2023-2024 Spring
MAT124 Final
(04/06/2024)

1. Using Lagrange multipliers, find the closest point of the plane $x + z + 1 = 0$ to the point $(1, 2, 0)$.

2. Sketch the region and reverse the order of the double integral

$$\int_0^1 \int_0^{x^2/2} dy\, dx + \int_1^{\sqrt{2}} \int_{x^2-1}^{x^2/2} dy\, dx$$

3. Sketch the region and use a double integral in polar coordinates to find the area inside the cardioid $r = 1 - \cos\theta$ outside the circle $r = 1$.

4. Sketch the region and use a double integral to find the volume of the solid bounded above by the plane $x = z$ and below in the xy-plane by the part of the disk $x^2 + y^2 \leq 4$ in the fourth quadrant.

5. Find the surface area of the portion of the paraboloid $z = 25 - x^2 - y^2$ that lies above the xy-plane.

6. Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral

$$\iint_{\mathcal{R}} (x - y) \sin\left(x^2 - y^2\right)\, dy\, dx$$

where $\mathcal{R}$ is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$.

7. Let R be the solid region bounded by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 9$. Let

$$\mathrm{I} = \iiint_R \left(x^2 + y^2\right)\, dV$$

(i) Express (but do not evaluate) I as a triple integral in spherical coordinates.

(ii) Express (but do not evaluate) I as a triple integral in cylindrical coordinates.

2023-2024 Spring Final (04/06/2024) Solutions
(Last update: 8/7/25 (7th of August) 2:29 PM)

1. Let $g(x, y, z) = x + z + 1$ and $f(x, y, z) = D^2 = (x-1)^2 + (y-2)^2 + (z-0)^2$. It is easier to work with the square of the distance. Finding the points will not change the result. Solve the system of equations below.

$$\left.\begin{array}{l}\nabla f = \lambda \nabla g \\ g(x,y,z) = 0\end{array}\right\} \qquad \begin{array}{l}\nabla f = \langle 2(x-1), 2(y-2), 2z\rangle = \lambda\,\langle 1,0,1\rangle = \lambda \nabla g \\ \therefore y = 2, \quad \lambda = 2x-2, \quad \lambda = 2z\end{array}$$

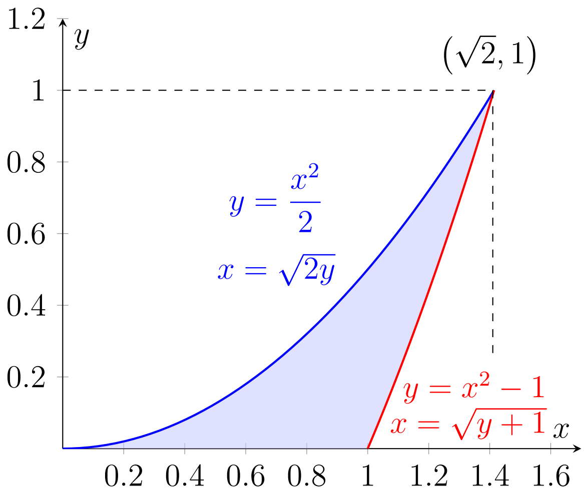
$$2x - 2 = 2z \implies z = x - 1 \tag{1}$$

Substitute (1) into the constraint.

$$x + z + 1 = 0 \implies x + x - 1 + 1 = 0 \implies 2x = 0 \implies x = 0$$
$$\therefore 0 + z + 1 = 0 \implies z = -1$$

The closest point is then $\boxed{(0, 2, -1)}$.

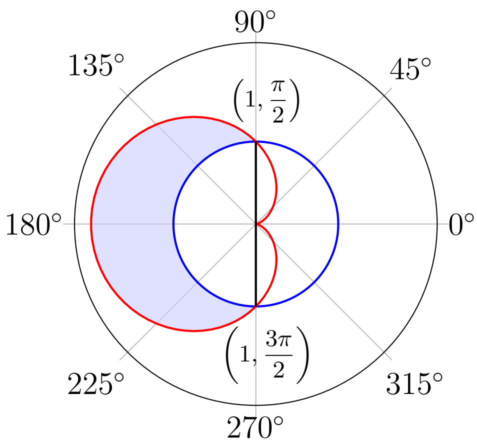
2.



$$\boxed{\int_0^1 \int_{\sqrt{2y}}^{\sqrt{y+1}} dx\, dy}$$

3. Find where these two curves intersect and then find the area.

$$1 = 1 - \cos\theta \implies \cos\theta = 0 \implies \theta = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}$$

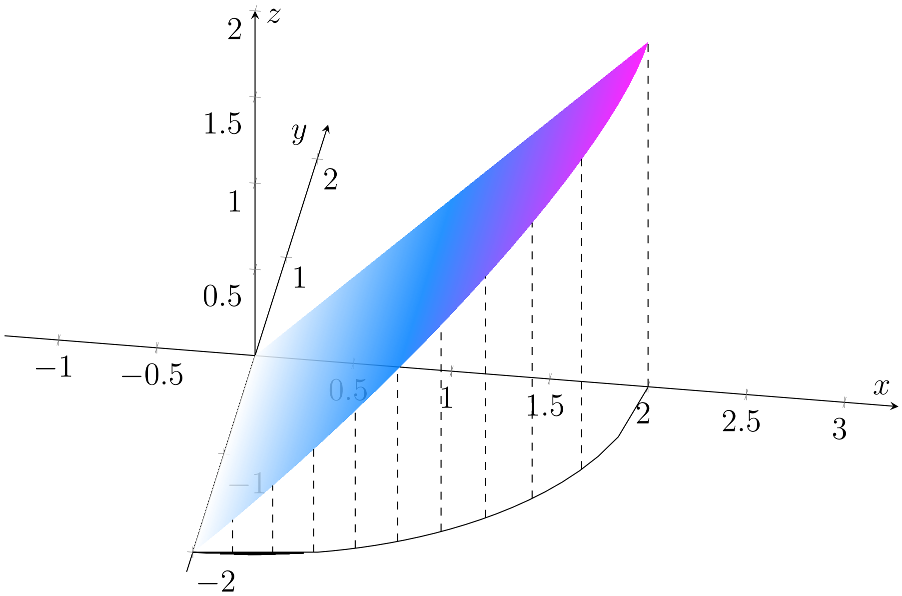


$$\text{Area} = \int_{\pi/2}^{3\pi/2} \int_{1}^{1-\cos\theta} r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2} \left[\frac{1}{2}r^2\right]_{r=1}^{r=1-\cos\theta} d\theta = \frac{1}{2}\int_{\pi/2}^{3\pi/2} \left[(1-\cos\theta)^2 - 1^2\right]\,d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{3\pi/2} \left(-2\cos\theta + \cos^2\theta\right)\,d\theta = \frac{1}{2}\int_{\pi/2}^{3\pi/2} \left(-2\cos\theta + \frac{\cos(2\theta)+1}{2}\right)\,d\theta$$

$$= \frac{1}{2}\left[-2\sin\theta + \frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right]_{\pi/2}^{3\pi/2} = \frac{1}{2}\left[\left(2+0+\frac{3\pi}{4}\right) - \left(-2+0+\frac{\pi}{4}\right)\right] = \boxed{2+\frac{\pi}{4}}$$

4.



$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2+y^2 &= r^2 \\ dA &= r\,dr\,d\theta \end{aligned} \quad \rightarrow \quad \begin{gathered} z = x \implies z_{\text{upper}} = r\cos\theta \\ z = 0 \implies z_{\text{lower}} = 0 \\ 0 \le \theta \le \frac{\pi}{2} \end{gathered}$$

The volume of this solid can be evaluated with the following integral.

$$\text{Volume} = \int_0^{\pi/2}\int_0^2 (r\cos\theta - 0)\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_{r=0}^{r=2}\cos\theta\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2}\cos\theta\,d\theta = \frac{8}{3}\sin\theta\Big|_0^{\pi/2} = \frac{8}{3}(1-0) = \boxed{\frac{8}{3}}$$

5. For $z = 0$, we get the circle $x^2 + y^2 = 25$. Therefore, the domain is $x^2 + y^2 \leq 25$. Using the double integral below, we find the surface area.

$$\text{Surface area} = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{1 + (-2x)^2 + (-2y)^2}\, dy\, dx$$

$$= \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

If we switch to polar coordinates, we can easily evaluate the integral.

$$\text{Surface area} = \int_0^{2\pi} \int_0^5 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{12}\left(1 + 4r^2\right)^{3/2}\right]_{r=0}^{r=5} d\theta$$

$$= \frac{1}{12} \int_0^{2\pi} \left[(1 + 4 \cdot 25)^{3/2} - (1 + 4 \cdot 0)^{3/2}\right] d\theta = \frac{101^{3/2} - 1}{12} \int_0^{2\pi} d\theta$$

$$= \boxed{\frac{101^{3/2} - 1}{6} \cdot \pi}$$

6. Rewrite x and y in terms of u and v and sketch the regions in both coordinates.

$$\left.\begin{array}{c} u = x - y \\ v = x + y \end{array}\right\} \quad x = \frac{u + v}{2}, \quad y = \frac{v - u}{2}$$

$$x + y = 3 \implies \frac{u + v}{2} + \frac{v - u}{2} = 3 \implies v = 3$$

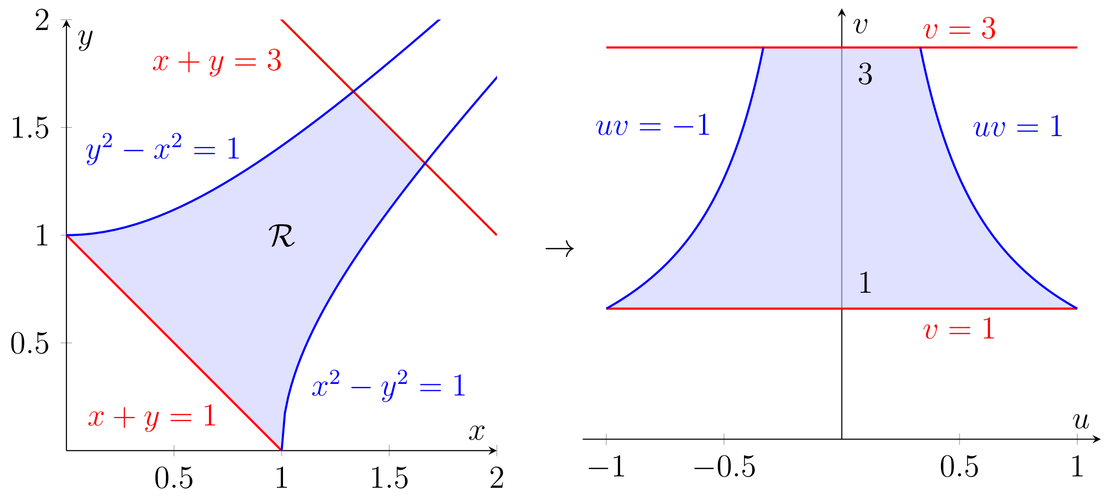
$$x + y = 3 \implies \frac{u + v}{2} + \frac{v - u}{2} = 1 \implies v = 1$$

$$x^2 - y^2 = 1 \implies \left(\frac{u + v}{2}\right)^2 - \left(\frac{v - u}{2}\right)^2 = \frac{1}{4}\left(u^2 + 2uv + v^2 - v^2 + 2uv - u^2\right) = 1$$

$$\implies u = \frac{1}{v}$$

$$y^2 - x^2 = 1 \implies \left(\frac{v - u}{2}\right)^2 - \left(\frac{u + v}{2}\right)^2 = \frac{1}{4}\left(v^2 - 2uv + u^2 - u^2 - 2uv - v^2\right) = 1$$

$$\implies u = -\frac{1}{v}$$



Calculate the Jacobian determinant to find the area in terms of u and v.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{-1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{2} \cdot \frac{1}{2} - \left(-\frac{1}{2} \cdot \frac{1}{2} \right) = \frac{1}{2}$$

The integral then becomes

$$\mathrm{I} = \iint_{\mathcal{R}} (x-y) \sin\left(x^2 - y^2\right)\, dy\, dx = \int_1^3 \int_{-1/v}^{1/v} u \cdot \sin(uv) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$= \int_1^3 \int_{-1/v}^{1/v} u \cdot \sin(uv) \cdot \frac{1}{2}\, du\, dv$$

Multiply each side by 2 to obviate the mess with the fraction $\dfrac{1}{2}$ and then change the order of integration.

$$2\mathrm{I} = \int_{-1}^{-\frac{1}{3}} \int_1^{-\frac{1}{u}} u \sin(uv)\, dv\, du + \int_{-\frac{1}{3}}^{\frac{1}{3}} \int_1^3 u \sin(uv)\, dv\, du + \int_{\frac{1}{3}}^{1} \int_1^{\frac{1}{u}} u \sin(uv)\, dv\, du$$

$$= \int_{-1}^{-\frac{1}{3}} (-\cos(uv)) \Big|_{v=1}^{v=-\frac{1}{u}} du + \int_{-\frac{1}{3}}^{\frac{1}{3}} (-\cos(uv)) \Big|_{v=1}^{v=3} du + \int_{\frac{1}{3}}^{1} (-\cos(uv)) \Big|_{v=1}^{v=\frac{1}{u}} du$$

$$= \int_{-1}^{-\frac{1}{3}} [-\cos(-1) + \cos u]\, du + \int_{-\frac{1}{3}}^{\frac{1}{3}} [-\cos(3u) + \cos u]\, du + \int_{\frac{1}{3}}^{1} [-\cos 1 + \cos u]\, du$$

$$= \Big[-u\cos(-1) + \sin u \Big]_{-1}^{-\frac{1}{3}} + \Big[-\frac{1}{3} \sin(3u) + \sin u \Big]_{-\frac{1}{3}}^{\frac{1}{3}} + \Big[-u\cos 1 + \sin u \Big]_{\frac{1}{3}}^{1}$$

$$2\mathrm{I} = \left[-\frac{2}{3}\cos(-1) + \sin\left(-\frac{1}{3}\right) - \sin(-1)\right] + \left[-\frac{1}{3}\left(\sin 1 - \sin(-1)\right) + \sin\frac{1}{3} - \sin\left(-\frac{1}{3}\right)\right]$$

$$+ \left[-\frac{2}{3}\cos 1 + \sin 1 - \sin\frac{1}{3}\right]$$

$$= \frac{4}{3}(\sin 1 - \cos 1)$$

This is the value of 2I. Therefore, $\boxed{\mathrm{I} = \frac{2}{3}(\sin 1 - \cos 1)}$.

7.

(i) For spherical coordinates, we have

$$\begin{array}{c} z = \rho\cos\phi \\ r = \rho\sin\phi \\ x^2+y^2 = r^2 \\ x^2+y^2+z^2 = \rho^2 \\ dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} z = \sqrt{3x^2+3y^2} \implies \rho\cos\phi = \sqrt{3}\rho\sin\phi \implies \phi = \frac{\pi}{6} \\ x^2+y^2 = r^2 = \rho^2\sin^2\phi \\ x^2+y^2+z^2 = 9 \implies \rho^2 = 9 \implies \rho = 3 \\ 0 \le \theta \le 2\pi \end{array}$$

The integral in spherical coordinates can be expressed as follows.

$$\boxed{\mathrm{I} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho^2\sin^2\phi \cdot \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta}$$

(ii) For cylindrical coordinates, we have

$$\begin{array}{c} z = z \\ r^2 = x^2+y^2 \\ dV = r\, dz\, dr\, d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} z = \sqrt{3x^2+3y^2} \implies z = r\sqrt{3} \\ x^2+y^2 = r^2 \\ x^2+y^2+z^2 = 9 \implies z = \sqrt{9-r^2} \\ 0 \le \theta \le 2\pi \end{array}$$

Find where the curves intersect to find the upper limit of r.

$$r\sqrt{3} = \sqrt{9-r^2} \implies 3r^2 = 9 - r^2 \implies r^2 = \frac{9}{4} \implies r = \frac{3}{2}$$

The integral in cylindrical coordinates can be expressed as follows.

$$\boxed{\mathrm{I} = \int_0^{2\pi}\int_0^{3/2}\int_{r\sqrt{3}}^{\sqrt{9-r^2}} r^2 \cdot r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{3/2}\int_{r\sqrt{3}}^{\sqrt{9-r^2}} r^3\, dz\, dr\, d\theta}$$