2023-2024 Spring
MAT124 Make-up
(27/06/2024)

1. Find the maximum and minimum values of $f(x, y, z) = x - y + z$ on the sphere $x^2 + y^2 + z^2 = 100$.

2. Sketch the region and reverse the order of the double integral

$$\int_0^{12} \int_{y/3}^{\sqrt{4-y}} dx\, dy$$

3. Using a polar double integral, find the volume of the sphere with radius 4.

4. Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies in the cylinder $x^2 + y^2 = 1$.

5. Using the change of variables, evaluate the area of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$.

6. Let R be the solid region bounded below by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 9$. Let

$$\mathrm{I} = \iiint_R (x^2 + y^2)\, dV.$$

(i) Express (but do not evaluate) I as a triple integral in spherical coordinates.

(ii) Express (but do not evaluate) I as a triple integral in cylindrical coordinates.

1. Let $g(x, y, z) = x^2 + y^2 + z^2 - 100$ for the constraint. Solve the system of equations below.

$$\begin{rcases} \nabla f = \lambda \nabla g \\ g(x, y, z) = 0 \end{rcases} \quad \nabla f = \langle 1, -1, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle = \lambda \nabla g$$

$$1 - 2\lambda x = 0 \implies x = \frac{1}{2\lambda}$$

$$-1 - 2\lambda y = 0 \implies y = -\frac{1}{2\lambda}$$

$$1 - 2\lambda z = 0 \implies z = \frac{1}{2\lambda}$$

Use the constraint to find the values of x, y, z.

$$x^2 + y^2 + z^2 - 100 = 0 \implies \left(\frac{1}{2\lambda} \right)^2 + \left(-\frac{1}{2\lambda} \right)^2 + \left(\frac{1}{2\lambda} \right)^2 = \frac{3}{4\lambda^2} = 100 \implies \lambda = \pm\frac{\sqrt{3}}{20}$$

$$x = \frac{1}{2\lambda} = \pm\frac{10\sqrt{3}}{3}, \quad y = -\frac{1}{2\lambda} = \pm\frac{10\sqrt{3}}{3}, \quad z = \frac{1}{2\lambda} = \pm\frac{10\sqrt{3}}{3}$$

To find the maximum and minimum values of f, use the points $\left(\frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3} \right)$ and $\left(-\frac{10\sqrt{3}}{3}, \frac{10\sqrt{3}}{3}, -\frac{10\sqrt{3}}{3} \right)$, respectively.

$$\boxed{\begin{aligned} f_{\max} &= \frac{10\sqrt{3}}{3} - \left(-\frac{10\sqrt{3}}{3} \right) + \frac{10\sqrt{3}}{3} = 10\sqrt{3} \\ f_{\min} &= -\frac{10\sqrt{3}}{3} - \frac{10\sqrt{3}}{3} - \frac{10\sqrt{3}}{3} = -10\sqrt{3} \end{aligned}}$$

2. **Remark**: It is counterintuitive that the area of the region is negative for $y > 3$. Therefore, let's assume that the upper bound for y is 3 rather than the value stated in the original question. The lecturer might have made a typo.
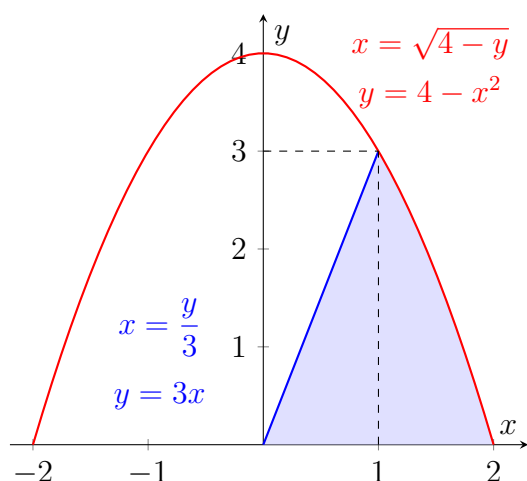
$$\boxed{\int_0^1 \int_0^{3x} dy \, dx + \int_1^2 \int_0^{4-x^2} dy \, dx}$$

3. The equation of this sphere is $x^2 + y^2 + z^2 = 16$. Solve for z to find the bounds of z.

$$z_{\text{lower}} = -\sqrt{16 - x^2 - y^2}, \quad z_{\text{upper}} = \sqrt{16 - x^2 - y^2}$$

If we project the sphere onto the xy-plane, we will notice that the domain is $x^2 + y^2 \leq 16$. Use the transformation for polar coordinates.

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
x^2 + y^2 &= r^2 \quad \rightarrow \\
dA &= r\, dr\, d\theta
\end{aligned}
\qquad
\begin{aligned}
z &= \sqrt{16 - x^2 - y^2} \implies z = \sqrt{16 - r^2} \\
z &= -\sqrt{16 - x^2 - y^2} \implies z = -\sqrt{16 - r^2} \\
x^2 + y^2 &\leq 16 \implies r^2 \leq 16 \implies 0 \leq r \leq 4, \quad 0 \leq \theta \leq 2\pi
\end{aligned}
$$

$$I = \int_0^{2\pi} \int_0^4 \left[\sqrt{16 - r^2} - \left(\sqrt{16 - r^2} \right) \right] r\, dr\, d\theta = 2 \int_0^{2\pi} \int_0^4 r\sqrt{16 - r^2}\, dr\, d\theta$$

$$= 2 \int_0^{2\pi} \left[-\frac{1}{3} \left(16 - r^2 \right)^{3/2} \right]_{r=0}^{r=4} d\theta = \frac{2}{3} \int_0^{2\pi} \left[0 - (-64) \right] d\theta = \frac{128}{3} \int_0^{2\pi} d\theta = \boxed{\frac{256\pi}{3}}$$

4. The domain is $x^2 + y^2 \leq 1$. Using the double integral below, we find the surface area.

$$\text{Surface area} = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2}\, dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + (2x)^2 + (2y)^2}\, dy\, dx$$

$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

If we switch to polar coordinates, we can easily evaluate the integral.

$$\text{Surface area} = \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \frac{1}{12} \int_0^{2\pi} \left[\left(1 + 4r^2 \right)^{3/2} \right]_{r=0}^{r=1} d\theta$$

$$= \frac{1}{12} \int_0^{2\pi} \left(5\sqrt{5} - 1 \right) d\theta = \boxed{\frac{\pi}{6} \left(5\sqrt{5} - 1 \right)}$$

5. Let $x = 4r\cos\theta$, $y = 5r\sin\theta$.

$$\frac{x^2}{16} + \frac{y^2}{25} = 1 \implies \frac{(4r\cos\theta)^2}{16} + \frac{(5r\sin\theta)^2}{25} = 1 \implies r^2 \left(\sin^2\theta + \cos^2\theta \right) = 1$$

$$r^2 = 1 \implies r = 1 \qquad 0 \leq \theta \leq 2\pi$$

Calculate the Jacobian determinant.

$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 4\cos\theta & -4r\sin\theta \\ 5\sin\theta & 5r\cos\theta \end{vmatrix} = 20r\cos^2\theta - \left(-20r\sin^2\theta\right) = 20r$$

Then we have the integral

$$\int_0^{2\pi}\int_0^1 \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\, d\theta = \int_0^{2\pi}\int_0^1 20r\, dr\, d\theta = 20\int_0^{2\pi}\left[\frac{1}{2}r^2\right]_{r=0}^{r=1} d\theta = 10\int_0^{2\pi} d\theta = \boxed{20\pi}$$

6.

(i) For spherical coordinates, we have

$$
\begin{aligned}
z &= \rho\cos\phi \\
r &= \rho\sin\phi \\
x^2 + y^2 &= r^2 \\
x^2 + y^2 + z^2 &= \rho^2 \\
dV &= \rho^2\sin\phi\, d\rho\, d\phi\, d\theta
\end{aligned}
\qquad \rightarrow
$$

$$z = \sqrt{3x^2 + 3y^2} \implies \rho\cos\phi = \sqrt{3}\rho\sin\phi \implies \phi = \frac{\pi}{6}$$
$$x^2 + y^2 = r^2 = \rho^2\sin^2\phi$$
$$x^2 + y^2 + z^2 = 9 \implies \rho^2 = 9 \implies \rho = 3$$
$$0 \le \theta \le 2\pi$$

The integral in spherical coordinates can be expressed as follows.

$$\boxed{I = \int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho^2\sin^2\phi \cdot \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta}$$

(ii) For cylindrical coordinates, we have

$$
\begin{aligned}
z &= z \\
r^2 &= x^2 + y^2 \\
dV &= r\, dz\, dr\, d\theta
\end{aligned}
\qquad \rightarrow
$$

$$z = \sqrt{3x^2 + 3y^2} \implies z = r\sqrt{3}$$
$$x^2 + y^2 = r^2$$
$$x^2 + y^2 + z^2 = 9 \implies z = \sqrt{9 - r^2}$$
$$0 \le \theta \le 2\pi$$

Find where the curves intersect to find the upper limit of r.

$$r\sqrt{3} = \sqrt{9 - r^2} \implies 3r^2 = 9 - r^2 \implies r^2 = \frac{9}{4} \implies r = \frac{3}{2}$$

The integral in cylindrical coordinates can be expressed as follows.

$$\boxed{I = \int_0^{2\pi}\int_0^{3/2}\int_{r\sqrt{3}}^{\sqrt{9-r^2}} r^2 \cdot r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{3/2}\int_{r\sqrt{3}}^{\sqrt{9-r^2}} r^3\, dz\, dr\, d\theta}$$