Q3 (a) $z = ax + by + c$.

$Z_i = ax_i + by_i + c + \varepsilon_i \quad \forall\, i \in \{1, 2, \ldots, n\}$

where x_i, y_i are known and also $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$.

All ε_i values are independent. as they are drawn from iid random variable.

$z_i \sim \mathcal{N}(ax_i + by_i + c, \sigma^2)$

$\therefore$ ~~$p(z_i; x_i, y_i, a, b, c) = \dfrac{e^{-\frac{(y_i - mx_i - c)^2}{2\sigma}}}{\sigma\sqrt{2\pi}}$~~

$$p(z_i; x_i, y_i, a, b, c) = \frac{e^{-\left(\frac{(z_i - ax_i - by_i - c)^2}{2\sigma^2}\right)}}{\sigma\sqrt{2\pi}}$$

$$p(\{z_i\}; \{x_i\}, \{y_i\}, a, b, c) = \prod_{i=1}^{n} \frac{e^{\left(-\frac{(z_i - ax_i - by_i - c)^2}{2\sigma^2}\right)}}{\sigma\sqrt{2\pi}}$$

$$\boxed{\log p = \sum_{i=1}^{n} -\left(\frac{(z_i - ax_i - by_i - c)^2}{2\sigma^2}\right) - \log(\sigma\sqrt{2\pi})}$$

$$\frac{1}{p} \cdot \frac{\partial p}{\partial a} = \sum_{i=1}^{n} + \frac{(z_i - ax_i - by_i - c)\, x_i}{\sigma^2} = 0.$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} x_i z_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} y_i x_i + c\sum_{i=1}^{n} x_i} \quad \text{—— (1)}$$

$$\frac{1}{p}\frac{\partial p}{\partial b} = 0 = \sum_{i=1}^{n} \frac{(z_i - ax_i - by_i - c)\, y_i}{\sigma^2} = 0$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} z_i y_i = a\sum_{i=1}^{n} x_i y_i + b\sum_{i=1}^{n} y_i^2 + c\sum_{i=1}^{n} y_i} \quad \text{—— (2)}$$

$$\frac{1}{p}\frac{\partial p}{\partial c} = \sum_{i=1}^{n} \frac{(z_i - ax_i - by_i - c)}{\sigma^2} = 0$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} z_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i + nc} \quad \text{—— (3)}$$

Matrix Form

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} y_i^2 & \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & \sum_{i=1}^{n} 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i z_i \\ \sum_{i=1}^{n} y_i z_i \\ \sum_{i=1}^{n} z_i \end{bmatrix}$$

Vector Form

I can represent every Row of the 3x3 matrix by a 3 dimentional vector. a_{ij} are elements of this 3x3 matrix.

$\cancel{(R_1 + R_2 + R_3) \cdot (A) = Z}$

$$R_i = (a_{i1}, a_{i2}, a_{i3})$$

$$A = (a, b, c)$$

$\cancel{Z = \left(\sum x_i z_i, \sum y_i z_i, \sum z_i\right)}$

$$Z_1 = \sum_{i=1}^{n} x_i z_i$$

$$Z_2 = \sum_{i=1}^{n} y_i z_i$$

$$Z_3 = \sum_{i=1}^{n} z_i$$

$$R_i \cdot A = Z_i \quad \forall\, i \in \{1, 2, 3\}$$

(b) $z = ax+by+c$

$\boxed{\text{Note } \sum_{i=1}^{n} \text{ will be represented as } \sum}$

$$z_i = a_1x_i^2 + a_2y_i^2 + a_3x_iy_i + a_4x_i + a_5y_i + a_6 + \varepsilon_i \quad \forall i \in \{1,2,3,\dots N\}$$

$\varepsilon_i \sim N(0,\sigma^2)$

All ε_i are independent.

$$z_i \sim N(a_1x_i^2+a_2y_i^2+a_3x_iy_i+a_4x_i+a_5y_i+a_6, \sigma^2)$$

$$\therefore p(z_i; x_i, y_i, a_1, a_2, a_3, a_4, a_5, a_6) = \frac{e^{-\frac{(z_i - a_1x_i^2 - a_2y_i^2 - a_3x_iy_i - a_4x_i - a_5y_i - a_6)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$p(\{z_i\}; \{x_i\}, \{y_i\}, a_1, a_2, a_3, a_4, a_5, a_6) = \prod_{i=1}^{n} \frac{e^{-\frac{(z_i - a_1x_i^2 - a_2y_i^2 - a_3x_iy_i - a_4x_i - a_5y_i - a_6)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$\boxed{\log p = \sum_{i=1}^{n} -\frac{(z_i - a_1x_i^2 - a_2y_i^2 - a_3x_iy_i - a_4x_i - a_5y_i - a_6)^2}{2\sigma^2} - \log(\sigma\sqrt{2\pi})}$$

$$\frac{1}{p}\frac{\partial p}{\partial a_1} = \sum_{i=1}^{n} \frac{(z_i - a_1x_i^2 - a_2y_i^2 - a_3x_iy_i - a_4x_i - a_5y_i - a_6)\,x_i^2}{\sigma^2} = 0$$

① $\boxed{\sum z_ix_i^2 = a_1\sum x_i^4 + a_2\sum y_i^2x_i^2 + a_3\sum x_i^3y_i + a_4\sum x_i^3 + a_5\sum x_i^2y_i + a_6\sum x_i^2}$

$$\frac{1}{p}\frac{\partial p}{\partial a_2} = \sum \frac{(z_i - a_1x_i^2 - a_2y_i^2 - a_3x_iy_i - a_4x_i - a_5y_i - a_6)(y_i^2)}{\sigma^2} = 0$$

② $\boxed{\sum z_iy_i^2 = a_1\sum x_i^2y_i^2 + a_2\sum y_i^4 + a_3\sum x_iy_i^3 + a_4\sum x_iy_i^2 + a_5\sum y_i^3 + a_6\sum y_i^2}$

$$\frac{1}{p}\frac{\partial p}{\partial a_3} = \sum \frac{(z_i - a_1x_i^2 - a_2y_i^2 - a_3x_iy_i - a_4x_i - a_5y_i - a_6)\,x_iy_i}{\sigma^2} = 0$$

③ $\boxed{\sum x_iy_iz_i = a_1\sum x_i^3y_i + a_2\sum x_iy_i^3 + a_3\sum x_i^2y_i^2 + a_4\sum x_i^2y_i + a_5\sum x_iy_i^2 + a_6\sum x_iy_i}$

$$\frac{1}{p}\frac{\partial p}{\partial a_4} = \sum \frac{(z_i - a_1x_i^2 - a_2y_i^2 - a_3x_iy_i - a_4x_i - a_5y_i - a_6)\,x_i}{\sigma^2} = 0$$

④ $\boxed{\sum x_iz_i = a_1\sum x_i^3 + a_2\sum x_iy_i^2 + a_3\sum x_i^2y_i + a_4\sum x_i^2 + a_5\sum x_iy_i + a_6\sum x_i}$

$$\frac{1}{p}\frac{\partial p}{\partial a_5} = \sum \frac{(z_i - a_1x_i^2 - a_2y_i^2 - a_3x_iy_i - a_4x_i - a_5y_i - a_6)\,y_i}{\sigma^2} = 0$$

⑤ $\boxed{\sum y_iz_i = a_1\sum x_i^2y_i + a_2\sum y_i^3 + a_3\sum x_iy_i^2 + a_4\sum x_iy_i + a_5\sum y_i^2 + a_6\sum y_i}$

$$\frac{1}{p}\frac{\partial p}{\partial a_6} = \sum \frac{(z_i - a_1x_i^2 - a_2y_i^2 - a_3x_iy_i - a_4x_i - a_5y_i - a_6)}{\sigma^2} = 0$$

⑥ $\boxed{\sum z_i = a_1\sum x_i^2 + a_2\sum y_i^2 + a_3\sum x_iy_i + a_4\sum x_i + a_5\sum y_i + a_6\sum 1.}$

Matrix Form

$$
\begin{bmatrix}
\sum x_i^4 & \sum x_i^2 y_i^2 & \sum x_i^3 y_i & \sum x_i^3 & \sum x_i^2 y_i & \sum x_i^2 \\
\sum y_i^2 x_i^2 & \sum y_i^4 & \sum x_i y_i^3 & \sum x_i y_i^2 & \sum y_i^3 & \sum y_i^2 \\
\sum x_i^3 y_i & \sum x_i y_i^2 & \sum x_i^2 y_i^2 & \sum x_i^2 y_i & \sum x_i y_i^2 & \sum x_i y_i \\
\sum x_i^3 & \sum x_i y_i^2 & \sum x_i^2 y_i & \sum x_i^2 & \sum x_i y_i & \sum x_i \\
\sum x_i^2 y_i & \sum y_i^3 & \sum x_i y_i^2 & \sum x_i y_i & \sum y_i^2 & \sum y_i \\
\sum x_i^2 & \sum y_i^2 & \sum x_i y_i & \sum x_i & \sum y_i & \sum 1
\end{bmatrix}_{6\times 6}
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}
=
\begin{bmatrix} \sum x_i^2 z_i \\ \sum y_i^2 z_i \\ \sum x_i y_i z_i \\ \sum x_i z_i \\ \sum y_i z_i \\ \sum z_i \end{bmatrix}
$$

Vector Form

$R_i = \{a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}, a_{i6}\}$ a_{ij} are elements of 6×6 matrix.

$A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$

$Z_1 = \sum x_i^2 z_i$; $Z_2 = \sum y_i^2 z_i$; $Z_3 = \sum x_i y_i z_i$; $Z_4 = \sum x_i z_i$

$Z_5 = \sum y_i z_i$ $Z_6 = \sum z_i$

$\therefore \forall i \in \{1, 2, 3, 4, 5, 6\}$

$R_i \cdot A = Z_i$