**2024-2025 Fall**
**MAT123-02,05 Final**
**(09/01/2025)**

**1.** Evaluate the following integrals.

**(a)** $\displaystyle\int \frac{dx}{2\cos x + 3}$

**(b)** $\displaystyle\int \frac{\sqrt{5-x^2}}{x}\,dx$

**(c)** $\displaystyle\int \frac{dx}{1-x^4}$

**(d)** $\displaystyle\int \ln\left(1+x^2\right)\,dx$

**2.** Consider the region $R$ bounded by the curves $y = x,\ y = x^2$ and $y = x^2/2$.

**(a)** Find the area of the region $R$.

**(b)** Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region $R$ about the $y$-axis.

**(c)** Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of the solid obtained by rotating the region $R$ about the line $y = 2$.

**3.** Use the Integral Test to determine the existence of the sum of the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$.

**4.** Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)^{1/n}$ is convergent or divergent.

**5.** Find the Maclaurin series of $f(x) = \sqrt{e^x}$ and determine the interval of convergence of this series.

**2024-2025 Final (09/01/2025) Solutions**
**(Last update: 21/08/2025 00:12)**

**1.** **(a)** Use the tangent half-angle substitution, which is also called the Weierstrass substitution. Let $t = \tan\left(\frac{x}{2}\right)$. Then

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2}{1+t^2}\,dt$$

$$\int \frac{dx}{2\cos x + 3} = \int \frac{1}{2\left(\frac{1-t^2}{1+t^2}\right)+3} \cdot \frac{2}{1+t^2}\,dt = \int \frac{2}{5+t^2}\,dt = \frac{2}{5}\int \frac{1}{\left(1+\left(\frac{t}{\sqrt{5}}\right)^2\right)}\,dt$$

Let $u = \frac{t}{\sqrt{5}}$, then $\sqrt{5}\,du = dt$.

$$\frac{2}{5}\int \frac{1}{\left(1+\left(\frac{t}{\sqrt{5}}\right)^2\right)}\,dt = \frac{2\sqrt{5}}{5}\int \frac{1}{1+u^2}\,du = \frac{2\sqrt{5}}{5}\arctan u + c = \frac{2\sqrt{5}}{5}\arctan\frac{t}{\sqrt{5}} + c$$

$$= \boxed{\frac{2\sqrt{5}}{5}\arctan\left(\frac{1}{\sqrt{5}}\cdot\tan\left(\frac{x}{2}\right)\right) + c, \quad c \in \mathbb{R}}$$

**(b)** Let $x = \sqrt{5}\sin u$ for $-\frac{\pi}{2} \le u < \frac{\pi}{2}$, then $dx = \sqrt{5}\cos u\,du$.

$$I = \int \frac{\sqrt{5-x^2}}{x}\,dx = \int \frac{\sqrt{5-5\sin^2 u}}{\sqrt{5}\sin u}\cdot\sqrt{5}\cos u\,du \qquad \left[\sin^2 u + \cos^2 u = 1\right]$$

$$= \sqrt{5}\int \sqrt{\cos^2 u}\cot u\,du \qquad [\cos u > 0]$$

$$= \sqrt{5}\int \cos u \cdot \cot u\,du = \sqrt{5}\int \frac{\cos^2 u}{\sin u}\,du = \sqrt{5}\int \frac{1-\sin^2 u}{\sin u}\,du$$

$$= \sqrt{5}\int (\csc u - \sin u)\,du = \sqrt{5}\left(-\ln|\cot u + \csc u| + \cos u\right) + c$$

Recall that $x = \sqrt{5}\sin u$.

$$\sin u = \frac{x}{\sqrt{5}} \implies \sin^2 u = \frac{x^2}{5} \implies \cos^2 u = \frac{5-x^2}{5} \implies \cos u = \frac{\sqrt{5-x^2}}{\sqrt{5}}$$

$$\cot u = \frac{\cos u}{\sin u} = \frac{\sqrt{5-x^2}}{x}, \qquad \csc u = \frac{1}{\sin u} = \frac{\sqrt{5}}{x}$$

Therefore,

$$I = \boxed{-\sqrt{5}\ln\left|\frac{\sqrt{5-x^2}}{x}+\frac{\sqrt{5}}{x}\right| + \sqrt{5-x^2}+c, \quad c\in\mathbb{R}}$$

**(c)** Use partial fractions to compute the integral.

$$I=\int\frac{1}{1-x^4}\,dx=\int\frac{1}{(1-x^2)\,(1+x^2)}\,dx=\int\frac{1}{(1-x)(1+x)\,(1+x^2)}\,dx$$

$$=\int\left(\frac{A}{1-x}+\frac{B}{1+x}+\frac{Cx+D}{1+x^2}\right)\,dx$$

$$\begin{gathered}A(1+x)\,(1+x^2)+B(1-x)\,(1+x^2)+(Cx+D)(1+x)(1-x)=1\\ x^3(A-B-C)+x^2\,(A+B-D)+x\,(A-B+C)+A+B+D=1\end{gathered}$$

Equate the coefficients of like terms.

$$\left.\begin{array}{ll}A-B-C=0 & (1)\\ A+B-D=0 & (2)\\ A-B+C=0 & (3)\\ A+B+D=1 & (4)\end{array}\right\} \qquad \begin{array}{c}(1)\ \&\ (3)\to 2A-2B=0\\ (2)\ \&\ (4)\to 2A+2B=1\\ \\ \therefore A=\dfrac{1}{4},\quad B=\dfrac{1}{4}\\ (1)\to C=0,\quad (2)\to D=\dfrac{1}{2}\end{array}$$

Rewrite the integral.

$$I=\int\left(\frac{A}{1-x}+\frac{B}{1+x}+\frac{Cx+D}{1+x^2}\right)\,dx=\int\left(\frac{1}{4(1-x)}+\frac{1}{4(1+x)}+\frac{1}{2\,(1+x^2)}\right)\,dx$$

$$=\boxed{-\frac{1}{4}\ln|1-x|+\frac{1}{4}\ln|1+x|+\frac{1}{2}\arctan(x)+c,\ c\in\mathbb{R}}$$

**(d)** Use the method of integration by parts.

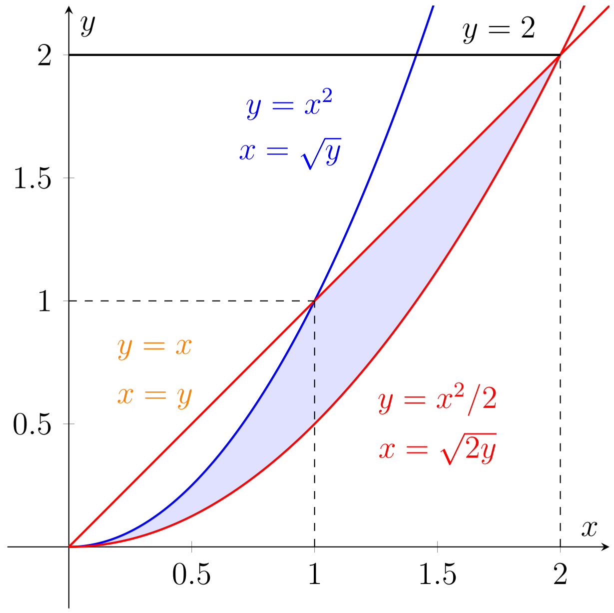
$$\left.\begin{array}{r}u=\ln(1+x^2)\implies du=\dfrac{2x}{1+x^2}\,dx\\ dv=dx\implies v=x\end{array}\right\}\to\int u\,dv=uv-\int v\,du$$

$$\int\ln\left(1+x^2\right)\,dx=x\ln\left(1+x^2\right)-\int\frac{2x^2}{1+x^2}\,dx=x\ln\left(1+x^2\right)-\int\frac{2x^2+2-2}{1+x^2}\,dx$$

$$=x\ln\left(1+x^2\right)-2\int dx+2\int\frac{dx}{1+x^2}$$

$$=\boxed{x\ln\left(1+x^2\right)-2x+2\arctan x+c,\quad c\in\mathbb{R}}$$

**2.**




**(a)**

$$A = \int_0^1 \left(x^2 - \frac{x^2}{2}\right)\, dx + \int_1^2 \left(x - \frac{x^2}{2}\right)\, dx = \left[\frac{x^3}{3} - \frac{x^3}{6}\right]_0^1 + \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_1^2$$

$$= \left[\left(\frac{1}{3} - \frac{1}{6}\right) - 0\right] + \left[\left(2 - \frac{8}{6}\right) - \left(\frac{1}{2} - \frac{1}{6}\right)\right] = \boxed{\frac{1}{2}}$$

**(b)**

$$V = \int_D \pi \left[r_2^2(y) - r_1^2(y)\right]\, dy = \boxed{\int_0^1 \pi \left[\left(\sqrt{2y}\right)^2 - (\sqrt{y})^2\right]\, dy + \int_1^2 \pi \left[\left(\sqrt{2y}\right)^2 - y^2\right]\, dy}$$

**(c)**

$$V = \int_D 2\pi \cdot h(y) \cdot r(y)\, dy = \boxed{\int_0^1 2\pi(2-y)\left(\sqrt{2y} - \sqrt{y}\right) dy + \int_1^2 2\pi(2-y)\left(\sqrt{2y} - y\right) dy}$$

**3.** Take $f(x) = xe^{-x^2}$. $f$ is continuous because the product of a polynomial and an exponential expression is still continuous. $f$ is positive and decreasing for $x \geq 1$. Verify the monotonicity of $f$ by taking the first derivative.

$$\frac{df}{dx} = 1 \cdot e^{-x^2} + xe^{-x^2} \cdot (-2x) = e^{-x^2}\left(1 - 2x^2\right)$$

$$f'(x) < 0 \quad \text{for} \quad x > \frac{\sqrt{2}}{2} \implies f'(x) < 0 \quad \text{for} \quad x \geq 1$$

We may now apply the Integral Test. Take the limit for the improper integral.

$$\int_1^\infty xe^{-x^2}\, dx = \lim_{R\to\infty} \int_1^R xe^{-x^2}\, dx = \lim_{R\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^R = \lim_{R\to\infty} -\frac{1}{2}\left(e^{-R^2} - \mathrm{e}^{-1}\right) = \frac{1}{2e}$$

The integral converges. Then the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ also converges.

**4.** Apply the $n$th Term Test for the non-alternating part. Let $L$ be the value of the limit.

$$L = \lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} \implies \ln(L) = \ln\left[\lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n}\right]$$

We can also take the limit inside the function because ln is continuous on its domain.

$$\ln(L) = \lim_{n\to\infty}\ln\left[\left(\frac{1}{n}\right)^{1/n}\right] = \lim_{n\to\infty}\frac{\ln\left(\frac{1}{n}\right)}{n}$$

To be able to use L'Hôpital's rule, take the corresponding function $f(x) = \dfrac{\ln\left(\frac{1}{x}\right)}{x}$.

$$\lim_{x\to\infty}\frac{\ln\left(\frac{1}{x}\right)}{x} \overset{\text{L'H.}}{=} \lim_{x\to\infty}\frac{x\cdot\left(-\frac{1}{x^2}\right)}{1} = \lim_{x\to\infty}-\frac{1}{x} = 0 \implies \ln(L) = 0 \implies L = 1$$

The limit $\displaystyle\lim_{n\to\infty}(-1)^{n+1}$ does not exist because of the oscillation. So, $\displaystyle\lim_{n\to\infty}(-1)^{n+1}\left(\frac{1}{n}\right)^{1/n}$ does not exist as well. Therefore, by the $n$th Term Test for divergence, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)^{1/n}$ is divergent.

**5.** The Maclaurin series of $f$ is given by $\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$. Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ to look for the pattern.

$$f'(x) = \frac{1}{2\sqrt{e^x}}\cdot e^x = \frac{1}{2}\sqrt{e^x},\quad f''(x) = \frac{1}{4\sqrt{e^x}}\cdot e^x = \frac{1}{4}\sqrt{e^x},\quad f'''(x) = \frac{1}{8\sqrt{e^x}}\cdot e^x = \frac{1}{8}\sqrt{e^x}$$

$$f(0) = 1,\quad f'(0) = \frac{1}{2},\quad f''(0) = \frac{1}{4},\quad f'''(0) = \frac{1}{8}$$

Therefore, $f^{(k)}(0) = \left(\dfrac{1}{2}\right)^k$. Rewrite the summation formula.

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)^k\cdot x^k}{k!} = \boxed{\sum_{k=0}^{\infty}\frac{x^k}{2^k\cdot k!} = 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \ldots}$$

Find the interval of convergence by applying the Ratio Test.

$$\lim_{k\to\infty}\left|\frac{x^{k+1}}{2^{k+1}\cdot(k+1)!}\cdot\frac{2^k\cdot k!}{x^k}\right| = \lim_{k\to\infty}\frac{x}{2k} = 0 < 1$$

$$\boxed{\text{The series is convergent on } \mathbb{R}.}$$