MAT123 24.10.2025

QUESTIONS

Q1: Evaluate $\lim\limits_{x\to 0} \dfrac{\sin ax - \tan ax}{x^3}$.

Q2: Does $f(x) = 2x^{75} + 5x^{49} + 4x^6 + 1$ have a real solution on its domain? That is, is there any x_0 such that $f(x_0) = 0$?

Q3: Find the first derivative of $\tan(e^{2x} \cdot \sin(3x))$.

Q4: Using differentials, approximate $3\sqrt[3]{66} + 2\sqrt{66}$.

ANSWERS

Q1: If we substitute $x = 0$, we get $\dfrac{\sin(a \cdot 0) - \tan(a \cdot 0)}{0^3} = \dfrac{0}{0}$, which is an indeterminate form. We will manipulate this expression in a way such that the indeterminate form disappears. Multiply each side by $\cos ax$.

$$L = \lim_{x\to 0} \frac{\sin ax - \tan ax}{x^3} = \lim_{x\to 0} \frac{\sin ax - \frac{\sin ax}{\cos ax}}{x^3} = \lim_{x\to 0} \frac{\sin ax \cdot \cos ax - \sin ax}{x^3 \cdot \cos ax}$$

Factor $\sin ax$ in the numerator.

$$L = \lim_{x\to 0} \frac{\sin ax \cdot (\cos ax - 1)}{x^3 \cos ax}$$

Recall the **standard limit** $\lim\limits_{u\to 0} \dfrac{\sin u}{u} = 1$. We will rewrite the original limit so that we can use this standard limit. Multiply the limit by $\frac{a}{a}$.

$$L = \lim_{x\to 0} \frac{\sin ax \cdot (\cos ax - 1)}{x^3 \cos ax} \cdot \frac{a}{a}$$

By using the **product rule** and **constant multiple rule** for limits, we may rewrite this limit as

$$L = \underbrace{\lim_{x\to 0} \frac{\sin ax}{ax}}_{1} \cdot \lim_{x\to 0} \frac{a \cdot (\cos ax - 1)}{x^2 \cos ax} = a \cdot \lim_{x\to 0} \frac{\cos ax - 1}{x^2 \cos ax}$$

If we substitute $x = 0$, we get $a \cdot \dfrac{1 - \cos(a \cdot 0)}{0^2 \cdot \cos(a \cdot 0)} = \dfrac{0}{0}$, which is still indeterminate. We now use the trigonometric formula $\cos x = 1 - 2\sin^2(\frac{x}{2})$ for the numerator. Since we have $\cos ax$, then $\cos ax = 1 - 2\sin^2(\frac{ax}{2})$.

$$L = a \cdot \lim_{x\to 0} \frac{(1 - 2\sin^2(\frac{ax}{2})) - 1}{x^2 \cos ax} = a \cdot \lim_{x\to 0} \frac{-2\sin^2(\frac{ax}{2})}{x^2 \cos ax}$$

Rewrite the limit to obtain $\lim\limits_{u\to 0}\dfrac{\sin u}{u}=1$. Multiply and divide the numerator by $\frac{a^2}{4}$.

$$\begin{aligned} L &= a\cdot\lim_{x\to 0}\frac{-2\sin^2(\frac{ax}{2})}{x^2\cos ax}\cdot\frac{\frac{a^2}{4}}{\frac{a^2}{4}} = a\cdot\lim_{x\to 0}\frac{-\frac{1}{2}a^2\sin^2(\frac{ax}{2})}{\frac{1}{4}a^2x^2\cos ax} \\ &= -\frac{a^3}{2}\cdot\underbrace{\lim_{x\to 0}\frac{1}{\cos ax}}_{\frac{1}{\cos 0}=1}\cdot\lim_{x\to 0}\frac{\sin^2(\frac{ax}{2})}{\frac{a^2x^2}{4}} = -\frac{a^3}{2}\cdot\underbrace{\lim_{x\to 0}\frac{\sin(\frac{ax}{2})}{\frac{ax}{2}}}_{1}\cdot\underbrace{\lim_{x\to 0}\frac{\sin(\frac{ax}{2})}{\frac{ax}{2}}}_{1} = \boxed{-\frac{a^3}{2}} \end{aligned}$$

Remark: The value of the limit $\lim\limits_{x\to 0}\dfrac{\sin ax}{x}$ is a. Similarly, $\lim\limits_{x\to 0}\dfrac{\sin(ax)}{\sin(bx)}=\dfrac{a}{b}$. As a shortcut, you can evaluate such limits by taking the ratio of the coefficients.

Q2: At first glance, it seems to be impossible to find an x_0 that satisfies $f(x_0)=0$. If we were able to find *only* one root, then the question would already be answered. For this case, we will use the **IVT (Intermediate Value Theorem)**.

IVT states that if f is a continuous function on $[a,b]$, then f takes any value on $[f(a),f(b)]$. That is, $f(a)\le f(x)\le f(b)$ for $x\in[a,b]$. We will demonstrate why IVT is useful.

We now choose two arbitrary x values. Let $x_0=-1$ and $x_1=0$. The value of f at x_0 is $f(-1)=2(-1)^{75}+5(-1)^{49}+4(-1)^6+1=-2$. The value of f at x_1 is $f(0)=1$. f is continuous everywhere (i.e., continuous on $\mathbb{R}$) because it is a polynomial function. By IVT, f takes any value on $I=[f(-1),f(0)]$ for $x\in[-1,0]$. Notice that 0 is an element of I, from which we can infer that at some point $x_2\in[-1,0]$, the value of $f(x_2)$ becomes 0. Therefore, by IVT, there exists at least one point such that $f(x)$ is zero there. That is, we have a root on $[-1,0]$.

Until now, we cannot conclude that f has different roots. Using IVT, we were able to show that at least one root exists. Since the question asks for one root, then we're done.

Q3: Let $f(x)=\tan x$, $g(x)=e^{2x}\cdot\sin(3x)$, then we have the composite function $f(g(x))$. The **chain rule** states that $[f(g(x))]'=f'(g(x))\cdot g'(x)$. Recall the derivatives of the following functions.

$$\frac{d}{dx}(\tan x)=\sec^2 x,\quad \frac{d}{dx}(e^x)=e^x,\quad \frac{d}{dx}(\sin(x))=\cos x$$

Calculate f' and g'. For g', we will use the product rule. Notice that we have e^{2x} and $\sin(3x)$ inside g. By the chain rule, we have the following derivatives.

$$\frac{d}{dx}(e^{2x})=e^{2x}\cdot 2,\quad \frac{d}{dx}(\sin(3x))=\cos(3x)\cdot 3$$

$$f'(x)=\sec^2 x,\quad g'(x)=(e^{2x}\cdot 2)\cdot\sin(3x)+e^{2x}\cdot(3\cos(3x))$$

Then $[f(g(x))]'$ is

$$f'(g(x))\cdot g'(x)=\boxed{\sec^2(e^{2x}\cdot\sin(3x))\cdot[(2e^{2x})\cdot\sin(3x)+e^{2x}\cdot(3\cos(3x)]}$$

Q4: If we were to calculate the exact value without using a calculator, we would not be able to find it. If we approximate it as $3\sqrt[3]{64} + 2\sqrt{64}$, we obtain 28, but this is a bit off the result. To approximate this value to decimal digits, we may use an approximation method called **approximation using differentials**.

Let $y = f(x)$ be a continuous function. Suppose that we want to calculate $f(x + \Delta x)$ for sufficiently small Δx. By taking $\Delta x = dx$, we can approximate $f(x + \Delta x)$ as

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)dx,$$

where dy is called the **differential** of y. Since y is a function of x, we can rewrite dy as an expression of the independent variable, which is $f'(x)dx$.

First off, calculate $\sqrt[3]{66}$. To approximate this value, rewrite it as $\sqrt[3]{64 + 2}$. Notice that if we take $x = 64$ and $\Delta x = dx = 2$, we can easily approximate this value. Let $g(x) = \sqrt[3]{x}$, then

$$g(64 + 2) \approx g(64) + g'(64)\Delta x = \sqrt[3]{4^3} + g'(64) \cdot 2 = 4 + 2g'(64)$$

Now, compute $g'(x)$ to evaluate $g'(64)$. Using the **power rule**, we get

$$g'(x) = (\sqrt[3]{x})' = (x^{1/3})' = \frac{1}{3}x^{-2/3} \implies g'(64) = \frac{1}{3}(64)^{-2/3} = \frac{1}{3}(4^3)^{-2/3} = \frac{1}{48}$$

$$g(66) \approx 4 + 2 \cdot \frac{1}{48} = 4 + \frac{1}{24}$$

Do the same for $\sqrt{66}$. Let $h(x) = \sqrt{x}$. Then

$$h(64 + 2) \approx h(64) + h'(64)\Delta x = \sqrt{8^2} + h'(64) \cdot 2 = 8 + 2h'(64)$$

Now, compute $h'(x)$ to evaluate $h'(64)$.

$$h'(x) = (\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{-1/2} \implies h'(64) = \frac{1}{2}(64)^{-1/2} = \frac{1}{2}(8^2)^{-1/2} = \frac{1}{16}$$

$$h(66) \approx 8 + 2 \cdot \frac{1}{16} = 8 + \frac{1}{8}$$

Eventually, we can approximate $3\sqrt[3]{66} + 2\sqrt{66}$ as

$$3g(66) + 2h(66) \approx 3 \cdot \left(4 + \frac{1}{24}\right) + 2\left(8 + \frac{1}{8}\right) = 12 + \frac{1}{8} + 16 + \frac{1}{4} = \boxed{\frac{227}{8} = 28.375}$$

The calculator gives $3\sqrt[3]{66} + 2\sqrt{66} = 28.3717...$, which confirms our approximation.