\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}}$$

Since we choose the minimum of the bounds for ρ, we can write the equivalent expression.

$$\boxed{V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\min\left(\cot\phi\csc\phi,\ \frac{-\cos\phi+\sqrt{\cos^2\phi+32\sin^2\phi}}{2\sin^2\phi}\right)} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

(ii) For cylindrical coordinates, we have

$$\begin{aligned} z &= z \\ r^2 &= x^2+y^2 \\ dV &= r\,dz\,dr\,d\theta \end{aligned} \quad \rightarrow \quad \begin{aligned} &z = 8 - x^2 - y^2 \implies z = 8 - r^2 && (3) \\ &z = x^2 + y^2 \implies z = r^2 && (4) \end{aligned}$$

Equate (3) and (4) to find the upper bound for r.

$$8 - r^2 = r^2 \implies r^2 = 4 \implies r_{\text{upper}} = 2$$

The lower bound for r is 0, and $0 \le \theta \le 2\pi$. The volume can be expressed as follows.

$$\boxed{V = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\,dz\,dr\,d\theta}$$