Solution for Q3

Information provided to us :-

Event $F = \{|Q_1 + Q_2| > \epsilon\}$, $E_1 = \{|Q_1| > \frac{\epsilon}{2}\}$ and $E_2 = \{|Q_2| > \frac{\epsilon}{2}\}$

Their Complementary forms :-

Event $F^c = \{|Q_1 + Q_2| \leq \epsilon\}$, $E_1^c = \{|Q_1| \leq \frac{\epsilon}{2}\}$ and $E_2^c = \{|Q_2| \leq \frac{\epsilon}{2}\}$

Important Inequality :-

$$\forall x, y : |x + y| \leq |x| + |y| \tag{1}$$

Apply equation (1) on Q_1 and Q_2, hence we get : $|Q_1 + Q_2| \leq |Q_1| + |Q_2|$ for any Q_1, Q_2 ... (A)

Important Observation :-

If $|Q_1| \leq \frac{\epsilon}{2}$ **AND** $|Q_2| \leq \frac{\epsilon}{2}$ then from equation (A) we see that :-

$$|Q_1 + Q_2| \leq |Q_1| + |Q_2| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon \implies |Q_1 + Q_2| \leq \epsilon \tag{2}$$

Hence, all values satisfying $|Q_1| \leq \frac{\epsilon}{2}$ **AND** $|Q_2| \leq \frac{\epsilon}{2}$ must also satisfy $|Q_1 + Q_2| \leq \epsilon$.

Using this Observation :-

We see that the set...

$$\{|Q_1| \leq \frac{\epsilon}{2} \mathbf{AND} |Q_2| \leq \frac{\epsilon}{2}\} = E_1^c \cap E_2^c \tag{3}$$

Because **AND** means that both the inequalities must follow at the same time, hence that set is **intersection** of the two complementary events E_1^c, E_2^c.

We already know $\{|Q_1 + Q_2| \leq \epsilon\} = F^c$.

Important Remark :-

All the values belonging to $E_1^c \cap E_2^c$ must also belong to the set F^c, from our above observation, hence the former set is **subset** of the latter, or...

$$E_1^c \cap E_2^c \subseteq F^c \tag{4}$$

Some Properties of Probability :-

We will use some properties of probability...

- a) If $A \subseteq B$ then $P(A) \leq P(B)$.
- b) $P(A^c) = 1 - P(A)$
- c) $P(A \cup B) = P(A) + P(B) - P(A \cap B) \implies P(A \cup B) \leq P(A) + P(B)$

Property (c) has been proved on the next page.

Final Calculations :-

We have

$$E_1^c \cap E_2^c \subseteq F^c$$

then using (a)

$$\implies P(E_1^c \cap E_2^c) \leq P(F^c)$$

By **De-Morgan's Law**, $E_1^c \cap E_2^c = (E_1 \cup E_2)^c$, hence,

$$\implies P((E_1 \cup E_2)^c) \leq P(F^c)$$

Using (b),

$$\implies 1 - P(E_1 \cup E_2) \leq 1 - P(F)$$

$$\implies P(F) \leq P(E_1 \cup E_2)$$

Using (c),

$$P(F) \leq P(E_1 \cup E_2) \leq P(E_1) + P(E_2)$$
$$\implies P(F) \leq P(E_1) + P(E_2)$$

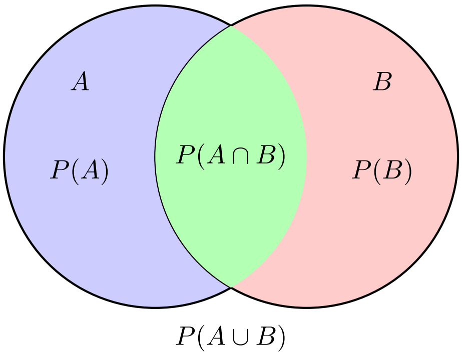
Hence Proved !!

Proof of $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Let A and B be two events. We want to prove that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Consider the Venn diagram below:



The total area of the two circles represents the probability $P(A \cup B)$.

The area of circle A represents $P(A)$, and the area of circle B represents $P(B)$.

The overlapping region represents the intersection $A \cap B$, and its area is $P(A \cap B)$.

According to the Venn diagram, we can see that the overlapping area is counted twice when calculating $P(A \cup B)$. Thus, we need to subtract the area of $A \cap B$ once to avoid double counting.

Therefore, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

which completes the proof.