2022-2023 Fall
MAT123-02,05 Final
(13/01/2023)

1. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$ **(b)** $\displaystyle\int_0^{\pi/4} x\sec^2 x\,dx$

2. (a) Consider the finite region between the curves $y = e^{-x}$, $y = x/e$, and the y-axis. Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of the solid obtained by rotating this region about the x-axis.

(b) Consider the infinite region between the curves $y = e^{-x}$, $y = x/e$ and the x-axis. Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of the solid obtained by rotating this region about the line $y = -1$.

(c) Evaluate the area of the region given in part (a).

3. Determine whether each series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\arctan(n^2)}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right)$ **(c)** $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

4. (a) Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^{n+1}\cdot(x+1)^n}{n\cdot 3^n}.$$

(b) Using the formula $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, find the Maclaurin series of the function

$$f(x) = \frac{x^{123}}{1+x^4}$$

2022-2023 Final (13/01/2023) Solutions
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1. (a) Let $x = 2\sin u$ for $-\dfrac{\pi}{2} < u < \dfrac{\pi}{2}$, then $dx = 2\cos u\, du$.

$$x = 0 \implies 2\sin u = 0 \implies u = 0$$

$$x = 1 \implies 2\sin u = 1 \implies \sin u = \frac{1}{2} \implies u = \arcsin\frac{1}{2} = \frac{\pi}{6}$$

$$I = \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_0^{\pi/6} \frac{4\sin^2 u}{\sqrt{4-4\sin^2 u}} \cdot 2\cos u\,du \int_0^{\pi/6} \frac{4\sin^2 u\cos u}{|\cos u|}\,du \quad [\cos u > 0]$$

$$= \int_0^{\pi/6} 4\sin^2 u\,du = 4\int_0^{\pi/6} \left(1-\cos^2 u\right)\,du = 4\int_0^{\pi/6} \frac{1-\cos 2u}{2}\,du$$

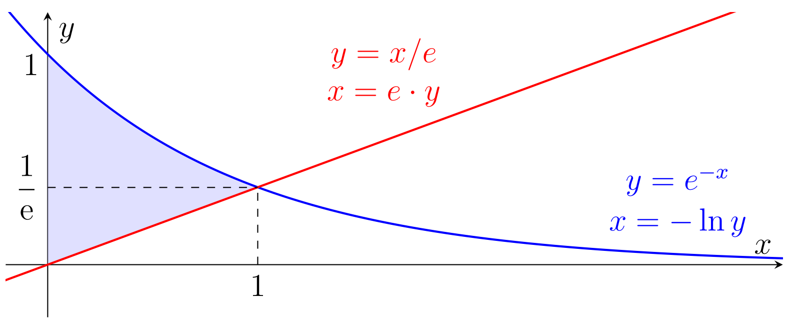
$$= 4\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\Bigg|_0^{\pi/6} = 2u - \sin 2u\Bigg|_0^{\pi/6} = \left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) - 0 = \boxed{\frac{\pi}{3} - \frac{\sqrt{3}}{2}}$$

(b) Use the method of integration by parts.

$$\left.\begin{array}{r} u = x \implies du = dx \\ dv = \sec^2 x\,dx \implies v = \tan x \end{array}\right\} \to \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

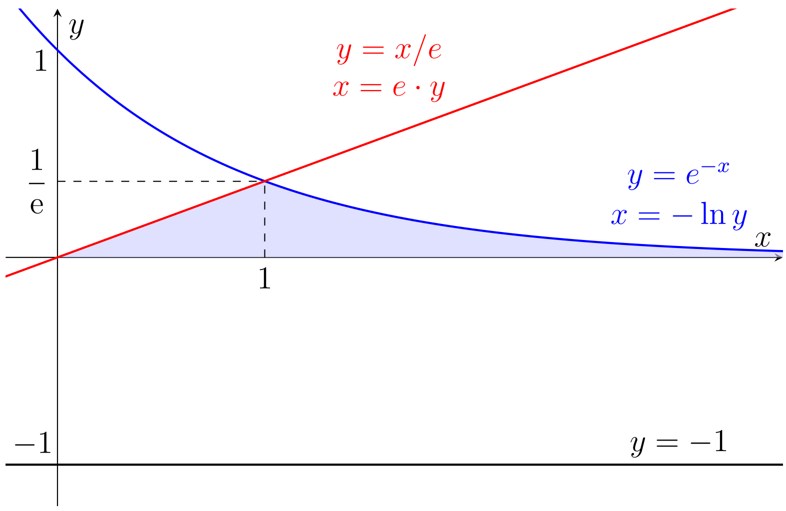
$$\int_0^{\pi/4} x\sec^2 x\,dx = x\tan x\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x\,dx = x\tan x + \ln|\cos x|\Big|_0^{\pi/4} = \boxed{\frac{\pi}{4} + \ln\frac{\sqrt{2}}{2}}$$

2. (a)



$$V = \int_D 2\pi \cdot h(y) \cdot r(y)\,dy = \boxed{\int_0^{1/\mathrm{e}} 2\pi \cdot y \cdot (\mathrm{e}\cdot y - 0)\,dy + \int_{1/\mathrm{e}}^{1} 2\pi \cdot y \cdot (-\ln y - 0)\,dy}$$

(b)



$$V = \int_D \pi \left[r_2^2(x) - r_1^2(x) \right] \, dx = \boxed{\int_0^1 \left[\left(\frac{x}{e} + 1 \right)^2 - 1^2 \right] \, dx + \int_1^\infty \left[\left(e^{-x} + 1 \right)^2 - 1^2 \right] \, dx}$$

(c)

$$V = 2\pi \left[\int_0^{1/e} ey^2 \, dy - \int_{1/e}^1 y \ln y \, dy \right] \tag{1}$$

Calculate the integral on the left in (1).

$$\int_0^{1/e} ey^2 \, dy = \left. \frac{ey^3}{3} \right|_0^{1/e} = \frac{1}{3e^2}$$

Calculate the integral on the right in (1) by using the method of integration by parts.

$$\left. \begin{array}{l} u = \ln x \implies du = \dfrac{1}{x} \, dx \\ dv = x \, dx \implies v = \dfrac{x^2}{2} \end{array} \right\} \to \int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

$$\int_{1/e}^1 x \ln x \, dx = \left. \frac{x^2 \ln x}{2} \right|_{1/e}^1 - \int_{1/e}^1 \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \left. \frac{x^2 \ln x}{2} - \frac{x^2}{4} \right|_{1/e}^1$$

$$= \left(0 - \frac{1}{4} \right) - \left(-\frac{1}{2e^2} - \frac{1}{4e^2} \right) = \frac{3}{4e^2} - \frac{1}{4}$$

Therefore,

$$V = 2\pi \left[\left(\frac{1}{3e^2} \right) - \left(\frac{3}{4e^2} - \frac{1}{4} \right) \right] = \boxed{\pi \left(\frac{1}{2} - \frac{5}{6e^2} \right)}$$

3. (a) Apply the nth Term Test for divergence. The inverse trigonometric function arctan is continuous on $\mathbb{R}$. Therefore, we may take the limit inside the function.

$$\lim_{n\to\infty} \frac{1}{\arctan\left(n^2\right)} = \frac{1}{\arctan\left(\lim\limits_{n\to\infty} n^2\right)} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \neq 1$$

By the nth Term Test for divergence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\arctan\left(n^2\right)}$ diverges.

(b) Recall the sine inequality $-\theta \leq \sin\theta \leq \theta$. Then for all $n \in \mathbb{R}$ except zero, we have $-\dfrac{1}{n^3} \leq \sin\left(\dfrac{1}{n^3}\right) \leq \dfrac{1}{n^3}$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because it is a p-series with $p = 3 > 1$. By the p-series Test, the series converges. The series $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right)$ also converges by the Direct Comparison Test because $\sin\left(\dfrac{1}{n^3}\right) < \dfrac{1}{n^3}$ for every $n \geq 1$.

(c) Take $f(x) = xe^{-x^2}$. f is continuous because the product of a polynomial and an exponential expression is still continuous. f is positive and decreasing for $x \geq 1$. Verify the monotonicity of f by taking the first derivative.

$$\frac{df}{dx} = 1 \cdot e^{-x^2} + xe^{-x^2} \cdot (-2x) = e^{-x^2}\left(1 - 2x^2\right)$$

$$f'(x) < 0 \quad \text{for} \quad x > \frac{\sqrt{2}}{2} \implies f'(x) < 0 \quad \text{for} \quad x \geq 1$$

We may now apply the Integral Test. Take the limit for the improper integral.

$$\int_1^{\infty} xe^{-x^2}\, dx = \lim_{R\to\infty} \int_1^{R} xe^{-x^2}\, dx = \lim_{R\to\infty} -\frac{1}{2}e^{-x^2}\Bigg|_1^R = \lim_{R\to\infty} -\frac{1}{2}\left(e^{-R^2} - \mathrm{e}^{-1}\right) = \frac{1}{2e}$$

The integral converges. Then the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ also converges.

4. (a) Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\lim_{n\to\infty} \left| \frac{2^{n+2} \cdot (x+1)^{n+1}}{(n+1) \cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x+1)^n \cdot 2^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2n \cdot (x+1)}{(n+1) \cdot 3} \right|$$

$$= \frac{2|x+1|}{3} \cdot \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = \frac{2|x+1|}{3}$$

$$\frac{2|x+1|}{3} < 1 \implies |x+1| < \frac{3}{2}$$

The radius of convergence is $\boxed{\dfrac{3}{2}}$.

$$|x+1| < \frac{3}{2} \implies -\frac{3}{2} < x+1 < \frac{3}{2} \implies -\frac{5}{2} < x < \frac{1}{2} \quad \text{(convergent)}$$

Investigate the convergence at the endpoints.

$$x = \frac{1}{2} \implies \sum_{n=1}^{\infty} \frac{2^{n+1} \cdot \left(\frac{3}{2}\right)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{2}{n} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n}$$

This is a p-series with $p = 1$, for which the series diverges by the p-series Test. Try the other endpoint.

$$x = -\frac{5}{2} \implies \sum_{n=1}^{\infty} \frac{2^{n+1} \cdot \left(-\frac{3}{2}\right)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^n}{n} = 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{n}$, is nonincreasing for $n \geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges.

The convergence set for the power series is $\boxed{\left[-\dfrac{5}{2}, \dfrac{1}{2}\right)}$.

(b)

$$f(x) = \frac{x^{123}}{1+x^4} = x^{123} \cdot \frac{1}{1-(-x^4)} = x^{123} \cdot \sum_{n=0}^{\infty} \left(-x^4\right) = x^{123} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot x^{4n}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \cdot x^{123+4n} = x^{123} - x^{127} + x^{131} - x^{135} + \ldots}$$