22B1053-22B1205-22B1025

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Q5 let Y be R.V st $(X \in [a,b]$ is ~~bounded~~ R.V$)$

$Y = e^{s(X - E[X])}$ where $s > 0$

now using markov's inequality on Y, (as Y is positive) (as $e^{st} > 0 \; \forall t \in R$)

$$P(Y = e^{s(X-E(X))} \geq e^{st}) \leq \frac{E(Y)}{e^{st}}$$

now by IR $E(Y) \leq e^{s^2(b-a)^2/8}$

$$\therefore P(e^{s(X-E(X))} \geq e^{st}) \leq \frac{e^{s^2(b-a)^2/8}}{e^{st}}$$

now as $e^{s(X-E(X))} \geq e^{st} \iff X - E(X) \geq t$ for $t > 0$

$$\therefore P(X - E(X) \geq t) \leq e^{\frac{s^2(b-a)^2}{8} - st} = e^{\lambda} \quad \text{for } t > 0$$

for $t > 0, s > 0$ we minimize R.H.S

~~which is obtained at~~

$$\lambda = \frac{s^2(b-a)^2}{8} - st = \left(\frac{s(b-a)}{2\sqrt{2}} - \frac{t\sqrt{2}}{b-a}\right)^2 + \frac{2t^2}{(b-a)^2}$$

$$\therefore \lambda \geq \frac{2t^2}{(b-a)^2}$$

$$\therefore P(X - E(X) \geq t) \leq e^{2t^2/(b-a)^2} \quad \text{for } t > 0$$

for $t < 0$,

let $Y = e^{s(E(X) - X)}$ where $s > 0$, ~~$t < 0$~~, $Y > 0$.

applying IR on $E(X) - X$ as R.V (note: if $X \in [a,b]$ then $E(X) - X \in [\mu - b, \mu - a]$ $\mu = E(X)$)

$$E(Y) \leq e^{s^2(b-a)^2/8}$$

now for $t < 0$ we use markov's equality as follows.

$$P(Y \geq e^{-st}) \leq \frac{e^{s^2(b-a)^2/8}}{e^{-st}}$$

now $e^{s(E(X)-X)} \geq e^{-st} \iff E(X) - X \geq -t$ for $t < 0$

$\iff X - E(X) \leq t$

$$1 - P(X - E(X) > t) = P(X - E(X) \leq t) \leq e^{\frac{s^2(b-a)^2}{8} + st}$$

$$\therefore P(X - E(X)) \geq 1 - e^{\frac{s^2(b-a)^2}{8} + st} = 1 - e^{\lambda}$$

we maximize R.H.S

now, $\lambda = \dfrac{s^2(b-a)^2}{8} + st = \left(\dfrac{s(b-a)}{2\sqrt{2}} + \dfrac{t\sqrt{2}}{b-a}\right)^2 - \dfrac{2t^2}{(b-a)^2} \geq \dfrac{-2t^2}{(b-a)^2}$

$\therefore$ maximum R.H.S $= 1 - e^{-2t^2/(b-a)^2}$

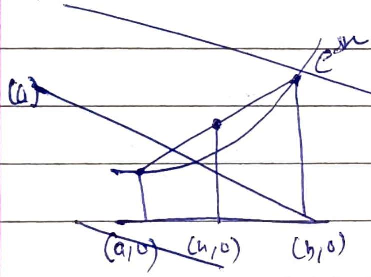
$\therefore \quad P(X-E(X) > t) \geq 1 - e^{-2t^2/(b-a)^2}$ for $t < 0$

$\because \quad P(X-E(X) > t) \in [1, 0]$

$\therefore$

$$P(X-E(X) \geq t) \leq e^{2t^2/(b-a)^2} \quad \text{for } t > 0$$

$$1 - e^{-2t^2/(b-a)^2} \leq P(X-E(X) > t) \leq 1 \quad \text{for } t < 0$$



by Jensen's inequality,

$$e^{su} \leq \frac{(e^{sb}-e^{sa})u}{b-a} - \frac{(ae^{sb}-e^{sa}b)}{b-a} = \frac{(b-u)e^{sa}}{b-a} + \frac{(u-a)e^{sb}}{b-a}$$

continue first part ☺

now we need $P(S_n - E(S_n) > t)$

now, we note that

if $x_1 \in [a_1, b_1]$, $x_2 \in [a_2, b_2]$

$x_1 + x_2 \in [a_1+a_2, b_1+b_2]$

$\therefore$ by inductive argument

$$S_n \in \left[\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right]$$

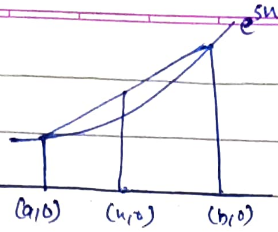
$\therefore$ (also assume X_i's to be continuous)

$\therefore$ from above arguments,

$$P(S_n - E(S_n) > t) \leq e^{2t^2/(\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i)^2} \quad \text{for } t > 0$$

$$1 - e^{-2t^2/(\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i)^2} \leq P(S_n - E(S_n) > t) \leq 1 \quad \text{for } t < 0$$

(a)



by Jensen's inequality

$$e^{su} \le \frac{(e^{sb}-e^{sa})u}{b-a} - \frac{(ae^{sb}-be^{sa})}{b-a} = \frac{(b-u)e^{sa}}{b-a} + \frac{(u-a)e^{sb}}{b-a}$$

(b)

$$E(e^{su}) \le E\left[\frac{(e^{sb}-e^{sb})u}{b-a}\right] + E\left[\frac{be^{sa}-ae^{sb}}{b-a}\right] = \frac{be^{sa}-ae^{sb}}{b-a} + \frac{e^{sb}-e^{sa}}{b-a}E[X]$$

$$E(e^{su}) \le \frac{be^{sa}-ae^{sb}}{b-a} = l$$

~~now its enough to show,~~

$$\cancel{e^{L(s(b-a))}} = \cancel{\frac{be^{sa}-ae^{sb}}{b-a}}$$

$$l = \frac{be^{sa}-ae^{sb}}{b-a} = e^{\log\frac{be^{sa}-ae^{sb}}{b-a}} = e^{\log e^{sa}\left(\frac{b-ae^{s(b-a)}}{b-a}\right)}$$

$$l = e^{sa+\log\left(\frac{b-ae^{s(b-a)}}{b-a}\right)}$$

$$= e^{sa+\log\left(1+\frac{a(1-e^{s(b-a)})}{b-a}\right)}$$

$$= e^{\frac{s(b-a)a}{b-a}+\log\left(1+\frac{a(1-e^{s(b-a)})}{b-a}\right)}$$

$$l = e^{L(s(b-a))}$$

where $L(h) = \frac{ha}{b-a} + \log\left(1+\frac{a-ae^{h}}{b-a}\right)$

$$\therefore E(e^{su}) \le e^{L(s(b-a))}$$

$$\cancel{L(0)=0}$$

(c) Note that we need only $h>0$ as $s(b-a)>0$.

now, clearly $L(h)$ is continuous and differentiable.

$$\therefore L'(h) = \frac{a}{b-a} + \frac{\frac{-ae^{h}}{b-a}}{1+\frac{a(1-e^{h})}{b-a}} = \frac{a}{b-a} - \frac{ae^{h}}{b-ae^{h}}$$

$L'(h)$ is differentiable for $h>0$ (as $a<0$), $L'(0)=0$ — ①

$$\therefore L''(h) = \frac{-a^2e^{2h}}{(b-ae^h)^2} - \frac{ae^h}{(b-ae^h)} = \frac{-ab\,e^h}{(b-ae^h)^2} = \frac{-ab}{(be^{-h/2}-ae^{h/2})^2}$$

for $x,y>0$ $\quad x+y \geq 2\sqrt{xy}$

$$\therefore be^{-h/2} - ae^{h/2} \geq 2\sqrt{-ab}$$

$$\therefore L''(h) \leq \frac{-ab}{4\,-ab} = 1/4$$

(d) now, we have if $f'(x) \leq g'(x)$ ~~or~~ $f'(x)-g'(x) \leq 0$ $\forall x>0$

and $f(x)-g(x) \leq 0$

$\Rightarrow f(x)-g(x) \leq 0 \;\forall x>0$ — ②

$\therefore$ as

$L''(h) - 1/4 \leq 0$

$L'(0)=0 \therefore$ $L'(0) - \frac{0}{4} \leq 0$ (here $g(x)=\frac{h}{4}$)

$$\therefore L'(h) - \frac{h}{4} \leq 0 \quad \forall h>0$$

now as $L(0)=0$,

$\therefore L(0) - \frac{0}{8} \leq 0$ (here $g(x) = \frac{h^2}{8}$)

$$\therefore L(h) - \frac{h^2}{8} \leq 0 \quad \forall h>0$$

$$\therefore L(h) \leq \frac{h^2}{8} \quad \forall h>0$$

$$\therefore P[e^{sx}] \leq e^{L(s(b-a))} \leq e^{\frac{s^2(b-a)^2}{8}}$$

<u>prove for ②</u>,

let $h(x) = f(x)-g(x)$ then given $h'(x) \leq 0$, $h(0) \leq 0$

Then by Taylor's theorem $\exists\, c \in (0,x)$ s.t

$$h(x) = h(0) + \underset{\shortparallel}{h'(c)}(x-0) = h(0) + xh'(c)\,x$$

as $h(0) \leq 0$ and $xh'(x) \leq 0$ for $x>0$ $\therefore$ $h(x) \leq 0 \;\forall x>0$.