MAT123 07.11.2025

QUESTIONS

Q1: For $0 < t \leq 1$, let $A(t)$ denote the area of the triangle bounded by the x-axis, the y-axis and the tangent line to the curve $y = \ln x$ at $(t, \ln t)$. Find the maximum value of $A(t)$.

Q2: A manufacturer will construct an open rectangular box with a square base and a volume of $6\,\mathrm{m}^3$. If material for the bottom costs 60 Turkish Liras/m^2 and material for the sides costs 40 Turkish Liras/m^2, what dimensions will result in the least expensive box? What is the minimum cost?

Q3: Find $y'(1)$ if $x^y + y^x = 3$.

Q4: Show that the equation $x^3 + x^2 + 2x + 5 = 0$ has a root, but no more than one.

Q5: Let C be the graph of the equation $3y^2x - x^3 - y^3 - y = 0$. Find all the points on C through which the tangent lines to C are horizontal.

Q6: How fast is the surface area of a cube changing when the volume of the cube is 64 cm^2 and is increasing at 2 cm^3/sec?

Q7: Evaluate $\lim_{x\to\infty} \left(\sqrt{x + \sqrt{x}} - \sqrt{x}\right)$.

Q8: Let $f(x) = \dfrac{x^2 - 2}{x^2 - 1}$.

- **a**. Find the entire domain.
- **b**. Find the x- and y- intercepts.
- **c**. Find the asymptotes.
- **d**. Determine $f'(x)$ and $f''(x)$.
- **e**. Make a table.
- **f**. Sketch the graph.

Q9: Find $\dfrac{dy}{dx}$ where $y = (\ln x)^{\ln x}$

Q10: Evaluate $\lim_{x\to 1^-} \dfrac{\ln \sin \pi x}{\csc \pi x}$.

ANSWERS

Q1: $\dfrac{2}{e}$ **Q2:** Dimensions: $2\text{m} \times 2\text{m} \times 1.5\text{m}$, Cost: 720 TL **Q3:** $y = -2\ln 2 - 2$.

Q4: Proof by means of IVT and MVT. **Q5:** $\{(-1,-1), (0,0), (1,1)\}$ **Q6:** 2 cm^2/sec

Q7: $\dfrac{1}{2}$

Q8: **a.** $\mathcal{D}_f = \mathbb{R} \setminus \{-1, 1\}$ **b.** $\{(0,2), (-\sqrt{2}, 0), (\sqrt{2}, 0)\}$

c. Horizontal: $y = 1$, vertical: $x = \pm 1$ **d.** $f'(x) = \dfrac{2x}{(x^2-1)^2}$, $f''(x) = \dfrac{-6x^2-2}{(x^2-1)^3}$

e. **& f.** on the paper below.

Q9: $y' = (\ln x)^{\ln x} \left[\dfrac{1}{x} \ln(\ln x) + \dfrac{1}{x} \right]$ **Q10:** 0

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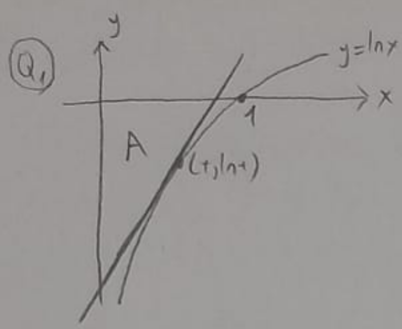
Q1



The derivative of y is $y' = \frac{1}{x}$. At $(t, \ln t)$, the derivative is $y'(t) = \frac{1}{t}$

Recall the straight line formula $y - y_0 = m(x - x_0)$. The slope of the tangent line is the derivative at the point on the curve. Therefore, $m = \frac{1}{t}$

We have $x_0 = t$, $y_0 = \ln t$. The equation of the tangent line is

$$y - \ln t = \frac{1}{t}(x - t)$$

Set $x=0$ to find the y-intercept. $\Rightarrow y - \ln t = \frac{1}{t}(0 - t) \Rightarrow y - \ln t = -1 \Rightarrow y = \ln t - 1$

Set $y=0$ to find the x-intercept. $\Rightarrow 0 - \ln t = \frac{1}{t}(x - t) \Rightarrow x = t - t\ln t$

The area can then be expressed as $A(t) = \frac{1}{2}\left[(t - t\ln t)\underbrace{(1 - \ln t)}_{1 - \ln t > 0}\right]$. Set $A'(t) = 0$ to find the extremum points.

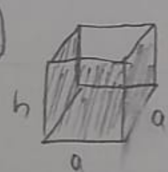
$$A'(t) = \frac{1}{2}\left[\left(1 - 1\cdot\ln t - t\cdot\frac{1}{t}\right)(1 - \ln t) + (t - t\ln t)\left(-\frac{1}{t}\right)\right] = \frac{1}{2}\left[-\ln t + \ln^2 t - 1 + \ln t\right] = \frac{1}{2}(\ln^2 t - 1) = 0$$

$$\Rightarrow \ln^2 t - 1 = 0 \Rightarrow (\ln t - 1)(\ln t + 1) = 0 \Rightarrow t_1 = e,\ t_2 = e^{-1}$$

But $t_1 > 1$. Therefore, we have $t_2 = e^{-1}$.

$$A(t = t_2) = \frac{1}{2}\left[\left(\frac{1}{e} - \left(-\frac{1}{e}\right)\right)\left(1 - (-1)\right)\right] = \boxed{\frac{2}{e}}$$

Q2



Let V be the volume and C be the total cost of material used.

$V = a^2 h = 6 \Rightarrow h = 6/a^2$

$C = 60a^2 + 40\cdot 4ah = 60a^2 + \frac{960}{a}$ Set $C'(a) = 0 \Rightarrow 120a - \frac{960}{a^2} = 0 \Rightarrow a^3 = 8 \Rightarrow a = 2m$

$a = 2 \Rightarrow h = \frac{6}{a^2} = \frac{6}{4} = 1.5m$

The dimensions for the box: $2m \times 2m \times 1.5m$ Total cost: $60\cdot 2^2 + \frac{960}{2} = 720€$

Q3 Take $A = x^y$, then $\ln A = \ln x^y = y\ln x$. Take $B = y^x$, then $\ln B = \ln y^x = x\ln y$

$\frac{d}{dx}(\ln A) = \frac{d}{dx}(y\ln x)$ $\quad$ $\frac{d}{dx}(\ln B) = \frac{d}{dx}(x\ln y)$

$\Rightarrow \frac{1}{A}\cdot A' = \frac{dy}{dx}\cdot\ln x + y\cdot\frac{1}{x}$ $\quad$ $\Rightarrow \frac{1}{B}\cdot B' = 1\cdot\ln y + x\cdot\frac{1}{y}\cdot y'$

$\Rightarrow A' = x^y\left(\frac{dy}{dx}\cdot\ln x + \frac{y}{x}\right)$ $\quad$ $\Rightarrow B' = y^x\left(\ln y + \frac{xy'}{y}\right)$

$$x^y + y^x = 3 \Rightarrow \underbrace{\frac{d}{dx}(x^y)}_{A'} + \underbrace{\frac{d}{dx}(y^x)}_{B'} = \underbrace{\frac{d}{dx}(3)}_{0} \Rightarrow x^y y'\ln x + yx^{y-1} + y^x\ln y + xy'y^{x-1} = 0$$

$$\Rightarrow y'\left[x^y\ln x + xy^{x-1}\right] = -y^x\ln y - yx^{y-1}$$

$$\Rightarrow y' = -\frac{y^x\ln y + yx^{y-1}}{x^y\ln x + xy^{x-1}}$$

We shall find $y'(x=1)$. Evaluate $y(1)$.

$1^y + y^1 = 3 \Rightarrow y = 2 \Rightarrow y'(1) = -\frac{2\ln 2 + 2}{1} = -2\ln 2 - 2$ $\quad$ $\boxed{y'(1) = -2\ln 2 - 2}$

(Q4) Let $f(x)=x^3+x^2+2x+5$. f is continuous and differentiable for all $x\in\mathbb{R}$. Choose two x, say, -2 and 0, $f(-1)=-8+4-4+5=-3$, $f(0)=5$. By IVT, there exists at least one point x_1 such that $f(x_1)=0$ because f takes on any value between $f(-1)$ and $f(0)$.

We now assume there are two roots: x_1 and x_2. Without loss of generality (WLOG), choose $x_1<x_2$. By MVT, there exists at least one point $d\in(x_1,x_2)$ such that

$$f'(d)=\frac{f(x_2)-f(x_1)}{x_2-x_1}=0 \quad [\because f(x_2)=f(x_1)=0]$$ Now, investigate the derivative of f at d.

$$f'(x)=3x^2+2x+2 \Rightarrow f'(d)=3d^2+2d+2 \Rightarrow d_{1,2}=\frac{-2\pm\sqrt{2^2-4\cdot3\cdot2}}{2\cdot3}\notin\mathbb{R} \quad [\because \sqrt{-20}\notin\mathbb{R}]$$

There is no real d that satisfies $f'(d)=0$. However, this is a contradiction. Therefore, our assumption was wrong. By IVT and MVT, there exists only one root.

(Q5) The slope of a horizontal tangent line is zero. Apply implicit differentiation

$$\frac{d}{dx}\left(3y^2x-x^3-y^3-y\right)=0\Rightarrow 3\cdot2y\frac{dy}{dx}x+3y^2-3x^2-3y^2\frac{dy}{dx}-\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}\left[6yx-3y^2-1\right]=3x^2-3y^2$$

$$\Rightarrow\frac{dy}{dx}=\frac{3x^2-3y^2}{6yx-3y^2-1} \quad \text{Set } \frac{dy}{dx}=0\Rightarrow 3x^2-3y^2=0\Rightarrow(x-y)(x+y)=0\Rightarrow \begin{matrix}x=y\\x=-y\end{matrix}$$

$$x=y\Rightarrow 3x^2\cdot x-x^3-x^3-x=0\Rightarrow x(x-1)(x+1)=0 \qquad x=-y\Rightarrow 3x^2\cdot x-y^3+x^3+x=0\Rightarrow x(3x^2+1)=0$$

$$x_1=0,\ x_2=1,\ x_3=-1 \qquad x_4=0$$

$x=0\Rightarrow y=0$, $x=1\Rightarrow y=1$, $x=-1\Rightarrow y=-1$ $\boxed{\text{Points: } \{(0,0),(-1,-1),(1,1)\}}$

(Q6) $V(x)=x^3$ for the cube. If $V(a)=64$, $a^3=64\Rightarrow a=4$. We also have $S(x)=6x^2$

$$\left.\begin{matrix}\dfrac{dV}{dt}=3x^2\cdot\dfrac{dx}{dt}, & \dfrac{dS}{dt}=12x\cdot\dfrac{dx}{dt}\end{matrix}\right\}\Rightarrow\frac{dS}{dt}=12x\cdot\frac{1}{3x^2}\cdot\frac{dV}{dt}=\frac{4}{x}\cdot\frac{dV}{dt},\ \left.\frac{dS}{dt}\right|_{x=4}=\frac{4}{4}\cdot2=\boxed{2\,cm^2/sec}$$

(Q7) $$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right)=\lim_{x\to\infty}\left[\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right)\cdot\frac{\sqrt{x+\sqrt{x}}+\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x}}\right]=\lim_{x\to\infty}\frac{x+\sqrt{x}-x}{\sqrt{x+\sqrt{x}}+\sqrt{x}}=\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{1}{\frac{\sqrt{x+\sqrt{x}}+\sqrt{x}}{\sqrt{x}}}=\lim_{x\to\infty}\frac{1}{\sqrt{1+x^{-1/2}}+1}=\frac{1}{\sqrt{1+0}+1}=\boxed{\frac{1}{2}}$$

Q8 a. $x^2-1\neq 0 \longleftrightarrow x^2\neq 1 \longleftrightarrow x\neq \pm 1$ $\quad D_f=\mathbb{R}\setminus\{-1,1\}$

b. $x=0 \Rightarrow y=\frac{0^2-2}{0^2-1}=2$, $y=0 \Rightarrow 0=\frac{x^2-2}{x^2-1} \Rightarrow x=\pm\sqrt{2}$ $\quad \{(0,2);(\sqrt{2},0),(-\sqrt{2},0)\}$

c. The degree of each polynomial in the equation is equal.

$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x^2-2}{x^2-1}=\lim_{x\to\infty}\frac{1-2/x^2}{1-1/x^2}=1$, $\underset{\text{horizontal}}{\lim_{x\to-\infty} f(x)=1}$ (similarly) horizontal

$\lim_{x\to 1^-} f(x)=\lim_{x\to -1^+} f(x)=\infty$

$\lim_{x\to 1^+} f(x)=\lim_{x\to -1^-} f(x)=-\infty$

vertical

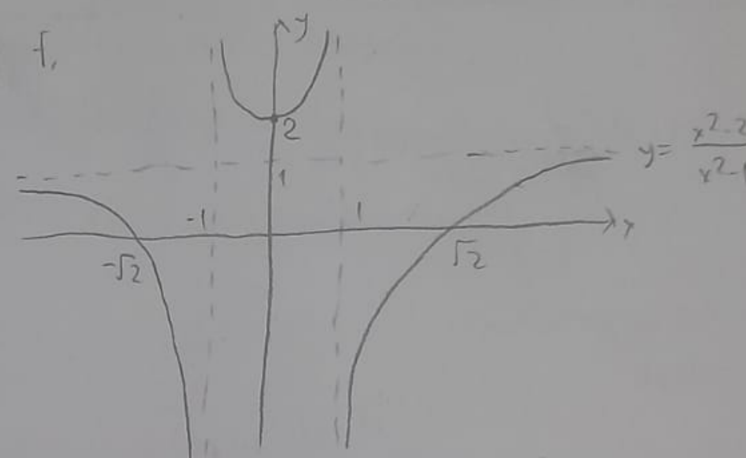
d. $f'(x)=\frac{2x(x^2-1)-(x^2-2)\cdot 2x}{(x^2-1)^2}=\frac{2x^3-2x-2x^3+4x}{(x^2-1)^2}=\frac{2x}{(x^2-1)^2}$

$f''(x)=\frac{2(x^2-1)^2-2x\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}=\frac{2x^2-2-8x^2}{(x^2-1)^3}=\frac{-6x^2-2}{(x^2-1)^3}$

e.

	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	
f'	$-$	$-$	$-$	$+$	$+$	$+$
f''	$-$	$-$	$+$	$-$	$+$	$+$
	↘	↘	↘	↗	↗	↗
	∩	∩	∪	∩	∪	∪

f.



Q9 Apply implicit differentiation and the chain rule.

$y=(\ln x)^{\ln x} \Rightarrow \ln y=\ln\left[(\ln x)^{\ln x}\right] \Rightarrow \ln y=\ln x\left[\ln(\ln x)\right] \Rightarrow \frac{1}{y}\cdot y'=\frac{1}{x}\cdot\ln(\ln x)+\ln x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}$

$\rightarrow y'=(\ln x)^{\ln x}\left[\frac{1}{x}\ln(\ln x)+\frac{1}{x}\right]$

Q10 $\lim_{x\to 1^-}\frac{\ln\sin\pi x}{\csc\pi x}\quad\left[\frac{\infty}{\infty}\right]$

$\overset{LH}{=}\lim_{x\to 1^-}\frac{\frac{1}{\sin\pi x}\cdot\cos\pi x\cdot\pi}{-\csc\pi x\cdot\cot\pi x\cdot\pi}=\lim_{x\to 1^-}-\sin\pi x=-\sin\pi=\boxed{0}$