

# Spatial log-Gaussian Cox process models and sampling paths: Toward optimal design

Kenneth Flagg<sup>a,\*</sup>, John Borkowski<sup>a</sup>, Andrew Hoegh<sup>a</sup>

<sup>a</sup>*Department of Mathematical Sciences, Montana State University, Bozeman, MT 59717*

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## Abstract

The log-Gaussian Cox (LGCP) process model is becoming the go-to model for spatial point pattern data as fast Bayesian computing methods are removing the barriers to its use. In many applications, such as unexploded ordnance remediation and species distribution studies, a sampling procedure is used in which only events near a one-dimensional path are observed. Very little work has been done regarding optimal design of these paths. We present a variety of path designs and user simulations to compare them in terms of model-based spatial prediction criteria. We also define a spatial coverage statistic for path designs, the average distance-to-path. We show that, when total length is limited, paths with regular spacing and direction changes provide small errors and low posterior variance while being relatively robust against computational artifacts. For longer paths, systematic line-transect designs perform well.

*Keywords:* spatial point process, log-Gaussian Cox process, optimal sampling, model-based design, spatial sampling design

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## 1. Introduction

Spatial point process models have generally been infeasible because of their computational demands, but recent advances in Bayesian computing have made the Log-Gaussian Cox process (LGCP) an attainable model in practice (Rue et al., 2009; Lindgren et al., 2011; Illian et al., 2012; Simpson et al., 2016).

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\*Corresponding author  
Email address: kenneth.flagg@montana.edu (Kenneth Flagg)

These advances make it possible to fit LGCP models easily and quickly, without time-consuming Monte Carlo methods.

In some spatial point process applications, the entire point pattern is not fully observed due to variable sampling effort. This is referred to as a partial  
10 or degraded point pattern (Chakraborty et al., 2011). It is relatively simple to accommodate variable sampling effort in these models using modern Bayesian computing tools (Yuan et al., 2017). However, the literature on optimal sampling for spatial point process models is in its infancy (Liu and Vanhatalo, 2020).

15 Point pattern data are routinely collected in species distribution studies and ordnance response projects. The data consist of the locations of events in some spatial region. In many cases it is too costly to observe the entire study region, so a sampling procedure is employed. These applications may use quadrat sampling or line-transect sampling, with transect sampling being more common. Software  
20 is now available to fit spatial point process models to data acquired via distance sampling and simultaneously estimate the detection function (Johnson et al., 2014; R Core Team, 2020). Various spatial mapping procedures have been used to map where events occur in space. However, these have traditionally involved aggregating the data to grid cell counts or computing moving averages.  
25 Aggregation has the downside of introducing arbitrary structure into the data by the choice of grid scheme or averaging window, and requires unnecessary computation effort (Simpson et al., 2016).

In ecological settings, sampling plans are often designed around the goal of estimating total abundance. Ordnance response surveys are typically de-  
30 signed to provide enough data to detect (but not necessarily map) intensity hotspots (USACE, 2015; Flagg et al., 2020). However, there has been very little work done in deciding *where* to collect data when the goal is to map the intensity using a spatial point process model. While some ideas about the characteristics  
35 of a good point design apply to paths, creating an optimal path design is not as simple as connecting the points of a point design with line segments. There are many ways to connect points into a path, so optimal design criteria must apply

to the whole path and not only to the waypoints.

In this paper, we develop a space-filling criterion for path designs, present a variety of designs, and assess their utility for spatial prediction using LGCP  
40 model fitting in terms of robustness to computational issues and model-based criteria.

### 1.1. Log-Gaussian Cox process

The log-Gaussian Cox process (LGCP) is an inhomogeneous Poisson process where the logarithm of the intensity function is a Gaussian process (GP; Møller  
45 et al. 1998). The LGCP model provides a flexible model for mapping event intensity over space using few parameters. Efficient Bayesian computation tools are available using INLA to approximate the posterior marginal distributions (Rue et al., 2009), a finite element approach to represent the GP (Lindgren et al.,  
50 2011), and pseudodata to approximate the point process likelihood (Simpson et al., 2016).

### 1.2. Spatial design

Most classical sampling and design work has been done for points, or for small quadrats approximated as points, rather than for paths. In two-dimensional geostatistical model-based design, regular spacing is optimal for spatial prediction  
55 but randomness and a variety of interpoint distances are best for parameter estimation (Diggle and Lophaven, 2006). Inhibitory plus close pairs designs are a good compromise when estimation and prediction are both important (Chipeta et al., 2017).

Design-based approaches exist to spread points through high-dimensional  
60 design spaces (Borkowski and Piepel, 2009), and Latin hypercube sampling has space-filling characteristics (McKay et al., 1979; Husslage et al., 2011). Minimax and maximin distance are popular criteria for ensuring design points represent the entirety of a large (or high-dimensional) design space (Johnson et al., 1990). These criteria are calculated from the distances between design points and arbitrary points in the design space, explicitly ensuring coverage by minimizing  
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the maximum distance between the design space and the design (minimax) or maximizing the minimum distance between pairs of design points (maximin).

### 1.3. Space-filling curves

A relevant area of mathematical research is in deterministic space-filling  
70 curves. These have been used in the design of dense or stretchable circuits (Ogorzałek,  
2009; Ma and Zhang, 2016) and high-dimensional data visualization in bioin-  
formatics (Anders, 2009). They are appealing as path designs because, given  
enough length, the path will visit every part of the study site.

Example space-filling curves include the Hilbert curve, which is simple to  
75 construct, and the Peano curve, which is very flexible for filling irregular shapes  
(Fan et al., 2014). Space-filling curves are one-dimensional paths constructed  
iteratively; as the number of iterations goes to infinity, the limiting path has  
nonzero area and actually fills the space (Sagan, 1994). For applications we stop  
after a finite number of iterations.

### 80 1.4. Paths as sampling designs

The small body of literature on spatial sampling design for point pattern data  
has focused on line transects. Pollard et al. (2002) began with line transects and  
adaptively added zigzags in a species abundance survey. The Visual Sample Plan  
software, commonly used for design and data analysis in ordnance remediation  
85 projects, includes features to create systematic transect plans and augment plans  
with additional transects in regions lacking spatial coverage (Matzke et al.,  
2014). It helps the user choose the transect spacing to maximize the probability  
of detecting the presence of a hotspot of specified size and intensity. However,  
it does not employ criteria to optimize spatial prediction. Liu and Vanhatalo  
90 (2020) provided one of the first explicit discussions of design in the context of  
spatial LGCP models. The study used narrow quadrats (swaths along line-  
transects) as the sampling units. The transects were short relative to the size  
of the study region and not connected into a path, so they were treated as  
disconnected points for the purposes of design construction.

95 1.5. *Goals of this paper*

We have two broad goals for this paper. First, we will illuminate the current state of practical computing for LGCP models fit to degraded point patterns observed via survey sampling. Second, we open discussion of path design for LGCP models by identifying design schemes that are promising for field use.

100 We accomplish these goals via a simulation study, using many designs each applied to the same simulated datasets. The simulated data collection procedure is based upon geomagnetic surveys for unexploded ordnance, where metal detectors traverse a path and detect events under their footprint, i.e. within a fixed, and assumed known, width centered on the path. The paths are long  
105 enough that they cannot be approximated by discrete design points.

## 2. Materials and methods

With an eye toward practical considerations of data collection, we present criteria to compare sampling strategies that impact LGCP model fitting and posterior inferences. We compare plans with (approximately) fixed path lengths,  
110 most of which avoid sharp turns. Data collection equipment (e.g. vehicle-towed metal detectors) may have limited maneuverability, requiring minimizing the number or angle of turns. This section describes the designs in detail, then introduces notation and design criteria, and finally summarizes the computational model-fitting approach.

115 2.1. *Sampling design schemes*

In this section, we present three variations of parallel line transect designs and three schemes that produce more complex designs. To clarify terminology, a *path* or *design* is a realized set of one or more connected components with length but not area. The paths considered in this work are constructed as  
120 sequences of line segments. A *design scheme*, or simply *scheme*, is a procedure for generating designs with some shared characteristics. Figure 1 illustrates a selection of designs from these schemes.

### *2.1.1. Parallel line transects*

Parallel straight-line transects are common in ordnance response studies and  
125 in ecological studies using distance sampling. Systematic designs are popular  
because they provide good spatial coverage in the sense that any point in the  
study site has an a priori known maximum distance from the path. For point  
designs, systematic designs are preferred for prediction, simple random samples  
are good for estimation, and inhibitory with close pairs designs are becoming  
130 a popular compromise (Chipeta et al., 2017). We adapt all three of these to  
the parallel line transect setting. We use line transects running north-south,  
with three methods of choosing the horizontal coordinate: simple random sample  
(SRS), systematic with a random starting point and fixed spacing, and  
inhibitory plus close pairs. Figure 1 (left column) shows an example of each  
135 scheme with 25 transects. Additional examples appear in Appendix Appendix  
C.

### *2.1.2. Parallel serpentine transects*

One simple way to observe a greater variety of locations and different directions  
is to add lateral “zigzags” to line transects. We include alternate right  
140 and left turns at right angles to create serpentine transects. This could decrease  
prediction variance because more of the path will be close to each point  
in the study area than would be under a line transect design with similar total  
distance. They will also improve estimation of the covariance function in the  
presence of anisotropy. Figure 1, top right, shows two examples.

### *145 2.1.3. Latin hypercube sampling*

Random Latin hypercube sampling (LHS) produces a design that spreads discrete points through a (potentially high-dimensional) design space, ensuring that the full range of each dimension is included while remaining balanced and keeping the number of points small (McKay et al., 1979). This is done by  
150 partitioning each dimension into a specified number  $k$  of intervals (thus stratifying a  $d$ -dimensional design space into  $k^d$  cells), selecting a Latin hypercube

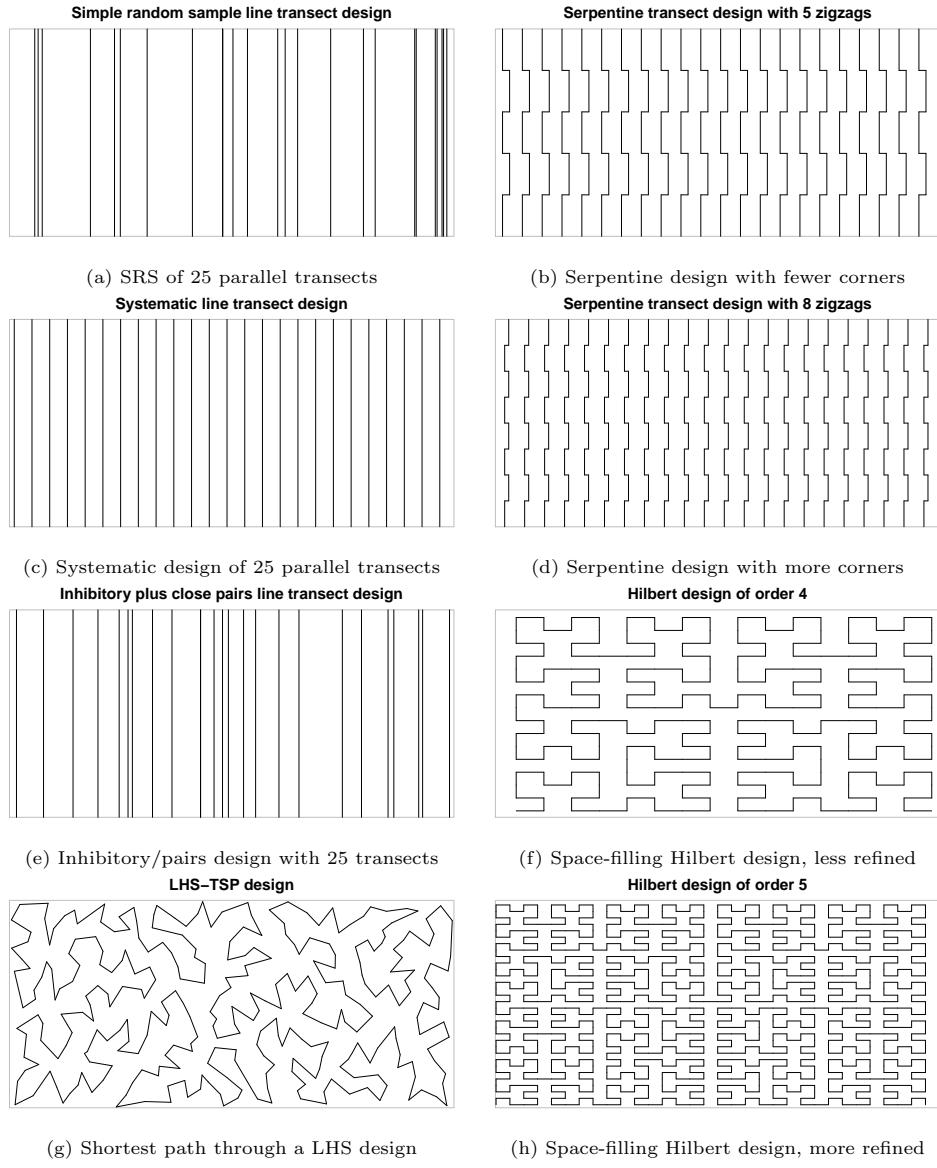


Figure 1: Examples of plans from six design schemes. Except for the Hilbert curve of order 5, all of these plans have approximately the same total length.

design to determine which  $k$  cells will contain a design point, and then drawing each design point from a uniform distribution over its cell. In two dimensions, this scheme produces point designs with good spatial coverage properties. We  
155 use the LHS design as waypoints for a path. Because longer distance typically brings increased costs, we treat this as a traveling salesperson problem (TSP) and use the shortest path through the waypoints as our design. This LHS-TSP scheme produces paths that have many sharp corners but leaves few large voids (example in Figure 1, bottom left). Waypoints are generated by the  
160 `lhs` R package (Carnell, 2020) and connected into a shortest path by the TSP package (Hahsler and Hornik, 2020). A downside of this design scheme is that the length cannot be specified directly, and only certain distances are possible depending on the number of bins used.

#### 2.1.4. Space-filling curves

165 As a representative of space-filling curves, we use the Hilbert curve scaled to fit the study site. These designs have many short segments meeting at right angles. The only parameter of this design scheme is the order, or number of iterations used in refining the curve. Each iteration increases the length and complexity of the design. This produces a deterministic design, so a random  
170 offset is added to vary which locations are visited. The Hilbert curve is generated by the `HilbertVis` R package (Anders, 2009).

#### 2.1.5. Spatial coverage criterion

One critically important characteristic of a spatial design is the concept of spatial coverage, or how well the design represents the study site in the sense  
175 that all points in the site are close to observed locations. The criterion we propose to measure this is the average distance to the path. We define the distance between a point and the path as the distance between that point and the nearest point on the path, then take the average over all points in the study region. This can be roughly understood as the radius of a typical unsurveyed  
180 void.

## 2.2. LGCP model and notation

The full point process is denoted  $\mathbf{X}$  and operates over the entire study site  $\mathcal{R}$ . The full realized point pattern is denoted  $\mathbf{x}$  and is a finite (countable) subset of  $\mathbf{R}$ . However, only a subregion of  $\mathcal{R}$  is surveyed. This subregion is  $\mathcal{S}$ , the set  
<sup>185</sup> of all points within a specified radius of the survey path. The observed point pattern is defined as  $\mathbf{x}_{\mathcal{S}} = \mathbf{x} \cap \mathcal{S}$ .

In the LGCP model,  $\mathbf{X}$  is a Poisson process with intensity function  $\lambda(u)$ ,  $u \in \mathcal{R}$ . This intensity function is itself a stochastic process, modeled on the natural logarithm scale as

$$\log[\lambda(u)] = \mu + \mathbf{e}(u), \quad (1)$$

with an intercept  $\mu$  and spatial error term  $\mathbf{e}(u)$ . The error term is a Gaussian process with mean 0 and covariance function  $C(u, v)$ . In our simulation study,  
<sup>190</sup>  $C$  is a stationary and isotropic Matérn covariance function with smoothness parameter  $\nu$ , spatial scale  $\rho$ , and variance  $\sigma^2$ .

### 2.2.1. Model-based criteria

For model-based comparisons, we compare the posterior distribution of the intercept  $\mu$  to the realized mean of the simulated GP, and evaluate two spatial prediction criteria. For the intercept, we simply record whether the 95% central  
<sup>195</sup> interval captures the realized mean. This is aggregated into the capture rate for the design scheme.

The spatial prediction criteria that we evaluate are the mean squared prediction error (MSPE) of the log intensity surface,

$$E[(\mu + \mathbf{e}(u) - \log[\lambda(u)])^2 | \mathbf{x}_{\mathcal{S}}], \quad (2)$$

and the average prediction variance (APV) of the latent Gaussian process,

$$\text{Var}[\mathbf{e}(u) | \mathbf{x}_{\mathcal{S}}]. \quad (3)$$

In the above notation,  $\lambda$  is the true realized intensity function, which is known in the simulations.

The capture rate of the mean describes accuracy in the overall scale of the  
200 intensity surface. Inaccuracy in the mean implies that that model will predict  
overall too many or too few events. The other criteria reflect two important  
characteristics of a useful spatial prediction. First, deviation from the true  
surface should be low, and is indicated by low MSPE. Second, the model's own  
assessment of prediction uncertainty will be used in practice to build trust in  
205 the inferences, so low APV is desirable.

### 2.2.2. Model fitting

We fit the spatial LGCP model using nested integrated Laplace approximations and the R-INLA package (Rue et al., 2009; Blangiardo and Cameletti, 2015). We use the `inla()` function with its default optimizer parameters. The  
210 Gaussian process is approximated using a finite element approach (Lindgren et al., 2011). The finite element approach models the GP at a finite set of nodes in a mesh, and requires an assumption that the realized GP surface is approximately linear between the nodes. The data analyst must choose the nodes so that this assumption holds. Then, the point pattern is modeled by pseudodata  
215 placed at the events and the finite element nodes (Simpson et al., 2016). When using an adequate finite element mesh, this procedure allows fast and accurate approximation of the posterior distribution.

## 3. Simulation study

We compare the design schemes by applying multiple designs from each  
220 scheme to the same simulated datasets. There are two data-generating models and five datasets generated from each, for a total of ten datasets (Section 3.1). We will focus on two illustrative datasets when presenting the results. We use 8 schemes at four levels of survey effort, and 100 designs generated from each scheme at each effort level (Section 3.2). All events within a 2 unit radius of  
225 the path are observed. The same model is fit to each observed point pattern (Section 3.3).

### 3.1. Study site

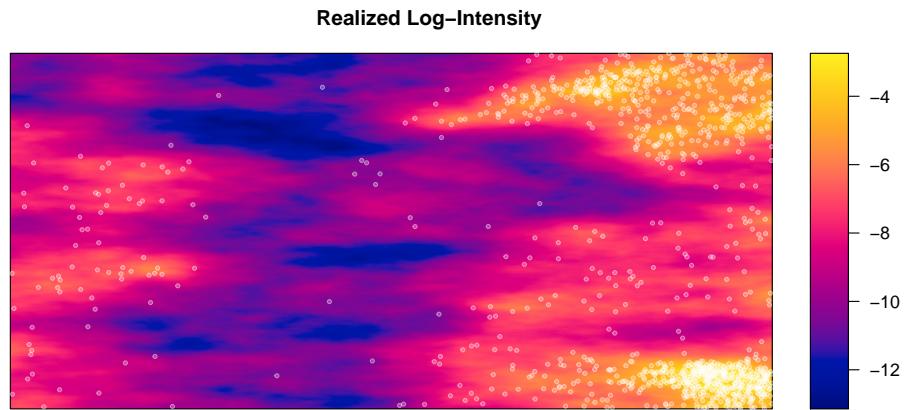
We consider a fictitious site  $\mathcal{R}$  with the simple shape of a 1500 unit by 700 unit rectangle. In this site, we simulate two data generating models that produce  
230 random intensity functions with hotspots. First, a LGCP with latent GP mean  $\mu = \log(250/|\mathcal{R}|) = -8.34$  and Matérn covariance function with  $\nu = 1$ ,  $\sigma = 2$ , and  $\rho = 200$ . This model produces relatively unstructured hotspots due to large variability in the GP. Figure 2a shows an example realization from this process.

Second, a two-stage cluster process and a LGCP are superposed. The cluster  
235 process (a Neyman-Scott or, more specifically, Thomas process) is constructed as follows. The number of clusters is Poisson-distributed with mean 3. The number of events per cluster is Poisson-distributed with mean 200. The cluster centers are distributed uniformly over  $\mathcal{R}$ . Events come from a bivariate normal distribution with mean equal to the cluster center and variance  $\Sigma = \tau^2 \mathbf{I}$ ,  $\tau = 50$ .  
240 The LGCP component has  $\mu = \log(250/|\mathcal{R}|) = -8.34$  and Matérn covariance with  $\nu = 1$ ,  $\sigma = 1$ , and  $\rho = 200$ . This model is based upon the typical conceptual model of a firing range, with a background process (represented by the LGCP) and a small number of higher-intensity foreground clusters containing the events of interest (example in Figure 2b). As a shorthand, we refer to this generative  
245 process as LGCP with Clusters.

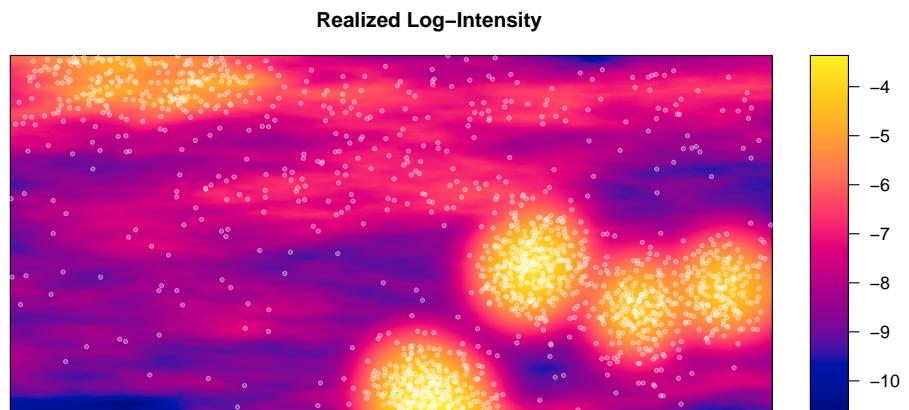
The LGCP generating process matches the model fit to the realized data. The model should both produce accurate predictions and have low bias in the posterior means of the parameters. On the other hand, the parameters of the LGCP with Clusters process will not correspond to the LGCP model parameters, but we anticipate that in general the latent GP in the LGCP model will still accurately predict the log-intensity function.  
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### 3.2. Path design schemes

The simulation uses each of the design schemes discussed in Section 2.1. The surveyed region  $\mathcal{S}$  consists of all points within 2 units of the design. That is,  
255  $\mathcal{S}$  is a collection of strips along each segment of the design. All events within



(a) Realized LGCP log-intensity



(b) Realized LGCP with Clusters log-intensity

Figure 2: The realized intensity function (natural log scale) and complete point pattern from a LGCP, and from a LGCP superposed with a cluster process.

Table 1: Design schemes used in the simulation study. The table presents the design parameters, distance travelled along the path, percent of the study area surveyed, and average distance to the path under each of four survey effort levels. The average distance to the path varies from design to design, and is presented as average (SD).

\* Variable area due to overlapping segments, reported as average (SD).

\*\* Variable distance surveyed, reported as average (SD).

Scheme	Low Effort	Medium Effort	High Effort	Very High Effort
SRS	10 transects Length: % Surveyed: Avg. Dist. to Path:	25 transects 7000 units 6.46% (0.202%)* 29.7 (6.01)	50 transects 35000 units 12.5% (0.331%)* 15.0 (1.99)	70 transects 49000 units 17.2% (0.459%)* 10.8 (1.44)
Systematic	10 transects Length: % Surveyed: Avg. Dist. to Path:	25 transects 7000 units 2.68% 6.70% 38.4 (0.866)	50 transects 35000 units 13.4% 7.56 (0.0307)	70 transects 49000 units 18.7% 5.40 (0.0164)
Inhibitory, 10% Close Pairs	9 transects 1 paired Length: % Surveyed: Avg. Dist. to Path:	23 transects 2 paired 7000 units 6.53% (0.172%)* 20.8 (1.82)	45 transects 5 paired 35000 units 12.8% (0.314%)* 10.6 (1.24)	63 transects 7 paired 49000 units 17.5% (0.427%)* 7.64 (0.576)
Inhibitory, 20% Close Pairs	8 transects 2 paired Length: % Surveyed: Avg. Dist. to Path:	20 transects 5 paired 7000 units 6.56% (0.157%)* 22.0 (2.97)	40 transects 10 paired 35000 units 12.8% (0.303%)* 11.4 (1.70)	56 transects 14 paired 49000 units 17.6% (0.368%)* 8.26 (0.826)
Serpentine, 5 Zigzags	7 transects 75 unit offset Length: % Surveyed: Avg. Dist. to Path:	22 transects 23.9 unit offset 7000 units 6.68% 43.2 (1.83)	47 transects 11.2 unit offset 35000 units 13.4% 7.79 (0.0508)	67 transects 7.84 unit offset 49000 units 18.7% 5.51 (0.0309)
Serpentine, 8 Zigzags	7 transects 42.9 unit offset Length: % Surveyed: Avg. Dist. to Path:	22 transects 13.6 unit offset 7000 units 6.67% 44.5 (2.03)	47 transects 6.38 unit offset 35000 units 13.3% 7.76 (0.0328)	67 transects 4.48 unit offset 49000 units 18.7% 5.50 (0.0271)
LHS-TSP	50 bins Length: % Surveyed: Avg. Dist. to Path:	300 bins 7177 (189) units** 2.73% (0.0722%)** 43.5 (1.44)	1200 bins 17191 (168) units** 6.51% (0.0632%)** 17.4 (0.186)	2400 bins 34196 (225) units** 12.9% (0.0835%)** 8.63 (0.0683) 48433 (342) units** 18.2% (0.127%)** 6.08 (0.0562)
Hilbert Curve	3rd Order Length: % Surveyed: Avg. Dist. to Path:	4th Order 8581 units 3.27% 37.3 (1.95)	5th Order 17442 units 6.634% 17.9 (0.480)	6th Order 35025 units 13.3% 8.77 (0.113) 70121 units 26.47% 4.37 (0.0382)

2 units of the design are observed and all events farther than 2 units from the path are unobservable.

The design parameters for each scheme are varied to create four different levels of survey effort: low, medium, high, and very high. Within each effort level, the total length or distance traveled is comparable across all schemes.  
260 The inhibitory plus close pairs scheme and the serpentine scheme are further varied to employ different numbers of pairs and different numbers of zigzags, respectively. Table 1 provides an overview of the different settings.

The parallel line transect schemes have transects running north-south, with  
265 the effort level determining the number of transects. We expect the simple random sample scheme to produce high prediction variance and large prediction error in big gaps between transects. The systematic sample scheme uses a uniformly-distributed starting point and constant spacing between adjacent transects. We expect systematic transects to provide accurate inferences for the  
270 intensity surface and moderate prediction variance.

For the inhibitory plus close pairs line transect scheme, we vary the numbers of paired and unpaired transects. The total number of transects is the same as in the SRS and systematic designs, but 10% and 20% of the transects (rounded to the nearest integer) are designated as redundant members of a pair. The  
275 remaining primary transects are placed according to a one-dimensional Strauss process (Strauss, 1975; Kelly and Ripley, 1976). The Strauss attraction parameter is set at  $\gamma = 0.05$  and the radius for counting pairs is 1500 units divided by the total number of transects. Then each redundant transect is randomly paired to a primary transect, and placed within the pair radius of the primary  
280 transect according to a uniform distribution. We expect this scheme to have intermediate performance between the simple random sample and the systematic line transect schemes.

The serpentine transect scheme has transects running north-south with constant east-west spacing and a random starting point for the first transect. The  
285 number of zigzags (north-south segments) is 5 or 8 per transect. The designs are constructed to have the same length as the line-transect designs: there are

three fewer serpentine transects, and the perpendicular offset is set so the total east-west distance equals the length of three north-south line transects. These designs should result in smaller prediction errors and lower variance farther from  
290 the path when compared to line-transect designs.

Our Latin hypercube sampling/traveling salesperson (LHS-TSP) scheme uses a different number of bins for each effort level (see Table 1). Preliminary experimentation found that these bin numbers produced total lengths similar to the transect schemes. The LHS-TSP scheme is expected to result in small  
295 prediction errors and low prediction variance per unit distance traveled. However, the designs will have many sharp corners and, on rare occasions, some large voids.

The Hilbert curve scheme uses a random starting point and a Hilbert curve of order 3, 4, 5, or 6. The path length is a deterministic function of the order  
300 and differs greatly among curves of different orders. These orders yield lengths similar to the lengths of the transect designs. Hilbert designs should provide low prediction variance, but have lots of short segments.

### 3.3. Model specification

The same Bayesian LGCP model is fit to each observed dataset. The observed point pattern  $\mathbf{x}_S$  is a partial realization of  $\mathbf{X}$ , a Poisson process on  $\mathcal{R}$  with intensity  $\lambda(u)$ . The intensity is modeled as  $\log[\lambda(u)] = \mu + \mathbf{e}(u)$ . The spatial error term  $\mathbf{e}$  is a Gaussian process with mean  $\mathbf{0}$  and a Matérn covariance function with fixed  $\nu = 1$ .

The intercept  $\mu$  has a  $\text{Unif}(-\infty, \infty)$  prior. The covariance parameters  $\sigma$  and  
310  $\rho$  have a PC prior with  $\Pr(\sigma > 3) = 0.1$  and  $\Pr(\rho < 100) = 0.1$  (Fuglstad et al., 2019; Simpson et al., 2017).

### 3.4. Finite element mesh

The Gaussian process prediction surface is approximated on the finite element mesh shown in Figure 3. The GP is predicted at the 298 nodes (black  
315 dots on the plot) and is linearly interpolated elsewhere. The nodes are weighted

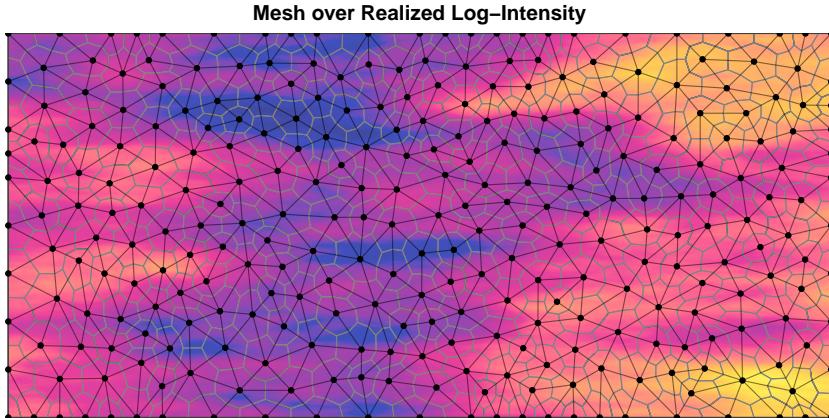


Figure 3: Illustration of the mesh used to approximate the latent GP. The mesh nodes are used in a numerical integration scheme where they are weighted by the area of their dual cells (outlined in grey).

according to the area of their dual cells (grey outlines) and used for numerical integration of the likelihood (Lindgren et al., 2011).

The choice of mesh is important because a mesh that is too coarse cannot represent small-scale structures and thus will incur large approximation errors. On the other hand, the mesh has implications for computation because the nodes and the observed events correspond to pseudodata points. In our sparsely-observed point pattern setting, the number of mesh nodes will be large relative to the number of observed events, making the mesh the primary driver of computational complexity in fitting the model.

A good approach in practice is to refit the model with several meshes and refine the mesh until the prediction surface has converged. However, for a simulation study we must choose one mesh that will perform reasonably well for a wide variety of datasets. This mesh is adequately fine to model the large-scale trends in the surface while keeping the computing time well under a minute for each model fit.

## 4. Results

In describing the results, we focus on one LGCP dataset and one LGCP with Clusters dataset (Figure 2). The results are similar for all datasets (see figures in Appendix A). These two datasets were chosen for illustration because the simulations using them exhibited a wide variety of outcomes. This included a mix of very good and very poor prediction performance.

Figure 4 shows an example where the model does well at predicting the intensity of the realized LGCP. This example uses one of the SRS paths. In the figure, the path appears in white and the observed events are shown as white dots. The posterior predicted mean of the log-intensity (top panel) accurately captures the large-scale features, but smooths out much of the small-scale variation. The bottom panel shows the prediction standard deviation for each mesh node. The SD ranges from 2.0 to 3.1, and is lowest near observed events. SD increases farther from observed events, including in places where the surveyed area was observed to contain no events.

This same survey plan also did well for the LGCP with Clusters dataset, with the model accurately capturing the large-scale details of the intensity function, including two of the circular hotspots corresponding to clusters (Figure 5a). However, it also smoothed the surface quite a bit, notably merging the two overlapping clusters into a single oblong hotspot.

Most plans yielded similar prediction surfaces, capturing the large-scale trends, and having the least uncertainty near observed events. Results varied in accuracy at the most extreme peaks and valleys of the intensity function and in overall SD across the study region.

However, a small number of model fits failed to converge, and others suffered from apparent edge effects. For example, Figure 6 shows the prediction surface resulting from a serpentine transect plan. The predicted log-intensity has a hotspot of extremely large values in the southeast corner (notice the color scale). The hotspot is driven by two nodes on the boundary with very large prediction values. Another, less extreme, edge effect is present in the northeast corner.

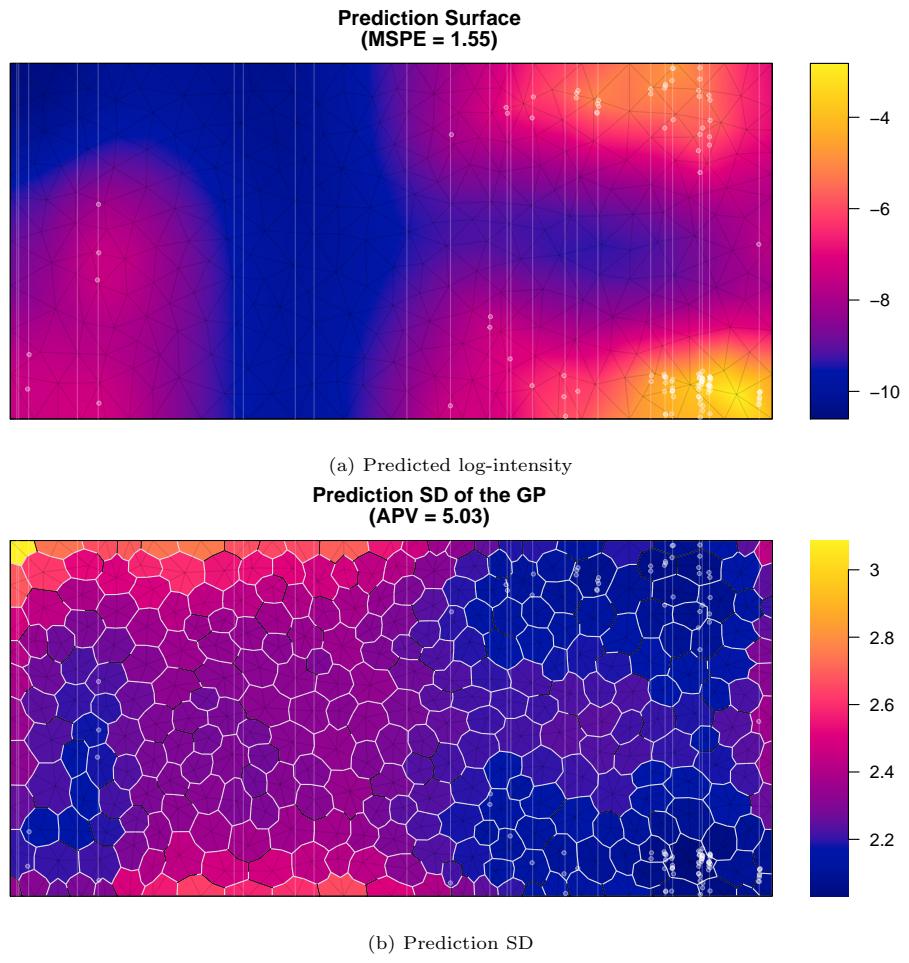


Figure 4: Predicted log-intensity function and prediction standard deviation using data observed via a SRS of line transects. The SD is shown for each finite element node. This example is a medium-effort plan applied to a LGCP dataset.

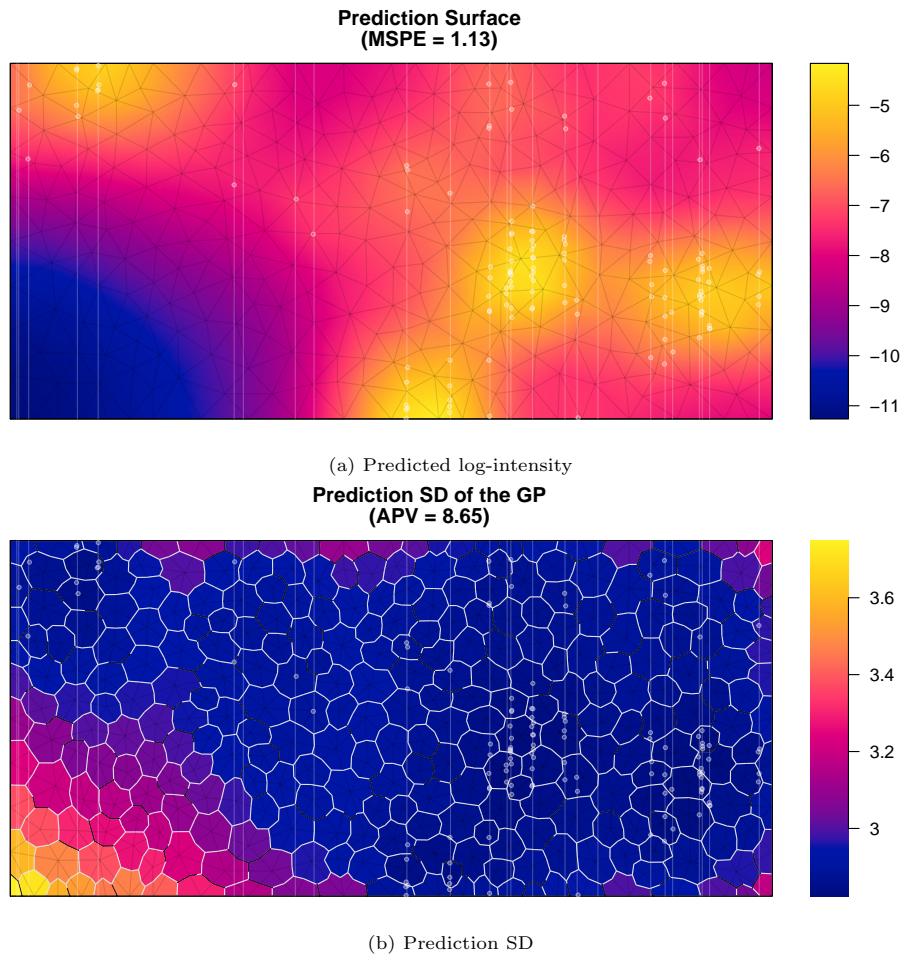


Figure 5: Predicted log-intensity function and prediction standard deviation using data observed via a SRS of line transects. The SD is shown for each finite element node. This example is a medium-effort plan applied to a LGCP with Clusters dataset.

The surface also contains many circular peaks centered on observed events. When computational problems occur, they may be fixed by refining the mesh at additional computational cost. We were not able to use a finer mesh for all simulations due to computational limitations, but we point out which schemes  
365 are robust to these problems (see Tables B.2 and B.3).

Section 4.1 illustrates refitting the model using a refined mesh. The sections after that compare the schemes in terms of model-based criteria applied to one realized LGCP dataset and one realized LGCP with Clusters dataset, and then discuss the results in the context of spatial coverage.

#### 370 *4.1. Dealing with computational problems*

A notable proportion of simulation runs resulted in surfaces with edge effects similar to Figure 6, especially at lower levels of sampling effort. This is most likely a symptom of the mesh not being able to represent the posterior surface implied by the observed data. In practice, data analysts should fit the model  
375 with several different meshes to assess adequacy of the approximation.

We constructed a finer mesh with 1102 nodes (Figure 8), then refit the model to the point pattern from Figure 6. The resulting log-intensity surface appears in Figure 9. The surface is free of edge effects or extreme values, has an MSPE of 2.31 (down from 1605), and looks generally similar to well-performing surfaces  
380 from other simulation runs using the same dataset (e.g. Figure 4, more examples in the Appendix Appendix C).

This refined mesh has nearly four times as many nodes and the mesh used for the full set of simulations, and consequently fitting the model takes several times longer. The time requirement precluded us from rerunning the full simulation  
385 study with this mesh, but the simulations using the coarse mesh provide helpful insight into which schemes are more likely to require more computation effort.

#### *4.2. LGCP simulation results*

Considered across all survey plans applied to one LGCP dataset, most 95% posterior intervals capture the true mean value, but those that did not were more

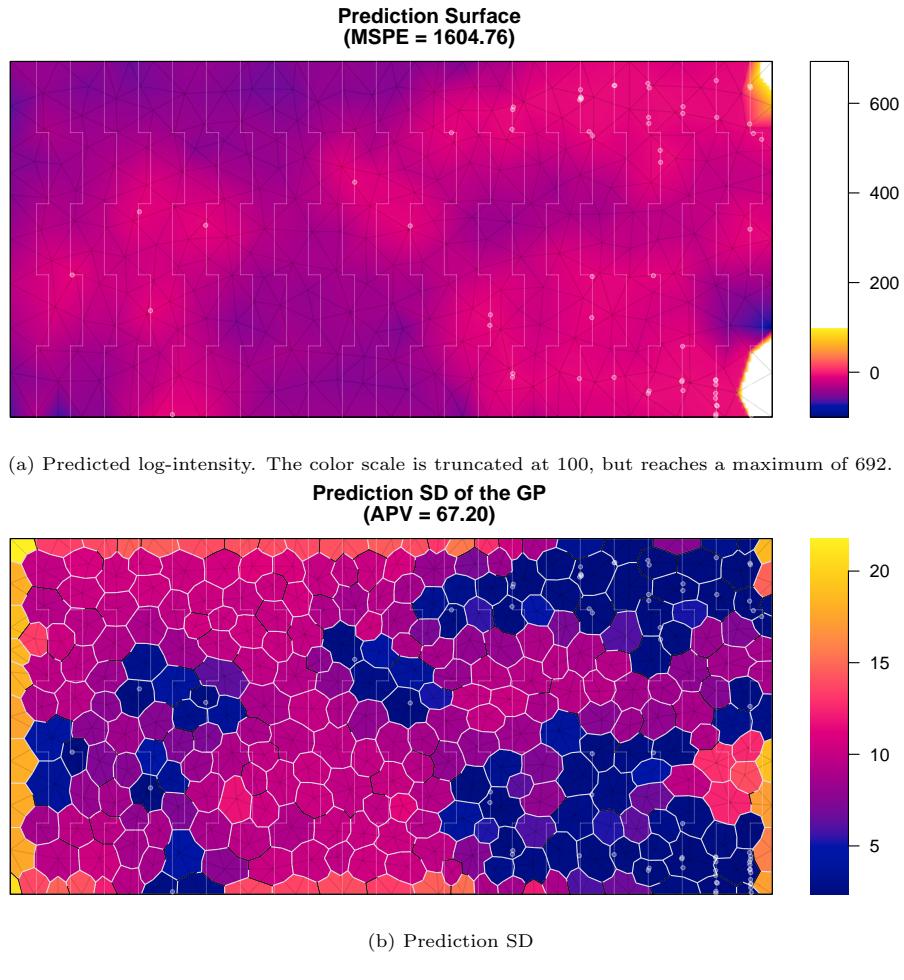
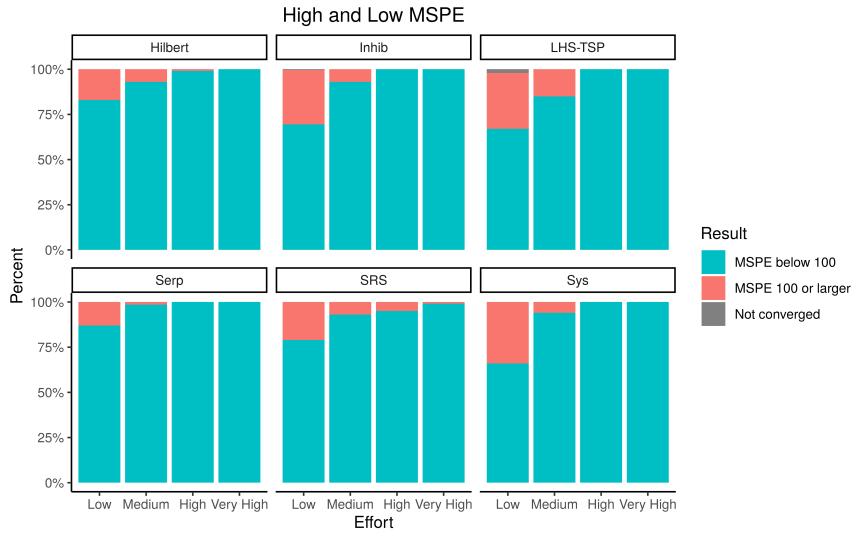
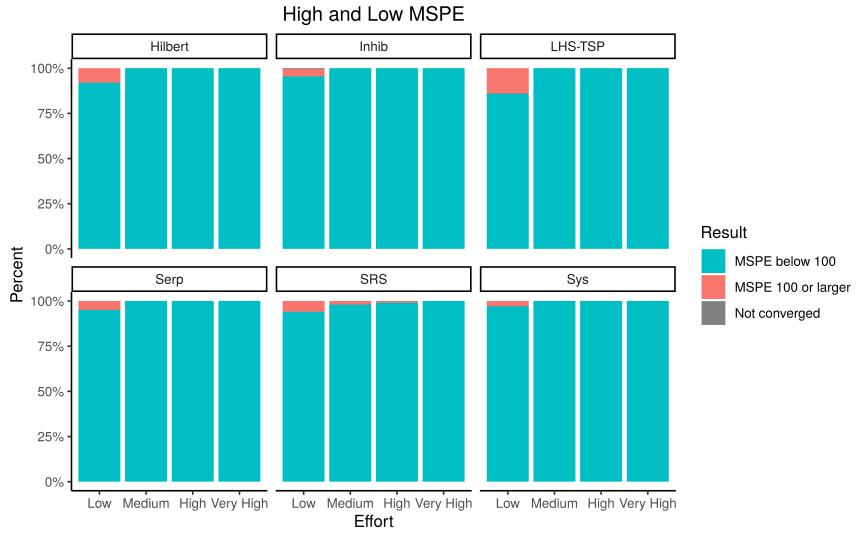


Figure 6: Predicted GP surface and prediction SD using data observed via a serpentine transect plan. The prediction has an apparent edge effect in the southeastern corner. The SD is high across much of the site. This example is a medium-effort plan applied to a LGCP dataset.



(a) MSPE distribution for one LGCP dataset



(b) MSPE distribution for one LGCP with Clusters dataset

Figure 7: Plots of the distribution of high and low mean squared prediction error (MSPE) vs survey effort for each plan applied to one realization of a LGCP and one realization of a LGCP with Clusters.

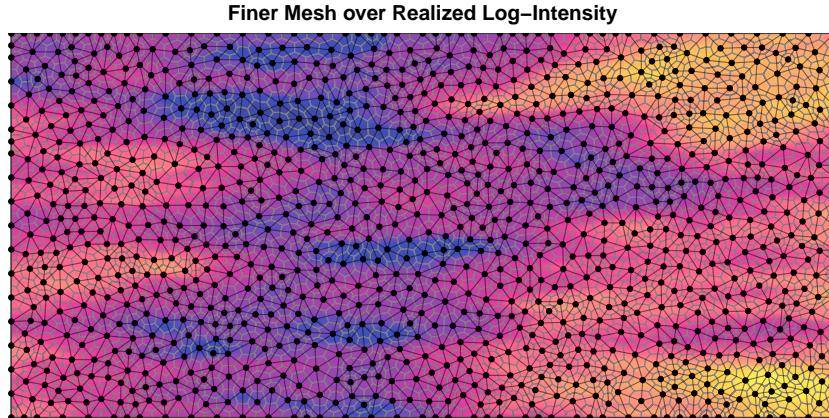


Figure 8: Illustration of a finer mesh than the one used in the full simulation. The mesh nodes are used in a numerical integration scheme where they are weighted by the area of their dual cells (outlined in grey).

<sup>390</sup> likely to underestimate than overestimate the mean (Figure 10a). The capture rate increases with survey effort, with only the SRS and Hilbert schemes being observed to miss the true mean at the highest effort level.

<sup>395</sup> Both MSPE and APV had right-skewed distributions. Thus we use logarithmic scales for plots and summarize them using the median and interquartile range (IQR). Median MSPE decreases with increasing path distance, leveling off between the medium and high effort levels (Figure 11a). Variability (IQR) of MSPE also decreases as effort increases. Prediction surfaces with edge effects form a cluster of large, outlying MSPE values. At all levels of effort, systematic designs had the least variability in MSPE. The lowest-MSPE prediction surfaces <sup>400</sup> result from the longest Hilbert designs, but the distributions of MSPE are similar for all schemes at the high and very effort levels. Overall, the differences among the different schemes with respect to median MSPE are much less than the variability in MSPE within each effort level. The results are largely the same for APV (Figure 11b).

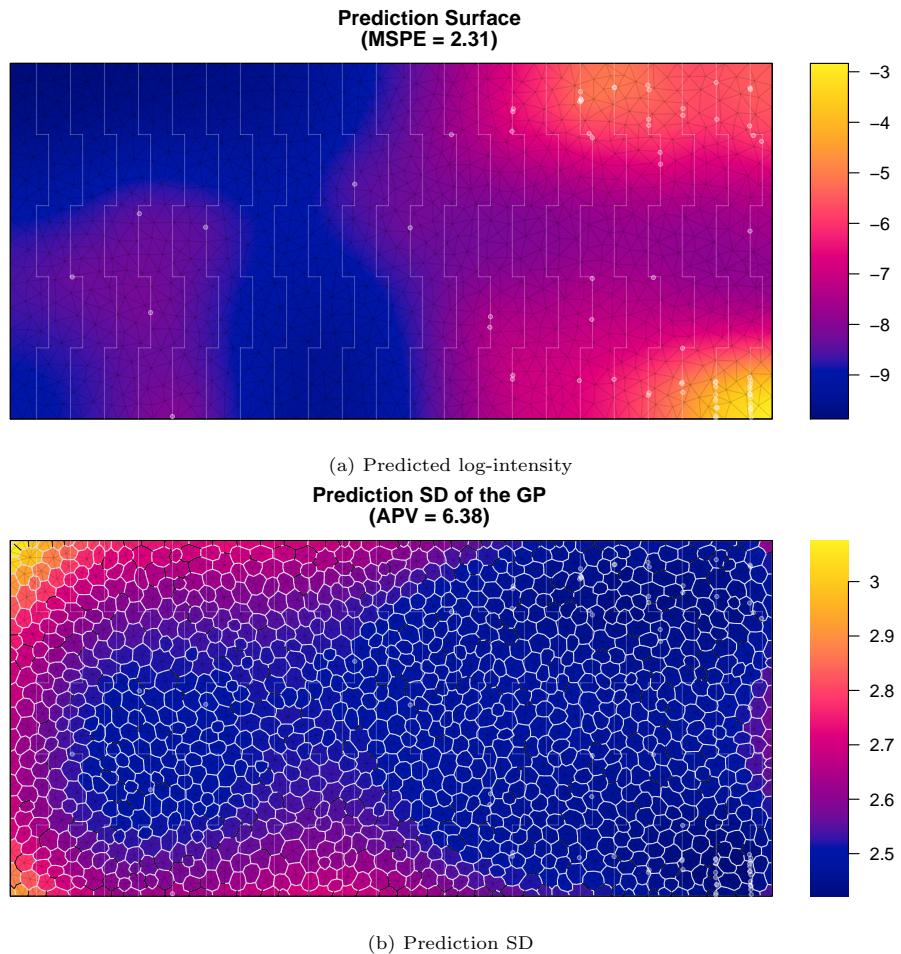
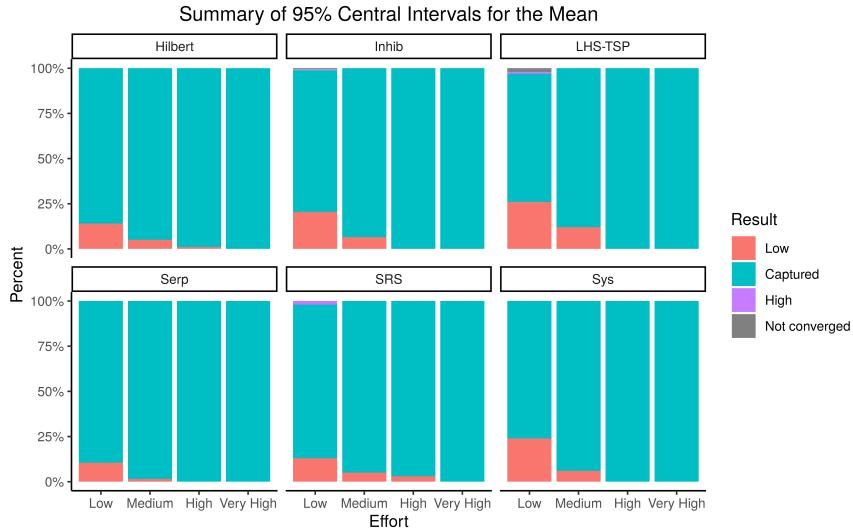
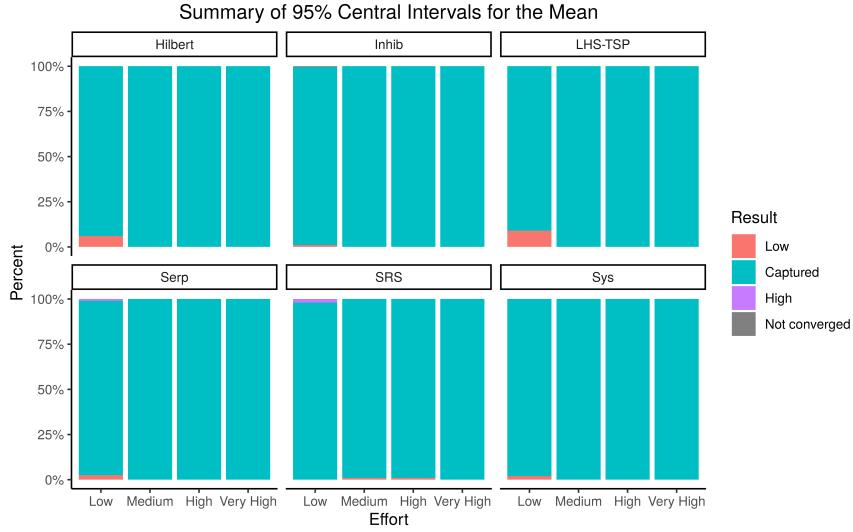


Figure 9: Predicted GP surface and prediction SD using data observed via a serpentine transect plan. When these data are modeled with a coarser mesh, the prediction surface exhibits numerical instability. This refined mesh results in a much-improved prediction surface.

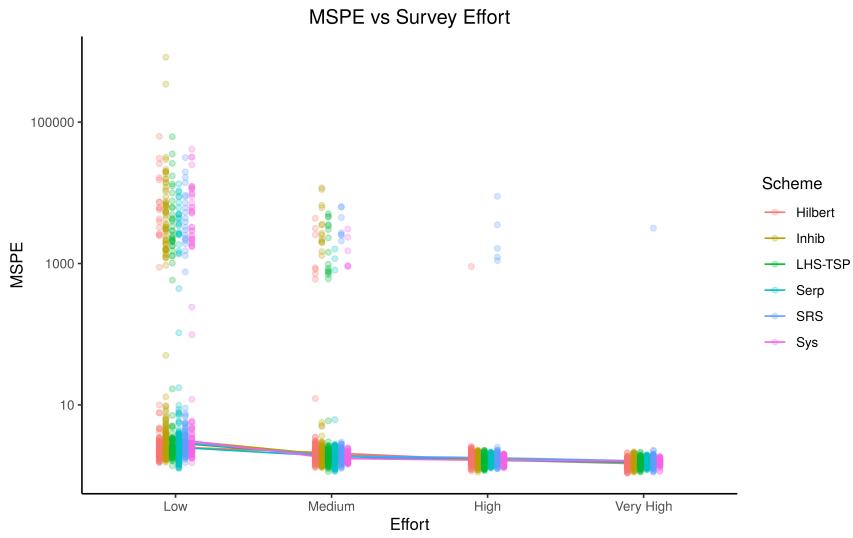


(a) Comparisons of posterior intervals for the intercept to the true mean for one LGCP dataset

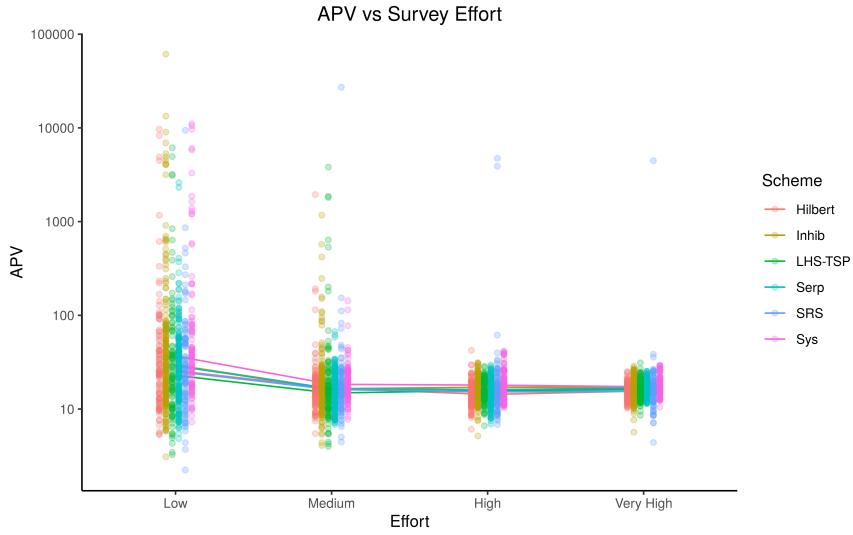


(b) Comparisons of posterior intervals for the intercept to the true mean for one LGCP with Clusters dataset

Figure 10: Plots summarizing comparisons between central 95% posterior intervals for the intercept and the true mean log-intensity of the realized GP. “Captured” indicates that the true mean is inside the interval. “High” and “Low” indicate that the entire interval is, respectively, above or below the true mean.



(a) MSPE vs survey effort for one LGCP dataset



(b) APV vs survey effort for one LGCP dataset

Figure 11: Plots of mean squared prediction error (MSPE) and average prediction variance (APV) vs survey effort for each plan applied to one realization of a LGCP. Line segments connect the median at each effort level.

405     4.3. *LGCP with Clusters simulation results*

For the LGCP with Clusters dataset, all schemes tended to do well at estimating the mean (Figure 10b). At low effort, all schemes occasionally fail to capture the true mean in a 95% central posterior interval. Only the SRS scheme is observed to miss the mean at medium or high effort.

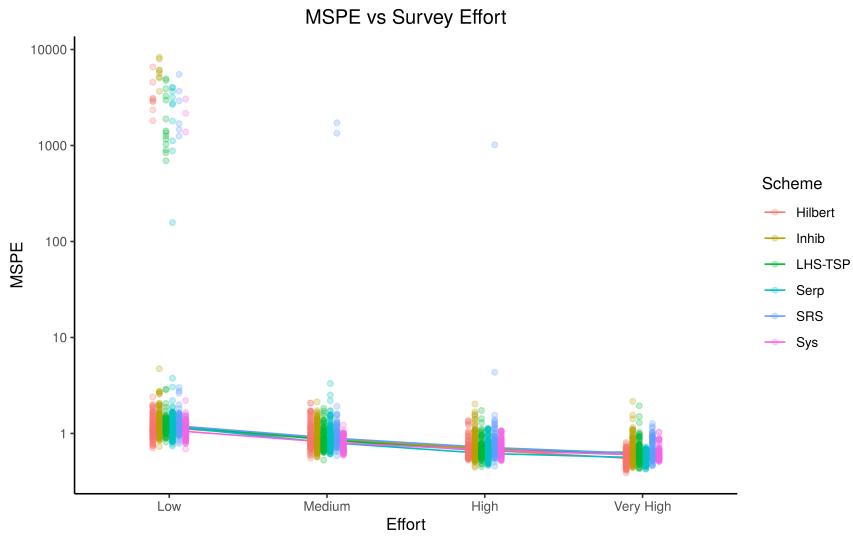
410     MSPE and APV again had right-skewed distributions. Median MSPE decreases as effort increases (Figure 12a). Variability in MSPE is roughly constant across effort levels. The systematic scheme had the lowest median MSPE for low and medium effort and low IQR for all effort levels, while the serpentine and Hilbert schemes have the lowest median at high and very high effort. However,  
415     differences between schemes are much less than differences across survey effort. At low effort, the distribution of APV has a long tail. Otherwise there is little difference in distribution of APV across schemes or effort (Figure 12b).

4.4. *Spatial coverage*

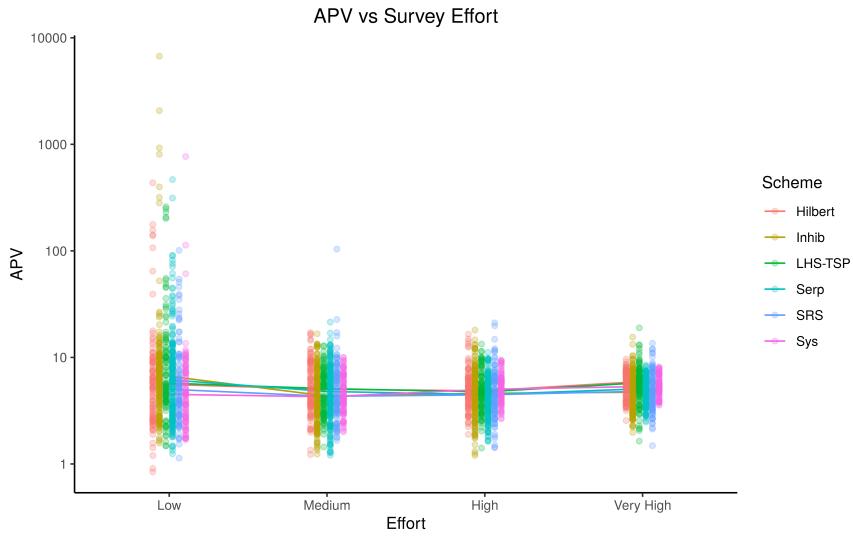
While the above results suggest the choice of design is relatively unimportant and the distance traveled is the main driver of the quality of the spatial predictions, it is important to consider that all of the design schemes ensure the path is distributed across the entire study region. Designs that leave large unexplored voids will not perform as well.

As an example, Figure 13 shows the results of using a systematic sample of  
425     50 parallel transects in the western 20% of the study site. This design is the same length as our high-effort designs, but it leaves most of the site far from the surveyed path. As a result, the predicted log-intensity is flat near the GP posterior mean of  $-8.71$  over most of the site, rendering the prediction mostly useless.

430     The problem with this example is easily explained in terms of the average distance between the path and arbitrary points in the study site. This example design has an average distance from the path of 486 units; the designs used in the simulation study all had average distances to the path under 130 (with most under 50). In the simulation study, there was little association between



(a) MSPE vs survey effort for one LGCP with Clusters dataset



(b) APV vs survey effort for one LGCP with Clusters dataset

Figure 12: Plots of mean squared prediction error (MSPE) and average prediction variance (APV) vs survey effort for each plan applied to one LGCP with Clusters dataset. Line segments connect the median at each effort level.

<sup>435</sup> MSPE and average distance to the path for paths of comparable length, but the average distance to path decreased as effort (length) increased (Figure 14). This means that all of these schemes do well at distributing the path around the site at all of these effort levels.

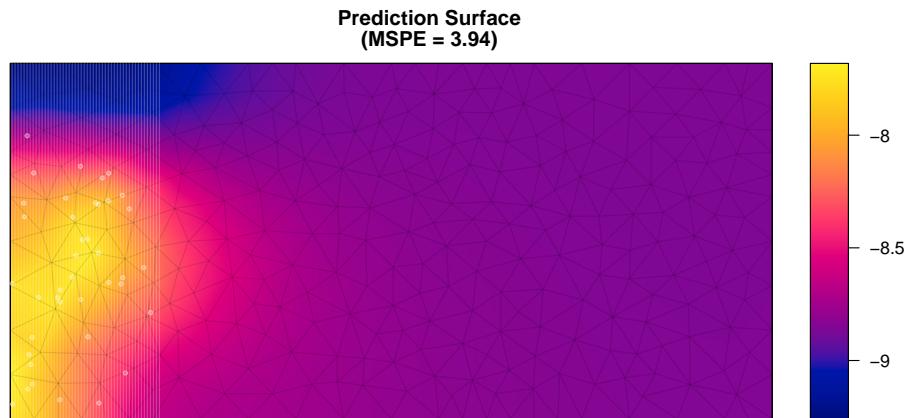
## 5. Discussion

### <sup>440</sup> 5.1. Implications for sampling

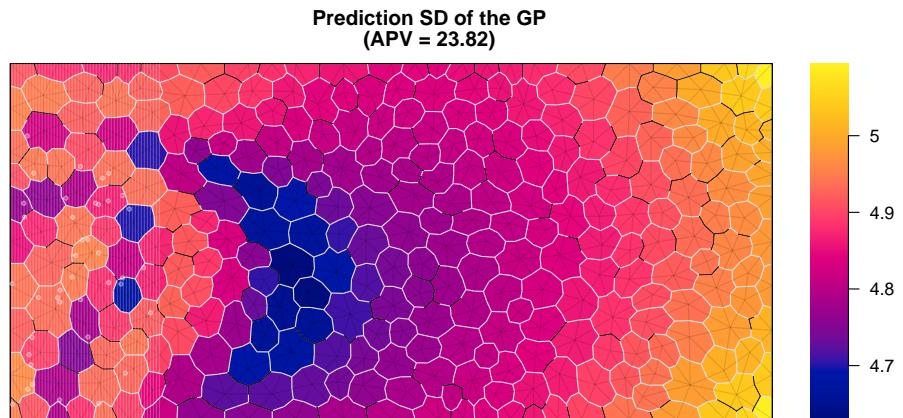
This study provides several insights about optimal sampling for LGCP models. First of all, prediction uncertainty increases with distance from observed *events*, even along the path (consider the region of low posterior mean and high standard deviation in the southwestern corner of Figure 5b). This behavior is <sup>445</sup> different from the behavior of Gaussian process models in geostatistical studies, where observed design points all provide information that reduces uncertainty of nearby predictions, regardless of the values observed. In the LGCP model setting, an "observed" strip that is observed to contain no events does not reduce the uncertainty in nearby predictions. This is reasonable behavior with sparse <sup>450</sup> sampling because there may be events just out of detection range. Intuitively, observing no events tells us the intensity should be "low" but says little about how low, especially when intensity is modeled on a logarithmic scale.

This has consequences for adaptive or sequential design. A greedy algorithm based entirely on minimizing variance may excessively sample regions <sup>455</sup> of low intensity, behavior that is unhelpful if the goal is to map hotspots of high intensity. We recommend including design-based considerations of spatial coverage in future studies on multi-stage design.

Additional considerations of applying path design to real-world studies include practical issues of data collection. Generally the path will be smoothed, <sup>460</sup> without the perfectly-straight segments and instantaneous direction changes of our simulated data collection. Equipment may have limitations which render impossible the sharp corners of LHS-TSP designs or the repetitious short segments and right angles of Hilbert curves. Future optimal design work could incorpo-

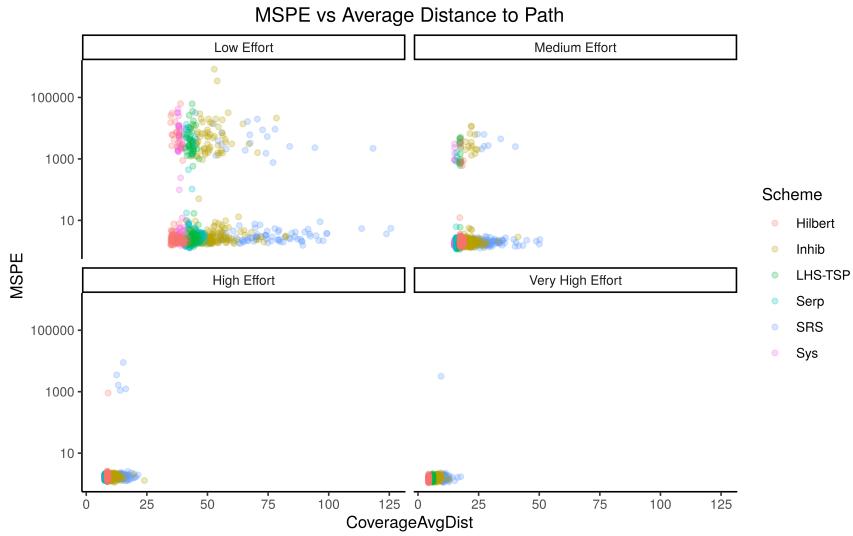


(a) Predicted log-intensity

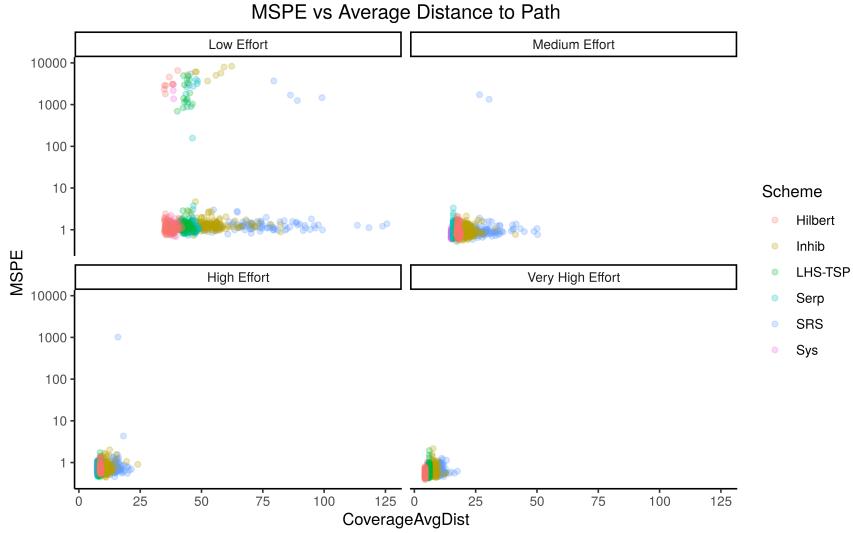


(b) Prediction SD

Figure 13: Predicted log-intensity function and prediction standard deviation using data observed via a systematic sample of a small section of the site. The SD is shown for each finite element node.



(a) MSPE vs coverage for one LGCP dataset



(b) MSPE vs coverage for one LGCP with Clusters dataset

Figure 14: MSPE plotted against the average distance from points in the site to the nearest point on the path. As survey effort increases, the cloud of points drifts down (lower MSPE) and to the left (lower average distance to the path).

rate turn angles into a loss function or use multi-objective optimization (Lark,  
465 2016).

### 5.2. Computing

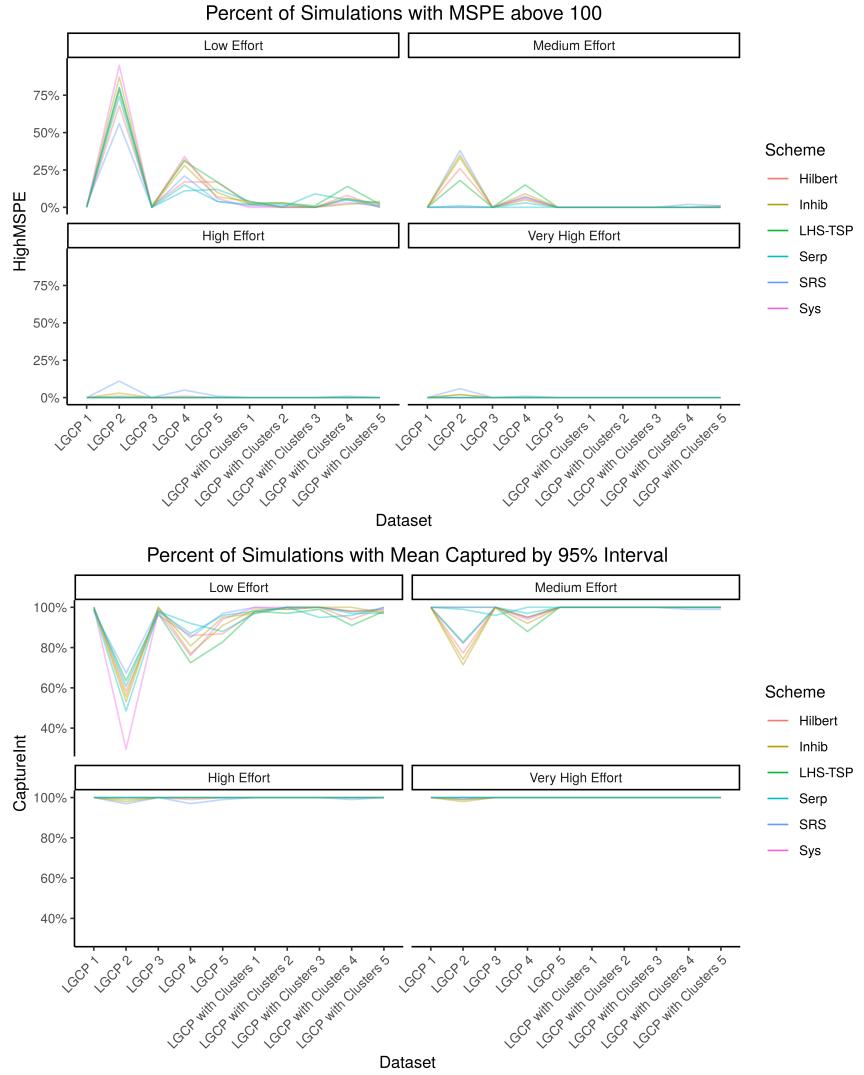
Bayesian computing (especially INLA) has brought LGCP models into the realm of practicality. However, this is not yet a fully “push button, get results” procedure; in a handful of our simulations, the gradient descent step of INLA  
470 failed to converge with the software’s default settings. In many other simulations, the posterior predictions appear numerically unstable at nodes on the boundary.

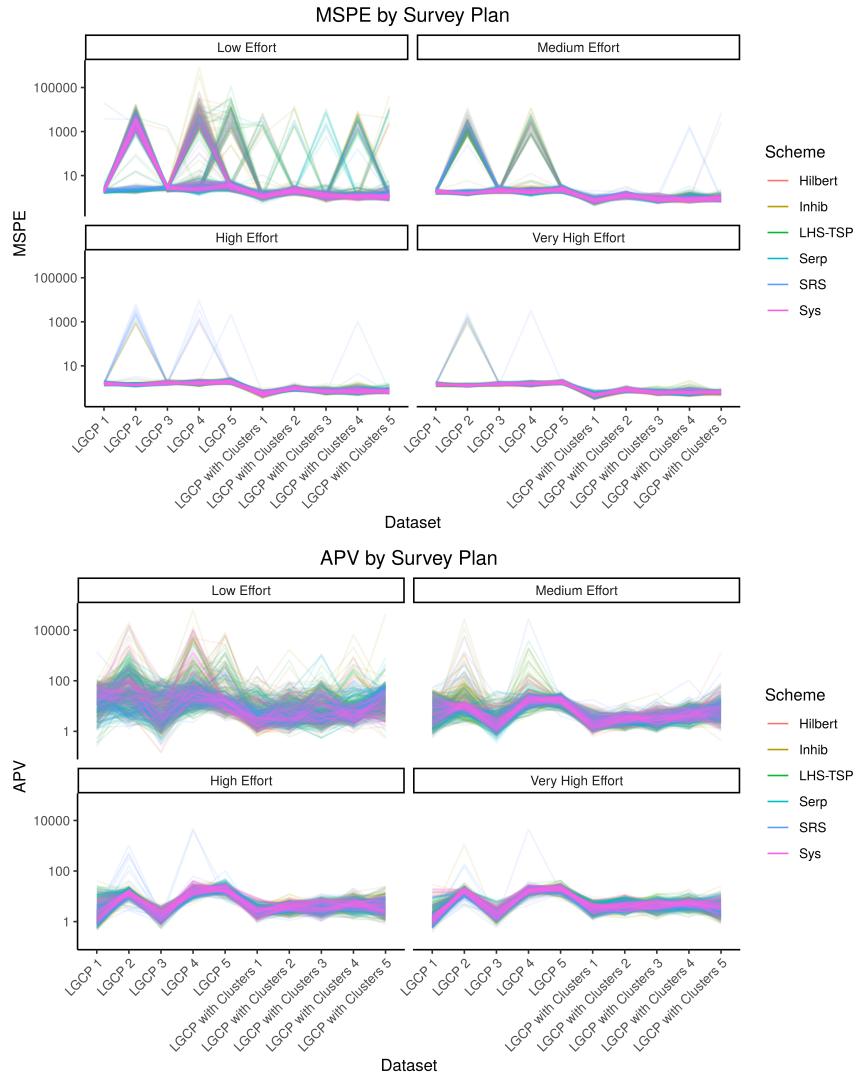
One aspect that would affect both of these issues is the choice of finite element mesh. A finer mesh than the one used in these simulations would have  
475 more flexibility to model steep changes in the GP surface. We hypothesize that this would make convergence problems and edge effects less likely. The downside of a finer mesh is increased computing time because each mesh node becomes a pseudodata point that must be processed by the model-fitting software. The choice of mesh is non-trivial and worthy of further study.

### 480 5.3. Conclusions

For optimal MSPE and APV, the choice of design scheme does not matter much as long as it provides spatial coverage. For short paths, including corners and regular spacing help avoid model-fitting problems, as in the Hilbert and serpentine transect designs. On the other hand, systematic line-transect designs  
485 provide the best spatial coverage but require many transects for good model performance. There is a tradeoff between obtaining a useable posterior and employing a simple design. We recommend the Hilbert or serpentine designs if path length is heavily constrained, but systematic line transects are fine if the path can be long. A simulation using available prior information for a specific  
490 study will be helpful in assessing if a given length is “long enough” to use a line-transect design.

## Appendix A. Profiles of designs across all datasets





## Appendix B. Tables summarizing results

Table B.2: Summary of model fitting and spatial prediction for each design applied to a single LGCP dataset.

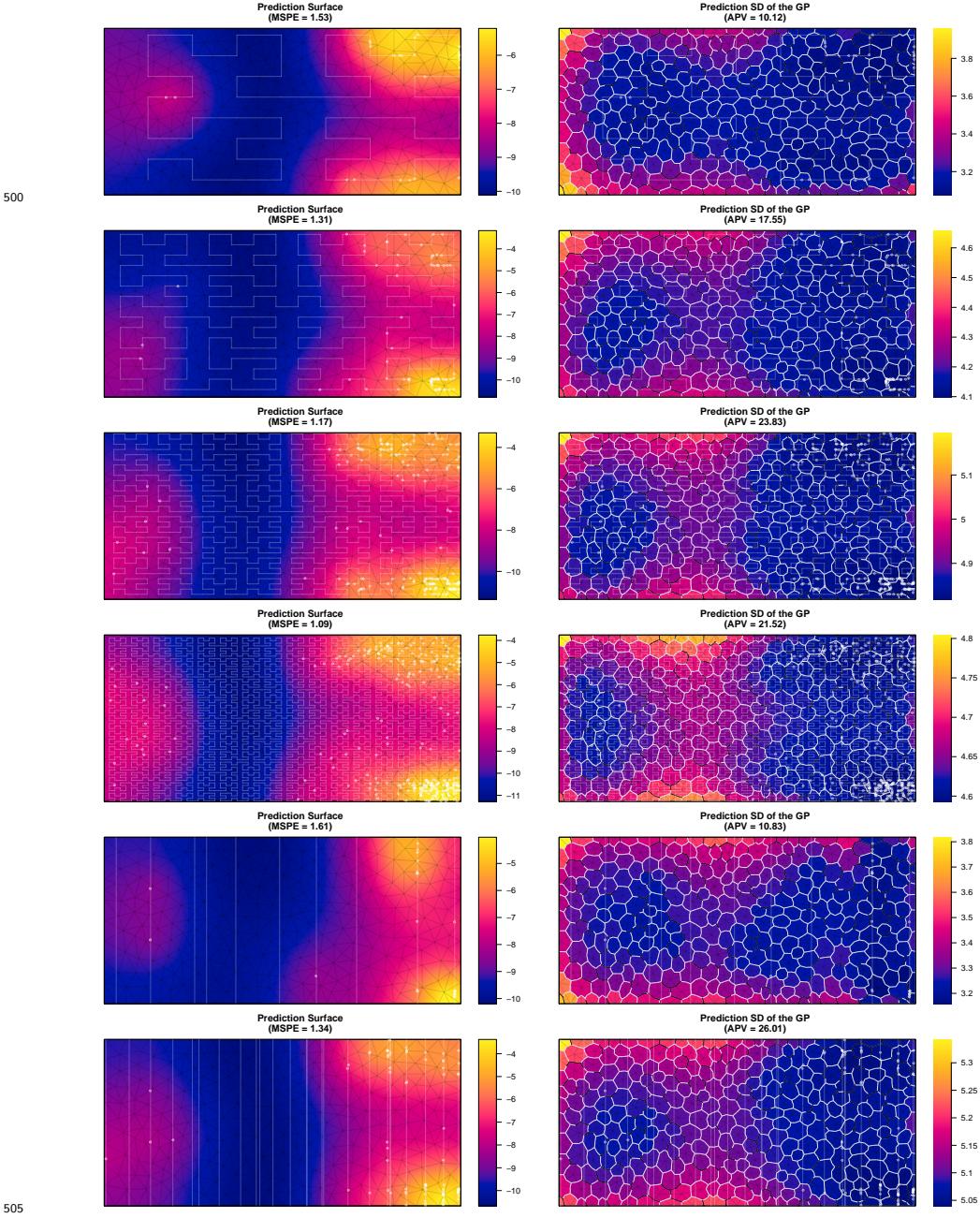
Scheme	Effort	Not Converged	MSPE > 100	Mean Captured	MSPE		APV	
					Median	IQR	Median	IQR
SRS	Low	0%	21%	85%	3.02	3.71	25.2	42.2
	Medium	0%	7%	95%	1.90	0.593	16.2	10.6
	High	0%	5%	97%	1.79	0.442	15.2	7.17
	Very High	0%	1%	100%	1.61	0.399	15.5	4.93
Systematic	Low	0%	34%	76%	3.29	2260	36.8	61.4
	Medium	0%	6%	94%	1.75	0.370	18.4	10.2
	High	0%	0%	100%	1.65	0.216	18.0	11.4
	Very High	0%	0%	100%	1.62	0.158	17.3	5.27
Inhibitory, 10% Close Pairs	Low	0%	32%	77%	3.02	2094	26.9	60.6
	Medium	0%	5%	95%	1.96	0.570	16.7	10.7
	High	0%	0%	100%	1.73	0.367	16.7	5.70
	Very High	0%	0%	100%	1.60	0.345	16.6	5.89
Inhibitory, 20% Close Pairs	Low	1%	28%	80%	3.57	1290	32.2	46.5
	Medium	0%	9%	92%	1.96	0.543	16.4	8.29
	High	0%	0%	100%	1.68	0.382	17.5	5.98
	Very High	0%	0%	100%	1.59	0.289	17.2	5.07
Serpentine, 5 zigzags	Low	0%	11%	92%	2.37	1.33	27.9	35.6
	Medium	0%	3%	97%	1.88	0.350	13.9	9.26
	High	0%	0%	100%	1.64	0.268	14.6	9.27
	Very High	0%	0%	100%	1.55	0.123	16.9	2.87
Serpentine 8 zigzags	Low	0%	15%	87%	2.61	2.07	29.2	33.3
	Medium	0%	0%	100%	1.88	0.408	18.3	8.15
	High	0%	0%	100%	1.64	0.245	17.6	6.25
	Very High	0%	0%	100%	1.53	0.272	16.4	5.41
LHS-TSP	Low	2%	31%	71%	2.90	1777	22.9	57.5
	Medium	0%	15%	88%	1.92	0.730	14.8	14.4
	High	0%	0%	100%	1.64	0.335	15.8	6.80
	Very High	0%	0%	100%	1.53	0.298	16.2	5.61
Hilbert	Low	0%	17%	86%	2.54	1.44	25.6	44.2
	Medium	0%	7%	95%	2.10	0.748	16.7	7.96
	High	0%	1%	99%	1.67	0.438	14.3	5.54
	Very High	0%	0%	100%	1.46	0.388	15.6	4.57

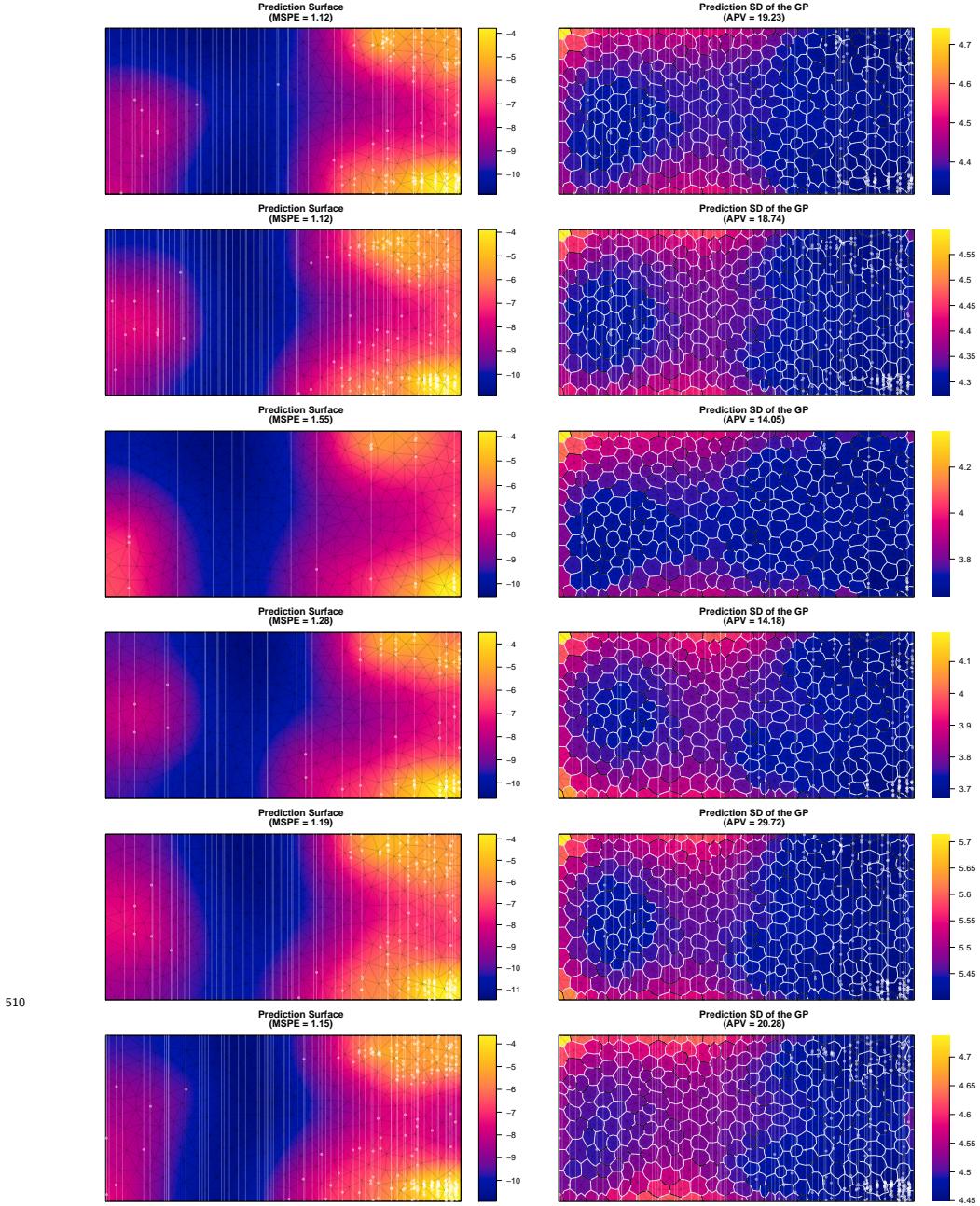
Table B.3: Summary of model fitting and spatial prediction for each design applied to a single LGCP with Clusters dataset.

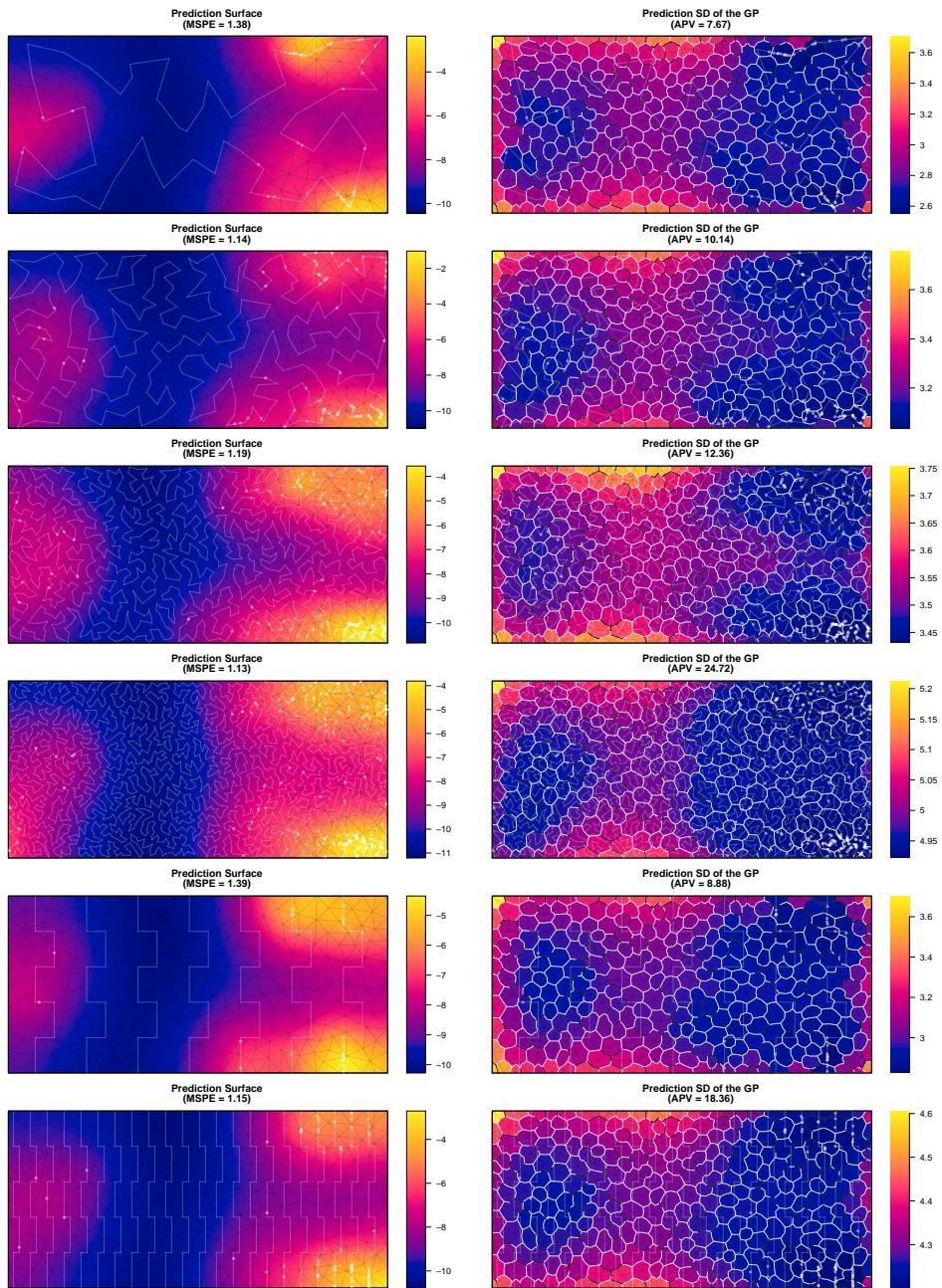
Scheme	Effort	Not Converged	MSPE > 100	Mean Captured	MSPE		APV	
					Median	IQR	Median	IQR
SRS	Low	0%	6%	98%	1.22	0.339	5.02	5.72
	Medium	0%	2%	99%	0.902	0.219	4.32	3.89
	High	0%	1%	99%	0.717	0.216	4.45	3.28
	Very High	0%	0%	100%	0.619	0.223	4.81	2.97
Systematic	Low	0%	3%	98%	1.08	0.338	4.50	4.67
	Medium	0%	0%	100%	0.812	0.199	4.27	2.85
	High	0%	0%	100%	0.671	0.189	5.01	2.02
	Very High	0%	0%	100%	0.596	0.0846	5.33	1.39
Inhibitory, 10% Close Pairs	Low	1%	6%	97%	1.20	0.326	6.93	6.78
	Medium	0%	0%	100%	0.887	0.245	4.92	3.33
	High	0%	0%	100%	0.694	0.167	4.64	2.27
	Very High	0%	0%	100%	0.599	0.144	4.47	2.21
Inhibitory 20% Close Pairs	Low	0%	2%	100%	1.23	0.287	6.42	3.59
	Medium	0%	0%	100%	0.832	0.222	4.19	3.82
	High	0%	0%	100%	0.701	0.253	4.53	3.57
	Very High	0%	0%	100%	0.630	0.204	5.08	2.91
Serpentine, 5 zigzags	Low	0%	5%	96%	1.22	0.364	6.35	8.95
	Medium	0%	0%	100%	0.808	0.259	4.32	4.28
	High	0%	0%	100%	0.597	0.137	4.23	1.73
	Very High	0%	0%	100%	0.530	0.0924	4.32	2.50
Serpentine, 8 zigzags	Low	0%	5%	97%	1.11	0.342	6.10	5.77
	Medium	0%	0%	100%	0.795	0.163	5.14	3.19
	High	0%	0%	100%	0.643	0.235	4.61	2.75
	Very High	0%	0%	100%	0.564	0.0621	5.48	1.65
LHS-TSP	Low	0%	14%	91%	1.21	0.374	5.73	8.92
	Medium	0%	0%	100%	0.858	0.281	5.09	3.81
	High	0%	0%	100%	0.658	0.160	4.75	3.12
	Very High	0%	0%	100%	0.635	0.198	5.79	2.96
Hilbert	Low	0%	8%	94%	1.14	0.451	5.57	6.15
	Medium	0%	0%	100%	0.871	0.358	5.02	5.03
	High	0%	0%	100%	0.669	0.152	4.86	2.64
	Very High	0%	0%	100%	0.542	0.138	5.95	2.82

## Appendix C. Additional plots of posterior predictions

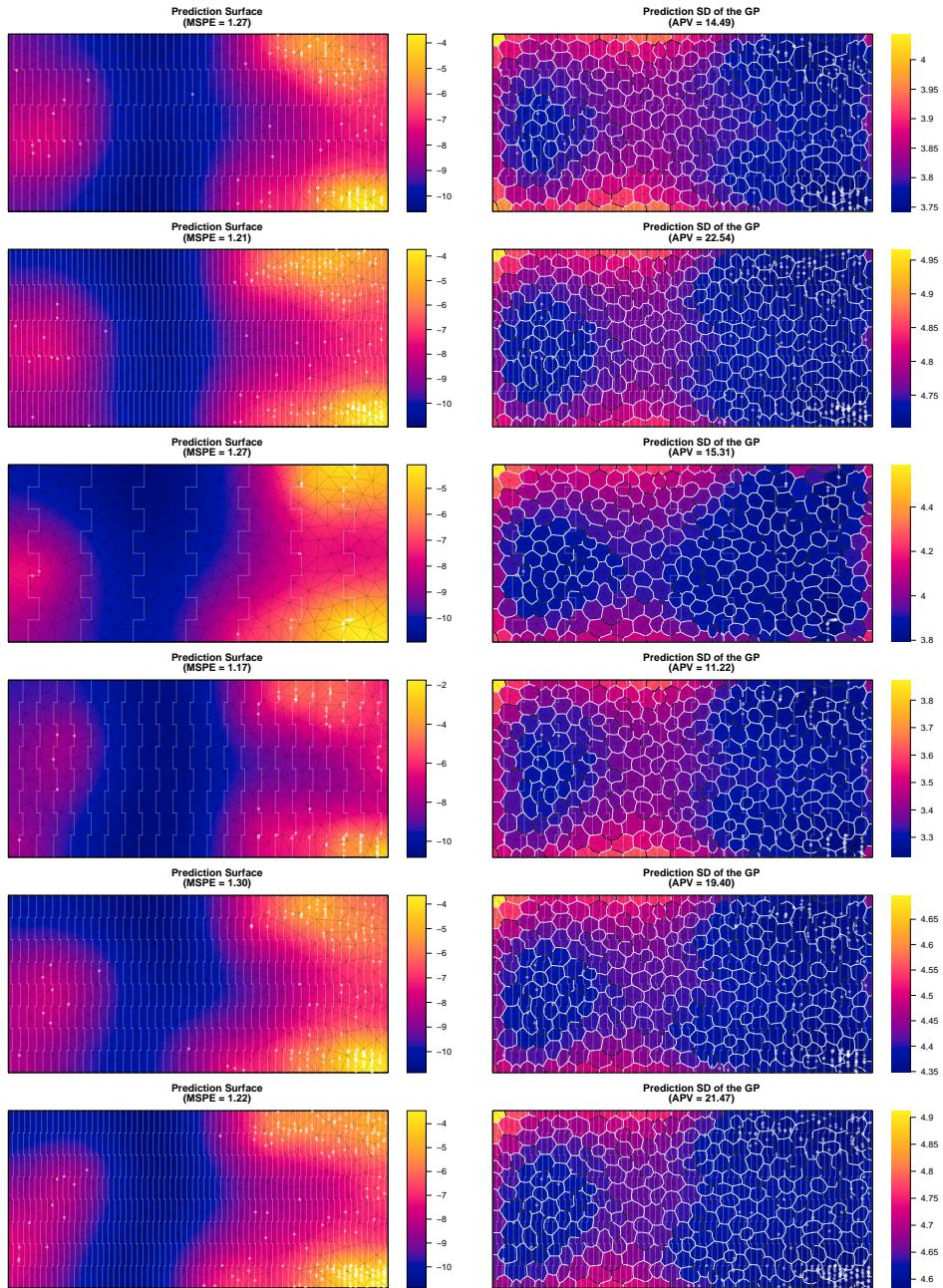
### Appendix C.1. Minimum-MSPE surfaces, LGCP dataset



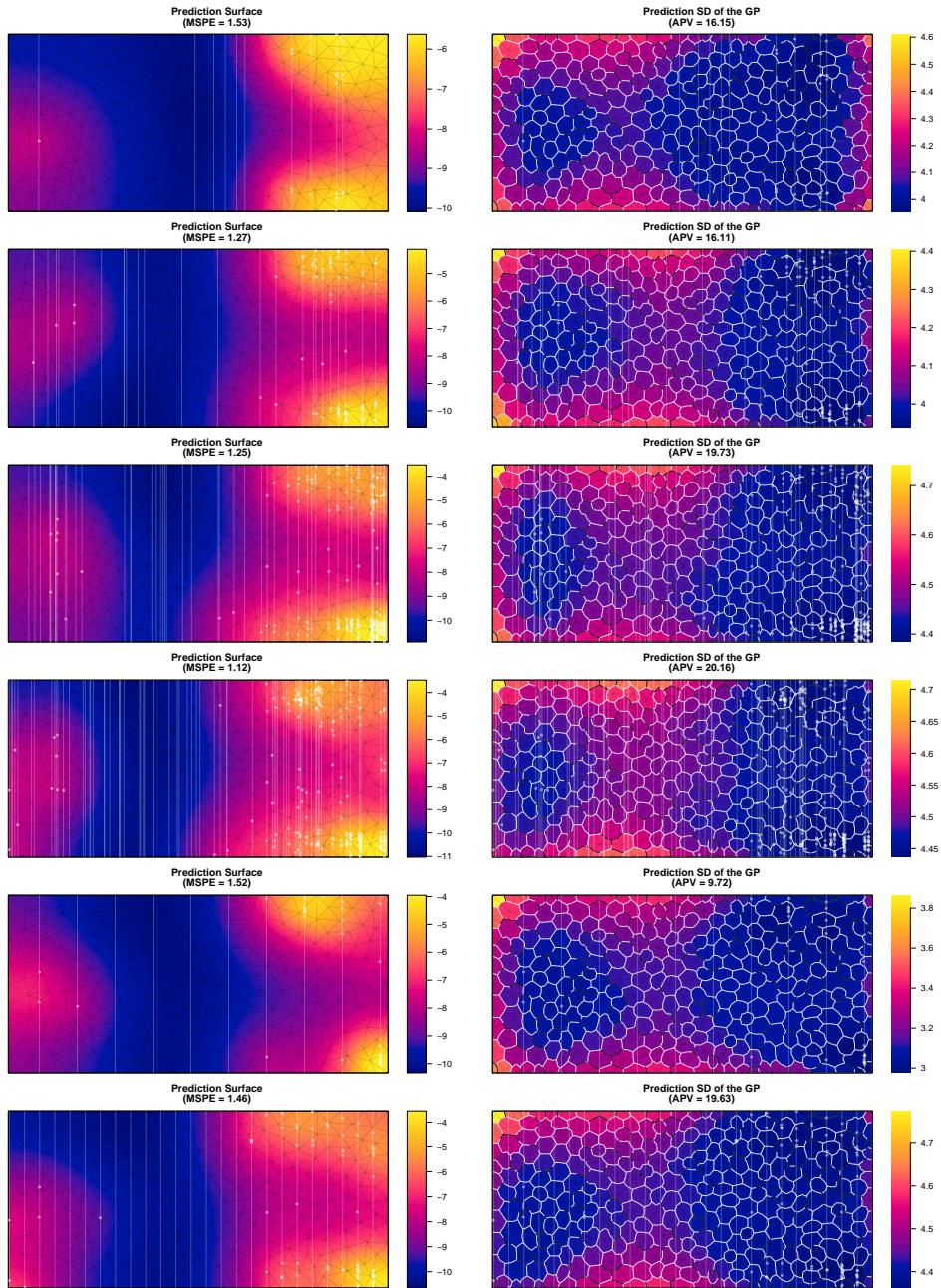




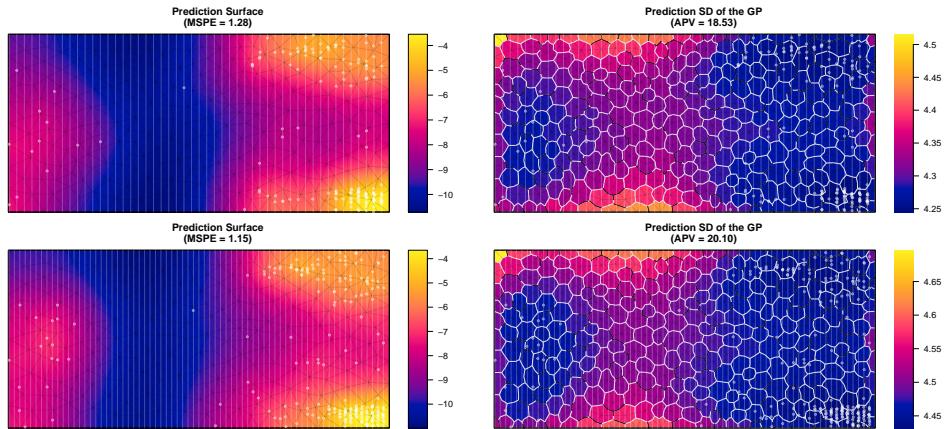
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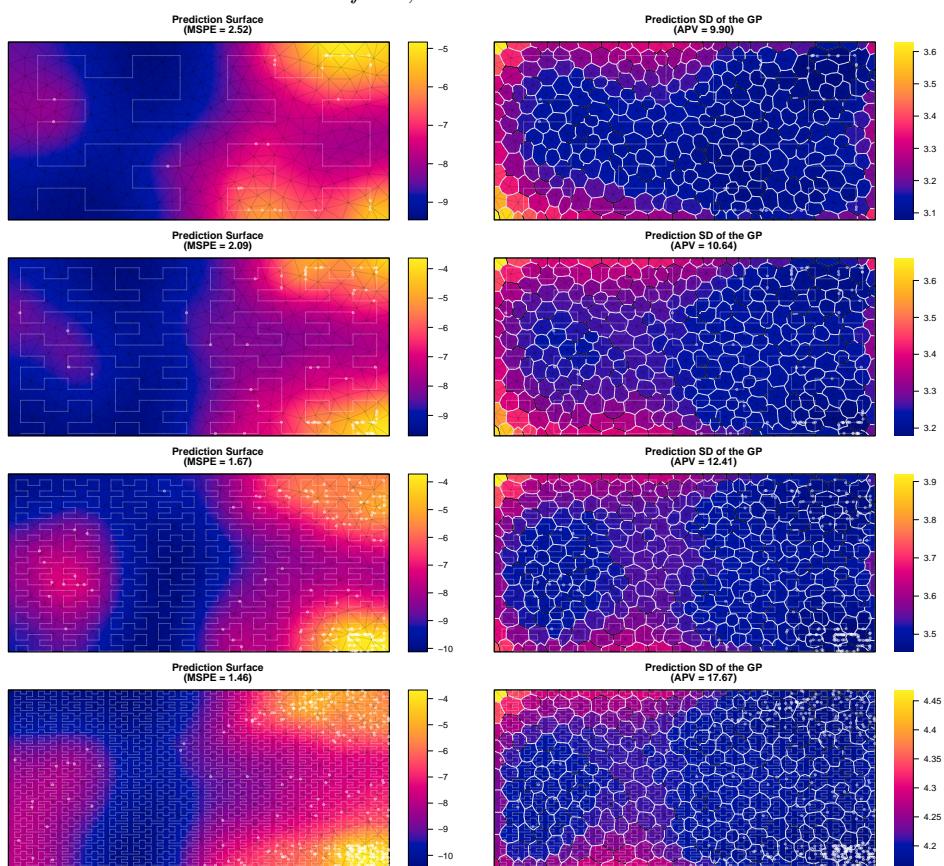


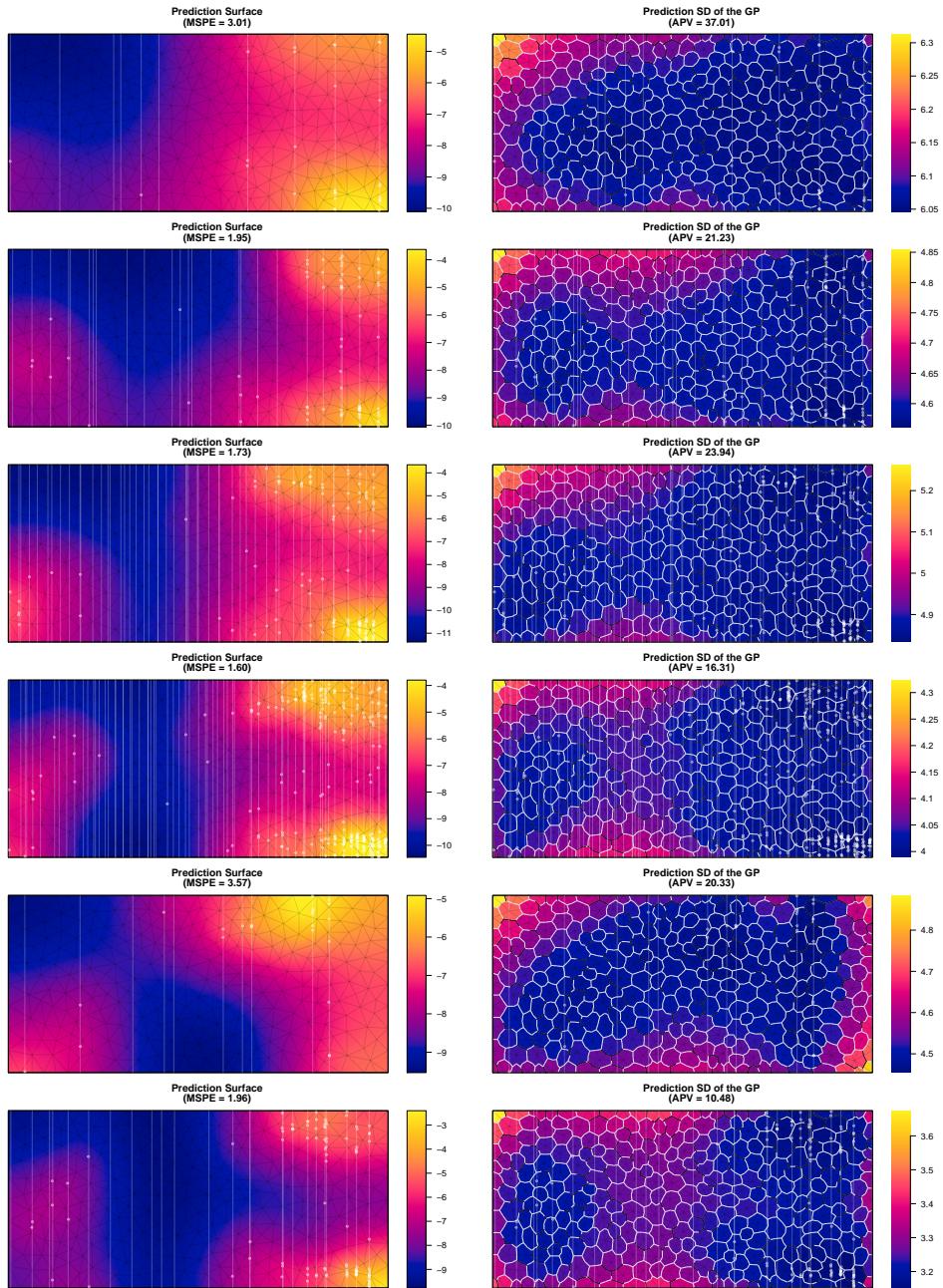
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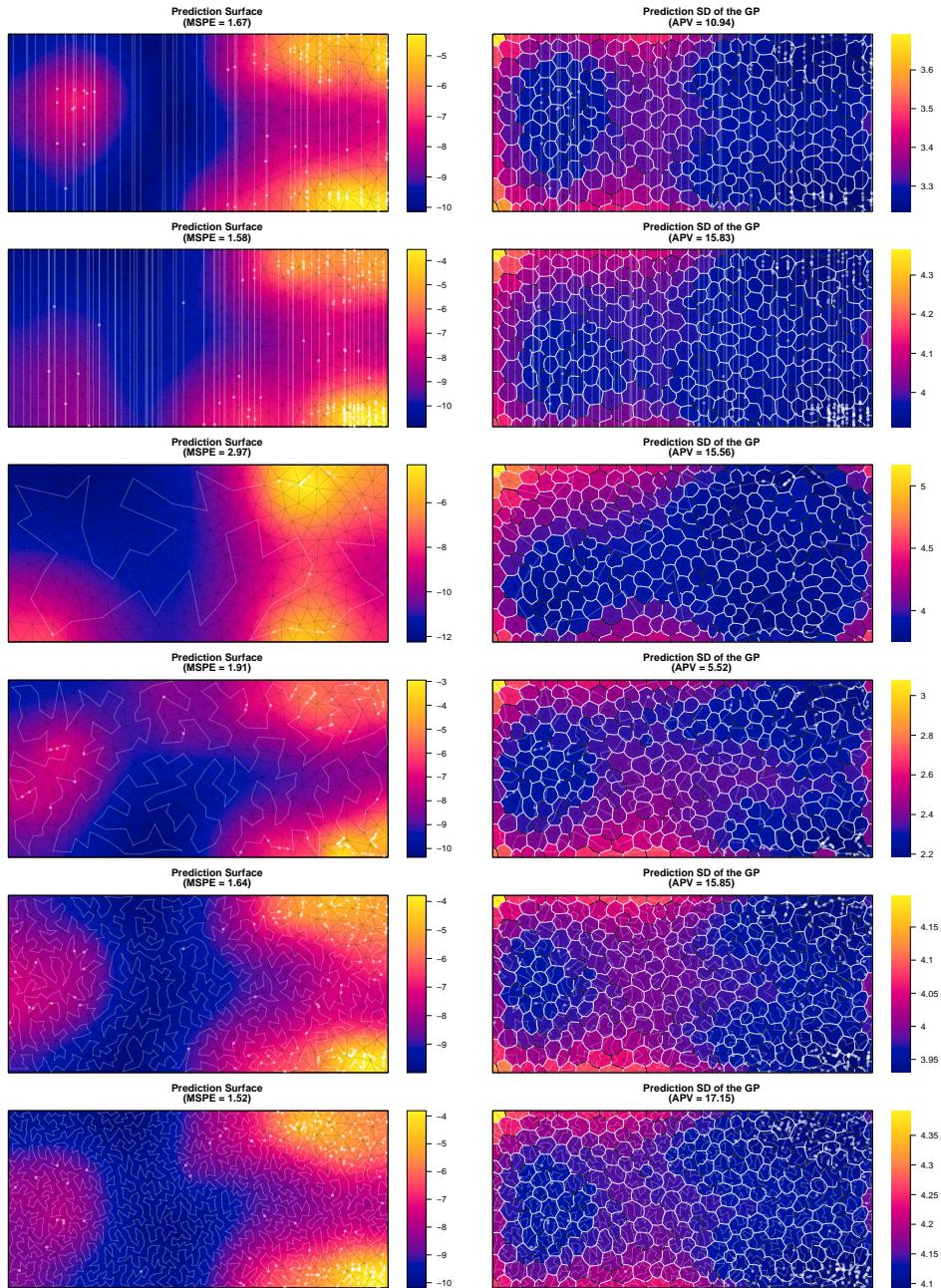
### Appendix C.2. Median-MSPE surfaces, LGCP dataset

535

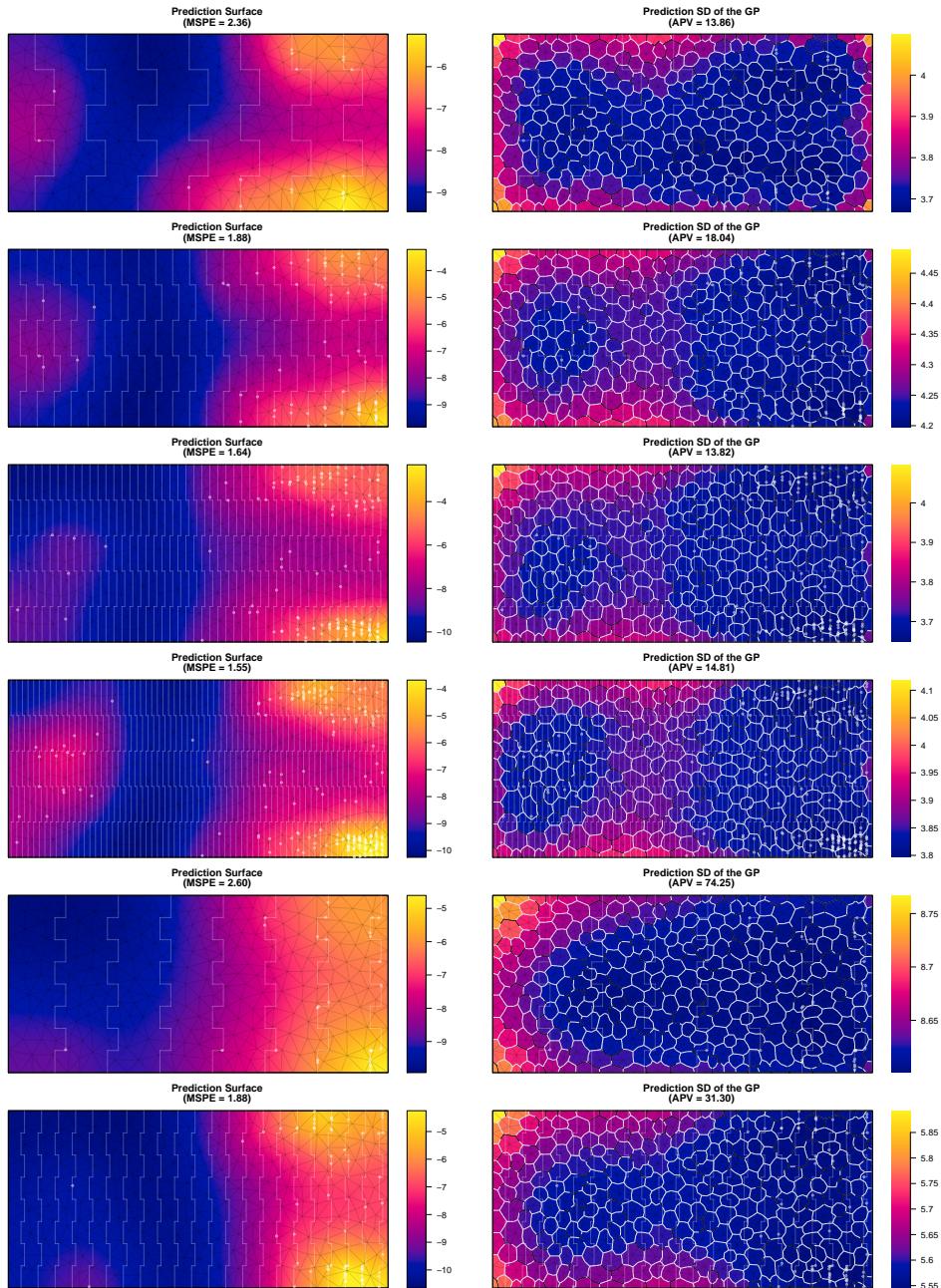




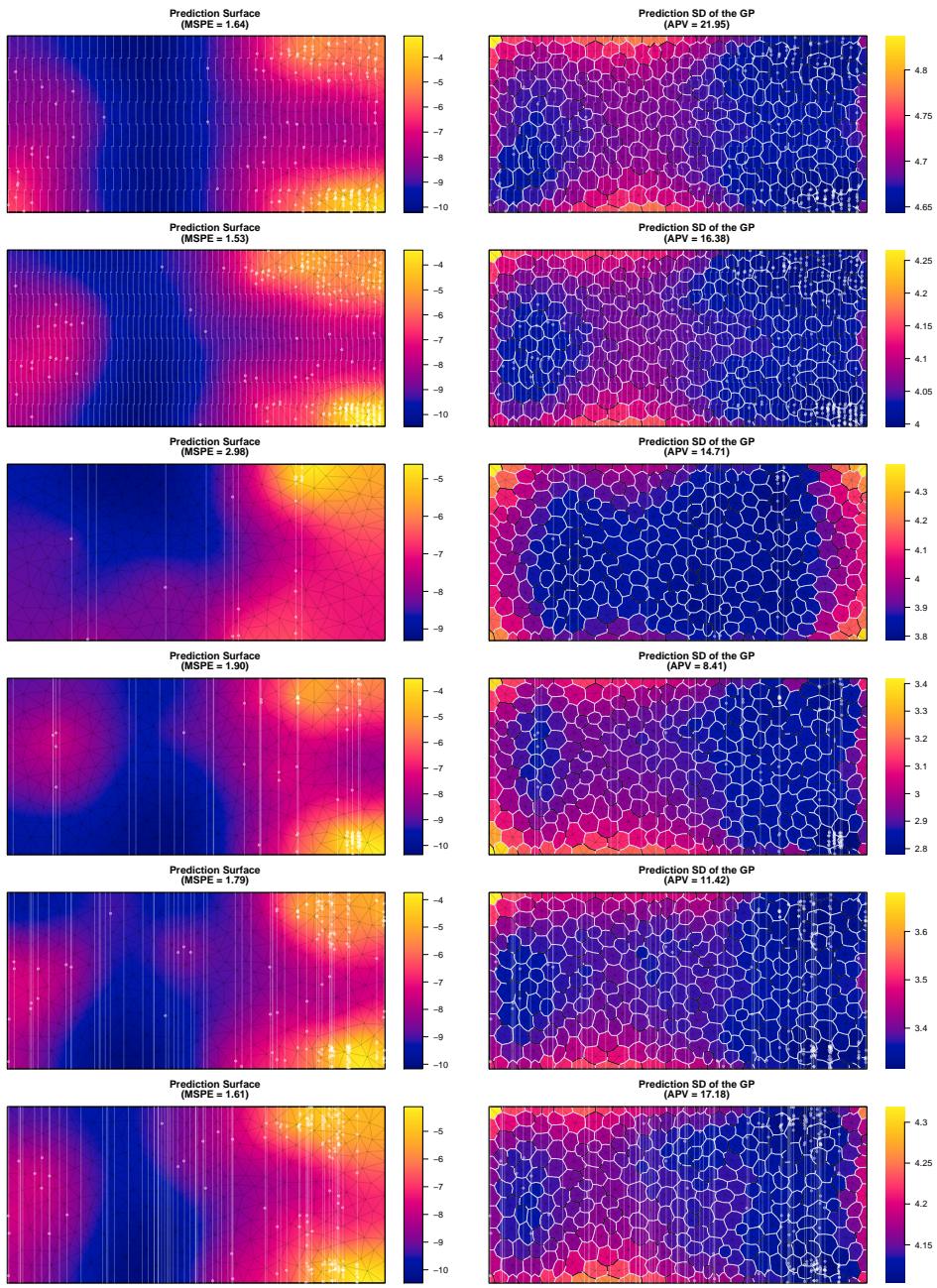
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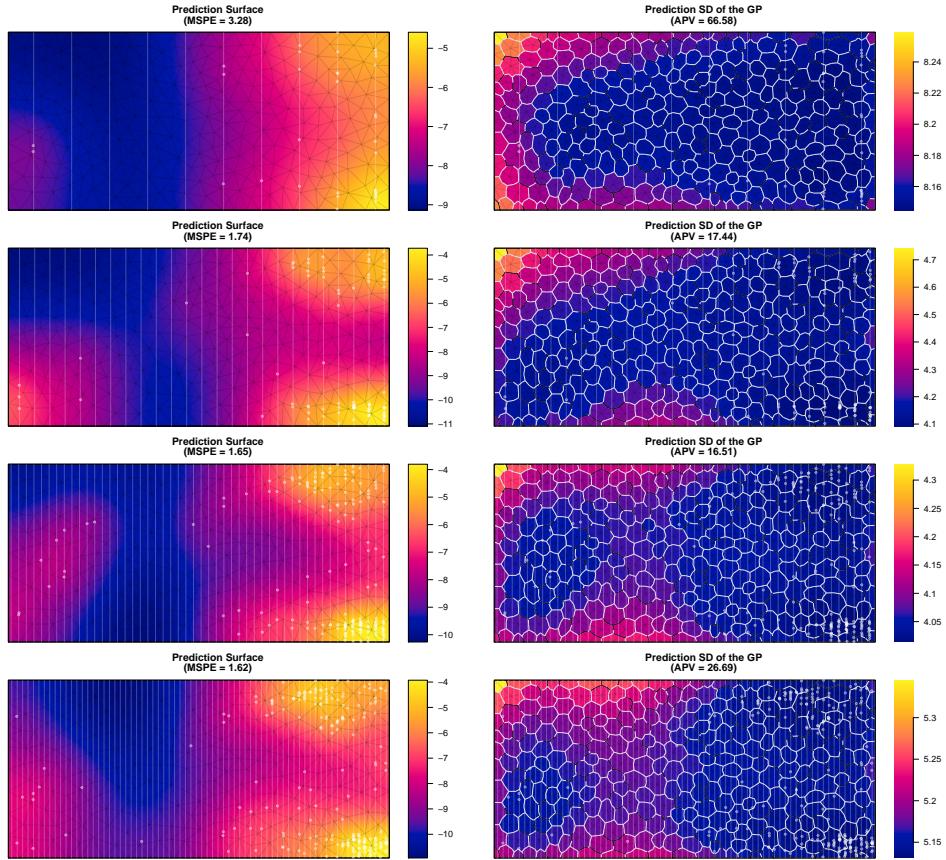
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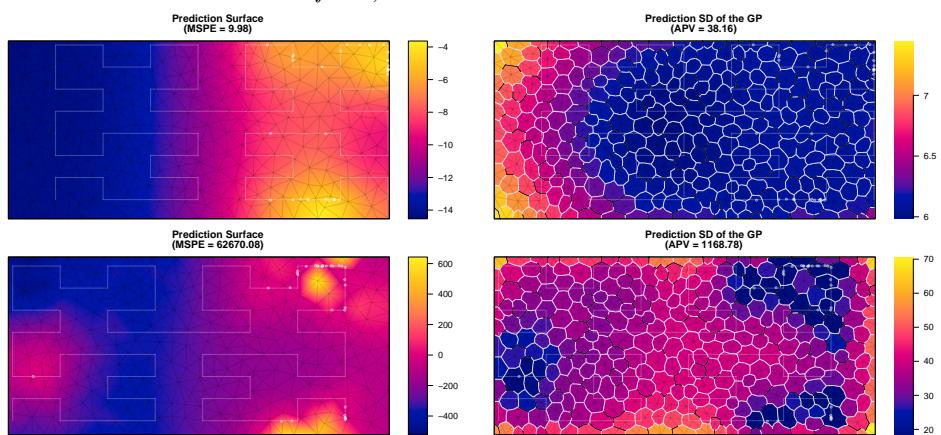
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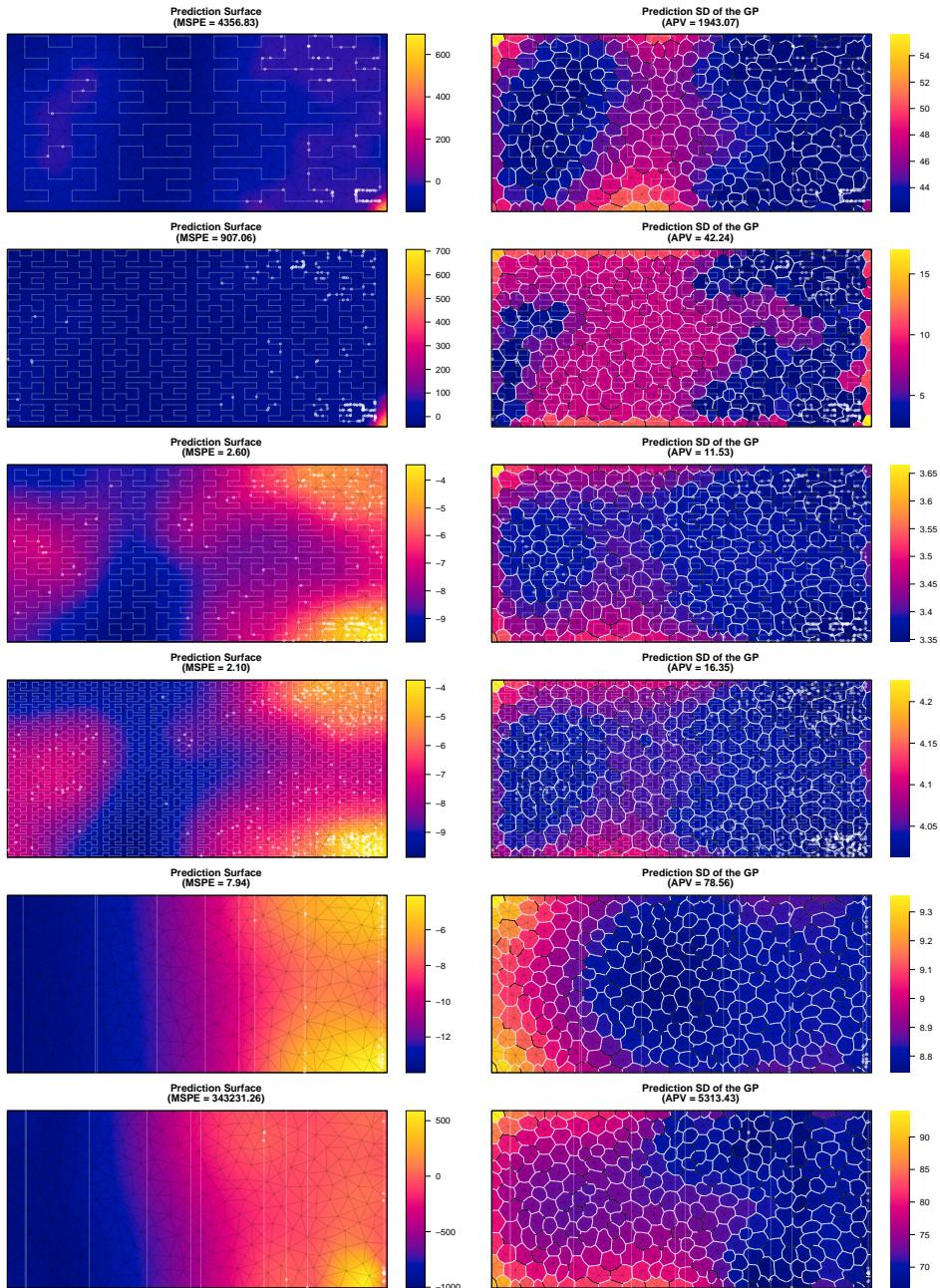
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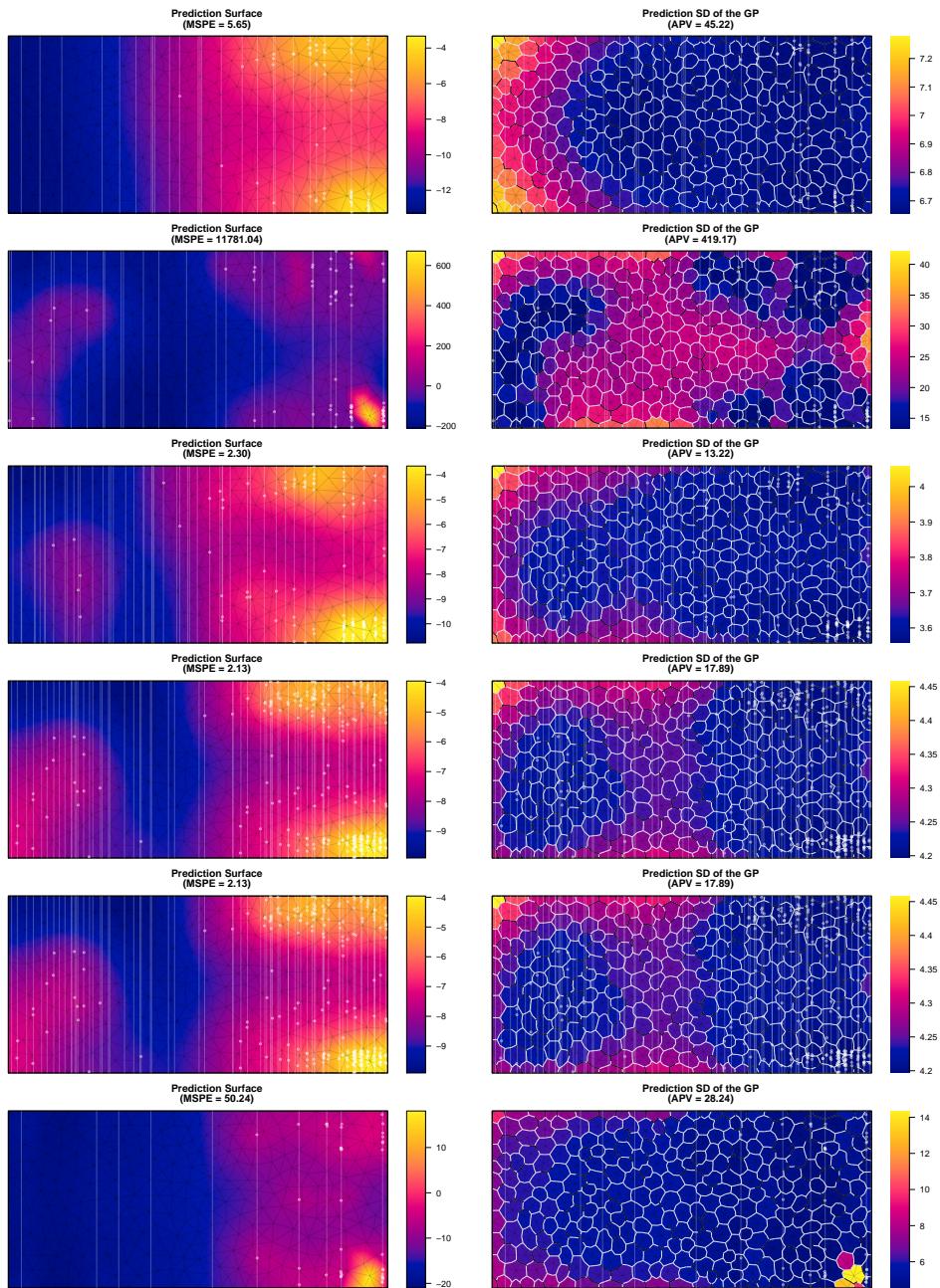
565      Appendix C.3. Max-MSPE surfaces, LGCP dataset



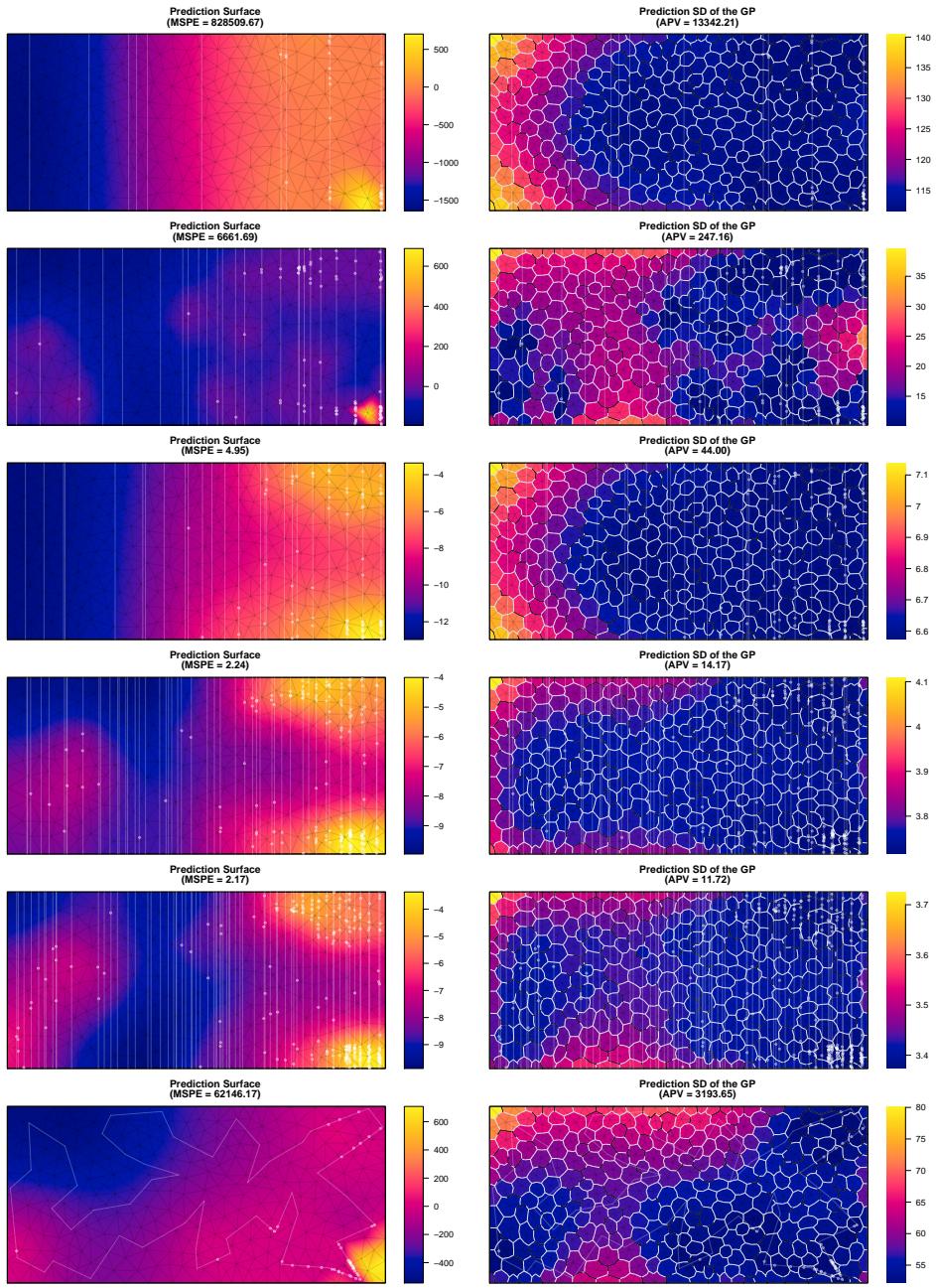
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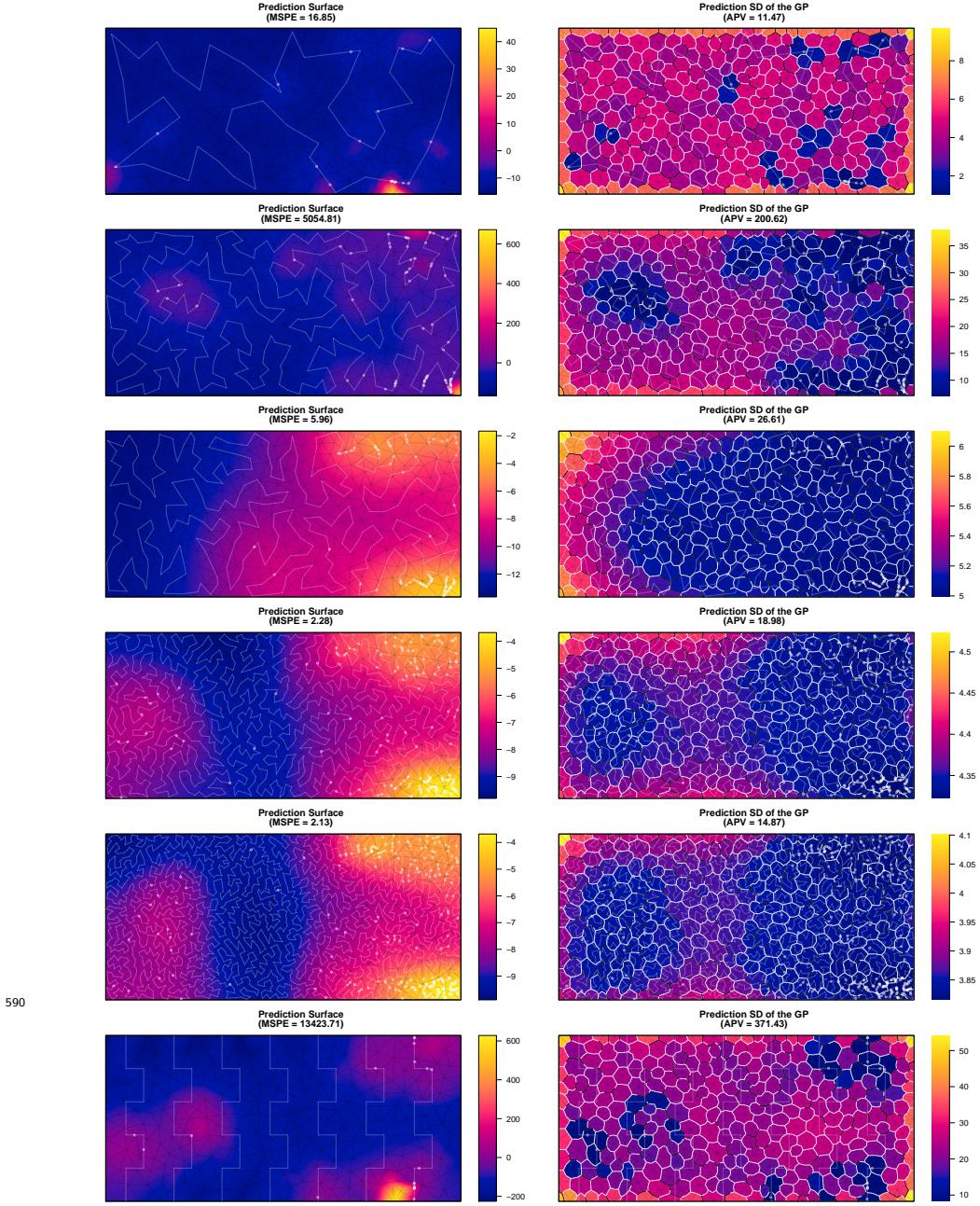
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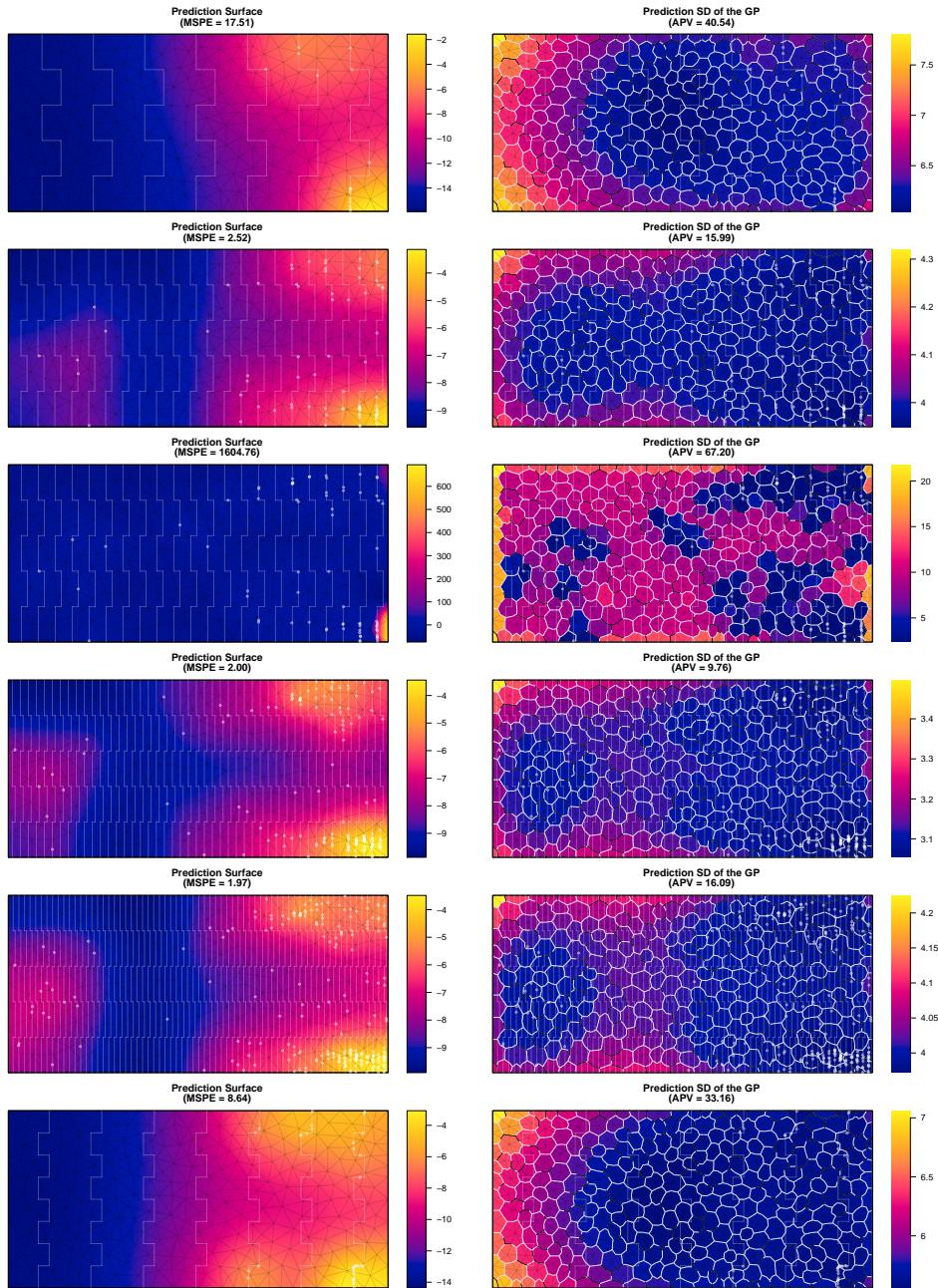
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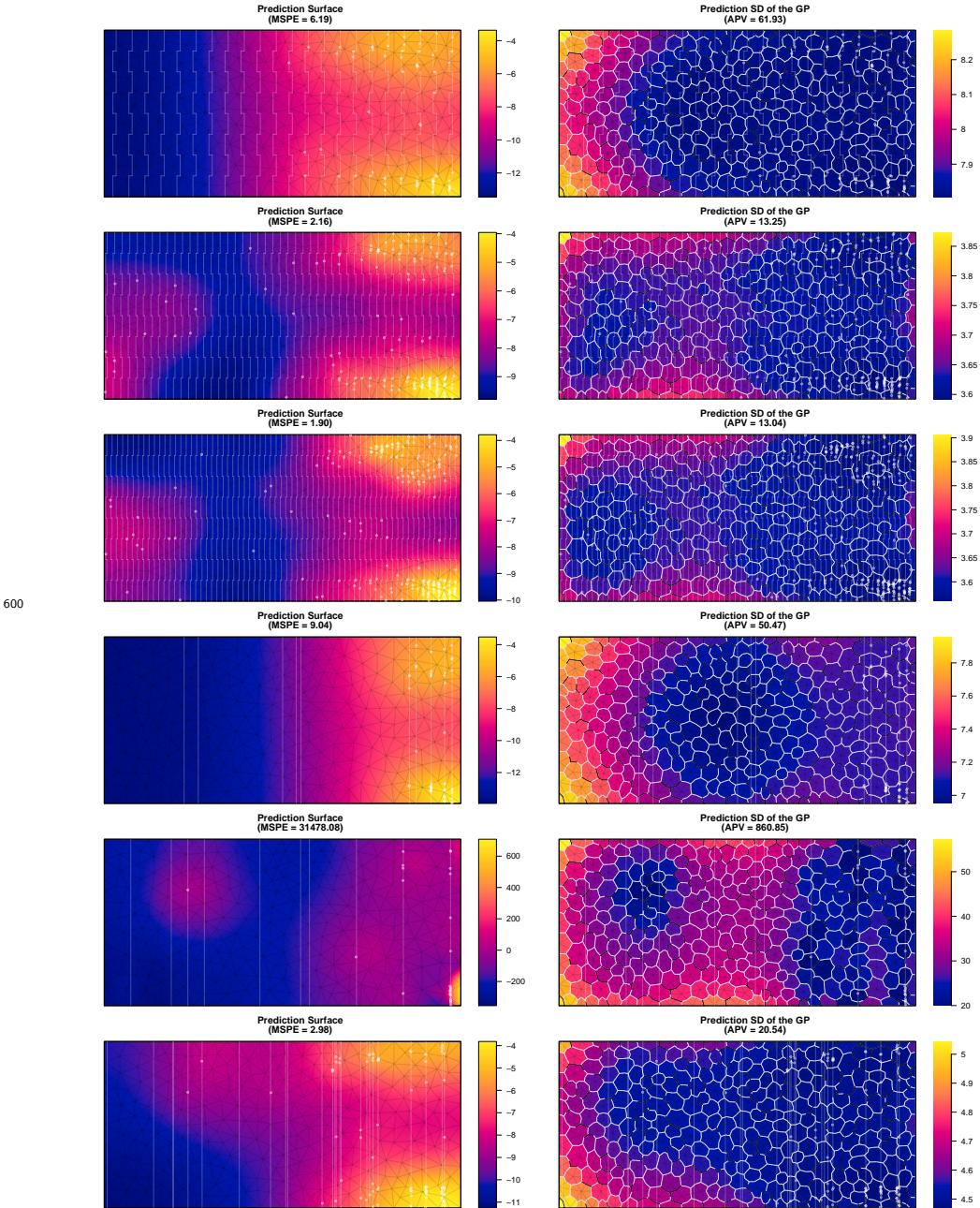


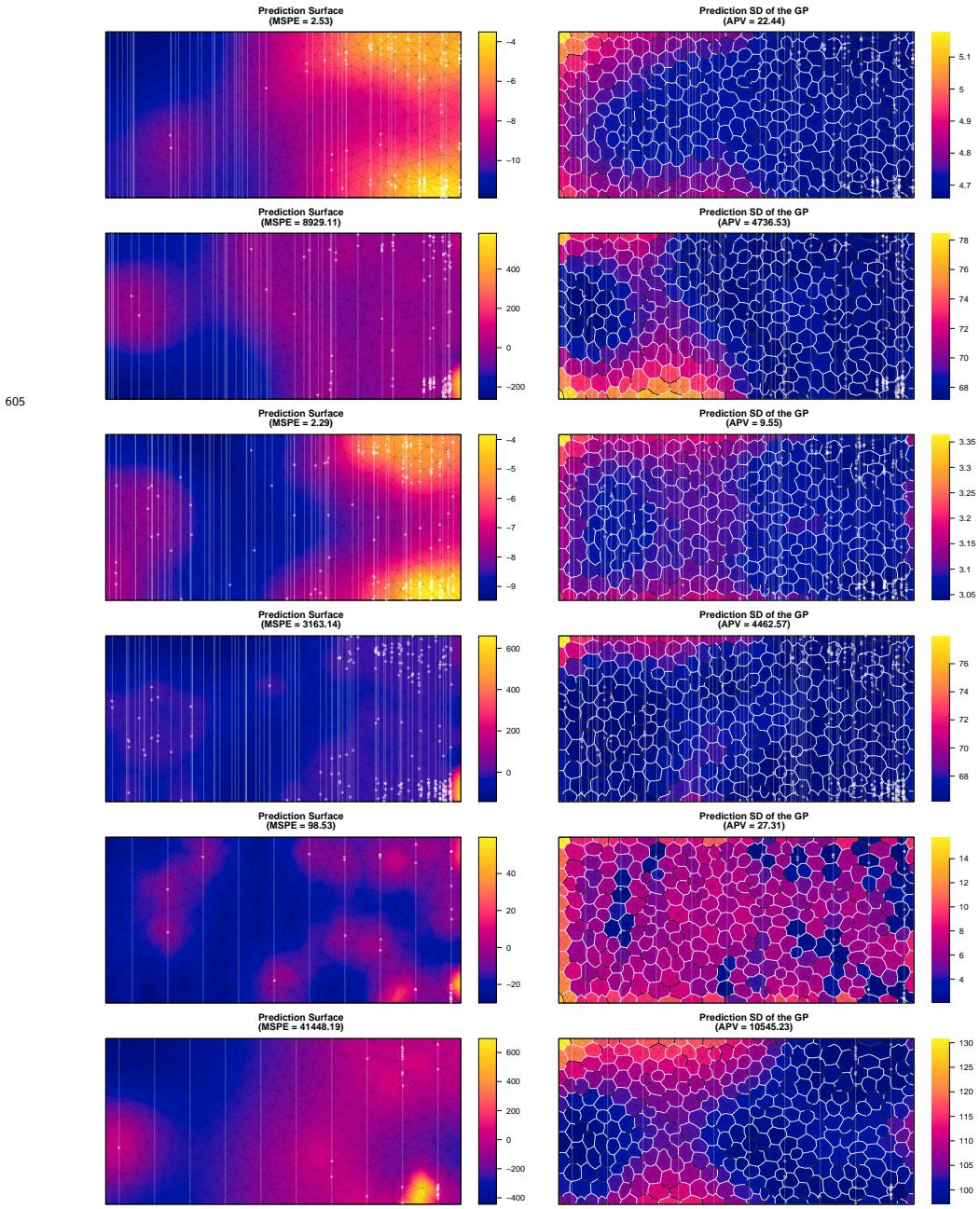
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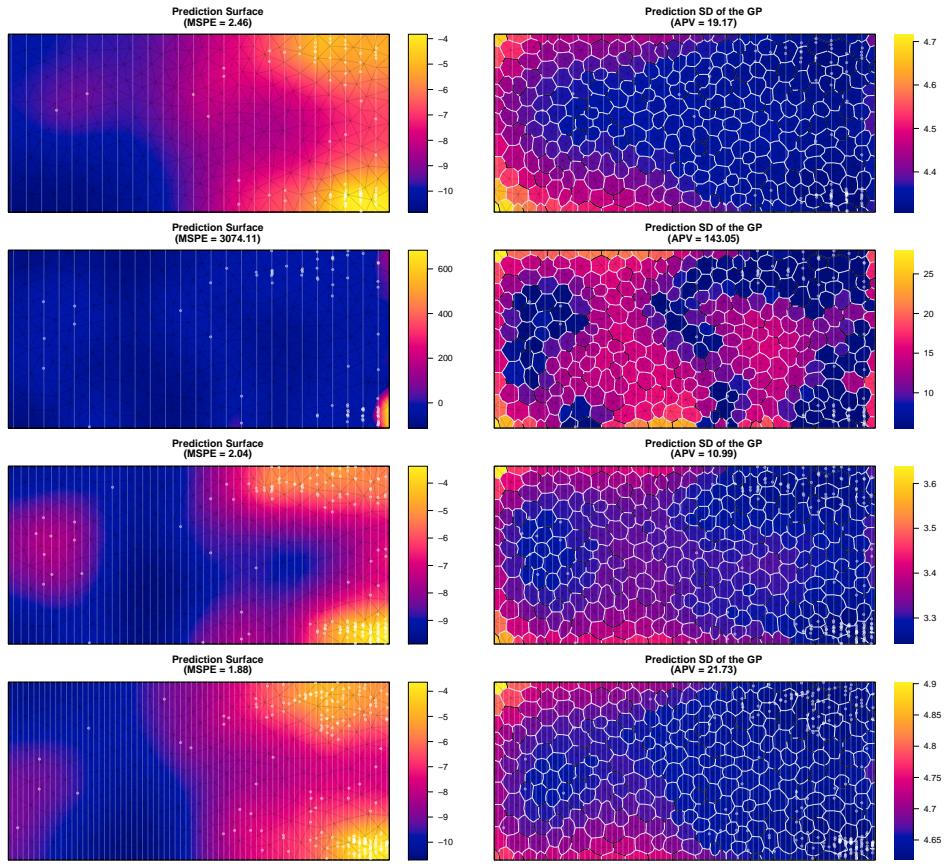
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