

2022-2023 Spring  
MAT124 Final  
(12/06/2023)

1. Maximize the function  $f(x, y) = xy^2z$  on the sphere  $x^2 + y^2 + z^2 = 4$ .
2. Sketch the region corresponding to the double integral

$$\int_0^1 \int_{x^{1/4}}^1 e^{y^5} dy dx$$

and evaluate it.

3. Sketch the region  $R$  inside the cardioid  $r = 1 + \cos \theta$  and outside the limaçon  $r = 2 - \cos \theta$ , and set up the polar double integral corresponding to the area of the region  $R$ .
4. Using a double integral, find the volume of the solid bounded above by the elliptic paraboloid  $z = 4 - x^2 - y^2$  and below by the circular region  $x^2 + y^2 \leq 2$  in the  $xy$ -plane where  $x \geq 0$  and  $y \geq 0$ .
5. Let us consider the frustum of the cone  $z = \sqrt{x^2 + y^2}$  between the planes  $z = 1$  and  $z = 2$ .
  - (i) Sketch the graph of the frustum.
  - (ii) Find the surface area of the frustum.
6. Let  $S$  be the region in the cylinder  $x^2 + y^2 = 1$  bounded above by the plane  $z = 4$  and below by the sphere  $x^2 + y^2 + z^2 = 1$ .
  - (i) Using the spherical coordinates, set up (but do not evaluate!) an integral for the volume of the solid  $S$ .
  - (ii) Using the cylindrical coordinates, find the volume of the solid  $S$ .

1) Let  $g(x, y, z) = x^2 + y^2 + z^2 - 4$  and then solve the system of equations below using the method of Lagrange multipliers.

$$\left. \begin{array}{l} \nabla f = \lambda \nabla g \\ g(x, y, z) = 0 \end{array} \right\} \quad \begin{array}{l} \nabla f = \langle y^2 z, 2xyz, xy^2 \rangle = \lambda \langle 2x, 2y, 2z \rangle = \lambda \nabla g \\ x^2 + y^2 + z^2 - 4 = 0 \end{array}$$

$$\left. \begin{array}{l} y^2 z = \lambda \cdot 2x \quad (1) \\ 2xyz = \lambda \cdot 2y \quad (2) \\ xy^2 = \lambda \cdot 2z \quad (3) \end{array} \right\} \quad \begin{array}{l} (1) \& (3) \rightarrow \frac{z}{x} = \frac{x}{z} \rightarrow x^2 = z^2 \rightarrow x = \pm z \quad (4) \\ (1) \& (2) \rightarrow \frac{y}{2x} = \frac{x}{y} \rightarrow y^2 = 2x^2 \rightarrow y = \pm \sqrt{2}x \quad (5) \\ (4) \& (5) \rightarrow y = \pm \sqrt{2}z \end{array}$$

Now, use the constraint to find the coordinates. Write  $y$  and  $z$  in terms of  $x$ .

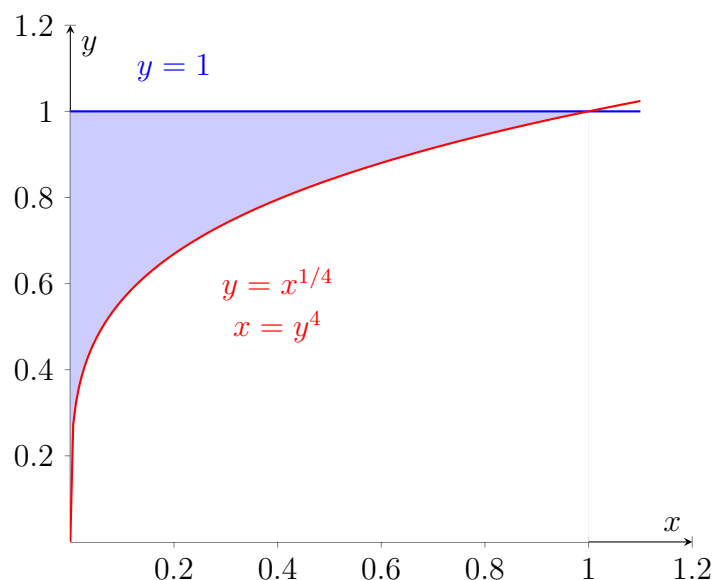
$$x^2 + y^2 + z^2 - 4 = 0 \implies x^2 + (\sqrt{2}x)^2 + x^2 - 4 = 0 \implies 4x^2 = 4 \implies x = \pm 1$$

$$\therefore y = \pm \sqrt{2}, \quad z = \pm 1$$

Evaluate  $f$  at either of these points:  $(1, \sqrt{2}, 1)$ ,  $(-1, \sqrt{2}, -1)$ ,  $(-1, -\sqrt{2}, -1)$ .

$$f_{\max} = f(1, \sqrt{2}, 1) = 1 \cdot (\sqrt{2})^2 \cdot 1 = \boxed{2}$$

2)

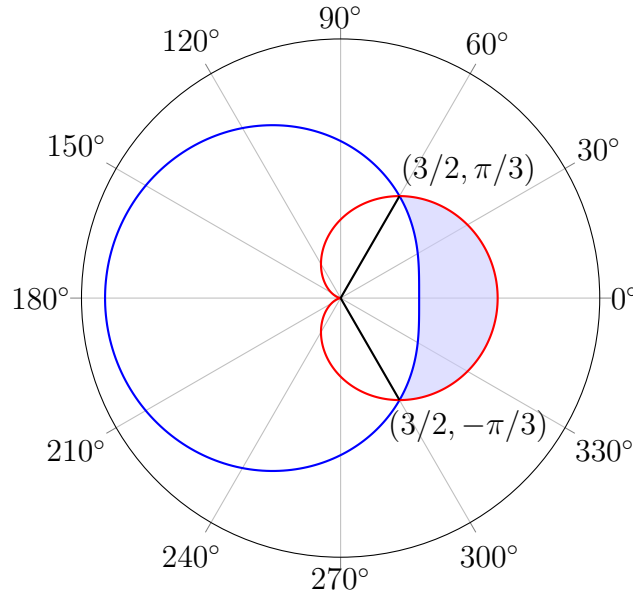


The integral given with this order is difficult to evaluate. Change the order of integration.

$$\int_0^1 \int_{x^{1/4}}^1 e^{y^5} dy dx = \int_0^1 \int_0^{y^4} e^{y^5} dx dy = \int_0^1 y^4 e^{y^5} dy = \left[ \frac{1}{5} e^{y^5} \right]_0^1 = \boxed{\frac{e-1}{5}}$$

3) Find where these two curves intersect and then find the area.

$$2 - \cos \theta = 1 + \cos \theta \implies 2 \cos \theta = 1 \implies \cos \theta = \frac{1}{2} \implies \theta = 2k\pi \pm \frac{\pi}{3}, \quad k \in \mathbb{Z}$$



$$\begin{aligned} \text{Area} &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} [(1 + \cos \theta)^2 - (2 - \cos \theta)^2] d\theta \\ &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} [1 + 2 \cos \theta + \cos^2 \theta - (4 - 4 \cos \theta + \cos^2 \theta)] d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (6 \cos \theta - 3) d\theta \\ &= \frac{1}{2} \left[ 6 \sin \theta - 3\theta \right]_{-\pi/2}^{\pi/2} = \frac{1}{2} \left[ 6 \sin \frac{\pi}{2} - 3 \cdot \frac{\pi}{2} - \left( 6 \sin \left( -\frac{\pi}{2} \right) + \frac{3\pi}{2} \right) \right] = \boxed{6 - \frac{3\pi}{2}} \end{aligned}$$

4) So, the upper bound is  $z = 4 - x^2 - y^2$ , while the lower bound is  $z = 0$ . If we project the domain onto the  $xy$ -plane, we see that the upper and lower bounds for  $y$  are  $\sqrt{2 - x^2}$  and 0, respectively. For  $x$ , the integration starts from 0 and ends at  $\sqrt{2}$ . The volume of the object can be evaluated using the following integral.

$$I = \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} [4 - x^2 - y^2 - 0] dy dx$$

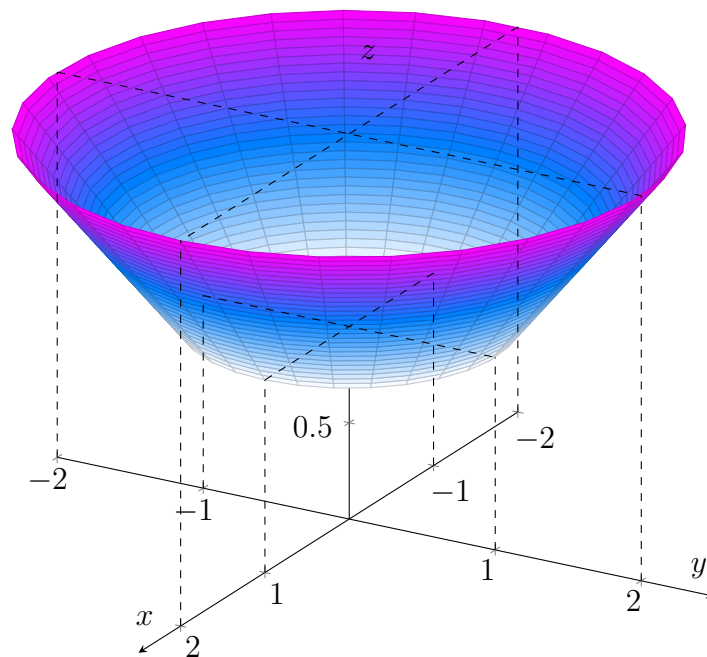
This integral seems a little bit hard. We can switch to polar coordinates for ease. Use the transformation below.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ dA &= dy dx = r dr d\theta \end{aligned} \quad \rightarrow \quad \begin{aligned} 0 &\leq z \leq 4 - r^2 \\ 0 &\leq r \leq \sqrt{2} \\ 0 &\leq \theta \leq \pi/2 \end{aligned}$$

$$\begin{aligned}
I &= \int_0^{\pi/2} \int_0^{\sqrt{2}} (4 - r^2) r \, dr \, d\theta = \int_0^{\pi/2} \int_0^{\sqrt{2}} (4r - r^3) \, dr \, d\theta = \int_0^{\pi/2} \left[ 2r^2 - \frac{r^4}{4} \right]_{r=0}^{r=\sqrt{2}} d\theta \\
&= \int_0^{\pi/2} 3 \, d\theta = \boxed{\frac{3\pi}{2}}
\end{aligned}$$

5)

(i)



(ii) Using the double integral below, we find the lateral surface area.

$$\begin{aligned}
A &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA = \iint_D \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2} \, dA \\
&= \iint_D \sqrt{1 + \frac{x^2 + y^2}{x^2 + y^2}} \, dA = \iint_D \sqrt{1 + 1} \, dA = \sqrt{2} \iint_D \, dA
\end{aligned}$$

If we switch to polar coordinates, we can easily evaluate the integral.

$$A = \sqrt{2} \int_0^{2\pi} \int_1^2 r \, dr \, d\theta = \sqrt{2} \int_0^{2\pi} \left[ \frac{r^2}{2} \right]_{r=1}^{r=2} d\theta = \sqrt{2} \int_0^{2\pi} \frac{3}{2} \, d\theta = \boxed{3\pi\sqrt{2}}$$

6)

(i) For spherical coordinates, we have

$$\begin{array}{lcl} \begin{array}{l} z = \rho \cos \theta \\ r = \rho \sin \theta \\ x^2 + y^2 + z^2 = \rho^2 \\ dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{array} & \rightarrow & \begin{array}{l} x^2 + y^2 = 1 \rightarrow \rho^2 \sin^2 \phi = 1 \\ z = 4 \rightarrow \rho \cos \phi = 4 \\ z = \sqrt{1 - x^2 - y^2} \rightarrow \rho \cos \phi = \sqrt{1 - \rho^2 \sin^2 \phi} \end{array} \end{array}$$

We have the lower bound for  $\rho$ , which is the solution of  $\rho \cos \phi = \sqrt{1 - \rho^2 \sin^2 \phi}$ .

$$\rho \cos \phi = \sqrt{1 - \rho^2 \sin^2 \phi} \implies \rho^2 \cos^2 \phi = 1 - \rho^2 \sin^2 \phi \implies \rho^2 (\cos^2 \phi + \sin^2 \phi) = 1 \\ \rho^2 = 1 \implies \rho = 1$$

However, we have two distinct upper bounds for  $\rho$ . We need to find the value of  $\phi$  where the surfaces  $\rho \cos \phi = 4$  and  $\rho^2 \sin^2 \phi = 1$  intersect.

$$\begin{array}{l} \rho^2 \sin^2 \phi = 1 \rightarrow \rho \sin \phi = 1 \\ \left. \begin{array}{l} \rho \cos \phi = 4 \\ \rho \sin \phi = 1 \end{array} \right\} \cot \phi = \implies \phi = \cot^{-1}(4) \end{array}$$

For  $\phi < \cot^{-1}(4)$ , the upper bound is  $\frac{4}{\cos \phi}$ . Meanwhile, for  $\phi > \cot^{-1}(4)$ , it is  $\frac{1}{\sin \phi}$ .

The region in the  $xy$ -plane is circular, therefore  $0 \leq \theta \leq 2\pi$ . As for  $\phi$ , we have  $0 \leq \phi \leq \frac{\pi}{2}$ . The volume of the object in spherical coordinates can be expressed as follows.

$$V = \int_0^{2\pi} \int_0^{\cot^{-1}(4)} \int_1^{\frac{4}{\cos \phi}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\cot^{-1}(4)}^{\pi/2} \int_1^{\frac{1}{\sin \phi}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

In fact, we are looking for the minimum value of the upper bound for  $\rho$  between  $\frac{1}{\sin \phi}$  and  $\frac{4}{\cos \phi}$ . Hence, we can write the equivalent expression as follows.

$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_1^{\min\left(\frac{4}{\cos \phi}, \frac{1}{\sin \phi}\right)} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

(ii) For cylindrical coordinates, we have

$$\begin{array}{lcl} \begin{array}{l} z = z \\ r^2 = x^2 + y^2 \\ dV = r \, dz \, dr \, d\theta \end{array} & \rightarrow & \begin{array}{l} x^2 + y^2 = 1 \rightarrow r^2 = 1 \rightarrow r = 1 \\ z = 4 \\ z = \sqrt{1 - x^2 - y^2} \rightarrow z = \sqrt{1 - r^2} \end{array} \end{array}$$

The volume can be expressed as follows.

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^1 \int_{\sqrt{1-r^2}}^4 r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 [z]_{z=\sqrt{1-r^2}}^{z=4} r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 (4r - r\sqrt{1-r^2}) \, dr \, d\theta = \int_0^{2\pi} \left[ 2r^2 + \frac{1}{3} (1-r^2)^{3/2} \right]_{r=0}^{r=1} d\theta \\
&= [\theta]_0^{2\pi} \cdot \left[ (2 \cdot 1^2 + 0) - \left( \frac{1}{3} + 0 \right) \right] = \boxed{\frac{10\pi}{3}}
\end{aligned}$$