MONASH University
Information Technology

FIT5226
Multi-agent System & Collective Behaviour

Wk 2: Macroscopic Stochastic Models, Markov Chains

Bernd Meyer, July 2024

Stochastic model

A model describing how the <u>probability</u> of a system being in different states changes over time.

Why do we need stochastic models?

Stochastic Process

A chance experiment takes outcome values in a sample space.

A random variable assigns a unique numerical value (typically real) to outcomes in a chance experiment.

An indexed family of random variables $X(t), t \in T$

Stochastic Process

a process where the rule for making a transition to a new state of the system at time t+1 (or t+dt, if time is continuous) from the current state at time t is a random variable

Unlike a deterministic process, **only** the **probability** of being in a given state *n* at time *t* **can be specified**.

In the following we will look at **discrete** states

Markov chain

http://setosa.io/ev/markov-chains/

- Set of states $S = \{s_1, s_2, s_3, ..., s_r\}$
- Probability of going from s_i to s_j is given by p_{ij} (transition probability)
- Transition matrix, $A = \{p_{ij}\}$
- For each i $\sum_j p_{ij} = 1$
- Initial distribution **u**, probability vector of size r

conditional probability distribution of future states depends *only* on the present state

Examples of Markov Chains

- Infection processes
- Population processes
- Evolution
- Decision Making
- **Decisions by independent agents**

MC Example

The Land of Oz is blessed by many things, but not by good weather.
They never have two nice days in a row.
If they have a **nice** day, they are just as likely to have **snow** as **rain** the next day.
If they have **snow** or **rain**, they have an even chance of having the same the next day.
If there is change from **snow** or **rain**, only half of the time is this a change to a **nice** day.

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{ccc} R & N & S \end{array} \\ \begin{array}{c} R \\ N \\ S \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \end{array}$$

Simulation

- At every time step there is a discrete random variable that needs to be sampled.
- While stopping criteria is not met, decide what is the next state.

$$\vec{u}_{t+1} = \vec{u}_t P$$

If transitions are explicit....

$$P\{X = x_j\} = p_j \qquad j = 0, 1, \ldots \sum_j p_j = 1$$

$$X = \begin{cases} x_0 & \text{if } U < p_0 \\ x_1 & \text{if } p_0 \leq U < p_0 + p_1 \\ \vdots & \\ x_j & \text{if } \sum_{i=0}^{j-1} p_i \leq U < \sum_{i=0}^{j} p_i \\ \vdots & \end{cases}$$

with U, uniformly distributed on (0,1)

Follows from: $0 < a < b < 1, P\{a \leq U < b\} = b - a$

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{ccc} R & N & S \end{array} \\ \begin{array}{c} R \\ N \\ S \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \end{array}$$

If it is raining today, what is the probability that it will be snowing in **two** days?

$$\mathbf{P} = \begin{array}{c} \\ R \\ N \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & N & S \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \end{array}$$

Given that today we are in state *i*, what's the chance that we are in state *j* in **two** steps...

$$p_{ij}^{(2)} = \sum_k p_{ik} p_{kj}$$

for a Markov chain given by transition matrix $\mathbf{P}$

$p_{ij}^{(n)}$ the probability to go from state s_i to state s_j

in n steps, is given by the *ij-th entry of* $\mathbf{P}^n$

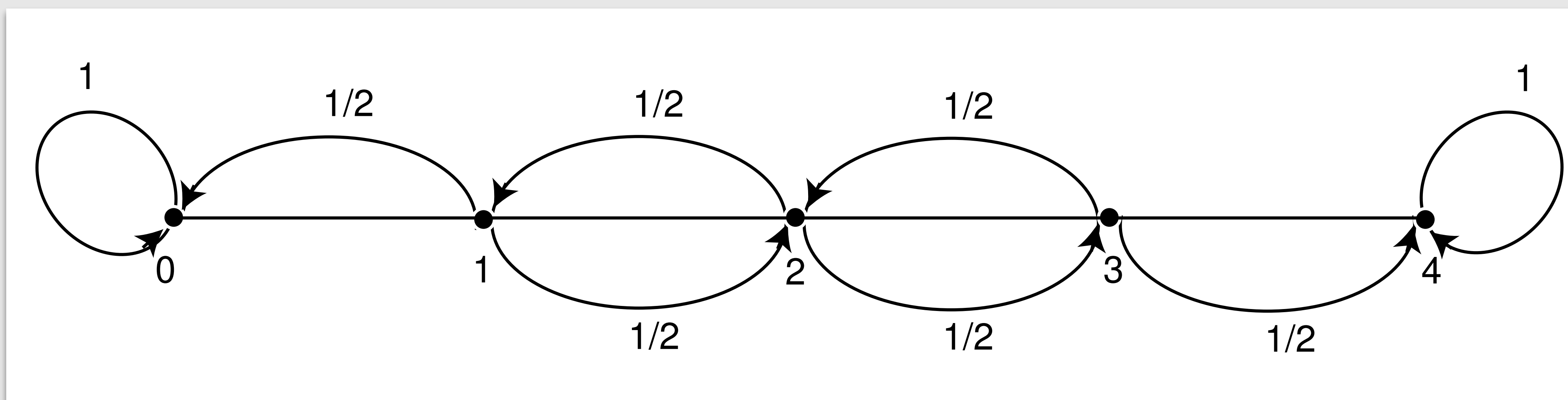
Long-term behaviour:

$$\mathbf{P}^* = \lim_{k \to \infty} \mathbf{P}^k$$

$$
P = \begin{array}{cc}
& \begin{array}{ccc} \text{Monash} & \text{MelbUni} & \text{RMIT} \end{array} \\
& \begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.7 \end{pmatrix}
\begin{array}{l} \text{Monash} \\ \text{MelbUni} \\ \text{RMIT} \end{array}
\end{array}
$$

What is the student distribution in the long term?

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1.0 \end{pmatrix}$$



What happens in the long term...

$$
\mathbf{P} = \begin{array}{c c}
 & \begin{array}{ccc} R & N & S \end{array} \\
\begin{array}{c} R \\ N \\ S \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}
\end{array}
$$

What happens in the long term...

Absorbing vs Ergodic

Absorbing chains.

- A state is absorbing if it is impossible to leave (i.e., $p_{ii} = 1.0$).
- A chain is absorbing, if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (possibly in several steps)
- A state that is non-absorbing is also called transient.

Which chains are absorbing?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.7 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

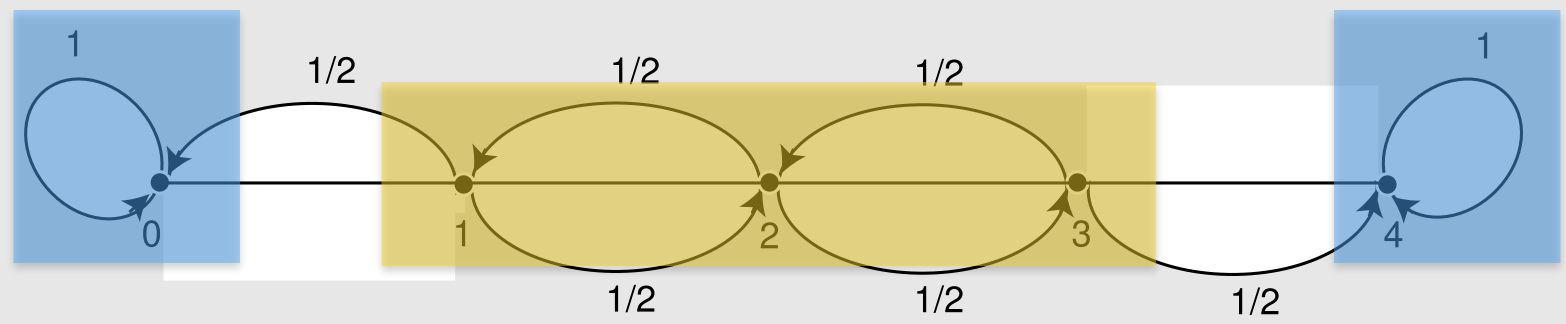
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1.0 \end{pmatrix}$$

Canonical form

$$
\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\ \begin{array}{c} \text{TR.} \\ \\ \text{ABS.} \end{array} & \left(\begin{array}{c|c} \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right) \end{array}
$$

Re-number states so that transient states come first

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\ \begin{array}{c} \text{TR.} \\ \\ \text{ABS.} \end{array} & \left(\begin{array}{c|c} \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right) \end{array}$$



$$\mathbf{P} = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 0 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{array} & \left(\begin{array}{ccc|cc} 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

Probability of being in a transient state after n steps

$$\mathbf{P}^n = \begin{array}{c c} & \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\ \begin{array}{c} \text{TR.} \\ \text{ABS.} \end{array} & \left(\begin{array}{c|c} \mathbf{Q}^n & * \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right) \end{array}$$

In an absorbing Markov chain the probability that the process will be absorbed is 1

$$\lim_{n\to\infty} \mathbf{Q}^n = \mathbf{0}$$

Expected Time to Absorption

$$F = \sum_{k=0}^{\infty} Q^k = (I_t - Q)^{-1}$$

fundamental matrix

expected time to absorption from a transient state *i* … *i-th* entry in

$$\tau = (I_t - Q)^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = F\mathbf{1}$$

Why??

Regular chains.

- A Markov chain is called **ergodic** if it is possible to go from every state to every state (not necessarily in one step).
- Ergodic chains have a unique long-run equilibrium that does not depend on history.
- A Markov chain is called **regular** if it has a positive power of the transition matrix such that all its elements are strictly positive. Every regular chain is ergodic.

Periodic chains.

- A Markov chain can cycle through a number of states (or a number of different subsets of states) in the long run.
- This is called periodic. Even though such a chain is not regular, it has a long run equilibrium.
- Computing this equilibrium is more complicated and the following only applies to regular chains.

Long-term behaviour:

$$\mathbf{P}^* = \lim_{k \to \infty} \mathbf{P}^k$$

$\mathbf{P}^* = \{p^*_{ij}\}$ if history does not matter $p^*_{ij} = p^*_{i'j}$

i.e. all rows of **P*** must be the same

$$\mathbf{P}\mathbf{P}^* = \mathbf{P} \lim_{k \to \infty} \mathbf{P}^k = \lim_{k \to \infty} \mathbf{P}^{k+1} = \mathbf{P}^*$$

$\mathbf{u} = (u_1, u_2, ..., u_n)$ is the common row of P*

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{ccc} R & N & S \end{array} \\ \begin{array}{c} R \\ N \\ S \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \end{array}$$

How to interpret **u**?

Stationary distribution.

Since u does not change with time, a single transition has no effect,

i.e. u = u P

$$\mathbf{u} = \mathbf{uP}$$

u is the left eigenvector for eigenvalue 1 of **P**

$$\vec{u}P = \lambda\vec{u}$$

$$(P - \lambda\ I)\vec{u} = 0$$

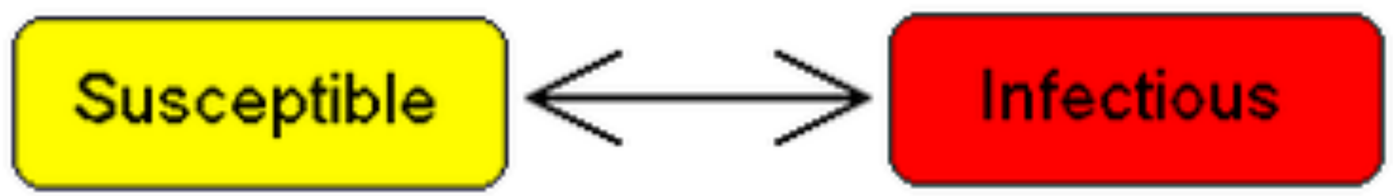
Markov Chains - biological application: SIS revisited

SIS - Infection Model

see Linda J.S. Allen,
Applications of continuous-time Markov chains and
comparison of discrete-time and continuous-time processes

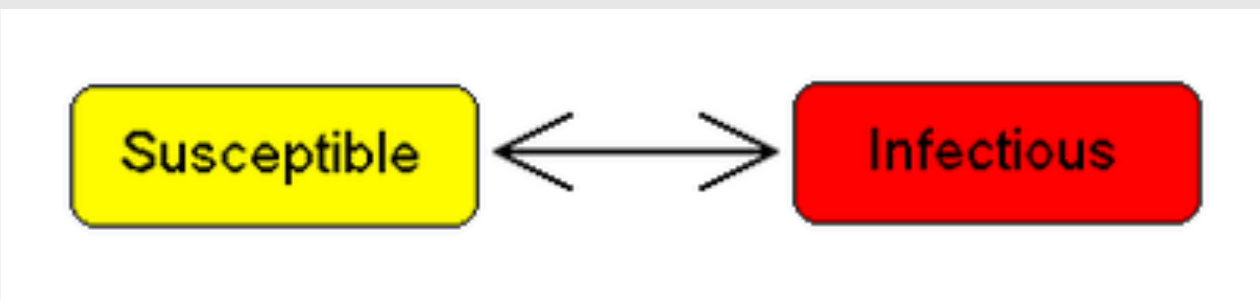
Other SD Models of Disease Spread

The (much simpler) SIS model captures diseases that do not have any relevant period of immunity.



see Wikipedia "Compartmental Models in Biology"

SIS model



$N = S + i$

$$\frac{di}{dt} = \beta S i - \gamma i$$

$$\beta S i = \beta (N - i) i = \beta (N i - i^2)$$

$$\frac{di}{dt} = \beta (N i - i^2) - \gamma i = i(\beta N - \gamma) - \beta i^2$$

"Logistic equation"

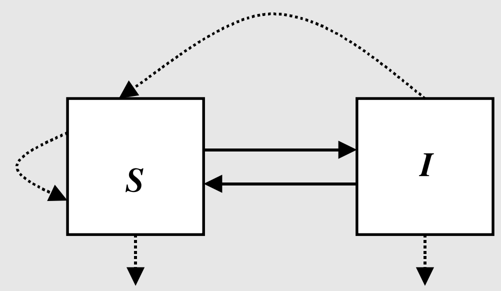
Steady state:

$$\frac{di}{dt} = 0 = (\beta N - \gamma) i - \beta i^2 \quad | \; i \neq 0$$

$$\beta N - \gamma = \beta i$$

$$i = N - \frac{\gamma}{\beta}$$

(4) SIS Epidemic Model



Deterministic Model:

$$\frac{dS}{dt} = -\frac{\beta}{N} IS + (b+\gamma) I$$

$$\frac{dI}{dt} = \frac{\beta}{N} IS - (b+\gamma) I = \frac{\beta}{N} I(N-I) - (b+\gamma) I$$

$S(t) = N - I(t)$, **where** $N =$ **constant total population size.**

with birth/death!

all births are non-infected

population size is constant

SIS Epidemic Process

Since $S(t) = N - I(t)$ and N is constant, only $I(t)$ is modeled.

Let $I(t)$ be the random variable for the number infectious at time $t = 0, \Delta t, 2\Delta t, \ldots$.

$$p_i(t) = \text{Prob}\{I(t) = i\},\ i = 0, 1, 2, \ldots, N.$$

Transition Probabilities:

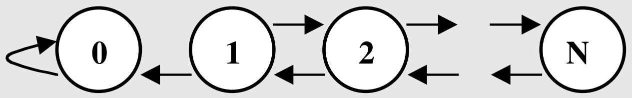
$$p_{ji}(\Delta t) = \text{Prob}\{I(t + \Delta t) = j | I(t) = i\}.$$

Transition Probabilities

$$p_{ji}(\Delta t) = \begin{cases} \dfrac{\beta i(N-i)}{N}\Delta t, & j = i+1 \\ (b+\gamma)i\Delta t, & j = i-1 \\ 1 - \left[\dfrac{\beta i(N-i)}{N} + (b+\gamma)i\right]\Delta t, & j = i \\ 0, & j \neq i+1, i, i-1. \end{cases}$$

Similar to a birth and death process:

$$p_{ji}(\Delta t) = \begin{cases} b(i)\Delta t, & j = i+1 \\ d(i)\Delta t, & j = i-1 \\ 1 - [b(i) + d(i)]\Delta t, & j = i \\ 0, & j \neq i+1, i, i-1. \end{cases}$$



Recurrent class: $\{0\}$ **Transient class:** $\{1, \ldots, N\}$.

$$\boxed{\lim_{t\to\infty} p_0(t) = 1.}$$

Three Sample Paths

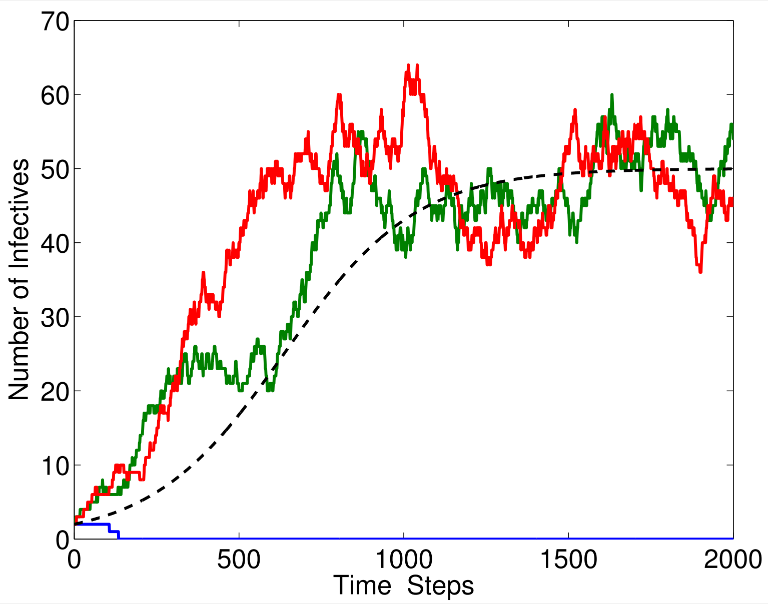


Figure 5: Three sample paths of the DTMC SIS epidemic model are graphed with the deterministic solution (*dashed curve*); $\Delta t = 0.01$, $N = 100$, $\beta = 1$, $b = 0.25$, $\gamma = 0.25$, and $I(0) = 2$.

Note the extinction path (blue)

Probability Distribution

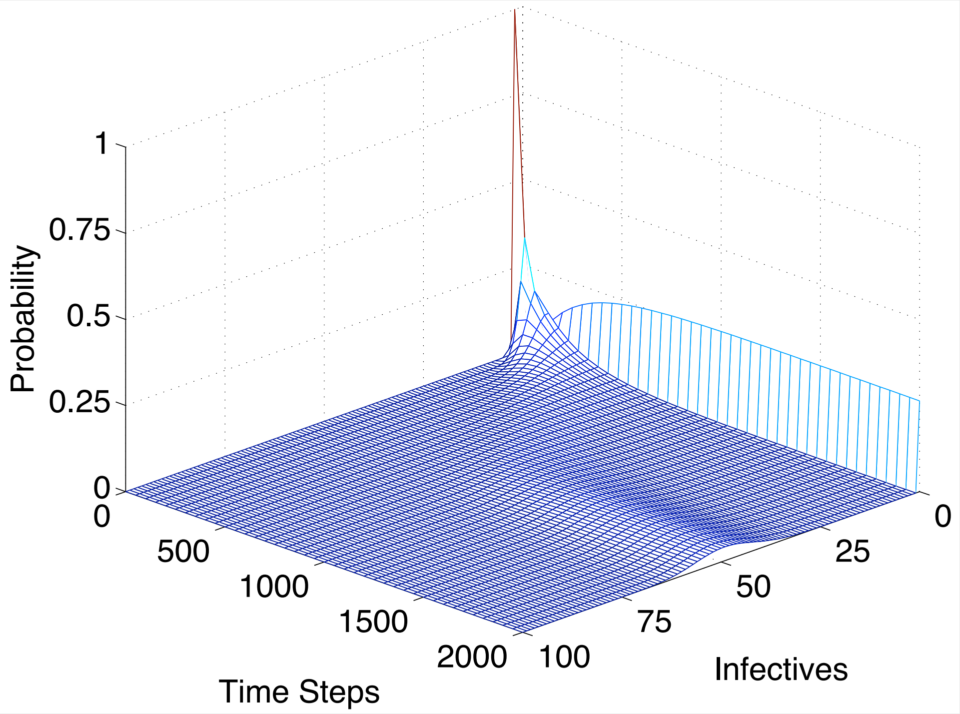
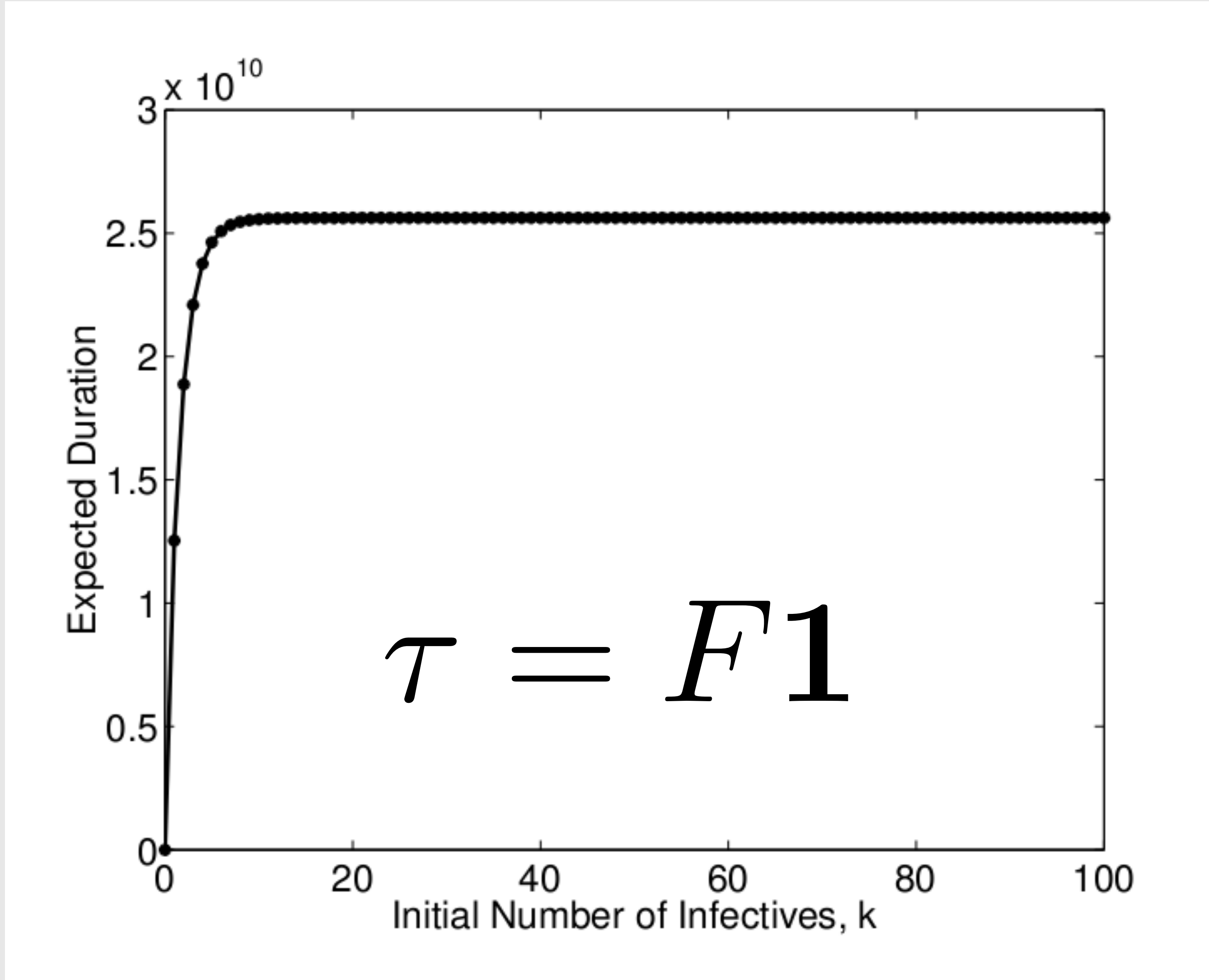


Figure 6: Probability distribution of the DTMC SIS epidemic model. $\boldsymbol{\Delta t = 0.01,\ N = 100,\ \beta = 1,\ b = 0.25,\ \gamma = 0.25,\ I(0) = 2,}$



The chain is absorbing (resulting in extinction of the disease) but expected time to absorption is on the order of 10^{10} if time is measured in seconds this is about 800 years.

We could not have calculated this with a deterministic mean-field model

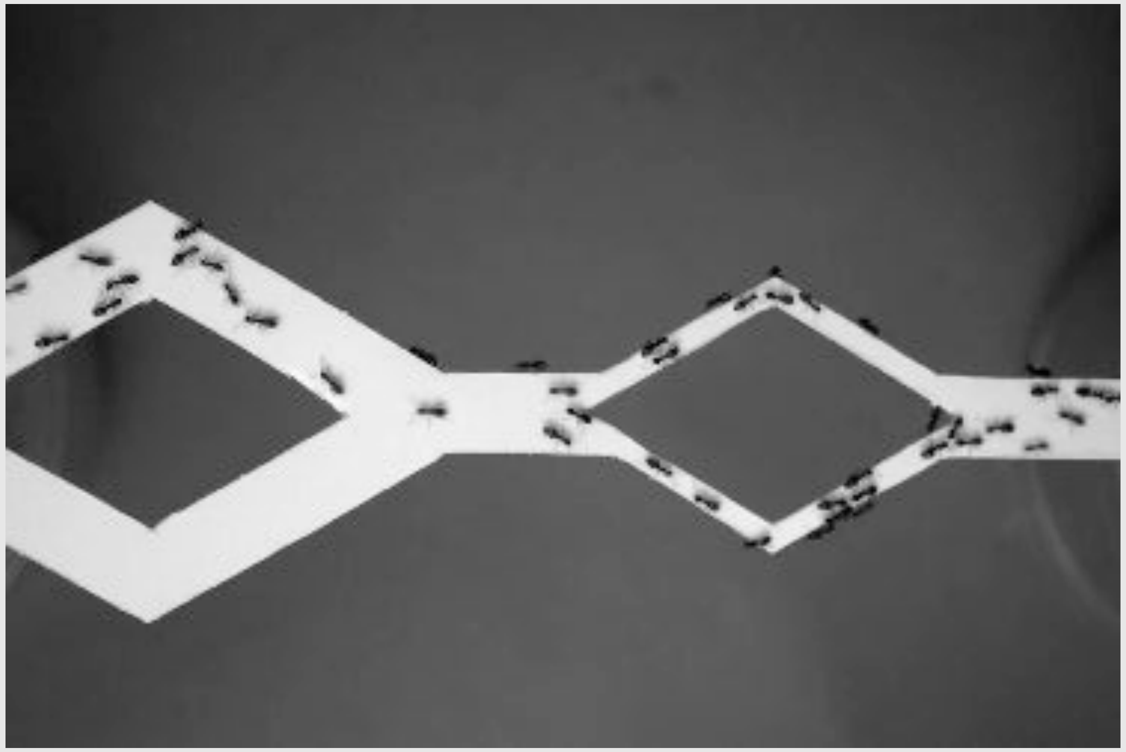
Stochastic Differential Equations

Note that Systems Dynamics / Differential Equation models can also be treated as stochastic systems by using stochastic differential equations instead of normal differential equations.

Akin to working with Markov Chains, we can then also calculate equilibrium distributions, transition times, etc.

time permitting, we will discuss this in the seminar … or else in next week's tutorial

Example: Ant Foraging by Mass Recruitment

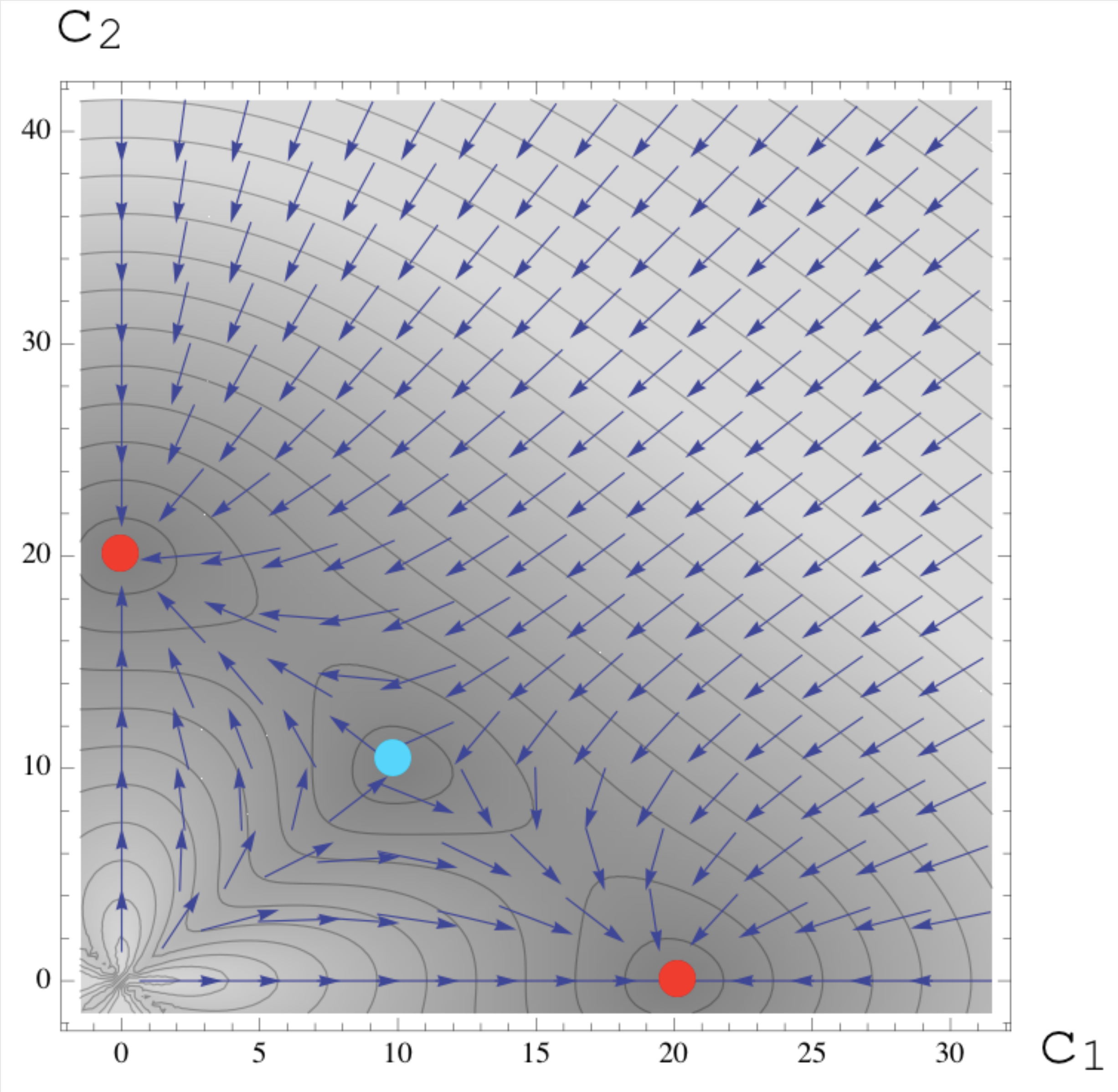


- Pheromone-mediated trails
- Pheromone deposit modulated by trail quality
- Individuals follow trail probabilistically
- Pheromone evaporates over time

➡ *Self-limiting positive feedback loop*

Note that ants collectively perform a (very simple) form of reinforcement learning

Predictions of the Deneubourg model



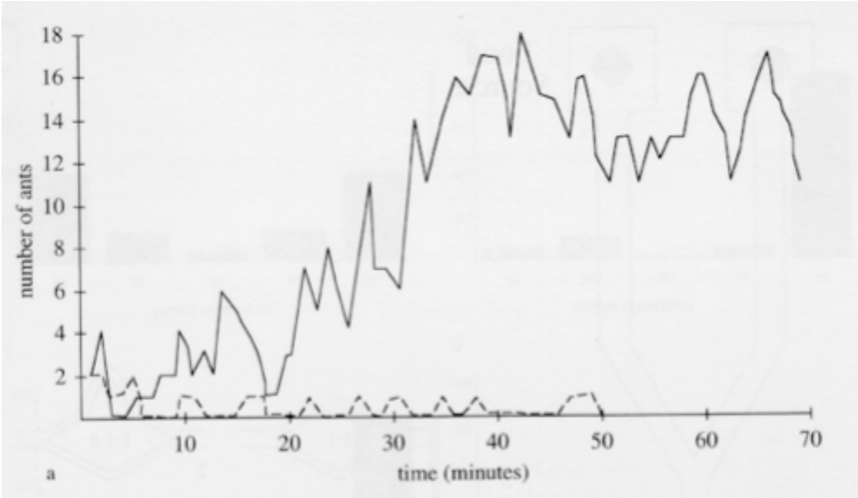
$$\frac{\partial C_i}{\partial t} = \frac{1}{l_i} \cdot \frac{f(C_i)}{f(C_1)+f(C_2)} - \rho C_i$$

$$f(x) = (k+x)^{\mu}$$

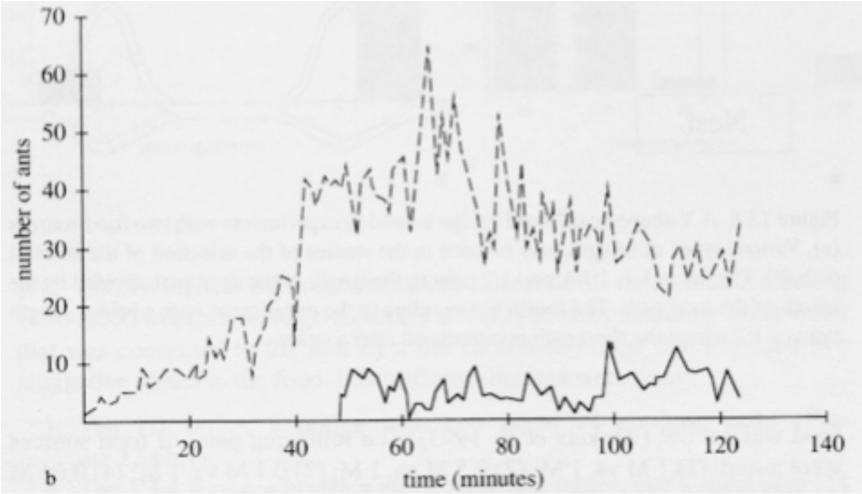
Streamline plot for pheromone model

The only stable fix-points are where one of the branches carries no pheromone (and no traffic). Note that this depends on the parameters setting. It applies for many commonly used parameter ranges but not always.

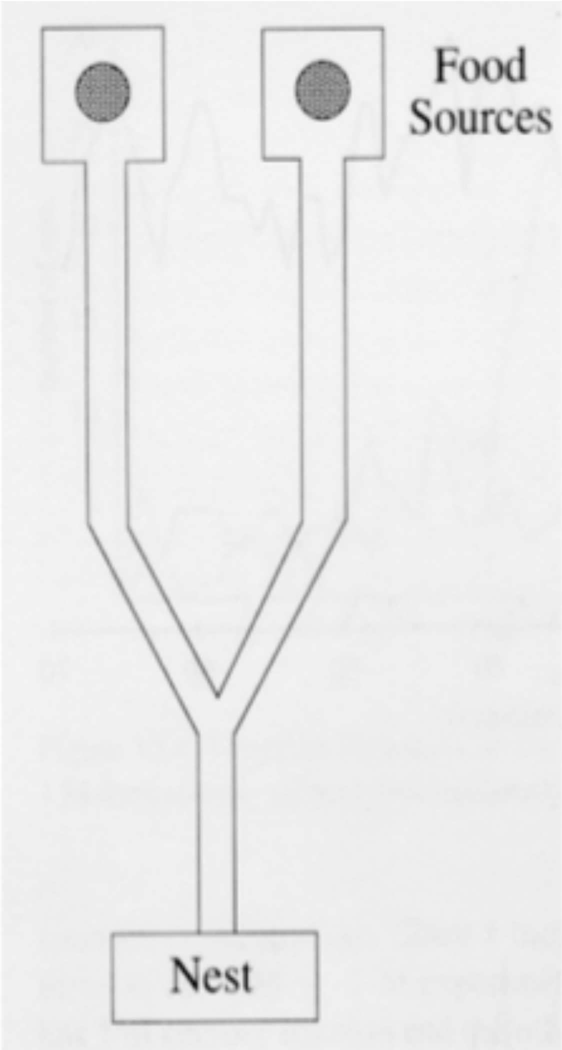
Mass foraging ants don't adapt



both sources present from start



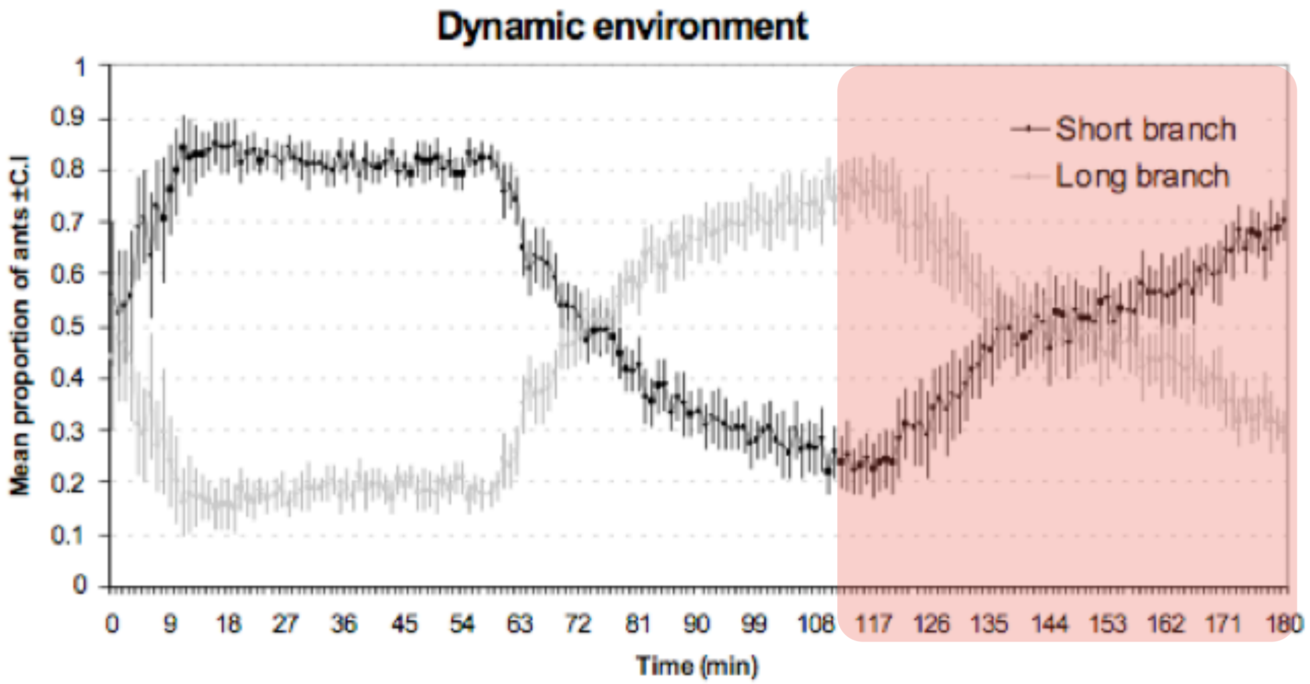
better source presented with delay

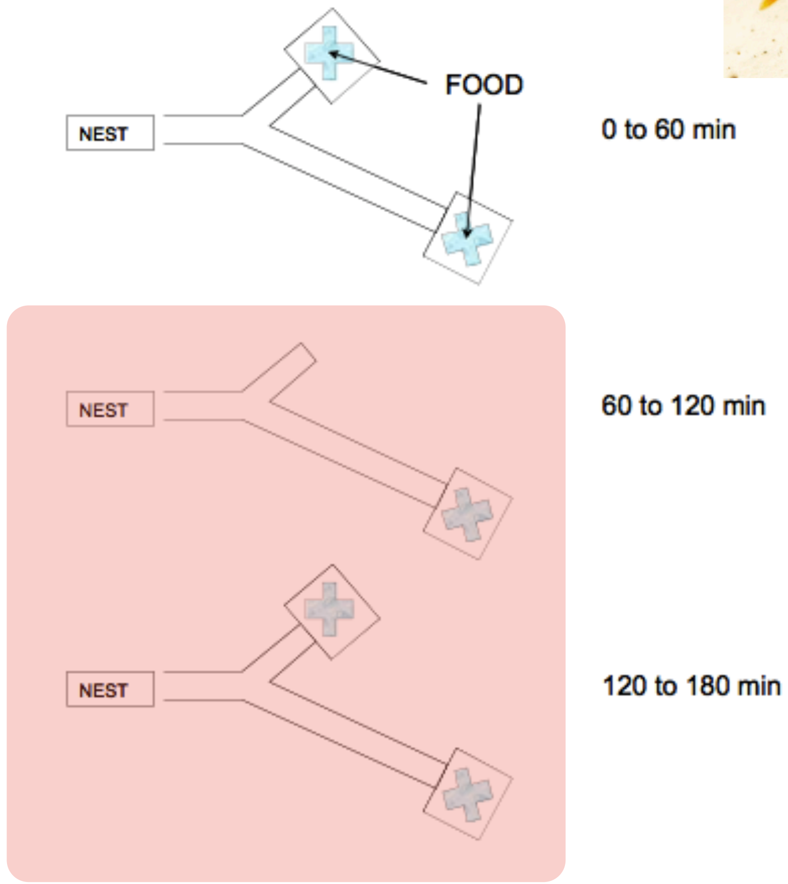


Lasius Niger

But some mass foraging ants do adapt!

Proportion of ants on each branch (N=20):

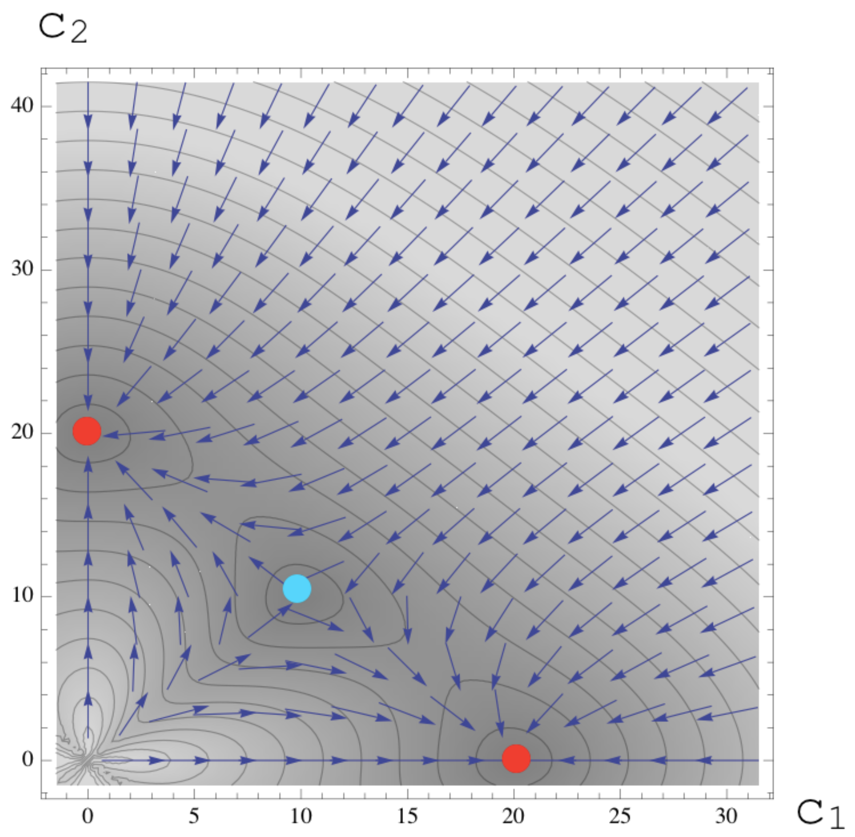




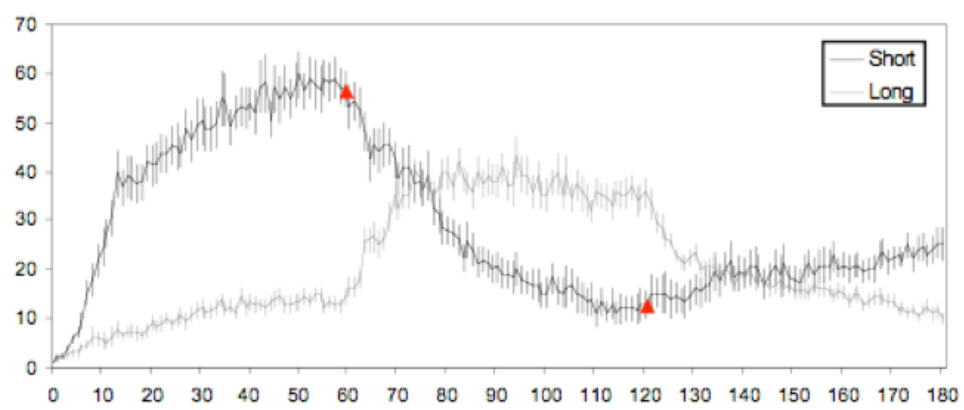
Non-Mean Field Model

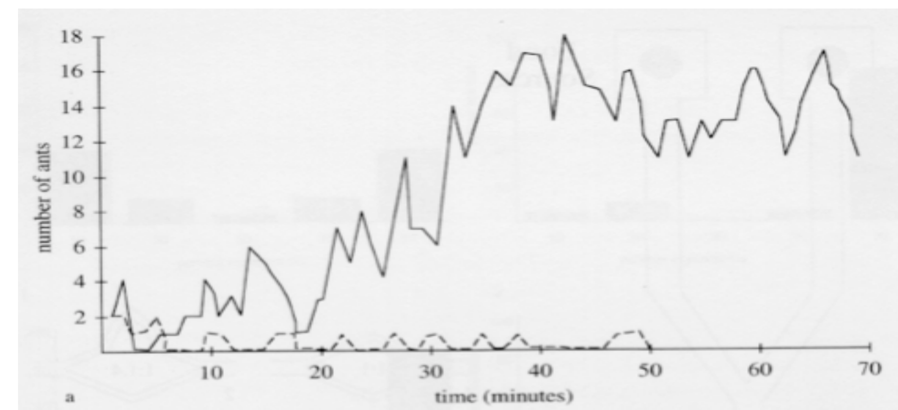
$$\frac{dC_i}{dt} = q_i \cdot \frac{C_i}{C_1 + C_2} - \rho C_i \quad + \sigma dW$$

$$t(x) = \int_{a}^{x} \left(\frac{2}{\psi(y)} \int_{y}^{b} \frac{\psi(z)}{\sigma^2(z)} dz \right) dy$$



Time to switch



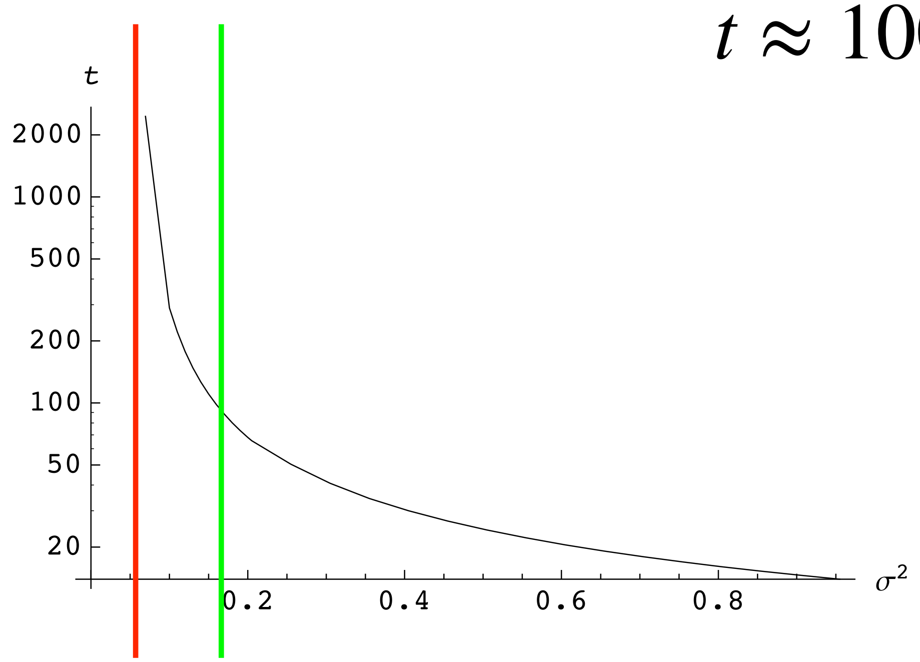


P. megacephala $\sigma \approx 0.16$

$t \approx 100$

L. niger $\sigma \approx 0.05$

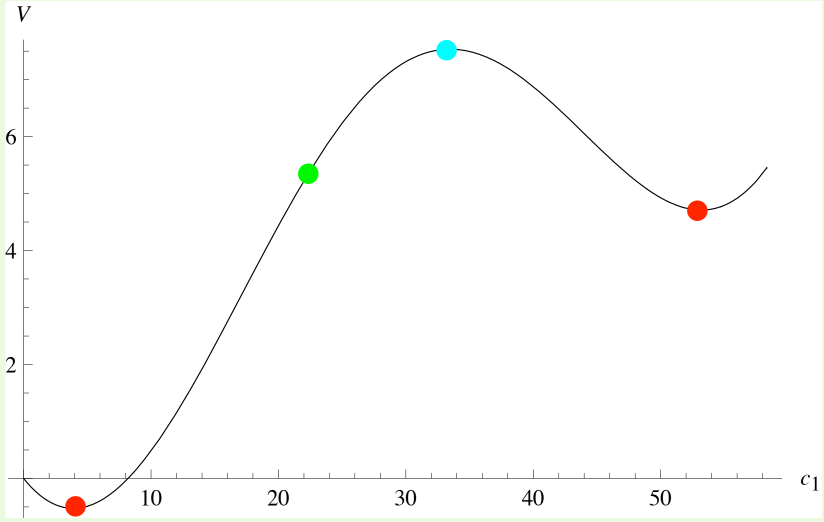
$t > 5000$

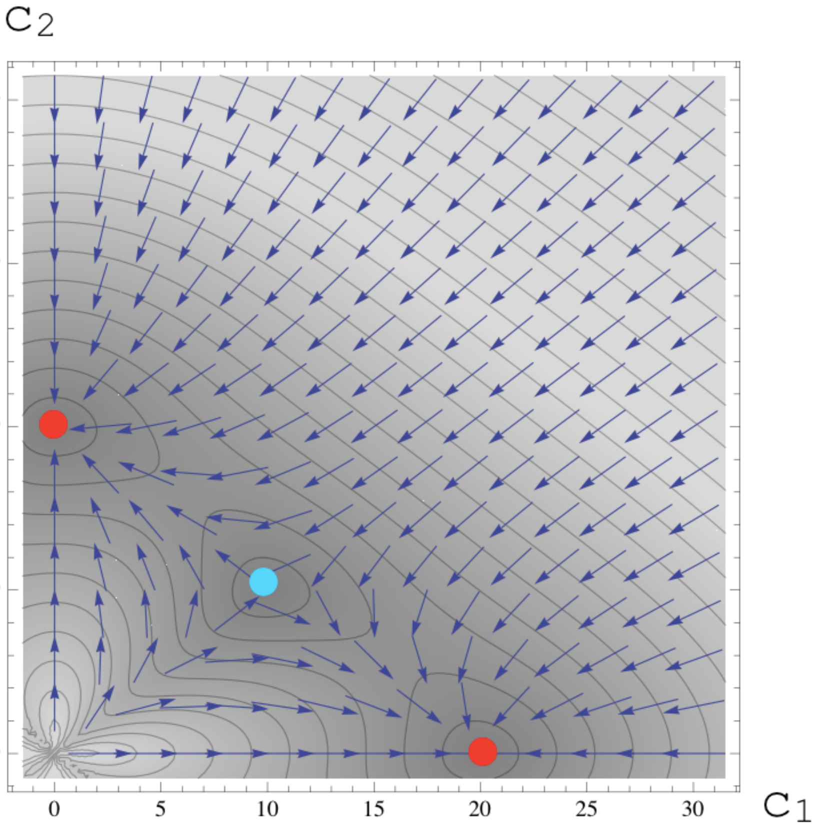


Note the time scale [minutes]

You cannot expect to observe the switch for *L. niger*!

Noise-induced Transitions





Social decision-making

requires "Noise"

- **Biological Group Behavior (e.g. Ant colonies, Slime molds)**
- **Human Collective Decision Making (e.g. recommender systems, fashion trends)**
- **Bio-inspired Algorithms (e.g. Ant algorithms)**

Take home lessons

- Deterministic mean-field (average) models and stochastic models of the same process can show *very different* outcomes.

- Markov Chains are the most fundamental modelling tool for discrete, stochastic processes

- They are based on step-wise transition probabilities between states of a system

- They allow us to determine periodicity, long term stationary distributions, time to reach these, etc. using simple matrix algebra.