2022-2023 Fall
MAT123-02,05 Final
(13/01/2023)

1. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$ (b) $\displaystyle\int_0^{\pi/4} x\sec^2 x\,dx$

2.

(a) Consider the finite region between the curves $y = \mathrm{e}^{-x}$, $y = x/\mathrm{e}$, and the y-axis. Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of the solid obtained by rotating this region about the x-axis.

(b) Consider the infinite region between the curves $y = \mathrm{e}^{-x}$, $y = x/\mathrm{e}$ and the x-axis. Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of the solid obtained by rotating this region about the line $y = -1$.

(c) Evaluate the area of the region given in part (a).

3. Determine whether each series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\arctan(n^2)}$ (b) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right)$ (c) $\displaystyle\sum_{n=1}^{\infty} n\mathrm{e}^{-n^2}$

4.

(a) Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^{n+1}\cdot(x+1)^n}{n\cdot 3^n}.$$

(b) Using the formula $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, find the Maclaurin series of the function $f(x) = \dfrac{x^{123}}{1+x^4}$.

2022-2023 Final (13/01/2023) Solutions
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1.

(a) Let $x = 2\sin u$ for $-\dfrac{\pi}{2} < u < \dfrac{\pi}{2}$, then $dx = 2\cos u\, du$.

$$x = 0 \implies 2\sin u = 0 \implies u = 0$$

$$x = 1 \implies 2\sin u = 1 \implies \sin u = \frac{1}{2} \implies u = \arcsin\frac{1}{2} = \frac{\pi}{6}$$

$$\mathrm{I} = \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\, dx = \int_0^{\pi/6} \frac{4\sin^2 u}{\sqrt{4-4\sin^2 u}} \cdot 2\cos u\, du \int_0^{\pi/6} \frac{4\sin^2 u\cos u}{|\cos u|}\, du \qquad [\cos u > 0]$$

$$= \int_0^{\pi/6} 4\sin^2 u\, du = 4\int_0^{\pi/6} \left(1-\cos^2 u\right)\, du = 4\int_0^{\pi/6} \frac{1-\cos 2u}{2}\, du$$

$$= 4\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\Bigg|_0^{\pi/6} = 2u - \sin 2u\,\Bigg|_0^{\pi/6} = \left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) - 0 = \boxed{\frac{\pi}{3} - \frac{\sqrt{3}}{2}}$$

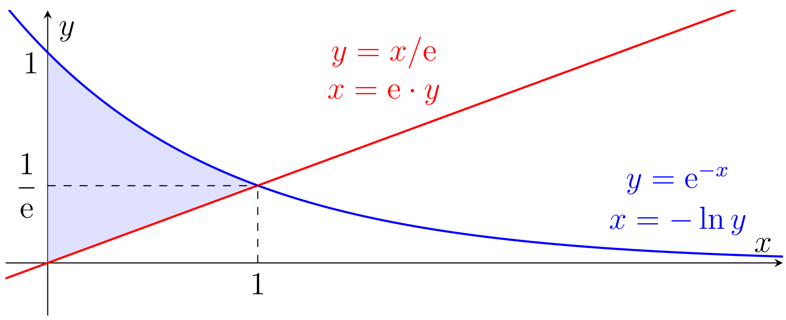
(b) Use the method of integration by parts.

$$\left.\begin{aligned} u = x &\implies du = dx \\ dv = \sec^2 x\, dx &\implies v = \tan x \end{aligned}\right\} \to \int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

$$\int_0^{\pi/4} x\sec^2 x\, dx = x\tan x\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x\, dx = x\tan x + \ln|\cos x|\,\Big|_0^{\pi/4} = \boxed{\frac{\pi}{4} + \ln\frac{\sqrt{2}}{2}}$$

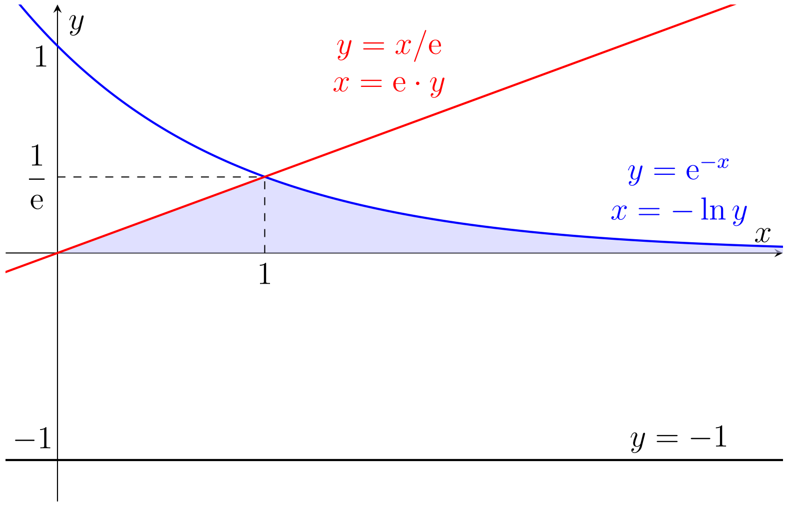
2.

(a)



$$V = \int_D 2\pi \cdot h(y) \cdot r(y)\, dy = \boxed{\int_0^{1/\mathrm{e}} 2\pi \cdot y \cdot (\mathrm{e}\cdot y - 0)\, dy + \int_{1/\mathrm{e}}^1 2\pi \cdot y \cdot (-\ln y - 0)\, dy}$$

(b)



$$V = \int_D \pi\left[r_2^2(x) - r_1^2(x)\right]\,dx = \boxed{\int_0^1 \left[\left(\frac{x}{\mathrm{e}} + 1\right)^2 - 1^2\right]\,dx + \int_1^\infty \left[\left(\mathrm{e}^{-x} + 1\right)^2 - 1^2\right]\,dx}$$

(c)

$$V = 2\pi\left[\int_0^{1/\mathrm{e}} \mathrm{e}y^2\,dy - \int_{1/\mathrm{e}}^1 y\ln y\,dy\right] \tag{1}$$

Calculate the integral on the left in (1).

$$\int_0^{1/\mathrm{e}} \mathrm{e}y^2\,dy = \left.\frac{\mathrm{e}y^3}{3}\right|_0^{1/\mathrm{e}} = \frac{1}{3\mathrm{e}^2}$$

Calculate the integral on the right in (1) by using the method of integration by parts.

$$\left.\begin{aligned} u = \ln x &\implies du = \frac{1}{x}\,dx \\ dv = x\,dx &\implies v = \frac{x^2}{2} \end{aligned}\right\} \to \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$\int_{1/\mathrm{e}}^1 x\ln x\,dx = \left.\frac{x^2\ln x}{2}\right|_{1/\mathrm{e}}^1 - \int_{1/\mathrm{e}}^1 \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \left.\frac{x^2\ln x}{2} - \frac{x^2}{4}\right|_{1/\mathrm{e}}^1$$

$$= \left(0 - \frac{1}{4}\right) - \left(-\frac{1}{2\mathrm{e}^2} - \frac{1}{4\mathrm{e}^2}\right) = \frac{3}{4\mathrm{e}^2} - \frac{1}{4}$$

Therefore,

$$V = 2\pi\left[\left(\frac{1}{3\mathrm{e}^2}\right) - \left(\frac{3}{4\mathrm{e}^2} - \frac{1}{4}\right)\right] = \boxed{\pi\left(\frac{1}{2} - \frac{5}{6\mathrm{e}^2}\right)}$$

3.

(a) Apply the nth Term Test for divergence. The inverse trigonometric function arctan is continuous on $\mathbb{R}$. Therefore, we may take the limit inside the function.

$$\lim_{n\to\infty} \frac{1}{\arctan\left(n^2\right)} = \frac{1}{\arctan\left(\lim\limits_{n\to\infty} n^2\right)} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \neq 1$$

By the nth Term Test for divergence, the series $\sum\limits_{n=1}^{\infty} \frac{1}{\arctan\left(n^2\right)}$ diverges.

(b) Recall the sine inequality $-\theta \leq \sin\theta \leq \theta$. Then for all $n \in \mathbb{R}$ except zero, we have $-\frac{1}{n^3} \leq \sin\left(\frac{1}{n^3}\right) \leq \frac{1}{n^3}$. The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ converges because it is a p-series with $p = 3 > 1$. By the p-series Test, the series converges. The series $\sum\limits_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right)$ also converges by the Direct Comparison Test because $\sin\left(\frac{1}{n^3}\right) < \frac{1}{n^3}$ for every $n \geq 1$.

(c) Take $f(x) = x\mathrm{e}^{-x^2}$. f is continuous because the product of a polynomial and an exponential expression is still continuous. f is positive and decreasing for $x \geq 1$. Verify the monotonicity of f by taking the first derivative.

$$\frac{df}{dx} = 1 \cdot \mathrm{e}^{-x^2} + x\mathrm{e}^{-x^2} \cdot (-2x) = \mathrm{e}^{-x^2}\left(1 - 2x^2\right)$$

$$f'(x) < 0 \quad \text{for} \quad x > \frac{\sqrt{2}}{2} \implies f'(x) < 0 \quad \text{for} \quad x \geq 1$$

We may now apply the Integral Test. Take the limit for the improper integral.

$$\int_1^{\infty} x\mathrm{e}^{-x^2}\,dx = \lim_{R\to\infty} \int_1^{R} x\mathrm{e}^{-x^2}\,dx = \lim_{R\to\infty} -\frac{1}{2}\mathrm{e}^{-x^2}\Bigg|_1^R = \lim_{R\to\infty} -\frac{1}{2}\left(\mathrm{e}^{-R^2} - \mathrm{e}^{-1}\right) = \frac{1}{2\mathrm{e}}$$

The integral converges. Then the series $\sum\limits_{n=1}^{\infty} n\mathrm{e}^{-n^2}$ also converges.

4.

(a) Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\lim_{n\to\infty} \left| \frac{2^{n+2} \cdot (x+1)^{n+1}}{(n+1) \cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x+1)^n \cdot 2^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2n \cdot (x+1)}{(n+1) \cdot 3} \right|$$

$$= \frac{2|x+1|}{3} \cdot \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = \frac{2|x+1|}{3}$$

$$\frac{2|x+1|}{3} < 1 \implies |x+1| < \frac{3}{2}$$

The radius of convergence is $\boxed{\frac{3}{2}}$.

$$|x+1| < \frac{3}{2} \implies -\frac{3}{2} < x+1 < \frac{3}{2} \implies -\frac{5}{2} < x < \frac{1}{2} \quad \text{(convergent)}$$

Investigate the convergence at the endpoints.

$$x = \frac{1}{2} \implies \sum_{n=1}^{\infty} \frac{2^{n+1} \cdot \left(\frac{3}{2}\right)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{2}{n} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n}$$

This is a p-series with $p = 1$, for which the series diverges by the p-series Test. Try the other endpoint.

$$x = -\frac{5}{2} \implies \sum_{n=1}^{\infty} \frac{2^{n+1} \cdot \left(-\frac{3}{2}\right)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^n}{n} = 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{n}$, is nonincreasing for $n \geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges.

The convergence set for the power series is $\boxed{\left[-\frac{5}{2}, \frac{1}{2}\right)}$.

(b)

$$f(x) = \frac{x^{123}}{1+x^4} = x^{123} \cdot \frac{1}{1-(-x^4)} = x^{123} \cdot \sum_{n=0}^{\infty} \left(-x^4\right) = x^{123} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot x^{4n}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \cdot x^{123+4n} = x^{123} - x^{127} + x^{131} - x^{135} + ...}$$