22B1053- 22B1205-22B1025

Page No	
Date	

Q3 Given X is random Variable With mean μ, and variance σ^2.

(a) we first prove that for $\tau > 0$,

$$P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

let Y is random variable given by

$$Y = X - \mu$$

then by

$$E(Y) = E(X-\mu) = E(X) - E(\mu) = E(X) - \mu = \mu - \mu = 0.$$

$$Var(Y) = E(Y-E(Y))^2 = E(Y^2) = E((X-\mu)^2) = Var(X) = \sigma^2.$$

let $b>0$, be arbitary positive real number.

let Z be random variable given by,

$$Z = (Y+b)^2$$

then,

$$E(Z) = E((Y+b)^2) = E(Y^2 + 2Yb + b^2)$$
$$= E(Y^2) + E(2Yb) + E(b^2)$$
$$= E(Y^2) + E(b^2) + 2bE(Y)$$
$$= \sigma^2 + b^2 + 0 = \sigma^2 + b^2$$

now we prove that $Y \geq \tau \Rightarrow Z \geq (\tau+b)^2$

Proof: Given $(Y-\tau) \geq 0$

now $Z - (\tau+b)^2 = (Y+b)^2 - (\tau+b)^2$

$$= (Y-\tau)(Y+\tau+2b) \geq 0$$

$$\Rightarrow Z \geq (\tau+b)^2$$

$$\therefore P(Y \geq \tau) \leq P(Z \geq (\tau+b)^2)$$

using Markov's inequality, (as Z is non-negative)

$$P(Y \geq \tau) \leq P(Z \geq (\tau+b)^2) \leq \frac{E(Z)}{(\tau+b)^2} = \frac{\sigma^2+b^2}{(\tau+b)^2}$$

$$P(Y \geq \tau) \leq \frac{\sigma^2+b^2}{(\tau+b)^2}$$

$\because$ b is arbitiary positive real number.

we can minimize $\frac{\sigma^2+b^2}{(\tau+b)^2}$, let $l = \frac{\sigma^2+b^2}{(\tau+b)^2}$

One way is ~~not~~ to see l as function of b and we differentiation

But, we minimize it we using ~~A.M.-G.M.~~ ~~inequality~~ subsitution and then differentiation,

$$l = \frac{\sigma^2 + b^2}{(\tau + b)^2}$$

put $\alpha = \tau + b$, (note: $b > 0 \Rightarrow \alpha > \tau$)

$$l(\alpha) = l = \frac{\sigma^2 + (\alpha - \tau)^2}{\alpha^2} = \frac{\sigma^2 + \tau^2 - 2\alpha\tau + \alpha^2}{\alpha^2} = \frac{\sigma^2 + \tau^2}{\alpha^2} - \frac{2\tau}{\alpha} + 1$$

now as $\alpha > \tau$ $\therefore$ l is differentiable w.r.t α.

$$l'(\alpha) = \frac{-2(\sigma^2 + \tau^2)}{\alpha^3} + \frac{2\tau}{\alpha^2} = \frac{2(\tau\alpha - (\sigma^2 + \tau^2))}{\alpha^3}$$

$$l'(\alpha) = 0 \Rightarrow \alpha = \frac{\sigma^2 + \tau^2}{\tau}$$

~~as~~ for $\alpha > \frac{\sigma^2 + \tau^2}{\tau}$, $l'(\alpha) > 0$

$\therefore$ $\alpha = \frac{\sigma^2 + \tau^2}{\tau}$ is point of minima

we put $\alpha = \frac{\sigma^2 + \tau^2}{\tau}$ ~~in $l(\alpha)$~~ in $\alpha = \tau + b$

$$\Rightarrow b = \frac{\sigma^2}{\tau}$$

now we put $b = \frac{\sigma^2}{\tau}$ in $l = \frac{\sigma^2 + b^2}{(\tau + b)^2}$

$$\therefore \quad l \geq \frac{\sigma^2\left(1 + \frac{\sigma^2}{\tau^2}\right)}{\left(\tau + \frac{\sigma^2}{\tau}\right)^2} = \frac{\sigma^2\left(1 + \frac{\sigma^2}{\tau^2}\right)}{\tau^2\left(1 + \frac{\sigma^2}{\tau^2}\right)^2}$$

$$l \geq \frac{\sigma^2}{\tau^2\left(1 + \frac{\sigma^2}{\tau^2}\right)} = \frac{\sigma^2}{\tau^2 + \sigma^2}$$

$$\therefore \quad P(Y \geq \tau) \leq \frac{\sigma^2}{\tau^2 + \sigma^2}$$ (as $b > 0$ was arbitrary)

$$\Rightarrow P(X - \mu \geq \tau) \leq \frac{\sigma^2}{\tau^2 + \sigma^2}$$

(b) Now we prove for $\tau < 0$, we have,

$$P(X-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$$

we first oberse that $X-\mu \geq \tau \Rightarrow \mu - X \leq -\tau$

now $-\tau > 0$

$\therefore$ ~~from~~

$E(\mu-X) = E(\mu) - E(X) = \mu - E(X) = 0$

$Var(\mu-X) = E[(\mu-X)^2] = E[(X-\mu)^2] = \sigma^2$

$\because Var(\mu-X) = \sigma^2$ and $-\tau > 0$ when $E(\mu-X) = 0$,

$\therefore$ from previous argument, ~~we have~~, applying on $\mu - X$,

$$\therefore P(\mu - X \geq -\tau) \leq \frac{\sigma^2}{\sigma^2+(-\tau)^2} = \frac{\sigma^2}{\sigma^2+\tau^2} \quad \text{— (1)}$$

now $P(\mu - X \geq -\tau) = P(X-\mu \leq \tau)$ — (2)

~~now $\because \{x = \mu + \tau$~~

now $\because$ $\{x \mid x-\mu \leq \tau\}$ and $\{x \mid x-\mu > \tau\}$

are mutually exclusive and exhaustive sets.

$\therefore$

$$P(X-\mu \leq \tau) + P(X-\mu > \tau) = 1$$

$$1 - P(X-\mu > \tau) = P(X-\mu \leq \tau)$$

from (1) and (2)

$$1 - P(X-\mu > \tau) = P(X-\mu \leq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$\Rightarrow P(X-\mu > \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$$

Now if we X to be continous in neighbourhood of $x = \mu + \tau$ then $P(X = \mu + \tau) = 0$

$$\Rightarrow P(X-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$$