Stat 541 Homework #2

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1.

Circuit 1	Circuit 2	Circuit 3	
9	20	6	
12	21	5	
10	23	8	
8	17	16	
15	30	7	
$y_{1.} = 54$	$y_{2.} = 111$	$y_{3.} = 42$	$y_{..} = 207$

(a) Normal equations:

$$\begin{aligned} 207 &= 15\widehat{\mu} + 5\widehat{\tau}_1 + 5\widehat{\tau}_2 + 5\widehat{\tau}_3 && (1) \\ 54 &= 5\widehat{\mu} + 5\widehat{\tau}_1 && (2) \\ 111 &= 5\widehat{\mu} + 5\widehat{\tau}_2 && (3) \\ 42 &= 5\widehat{\mu} + 5\widehat{\tau}_3 && (4) \end{aligned}$$

(b) If $\tau_1 + \tau_2 + \tau_3 = 0$, equation (1) becomes

$$207 = 15\widehat{\mu}$$

so

$$\widehat{\mu} = \frac{207}{15} = 13.8$$

Equation (2) becomes

$$54 = 5 \times 13.8 + \widehat{\tau}_1$$

so

$$\widehat{\tau}_1 = \frac{54 - 69}{5} = -3$$

Equation (3) becomes

$$111 = 5 \times 13.8 + 5\widehat{\tau}_2$$

so

$$\widehat{\tau}_2 = \frac{111 - 69}{5} = 8.4$$

Finally, equation (4) becomes

$$42 = 5 \times 13.8 + 5\widehat{\tau}_3$$

so

$$\widehat{\tau}_3 = \frac{42 - 69}{5} = -5.4$$

(c) If $\tau_2 = 0$, equation (3) becomes

$$111 = 5\widehat{\mu}$$

so

$$\widehat{\mu} = \frac{111}{5} = 22.2$$

Equation (2) becomes

$$54 = 5 \times 22.2 + \widehat{\tau}_1$$

so

$$\widehat{\tau}_1 = \frac{54 - 111}{5} = -11.4$$

And equation (4) becomes

$$42 = 5 \times 22.2 + 5\widehat{\tau}_3$$

so

$$\widehat{\tau}_3 = \frac{42 - 111}{5} = -13.8$$

(d) If $\mu = 5$, equation (2) becomes

$$54 = 5 \times 5 + 5\widehat{\tau}_1$$

so

$$\widehat{\tau}_1 = \frac{54 - 25}{5} = 5.8$$

Equation (3) becomes

$$111 = 5 \times 5 + \widehat{\tau}_2$$

so

$$\widehat{\tau}_2 = \frac{111 - 25}{5} = 17.2$$

And equation (4) becomes

$$42 = 5 \times 5 + 5\widehat{\tau}_3$$

so

$$\widehat{\tau}_3 = \frac{42 - 25}{5} = 3.4$$

(e)

Constraint	$\widehat{\tau}_1 + \widehat{\tau}_2 - 2\widehat{\tau}_3$	$\widehat{\mu} + \widehat{\tau}_1 + \widehat{\tau}_3$
(b)	$-3 + 8.4 - 2 \times (-5.4) = 16.2$	$13.8 - 3 - 5.4 = 5.4$
(c)	$-11.4 + 0 - 2 \times (-13.8) = 16.2$	$22.2 - 11.4 - 13.8 = -3$
(d)	$5.8 + 17.2 - 2 \times (3.4) = 16.2$	$5 + 5.8 + 3.4 = 14.2$

(f) The estimates of $\widehat{\mu} + \widehat{\tau}_1 + \widehat{\tau}_3$ differ because $\widehat{\tau}_3$ is not uniquely estimible. $\widehat{\mu} + \widehat{\tau}_1$ is uniquely estimible, but the value of $\widehat{\tau}_3$ alone depends upon the constraint used.

2. (a)

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad X'X = \begin{bmatrix} 15 & 5 & 5 \\ 5 & 5 & 0 \\ 5 & 0 & 5 \end{bmatrix} \qquad (X'X)^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$X'y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 12 \\ 10 \\ 8 \\ 15 \\ 20 \\ 21 \\ 23 \\ 17 \\ 30 \\ 6 \\ 5 \\ 8 \\ 16 \\ 7 \end{bmatrix} = \begin{bmatrix} 207 \\ 54 \\ 42 \end{bmatrix}$$

(b)

$$\theta = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_3 \end{bmatrix}$$

(c)

$$\widehat{\theta} = \begin{bmatrix} \widehat{\mu} \\ \widehat{\tau}_1 \\ \widehat{\tau}_3 \end{bmatrix} = (X'X)^{-1}X'y = \frac{1}{5}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 207 \\ 54 \\ 42 \end{bmatrix} = \begin{bmatrix} 22.2 \\ -11.4 \\ -13.8 \end{bmatrix}$$

(d)

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \qquad X'X = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 10 & 5 \\ 0 & 5 & 10 \end{bmatrix} \qquad (X'X)^{-1} = \frac{1}{15}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$X'y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 12 \\ 10 \\ 8 \\ 15 \\ 20 \\ 21 \\ 23 \\ 17 \\ 30 \\ 6 \\ 5 \\ 8 \\ 16 \\ 7 \end{bmatrix} = \begin{bmatrix} 207 \\ 12 \\ 69 \end{bmatrix}$$

(e)

$$\theta = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \end{bmatrix}$$

(f)

$$\widehat{\theta} = \begin{bmatrix} \widehat{\mu} \\ \widehat{\tau}_1 \\ \widehat{\tau}_2 \end{bmatrix} = (X'X)^{-1}X'y = \frac{1}{15}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 207 \\ 12 \\ 69 \end{bmatrix} = \begin{bmatrix} 13.8 \\ -3 \\ 8.4 \end{bmatrix}$$

$\widehat{\tau}_3 = -\widehat{\tau}_1 - \widehat{\tau}_2 = 3 - 8.4 = -5.4$

3. (a) The new data are:

Circuit 1	Circuit 2	Circuit 3	
9	20	6	
12	21	5	
10	23	8	
8	17	16	
15	30		
$y_{1.} = 54$	$y_{2.} = 111$	$y_{3.} = 35$	$y_{..} = 200$

The new normal equations are:

$$\begin{aligned} 200 &= 14\widehat{\mu} + 5\widehat{\tau}_1 + 5\widehat{\tau}_2 + 4\widehat{\tau}_3 && (5) \\ 54 &= 5\widehat{\mu} + 5\widehat{\tau}_1 && (6) \\ 111 &= 5\widehat{\mu} \qquad\quad + 5\widehat{\tau}_2 && (7) \\ 35 &= 4\widehat{\mu} \qquad\qquad\qquad + 4\widehat{\tau}_3 && (8) \end{aligned}$$

(b) If $5\tau_1 + 5\tau_2 + 4\tau_3 = 0$, equation (5) becomes

$$200 = 14\widehat{\mu}$$

so

$$\widehat{\mu} = \frac{200}{14} = 14.286$$

Equation (6) becomes

$$54 = 5 \times 14.286 + \widehat{\tau}_1$$

so

$$\widehat{\tau}_1 = \frac{54 - 71.429}{5} = -3.486$$

Equation (7) becomes

$$111 = 5 \times 14.286 + 5\widehat{\tau}_2$$

so

$$\widehat{\tau}_2 = \frac{111 - 71.429}{5} = 7.914$$

Lastly, equation (8) becomes

$$35 = 4 \times 14.286 + 4\widehat{\tau}_3$$

so

$$\widehat{\tau}_3 = \frac{35 - 57.143}{4} = -5.536$$

5.

$$
\begin{aligned}
\frac{-\sum_{i=1}^{a-1} n_i\left(\bar{y}_{i.}-\bar{y}_{..}\right)}{n_a} &= \frac{n_a\left(\bar{y}_{a.}-\bar{y}_{..}\right)-\sum_{i=1}^{a} n_i\left(\bar{y}_{i.}-\bar{y}_{..}\right)}{n_a} \\
&= \left(\bar{y}_{a.}-\bar{y}_{..}\right)-\frac{\left(\sum_{i=1}^{a} n_i\bar{y}_{i.}-\sum_{i=1}^{a} n_i\bar{y}_{..}\right)}{n_a} \\
&= \bar{y}_{a.}-\bar{y}_{..}-\frac{\left(\sum_{i=1}^{a} y_{i.}-N\bar{y}_{..}\right)}{n_a} \\
&= \bar{y}_{a.}-\bar{y}_{..}-\frac{\left(y_{..}-y_{..}\right)}{n_a} \\
&= \bar{y}_{a.}-\bar{y}_{..}
\end{aligned}
$$