Investigating the Distribution of Market Returns

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Bio of The Author

Kenneth Potts is a senior student of Mathematics at Ithaca College, graduating in May of 2017. He entered Ithaca College as a Television and Radio major, but his interest in math soon developed and took over his academic interests. After taking many mathematics courses, he developed an interest in computational, statistical, and financial mathematics, as well as computer programming and data science. Apart from his mathematical interests, he loves playing guitar in his band, reading new books, and learning new concepts.

Abstract

Many academic financial models are built on the assumption that market returns are normally distributed, however, this is not often the case. The data shows that currency and equity market returns are leptokurtic, or fat tailed, especially for negative returns. A substitution of the normal distribution for the Laplace distribution can yield a more accurate model of market returns, especially at smaller, or finer, time frames. A T-distribution with 4-5 degrees of freedom is also a potential representative distribution. The fat tailed behavior of markets is not likely due to autocorrelation of returns, although negative autocorrelation is present in the currency market.

1 Introduction

An understanding of financial market returns is critical whether one is planning to hedge risks or or designing systems intended to capture market returns. Models such as Value at Risk, or VAR, aim to calculate the probability, with confidence intervals, of an event which would cause a portfolio to experience a large loss of a specified size [2]. VAR and similar style models require an understanding of the distribution of returns for the investment, or multiple investments within a portfolio, to determine the maximum likely loss of such a portfolio. Then there are asset pricing models such as the Black-Scholes pricing model for options. This model assumes that asset prices follow an infinitesimal log-normal random walk [6]. Even Harry Markowitz's famed mean variance theory, which describes how to optimally construct a portfolio of assets, assumes that market returns are independent and normally distributed between periods [10]. The normal distribution of returns is a core assumption underlying many prominent academic financial models.

There have been many large scale financial events which triggered wild moves in markets that would suggest that market returns are non-normal. Some of these events include the crash on Black Monday in 1987, the breaking of the peg on the Pound Sterling in 1992, the Russian default crisis and the fall of LTCM–Long Term Capital Management, a hedge fund overseen by two Nobel laureates–in 1998, the bursting of the technology bubble in 2000-2001, the bursting of the real estate and debt bubbles in 2007-2008, the flash crash of 2010, the break of the peg on the Swiss Franc in 2015, the Brexit vote in 2016, and the British Pound flash crash of 2016. These are all examples of large and nearly unpredictable market events, many of which shook entire markets. This paper aims to analyze returns from the equity and currency markets from the beginning of the 1990s to the present day, which encompasses many of these events. Although present, this study extends beyond these events to analyze returns across various time frames, looking at individual assets and also the equity market as a whole.

2 Market Data

2.1 Equities

All equity market data used in this study was retrieved from Yahoo! Finance, which provides open, high, low, close, adjusted close, and volume data for the roughly 6500 equities currently listed on either the NASDAQ exchange or NYSE. For the purpose of multiple statistical test in this study, an algorithm has been written to randomly select 1 or more of these securities. Throughout the study, the S&P 500 index is used as the primary benchmark of the equity market as a whole. This is done via the SPY ETF–exchange traded fund–which follows the S&P 500. The Russel 2000 ETF (IWM), PowerShares QQQ ETF (QQQ), Dow Jones Industrial Average ETF (DIA), and Vanguard Total Stock Market ETF (VTI) are used to represent the market as a whole. Split and dividend adjusted prices have also been calculated for all equities where applicable. The beginning and end of the pricing data has been set as January 1^{st}, 1990 and January 1^{st}, 2017 respectively. In the case where the asset existed before January 1^{st}, 1990, only data from that day forward was retrieved. In the case where the asset did not exist on January 1^{st}, 1990, all data available for the asset to January 1^{st}, 2017 has been retrieved. Only equities listed on the NASDAQ and NYSE exchanges as of January 1^{st}, 2017 are used, as data for delisted securities is not available on Yahoo! Finance. As a result, all calculations done with equities data exhibit survivorship bias, as any company that was delisted, failed,or acquired during the period from 1990 to 2017 are not present in the data. If it is the case that a majority of missing companies were delisted due to company failure or market crash effects, then we would expect to see fewer negative extreme events in the data. A follow up study is being considered, where data for the missing companies would be used in order to eliminate this survivorship bias. The survivorship bias-free results would then be compared to current results, which could give insight into the hidden effect of such a bias.

2.2 Currencies

All currency data in this study was retrieved from Oanda, which provides foreign exchange services. Hourly open, high, low, and close pricing data was retrieved for selected currency pairs consisting of cross rates for the following major currencies: the Euro (EUR), Pound sterling (GBP), Australian dollar (AUD), New Zealand dollar (NZD), United States dollar (USD), Canadian dollar (CAD), Swiss franc (CHF), and Japanese yen (JPY). All pair rates are calculated with standard cross rates using the base currency in prioritized order of the previous list, e.g., EUR/USD is calculated instead of USD/EUR because the Euro has the highest priority as a base rate.

2.3 Calculating Returns

It is difficult to compare asset prices directly because a large price in one asset may be comparatively small for another. This is because each asset's prices vary greatly in magnitude, e.g. a \$1 increase in a \$10 stock is comparable to a \$10 increase in a \$100 stock. It is more useful to compare returns rather than prices. We can look at r_i, the return for period i, as the percent change in price from period $i-1$ to period i.

$$r_i = \frac{P_i - P_{i-1}}{P_{i-1}}$$

where P_i is the price at the close of period i. This is a geometric calculation of returns because calculating the return over multiple periods requires multiplication. Consider the returns for periods 1 to n, written as $r_1, r_2, r_3, ..., r_n$, then to calculate the total return, r, over the we have:

$$r = (1 + r_1)(1 + r_2)(1 + r_3)...(1 + r_n) - 1$$

Another way to measure returns is logarithmically, known as log returns, which represent the rate of continuously compounding returns. The calculation:

$$r_i = ln\left(\frac{P_i}{P_{i-1}}\right)$$

This is beneficial due to the additive property of logarithms, by which we can calculate the return from period 1 to n, r_N, as

$$\sum_{j=1}^{n} r_j = r_1 + r_2 + ... + r_n = r_N$$

bb The returns are additive rather than multiplicative and therefore a logarithmic return of k is negated by a logaritmic return of $-k$. This is not the case for geometric returns, e.g. $(1+1)(1-\frac{1}{2}) = 1$, a 100% return is negated when followed by a -50% return. Because of their additive property, log returns are best to use when measuring normality [5]. For this reason, log returns will be used for this study.

3 The Normal Distribution

3.1 Definition

The normal distribution is an example of a probability distribution. We defined the probability density function of the normal distribution with a mean μ and standard deviation σ as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

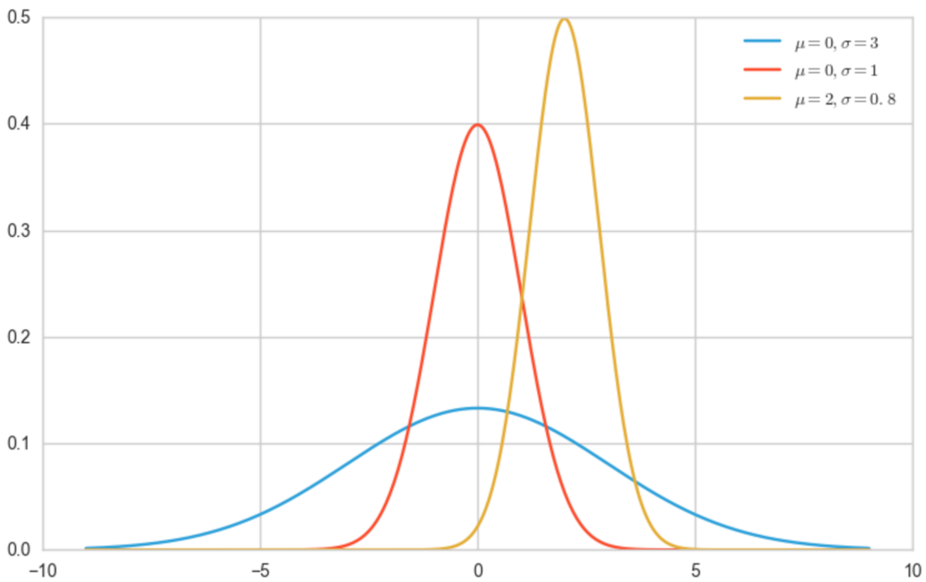

Figure 1: Normal Distribution

3.2 Skewness

The normal distribution is symmetric. Probability distributions do not have to be symmetric. A distribution may have a longer and fatter tail on the right side and not many large values on the left (negatively skewed), or vice versa (positively skewed) irrespective of the mean and variance. The sample skewness describes the symmetry of the distribution and is explicitly defined as:

$$S_K = \frac{n}{(n-1)(n-2)} \sum_{i=1}^{n} \left(\frac{(X_i - \overline{X})}{\sigma} \right)^3$$

where $\overline{X}$ is the sample mean and n is the number of observations in the sample [3].

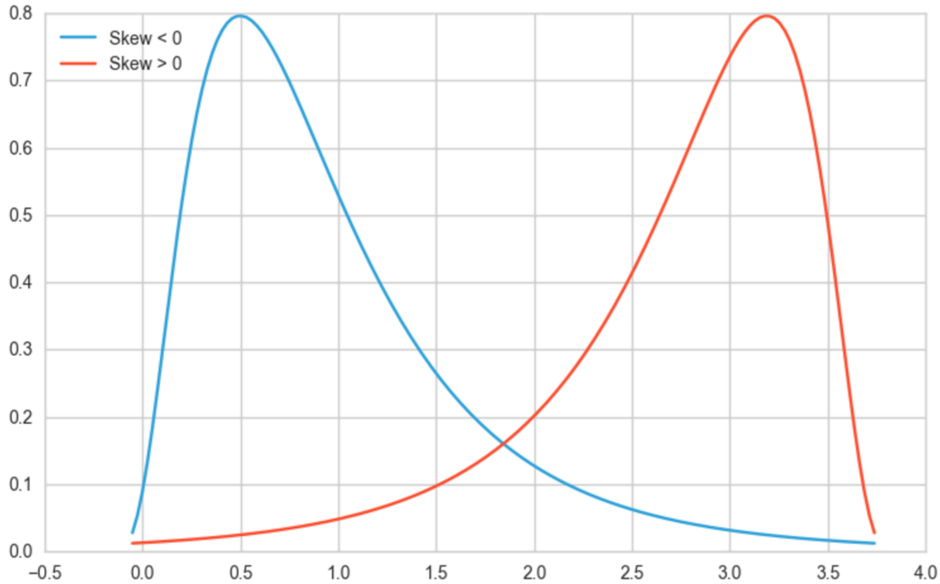

Figure 2: Skewed Log-Normal Distribution [8]

3.3 Kurtosis

Another measure of the shape of a distribution is kurtosis, which describes the shape of the deviation from the mean, and more specifically how the center and tails of the distribution behave. For large enough values of n, kurtosis is calculated as [7]:

$$K = \frac{1}{n} \frac{\sum_{i=1}^{n} (X - \overline{X})^4)}{s^4}$$

for small values of n:

$$K = \left(\frac{n(n+1)}{(n-1)(n-2)(n-3)} \frac{\sum_{i=1}^{n} (X_i - \overline{X})^4}{s^4} \right)$$

A normal distribution always has a kurtosis of 3. A deviation from 3 is called excess curtosis. We can subtract 3 from kurtosis to calculate excess kurtosis. For large enough values of n, excess kurtosis is calculated as:

$$K_e = \frac{1}{n} \frac{\sum_{i=1}^{n} (X - \overline{X})^4)}{s^4} - 3$$

for small values of n:

$$K_E = \left(\frac{n(n+1)}{(n-1)(n-2)(n-3)} \frac{\sum_{i=1}^{n} (X_i - \overline{X})^4}{s^4} \right) - \frac{3(n-1)^2}{(n-2)(n-3)}$$

A leptokurtic distribution has a kurtosis greater than 3, which means it has a very tall peak and fat tails as compaired to the normal distribution. A platykurtic distribution has as kurtosis less than 3, which means it has a very short peak and small tails as compaired to the normal distribution. A distribution with a kurtosis of 3, such as the normal distribution, is called mesokurtic [8][9].

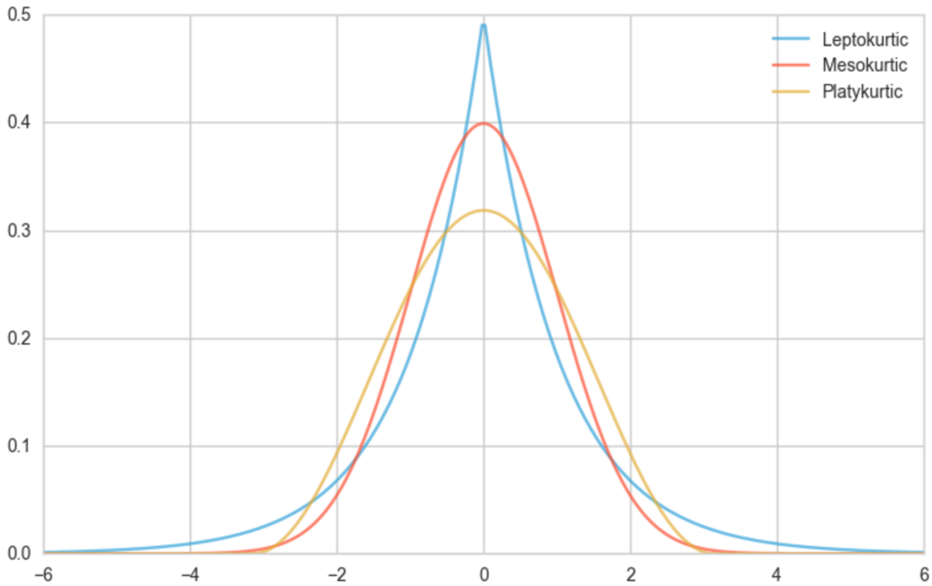

Figure 3: Kurtosis [8]

3.4 Fitting The Normal Distribution

We can easily compare the empirical equity market returns with the theoretical normal distribution by calculating the standard deviation and mean of the daily returns of the SPY, then fitting a normal distribution with the same mean and standard deviation. We can see this visually by plotting the fitted distribution against a histogram of the SPY returns.

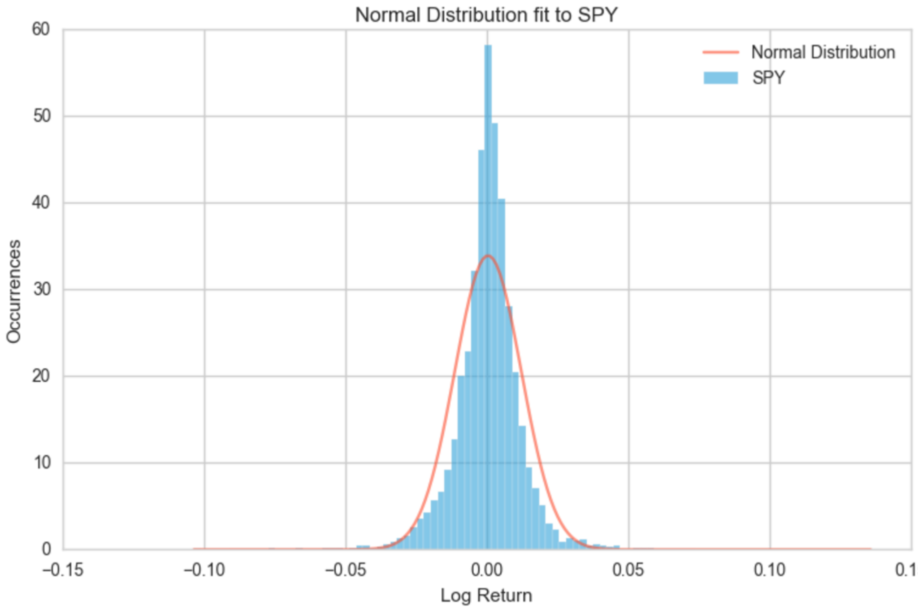

Figure 4: SPY Returns Comparison

The histogram has been normalized, i.e., scaled so that the areas of the rectangles sum to 1, in order to make the visual comparison. It appears that the daily returns of the SPY are leptokurtic. In fact is has a kurtosis of $K \approx 10.1033$ which is well above the $K = 3$ of the normal distribution. See appendix A for more examples with randomly selected stocks.

4 Tests for Normality

4.1 Skewness Test

We can test for normallity by producing a Z-score using the skewness of the distribution [1]. We test for normality of the log returns of SPY with the null hypothesis that the distribution of returns is normal using a significance level of $p = 0.05$. This will be a two sided hypothesis test. With a test statistic of $Z \approx -3.74611$ and $p \approx 0.00018 < 0.05$, we reject the null hypothesis and accept the alternative hypothesis that the returns are non normally distributed. See appendix B for another example with a randomly selected stock.

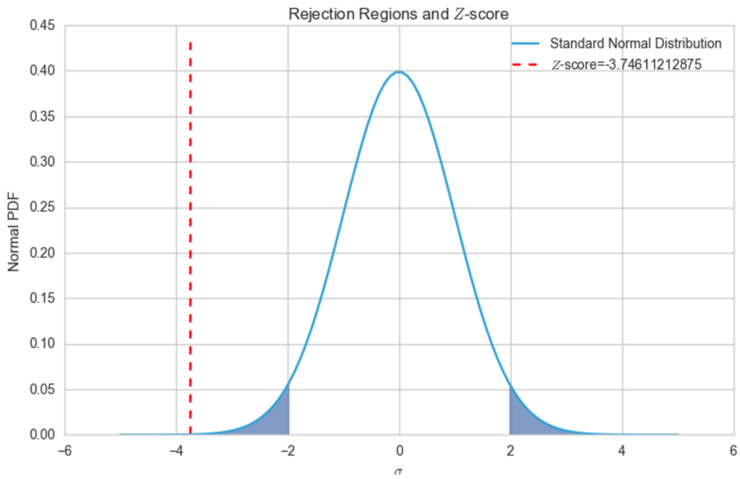

Figure 5: SPY Skewness Test Results

4.2 Kurtosis Test

We can perform an identical test to the skewness test with the exception that now we will use the kurtosis calculation instead of skewness [1]. We test for normality of the log returns of SPY with the null hypothesis that the distribution of returns is normal using a significance level of $p = 0.05$. This will be a two sided hypothesis test. With a test statistic of $Z \approx 32.6187$ and $p \approx 2.22187 \cdot 10^{-233} < 0.05$, we reject the null hypothesis and accept the alternative hypothesis that the returns are non normally distributed.

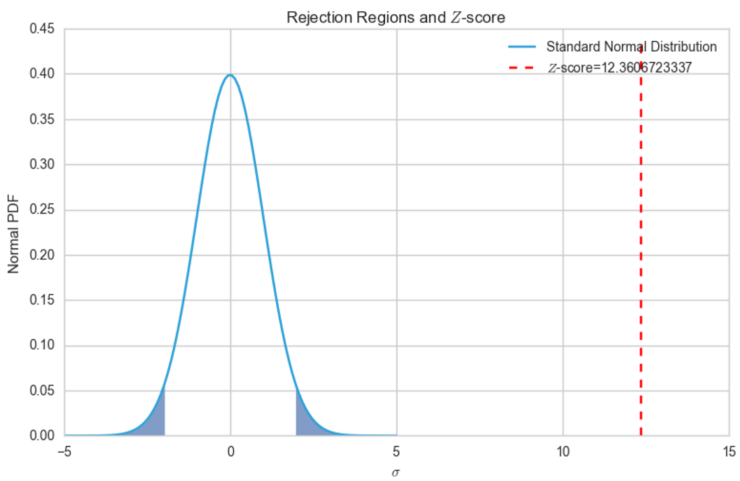

Figure 6: SPY Kurtosis Test Results

4.3 Jarque-Bera Test

The Jarque-Bera Test assesses for normality by using both the skewness and the excess kurtosis. It uses a JB statistc which is calculated as:

$$JB = \frac{n-k+1}{6}\left(S^2 + \frac{1}{4}(C-3)^2\right)$$

where S is the sample skewness, C is the sample kurtosis, and k is the number of regressors [4]. The JB asymptotically has a χ^2 distribution with 2 degrees of freedom.

We test for normality of the log returns of SPY with the null hypothesis that the distribution of returns is normal using a significance level of $p = 0.05$. With a Jarque-Bera statistic of $JB \approx 25644.04$ and $p \approx 4.26573 \cdot 10^{-35} < 0.05$, we reject the null hypothesis and accept the alternative hypothesis that the returns are non normally distributed. See Appendix B for visual representations of the Jarque-Bera test.

Performing a Jarque-Bera test for the daily returns of 150 randomly selected equities with a cutoff of $p = 0.05/150$–p-value adjusted via a Bonferroni Correction due to multiple (150) comparisons–resulted in a $97.\overline{33}\%$ failure rate. Similarly, the same test performed on the monthly returns of 50 equities yielded a 64% failure rate. The same test can be performed on the returns of the 28 currency pair, now instead using a corrected p-value of 0.05/28. Running the test for the hourly and weekly returns of all currency pairs resulted in a failure rate of 100%. There was a decrease in the failure rate, at 71.43%, for monthly currency returns.

The Jarque-Bera test results suggest that equity and currency market returns are not normally distributed. As the sample time frame for the returns get longer, e.g., days, to weeks, to months, the returns seem to be less extreme, but still relatively fat tailed compared to the normal distribution. The next step is to determine whether we can model market returns with something else, say another type of distribution.

4.4 Quantile-Quantile Plot

So far only histograms and statistical tests have been used to show that markets are non-normal. Histograms, however, make it hard to view the discrepancies between a normal distribution and empirical data. For this purpose, there is a better option.

The Q-Q plot, or quantile-quantile plot, plots the quantiles of two distributions against one another. In this case one of the distributions is observed data. If the two distributions are a close match, they will have a linear relationsip between quantiles and therefore show up as a straight line on the plot [5]. See Appendix C for an example. It is easy to see the differences in shape of the distributions from a Q-Q plot. If the sample distribution is leptokurtic, when plotted against the normal distribution it will curve down on the left and curve up on the right, showing a greater frequency of extreme events [5].

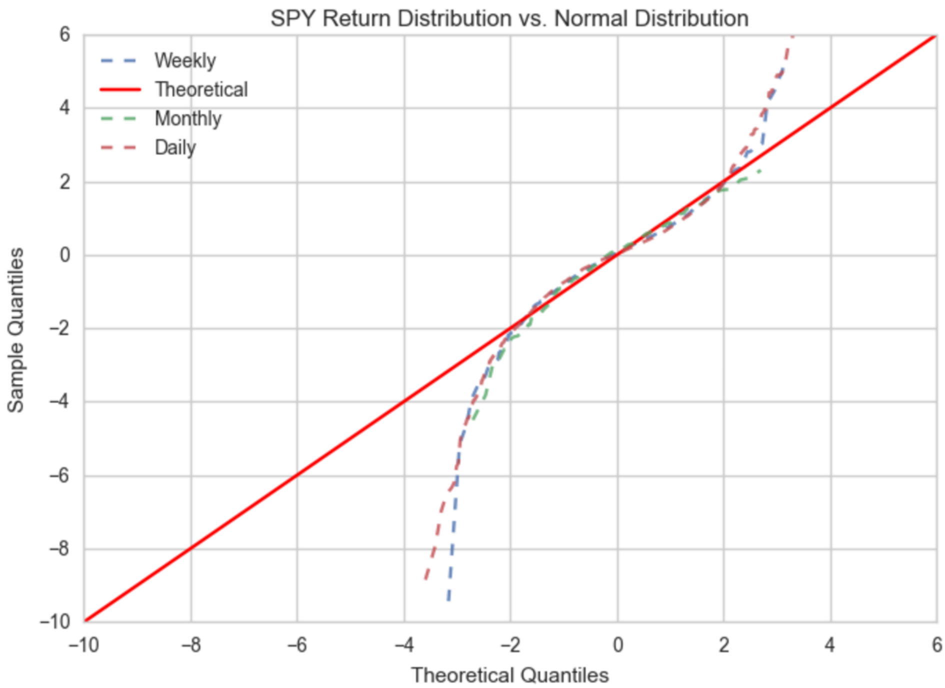

Figure 7: SPY Daily, Weekly, and Monthly Returns

The shape of the dashed red line, daily returns, is indicative of a fat tail distribution relative to the normal distribution. The distribution for SPY returns shows 8σ - 9σ data points that correspond to 3σ or 4σ data points from a normal distribution. The Q-Q plot also shows that the positive fat tail disappears as we move to a larger time frame. We can make the same comparison across market ETFs and Currencies.

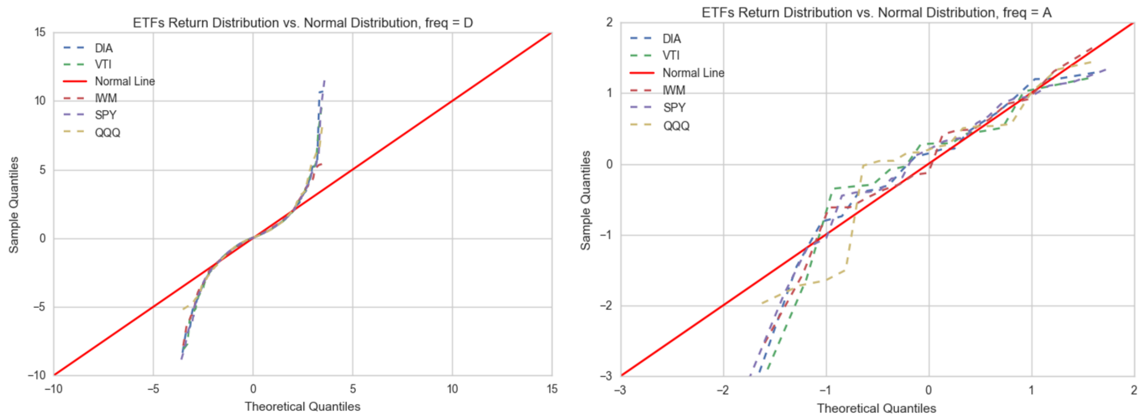

Figure 8: ETF Daily and Annual Returns

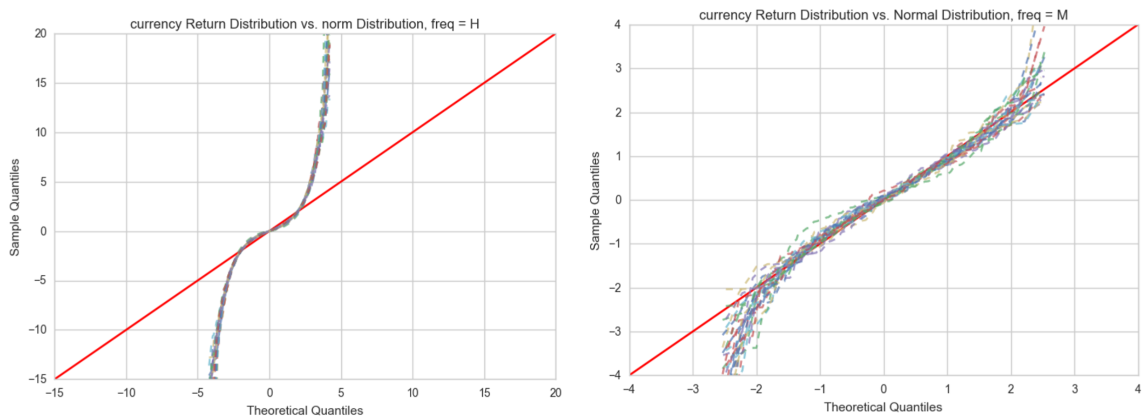

Figure 9: Currency Hourly and Monthly Returns

For the intermediate time frames see Appendix C. The distribution for hourly Currency returns shows 15σ - 20σ data points that correspond to about 4σ data points from a normal distribution. This shows that the currency market on the hourly time frame frequently exhibits severe market moves, far more than predicted by a normal distribution. The positive fat tail does not exist at the annual time frame for the ETFs while the negative fat tail does. The ETF returns of the larger time frames appear to be negatively skewed. This thought is confirmed by the following plots which show a few statistical moments across various time frames.

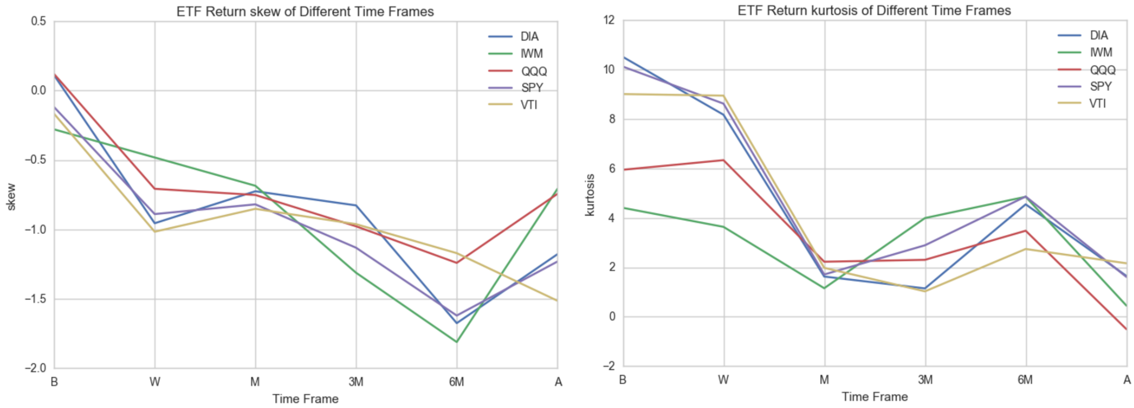

Figure 10: ETF Skewness and Kurtosis Across Time Frames

This relationship does not appear to be present it currency pair returns, see Appendix D.

5 Finding A Better Fit

We have seen that the normal distribution is a poor fit for equity and currency market returns, especially at the smaller time frames, e.g., hourly, daily. When various assets are plotted on a QQ-plot with a normal distribution, it creates a curve that is indicative of a fat tail distribution. The QQ-plot can also be used with other distributions. In this way, we can use it to see what distribution fits best with the observed distribution of market returns.

5.1 The Laplace Distribution

The Laplace Distribution is essentially a double exponential distribution. The Laplace distribution is leptokurtic and features power-law decay in the tails, expressed via fat tails [9]. These characteristics make it a potential fit with market returns.

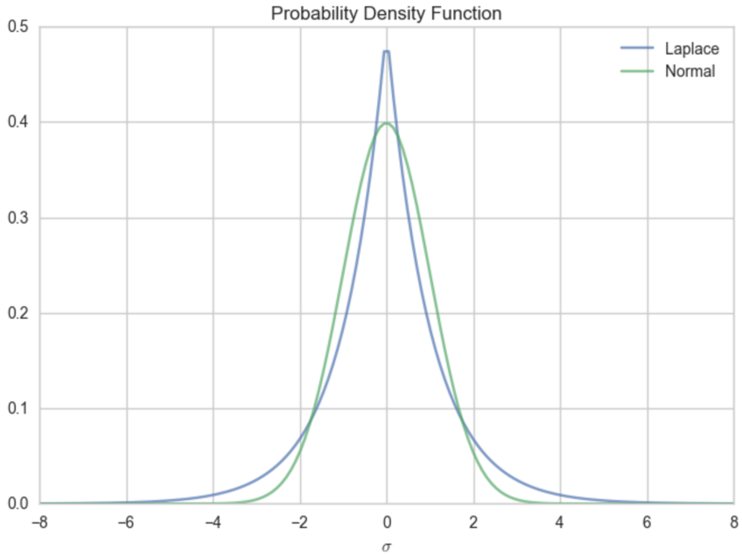

Figure 11: Laplace vs. Normal Distribution

We can plot multiple time frames of ETF returns against the Laplace distribution, which will give us a sense for its goodness of fit with empirical ETF returns.

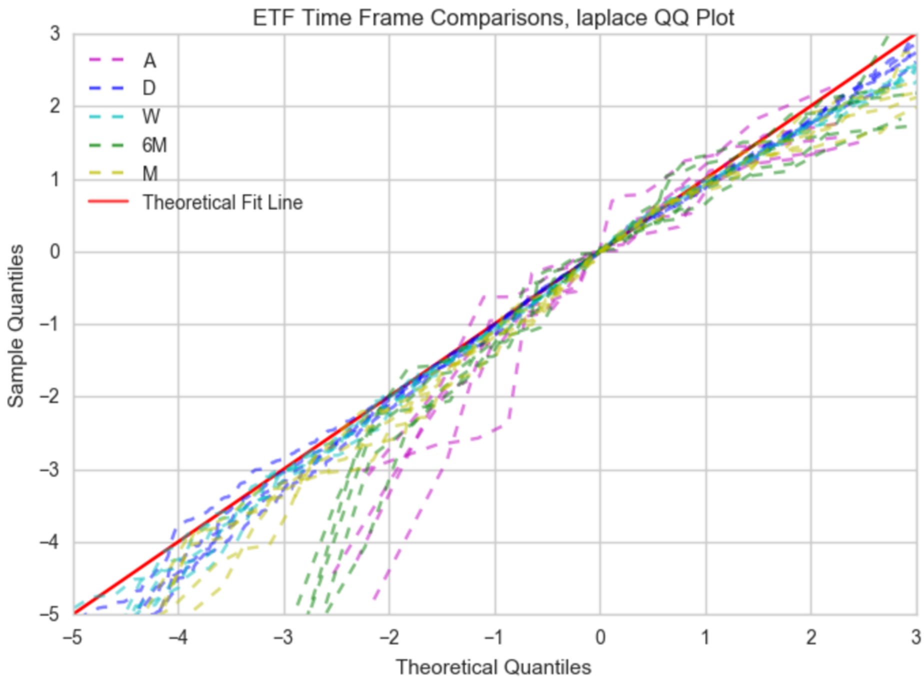

Figure 12: ETF Returns Across Time Frames vs. Laplace Distribution

From the chart above, we can see that the Laplace distribution is a relatively good fit for market ETF returns for the day, week, and month time frames as compared to the Normal distribution. Although the fit is better, it is not perfect. There is still a relatively fat negative tail in the market returns compare to that of the Laplace distribution. This is shown as the downward sloping lines on the bottom left of the plot, below the theoretical fit line. Currency and equity returns are also well fit with the Laplace distribution at the smaller time frames.

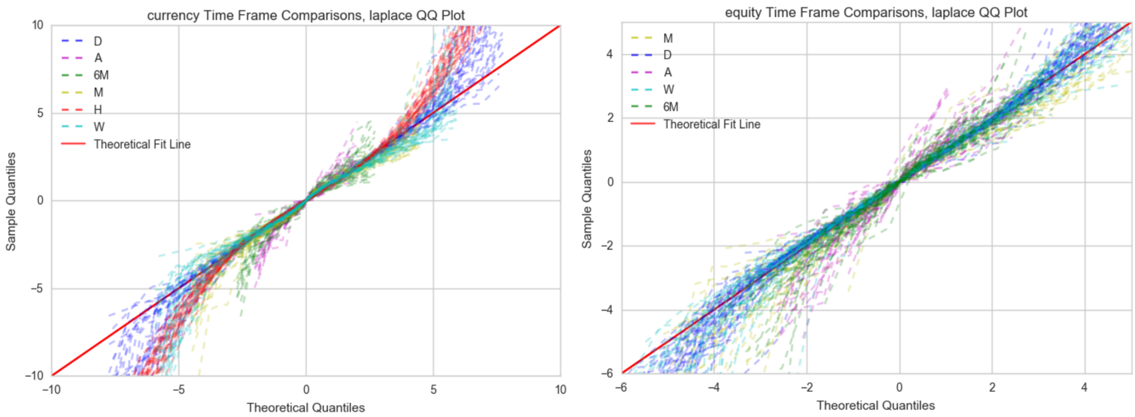

Figure 13: Currency and Equity Returns Across Time Frames vs. Laplace Distribution

5.2 Comparing Distributions

The Cauchy distribution and the T-distribution are two standard relatively fat tailed (leptokurtic) continuous random variable distributions. We can plot the normal, Laplace, Cauchy, and T distributions all at once against Currency returns to compare their relative fit. See Appendix C for corresponding ETF and Equity plots.

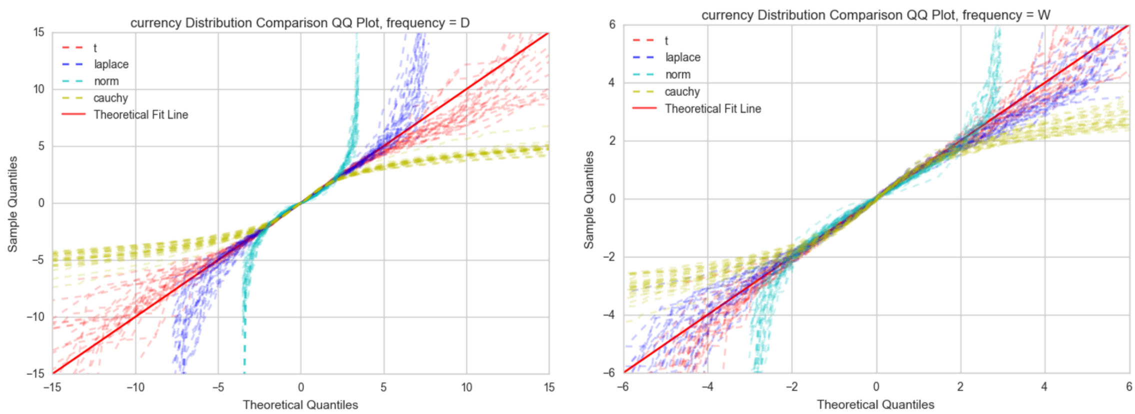

Figure 14: Currency Multi-Distribution Fit, Daily and Weekly Returns

5.3 T-Distributions

Among those distributions explored, the two which seemed to yield the best fit are the T-distribution and Laplace distribution. T-distributions are defined by their degrees of freedom as well as mean and standard deviation. See Appendix E. It is then necessary to determine which T-distribution, i.e., how many degrees of freedom, yields the best fit. To avoid over-fitting, it is best to find a single T-Distribution with a specific number of degrees of freedom that in general fits well with all the data. We can determine the best fit by examining the Q-Q plot of currency returns against T-distributions of varying degrees of freedom. See Appendix F for corresponding ETF and Equity plots.

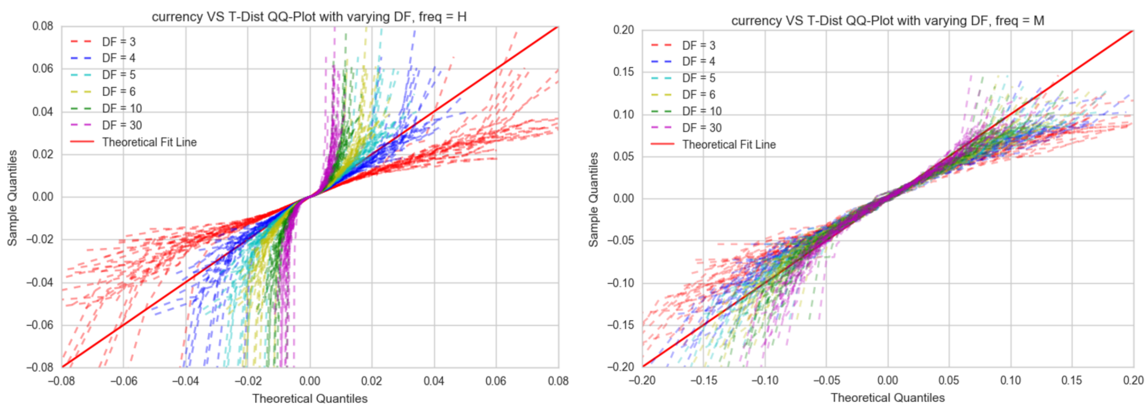

Figure 15: Hourly and Monthly Currency Returns against T-Distributions

It looks like a T-Distribution with 4 or 5 degrees of freedom has the Best fit. However, the distribution is primarily well fitting at higher time frames, such as a week or greater. The Laplace distribution has a better fit for the smaller time frames.

6 Autocorrelation

The presence of positive autocorrelated returns is a potential contributing factor to why equity and currency markets exhibit fat tailed return distributions. Autocorrelation is present when values in a time series are correlated with other values in the series at other points in time, i.e., lagging self-correlation. For example, we could say that a time series of returns shows autocorrelation if the previous 2 periods showed positive returns and if as a result the current period's returns were on average or expected to be positive. An example of negative autocorrelation would be if the previous 4 periods showed positive returns, then the current period's returns were more likely to be negative. This has the potential to cause a leptokurtik distribution of returns because a high magnitude outlier will increase the likelihood of another outlier, creating more densely packed tails [5]. We can measure the amount of autocorrelation by plotting a lagging period correlation.

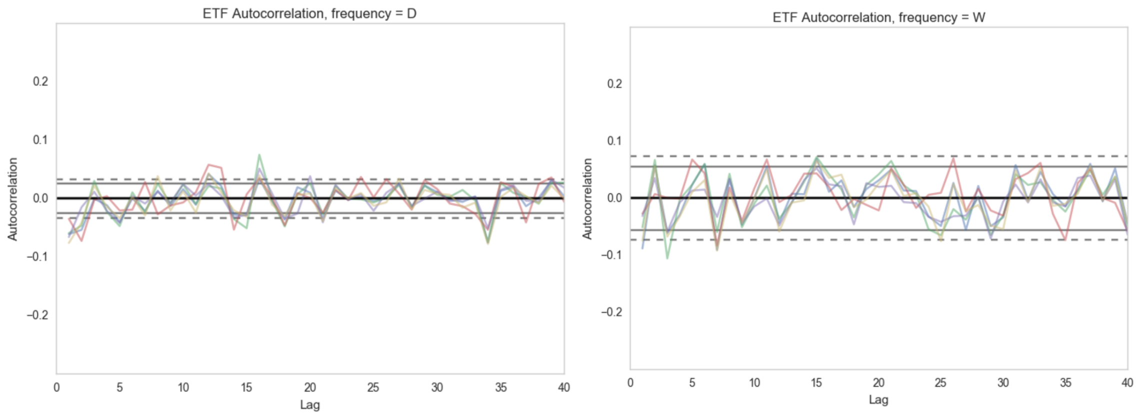

Figure 16: Daily and Weekly Autocorrelation for ETF Returns

Under the null hypothesis, there is no, or zero, autocorrelation. The autocorrelation value is significant if it falls within the rejection region. The 95% rejection region is shown with the outer, solid black lines, and the 99% rejection region in the outer, dotted black lines. All of these ETFs represent the market as a whole so it is expected that they would show rather similar results. It would be expected that the smaller the lag, the more potential there is for autocorrelation. The 1 and 2 period lag for the daily return time frame shows a potential negative correlation but it could also be due to chance. There is potential for multiple comparisons bias here. If enough correlations are checked, by definition we should expect to see some that are significant only due to chance. Equity returns show similar results. See Appendix G. Currency returns on the hourly time frame may exhibit some slight negative autocorrelation with the single period lag.

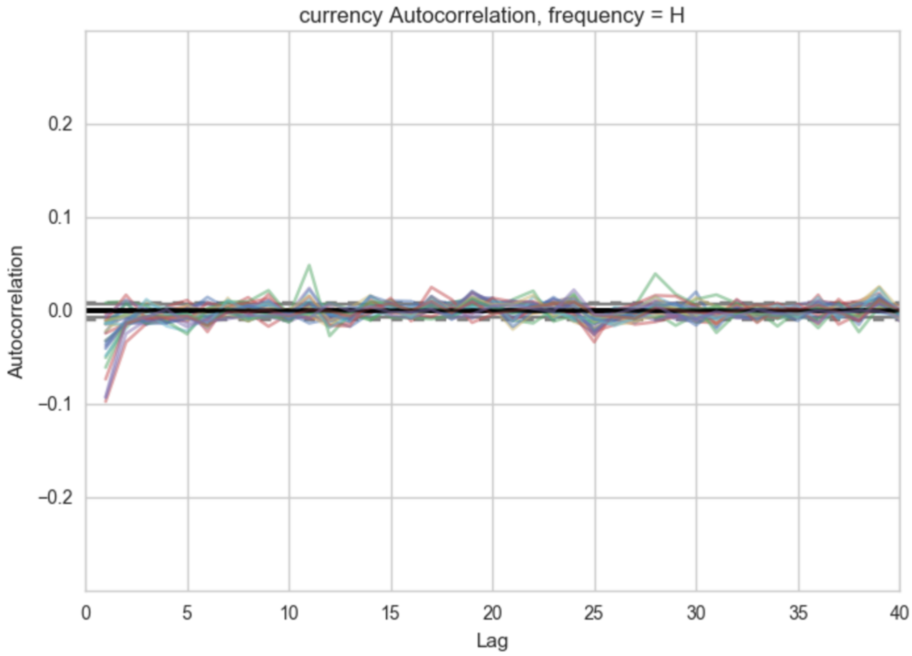

Figure 17: Hourly Autocorrelation for Currencies

There is a lot of noise present in this plot. One way to smooth out the noise is to calculate autocorrelation with a trailing moving/rolling average. In this way, autocorrelation is calculated with the average of multiple periods, rather than a single lagged period.

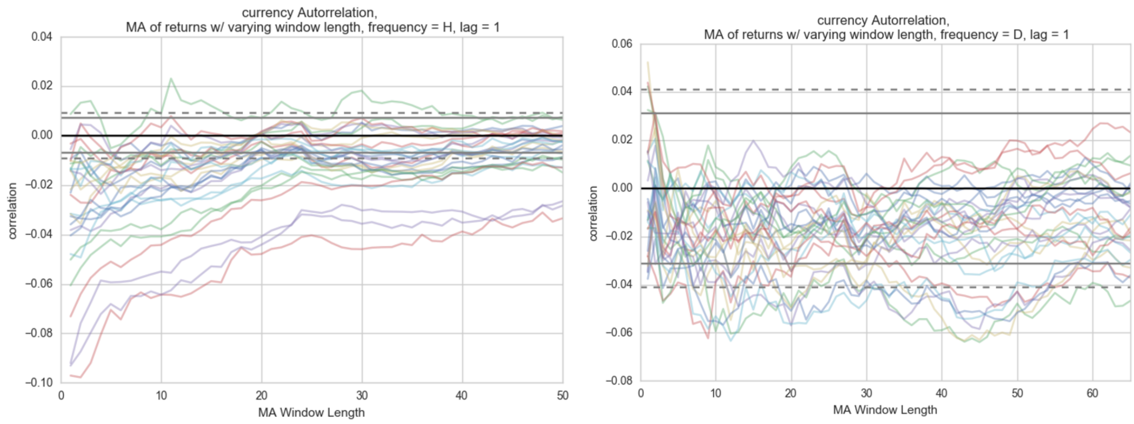

Figure 18: Hourly and daily Autocorrelation for Currencies with a Trailing Moving Average

It appears that without the noise that there is a negative autocorrelation for currency returns on the hourly time frame. This relationship may be significant, however it is very small. The same relationship is not present at the daily time frame. A negative correlation is the opposite of my original hypothesis, which stated that positive autocorrelated returns may contribute to fat tailed market behavior. At the very least it gives insight into the short term behavior of the currency market.

7 Conclusion

It is very apparent that market returns do not exhibit a normal distribution. There are, however, time frames where the returns become more normal, especially the higher time frames, e.g., annual rates of returns. At the same time, that statement is a stretch because there are only about 25 data points for the annual time frame in the data set used in this study. The data shows that market returns are leptokurtic (fat tailed) especially at smaller time frames; hours, days, weeks. The data also indicates that market returns are also somewhat negatively skewed, especially at the larger time frames; monthly, semi-annually, and annually. This means that large negative returns are more frequent than large positive returns. This is unfavorable for many investors who primarily trade long in the market. Long in this context means to purchase or buy an asset to gain from its increase in value. The Laplace distribution more closely represents market returns than does the normal distribution, and therefore is a more adequate model. The data does not point towards autocorrelated returns except at the hourly time frame for currencies. Autocorrelation is likely not contributing to fat tailed behavior. It is recommended that investors position themselves to cut their losses as soon as possible to prevent becoming a victim of negative fat tail occurrences, while not eliminating the possibility of benefiting from positive fat tail events.

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Appendices

Appendix A

The following plots compare the empirical return distributions of two randomly selected stocks to respective fitted normal distributions:

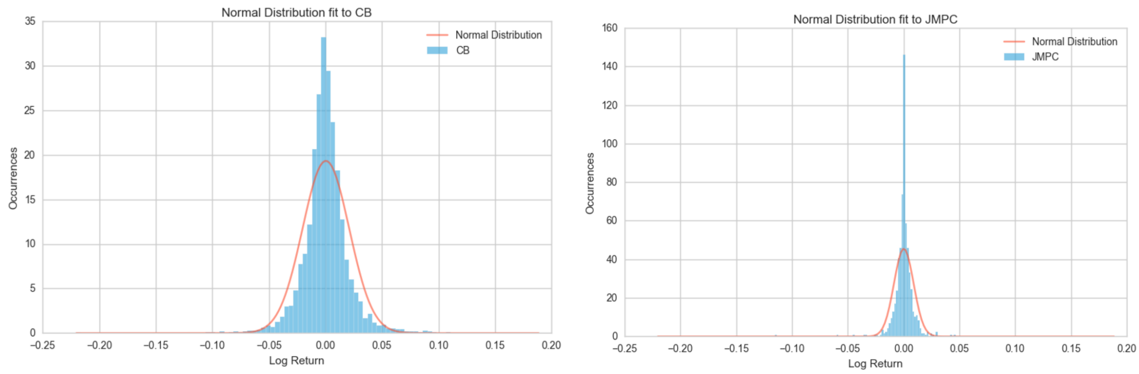

Figure 19: CB and JMPC Returns Comparison with Normal Distribution

Distribution of SPY returns across multiple time frames:

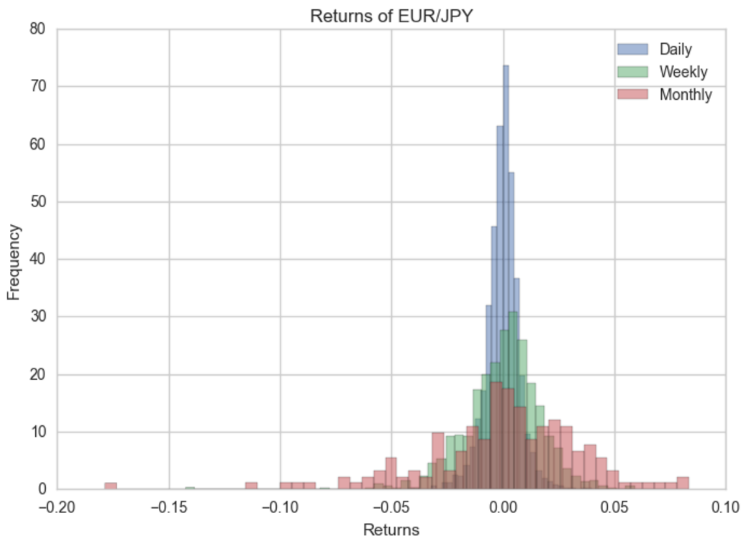

Figure 20: SPY Returns Across Time Frames

Appendix B

Skewness test results for the randomly selected stock FATE:

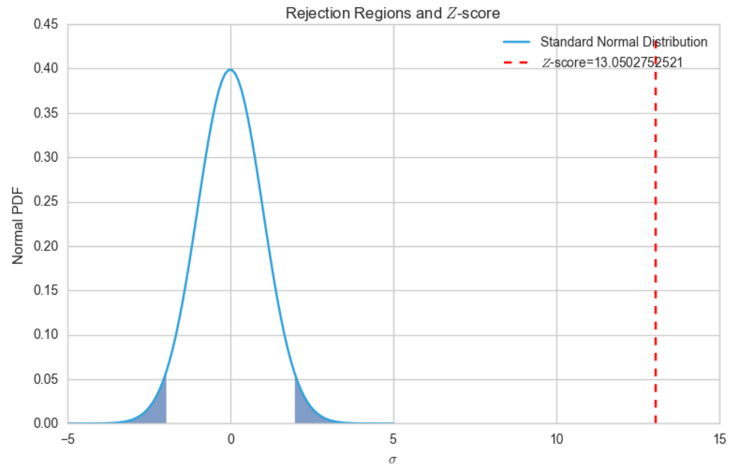

Figure 21: FATE Skewness Test Results

Jarque-Bera test results for SPY daily returns:

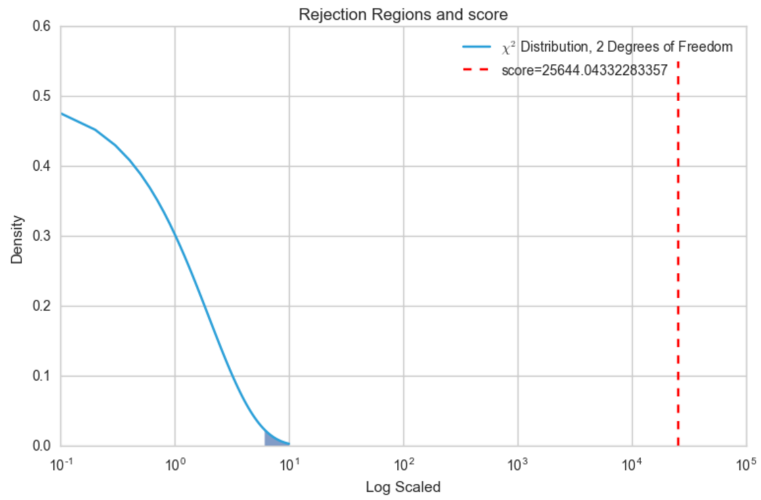

Figure 22: SPY Jarque-Bera Test Results

Appendix C

We expect to see a straight line on the Q-Q plot if a sample was taken from a normally distributed random variable. The following Q-Q plot uses synthetic data sample from a normal distribution:

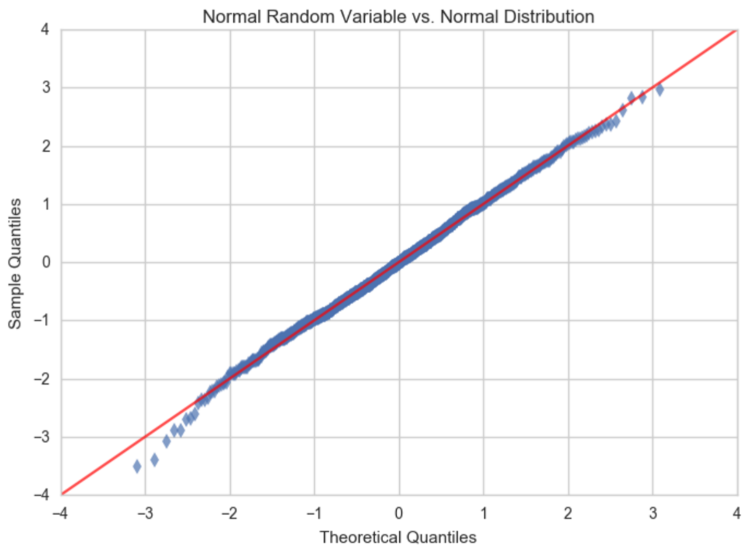

Figure 23: Normally Distributed RV vs. Normal Distribution

ETF returns Q-Q plots for various time frames:

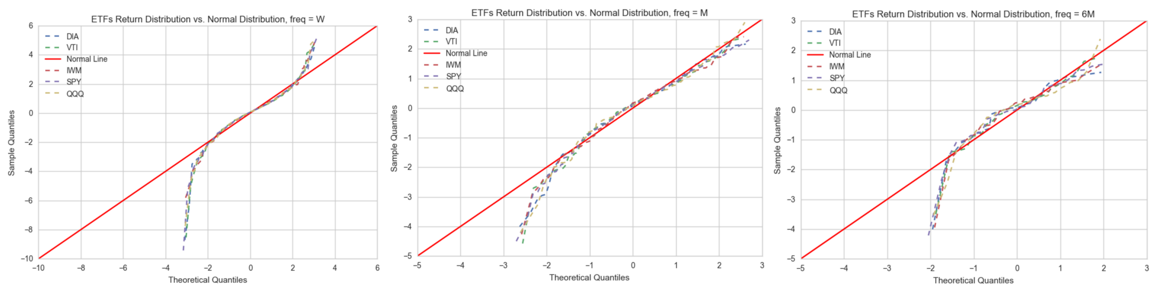

Figure 24: ETF Weekly, Monthly, and Semi-Annual Returns

Currency returns Q-Q plots for various time frames:

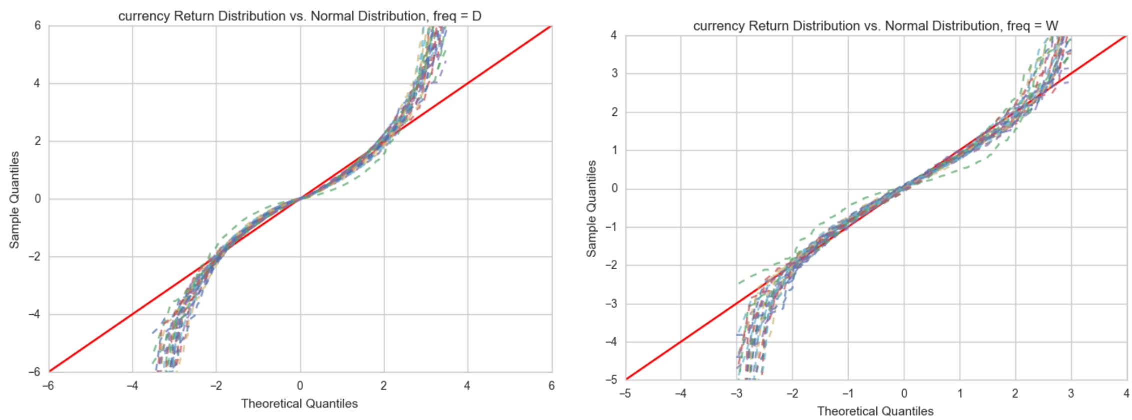

Figure 25: Currency Daily and Weekly Returns

Comparing Fit across time frames on the Q-Q plot:

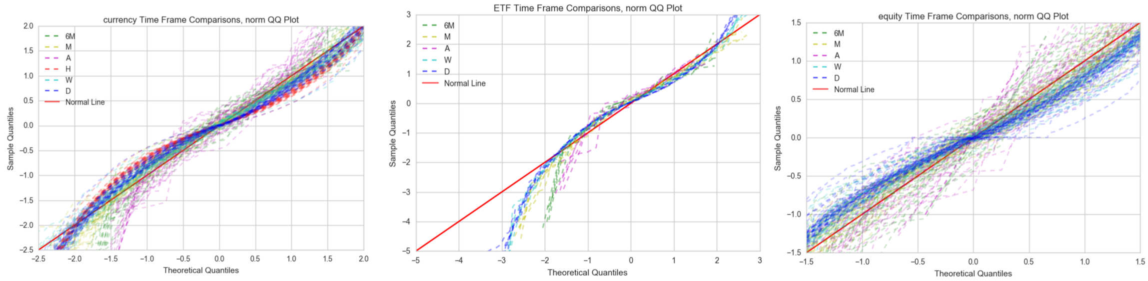

Figure 26: Q-Q Plot Time Frame Comparison; Currencies, ETFs, Equities

Plots of the normal, Laplace, Cauchy, and T distributions against ETF and Equity returns to compare their relative fit:

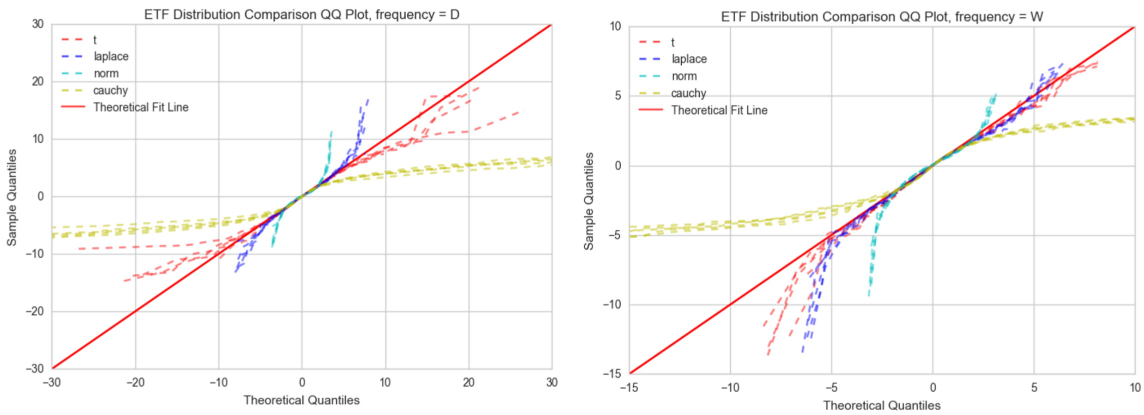

Figure 27: ETF Multi-Distribution Fit, Daily and Weekly Returns

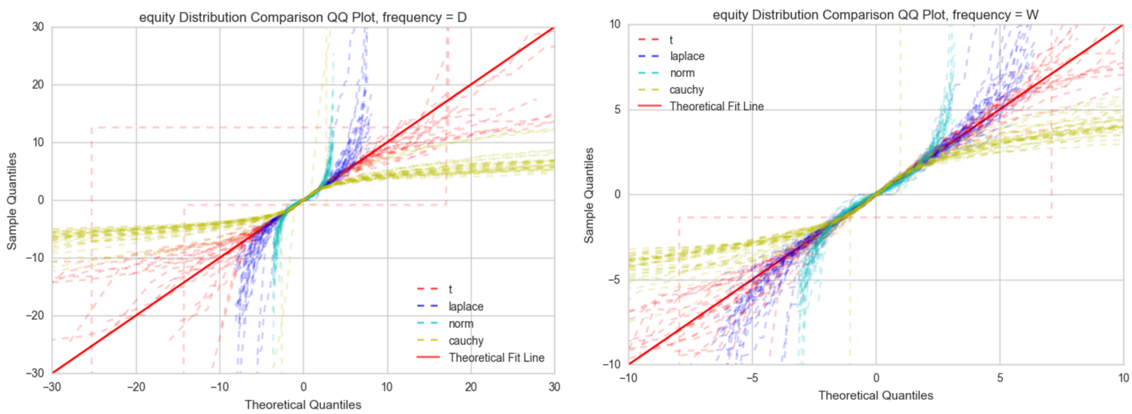

Figure 28: Equity Multi-Distribution Fit, Daily and Weekly Returns

Appendix D

The following plots show skewness and kurtosis of currency market returns as the time frame for calculation increases:

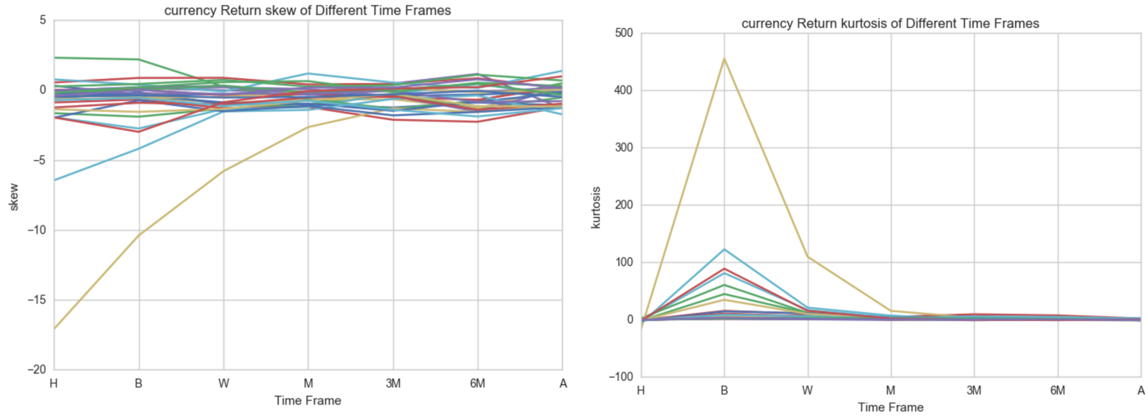

Figure 29: Currency Skewness and Kurtosis Across Time Frames

Appendix E

T-distributions with varying degrees of freedom compared to a normal distribution:

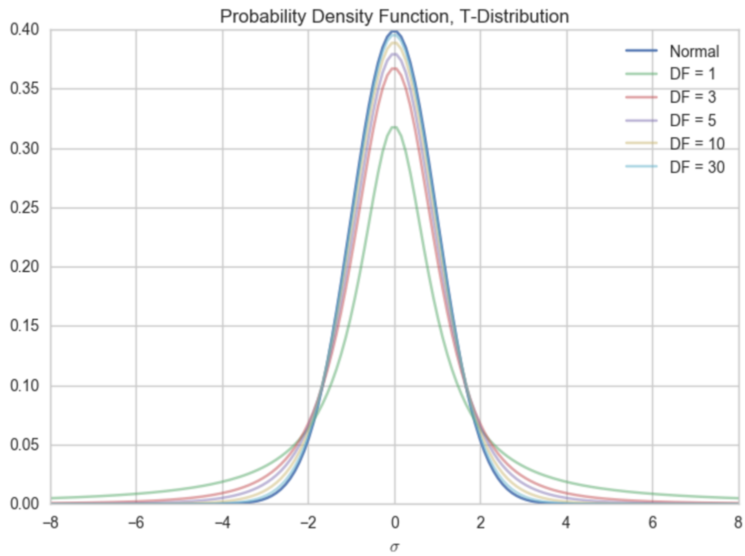

Figure 30: T-Distributions

Appendix F

T-distributions with varying degrees of freedom compared to Equity and ETF returns:

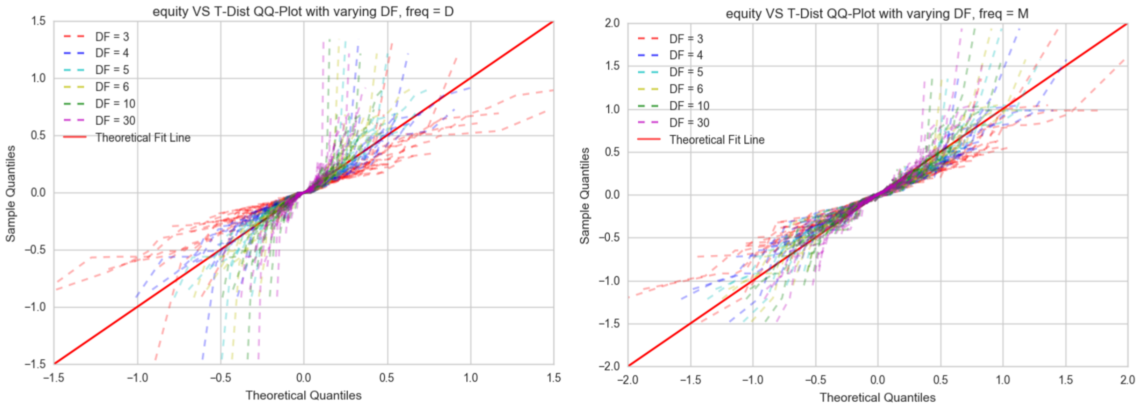

Figure 31: Hourly and Monthly Equity Returns against T-Distributions

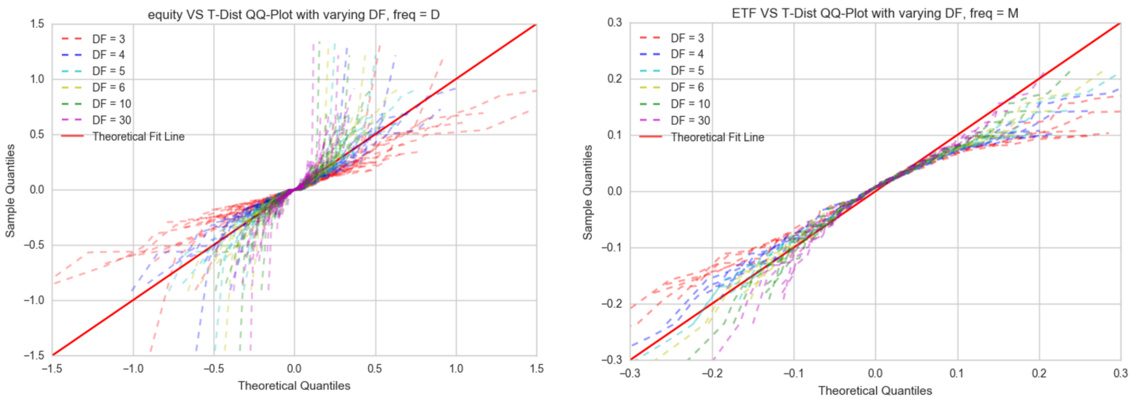

Figure 32: Hourly and Monthly ETF Returns against T-Distributions

Appendix G

Autocorrelation of Equity returns on various time frames:

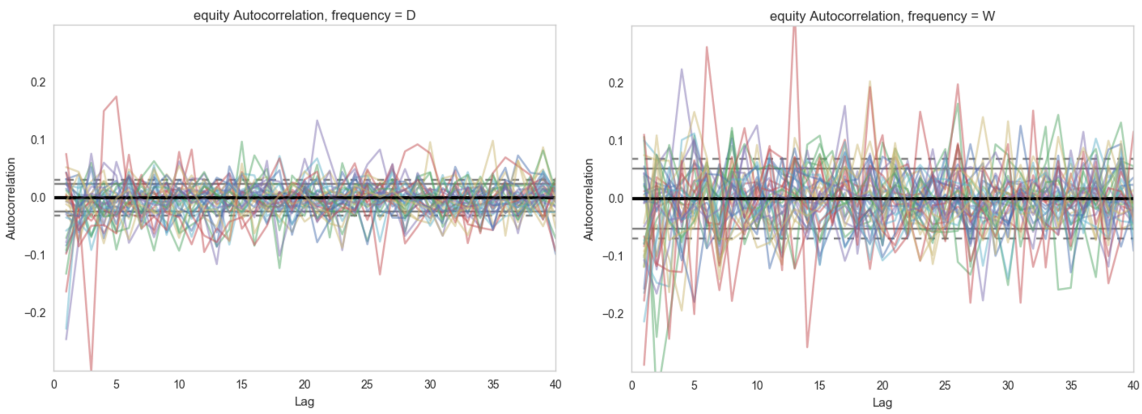

Figure 33: Daily and Weekly Autocorrelation for Equity Returns

Appendix H

All calculations were performed using python and its statistical packages in Jupyter notebooks which can be viewed on GitHub at: https://github.com/kfpotts1/Potts-Portfolio/tree/master/data_science/Market_Normality_Notebooks. These Notebooks were also used to produce the corresponding charts and visualizations from the data.