

2022-2023 Spring
MAT124 Makeup
(12/06/2023)

1. Find the absolute extrema of $f(x, y) = 2x^2 - y^2$ on the closed, bounded set $x^2 + y^2 \leq 1$ in the plane.

2. Sketch the region corresponding to the double integral

$$\int_{-3}^2 \int_{x^2}^{6-x} dy \, dx$$

and evaluate the integral by writing the equivalent integral with the order of integration reversed.

3. Using a double integral, find the area enclosed by the upper half of the cardioid $r = 1 + \sin \theta$.

4. Sketch the region R bounded above by the elliptic paraboloid $z = 2 - x^2 - y^2$ and below by the paraboloid $z = x^2 + y^2$. Using a double integral find the volume of R .

5. Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 4$ that is above the xy -plane and within the cylinder $x^2 + y^2 = 1$.

6.

(i) Sketch the graph of the region R bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by the plane $z = 4 - 2x$.

(ii) Evaluate the volume of R .

2022-2023 Spring Makeup (12/06/2023) Solutions
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1. Let $g(x, y, z) = x^2 + y^2 - 1$ and then solve the system of equations below using the method of Lagrange multipliers.

$$\left. \begin{array}{l} \nabla f = \lambda \nabla g \\ g(x, y) = 0 \end{array} \right\} \quad \begin{array}{l} \nabla f = \langle 4x, -2y \rangle = \lambda \langle 2x, 2y \rangle = \lambda \nabla g \\ x^2 + y^2 - 1 = 0 \end{array}$$

$$\begin{aligned} 4x &= \lambda(2x) \implies 2x(2 - \lambda) = 0 \implies \lambda = 2 \text{ or } x = 0 \\ -2y &= \lambda(2y) \implies -2y(1 + \lambda) = 0 \implies \lambda = -1 \text{ or } y = 0 \end{aligned}$$

Now, use the constraint to find the coordinates.

$$\begin{aligned} \lambda = 2 &\implies y = 0 \implies x^2 + 0^2 - 1 = 0 \implies x = \pm 1 \\ \lambda = -1 &\implies x = 0 \implies 0^2 + y^2 - 1 = 0 \implies y = \pm 1 \end{aligned}$$

Evaluate f at these points: $(0, 1)$, $(0, -1)$, $(1, 0)$, or $(-1, 0)$.

$$\begin{aligned} f(0, 1) &= 2 \cdot 0^2 - 1^2 = -1, & f(0, -1) &= 2 \cdot 0^2 - (-1)^2 = -1, \\ f(1, 0) &= 2 \cdot 1^2 - 0^2 = 2, & f(-1, 0) &= 2 \cdot (-1)^2 - 0^2 = 2 \end{aligned}$$

The *only* critical point occurs at $(0, 0)$.

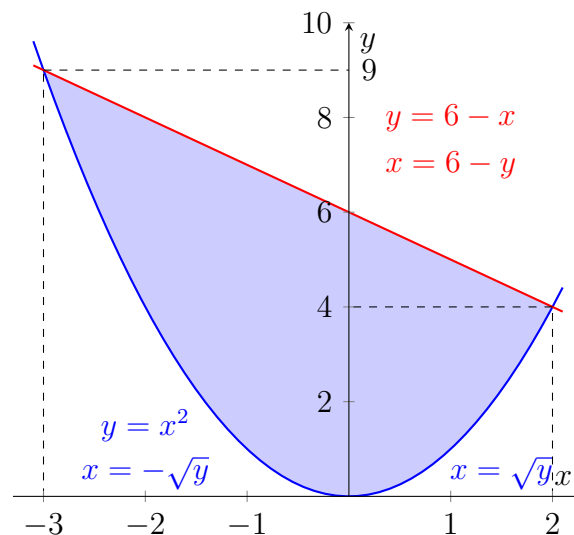
$$\frac{\partial f}{\partial x} = 4x = 0, \quad \frac{\partial f}{\partial y} = -2y = 0 \implies (x, y) = (0, 0) \rightarrow f(0, 0) = 0$$

Compare all the values.

$$f(0, 0) = 0, \quad f(0, 1) = f(0, -1) = -1, \quad f(1, 0) = f(-1, 0) = 2$$

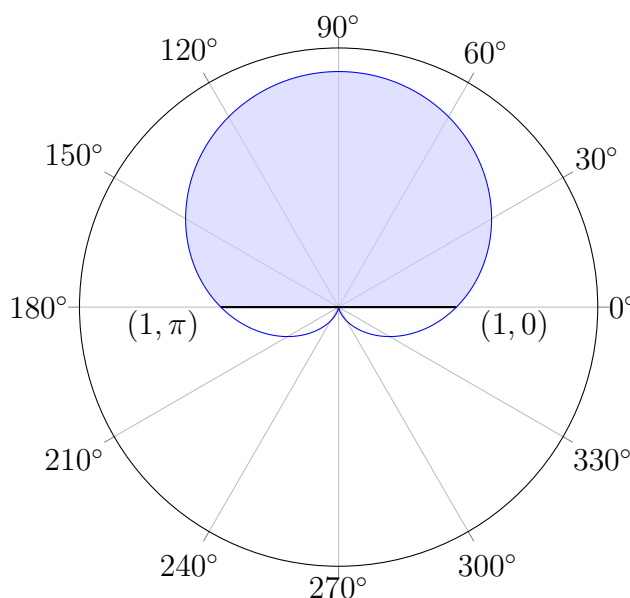
$$\boxed{f(0, 1) = f(0, -1) = -1 \implies \text{abs. min.} \quad f(1, 0) = f(-1, 0) = 2 \implies \text{abs. max.}}$$

2.



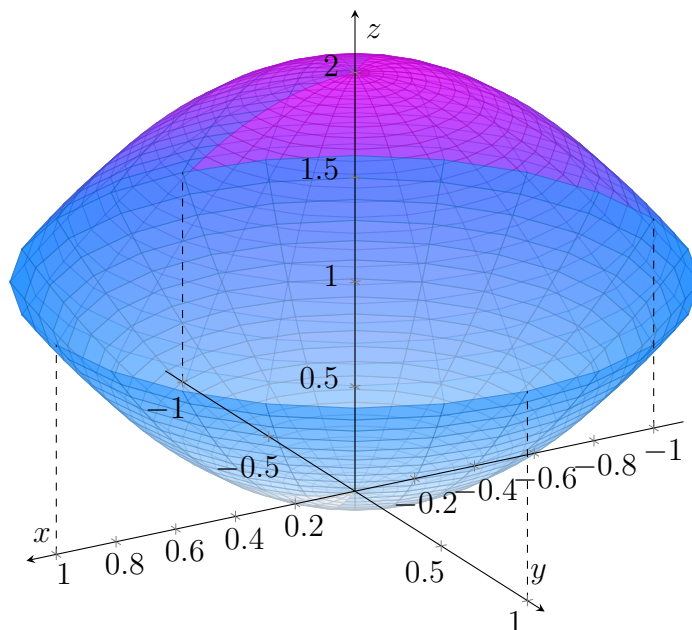
$$\begin{aligned}
I &= \int_{-3}^2 \int_{x^2}^{6-x} dy \, dx = \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dy + \int_4^9 \int_{-\sqrt{y}}^{6-y} dx \, dy \\
&= \int_0^4 \left[\sqrt{y} - (-\sqrt{y}) \right] dy + \int_4^9 \left[(6-y) - (-\sqrt{y}) \right] dy \\
&= 2 \int_0^4 \sqrt{y} \, dy + \int_4^9 (6-y+\sqrt{y}) \, dy = 2 \left[\frac{2}{3} y^{3/2} \right]_0^4 + \left[6y - \frac{y^2}{2} + \frac{2}{3} y^{3/2} \right]_4^9 \\
&= \frac{4}{3} [4^{3/2} - 0^{3/2}] + \left[\left(6 \cdot 9 - \frac{9^2}{2} + \frac{2}{3} \cdot 9^{3/2} \right) - \left(6 \cdot 4 - \frac{4^2}{2} + \frac{2}{3} \cdot 4^{3/2} \right) \right] = \boxed{\frac{125}{6}}
\end{aligned}$$

3. We are interested in the upper half of the cardioid. Therefore, $0 \leq \theta \leq \pi$.



$$\begin{aligned}
\text{Area} &= \int_0^\pi \int_0^{1+\sin\theta} r \, dr \, d\theta = \frac{1}{2} \int_0^\pi [(1+\sin\theta)^2 - 0^2] \, d\theta = \frac{1}{2} \int_0^\pi (1 + 2\sin\theta + \sin^2\theta) \, d\theta \\
&= \frac{1}{2} \int_0^\pi (1 + 2\sin\theta + 1 - \cos^2\theta) \, d\theta = \frac{1}{2} \int_0^\pi \left(1 + 2\sin\theta + \frac{1}{2} - \frac{\cos(2\theta)}{2} \right) \, d\theta \\
&= \frac{1}{2} \left[\theta - 2\cos\theta + \frac{\theta}{2} - \frac{\sin(2\theta)}{4} \right]_0^\pi \\
&= \frac{1}{2} \left[\left(\pi - 2\cos\pi + \frac{\pi}{2} - \frac{\sin(2\pi)}{4} \right) - (0 - 2\cos 0 + 0 - \sin 0) \right] = \boxed{\frac{3\pi}{4} + 2}
\end{aligned}$$

4.



$$I = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [2 - x^2 - y^2 - (x^2 + y^2)] dy dx$$

This integral seems difficult. Use the transformation below to switch to polar coordinates.

$$\begin{aligned} x^2 + y^2 = r^2 & \quad \rightarrow \quad r^2 \leq z \leq 2 - r^2 \\ dA = dy dx = r dr d\theta & \quad \rightarrow \quad 0 \leq r \leq 1 \\ & \quad \rightarrow \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^1 (2 - 2r^2) r dr d\theta = \int_0^{2\pi} \int_0^1 (2r - 2r^3) dr d\theta = \int_0^{2\pi} \left[r^2 - \frac{r^4}{2} \right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \frac{1}{2} d\theta = \boxed{\pi} \end{aligned}$$

5. For the upper hemisphere, we have $z = \sqrt{4 - x^2 - y^2}$. The projection of the surface onto the xy -plane gives us the region $x^2 + y^2 \leq 1$. Find the surface area.

$$\begin{aligned} \text{Surface area} &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} dA \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + \left(\frac{x}{\sqrt{4-x^2-y^2}} \right)^2 + \left(\frac{y}{\sqrt{4-x^2-y^2}} \right)^2} dy dx \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + \frac{x^2 + y^2}{4 - x^2 - y^2}} dy dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\sqrt{4 - x^2 - y^2}} dy dx \end{aligned}$$

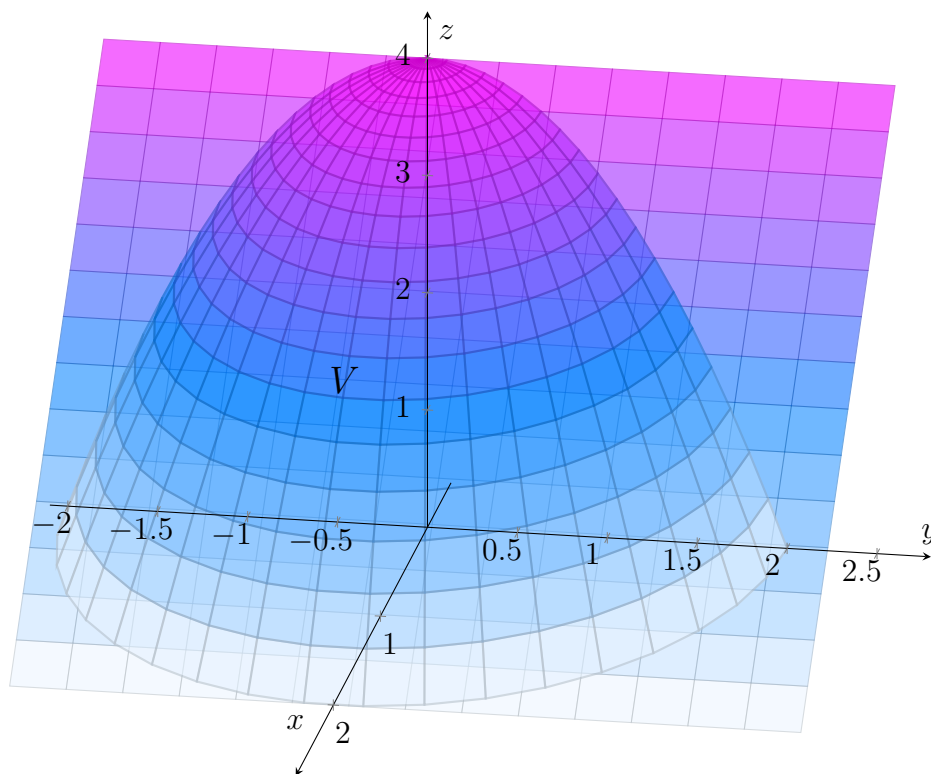
From this point on, we can switch to polar coordinates using the transformation below.

$$\begin{aligned} \sqrt{4-x^2-y^2} &= \sqrt{4-r^2} \\ r^2 = x^2 + y^2 &\rightarrow x^2 + y^2 \leq 1 \rightarrow r^2 \leq 1 \implies 0 \leq r \leq 1 \\ dA = r \, dr \, d\theta & \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

$$\begin{aligned} \text{Surface area} &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\sqrt{4-x^2-y^2}} \, dy \, dx = \int_0^{2\pi} \int_0^1 \frac{2}{\sqrt{4-r^2}} \cdot r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[-2\sqrt{4-r^2} \right]_{r=0}^{r=1} d\theta = \int_0^{2\pi} \left[-2\sqrt{3} - (-4) \right] d\theta = \boxed{4\pi(2-\sqrt{3})} \end{aligned}$$

6.

(i)



(ii) Using cylindrical coordinates, we may find the volume.

$$\begin{aligned} z &= z & z &= 4 - x^2 - y^2 \implies z = 4 - r^2 \\ x &= r \cos \theta & z &= 4 - 2x \implies z = 4 - 2r \cos \theta \\ y &= r \sin \theta & 4 - 2x &= 4 - x^2 - y^2 \implies r = 2 \cos \theta \\ r^2 &= x^2 + y^2 & & \\ dV &= r \, dz \, dr \, d\theta & 4 - 2r \cos \theta &\leq z \leq 4 - r^2, \quad 0 \leq r \leq 2 \cos \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned}
V &= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \int_{4-2r \cos \theta}^{4-r^2} r \, dz \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} [4 - r^2 - (4 - 2r \cos \theta)] \, r \, dr \, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} (2r^2 \cos \theta - r^3) \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{2r^3}{3} \cos \theta - \frac{r^4}{4} \right]_{r=0}^{r=2 \cos \theta} d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left[\left(\frac{16}{3} \cos^4 \theta - \frac{16 \cos^4 \theta}{4} \right) - 0 \right] d\theta = \frac{4}{3} \int_{-\pi/2}^{\pi/2} \cos^4 \theta \, d\theta \\
&= \frac{4}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{\cos(2\theta) + 1}{2} \right)^2 d\theta = \frac{1}{3} \int_{-\pi/2}^{\pi/2} (\cos^2(2\theta) + 2 \cos(2\theta) + 1) \, d\theta \\
&= \frac{1}{3} \int_{-\pi/2}^{\pi/2} \left[\left(\frac{\cos(4\theta) + 1}{2} \right) + 2 \cos(2\theta) + 1 \right] d\theta = \frac{1}{3} \left[\frac{\sin(4\theta)}{8} + \frac{\theta}{2} + \sin(2\theta) + \theta \right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{3} \left[\left(\frac{\sin(2\pi)}{8} + \frac{\pi}{4} + \sin \pi + \frac{\pi}{2} \right) - \left(\frac{\sin(-2\pi)}{8} - \frac{\pi}{4} + \sin(-\pi) - \frac{\pi}{2} \right) \right] = \boxed{\frac{\pi}{2}}
\end{aligned}$$