

2017-2018 Summer
MAT123-02 Final
(13/08/2018)

1. Let us consider the area A of the region bounded by the curves $y = e^x$, $y = x^2 - 1$ and the straight lines $x = -1$, $x = 1$. Write an integral (but do not evaluate) corresponding to the area A with respect to the y -axis.

2. Sketch the graph of

$$f(x) = \frac{4x}{x^2 + 1}.$$

3. Evaluate the following integrals.

(a) $\int \frac{\tan x}{\sec^4 x} dx$

(b) $\int \frac{\ln x}{x^3} dx$

(c) $\int \frac{1}{1 + \sin x} dx$

(d) $\int \frac{x + 7}{x^2(x + 2)} dx$

4. Use the Shell Method to determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$, $y = x$ about the line $y = 4$.

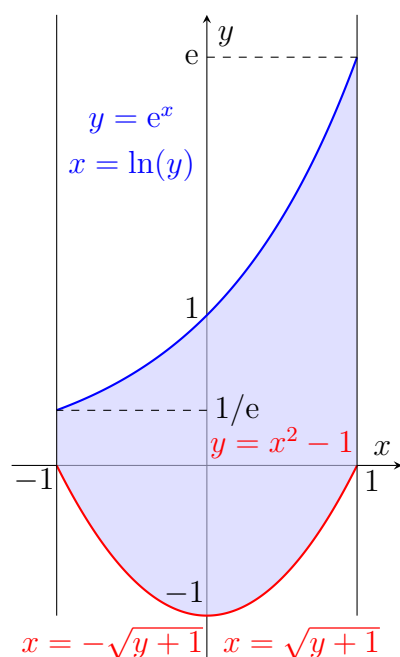
5. Find the area of the surface generated by rotating the curve $y = e^x$, $0 \leq x \leq 1$ about the x -axis.

6. Use the Integral Test to investigate the convergence of the series $\sum_{k=0}^{\infty} ke^{-k}$.

7. Find the Maclaurin series of the function $f(x) = \ln(1 + x)$.

8. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x - 2)^n}{n \cdot 5^n}$.

1.



$$A = \int_{-1}^0 \left[\left(\sqrt{y+1} \right) - \left(-\sqrt{y+1} \right) \right] dy + \int_0^{1/e} [(1) - (-1)] dy + \int_{1/e}^e [(1) - (\ln y)] dy$$

2. The function f is defined where the denominator is not equal to zero. However, the denominator is always positive. Therefore, the domain of f is \mathbb{R} .

Let us find the limit at infinity and negative infinity.

$$\lim_{x \rightarrow \infty} \frac{4x}{x^2 + 1} = \lim_{x \rightarrow \infty} \frac{4}{x + \frac{1}{x}} = 0$$

Similarly, at negative infinity, the limit is zero. Therefore, the x -axis is a horizontal asymptote. There is no vertical asymptote.

Take the first derivative and find the critical points. Apply the quotient rule.

$$y' = \frac{d}{dx} \left(\frac{4x}{x^2 + 1} \right) = \frac{4 \cdot (x^2 + 1) - 4x \cdot (2x)}{(x^2 + 1)^2} = \frac{-4x^2 + 4}{(x^2 + 1)^2}$$

The critical points occur at $x = \pm 1$. At these points, the first derivative is 0.

Take the second derivative. Apply the quotient rule again.

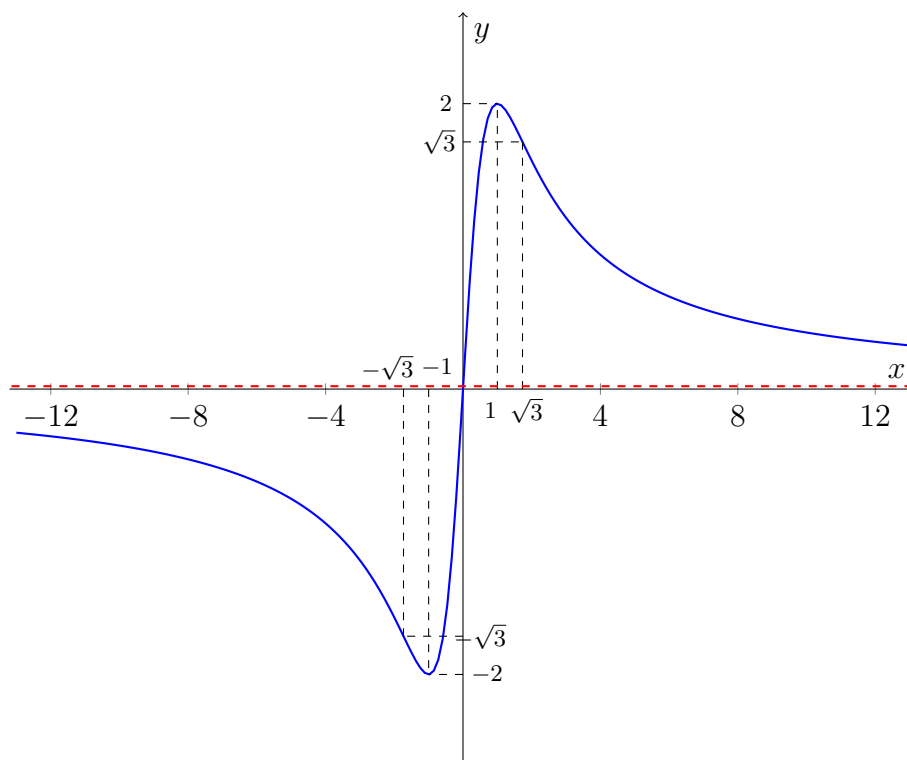
$$y'' = \frac{d}{dx} \left(\frac{-4x^2 + 4}{(x^2 + 1)^2} \right) = \frac{(-8x) \cdot (x^2 + 1)^2 - (-4x^2 + 4) \cdot 2(x^2 + 1) \cdot (2x)}{(x^2 + 1)^4} = \frac{8x(x^2 - 3)}{(x^2 + 1)^3}$$

The inflection points occur at $x = 0$ and $x = \pm\sqrt{3}$. At these points, the direction of the curvature changes.

Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(-\sqrt{3}) = -\sqrt{3}, \quad f(-1) = -2, \quad f(0) = 0, \quad f(1) = 2, \quad f(\sqrt{3}) = \sqrt{3}$$

x	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
y	$(-\sqrt{3}, 0)$	$(-2, -4\sqrt{3})$	$(-2, 0)$	$(0, 2)$	$(\sqrt{3}, 2)$	$(0, \sqrt{3})$
y' sign	-	-	+	+	-	-
y'' sign	-	+	+	-	-	+



3.

(a) Try to rewrite in the form of $\sin x$ and $\cos x$.

$$\int \frac{\tan x}{\sec^4 x} dx = \int \frac{\sin x}{\cos x} \cdot \cos^4 x dx = \int \sin x \cdot \cos^3 x dx = \int \sin x \cdot (1 - \sin^2 x) \cdot \cos x dx$$

Let $u = \sin x$, then $du = \cos x dx$.

$$\int \sin x \cdot (1 - \sin^2 x) \cdot \cos x dx = \int u (1 - u^2) du = \int (u - u^3) du = \frac{u^2}{2} - \frac{u^4}{4} + c$$

$$= \boxed{\frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + c, \quad c \in \mathbb{R}}$$

(b) Use the method of integration by parts.

$$\left. \begin{aligned} u = \ln x &\implies du = \frac{1}{x} dx \\ dv = \frac{1}{x^3} dx &\implies v = -\frac{1}{2x^2} \end{aligned} \right\} \rightarrow \int u dv = uv - \int v du$$

$$\int \frac{\ln x}{x^3} dx = -\frac{\ln x}{2x^2} - \int -\frac{1}{2x^3} dx = \boxed{-\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c, \quad c \in \mathbb{R}}$$

(c) Expand the expression by multiplying and dividing by the conjugate of the denominator.

$$\int \frac{1}{1 + \sin x} dx = \int \frac{1 - \sin x}{(1 + \sin x)(1 - \sin x)} dx = \int \frac{1 - \sin x}{1 - \sin^2 x} dx = \int \frac{1 - \sin x}{\cos^2 x} dx$$

$$= \int (\sec^2 x - \tan x \sec x) dx = \boxed{\tan x - \sec x + c, \quad c \in \mathbb{R}}$$

(d) Use the method of partial fraction decomposition.

$$\int \frac{x+7}{x^2(x+2)} dx = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} \right) dx$$

$$\begin{aligned} A(x)(x+2) + B(x+2) + C(x^2) &= x+7 \\ x^2(A+C) + x(2A+B) + 2B &= x+7 \end{aligned}$$

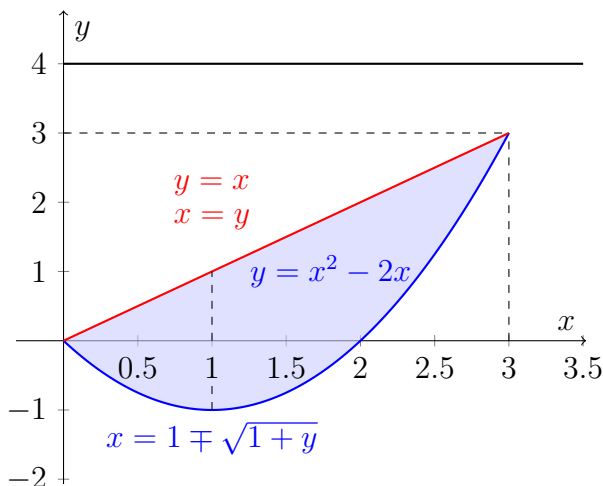
Equate the coefficients of like terms.

$$\left. \begin{aligned} A+C &= 0 \\ 2A+B &= 1 \\ 2B &= 7 \end{aligned} \right\} \implies A = -\frac{5}{4}, \quad B = \frac{7}{2}, \quad C = \frac{5}{4}$$

Rewrite the integral by substituting the values into the unknowns.

$$\int \left(-\frac{5}{4x} + \frac{7}{2x^2} + \frac{5}{4(x+2)} \right) dx = \boxed{-\frac{5}{4} \ln |x| - \frac{7}{2x} + \frac{5}{4} \ln |x+2| + c, \quad c \in \mathbb{R}}$$

4.



$$V = \int_D 2\pi \cdot h(y) \cdot r(y) dy$$

$$\begin{aligned}
V &= \int_{-1}^0 2\pi(4-y) \left[(1+\sqrt{1+y}) - (1-\sqrt{1+y}) \right] dy \\
&\quad + \int_0^3 2\pi(4-y) \left[(1+\sqrt{1+y}) - (y) \right] dy \\
&= 2\pi \int_{-1}^0 (4-y) (2\sqrt{1+y}) dy + 2\pi \int_0^3 (4-y) (1+\sqrt{1+y}-y) dy \quad (1)
\end{aligned}$$

Evaluate the first integral in (1). Let $u = 1 + y$, then $du = dy$.

$$y = -1 \implies u = 0, \quad y = 0 \implies u = 1$$

$$\begin{aligned}
I_1 &= \int_{-1}^0 (4-y) (2\sqrt{1+y}) dy = 2 \int_0^1 (5-u)\sqrt{u} du = 10 \int_0^1 \sqrt{u} du - 2 \int_0^1 u^{3/2} du \\
&= \frac{20}{3} u^{3/2} - \frac{4}{5} u^{5/2} \Big|_0^1 = \frac{20}{3} - \frac{4}{5} - 0 = \frac{88}{15}
\end{aligned}$$

Evaluate the second integral in (1). Let $u = 1 + y$, then $du = dy$.

$$y = 0 \implies u = 1, \quad y = 3 \implies u = 4$$

$$\begin{aligned}
I_2 &= \int_0^3 (4-y) (1+\sqrt{1+y}-y) dy = \int_1^4 (5-u) (-u+2+\sqrt{u}) du \\
&= \int_1^4 (-7u+10+u^2+5\sqrt{u}-u^{3/2}) du = \left[-\frac{7u^2}{2} + 10u + \frac{u^3}{3} + \frac{10}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right]_1^4 \\
&= \left[\left(-56 + 40 + \frac{64}{3} + \frac{80}{3} - \frac{64}{5} \right) - \left(-\frac{7}{2} + 10 + \frac{1}{3} + \frac{10}{3} - \frac{2}{5} \right) \right] = \frac{283}{30}
\end{aligned}$$

Therefore, the result is

$$2\pi \left(\frac{88}{15} + \frac{283}{30} \right) = \boxed{\frac{153\pi}{5}}$$

5. If the function $y = f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the x -axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

$\frac{dy}{dx} = e^x$. Set $a = 0$, $b = 1$, and then evaluate the integral.

$$S = \int_0^1 2\pi e^x \sqrt{1 + (e^x)^2} dx$$

Let $u = e^x$, then $du = e^x dx$.

$$x = 0 \implies u = e^0 = 1, \quad x = 1 \implies u = e^1 = e$$

$$S = \int_0^1 2\pi e^x \sqrt{1 + (e^x)^2} dx = 2\pi \int_1^e \sqrt{1 + u^2} du$$

We will now use a trigonometric substitution. Let $u = \tan t$ for $0 < t < \frac{\pi}{2}$, then $du = \sec^2 t dt$.

$$\begin{aligned} S &= 2\pi \int_1^e \sqrt{1 + u^2} du = 2\pi \int \sqrt{1 + \tan^2 t} \cdot \sec^2 t dt = 2\pi \int \sqrt{\sec^2 t} \cdot \sec^2 t dt \\ &= 2\pi \int |\sec t| \sec^2 t dt = 2\pi \int \sec^3 t dt \quad [\sec t > 0] \end{aligned}$$

Find the antiderivative of $\sec^3 t$ with the help of integration by parts.

$$\begin{aligned} w &= \sec t \rightarrow dw = \sec t \tan t dt \\ dz &= \sec^2 t dt \rightarrow z = \tan t \end{aligned}$$

$$\begin{aligned} \int \sec^3 t du &= \tan t \cdot \sec t - \int \tan^2 t \sec t dt = \tan t \cdot \sec t - \int \frac{1 - \cos^2 t}{\cos^3 t} dt \\ &= \tan t \cdot \sec t - \int \sec^3 t dt + \int \sec t dt \end{aligned}$$

Notice that the integral appears on the right side of the equation. Therefore,

$$\int \sec^3 t dt = \frac{1}{2} \cdot \tan t \cdot \sec t + \frac{1}{2} \cdot \int \sec t dt$$

The integral of $\sec t$ with respect to t is

$$\int \sec t dt = \ln |\tan t + \sec t| + c_1, \quad c_1 \in \mathbb{R}$$

Recall $u = \tan t$.

$$u = \tan t \implies u^2 = \tan^2 t = \sec^2 t - 1 \implies \sec t = \sqrt{u^2 + 1}$$

$$S = 2\pi \cdot \frac{1}{2} (\tan t \cdot \sec t + \ln |\tan t + \sec t|) + c = \pi \left[u \cdot \sqrt{u^2 + 1} + \ln \left| t + \sqrt{u^2 + 1} \right| \right]_1^e$$

$$S = \pi \left[\left(e \cdot \sqrt{e^2 + 1} + \ln \left| e + \sqrt{e^2 + 1} \right| \right) - \left(\sqrt{2} + \ln \left| 1 + \sqrt{2} \right| \right) \right]$$

$$= \pi \left[e \cdot \sqrt{e^2 + 1} - \sqrt{2} + \ln \left(\frac{e + \sqrt{e^2 + 1}}{1 + \sqrt{2}} \right) \right]$$

6. Let the corresponding function be $f(x) = xe^{-x}$. f is continuous for $x \geq 0$. f is positive for $x \geq 0$ because $x \geq 0$ and e^{-x} is positive everywhere. The function is also decreasing for $x \geq 1$. Verify this behavior by taking the first derivative of f . Apply the product rule.

$$f'(x) = 1 \cdot e^{-x} - xe^{-x} = (1 - x)e^{-x}$$

$f'(x) > 0$ for $x \geq 1$. The Integral Test states that all the conditions must be satisfied for and after a specific value, for instance $x = 1$. Therefore, set the lower bound $x = 1$ and evaluate the integral. We will exclusively evaluate the first term of the sequence thereafter.

$$\int_1^{\infty} xe^{-x} dx$$

Apply integration by parts and then evaluate the improper integrals by taking the limit.

$$\left. \begin{array}{l} u = x \implies du = dx \\ dv = e^{-x} dx \implies v = -e^{-x} \end{array} \right\} \rightarrow \int u dv = uv - \int v du$$

$$\begin{aligned} \int_1^{\infty} xe^{-x} dx &= \lim_{R \rightarrow \infty} [-xe^{-x}]_1^R - \lim_{P \rightarrow \infty} \int_1^P -e^{-x} dx = \lim_{R \rightarrow \infty} [-xe^{-x}]_1^R - \lim_{P \rightarrow \infty} e^{-x} \Big|_1^P \\ &= \lim_{R \rightarrow \infty} (-Re^{-R} + e^{-1}) - \lim_{P \rightarrow \infty} (e^{-P} - e^{-1}) = \lim_{R \rightarrow \infty} (-Re^{-R}) + 2e^{-1} \end{aligned}$$

To evaluate the limit, we assume that $-Re^{-R}$ is a function of R . After that, put the expression in a form that we can apply L'Hôpital's rule in order to eliminate the form ∞/∞ .

$$\lim_{R \rightarrow \infty} (-Re^{-R}) = \lim_{R \rightarrow \infty} \frac{-R}{\frac{1}{e^{-R}}} \stackrel{\text{L'H.}}{=} \lim_{R \rightarrow \infty} \frac{1}{\frac{1}{e^{-2R}} \cdot (-e^{-R})} = \lim_{R \rightarrow \infty} (-e^{-R}) = 0$$

Since the integral converges to $2e^{-1}$, the series $\sum_{k=1}^{\infty} ke^{-k}$ also converges. The first term of the series in the original question is $0 \cdot e^0 = 0$. The sum of a convergent series and a finite number is still finite. Therefore, the sum $\sum_{k=0}^{\infty} ke^{-k}$ converges.

7. The Maclaurin series of f is as follows.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, $f^{(4)}(0)$ to look for the pattern.

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}, \quad f^{(4)}(x) = -\frac{6}{(1+x)^4}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2, \quad f^{(4)}(0) = -6$$

This is an alternating sequence where the coefficient of each term is the factorial of the subsequent number starting from 0 except for $k = 0$, that is, the first term of the series. At $k = 0$, the first term is 0. So,

$$f^{(k)}(0) = \begin{cases} (-1)^{k-1} \cdot (k-1)!, & \text{if } k > 0 \\ 0, & \text{if } k = 0 \end{cases}$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 + \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot (k-1)!}{k \cdot (k-1)!} x^k$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

8. Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1) \cdot (5)^{n+1}} \cdot \frac{n \cdot 5^n}{(x-2)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2) \cdot n}{(n+1) \cdot 5} \right| = \frac{|x-2|}{5} \cdot \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = \frac{|x-2|}{5}$$

$$\frac{|x-2|}{5} < 1 \implies |x-2| < 5 \implies -5 < x-2 < 5 \implies -3 < x < 7 \quad (\text{convergent})$$

Now, take a look at the endpoints.

$$x = -3 \implies \sum_{n=1}^{\infty} \frac{(-5)^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{n}$, is nonincreasing for $n \geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges. Try $x = 7$.

$$x = 7 \implies \sum_{n=1}^{\infty} \frac{(5)^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This is a p -series with $p = 1$, for which the series diverges by the p -series Test.

The convergence set for the power series is $\boxed{[-3, 7)}$.