2012-2013 Fall
MAT123-[Instructor02]-02, [Instructor05]-05 Midterm II
(19/12/2012)
Time: 09:00 - 11:00
Duration: 120 minutes

1. A 10 m long wire is cut into two pieces. One piece is bent into an equilateral triangle and the other piece is bent into a circle. If the sum of the areas enclosed by each part is minimum, what is the length of each piece?

2. Integrate the following functions, and write each step in detail.

(a) $\displaystyle\int \frac{dy}{\sqrt{y}\left(1+\sqrt{y}\right)^2}$ (b) $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\cot x}{\ln(\sin x)}\,dx$ (c) $\displaystyle\int \frac{x^3-4x-10}{x^2-x-6}\,dx$

(d) $\displaystyle\int \tan x \sec^{123} x\,dx$ (e) $\displaystyle\int \sqrt{t-t^2}\,dt$ (f) $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx$

3. Integrate the following functions, and write each step in detail.

(a) $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$ (b) $\displaystyle\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-x^3}\,dx$

4. By using the Washer Method, find the volume of the solid generated by revolving the region bounded by the curves $x=y^3$ and $y+x^2=0$ about the x-axis.

Solutions (Last update: 7/24/25 (24th of July) 2:33 AM)

1. Let x be the length of the wire that is used to form the equilateral triangle. So, $10 - x$ is the length of the other piece. The area of the triangle is

$$A_1 = \frac{x^2\sqrt{3}}{4}$$

The perimeter of the circle is $10 - x$. Therefore, $2\pi r = 10 - x$, where r is the radius of the circle. The area of the circle can be extracted from the formula $A_2 = \pi r^2$. Solve the former equation for r, and express the area in terms of x.

$$r = \frac{10 - x}{2\pi} \rightarrow A_2 = \pi\left(\frac{10 - x}{2\pi}\right)^2 = \frac{100 - 20x + x^2}{4\pi}$$

Let $A(x)$ be the function of length representing the sum of the areas.

$$A(x) = \frac{x^2\sqrt{3}}{4} + \frac{100 - 20x + x^2}{4\pi}$$

Minimize $A(x)$ by taking the first derivative and setting it to 0.

$$A'(x) = \frac{x\sqrt{3}}{2} + \frac{x - 10}{2\pi} = 0 \rightarrow 10 = x + x \cdot \pi\sqrt{3} \rightarrow x = \frac{10}{1 + \pi\sqrt{3}}$$

The length of the piece used to form the triangle is $\dfrac{10}{1 + \pi\sqrt{3}}$, the length of the other piece is $\dfrac{10\pi\sqrt{3}}{1 + \pi\sqrt{3}}$.

2.

(a) Let $u = 1 + \sqrt{y}$, then $du = \dfrac{dy}{2\sqrt{y}}$.

$$\int \frac{dy}{\sqrt{y}\left(1 + \sqrt{y}\right)^2} = \int \frac{1}{u^2} \cdot 2\, du = -\frac{2}{u} + c = \boxed{-\frac{2}{1 + \sqrt{y}} + c,\ c \in \mathbb{R}}$$

(b) Let $u = \ln(\sin x)$, then $du = \dfrac{1}{\sin x} \cdot \cos x\, dx$.

$$\int_{\pi/6}^{\pi/4} \frac{\cot x}{\ln(\sin x)}\, dx = \int \frac{1}{u}\, du = \ln|u| + c = \Big[\ln|\ln(\sin x)|\Big]_{\pi/6}^{\pi/4}$$

$$= \ln\left|\ln\frac{\sqrt{2}}{2}\right| - \ln\left|\ln\frac{1}{2}\right| = \ln\frac{\left|\ln\frac{\sqrt{2}}{2}\right|}{\left|\ln\frac{1}{2}\right|}$$

$$= \ln\left(\frac{\ln 2 - \ln\sqrt{2}}{\ln 2 - \ln 1}\right) = \ln\frac{1}{2} = \boxed{-\ln 2}$$

(c) Attempt a long polynomial division and split into two integrals.

$$\mathrm{I} = \int \frac{x^3 - 4x - 10}{x^2 - x - 6}\, dx = \int (x+1)\, dx + \int \frac{3x-4}{(x-3)(x+2)}\, dx$$

$$= \frac{x^2}{2} + x + \int \left(\frac{A}{x-3} + \frac{B}{x+2} \right) dx$$

$$A(x+2) + B(x-3) = 3x - 4$$
$$x(A+B) + 2A - 3B = 3x - 4$$

$$\left.\begin{array}{l} A + B = 3 \\ 2A - 3B = -4 \end{array}\right\} \quad A = 1, \quad B = 2$$

$$\mathrm{I} = \frac{x^2}{2} + x + \int \left(\frac{1}{x-3} + \frac{2}{x+2} \right) dx = \boxed{\frac{x^2}{2} + x + \ln|x-3| + 2\ln|x+2| + c,\ c \in \mathbb{R}}$$

(d) Let $u = \sec x$, then $du = \tan x \sec x\, dx$.

$$\int \tan x \sec^{123} x\, dx = \int u^{122}\, du = \frac{u^{123}}{123} + c = \boxed{\frac{\sec^{123} x}{123} + c,\ c \in \mathbb{R}}$$

(e) Let $t = \sin^2 x$, then $dt = 2\sin x \cos x\, dx$.

$$\int \sqrt{t - t^2}\, dt = \int \sqrt{\sin^2 x - \sin^4 x} \cdot 2\sin x \cos x\, dx = \int \sqrt{\sin^2 x \cdot (1 - \sin^2 x)} \cdot \sin(2x)\, dx$$

$$= \int \sin x \cos x \cdot \sin(2x)\, dx = \frac{1}{2} \int \sin^2(2x)\, dx = \frac{1}{2} \int \left(1 - \cos^2(2x)\right) dx$$

$$= \frac{1}{2} \int \frac{1 - \cos(4x)}{2}\, dx = \frac{1}{4} \left[x - \frac{1}{4} \sin(4x) \right] + c,\ c \in \mathbb{R}$$

We will try to rewrite the result in terms of t.

$$t - t^2 \geq 0 \implies 0 \leq t \leq 1$$
$$t = \sin^2 x \implies \sqrt{t} = \sin x \implies \arcsin\left(\sqrt{t}\right) = x$$
$$t = \sin^2 x = 1 - \cos^2 x \implies \cos x = \sqrt{1-t}$$
$$\sin(4x) = 2\sin(2x)\cos(2x) = 4\sin x \cos x \left(2\cos^2 x - 1\right) = 4\sqrt{t} \cdot \sqrt{1-t} \cdot (1-2t)$$

$$\int \sqrt{t - t^2}\, dt = \boxed{\frac{1}{4}\left[\arcsin\left(\sqrt{t}\right) - \sqrt{t(1-t)} \cdot (1-2t)\right] + c,\ c \in \mathbb{R}}$$

(f) Let $u = \sqrt{1+\sqrt{x}}$.

$$u^2 = 1+\sqrt{x} \implies u^2 - 1 = \sqrt{x} \implies \left(u^2-1\right)^2 = x \implies 2\left(u^2-1\right)\cdot 2u\,du = dx$$

$$\int \sqrt{1+\sqrt{x}}\,dx = \int u\cdot(2u^2-2)\cdot 2u\,du = \int \left(4u^4-4u^2\right)\,du = \frac{4u^5}{5} - \frac{4u^3}{3} + c$$

$$= \boxed{\frac{4\left(\sqrt{1+\sqrt{x}}\right)^5}{5} - \frac{4\left(\sqrt{1+\sqrt{x}}\right)^3}{3} + c,\ c\in\mathbb{R}}$$

3.

(a)

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{S\to 1^-}\int_0^S \frac{dx}{\sqrt{1-x}} + \lim_{P\to 1^+}\int_P^2 \frac{dx}{\sqrt{x-1}}$$

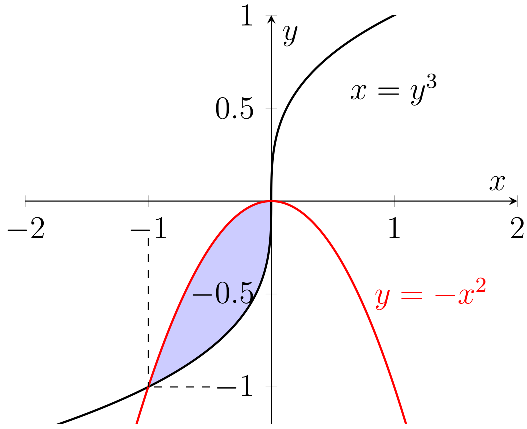
$$= \lim_{S\to 1^-}\left[-2\sqrt{1-x}\right]_0^S + \lim_{P\to 1^+}\left[2\sqrt{x-1}\right]_P^2$$

$$= \lim_{S\to 1^-}\left[-2\sqrt{1-S}\right] + 2\sqrt{1-0} + 2\sqrt{2-1} - \lim_{P\to 1^+}\left[2\sqrt{P-1}\right] = \boxed{4}$$

(b)

$$\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-x^3}\,dx = \lim_{a\to\infty}\int_{-a}^{a} x^2\mathrm{e}^{-x^3}\,dx = \lim_{a\to\infty} -\frac{1}{3}\left[\mathrm{e}^{-x^3}\right]_{-a}^{a} = \lim_{a\to\infty} -\frac{1}{3}\left[\mathrm{e}^{-a^3} - \mathrm{e}^{a^3}\right] = \boxed{\infty}$$

4)



$$V = \int_{-1}^{0} \pi\left[\left(x^{1/3}\right)^2 - \left(x^2\right)^2\right]\,dx$$

$$= \pi\int_{-1}^{0}\left(x^{2/3} - x^4\right)\,dx$$

$$= \pi\left[\frac{3}{5}x^{5/3} - \frac{x^5}{5}\right]_{-1}^{0}$$

$$= \pi\cdot\left[0 - \left(-\frac{3}{5}+\frac{1}{5}\right)\right] = \boxed{\frac{2\pi}{5}}$$