Summary of Important Formulas

1 Solving $ay'' + by' + cy = 0$

Work with the characteristic equation: $ar^2 + br + c = 0$

Roots: $r_1, r_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Case I: $b^2 - 4ac > 0$ (overdamped spring-mass system)

Distinct real roots $r_1 \neq r_2$

General solution: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case II: $b^2 - 4ac = 0$ (critically damped spring-mass system)

Repeated real roots $r_1 = r_2 = -b/2a$

General solution: $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

Case III: $b^2 - 4ac < 0$ (underdamped spring-mass system)

Complex conjugate roots

$$r_1, r_2 = \frac{-b}{2a} \pm i \frac{\sqrt{4ac - b^2}}{2a} = \alpha \pm i\beta$$

General solution: $y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

Special Cases:

(i) $y'' + k^2 y = 0$: $\quad y = c_1 \cos(kx) + c_2 \sin(kx)$

(ii) $y'' - k^2 y = 0$: $\quad y = c_1 e^{kx} + c_2 e^{-kx}$

Special case (i) also yields the solution for a *free undamped* spring mass system:

$$\ddot{x} + \omega^2 x = 0 \text{ has general solution } x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

2 Gradients and Applications

- **Gradient** of a function $z = f(x, y)$:

$$\nabla f(x, y) = f_x\,\mathbf{i} + f_y\,\mathbf{j} = \langle f_x, f_y \rangle$$

- **Directional derivative** of a function $z = f(x, y)$ in the direction of a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$:

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = f_x\, u_1 + f_y\, u_2$$

- **directions of steepest ascent/descent:**

$$\pm\frac{\nabla f(x,y)}{|\nabla f(x,y)|}$$

- **Rate of steepest ascent/descent:**

$$\pm|\nabla f(x,y)|$$

- If (x_0, y_0, z_0) is a point on the level surface $F(x,y,z) = C$ and $\langle a, b, c\rangle = \nabla F(x_0, y_0, z_0)$ is the gradient at this point, then
  - the **tangent plane** is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

  - the **normal line** has parametric equations

$$x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$$

- **Tangent plane/normal line to a graph** $z = f(x,y)$:
  Take
$$F(x,y,z) = f(x,y) - z = 0$$
  and then use the same method as above to find tangent plane and normal line.

3 Line Integrals and Work

- **Parameterization** of a 2D curve C:

$$x = f(t),\ y = g(t),\ a \le t \le b$$

  Alternatively the curve can be described by a **vector function**:

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j},\ a \le t \le b$$

- Evaluation of **line integrals** (using parameterization):

$$\begin{aligned}
\int_C G(x,y)\,dx &= \int_a^b G(f(t), g(t))\,f'(t)\,dt \\
\int_C G(x,y)\,dy &= \int_a^b G(f(t), g(t))\,g'(t)\,dt \\
\int_C G(x,y)\,ds &= \int_a^b G(f(t), g(t))\,\sqrt{(f'(t))^2 + (g'(t))^2}\,dt
\end{aligned}$$

- **Work** in a 2D force field $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ along a curve C:

$$
\begin{aligned}
W &= \int_C P(x,y)\,dx + Q(x,y)\,dy \\
&= \int_a^b P(f(t),g(t))f'(t)dt + Q(f(t),g(t))g'(t)dt
\end{aligned}
$$

- A force field $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is called **conservative** if it has a **potential** $\phi(x,y)$, meaning that

$$\mathbf{F}(x,y) = \nabla\phi(x,y) \quad (\text{or } P(x,y) = \phi_x,\ Q(x,y) = \phi_y)$$

- **Fundamental Theorem for Line Integrals:** If $\mathbf{F}(x,y)$ is conservative with potential $\phi(x,y)$ and the curve C (parameterized as $x = f(t)$, $y = g(t)$, $a \le t \le b$) has starting point $A = (f(a), g(a))$ and endpoint $B = (f(b), g(b))$, then

$$W = \phi(B) - \phi(A) = \phi(f(b), g(b)) - \phi(f(a), g(a))$$

- **Test for conservative forces:** A force $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad (\text{Short: } P_y = Q_x)$$

- **Finding a potential** for a conservative force $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$: (i) Find the x-antiderivative of $P(x,y)$ and add an arbitrary function $f(y)$. (ii) Find the y-antiderivative of $Q(x,y)$ and add an arbitrary function $g(x)$. (iii) Choose $f(y)$ and $g(x)$ so that the two previous steps both give the same result. This result is the potential $\phi(x,y)$.

4 Double Integrals and Applications

- **Double Integrals**
  - **Type I** region $R = \{(x,y) |\, a \le x \le b, g_1(x) \le y \le g_2(x)\}$:

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

  - **Type II** region $R = \{(x,y) |\, c \le y \le d, h_1(y) \le x \le h_2(y)\}$:

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

- **Center of mass** of a lamina of shape R and density $\rho(x,y)$:

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

Here:

$$m = \iint_R \rho(x,y)\,dA \quad (\textbf{mass})$$

$$M_y = \iint_R x\rho(x,y)\,dA, \quad M_x = \iint y\rho(x,y)\,dA \quad (\textbf{first moments})$$

- **Moments of inertia:**

$$I_x = \iint_R y^2\rho(x,y)\,dA \quad I_y = \iint_R x^2\rho(x,y)\,dA$$

- **Double integral in polar coordinates:** If

$$R = \{(r,\theta)|\, r_1 \le r \le r_2, \alpha \le \theta \le \beta\},$$

then

$$\int_R f(x,y)\,dA = \int_\alpha^\beta \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

5 Green's Theorem

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$$