2019-2020 Spring
MAT124 Midterm
(08/06/2020)

1. Find an equation for the line L that contains the point $P = (-1, 3, 1)$ and is orthogonal to the line

$$\frac{x-2}{-1} = \frac{y-1}{-2} = \frac{z-5}{1} = \lambda, \quad \lambda \in \mathbb{R}$$

2. Sketch the graph of the following surfaces.

(a) $z = e^y$ (b) $y = z^2 - x^2$

3. The position vector for a particle in space is given as

$$\mathbf{R}(t) = (2\cos t)\mathbf{i} + t^2\mathbf{j} + (2\sin t)\mathbf{k}.$$

Find the velocity and acceleration vectors of the particle and find the speed and direction of motion at t $= \pi/2$.

4. Let f be a function defined by

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Is f continuous at $(0,0)$? Explain.

5. Use the $\epsilon - \delta$ definition and show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^6}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at the origin.

6. Note that in Cartesian coordinates, the Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $u = (x, y)$ and $v = v(x, y)$.

Let $x = r\cos\theta$ and $y = r\sin\theta$. Show that the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

7. According to Poiseuille's law, the resistance to the flow of blood offered by a cylindrical blood vessel of radius r and length x is

$$R(x, r) = \frac{cx}{r^4}$$

for a constant $c > 0$. A certain blood vessel in the body is 8 cm long and has a radius of 2 mm. Estimate the percent change in R when x is increased by 3% and r is decreased by 2%.

8. Find the critical points of $f(x, y) = -x^3 + 9x - 4y^2$ and classify each point as a relative maximum, a relative minimum, or a saddle point.

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1. Let M be the line $\dfrac{x-2}{-1} = \dfrac{y-1}{-2} = \dfrac{z-5}{1} = \lambda, \quad \lambda \in \mathbb{R}$. The direction vector of M is $\mathbf{u} = \langle -1, -2, 1 \rangle$.

Let $\mathbf{v} = \langle a, b, c \rangle$, where $a, b, c \in \mathbb{R}$, be the direction vector of L. If M and L are orthogonal, the dot product of the direction vectors is zero.

$$\mathbf{u} \cdot \mathbf{v} = \langle 1, -2, 1 \rangle \cdot \langle a, b, c \rangle = a - 2b + c = 0$$

a, b, c could be any number with the relation $a - 2b + c = 0$. Let $a = 1$, $b = 1$. Then $c = 2b - a = 2 - 1 = 1$. The direction vector $\mathbf{v}$ is then $\mathbf{v} = \langle 1, 1, 3 \rangle$.

The parametric equations for a line that passes through the point $P_0(x_0, y_0, z_0)$ is given by

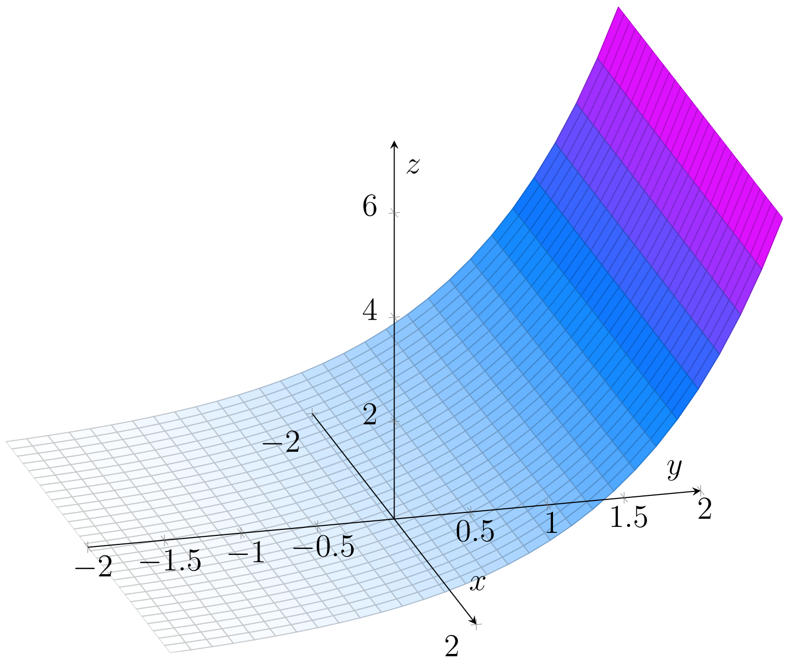
$$\left.\begin{aligned} x &= x_0 + v_1 t \\ y &= y_0 + v_2 t \\ z &= z_0 + v_3 t \end{aligned}\right\} \quad t \in \mathbb{R}$$

Therefore, using the point P, the parametric equations for the line L is

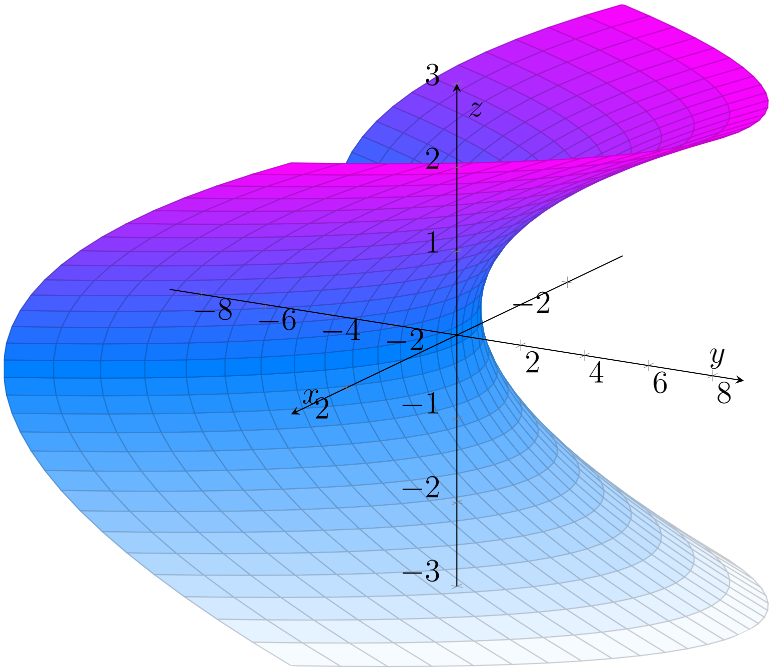
$$\boxed{\left.\begin{aligned} x &= -1 + t \\ y &= 3 + t \\ z &= 1 + 3t \end{aligned}\right\} \quad t \in \mathbb{R}}$$

2.

(a)



(b)



3. The velocity and acceleration vectors can be obtained by taking the first and the second derivatives of the vector function with respect to the parametrization variable, respectively.

$$\text{Velocity vector} : \mathbf{v}(t) = \frac{d\mathbf{R}}{dt} = \langle -2\sin t,\, 2t,\, 2\cos t\rangle$$

$$\text{Acceleration vector} : \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \langle -2\cos t,\, 2,\, -2\sin t\rangle$$

The speed of motion is the magnitude of the velocity vector, and the direction is the normalized velocity vector.

$$\text{Speed} = |\mathbf{v}(t=\pi/2)| = \sqrt{\left(-2\sin\frac{\pi}{2}\right)^2 + \left(2\cdot\frac{\pi}{2}\right)^2 + \left(2\cos\frac{\pi}{2}\right)^2} = \sqrt{4+\pi^2}$$

$$\text{Direction} = \frac{\mathbf{v}(t=\pi/2)}{|\mathbf{v}(t=\pi/2)|} = \frac{\left\langle -2\sin\frac{\pi}{2},\, 2\cdot\frac{\pi}{2},\, 2\cos\frac{\pi}{2}\right\rangle}{\sqrt{4+\pi^2}} = \left\langle -\frac{2}{\sqrt{4+\pi^2}},\, \frac{\pi}{\sqrt{4+\pi^2}}, 0\right\rangle$$

$$\boxed{\begin{gathered}\mathbf{v}(t) = \langle -2\sin t,\, 2t,\, 2\cos t\rangle \\ \mathbf{a}(t) = \langle -2\cos t,\, 2,\, -2\sin t\rangle \\[1em] \text{Speed of motion at } t=\frac{\pi}{2} : \sqrt{4+\pi^2} \\ \text{Direction of motion at } t=\frac{\pi}{2} : \left\langle -\frac{2}{\sqrt{4+\pi^2}},\, \frac{\pi}{\sqrt{4+\pi^2}}, 0\right\rangle\end{gathered}}$$

4. Apply the Two-Path Test.

$$x = y \implies \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{y\to 0} \frac{y^4}{y^2+y^6} = \lim_{y\to 0} \frac{y^2}{1+y^4} = \frac{0}{1} = 0$$

$$x = y^3 \implies \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{y\to 0} \frac{y^6}{y^6+y^6} = \lim_{y\to 0} \frac{y^6}{2y^6} = \frac{1}{2}$$

Since $0 \neq \frac{1}{2}$, by the Two-Path Test, the limit does not exist. Therefore, the function f is not continuous at $(0,0)$.

5. The value of the function at the point $(0,0)$ is 0. Therefore, we will show that the limit L is also 0. For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < \sqrt{(x-0)^2+(y-0)^2} < \delta \implies |f(x,y) - L| < \epsilon$$

$$\left|\frac{x^2y}{x^2+y^6} - 0\right| = |y| \cdot \left|\frac{x^2}{x^2+y^6}\right| \leq |y| \cdot 1 = |y| \qquad \left[\frac{x^2}{x^2+y^6} \leq \frac{x^2}{x^2} = 1\right]$$

$$\leq |x| + |y| < 2\delta \qquad \left[x^2 \geq 0,\ y^2 \geq 0,\ \sqrt{x^2+y^2} < \delta \implies |x| < \delta,\ |y| < \delta\right]$$

Let $\delta = \frac{\epsilon}{2}$.

$$\left|\frac{x^2y}{x^2+y^6}\right| \leq |x| + |y| = 2 \cdot \frac{\epsilon}{2} = \epsilon$$

Since the limit is equal to the value of the function at $(0,0)$, f is continuous at $(0,0)$.

6. u and v are functions of x and y. x and y are functions of r and θ. Use the chain rule.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = u_x \cdot \cos\theta + u_y \cdot \sin\theta = v_y \cos\theta - v_x \sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta} \implies \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}(-v_x \cdot r\sin\theta + v_y \cdot r\cos\theta) = -v_x \sin\theta + v_y \cos\theta$$

$$\frac{\partial u}{\partial r} = -v_x \sin\theta + v_y \cos\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r} = v_x \cdot \cos\theta + v_y \cdot \sin\theta = -u_y \cos\theta + u_x \sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} \implies -\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{1}{r}(-u_x r\sin\theta + u_y r\cos\theta) = u_x \sin\theta - u_y \cos\theta$$

$$\frac{\partial v}{\partial r} = u_x \sin\theta - u_y \cos\theta = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

7. The total differential of R is

$$dR = \frac{\partial R}{\partial x}\,dx + \frac{\partial R}{\partial r}\,dr = \frac{c}{r^4}\,dx - \frac{4cx}{r^5}\,dr$$

Since we estimate the percent change in R, we may take $\Delta R \approx dR$. Therefore, $dx = x \cdot 3\%$, $dr = r \cdot (-2\%)$. Given also $x = 8\,\mathrm{cm}$, $r = 2\,\mathrm{mm} = 0.2\,\mathrm{cm}$, the percent change can be estimated as

$$\frac{dR}{R} \cdot 100\% = \frac{1}{\frac{cx}{r^4}} \cdot \left(\frac{c}{r^4} \cdot x \cdot 3\% - \frac{4cx}{r^5} \cdot r \cdot (-2\%)\right) \cdot 100\% = \boxed{11\%}$$

8. To identify the critical points, find where both $f_x = f_y = 0$ or one of the partial derivatives does not exist.

$$f_x = -3x^2 + 9, \quad f_y = -8y$$

$$\left.\begin{array}{l} f_x = 0 \implies 9 = 3x^2 \\ f_y = 0 \implies -8y = 0 \end{array}\right\} \quad y = 0,\ x = \pm\sqrt{3}$$

The critical points are $\left(\sqrt{3}, 0\right)$ and $\left(-\sqrt{3}, 0\right)$. To classify these points, calculate the second partial derivatives and then find the Hessian determinants.

$$f_{xx} = -6x, \quad f_{xy} = f_{yx} = 0, \quad f_{yy} = -8$$

$$\left(\sqrt{3}, 0\right) \to \left\{\begin{array}{l} f_{xx} = -6\sqrt{3}, \quad f_{xy} = 0, \quad f_{yy} = -8 \\ \begin{vmatrix} -6\sqrt{3} & 0 \\ 0 & -8 \end{vmatrix} = \left(-6\sqrt{3}\right) \cdot (-8) - 0 \cdot 0 = 48\sqrt{3} > 0, \quad f_{xx} < 0 \end{array}\right.$$

$$\left(-\sqrt{3}, 0\right) \to \left\{\begin{array}{l} f_{xx} = 6\sqrt{3}, \quad f_{xy} = 0, \quad f_{yy} = -8 \\ \begin{vmatrix} 6\sqrt{3} & 0 \\ 0 & -8 \end{vmatrix} = \left(-6\sqrt{3}\right) \cdot (-8) - 0 \cdot 0 = -48\sqrt{3} < 0 \end{array}\right.$$

$$\boxed{\text{A local maximum occurs at } \left(\sqrt{3}, 0\right) \text{ and a saddle point occurs at } \left(-\sqrt{3}, 0\right).}$$