2024-2025 Fall
MAT123 Midterm
(02/12/2024)

1) Let

$$f(x) = \begin{cases} \dfrac{\tan ax}{\tan bx}, & \text{if } x < 0 \\ 4, & \text{if } x = 0 \\ ax + b, & \text{if } x > 0 \end{cases}$$

Determine the values of $a$ and $b$ such that $f$ is continuous at the point $x = 0$.

2. Use differential to approximate $3\sqrt[3]{66} + 2\sqrt{66}$.

3. (a) Without using L'Hôpital's rule, evaluate $\lim\limits_{x\to 0} \dfrac{5 - 6\cos x + \cos^2 x}{x \sin x}$.

(b) Prove that $\lim\limits_{x\to -3} \sqrt{-x-2} = 1$ by using the formal definition of limit.

(c) Evaluate $\lim\limits_{x\to 1^+} (\sqrt{x})^{\ln(x-1)}$.

4. Coffee is draining out of a conical filter at a rate of 2.25 in.$^3$/min. If the cone is 5 in. tall and has a radius of 2 in., how fast is the coffee level dropping when the coffee is 3 in. deep?

5. Using the mean value theorem, show that $\ln(x+1) < x$ for $x > 0$.

6. Let $f(x) = \dfrac{x^2 - 2}{(x-1)^2}$

(a) Determine the interval of increase, decrease and concavity of $f$.
(b) Construct a table.
(c) Sketch the graph of $f$.

2024-2025 Fall Midterm (02/12/2024) Solutions
(Last update: 8/21/25 (21st of August) 4:34 PM)

1) To ensure continuity at $x = 0$, the one-sided limit values must be equal to the value of the function at that point.

$$\lim_{x\to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x\to 0^+} (ax+b) = f(0) = 4$$

The easy part is that we can calculate the limit from the right.

$$\lim_{x\to 0^+} (ax+b) = 0 + b = b$$

Hence, $b = 4$. To calculate from the left, we need another technique.

$$\lim_{x\to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x\to 0^-} \left( \frac{\sin ax}{\cos ax} \cdot \frac{\cos bx}{\sin bx} \cdot \frac{bx}{bx} \cdot \frac{ax}{ax} \right)$$

$$= \lim_{x\to 0^-} \left( \frac{\sin ax}{ax} \right) \cdot \lim_{x\to 0^-} \left( \frac{1}{\frac{\sin bx}{bx}} \right) \cdot \lim_{x\to 0^-} \left( \frac{\cos(bx)\cdot ax}{\cos(ax)\cdot bx} \right)$$

$$= 1 \cdot \frac{1}{\lim\limits_{x\to 0^-} \frac{\sin bx}{bx}} \cdot \lim_{x\to 0^-} \left( \frac{\cos(bx)\cdot a}{\cos(ax)\cdot b} \right) = 1 \cdot 1 \cdot \left( \frac{\cos(0)\cdot a}{\cos(0)\cdot b} \right)$$

$$= \frac{a}{b}$$

Now, set $\frac{a}{b} = b \to a = 16$. $\boxed{a = 16,\ b = 4}$

2) Let $f(x) = x^{1/3}$ and $g(x) = x^{1/2}$. Using the differential approximation, we get

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x = x^{1/3} + \frac{1}{3}x^{-2/3}\Delta x$$
$$g(x+\Delta x) \approx g(x) + g'(x)\Delta x = x^{1/2} + \frac{1}{2}x^{-1/2}\Delta x$$

Set $x = 64$ and $\Delta x = 2$.

$$3\sqrt[3]{66} + 2\sqrt{66} \approx 3\left(64^{1/3} + \frac{1}{3}\cdot 64^{-2/3}\cdot 2\right) + 2\left(64^{1/2} + \frac{1}{2}\cdot 64^{-1/2}\cdot 2\right)$$
$$= 3\left(4 + \frac{1}{24}\right) + 2\left(8 + \frac{1}{8}\right) = \boxed{28.375}$$

3)

(a)

$$\lim_{x\to 0}\frac{5-6\cos x+\cos^2 x}{x\sin x}=\lim_{x\to 0}\frac{(\cos x-1)(\cos x-5)}{x\sin x}$$

$$=\lim_{x\to 0}\frac{(\cos x-1)(\cos x-5)(\cos x+1)}{(x\sin x)(\cos x+1)}=\lim_{x\to 0}\left(-\frac{\sin^2 x\cdot(\cos x-5)}{x\sin x\cdot(\cos x+1)}\right)$$

$$=\lim_{x\to 0}\left(-\frac{\sin x\cdot(\cos x-5)}{x\cdot(\cos x+1)}\right)=-\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\cos x-5}{\cos x+1}=-1\cdot\frac{\cos 0-5}{\cos 0+1}=\boxed{2}$$

(b) For all $\epsilon>0$, there exists a $\delta>0$ such that

$$0<|x+3|<\delta \implies |f(x)-1|<\epsilon$$

$$|f(x)-1|=\left|\sqrt{-x-2}-1\right|=\left|\sqrt{-x-2}-1\cdot\frac{\sqrt{-x-2}+1}{\sqrt{-x-2}+1}\right|$$

$$=\left|\frac{-x-3}{\sqrt{-x-2}+1}\right|\leq|-x-3|\quad\left[\sqrt{-x-2}+1\geq 0+1=1\right]$$

$$=|x+3|<\delta=\epsilon$$

If we set $\delta=\epsilon$, the proof is complete.

(c) Let $L$ be the value of the limit. Then, take the logarithm of both sides. Since the expression is continuous for $x>1$, we can take the logarithm function inside the limit.

$$L=\lim_{x\to 1^+}(\sqrt{x})^{\ln(x-1)}\quad\left[1^{-\infty}\right]$$
$$\ln(L)=\ln\left[\lim_{x\to 1^+}(\sqrt{x})^{\ln(x-1)}\right]$$

$$\ln(L) = \lim_{x\to 1^+} \ln\left[(\sqrt{x})^{\ln(x-1)}\right] = \lim_{x\to 1^+} \left[\ln(x-1)\cdot\ln(\sqrt{x})\right] \quad [\infty\cdot 0]$$

$$= \lim_{x\to 1^+} \left[\frac{\ln(x-1)}{\frac{1}{\ln\left(\sqrt{x}\right)}}\right] \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 1^+} \left[\frac{\frac{1}{x-1}}{\frac{1}{-\ln^2\left(\sqrt{x}\right)}\cdot\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}\right] = \lim_{x\to 1^+} \left[-\frac{\ln^2(\sqrt{x})\cdot 2x}{x-1}\right] \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 1^+} \left(-\frac{2\ln\left(\sqrt{x}\right)\cdot\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\cdot 2x + \ln^2(\sqrt{x})\cdot 2}{1}\right) = \lim_{x\to 1^+} \left[2\ln\left(\sqrt{x}\right) + 2\ln^2\left(\sqrt{x}\right)\right]$$

$$= \left[2\ln\left(\sqrt{1}\right) + 2\ln^2\left(\sqrt{1}\right)\right] = 0$$

$\ln(L) = 0$. Therefore, $\boxed{L = 1}$.

4) Let $f(x)$ represent the volume of coffee in the cone in cubic inches. The coffee in the cone will have a conical shape while draining. We may set up the equation below using the formula of the volume of a cone.

$$f(t) = \frac{1}{3}\cdot h(t)\cdot \pi r^2(t)$$

$h(t), r(t)$ represent the height and radius of the circular area that coffee forms, respectively, in inches. We can eliminate $r$ to proceed with $h$. $r$ and $h$ are proportional.

$$\frac{r}{h} = \frac{2}{5} \to r = \frac{2h}{5}$$

$$f(t) = \frac{4\pi h^3(t)}{75}$$

Take the derivative of both sides.

$$f'(t) = \frac{4\pi}{25}\cdot h^2(t)\cdot h'(t)$$

Given that at $t = t_0$, $f'(t_0) = -2.25$, $h(t_0) = 3$. We may now find $h'(t_0)$. Solve for $h'(t_0)$.

$$h'(t_0) = \frac{25f'(t_0)}{4\pi h^2(t_0)} = \frac{25\cdot(-2.25)}{4\pi\cdot(3)^2} = \boxed{-\frac{1.5625}{\pi}\text{ inches/minute}}$$

5) Let $f(x) = \ln(1+x) - x$. We have $f(0) = \ln(1+0) - 0 = 0$. The mean value theorem (MVT) states that if a function $g(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists a point $c$ such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$

$f$ is continuous on $[0, x]$ and differentiable on $(0, x)$. By MVT, $\dfrac{f(x) - f(0)}{x - 0} = f'(c)$ provided for some point $c$ such that $0 < c < x$.

$$f'(c) = \frac{1}{c+1} - 1 = \frac{\ln(x+1) - x}{x} = \frac{f(x) - f(0)}{x - 0}$$

$$\frac{1}{c+1} = \frac{\ln(x+1)}{x} \to c + 1 = \frac{x}{\ln(x+1)}$$

$$c = \frac{x - \ln(x+1)}{\ln(x+1)}$$

From the inequality $0 < c < x$,

$$0 < \frac{x - \ln(x+1)}{\ln(x+1)}$$

$$0 < x - \ln(x+1)$$

$$\ln(x+1) < x$$

6)

(a) First off, find the domain. The expression is undefined when the denominator is zero. Therefore, $(x-1)^2 \neq 0 \to x \neq 1$. The only vertical asymptote occurs at $x = 1$.

$$\mathcal{D} = \mathbb{R} - \{1\}$$

Let us find the limit at infinity.

$$\lim_{x\to\infty} \frac{x^2 - 2}{(x-1)^2} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2x}{2(x-1)} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2}{2} = 1$$

Similarly,

$$\lim_{x\to-\infty} \frac{x^2 - 2}{(x-1)^2} = 1$$

The horizontal asymptote occurs only at $y = 0$.

Take the first derivative by applying the quotient rule.

$$y' = \frac{(2x) \cdot (x-1)^2 - (x^2 - 2) \cdot 2(x-1)}{(x-1)^4} = \frac{4 - 2x}{(x-1)^3}$$

$y'$ is undefined for $x = 1$, and $y' = 0$ for $x = 2$. Since 1 is not in the domain, the *only* critical point is $x = 2$.

Take the second derivative.

$$y'' = \frac{(-2)\cdot(x-1)^3 - (4-2x)\cdot 3(x-1)^2}{(x-1)^6} = \frac{4x-10}{(x-1)^4}$$

The only inflection point occurs at $x = \dfrac{5}{2}$.

(b) Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f\left(-\sqrt{2}\right) = f\left(\sqrt{2}\right) = 0,\ f(0) = -2,\ f(2) = 2,\ f(5/2) = 17/9$$

| $x$ | $\left(-\infty, -\sqrt{2}\right)$ | $\left(-\sqrt{2}, 0\right)$ | $(0,1)$ | $\left(1, \sqrt{2}\right)$ | $\left(\sqrt{2}, 2\right)$ | $\left(2, \frac{5}{2}\right)$ | $\left(\frac{5}{2}, \infty\right)$ |
|---|---|---|---|---|---|---|---|
| $y$ | $(1,0)$ | $(-2,0)$ | $(-\infty,-2)$ | $(-\infty,0)$ | $(0,2)$ | $\left(2, \frac{17}{9}\right)$ | $\left(\frac{17}{9}, 1\right)$ |
| $y'$ sign | - | - | - | $+$ | $+$ | - | - |
| $y''$ sign | - | - | - | - | - | - | $+$ |

(c)