2024-2025 Spring
MAT124 Final
(18/06/2025)

1. Let $f(x,y) = 6 - x^2 - y^2$. Find the tangent plane and normal line equations (symmetric or parametric equations) of the graph of f at the point $P(1,2,1)$.

2. Find all critical points of the given function $f(x,y) = x^3 + y^3 - 3xy + 2$ and classify them (i.e., determine whether each critical point corresponds to a local maximum, local minimum, or saddle point).

3. Find the maximum and minimum of the function $f(x,y) = 81x^2 + y^2$ subject to the given constraint $4x^2 + y^2 = 9$ using Lagrange multipliers.

4. The equation $z^3 - zy^2 + yx = 3$ defines z implicitly as a function of x and y. It is given that $z = 2$ when $(x,y) = (-3,1)$. Evaluate $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(-3,1)$.

5.

(a) Reverse the order of integration and evaluate the double integral.

$$\int_0^{\sqrt{2}} \int_0^{2-x^2} x\mathrm{e}^{x^2}\, dy\, dx$$

(b) Evaluate the double integral

$$\iint_R \frac{2xy}{x^2+y^2}\, dA$$

where the region $R = \{(x,y) : x \geq 0,\ y \geq 0,\ z \geq 0,\ x^2 + y^2 \leq a^2\}$

6.

(a) Convert the triple integral into cylindrical coordinates. Do not evaluate the integral.

$$\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy$$

(b) Use a triple integral in spherical coordinates in order to find the volume of the sphere centered at $(0,0,0)$ with radius 3.

7.

(a) Determine whether the vector field $\mathbf{F}(x,y,z) = y\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ is conservative and find a potential if it is conservative.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F}(x,y,z) = z\,\mathbf{i} - y\,\mathbf{j} + 2x\,\mathbf{k}$ along the curve $x = t,\ y = t^2\ z = t^3$ from $(0,0,0)$ to $(1,1,1)$.

2024-2025 Spring Final (18/06/2025) Solutions
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1. The equation of the tangent plane at a point on the surface and the normal line to this plane are given, respectively, by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{1}$$

$$\left.\begin{array}{c} x = x_0 + f_x(x_0, y_0) \cdot t \\ y = y_0 + f_y(x_0, y_0) \cdot t \\ z = z_0 - t \end{array}\right\} \quad -\infty < t < \infty \tag{2}$$

Calculate the first partial derivatives of f.

$$f_x = -2x, \quad f_y = -2y \quad \Longrightarrow \quad f_x(1,2) = -2, \quad f_y(1,2) = -4$$

Using (1) and (2) determine the equations.

$$\boxed{\begin{array}{l} \text{Tangent plane equation: } z - 1 = -2(x-1) - 4(y-2) \implies z = -2x - 4y + 11 \\ \text{Normal line equation: } \left.\begin{array}{l} x = 1 - 2t \\ y = 2 - 4t \\ z = 1 - t \end{array}\right\} \quad -\infty < t < \infty \end{array}}$$

2. To identify the critical points, find where both $f_x = f_y = 0$ or one of the partial derivatives does not exist.

$$f_x = 3x^2 - 3y, \quad f_y = 3y^2 - 3x$$

$$\left.\begin{array}{l} f_x = 0 \implies 3y = 3x^2 \implies y = x^2 \\ f_y = 0 \implies 3y^2 = 3x \implies x = y^2 \end{array}\right\} x = x^4 \implies x(x^3 - 1) = 0 \implies x_1 = 0, \quad x_2 = 1$$

$$x_1 = 0 \implies y_1 = 0, \quad x_2 = 1 \implies y_2 = 1$$

The critical points are $(0,0)$ and $(1,1)$. To classify these points, calculate the second partial derivatives and then find the Hessian determinants.

$$f_{xx} = 6x, \quad f_{xy} = f_{yx} = -3, \quad f_{yy} = 6y$$

$$(0,0) \to \begin{cases} f_{xx} = 0, \quad f_{xy} = -3, \quad f_{yy} = 0 \\ \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = 0 \cdot 0 - (-3) \cdot (-3) = -9 < 0 \end{cases}$$

$$(1,1) \to \begin{cases} f_{xx} = 6, \quad f_{xy} = -3, \quad f_{yy} = 6 \\ \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 6 \cdot 6 - (-3) \cdot (-3) = 27 > 0, \quad f_{xx} > 0 \end{cases}$$

$$\boxed{\text{A local minimum occurs at } (1,1) \text{ and a saddle point occurs at } (0,0).}$$

3. Let $g(x,y)=4x^2+y^2-9$ be the constraint. Solve the system of equations below.

$$\left.\begin{array}{l}\nabla f=\lambda\nabla g\\ g(x,y)=0\end{array}\right\}\qquad \nabla f=\langle 162x,2y\rangle=\lambda\langle 8x,2y\rangle=\lambda\nabla g$$

$$2y=2\lambda y \implies 2y(1-\lambda)=0 \implies y=0 \quad \text{or} \quad \lambda=1$$
$$162x=8\lambda x \implies 2x(81-4\lambda)=0 \implies x=0 \quad \text{or} \quad \lambda=\frac{81}{4}$$

$$y=0 \implies g(x,0)=4x^2+0^2-9=0 \implies x=\pm\frac{3}{2}$$
$$x=0 \implies g(0,y)=4\cdot 0^2+y^2=9 \implies y=\pm 3$$

Evaluate f at the points $(0,3)$, $(0,-3)$, $\left(\frac{3}{2},0\right)$, $\left(-\frac{3}{2},0\right)$ and compare them all.

$$f(0,3)=81\cdot 0^2+3^2=9,\quad f(0,-3)=81\cdot 0^2+(-3)^2=9$$

$$f\left(\frac{3}{2},0\right)=81\cdot\left(\frac{3}{2}\right)^2+0^2=\frac{729}{4},\quad f\left(-\frac{3}{2},0\right)=81\cdot\left(-\frac{3}{2}\right)^2+0^2=\frac{729}{4}$$

$$\boxed{\text{The minimum value is } 9,\ \text{the maximum value is } \frac{729}{4}.}$$

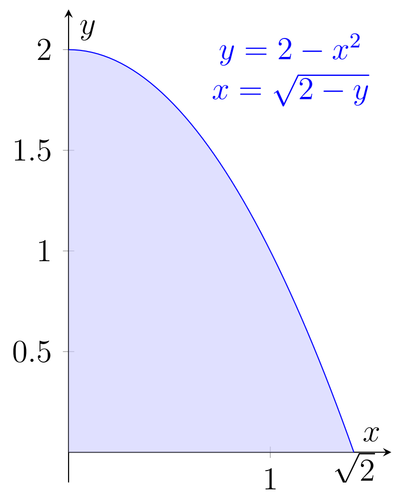
4. Obtain the partial derivatives of both sides.

$$3z^2\cdot z_x-z_x\cdot y^2+y=0 \implies z_x=\left.\frac{-y}{3z^2-y^2}\right|_{(-3,1,2)}=-\frac{1}{11}$$
$$3z^2\cdot z_y-z_y\cdot y^2-2zy+x=0 \implies z_y=\left.\frac{2zy-x}{3z^2-y^2}\right|_{(-3,1,2)}=\frac{7}{11}$$

$$\boxed{\frac{\partial z}{\partial x}=-\frac{1}{11},\quad \frac{\partial z}{\partial y}=\frac{7}{11}}$$

5.

(a)

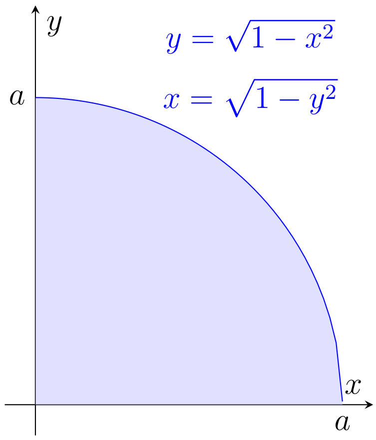


$$\mathrm{I}=\int_0^{\sqrt{2}}\int_0^{2-x^2} x\mathrm{e}^{x^2}\,dy\,dx=\int_0^2\int_0^{\sqrt{2-y}} x\mathrm{e}^{x^2}\,dx\,dy$$

$$=\int_0^2\left[\frac{\mathrm{e}^{x^2}}{2}\right]_{x=0}^{x=\sqrt{2-y}}dy=\frac{1}{2}\int_0^2\left(\mathrm{e}^{2-y}-1\right)dy$$

$$=\frac{1}{2}\left[-\mathrm{e}^{2-y}-y\right]_0^2=\boxed{\frac{\mathrm{e}^2-3}{2}}$$

(b)



$$\iint_R \frac{2xy}{x^2+y^2}\,dA = \int_0^a \int_0^{\sqrt{1-x^2}} \frac{2xy}{x^2+y^2}\,dy\,dx$$

Notice that it would be difficult to solve the integral in rectangular coordinates. We may switch to polar coordinates to easily evaluate the integral.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2+y^2 = r^2, \quad dA = r\,dr\,d\theta$$

$$\mathrm{I} = \iint_R \frac{2xy}{x^2+y^2}\,dA = \int_0^{\frac{\pi}{2}} \int_0^a \frac{2r\cos\theta \cdot r\sin\theta}{(r\cos\theta)^2 + (r\sin\theta)^2}\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a \frac{2r^3\cos\theta\sin\theta}{r^2(\sin^2\theta+\cos^2\theta)}\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^a r\sin(2\theta)\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \sin(2\theta)\left[\frac{r^2}{2}\right]_{r=0}^{r=a} d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta)\,d\theta = \frac{a^2}{2}\left[\frac{-\cos(2\theta)}{2}\right]_0^{\frac{\pi}{2}} = \frac{a^2}{4}(-\cos\pi + \cos 0) = \boxed{\frac{a^2}{2}}$$

6.

(a) Use the transformation below.

$$\begin{array}{c} z = z \\ x = r\cos\theta \\ y = r\sin\theta \\ r^2 = x^2+y^2 \\ dV = r\,dz\,dr\,d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} \sqrt{x^2+y^2} = \sqrt{r^2} = r = z_{\text{upper}} \\ x^2+y^2 = r^2 = z_{\text{lower}} \\ -1 < y < 1, < x < \sqrt{1-y^2} \implies 0 \le r \le 1,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ xyz = r\cos\theta \cdot r\sin\theta \cdot z \end{array}$$

$$\boxed{\int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^{r} zr^3 \sin\theta\cos\theta\,dz\,dr\,d\theta}$$

(b) We have the sphere $x^2+y^2+z^2 = 9$. Use the transformation below.

$$\begin{array}{c} z = \rho\cos\phi \\ r = \rho\sin\phi \\ x^2+y^2+z^2 = \rho^2 \\ dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} x^2+y^2+z^2 = 9 \implies \rho^2 = 9 \implies \rho = 3 \\ 0 \le \theta \le 2\pi, \quad 0 \le \phi \le 2\pi \end{array}$$

$$\text{Volume} = \int_0^{2\pi}\int_0^{2\pi}\int_0^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{2\pi} \sin\phi \cdot \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=3} d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{2\pi} 9\sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \Big[-9\cos\phi\Big]_{\phi=0}^{\phi=2\pi} d\theta$$

$$= \int_0^{2\pi} \left(-9\cos(2\pi) + 9\cos 0\right) \, d\theta = \int_0^{2\pi} 18 \, d\theta = 18\Big[\theta\Big]_0^{2\pi} = \boxed{36\pi}$$

7.

(a) Assume that $\nabla f = \mathbf{F}$ for some potential function f. Then, the mixed partial derivatives of the components must be equal.

$$\frac{\partial \mathrm{F}_1}{\partial y} = 1 = \frac{\partial \mathrm{F}_2}{\partial x}, \quad \frac{\partial \mathrm{F}_1}{\partial z} = 0 = \frac{\partial \mathrm{F}_3}{\partial x}, \quad \frac{\partial \mathrm{F}_2}{\partial z} = 0 = \frac{\partial \mathrm{F}_3}{\partial y}$$

This means $\mathbf{F}$ is conservative on its domain.

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = z^2$$

$$\int \frac{\partial f}{\partial x} \, dx = \int y \, dx = xy + g(y,z) = f(x,y,z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xy + g(y,z)\right) = x + g_y(y,z) = x \implies g_y(y,z) = 0$$

$$\int \frac{\partial f}{\partial y} \, dy = \int x \, dy = xy + h(z) = f(x,y,z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(xy + h(z)\right) = h_z(z) = z^2 \implies h_z(z) = z^2$$

$$\int \frac{\partial f}{\partial z} \, dz = \int z^2 \, dz = xy + \frac{z^3}{3} + c = f(x,y,z)$$

The potential function for $\mathbf{F}$ is

$$\boxed{f(x,y,z) = xy + \frac{z^3}{3} + c, \quad c \in \mathbb{R}}$$

(b) Parametrize the curve and then evaluate the integral.

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \quad 0 \le t \le 1 \implies \mathbf{r}'(t) = dt\,\mathbf{i} + 2t\,dt\,\mathbf{j} + 3t^2\,dt\,\mathbf{k}, \quad 0 \le t \le 1$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_0^1 \left(t^3\,\mathbf{i} - t^2\,\mathbf{j} + 2t\,\mathbf{k}\right) \cdot \left(\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}\right) dt$$

$$= \int_0^1 \left(t^3 - 2t^3 + 6t^3\right) \, dt = \int_0^1 5t^3 \, dt = \left[\frac{5t^4}{4}\right]_0^1 = \boxed{\frac{5}{4}}$$