2020-2021 Spring
MAT124 Midterm
(28/04/2022)

1. Consider the straight lines $x-1=\frac{z}{3}$, $y=0$ and $x=\frac{y+1}{2}=z$.

(a) Show that the lines are skew.

(b) Find an equation for the plane passing through the point $(1,3,5)$ which is parallel to the skew lines.

2. Sketch the graphs of the following surfaces in $\mathbb{R}^3$.

(a) $z^2=y^2+x^2$

(b) $z=y^3$

3. Let C be the space curve given by the vector-valued function

$$\mathbf{R}(t)=(1+2\sin(2\pi t))\mathbf{i}+(\ln t)\mathbf{j}+(\cos(\pi t))\mathbf{k}$$

Find an equation of the line which is tangent to C at $\mathbf{R}(1)$.

4. Use the $\epsilon-\delta$ definition and show that the function

$$f(x,y)=\begin{cases}\dfrac{4x^2-y^2}{2x+y}, & \text{if } (x,y)\neq(1,-2)\\ 4, & \text{if } (x,y)=(1,-2)\end{cases}$$

is continuous at the point $(1,-2)$.

5. Calculate $\dfrac{\partial w}{\partial u}$ if $w=f(x,y,z)=\dfrac{x+\sin y}{z}$ and

$$x=x(u,v)=\ln(u+v),\quad y=y(u,v)=\mathrm{e}^{uv},\quad z=z(u,v)=\frac{1}{u}.$$

6. The opposite and adjacent sides of a base and the height of a right triangular prism are measured, and measurements are known to have errors of at most 0.4 cm. If the opposite and adjacent sides of the base and the height are taken to be 5 cm, 3 cm, and 6 cm, respectively, find the bounds for the propagated error in the volume of the triangular prism.

7. Find the maximum and minimum values of $f(x,y,z)=x-z$ on the ellipsoid

$$x^2+2y^2+z^2=1.$$

8. Find the critical points of $f(x,y)=x^3-y^3+xy$ and classify each point as a relative maximum, a relative minimum, or a saddle point.

2020-2021 Spring Midterm (28/04/2021) Solutions
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1.

(a) Two lines are skew if they do not intersect and are not parallel.

Let L be the line $x - 1 = \dfrac{z}{3}$, $y = 0$ and M be the line $x = \dfrac{y+1}{2} = z$. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors that are parallel to these lines, respectively. Using the coefficients from the symmetric equations, we have $\mathbf{u} = \langle 1, 0, 3 \rangle$, $\mathbf{v} = \langle 1, 2, 1 \rangle$. If the cross product of these vectors is a non-zero vector, they are not parallel.

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \neq \mathbf{0}$$

Parametrize the lines and determine whether they intersect.

$$L = \begin{cases} x = 1 + t \\ y = 0 \\ z = 3t \end{cases} \quad t \in \mathbb{R} \qquad\qquad M = \begin{cases} x = s \\ y = 2s - 1 \\ z = s \end{cases} \quad s \in \mathbb{R}$$

Compare the y-components. If $2s - 1 = 0$, then $s = \dfrac{1}{2}$. If we substitute the value for s in the x- and z- components and compare, we obtain $1 + t = \dfrac{1}{2}$ and $3t = \dfrac{1}{2}$.

$$1 + t = \frac{1}{2} \implies t = \frac{1}{2}, \qquad 3t = \frac{1}{2} \implies t = \frac{3}{2}$$
$$\frac{1}{2} \neq \frac{3}{2}$$

This leads to a contradiction that the lines intersect. Since the lines do not intersect and are not parallel, the lines are skew.

(b) From (a), we have the normal vector $\mathbf{n}$ of the plane. The equation for the tangent plane with the normal vector $\mathbf{n}$ containing the point $P_0(x_0, y_0, z_0)$ is given by $n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$.

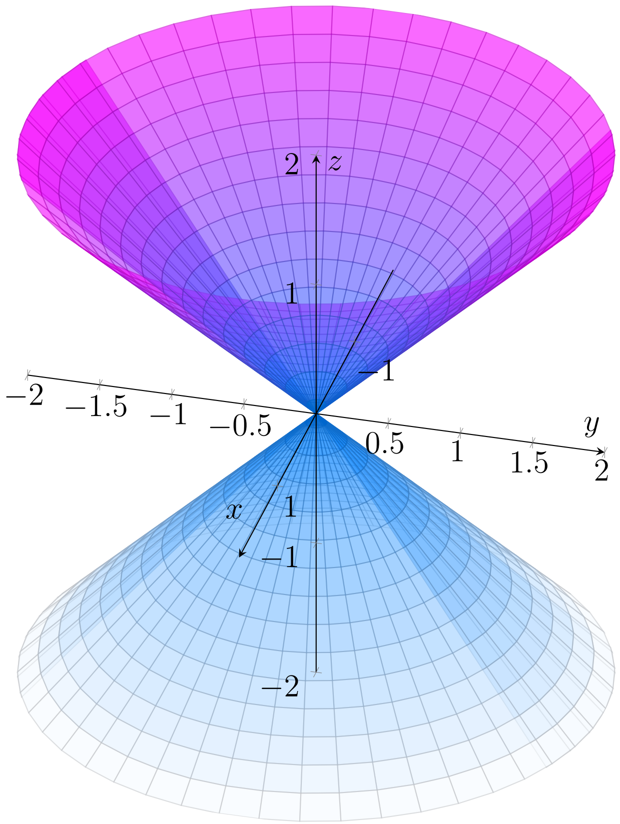
$$x_0 = 1, \quad y_0 = 3, \quad z = 5, \quad \mathbf{n} = -6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

The equation for the tangent plane is

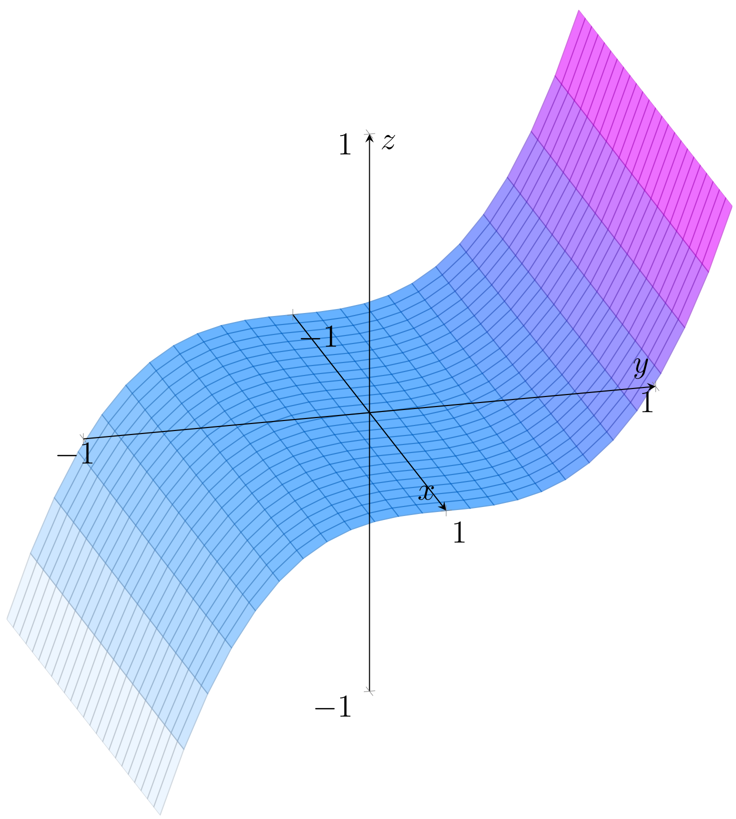
$$-6(x - 1) + 2(y - 3) + 2(z - 5) = 0 \implies \boxed{-3x + y + z = 5}$$

2.

(a)



(b)



3. Find $\mathbf{R}(1)$.

$$\mathbf{R}(1) = \mathbf{i} - \mathbf{k}$$

For $t = 1$, we have the point $(1, 0, -1)$ on the curve. The tangent vector of this curve is the first derivative of $\mathbf{R}$ with respect to the parametrization variable.

$$\mathbf{T} = \frac{d\mathbf{R}}{dt} = \left(\frac{d}{dt}\left(1 + 2\sin(2\pi t)\right)\right)\mathbf{i} + \left(\frac{d}{dt}\ln t\right)\mathbf{j} + \left(\frac{d}{dt}\cos(\pi t)\right)\mathbf{k}$$

$$= (4\pi\cos(2\pi t))\mathbf{i} + \left(\frac{1}{t}\right)\mathbf{j} + (-\pi\sin(\pi t))\mathbf{k}$$

At $t = 1$, the tangent vector is $\mathbf{T}(1) = (4\pi)\,\mathbf{i} + \mathbf{j}$. The parametric equations for a line that passes through the point $P_0(x_0, y_0, z_0)$ is given by

$$\left.\begin{array}{l} x = x_0 + T_1 u \\ y = y_0 + T_2 u \\ z = z_0 + T_3 u \end{array}\right\} \quad u \in \mathbb{R}$$

Therefore, the equation of the line that is tangent to C is

$$\boxed{\left.\begin{array}{l} x = 1 + 4\pi u \\ y = u \\ z = -1 \end{array}\right\} \quad u \in \mathbb{R}}$$

4. The value of the function at the point $(1, -2)$ is 4. Therefore, we will show that the limit is also 4. For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < \sqrt{(x-1)^2 + (y+2)^2} < \delta \implies |f(x) - L| < \epsilon$$

$$\left|\frac{4x^2 - y}{2x + y} - 4\right| = \left|\frac{(2x-y)(2x+y)}{2x+y} - 4\right| = |2x - y - 4| = |2(x-1) + (-y-2)|$$

$$\leq 2|x-1| + |-y-2| \quad (|a+b| \leq |a| + |b| \rightarrow \text{triangle inequality})$$

$$= 2|x-1| + |y+2|$$

Since $\sqrt{(x-1)^2 + (y+2)^2} < \delta \implies (x-1)^2 + (y+2)^2 < \delta^2$ and $(x-1)^2 \geq 0$ and $(y+2)^2 \geq 0$, we have $|x-1| \leq \delta$ and $|y+2| \leq \delta$.

$$2|x-1| + |y+2| \leq 2\delta + \delta = 3\delta$$

Let $\delta = \dfrac{\epsilon}{3}$.

$$\left|\frac{4x^2-y}{2x+y}-4\right| \leq 2|x-1|+|y+2| \leq 3\cdot\frac{\epsilon}{3}=\epsilon$$

Since the limit is equal to the value of the function at $(-1,2)$, f is continuous at $(-1,2)$.

5. Apply the chain rule.

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial x}=\frac{1}{z},\quad \frac{\partial w}{\partial y}=\frac{\cos y}{z},\quad \frac{\partial w}{\partial z}=-\frac{x+\sin y}{z^2}$$

$$\frac{\partial x}{\partial u}=\frac{1}{u+v},\quad \frac{\partial y}{\partial u}=v\mathrm{e}^{uv},\quad \frac{\partial z}{\partial u}=-\frac{1}{u^2}$$

$$\frac{\partial w}{\partial u}=\frac{1}{z}\cdot\frac{1}{u+v}+\frac{\cos y}{z}\cdot v\mathrm{e}^{uv}+\frac{x+\sin y}{z^2}\cdot\frac{1}{u^2}$$

$$=\boxed{\frac{u}{u+v}+\mathrm{e}^{uv}uv\cos\left(\mathrm{e}^{uv}\right)+\ln(u+v)+\sin\left(\mathrm{e}^{uv}\right)}$$

6. The volume of a right triangular prism with opposite and adjacent sides and height $a=5,\ b=3,\ h=6$ is given by

$$V(a,b,h)=\frac{1}{2}abh$$

For small errors, $\Delta V\approx dV$. The total differential of V is

$$dV=V_a da+V_b db+V_h dh$$

Compute $V_a,\ V_b,\ V_c$.

$$V_a=\frac{1}{2}bh=\frac{1}{2}\cdot 3\cdot 6=9,\quad V_b=\frac{1}{2}ah=\frac{1}{2}\cdot 5\cdot 6=15,\quad V_h=\frac{1}{2}ab=\frac{1}{2}\cdot 5\cdot 3=\frac{15}{2}$$

Given that $|da|\leq 0.4,\ |db|\leq 0.4,\ |dh|\leq 0.4$. Calculate the bounds for the propagated error.

$$|dV|\leq 9\cdot 0.4+15\cdot 0.4+\frac{15}{2}\cdot 0.4=\boxed{\frac{63}{5}}$$

7. Let $g(x,y,z)=x^2+2y^2+z^2-1$ be the constraint. Solve the system of equations below.

$$\left.\begin{aligned}\nabla f&=\lambda\nabla g\\ g(x,y,z)&=0\end{aligned}\right\}\qquad \begin{aligned}&\nabla f=\langle 1,0,-1\rangle=\lambda\langle 2x,4y,2z\rangle=\lambda\nabla g\\ &x=\frac{1}{2\lambda},\quad y=0\ \text{ or }\ \lambda=0,\quad z=-\frac{1}{2\lambda}\end{aligned}$$

$$\lambda = 0 \implies x = y = z = 0 \implies f(0,0,0) = 0$$

$$x = \frac{1}{2\lambda}, \quad y = 0, \quad z = -\frac{1}{2\lambda} \implies g\left(\frac{1}{2\lambda}, 0, -\frac{1}{2\lambda}\right) = \left(\frac{1}{2\lambda}\right)^2 + 2 \cdot 0^2 + \left(-\frac{1}{2\lambda}\right)^2 = 1$$

$$\frac{1}{2\lambda^2} = 1 \implies \lambda = \pm\frac{1}{\sqrt{2}}$$

$$\lambda = \frac{1}{\sqrt{2}} \implies x = \frac{1}{\sqrt{2}},\ y = 0,\ z = -\frac{1}{\sqrt{2}} \qquad \lambda = -\frac{1}{\sqrt{2}} \implies x = -\frac{1}{\sqrt{2}},\ y = 0,\ z = \frac{1}{\sqrt{2}}$$

$$f\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \qquad f\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

Compare the values $f(0,0,0)$, $f\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, $f\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$.

$$\boxed{\text{The minimum value is } -\sqrt{2}, \text{ the maximum value is } \sqrt{2}.}$$

8. To identify the critical points, find where both $f_x = f_y = 0$ or one of the partial derivatives does not exist.

$$f_x = 3x^2 + y, \quad f_y = -3y^2 + x$$

$$\left.\begin{array}{r} f_x = 0 \implies -y = 3x^2 \\ f_y = 0 \implies 3y^2 = x \end{array}\right\} -y = 27y^4 \implies y(27y^3 + 1) = 0 \implies y_1 = 0, \quad y_2 = -\frac{1}{3}$$

$$y_1 = 0 \implies x_1 = 0, \quad y_2 = -\frac{1}{3} \implies x_2 = \frac{1}{3}$$

The critical points are $(0,0)$ and $\left(\frac{1}{3}, -\frac{1}{3}\right)$. To classify these points, calculate the second partial derivatives and then find the Hessian determinants.

$$f_{xx} = 6x, \quad f_{xy} = f_{yx} = 1, \quad f_{yy} = -6y$$

$$(0,0) \to \begin{cases} f_{xx} = 0, \quad f_{xy} = 1, \quad f_{yy} = 0 \\ \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 1 = -1 < 0 \end{cases}$$

$$\left(\frac{1}{3}, -\frac{1}{3}\right) \to \begin{cases} f_{xx} = 2, \quad f_{xy} = 1, \quad f_{yy} = 2 \\ \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 1 = 3 > 0, \quad f_{xx} > 0 \end{cases}$$

$$\boxed{\text{A local minimum occurs at } \left(\frac{1}{3}, -\frac{1}{3}\right) \text{ and a saddle point occurs at } (0,0).}$$