

# Dependent Type Theory for Absolute Beginners

Created with the help of GPT-5 Pro

Kimion Fountoulakis

October 8, 2025

## 1 Types and universes

### The universe of types

In dependent type theory, all objects we manipulate are *terms*, and each term has a *type*. For example, a natural number 0 has type `Nat`, and a boolean value `true` has type `Bool`. We use the colon notation  $a : A$  to mean that  $a$  is a term of the type  $A$ . For example:

$$0 : \text{Nat}, \quad \text{false} : \text{Bool}, \quad \text{"hello world"} : \text{String}.$$

But what about `Nat` and `Bool` themselves? They too are objects in the theory and therefore must have a type. The type to which all ordinary types belong is called the *universe of types* and is written `Type`. Thus we write

$$\text{Nat} : \text{Type}, \quad \text{Bool} : \text{Type}, \quad \text{String} : \text{Type}.$$

Intuitively, `Type` is “the type of all types.” However, if we were to write  $\text{Type} : \text{Type}$ , we would obtain an inconsistency (Girard’s paradox). To avoid this, type theory introduces an infinite hierarchy of universes:

$$\text{Type}_0 : \text{Type}_1 : \text{Type}_2 : \dots$$

Each universe  $\text{Type}_i$  is an element of the next one, but not of itself. In most practical situations, we omit universe index that a term belongs to and simply write `Type`.

*Examples:*

$$\begin{array}{ll} \text{Nat} : \text{Type}_0, & \text{the type of natural numbers;} \\ \text{Bool} : \text{Type}_0, & \text{the type of booleans;} \\ \text{String} : \text{Type}_0, & \text{the type of strings;} \end{array}$$

Generally, writing  $A : \text{Type}$  means that  $A$  is itself a type.

## The simplest types: unit and empty types

Two fundamental base types in dependent type theory are the *unit type* and the *empty type*.

**The unit type.** The unit type, written  $\mathbf{1} : \text{Type}$ , is a type with exactly one canonical inhabitant, commonly written  $\star$ .

$$\mathbf{1} : \text{Type}, \quad \star : \mathbf{1}.$$

**The empty type.** The empty type, written  $\mathbf{0} : \text{Type}$ , is a type with no inhabitants. There are no terms  $a : \mathbf{0}$ .

## Product, sum and function types

More generally, if  $A, B : \text{Type}$ , then:

$$\begin{aligned} A \times B : \text{Type} & \quad (\text{product type, pairs } (a, b)); \\ A + B : \text{Type} & \quad (\text{sum type, disjoint union of } A \text{ and } B); \\ A \rightarrow B : \text{Type} & \quad (\text{function type, functions from } A \text{ to } B). \end{aligned}$$

Each of these is itself an inhabitant of the universe  $\text{Type}$ :

$$(A \rightarrow B) : \text{Type}, \quad (A \times B) : \text{Type}, \quad (A + B) : \text{Type}.$$

We will provide details in subsequent sections on how these three types are constructed from types  $A$  and  $B$ .

## 2 Contexts

### The empty context

The symbol  $\cdot$ , often written simply as  $\cdot$ , denotes the *empty context*. It represents the starting point of all contexts—a situation with no assumptions.

### General contexts

First, some terminology,  $x : A$  is called a *variable declaration*. A context is simply a finite list of typed variable declarations. For example:

$$\cdot, \quad x : \text{Nat}, \quad x : \text{Nat}, y : \text{Bool}, \quad x : \text{Nat}, y : \text{Bool}, z : \text{String}.$$

These are four different contexts. The first, i.e.,  $\cdot$ , denotes the empty context. The second includes only one declaration, the third includes two declarations and the fourth includes three declarations. Each larger context is obtained from a smaller one by adding one more declaration.

## Extension of a context

Let  $\Gamma$  be a context, then  $\Gamma, x : A$  is the *extension* of  $\Gamma$  with a new declaration.

### Example of construction.

$$\Gamma_0 = \cdot, \quad \Gamma_1 = \Gamma_0, x : \text{Nat}, \quad \Gamma_2 = \Gamma_1, y : \text{Bool}, \quad \Gamma_3 = \Gamma_2, z : \text{String}.$$

Hence

$$\Gamma_3 = x : \text{Nat}, y : \text{Bool}, z : \text{String}.$$

This shows how a recursive definition generates all finite contexts.

## Inductive definition of contexts

Formally, we define the *syntax of contexts* using the notation

$$\Gamma ::= \cdot \mid \Gamma, x : A.$$

The symbol  $::=$  is read as “is defined as,” and the vertical bar “ $\mid$ ” means “or.” Thus, this definition should be read as:

A context  $\Gamma$  is either the empty context  $\cdot$ , or an existing context extended by a new variable declaration  $x : A$ .

This is a *recursive* or *inductive* definition:

- The *base case* says that the empty context  $\cdot$  is a valid context.
- The *inductive case* says that if  $\Gamma$  is a valid context and  $A$  is a type, then  $\Gamma, x : A$  is also a valid context.

The domain of a context  $\Gamma$  is denoted by  $\text{dom}(\Gamma)$ , and it is the set of variables declared in  $\Gamma$ .

## 3 Judgments

A *judgment* is a basic assertion of the form we can derive in the theory. We will use the following judgment forms throughout:

$\Gamma \vdash t : A$ , reads “under context  $\Gamma$ , the term  $t$  has type  $A$ .”

Here,  $\vdash$  is the *turnstile* symbol separating assumptions (on the left) from a conclusion (on the right).

**Example (typing).** With  $\text{Nat} : \text{Type}$  and  $\mathbf{1} : \text{Type}$  (unit), the following are typing judgments:

$$\cdot \vdash \star : \mathbf{1} \quad \text{and} \quad x : \text{Nat} \vdash x : \text{Nat}.$$

The left judgement means that from the empty context we conclude that  $\star$  is a term of the unit type. The right judgement is trivial. It means that if we assume that  $x$  is a natural number, then we can conclude that  $x$  is a natural number.

## 4 Reading inference rules and the rule bar

An *inference rule* has the schematic form

$$\frac{\text{premise}_1 \quad \cdots \quad \text{premise}_n}{\text{conclusion}} \text{NAME}$$

The long *horizontal rule bar* separates premises (above) from the conclusion (below). If there are no premises, the rule is an *axiom* (always available).

**Example.**

- *Unit introduction.*

$$\frac{}{\Gamma \vdash \star : \mathbf{1}} \text{1-INTRO}$$

The Unit introduction example reads as “without any assumptions and under any context  $\Gamma$  we conclude that  $\star$  is of the unit type”.

## 5 Constructors and canonical forms

In type theory, each type is defined by specifying its *constructors*—the canonical ways of producing elements (terms) of that type. Constructors determine how we can *build* terms of a given type, and by extension, how we can reason about or eliminate them.

Formally, for a type  $A : \text{Type}$ , a constructor is a term-forming rule of the form

$$\frac{\Gamma \vdash t_1 : A_1 \quad \cdots \quad \Gamma \vdash t_n : A_n}{\Gamma \vdash c(t_1, \dots, t_n) : A},$$

which specifies how a new term  $c(t_1, \dots, t_n)$  of type  $A$  can be formed from existing terms of other types  $A_1, \dots, A_n$ . Each type comes with one or more constructors that uniquely determine its canonical inhabitants.

**Examples.**

- **Unit type.** The unit type  $\mathbf{1}$  has exactly one constructor:

$$\frac{}{\star : \mathbf{1}}.$$

Thus,  $\star$  is the only canonical inhabitant of  $\mathbf{1}$ . Note that this is equivalent to the “unit introduction” example. We just omitted the context  $\Gamma$ , since the construction holds for any  $\Gamma$ .

- **Empty type.** The empty type  $\mathbf{0}$  has *no* constructors. Therefore, there are no canonical terms  $a : \mathbf{0}$ . This makes sense, since by definition the empty type is empty.

- **Sum type.** For two types  $A, B : \text{Type}$ , the sum type  $A + B$  has two constructors:

$$\frac{a : A}{\text{inl}(a) : A + B} \quad \text{and} \quad \frac{b : B}{\text{inr}(b) : A + B}.$$

The first constructor  $\text{inl}$  injects values from the left component  $A$ , and the second constructor  $\text{inr}$  injects values from the right component  $B$ . These two rules fully describe the canonical forms of elements of  $A + B$ .

**Intuition.** Constructors are the primitive building blocks of each type. For instance, every term of  $A + B$  is either constructed as  $\text{inl}(a)$  for some  $a : A$ , or as  $\text{inr}(b)$  for some  $b : B$ ; there are no other canonical ways to obtain a term of this type. This property allows us to reason about sum types by *case analysis*, which we will define later.

## 6 Well-formed contexts ( $\Gamma \vdash \text{ctx}$ )

We write the *well-formedness* judgment

$$\Gamma \vdash \text{ctx}$$

and read it as “ $\Gamma$  is a well-formed context.” The symbol  $\text{ctx}$  is a fixed tag (a nullary predicate) used only on the right of  $\vdash$  to denote this judgment; it is not a type.

**Domain and freshness.** The set of variables declared in  $\Gamma$  is its *domain*, written  $\text{dom}(\Gamma)$ . It is defined inductively by

$$\text{dom}(\cdot) \equiv \emptyset, \quad \text{dom}(\Gamma, x : A) \equiv \text{dom}(\Gamma) \cup \{x\}.$$

We write  $x \notin \text{dom}(\Gamma)$  to express that  $x$  is *fresh* for  $\Gamma$ . We use the usual membership notation  $(x : A) \in \Gamma$  to mean that the declaration  $x : A$  occurs somewhere in  $\Gamma$ .

### What does it mean for a context to be well-formed?

Intuitively, a context  $\Gamma$  is *well-formed* if every declaration inside it makes sense: each type appearing in a declaration is itself already a valid type in the smaller context preceding it. Formally, recall that a context is a sequence of variable declarations:

$$\Gamma \equiv x_1 : A_1, x_2 : A_2, \dots, x_n : A_n.$$

We say that such a context is *well-formed*, written

$$\Gamma \vdash \text{ctx},$$

if and only if the following recursive conditions hold:

- The empty context is well-formed:

$$\cdot \vdash \text{ctx}.$$

- If  $\Gamma \vdash \text{ctx}$  and the type  $A$  is well-formed under  $\Gamma$ , i.e.  $\Gamma \vdash A : \text{Type}$ , and the variable  $x$  is fresh ( $x \notin \text{dom}(\Gamma)$ ), then the extended context  $\Gamma, x : A$  is well-formed:

$$\Gamma, x : A \vdash \text{ctx}.$$

Intuitively, this means that the declarations in a context must be arranged so that each type depends only on variables that have been declared earlier.

**Example.**

$$x : \text{Nat}, y : \text{Bool}, z : \text{String} \vdash \text{ctx}$$

is well-formed, since each type ( $\text{Nat}$ ,  $\text{Bool}$ ,  $\text{String}$ ) is already a valid type in the previous context.

**Dependent example.**

$$x : \text{Nat}, y : (\text{Bool}, x) \vdash \text{ctx}$$

is also well-formed, because  $(\text{Bool}, x)$  is a type depending on  $x$ , and  $x$  has already been declared.

**Non-example.**

$$y : (\text{Bool}, x), x : \text{Nat} \not\vdash \text{ctx},$$

because  $y$ 's type refers to  $x$ , but  $x$  has not yet been declared at that point.

**In summary.** A context is well-formed precisely when every variable declaration in it is meaningful in the smaller context built from the declarations before it. This ensures that type dependencies are acyclic and well-scoped.

## Formation rules for contexts

We inductively generate well-formed contexts with two rules.

*Empty context.*

$$\frac{}{\cdot \vdash \text{ctx}} \text{CTX-EMPTY}$$

*Extension.*

$$\frac{\Gamma \vdash \text{ctx} \quad \Gamma \vdash A : \text{Type} \quad x \notin \text{dom}(\Gamma)}{\Gamma, x : A \vdash \text{ctx}} \text{CTX-EXT}$$

The side condition  $x \notin \text{dom}(\Gamma)$  enforces that variable names in a context are pairwise distinct.

## Basic facts

From  $\Gamma \vdash \text{ctx}$  and  $(x : A) \in \Gamma$  we may derive the trivial *variable rule*:

$$\frac{\Gamma \vdash \text{ctx} \quad (x : A) \in \Gamma}{\Gamma \vdash x : A} \text{VAR}$$

## Examples and a non-example

Assume we have the closed type declarations  $\cdot \vdash \text{Nat} : \text{Type}$ ,  $\cdot \vdash \text{Bool} : \text{Type}$ , and  $\cdot \vdash \text{String} : \text{Type}$ . Then the following derivations show well-formed contexts:

*Example 1.*  $\cdot \vdash \text{ctx}$  by CTX-EMPTY.

*Example 2.*

$$\frac{\cdot \vdash \text{ctx} \quad \cdot \vdash \text{Nat} : \text{Type} \quad x \notin \text{dom}(\cdot)}{x : \text{Nat} \vdash \text{ctx}} \text{CTX-EXT}$$

*Example 3.*

$$\frac{x : \text{Nat} \vdash \text{ctx} \quad x : \text{Nat} \vdash \text{Bool} : \text{Type} \quad y \notin \text{dom}(x : \text{Nat})}{x : \text{Nat}, y : \text{Bool} \vdash \text{ctx}} \text{CTX-EXT}$$

*Non-example (duplicate name).*

$$x : \text{Nat}, x : \text{Bool} \not\vdash \text{ctx}$$

since  $x \in \text{dom}(x : \text{Bool})$  violates  $x \notin \text{dom}(x : \text{Nat})$ , meaning that  $x$  is not fresh.

## Worked derivations: recursively checking well-formedness

We illustrate how to *recursively* apply the context rules

$$\frac{}{\cdot \vdash \text{ctx}} \text{CTX-EMPTY} \quad \frac{\Gamma \vdash \text{ctx} \quad \Gamma \vdash A : \text{Type} \quad x \notin \text{dom}(\Gamma)}{\Gamma, x : A \vdash \text{ctx}} \text{CTX-EXT}$$

to decide whether a given sequence of declarations forms a well-formed context. Throughout, assume the base types are available in the empty context:

$$\cdot \vdash \text{Nat} : \text{Type}, \quad \cdot \vdash \text{Bool} : \text{Type}, \quad \cdot \vdash \text{String} : \text{Type}.$$

**Example 1 (non-dependent).** Check  $x : \text{Nat}, y : \text{Bool} \vdash \text{ctx}$ .

$$\frac{\frac{\cdot \vdash \text{ctx} \text{ CTX-EMPTY} \quad \cdot \vdash \text{Nat} : \text{Type} \quad x \notin \text{dom}(\cdot)}{x : \text{Nat} \vdash \text{ctx}} \text{CTX-EXT} \quad \frac{x : \text{Nat} \vdash \text{Bool} : \text{Type} \quad y \notin \text{dom}(x : \text{Nat})}{x : \text{Nat}, y : \text{Bool} \vdash \text{ctx}} \text{CTX-EXT}}$$

Reading bottom-up: start with  $\cdot$  via CTX-EMPTY; extend by  $x : \text{Nat}$  using  $\cdot \vdash \text{Nat} : \text{Type}$ ; then extend by  $y : \text{Bool}$  using  $x : \text{Nat} \vdash \text{Bool} : \text{Type}$  and freshness.

**Non-example (dependency out of order).** Consider  $y : (\text{Bool}, x), x : \text{Nat}$ . Attempting CTX-EXT on the first declaration requires  $\cdot \vdash (\text{Bool}, x) : \text{Type}$ . But in  $\cdot$  there is no variable  $x : \text{Nat}$ , so we cannot derive  $\cdot \vdash x : \text{Nat}$ , hence  $\cdot \vdash (\text{Bool}, x) : \text{Type}$  fails. Therefore

$$y : (\text{Bool}, x), x : \text{Nat} \not\vdash \text{ctx}.$$

**Non-example (duplicate name).** Consider  $x : \text{Nat}, x : \text{Bool}$ . The second extension violates freshness since  $x \in \text{dom}(x : \text{Nat})$ . Thus the side condition  $x \notin \text{dom}(\Gamma)$  fails and

$$x : \text{Nat}, x : \text{Bool} \not\vdash \text{ctx}.$$

These derivations show that checking  $\Gamma \vdash \text{ctx}$  reduces *recursively* to (i) checking the smaller prefix is well-formed, (ii) checking the new declaration's type is a type in that prefix, and (iii) enforcing freshness.

## 7 Sum types and the inl and inr constructors

### Definition of the sum type

Given two types  $A : \text{Type}$  and  $B : \text{Type}$ , their *sum type* is written

$$A + B : \text{Type}.$$

But how is  $A + B$  constructed? We first need to introduce the constructors of the type  $A + B$ . For two types  $A, B : \text{Type}$ , the sum type  $A + B$  has two constructors:

$$\frac{a : A}{\text{inl}(a) : A + B} \quad \text{and} \quad \frac{b : B}{\text{inr}(b) : A + B}.$$

The first constructor `inl` injects values into  $A + B$  from the left component  $A$ , and the second constructor `inr` injects values into  $A + B$  from the right component  $B$ . These two rules fully describe the canonical forms of elements of  $A + B$ .

**Intuition.** Constructors are the primitive building blocks of each type. For instance, every term of  $A + B$  is either `inl(a)` for some  $a : A$ , or `inr(b)` for some  $b : B$ ; there are no other canonical ways to obtain a term of this type.

**Comment.** A potential realization of the `inl` and `inr` constructors is

$$\text{inl}(a) \equiv (\text{true}, a), \quad \text{inr}(b) \equiv (\text{false}, b).$$

However, they are *not* needed to use sums soundly: the inductive rules already give full computational content. This is something which will be discussed later in more depth.



**Example: How  $A + B$  looks like?** To make the idea of a sum type concrete, let us take two small finite types:

$$A \equiv \{\text{red}, \text{green}\}, \quad B \equiv \{0, 1\}.$$

Then the sum type  $A + B$  consists of all elements of  $A$  tagged by  $\text{inl}$ , and all elements of  $B$  tagged by  $\text{inr}$ :

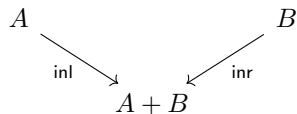
$$A + B = \{\text{inl}(\text{red}), \text{inl}(\text{green}), \text{inr}(0), \text{inr}(1)\}.$$

Intuitively,  $\text{inl}(a)$  means “a value coming from  $A$ , left side of the sum,” and  $\text{inr}(b)$  means “a value coming from  $B$ , right side of the sum.”

This can be illustrated in tabular form:

Element of $A$	Constructor	Element of $A + B$
red	$\text{inl}$	$\text{inl}(\text{red})$
green	$\text{inl}$	$\text{inl}(\text{green})$
0	$\text{inr}$	$\text{inr}(0)$
1	$\text{inr}$	$\text{inr}(1)$

In general, every element of  $A + B$  is either of the form  $\text{inl}(a)$  for some  $a : A$ , or  $\text{inr}(b)$  for some  $b : B$ . If desired, we can also represent the injections diagrammatically:



This expresses that both  $A$  and  $B$  “feed into” the coproduct  $A + B$  using the constructors  $\text{inl}$  and  $\text{inr}$ .