2021-2022 Spring
MAT124 Midterm
(20/04/2022)

1. Find the convergence set of the power series $\sum_{k=1}^{\infty} \frac{(\ln k)x^k}{k}$.

2. Show that the vectors $\mathbf{u} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} + 3\mathbf{k}$ are coplanar (all in the same plane).

3. Find the point where the line

$$x - 1 = z,\ y = 3$$

intersects the plane $2y - z = 5$.

4. Sketch the graphs of the following surfaces.

(a) $z = \sin x$

(b) $y = z^2 - x^2$

5. Let f be the function $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ for $(x, y) \neq (0, 0)$. Show that f has no limit at $(0, 0)$.

6. Van der Waal's equation in physical chemistry states that a gas occupying volume V at temperature T (Kelvin) exerts pressure P, where

$$\left(P + \frac{A}{V^2}\right)(V - B) = kT$$

for physical constants A, B and k. Compute the following rates.

(a) the rate of change of volume with respect to temperature.

(b) the rate of change of pressure with respect to volume.

7. Consider the surface $z = x^2 + y^2 - xy$, a paraboloid, on which a particle moves with x and y and coordinates given by $x = \cos t$ and $y = \sin t$. Using the chain rule, find $\dfrac{dz}{dt}$ when $t = 0$.

8. Find the parametric equations of the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $2x^2 + 2y^2 + z^2 = 8$ at $P = (-1, 1, 2)$.

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1. Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\lim_{k\to\infty}\left|\frac{\ln(k+1)\cdot x^{k+1}}{k+1}\cdot\frac{k}{\ln(k)\cdot x^k}\right| = \lim_{k\to\infty}\left|x\cdot\frac{k}{k+1}\cdot\frac{\ln(k+1)}{\ln(k)}\right| = |x|\cdot 1 = |x|$$

$k/(k+1)\to 1$ and $\ln(k)/\ln(k+1)\to 1$ as $k\to\infty$. Therefore, the limit is $|x|$.

$$|x|<1 \implies -1<x<1 \quad \text{(convergent)}$$

Investigate the convergence at the endpoints.

$$x=1 \implies \sum_{k=1}^{\infty}\frac{(\ln k)\cdot 1^k}{k} = \sum_{k=1}^{\infty}\frac{\ln k}{k}$$

Take the corresponding function $f(x)=\dfrac{\ln x}{x}$. The function is continuous and positive for $x>1$. It is also decreasing for $x>\mathrm{e}$ because x grows faster than $\ln x$. Confirm the behavior by taking the first derivative.

$$f'(x)=\frac{1-\ln x}{x^2}<0 \quad \text{for} \quad x>\mathrm{e}$$

We may now apply the Integral Test.

$$\int_1^{\infty}\frac{\ln x}{x}\,dx = \lim_{R\to\infty}\int_1^{R}\frac{\ln x}{x}\,dx = \lim_{R\to\infty}\frac{1}{2}(\ln x)^2\Big|_1^R = \frac{1}{2}\lim_{R\to\infty}\left((\ln R)^2-(\ln 1)^2\right)=\infty$$

Since the integral diverges, by the Integral Test, the series $\displaystyle\sum_{k=1}^{\infty}\frac{\ln k}{k}$ also diverges. Try $x=-1$.

$$x=-1 \implies \sum_{k=1}^{\infty}\frac{(\ln k)\cdot(-1)^k}{k}$$

This is an alternating series. The non-alternating part, which is $\dfrac{\ln k}{k}$, is nonincreasing for $k>\mathrm{e}$ and it is positive. The previous cases are already confirmed for $x=1$. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges.

Thus, the convergence set for the power series is $\boxed{[-1,1)}$.

2. If $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})=0$, the vectors are coplanar. Because $\mathbf{u}\times\mathbf{v}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$, and the resulting vector is also perpendicular to $\mathbf{w}$. Compute the cross product.

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&3&1\\2&-1&-1\end{vmatrix}=\mathbf{i}\begin{vmatrix}3&1\\-1&-1\end{vmatrix}-\mathbf{j}\begin{vmatrix}1&1\\2&-1\end{vmatrix}+\mathbf{k}\begin{vmatrix}1&3\\2&-1\end{vmatrix}$$

$$=(-1\cdot 3-1\cdot(-1))\mathbf{i}-(-1\cdot 1-2\cdot 1)\mathbf{j}+(-1\cdot 1-3\cdot 2)\mathbf{k}=-2\mathbf{i}+3\mathbf{j}-7\mathbf{k}$$

Compute the dot product.

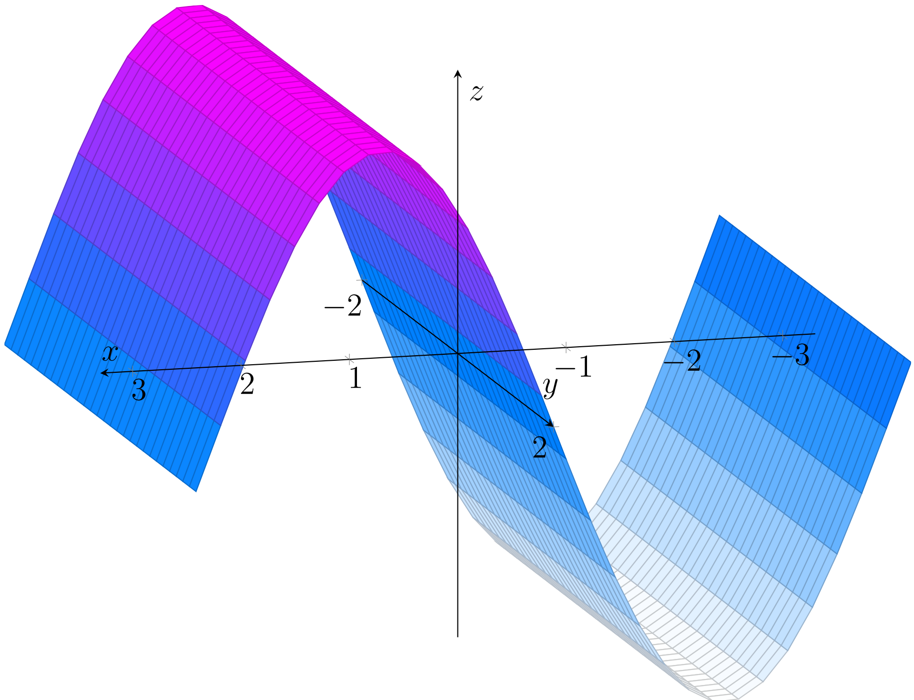
$$\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})=(7\mathbf{j}+3\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}-7\mathbf{k})=0\cdot 2+7\cdot 3+3\cdot(-7)=0$$

Therefore, all the vectors are coplanar.

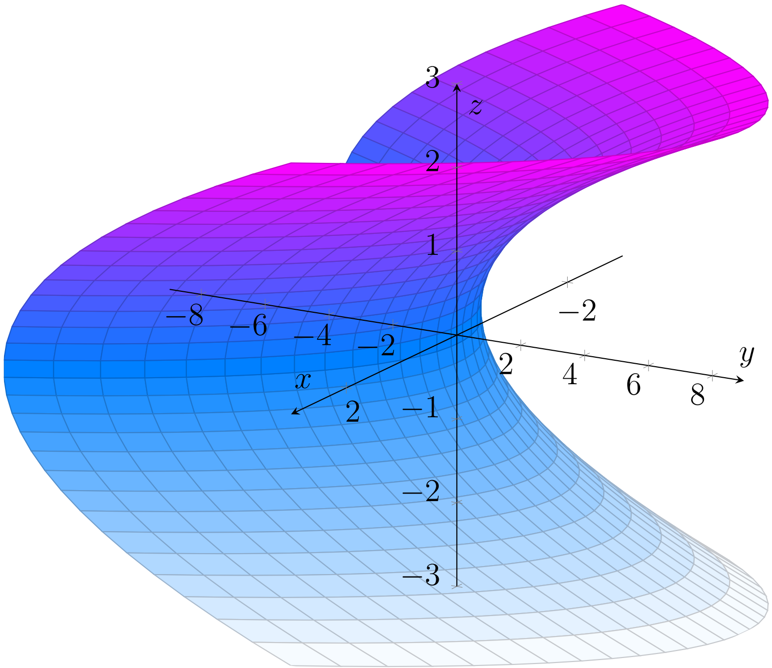
3. For $y=3$, $2y-z=5 \implies 6-z=5 \implies z=1$. For $z=1$, $x-1=z \implies x-1=1 \implies x=2$. Therefore, the point where the line and the plane intersect is $\boxed{(2,3,1)}$.

4.

(a)



(b)



5. Apply the Two-Path Test.

$$y = x \implies \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x^3}{x^2+x^4} = \lim_{x\to 0} \frac{x}{1+x^2} = \frac{0}{1} = 0$$

$$x = y^2 \implies \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

Since $0 \neq \frac{1}{2}$, by the Two-Path Test, the limit does not exist.

6.

(a) Apply the product rule.

$$\frac{\partial}{\partial T}\left[\left(P+\frac{A}{V^2}\right)(V-B)\right] = \frac{\partial}{\partial T}(kT)$$

$$\left(\frac{\partial P}{\partial T} - \frac{2A}{V^3}\cdot\frac{\partial V}{\partial T}\right)(V-B) + \left(P+\frac{A}{V^2}\right)\left(\frac{\partial V}{\partial T}\right) = k$$

$$(V-B)\frac{\partial P}{\partial T} + \frac{\partial V}{\partial T}\left(-\frac{2A}{V^2}+\frac{2AB}{V^3}\right) + \frac{\partial V}{\partial T}\left(P+\frac{A}{V^2}\right) = k$$

$$\boxed{\frac{\partial V}{\partial T} = \frac{k+(B-V)\dfrac{\partial P}{\partial T}}{P+\dfrac{2AB}{V^3}-\dfrac{A}{V^2}}}$$

(b) Apply the product rule again.

$$\frac{\partial}{\partial V}\left[\left(P+\frac{A}{V^2}\right)(V-B)\right]=\frac{\partial}{\partial V}(kT)$$

$$\left(\frac{\partial P}{\partial V}-\frac{2A}{V^3}\right)(V-B)+\left(P+\frac{A}{V^2}\right)\cdot 1=k\frac{\partial T}{\partial V}$$

$$(V-B)\frac{\partial P}{\partial V}-\frac{2A}{V^2}+\frac{2AB}{V^3}=k\frac{\partial T}{\partial V}-P-\frac{A}{V^2}$$

$$\boxed{\frac{\partial P}{\partial V}=\frac{k\frac{\partial T}{\partial V}-P+\frac{A}{V^2}-\frac{2AB}{V^3}}{V-B}}$$

7. $z=z(x,y)$, $x=x(t)$ and $y=y(t)$. Apply the chain rule.

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}=(2x-y)(-\sin t)+(2y-x)(\cos t)$$

$$=(2\cos t-\sin t)(-\sin t)+(2\sin t-\cos t)(\cos t)=\sin^2 t-\cos^2 t=-\cos 2t$$

The result is then

$$\left.\frac{dz}{dt}\right|_{t=0}=-\cos 0=\boxed{-1}$$

8. Let $f(x,y,z)=x^2+y^2-z=0$ and $g(x,y,z)=2x^2+2y^2+z^2=8$ be level surfaces. The tangent line is perpendicular to both ∇f and ∇g at P.

$$\nabla f=\langle 2x,2y,-1\rangle, \qquad \nabla g=\langle 4x,4y,2z\rangle$$
$$\nabla f(P)=\langle -2,2,-1\rangle, \qquad \nabla g(P)=\langle -4,4,4\rangle$$

$$\mathbf{T}=\nabla f\times\nabla g=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2&2&-1\\-4&4&4\end{vmatrix}=\mathbf{i}\begin{vmatrix}2&-1\\4&4\end{vmatrix}-\mathbf{j}\begin{vmatrix}-2&-1\\-4&4\end{vmatrix}+\mathbf{k}\begin{vmatrix}-2&2\\-4&4\end{vmatrix}$$

$$=(2\cdot 4-4\cdot(-1))\mathbf{i}-(-2\cdot 4-(-4)\cdot(-1))\mathbf{j}+(-2\cdot 4-2\cdot(-4))\mathbf{k}=12\mathbf{i}+12\mathbf{j}$$

The parametric equations for the tangent line is

$$\boxed{\left.\begin{array}{l}x=-1+12t\\y=1+12t\\z=2\end{array}\right\}\quad t\in\mathbb{R}}$$