2022-2023 Spring
MAT124 Final
(12/06/2023)

1. Maximize the function $f(x, y) = xy^2z$ on the sphere $x^2 + y^2 + z^2 = 4$.

2. Sketch the region corresponding to the double integral

$$\int_0^1 \int_{x^{1/4}}^1 \mathrm{e}^{y^5}\, dy\, dx$$

and evaluate it.

3. Sketch the region R inside the cardioid $r = 1 + \cos\theta$ and outside the limaçon $r = 2 - \cos\theta$, and set up the polar double integral corresponding to the area of the region R.

4. Using a double integral, find the volume of the solid bounded above by the elliptic paraboloid $z = 4 - x^2 - y^2$ and below by the circular region $x^2 + y^2 \leq 2$ in the xy-plane where $x \geq 0$ and $y \geq 0$.

5. Let us consider the frustum of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.

(i) Sketch the graph of the frustum.

(ii) Find the surface area of the frustum.

6. Let S be the region in the cylinder $x^2 + y^2 = 1$ bounded above by the plane $z = 4$ and below by the sphere $x^2 + y^2 + z^2 = 1$.

(i) Using the spherical coordinates, set up (but do not evaluate!) an integral for the volume of the solid S.

(ii) Using the cylindrical coordinates, find the volume of the solid S.

2022-2023 Spring Final (12/06/2023) Solutions
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1) Let $g(x,y,z) = x^2 + y^2 + z^2 - 4$ and then solve the system of equations below using the method of Lagrange multipliers.

$$\left.\begin{array}{l}\nabla f = \lambda \nabla g \\ g(x,y,z) = 0\end{array}\right\} \quad \begin{array}{l}\nabla f = \langle y^2 z, 2xyz, xy^2\rangle = \lambda\,\langle 2x, 2y, 2z\rangle = \lambda \nabla g \\ x^2 + y^2 + z^2 - 4 = 0\end{array}$$

$$\left.\begin{array}{ll}y^2 z = \lambda \cdot 2x & (1) \\ 2xyz = \lambda \cdot 2y & (2) \\ xy^2 = \lambda \cdot 2z & (3)\end{array}\right\} \quad \begin{array}{ll}(1)\,\&\,(3) \to \dfrac{z}{x} = \dfrac{x}{z} \to x^2 = z^2 \to x = \pm z & (4) \\ (1)\,\&\,(2) \to \dfrac{y}{2x} = \dfrac{x}{y} \to y^2 = 2x^2 \to y = \pm\sqrt{2}x & (5) \\ (4)\,\&\,(5) \to y = \pm\sqrt{2}z & \end{array}$$

Now, use the constraint to find the coordinates. Write y and z in terms of x.

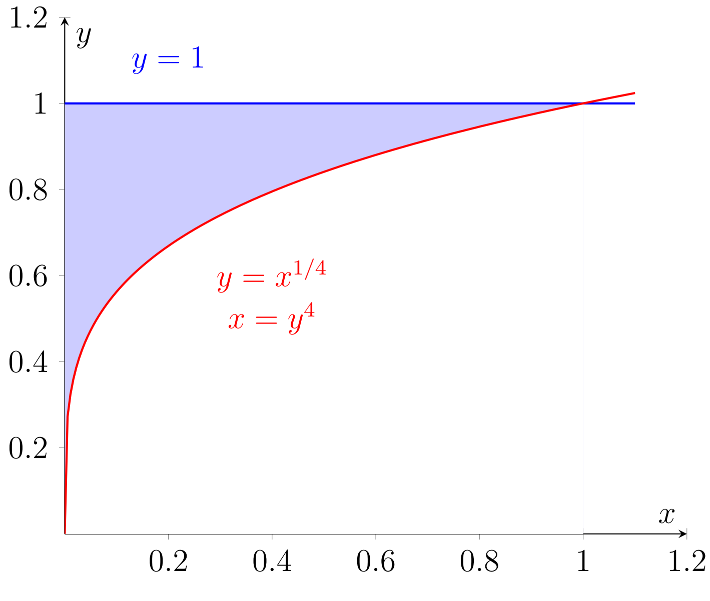
$$x^2 + y^2 + z^2 - 4 = 0 \implies x^2 + \left(\sqrt{2}x\right)^2 + x^2 - 4 = 0 \implies 4x^2 = 4 \implies x = \pm 1$$

$$\therefore y = \pm\sqrt{2}, \quad z = \pm 1$$

Evaluate f at either of these points: $(1, \sqrt{2}, 1)$, $(-1, \sqrt{2}, -1)$, $(-1, -\sqrt{2}, -1)$.

$$f_{\max} = f(1, \sqrt{2}, 1) = 1 \cdot \left(\sqrt{2}\right)^2 \cdot 1 = \boxed{2}$$

2)

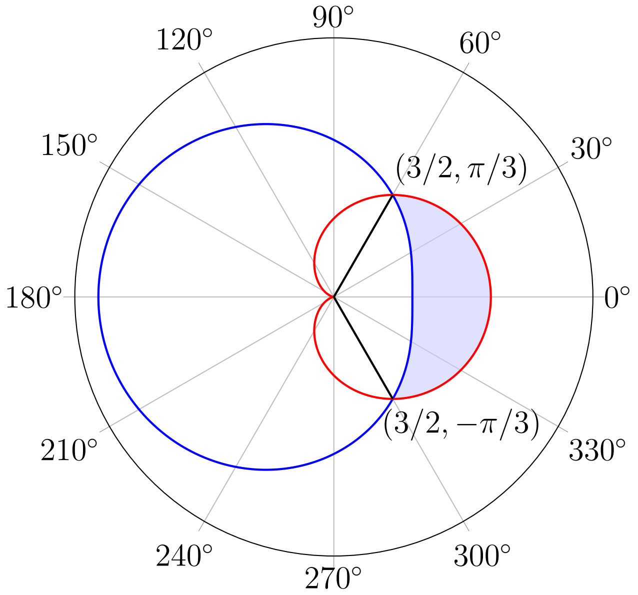


The integral given with this order is difficult to evaluate. Change the order of integration.

$$\int_0^1 \int_{x^{1/4}}^1 \mathrm{e}^{y^5}\, dy\, dx = \int_0^1 \int_0^{y^4} \mathrm{e}^{y^5}\, dx\, dy = \int_0^1 y^4 \mathrm{e}^{y^5}\, dy = \left[\frac{1}{5}\mathrm{e}^{y^5}\right]_0^1 = \boxed{\dfrac{\mathrm{e}-1}{5}}$$

3) Find where these two curves intersect and then find the area.

$$2 - \cos\theta = 1 + \cos\theta \implies 2\cos\theta = 1 \implies \cos\theta = \frac{1}{2} \implies \theta = 2k\pi \pm \frac{\pi}{3}, \quad k \in \mathbb{Z}$$



$$\text{Area} = \int_{-\pi/3}^{\pi/3}\int_{2-\cos\theta}^{1+\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(1+\cos\theta)^2 - (2-\cos\theta)^2\right]\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[1 + 2\cos\theta + \cos^2\theta - \left(4 - 4\cos\theta + \cos^2\theta\right)\right]\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(6\cos\theta - 3)\,d\theta$$

$$= \frac{1}{2}\Big[6\sin\theta - 3\theta\Big]_{-\pi/3}^{\pi/3} = \frac{1}{2}\left[6\sin\frac{\pi}{3} - 3\cdot\frac{\pi}{3} - \left(6\sin\left(-\frac{\pi}{3}\right) + 3\cdot\frac{\pi}{3}\right)\right] = \boxed{3\sqrt{3} - \pi}$$

4) The upper bound is $z = 4 - x^2 - y^2$, while the lower bound is $z = 0$. If we project the domain onto the xy-plane, we see that the upper and lower bounds for y are $\sqrt{2 - x^2}$ and 0, respectively. For x, the integration starts from 0 and ends at $\sqrt{2}$. The volume of the object can be evaluated using the following integral.

$$\mathrm{I} = \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\left[4 - x^2 - y^2 - 0\right]\,dy\,dx$$

This integral seems a little bit hard. We can switch to polar coordinates for ease. Use the transformation below.

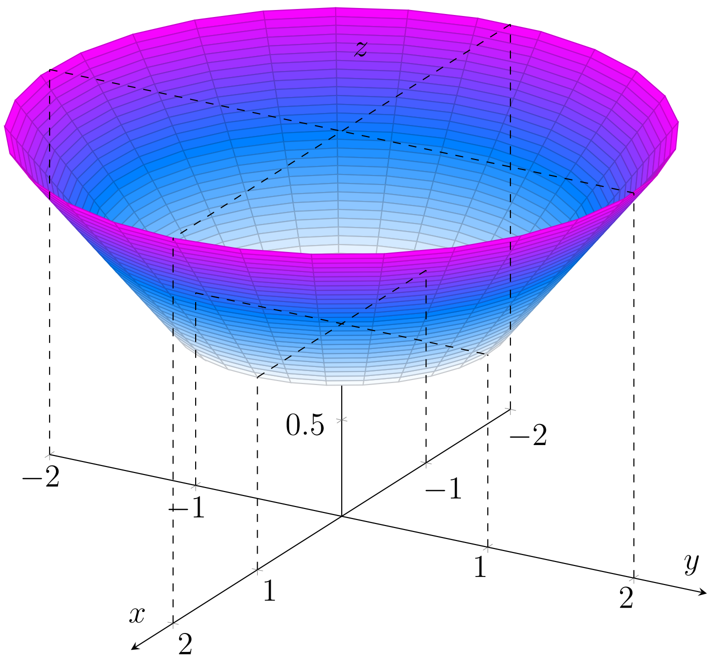
$$\begin{array}{c} x^2 + y^2 = r^2 \\ dA = dy\,dx = r\,dr\,d\theta \end{array} \quad \rightarrow \quad \begin{array}{c} 0 \le z \le 4 - r^2 \\ 0 \le r \le \sqrt{2} \\ 0 \le \theta \le \pi/2 \end{array}$$

$$\mathrm{I} = \int_0^{\pi/2} \int_0^{\sqrt{2}} \left(4 - r^2\right) r \, dr \, d\theta = \int_0^{\pi/2} \int_0^{\sqrt{2}} \left(4r - r^3\right) dr \, d\theta = \int_0^{\pi/2} \left[2r^2 - \frac{r^4}{4}\right]_{r=0}^{r=\sqrt{2}} d\theta$$

$$= \int_0^{\pi/2} 3 \, d\theta = \boxed{\frac{3\pi}{2}}$$

5)

(i)



(ii) Using the double integral below, we find the lateral surface area.

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA = \iint_D \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2} \, dA$$

$$= \iint_D \sqrt{1 + \left(\frac{x^2+y^2}{x^2+y^2}\right)} \, dA = \iint_D \sqrt{1+1} \, dA = \sqrt{2} \iint_D dA$$

If we switch to polar coordinates, we can easily evaluate the integral.

$$A = \sqrt{2} \int_0^{2\pi} \int_1^2 r \, dr \, d\theta = \sqrt{2} \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r=1}^{r=2} d\theta = \sqrt{2} \int_0^{2\pi} \frac{3}{2} \, d\theta = \boxed{3\pi\sqrt{2}}$$

6)

(i) For spherical coordinates, we have

$$\begin{matrix} z = \rho\cos\phi \\ r = \rho\sin\phi \\ x^2+y^2+z^2 = \rho^2 \\ dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta \end{matrix} \quad \rightarrow \quad \begin{matrix} x^2+y^2 = 1 \rightarrow \rho^2\sin^2\phi = 1 \\ z = 4 \rightarrow \rho\cos\phi = 4 \\ z = \sqrt{1-x^2-y^2} \rightarrow \rho\cos\phi = \sqrt{1-\rho^2\sin^2\phi} \end{matrix}$$

We have the lower bound for ρ, which is the solution of $\rho\cos\phi = \sqrt{1-\rho^2\sin^2\phi}$.

$$\rho\cos\phi = \sqrt{1-\rho^2\sin^2\phi} \implies \rho^2\cos^2\phi = 1-\rho^2\sin^2\phi \implies \rho^2\left(\cos^2\phi+\sin^2\phi\right) = 1$$
$$\rho^2 = 1 \implies \rho = 1$$

However, we have two distinct upper bounds for ρ. We need to find the value of ϕ where the surfaces $\rho\cos\phi = 4$ and $\rho^2\sin^2\phi = 1$ intersect.

$$\rho^2\sin^2\phi = 1 \rightarrow \rho\sin\phi = 1$$
$$\left.\begin{matrix} \rho\cos\phi = 4 \\ \rho\sin\phi = 1 \end{matrix}\right\} \quad \cot\phi = \implies \phi = \cot^{-1}(4)$$

For $\phi < \cot^{-1}(4)$, the upper bound is $\dfrac{4}{\cos\phi}$. Meanwhile, for $\phi > \cot^{-1}(4)$, it is $\dfrac{1}{\sin\phi}$.

The region in the $xy-$plane is circular, therefore $0 \le \theta \le 2\pi$. As for ϕ, we have $0 \le \phi \le \dfrac{\pi}{2}$. The volume of the object in spherical coordinates can be expressed as follows.

$$\boxed{V = \int_0^{2\pi}\int_0^{\cot^{-1}(4)}\int_1^{\frac{4}{\cos\phi}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\cot^{-1}(4)}^{\pi/2}\int_1^{\frac{1}{\sin\phi}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

In fact, we are looking for the minimum value of the upper bound for ρ between $\dfrac{1}{\sin\phi}$ and $\dfrac{4}{\cos\phi}$. Hence, we can write the equivalent expression as follows.

$$\boxed{V = \int_0^{2\pi}\int_0^{\pi/2}\int_1^{\min\left(\frac{4}{\cos\phi},\,\frac{1}{\sin\phi}\right)} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

(ii) For cylindrical coordinates, we have

$$\begin{matrix} z = z \\ r^2 = x^2+y^2 \\ dV = r\, dz\, dr\, d\theta \end{matrix} \quad \rightarrow \quad \begin{matrix} x^2+y^2 = 1 \rightarrow r^2 = 1 \rightarrow r = 1 \\ z = 4 \\ z = \sqrt{1-x^2-y^2} \rightarrow z = \sqrt{1-r^2} \end{matrix}$$

The volume can be expressed as follows.

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^1 \int_{\sqrt{1-r^2}}^4 r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \big[z\big]_{z=\sqrt{1-r^2}}^{z=4} r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 \left(4r - r\sqrt{1-r^2}\right) \, dr \, d\theta = \int_0^{2\pi} \left[2r^2 + \frac{1}{3}\left(1-r^2\right)^{3/2}\right]_{r=0}^{r=1} d\theta \\
&= \big[\theta\big]_0^{2\pi} \cdot \left[\left(2 \cdot 1^2 + 0\right) - \left(\frac{1}{3} + 0\right)\right] = \boxed{\frac{10\pi}{3}}
\end{aligned}
$$