2019-2020 Spring
MAT124-02,05 Final
(01/07/2020)

1. Find the maximum and minimum values of the function $f(x, y, z) = x - y + z$ on the sphere $x^2 + y^2 + z^2 = 100$.

2. A cylindrical tank is 4 ft high and has an outer diameter of 2 ft. The walls of the tank are 0.2 in. thick. Approximate the volume of the interior of the tank assuming the tank has a top and bottom that are both also 0.2 in. thick.

3. Let $z = f(x, y)$ be a differentiable function of x and y, and let $x = r\cos\theta$ and $y = r\sin\theta$ for $r > 0$ and $0 < \theta < 2\pi$. Show that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

4. Reverse the order of integration in

$$\int_1^2 \int_x^{x^3} f(x, y)\, dy\, dx + \int_2^8 \int_x^8 f(x, y)\, dy\, dx.$$

5. Evaluate the double integral

$$\int_1^2 \int_{y^2}^{y^5} e^{x/y^2}\, \cancel{dy\, dx}\, dx\, dy.$$

6. Use a double integral to find the area of the region that lies inside the circle $r = \cos\theta$ and outside the cardioid $r = 1 - \cos\theta$.

7. Evaluate the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$, by using a triple integral in cylindrical coordinates.

8. Let T be the solid in the first octant bounded above by the sphere $x^2 + y^2 + z^2 = 7$ and below by the paraboloid $z = x^2 + y^2$. Express (do not evalaute) the integral

$$\iiint_T \sin\left(\sqrt{x^2 + y^2 + z^2}\right)\, dV$$

in spherical coordinates.

2019-2020 Spring Final (01/07/2020) Solutions
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1. Let $g(x, y, z) = x^2 + y^2 + z^2 - 100$ and then, solve the system of equations below using the method of Lagrange multipliers.

$$\left.\begin{array}{l} \nabla f = \lambda \nabla g \\ g(x, y, z) = 0 \end{array}\right\} \qquad \begin{array}{c} \nabla f = \langle 1, -1, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle = \lambda \nabla g \\ \therefore\ x = \dfrac{1}{2\lambda}, \quad y = -\dfrac{1}{2\lambda}, \quad z = \dfrac{1}{2\lambda} \end{array}$$

Use the constraint.

$$g(x, y, z) = 0 \implies \left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 100 \implies \frac{3}{4\lambda^2} = 100 \implies \lambda = \pm\frac{\sqrt{3}}{20}$$

$$\lambda = \pm\frac{\sqrt{3}}{20\lambda} \implies x = \pm\frac{10\sqrt{3}}{3}, \quad y = \mp\frac{10\sqrt{3}}{3}, \quad z = \pm\frac{10\sqrt{3}}{3}$$

The absolute extrema occur at $\left(\frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}\right)$ and $\left(-\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}\right)$.

$$f\left(\frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}\right) = 10\sqrt{3}, \quad f\left(-\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}\right) = -10\sqrt{3}$$

$$\boxed{\text{The minimum value is } -10\sqrt{3} \text{ and the maximum value is } 10\sqrt{3}.}$$

2. Recall that 12 inches is 1 foot. The volume of a right circular cylinder is

$$V(r, h) = \pi r^2 h$$

The total differential is

$$dV = V_r\, dr + V_h\, dh = \pi\,(2r \cdot h)\; dr + \pi\,\left(r^2 \cdot 1\right)\; dh$$

Set $r = 1,\ h = 4,\ dr = -0.2/12 = -1/60,\ dh = -0.4/12 = -1/30.$

$$dV = \pi\,(2 \cdot 1 \cdot 4) \cdot \left(-\frac{1}{60}\right) + \pi\,(1^2)\left(-\frac{1}{30}\right) = -\frac{\pi}{6}$$

Calculate the volume of the outer cylinder.

$$V(1, 4) = \pi \cdot 1^2 \cdot 4 = 4\pi$$

Take $\Delta V \approx dV = -\dfrac{\pi}{6}$. Therefore, the volume of the interior can be approximated as follows.

$$\boxed{V \approx 4\pi - \frac{\pi}{6} = \frac{23\pi}{6}}$$

3. We have $x = r\cos\theta$ and $y = r\sin\theta$. Compute the first-order partial derivatives.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}, \qquad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial y}{\partial r} = \sin\theta, \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta, \qquad \frac{\partial y}{\partial \theta} = r\cos\theta,$$

Rewrite $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\cdot\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot(-r\sin\theta) + \frac{\partial z}{\partial y}\cdot(r\cos\theta) \implies \frac{1}{r}\cdot\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot(-\sin\theta) + \frac{\partial z}{\partial y}\cdot\cos\theta$$

Take the squares of both sides of the equations and add up side by side.

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\cdot\sin\theta\right)^2, \quad \left(\frac{1}{r}\cdot\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\cdot(-\sin\theta) + \frac{\partial z}{\partial y}\cdot\cos\theta\right)^2$$

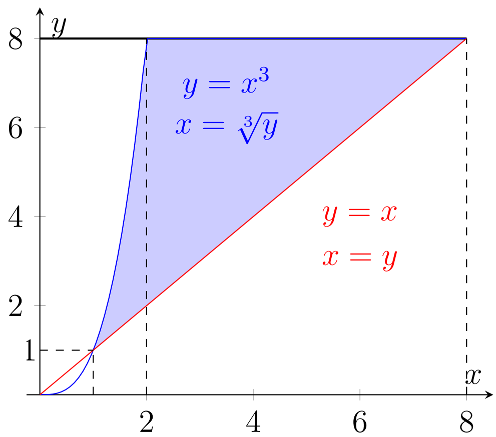
$$\begin{aligned}\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cdot\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta \\ &\quad + \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cdot\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta\end{aligned}$$

The terms with $\sin\theta\cos\theta$ cancel each other. Recall the equation $\sin^2 x + \cos^2 x = 1$. The equation then becomes

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2,$$

which we set out to demonstrate.

4.



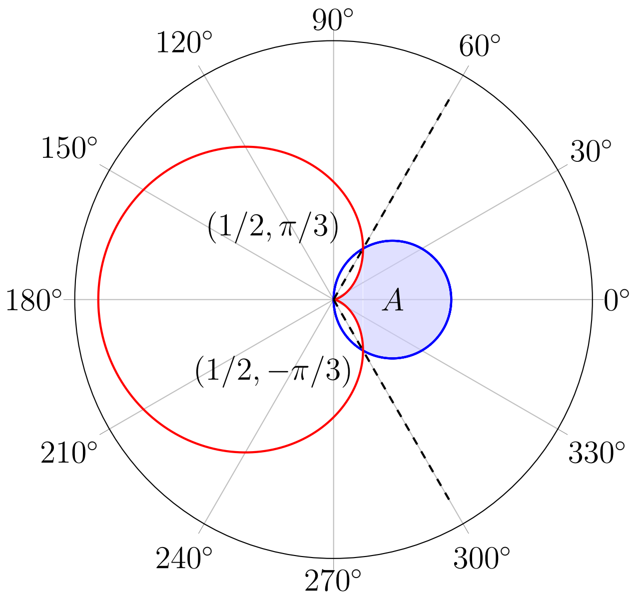
$$\boxed{\int_1^8 \int_{\sqrt[3]{y}}^{y} f(x,y)\,dx\,dy}$$

5. **Remark**: There was a mistake in the original question. The order of integration must be reversed.

$$\int_1^2 \int_{y^2}^{y^5} \mathrm{e}^{x/y^2}\,dx\,dy = \int_1^2 \left[y^2 \cdot \mathrm{e}^{x/y^2}\right]_{x=y^2}^{x=y^5} dy = \int_1^2 \left(y^2 \cdot \mathrm{e}^{y^3} - y^2 \cdot \mathrm{e}\right) dy = \left[\frac{1}{3}\mathrm{e}^{y^3} - \frac{1}{3}\mathrm{e}y^3\right]_1^2$$

$$= \frac{1}{3}\mathrm{e}^8 - \frac{8\mathrm{e}}{3} - \left(\frac{1}{3}\mathrm{e}^1 - \frac{1}{3}\mathrm{e}\right) = \boxed{\frac{\mathrm{e}}{3}\left(\mathrm{e}^7 - 8\right)}$$

6.



$$A = \int_{-\pi/3}^{\pi/3} \int_{1-\cos\theta}^{\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[\cos^2\theta - (1-\cos\theta)^2\right] d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta - 1)\,d\theta$$

$$= \frac{1}{2}\Big[2\sin\theta - \theta\Big]_{-\pi/3}^{\pi/3} = \frac{1}{2}\left[\left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(2\sin\left(-\frac{\pi}{3}\right) + \frac{\pi}{3}\right)\right] = \boxed{\sqrt{3} - \frac{\pi}{3}}$$

7. For cylindrical coordinates, we have

$$\begin{aligned} z &= z \\ r^2 &= x^2 + y^2 \\ dV &= r\,dz\,dr\,d\theta \end{aligned} \quad \rightarrow \quad \begin{gathered} z = \sqrt{x^2+y^2} \implies z = \sqrt{r^2} \implies z_{\text{lower}} = r \\ z = 2 - x^2 - y^2 \implies z_{\text{upper}} = 2 - r^2 \\ 0 \le \theta \le 2\pi \end{gathered}$$

Find where the surfaces $z = r$ and $z = 2 - r^2$ intersect to determine the upper bound of r.

$$\left.\begin{aligned} z &= r \\ z &= 2 - r^2 \end{aligned}\right\} \quad r^2 + r - 2 = 0 \implies (r+2)(r-1) = 0 \implies r_{\text{upper}} = 1$$

$$\mathrm{I} = \int_0^{2\pi}\int_0^1\int_r^{2-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \Big[z\Big]_{z=r}^{z=2-r^2} r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(2-r^2-r\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(2r-r^3-r^2\right) dr\,d\theta = \int_0^{2\pi}\left[r^2-\frac{r^4}{4}-\frac{r^3}{3}\right]_{r=0}^{r=1} d\theta = \int_0^{2\pi}\frac{5}{12}\,d\theta = \frac{5}{12}\cdot\theta\Big|_0^{2\pi}$$

$$= \boxed{\frac{5\pi}{6}}$$

8. For spherical coordinates, we have

$$\begin{aligned} z &= \rho\cos\phi \\ r &= \rho\sin\phi \\ x^2+y^2+z^2 &= \rho^2 \\ dV &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

$$x^2+y^2+z^2=7 \implies \rho^2=7 \implies \rho_{\text{upper},1}=\sqrt{7}$$
$$z=x^2+y^2 \implies \rho\cos\phi=\rho^2\sin^2\phi \implies \rho_{\text{upper},2}=\cot\phi\csc\phi$$
$$\sin\left(\sqrt{x^2+y^2+z^2}\right)=\sin\left(\sqrt{\rho^2}\right)=\sin\rho$$
$$0\le\theta\le\frac{\pi}{2}$$

Find where the surfaces $z=x^2+y^2$ and $x^2+y^2+z^2=7$ intersect to find the bounds of ϕ.

$$z^2+z-7=0 \implies z_{1,2}=\frac{-1\pm\sqrt{1^2-4\cdot1\cdot(-7)}}{2}$$

$$z>0 \implies z=\rho\cos\phi=\sqrt{7}\cos\phi=\frac{-1+\sqrt{29}}{2}$$

$$\cos\phi=\frac{-1+\sqrt{29}}{2\sqrt{7}} \implies \phi=\arccos\left(\frac{-1+\sqrt{29}}{2\sqrt{7}}\right)$$

For $\phi<\arccos\left(\frac{-1+\sqrt{29}}{2\sqrt{7}}\right)$, the upper bound for ρ is $\sqrt{7}$. For $\phi>\arccos\left(\frac{-1+\sqrt{29}}{2\sqrt{7}}\right)$, the lower bound is $\cot\phi\csc\phi$.

$$\boxed{\begin{aligned}&\int_0^{\pi/2}\int_0^{\arccos\left(\frac{-1+\sqrt{29}}{2\sqrt{7}}\right)}\int_0^{\sqrt{7}}\sin\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\ &+\int_0^{\pi/2}\int_{\arccos\left(\frac{-1+\sqrt{29}}{2\sqrt{7}}\right)}^{\pi/2}\int_0^{\cot\phi\csc\phi}\sin\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\end{aligned}}$$

Since we choose the minimum of the upper bounds of ρ, we can write the equivalent expression.

$$\boxed{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\min\left(\sqrt{7},\cot\phi\csc\phi\right)}\sin\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$