2012-2013 Fall
MAT123-[Instructor02]-02, [Instructor05]-05, Final
(23/01/2013)

1. Determine whether each given sequence with nth term converges or diverges. Evaluate the limit of each convergent sequence. Explain all your work, and write clearly.

(a) $a_n = (-1)^n \, n \sin\left(\dfrac{1}{n}\right)$ (b) $a_n = \mathrm{e}^{\cos\left(\frac{1}{n}\right)}$

2. Determine whether the following series converge or diverge. Give reasons for your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(3+n^2)^{3/4}}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \ln\left(\frac{n+1}{n-1}\right)$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{5^n (n!)^2}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\mathrm{e}^{1/n}$

3. Integrate the following functions and write each step in detail.

(a) $\displaystyle\int \frac{dx}{\mathrm{e}^x + 1}$ (b) $\displaystyle\int x \arcsin x \, dx$

4. Find the length of the curve $y = \displaystyle\int_0^x \sqrt{\cos(4t)}\, dt$ for $0 \le x \le \pi/8$.

5. For the function $f(x) = \dfrac{x}{x^2-4}$,

(a) Find all the asymptotes of f.

(b) Find the intervals of increase or decrease.

(c) Find the local maximum and minimum values, if any.

(d) Find the intervals of concavity and the inflection points, if any.

(e) Sketch the graph of f.

2012-2013 Fall Final (23/01/2013) Solutions
(Last update: 30/08/2025 00:29)

1.

(a) Evaluate the limit of the non-alternating part at infinity.

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \overset{n=\frac{1}{u}}{=} \lim_{u\to 0}\frac{\sin u}{u} = 1$$

However, for odd values of n, the limit at infinity becomes negative. On the other hand, for even values of n, the limit at infinity becomes positive. Therefore, the limit at infinity does not exist. So, the sequence diverges.

(b) The exponential function e^x and the trigonometric function $\cos x$ are continuous everywhere.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \mathrm{e}^{\cos\left(\frac{1}{n}\right)} = \mathrm{e}^{\cos\left(\lim\limits_{n\to\infty}\frac{1}{n}\right)} = \mathrm{e}^{\cos 0} = \mathrm{e}^1 = \mathrm{e}$$

The sequence converges to $\boxed{\mathrm{e}}$.

2.

(a) Let $a_n = f(n)$. Define $f(x) = \dfrac{x}{(3+x^2)^{3/4}}$. The function is continuous for $x > 1$ because the numerator and the denominator are continuous for $x > 1$ and $(3+x^2)^{3/4} \neq 0$, $\forall x \in \mathbb{R}$.

$$\left.\begin{array}{c} x > 0 \\ (3+x^2)^{3/4} > 0 \end{array}\right\} \text{ for } x > 1 \implies \frac{x}{(3+x^2)^{3/4}} > 0$$

For $x > 1$, $x < x^{3/2} = (x^2)^{3/4} < (3+x^2)^{3/4}$. The denominator grows faster than the numerator. Therefore, the function is decreasing.

We may now apply the Integral Test. Handle the improper integral by taking the limit.

$$\int_1^\infty \frac{x}{(3+x^2)^{3/4}}\,dx = \lim_{R\to\infty} 2\left(3+x^2\right)^{1/4}\Big|_1^R = 2\lim_{R\to\infty}\left[(3+R^2)^{1/4} - \left(3+1^2\right)^{1/4}\right] = \infty$$

Since the integral diverges, the series also diverges.

(b) $\ln(1+x) < x$ for $x > -1$. Therefore,

$$\frac{1}{\sqrt{n}}\ln\left(\frac{n+1}{n-1}\right) = \frac{1}{\sqrt{n}}\ln\left(1+\frac{2}{n-1}\right) < \frac{1}{\sqrt{n}}\cdot\frac{2}{n-1}$$

Since $n \geq 2$, we have the inequality $2n-2 \geq n \implies \dfrac{2}{n} \geq \dfrac{1}{n-1}$.

$$\frac{1}{\sqrt{n}}\cdot\frac{2}{n-1}\leq\frac{1}{\sqrt{n}}\cdot\frac{4}{n}=\frac{4}{n^{3/2}}$$

Now, let $a_n=\frac{1}{\sqrt{n}}\ln\left(\frac{n+1}{n-1}\right)$ and $b_n=\frac{4}{n^{3/2}}$. Apply the Direct Comparison Test.

$$0<a_n<b_n\implies 0<\frac{1}{\sqrt{n}}\ln\left(\frac{n+1}{n-1}\right)<\frac{4}{n^{3/2}}$$

b_n converges by the p-series test because $3/2>1$. Since b_n converges, by the Direct Comparison Test, a_n also converges.

(c) Apply the Ratio Test. Let $a_n=\frac{(2n)!}{5^n\left(n!\right)^2}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(2(n+1))!}{5^{n+1}\left((n+1)!\right)^2}\cdot\frac{5^n\left(n!\right)^2}{(2n)!}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(2n+2)\cdot(2n+1)\cdot((2n)!)\cdot(n!)^2\cdot 5^n}{5^n\cdot 5\cdot(n+1)^2\cdot(n!)^2\cdot((2n)!)}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)}{5(n+1)^2}\right|=\lim_{n\to\infty}\left|\frac{4n^2+6n+2}{5n^2+10n+5}\right|$$

Now, take the corresponding function and evaluate the limit using L'Hôpital's rule.

$$\lim_{x\to\infty}\left|\frac{4x^2+6x+2}{5x^2+10x+5}\right|\overset{\text{L'H.}}{=}\lim_{x\to\infty}\left|\frac{8x+6}{10x+10}\right|\overset{\text{L'H.}}{=}\lim_{x\to\infty}\left|\frac{8}{10}\right|=\frac{4}{5}<1$$

By the Ratio Test, the series converges absolutely. Since the series converges absolutely, the series converges.

(d) Let $a_n=\frac{1}{n^2}\mathrm{e}^{1/n}$. Define $f(x)=\frac{1}{x^2}\mathrm{e}^{1/x}$. The function is continuous for $x>1$ because the expressions $\frac{1}{x^2}$ and $\mathrm{e}^{1/x}$ are continuous for $x>1$ and $x^2\neq 0$, $\forall x>1$.

$$\left.\begin{array}{r}x^2>0\\ \mathrm{e}^{1/x}>0\end{array}\right\}\text{ for }x>1\implies\frac{\mathrm{e}^{1/x}}{x^2}>0$$

For $x>1$, $\mathrm{e}^{1/x}$ tends to 1 and x^2 is increasing. Therefore, the function is decreasing for $x>1$.

We may now apply the Integral Test. Handle the improper integral by taking the limit.

$$\int_1^\infty\frac{\mathrm{e}^{1/x}}{x^2}\,dx-=\lim_{R\to\infty}-\mathrm{e}^{1/x}\bigg|_1^\infty=\lim_{R\to\infty}\left(-\mathrm{e}^{1/R}+\mathrm{e}^1\right)=\mathrm{e}-1\quad\text{(convergent)}$$

Since the integral converges, by the Integral Test, the series also converges.

3.

(a) Add and subtract e^x in the numerator.

$$\int \frac{dx}{\mathrm{e}^x+1} = \int \frac{1+\mathrm{e}^x-\mathrm{e}^x}{\mathrm{e}^x+1}\,dx = \int \frac{\mathrm{e}^x+1}{\mathrm{e}^x+1}\,dx - \int \frac{\mathrm{e}^x}{\mathrm{e}^x+1}\,dx$$

The result of the integral on the left is x. To calculate the integral on the right, use the u-substitution. Let $u = \mathrm{e}^x+1$, then $du = \mathrm{e}^x\,dx$.

$$x - \int \frac{du}{u} = x - \ln|u| + c = \boxed{x - \ln|\mathrm{e}^x+1| + c = x - \ln\left(\mathrm{e}^x+1\right) + c,\ c \in \mathbb{R}}\ \left[\mathrm{e}^x+1>0\right]$$

(b) Solve the integral using the method of integration by parts.

$$\left.\begin{aligned} u = \arcsin x &\implies du = \frac{1}{\sqrt{1-x^2}}\,dx \\ dv = x\,dx &\implies v = \frac{x^2}{2} \end{aligned}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int x\arcsin x\,dx = \frac{x^2}{2}\arcsin x - \int \frac{x^2}{2\sqrt{1-x^2}}\,dx \tag{1}$$

Now, we need to find the result of the integral on the right. Let $x = \sin u$ for $\frac{\pi}{2} < u < \frac{\pi}{2}$. Then $dx = \cos u\,du$.

$$\int \frac{x^2}{2\sqrt{1-x^2}}\,dx = \int \frac{\sin^2 u}{2\sqrt{1-\sin^2 u}}\cdot\cos u\,du = \int \frac{\sin^2 u\cos u}{2\sqrt{\cos^2 u}}\,du \quad \left[\sin^2 u + \cos^2 u = 1\right]$$

$$= \int \frac{\sin^2 u\cos u}{2\left|\cos u\right|}\,du \quad \left[\cos u > 0\right]$$

$$= \frac{1}{2}\int \sin^2 u\,du = \frac{1}{2}\int\left(1-\cos^2 u\right)\,du = \frac{1}{2}\int\left(\frac{1-\cos 2u}{2}\right)\,du$$

$$= \frac{u}{4} - \frac{\sin 2u}{8} + c = \frac{u}{4} - \frac{\sin u\cos u}{4} + c, \quad c \in \mathbb{R}$$

Recall the equation $x = \sin u$.

$$x = \sin u \implies \arcsin x = u$$

$$x = \sin u \implies x^2 = \sin^2 u = 1 - \cos^2 u \implies \cos u = \sqrt{1-x^2}$$

Rewrite the result of the last integral.

$$\int \frac{x^2}{2\sqrt{1-x^2}}\,dx = \frac{1}{4}\left(\arcsin x - x\sqrt{1-x^2}\right)$$

Rewrite (1).

$$\int x \arcsin x \, dx = \boxed{\frac{x^2}{2} \arcsin x - \frac{1}{4} \arcsin x + \frac{1}{4} x\sqrt{1-x^2} + c, \quad c \in \mathbb{R}}$$

4. The length of a curve defined by $y = f(x)$ whose derivative is continuous on the interval $a \le x \le b$ can be evaluated using the integral.

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$

Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx} \int_0^x \sqrt{\cos(4t)} \, dt$$

By the Fundamental Theorem of Calculus, $\dfrac{dy}{dx}$ can be rewritten as

$$\frac{dy}{dx} = \sqrt{\cos(4x)}$$

Set $a = 0, \ b = \dfrac{\pi}{8}, \ \dfrac{dy}{dx} = \sqrt{\cos(4x)}$ and then find the length.

$$S = \int_0^{\pi/8} \sqrt{1 + \left(\sqrt{\cos(4x)}\right)^2} \, dx = \int_0^{\pi/8} \sqrt{1 + \cos(4x)} \, dx \quad \left[\cos(4x) = 2\cos^2(2x) - 1\right]$$

$$= \int_0^{\pi/8} \sqrt{2\cos^2(2x)} \, dx = \sqrt{2} \int_0^{\pi/8} |\cos(2x)| \, dx \quad [\cos(2x) > 0]$$

$$= \sqrt{2} \int_0^{\pi/8} \cos(2x) \, dx = \frac{\sqrt{2}}{2} \sin(2x) \Big|_0^{\pi/8} = \frac{\sqrt{2}}{2} \left(\sin \frac{\pi}{4} - \sin 0\right) = \boxed{\frac{1}{2}}$$

5.

(a) Find the horizontal asymptotes.

$$\lim_{x \to \pm\infty} \frac{1}{x^2 - 4} = 0$$

Find the vertical asymptotes. The expression is undefined for $x = \pm 2$.

$$\lim_{x \to 2^+} \frac{1}{x^2 - 4} = \lim_{x \to -2^+} \frac{1}{x^2 - 4} = \infty$$

$$\lim_{x \to 2^-} \frac{1}{x^2 - 4} = \lim_{x \to -2^-} \frac{1}{x^2 - 4} = -\infty$$

$$\boxed{\text{The horizontal asymptote is } y = 0. \text{ The vertical asymptotes are } x = \pm 2.}$$

(b) Compute the first derivative and set it to 0 to find the critical points. Apply the product rule appropriately.

$$f'(x) = \frac{1 \cdot (x^2 - 4) - x \cdot (2x)}{(x^2 - 4)^2} = -\frac{x^2 + 4}{(x^2 - 4)^2}$$

f is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$. Therefore,

$$\boxed{\text{f is decreasing everywhere except at the undefined points.}}$$

(c) $\boxed{\text{No local maximum or minimum values exist.}}$

(d) Compute the second derivative.

$$f''(x) = -\frac{2x \cdot (x^2 - 4)^2 - (x^2 + 4) \cdot 2 \cdot (x^2 - 4) \cdot (2x)}{(x^2 - 4)^4} = \frac{2x^3 + 24x}{(x^2 - 4)^3}$$

$$\boxed{\begin{array}{c}\text{An inflection point occurs at } x = 0. \\ \text{f is concave up for } -2 < x < 0 \cup x > 2. \text{ f is concave down for } x < -2 \cup 0 < x < 2.\end{array}}$$

(e)