Mean Value & Max/Min

Wednesday, September 30, 2020 12:34 AM

- **Mean Value Theorem**

  Suppose $f(x)$ is a function that satisfies both of the following.

  1. $f(x)$ is continuous on the closed interval $[a, b]$.
  2. $f(x)$ is differentiable on the open interval (a, b).

  Then there is a number c such that $a < c < b$ and

  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Simple definition: If a and b (variables) exist on a graph with either defined points or undefined points, then c should exist too :)

- **Steps**
  1) Take the derivative of x
  2) Using equation above, solve for f'(c) by plugging in given (a,b) values
  3) Set f'(x) and f'(c) equal to each other
  4) Use the **quadratic formula** to find c
  5) From the two c answers you obtain, choose the one within the interval

- **Example**

  Determine all the numbers of c that will satisfy the mean value theorem of the following function.

  $$f(x) = x^3 + 2x^2 - x \quad \text{on} \quad \overset{a\ b}{[-1, 2]}$$

  1) Find f'(x)

  $$f'(x) = 3x^2 + 4x - 1$$

  2) Find f'(c)

  $$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

  3) f'(x) = f'(c)

  $$\underbrace{3c^2 + 4c - 1}_{f'(x)} = \cancel{\frac{14}{3} - \frac{2}{3}} = \cancel{\frac{12}{3}} = \underbrace{4}_{f'(c)}$$

  $$3c^2 + 4c - 1 = 4$$
  $$\overset{a}{3c^2} + \overset{b}{4c} - \overset{c}{5} = 0$$

  4) Quadratic Formula

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  $$c = \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

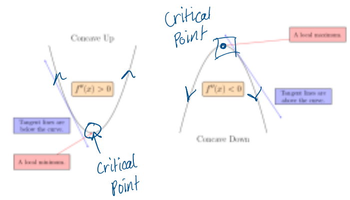
  $$c = \frac{-4 + \sqrt{76}}{6} = 0.7863 \qquad c = \frac{-4 - \sqrt{76}}{6} = -2.1196$$

  5) Find C
  - The two answer choices we are get are 0.7863 and -2.1196
  - Since -2.1196 is out of our interval [-1,2], the answer is 0.7863

  $$c = 0.7863$$

- **Maximum and Minimum Values**
  - The turning points on a graph
  - Can be classified as local (min or max) or absolute (min or max)
  - Turns make concave shapes meaning that the graph turns inward

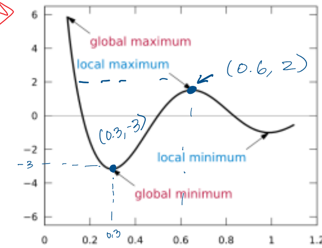
- **Concavity**
  - Can be either upward (increasing) or downward (decreasing)



- **Local/ Absolute Min and Max**
  - The x value at which the function has a max or min is called a critical value

- **Practice Problems**

  Example: Find the absolute maxima and minima for the following function



$$y = 5x^3 + 2x^2 - 3x$$

1) $y'(x)$

$$y' = \overset{a}{15x^2} + \overset{b}{4x} - \overset{c}{3}$$

2) Quadratic Formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(4) \pm \sqrt{4^2 - 4(15)(-3)}}{2(15)}$$

$$\boxed{\begin{aligned} x &= -3/5 \\ x &= 1/3 \end{aligned}}$$

3) y''

$$y = 5x^3 + 2x^2 - 3x$$
$$y' = 15x^2 + 4x - 3$$
$$y'' = 30x + 4$$

4) Plug in both x values into y''

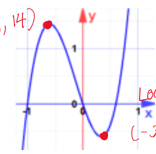
$$y'' = 30(-3/5) + 4 = -14$$
$$y'' = 30(1/3) + 4 = 14$$

Do steps 3-4 only if (x, y) are not given in problem

5) Plug in x values given and found $(-3/5, -14)$ $(1/3, 14)$

Since the answer (-14) is $< 0 \rightarrow$ local min

Since the answer (14) is $> 0 \rightarrow$ local max



- **Practice Problem**

  Determine all the numbers of c that will satisfy the mean value theorem of the following function.

  $$g(t) = 2t - t^2 - t^3 \text{ on } [-2, 1].$$

$$g'(t) = 2 - 2t - 3t^2$$

$$g'(c) = \frac{g(b) - g(a)}{b - a} = \frac{g(1) - g(-2)}{1 - (-2)}$$

$$= \frac{0 + 0}{3} = 0$$

$$g'(t) = g'(c) \Rightarrow 2 - 2t - 3t^2 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{aligned} c &= -1.21 \\ c &= 0.54 \end{aligned}$$

$[-2, 1]$ $c = -1.21$ ✓ $c = 0.54$ ✓

Horizontal Asymptote

$$\lim_{x \to \infty} \frac{x^2 + 2}{x^2 - 1} \qquad \frac{\# 4^a}{\# 6^b} \quad a = b$$

$$\frac{x^2 + 2}{x^2 - 1} = \frac{x^2}{x^2} = \frac{\# x^2}{\# x^2} = 1$$

Notes:

top deg < bottom deg → answer is $y = 0$

top deg = bottom deg → $y = \frac{n}{m}$ (n and m are coefficients)

top deg > bottom deg → No horiz. asymp.

Vertical Asymptote

$$f(x) = \frac{x + 1}{x^2 - 1} \leftarrow b(x)$$

$$b(x) = 0$$
$$x^2 - 1 = 0$$

Notes:

$$f(x) = \frac{a(x)}{b(x)}$$

$b(x) \neq 0$

$b(x) = 0$ ← Find the zeros

Faster

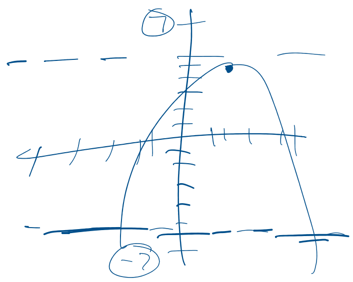
$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$\sqrt{x^2} = \sqrt{1}$$
$$x = \pm 1$$

Easier

$$x^2 - 1 = 0$$
$$(x - 1)(x + 1)$$
$$x - 1 = 0 \qquad x + 1 = 0$$
$$x = 1 \qquad x = -1$$

$$x = 1 \text{ and } -1$$

Graphing Example



Known:

vasym $= \pm 7$

hasym = none

$f(x) = -x^2 + 1$

Quadratic

V or ∩

Increase by 1

Interval

$[-2, 1]$

$f(x) = (-2)^2 + 1$

$= 4 + 1$

$y = 5$

$x = -2$ → $(-2, 5)$