Exam I Review

Tuesday, September 15, 2020 9:33 AM

Questions

Question 1

Use the **definition** of the derivative to find the derivative of $f(x) = \sqrt{x}$.

Question 2

Find the derivative of $f(x) = \sqrt{x}\cos(x)$

Question 3

Find the derivative of $f(x) = \frac{x^2+x+1}{x^2+1}$.

Question 4

Find the derivative of $f(x) = \sqrt[4]{x}(x^2+x-1)$.

Question 5

Find the equation of the tangent line to the graph of $y = f(x) = \tan(x) + x$ at the point $a = \pi/4$.

Question 6

Evaluate $\lim_{x\to\infty} \frac{\sin(x)+\cos(x)}{x^2}$

Question 7

Find $\lim_{x\to 0} \frac{\cos(x)}{x^2}$; in doing so allow for any limits, including infinite.

Question 8

Evaluate $\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$

Problem 1 (12 points)

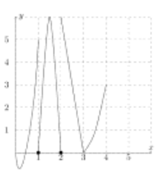
Suppose that $S(t) = \sin^2(t) + t$ is the position of a particle at time t (in seconds) on a line. Find:
(a) the velocity at time t
(b) the displacement from $t = 0$ to $t = \pi/2$
(c) the displacement from $t = \pi/2$ to $t = \pi$
(d) the displacement from $t = 0$ to $t = \pi$.
Recall that the displacement could be positive or negative depending on the direction of movement.

Problem 2 (10 points)

Given the graph of the function f below, answer the following questions.

1. What are the points of discontinuity of f?
2. What are the points at which f' does not exist?
3. What is the value of $f'(2.5)$?

A black bullet indicates that the corresponding point on the graph does NOT reflect the value of the function.



Problem 3 (10 points)

Find all points on the graph of $f(x) = x^3 + 6x^2 + 3x + 1$ where the tangent line is parallel to the line $y = -6x + \pi$.

Problem 4 (12 points)

Define a function

$$f(x) = \begin{cases} \sin(x), & x < \pi, \\ x - k, & x \geq \pi. \end{cases}$$

a) (9 points) Find a value of k so that $f(x)$ is a continuous function for all real numbers x.

b) (3 points) For the value of k you found, is $f(x)$ differentiable at $x = \pi$? Explain your conclusion.

Answers

Question 1

Use the **definition** of the derivative to find the derivative of $f(x) = \sqrt{x}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x-h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(\sqrt{x}+h) - \sqrt{x}}{h} \Rightarrow \lim_{h\to 0} \frac{\cancel{(\sqrt{x}+h)} \cancel{-\sqrt{x}}}{h} + \frac{-\cancel{(\sqrt{x}+h)} \cancel{+\sqrt{x}}}{(\sqrt{x}-h)+\sqrt{x}}$$

1) Formula (blue highlight)
2) Input $f(x)$
3) Reciprocal
4) Cancel like terms

$$\frac{-1}{\cancel{h}+(\sqrt{x}-\cancel{h})+\sqrt{x}} = \frac{-1}{\sqrt{x}+\sqrt{x}} = \frac{-1}{2\sqrt{x}}$$

Question 3

Find the derivative of $f(x) = \frac{x^2+x+1}{x^2+1}$.

Quotient Rule $\frac{(f'g)-(g'f)}{g^2}$

$$\frac{((2x+1)\,x^2+1) - (x)(x^2+x+1)}{(x^2+1)^2} = \frac{-x+1}{(x^2+1)^2}$$

Question 8

Evaluate $\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$

Using L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1}$$

1) Input x value
2) Rule that help get derivative (ex: L'Hospital Rule)
3) Input x

$$\frac{\sin(2\cdot 0)}{\sin(3\cdot 0)} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{\boxed{\sin(2x)}}{\boxed{\sin(3x)}} = \frac{\sin 2x}{2x} = \frac{\cos 2x}{2} + \frac{\sin(3x)}{3x} = \frac{3x}{\sin 3x} = \frac{3}{\cos 3x}$$

$$\frac{\cos 2x}{2} + \frac{3}{\cos 3x} = \frac{3\cos 2x}{2\cos 3x}$$

$$\lim_{x\to 0} \frac{3\cos 2x}{2\cos 3x} = \frac{3\cos(2\cdot 0)}{2\cos(3\cdot 0)}$$

$$= \frac{3\cos(0)}{2\cos(0)} = \frac{3}{2}$$

Problem 3 (10 points)

Find all points on the graph of $f(x) = x^3 + 6x^2 + 3x + 1$ where the tangent line is parallel to the line $y = -6x + \pi$. = Slope = $-6+\pi$

1) Take derivative

$f'(x) = 3x^2 + 12x + 3$ = tangent line

2) Set equal $f'(x)$ to parallel tangent line

$3x^2 + 12x + 3 = -6 + \pi$

$3x^2 + 12x + 3 = -3.14$

$3x^2 + 12x + 6.14 = 0$

3) Use quadratic formula

$x = -0.602$

$x = -3.39$

4) Input x values into tangent $F(x)$ to get y

$3(-0.602)^2 + 12(-0.602) + 3 = -3.136 = y$

$3(-3.39)^2 + 12(-3.39) + 3 = -3.204 = y$

$(-0.602, -3.136)$ $(-3.39, -3.204)$

Question 5

Find the equation of the tangent line to the graph of $y = f(x) = \tan(x) + x$ at the point $a = \pi/4$. $(\pi/4, 0) \leftarrow (a, b)$

$f(x) = \tan(x) + x$ ← given

$f'(x) = \sec^2(x) + 1$ ← tangent

$y - y_1 = m(x - x_1)$ ← point-intercept equation

0 ? $\pi/4$

$f'(x) = \sec^2(\pi/4) + 1 = m = 3$

$y - 0 = 3\left(x - \frac{\pi}{4}\right) \rightarrow y = 3x - \frac{3\pi}{4}$

Question 6

Evaluate $\lim_{x\to\infty} \frac{\sin(x) - \cos(x)}{x^2}$

$\lim_{x\to\infty} \frac{\sin(\infty) + \cos(\infty)}{\infty^2} = \frac{\infty}{\infty}$

L'Hôspital's Rule
- If $\lim = \infty, \frac{\infty}{\infty}$, or 0
- use rule by simplifying our limit

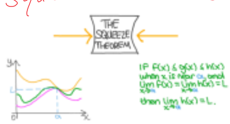
$\lim_{x\to\infty} \frac{\sin x + \cos x}{x^2} \rightarrow \lim_{x\to\infty} \frac{\sin x}{x^2} + \lim_{x\to\infty} \frac{\cos x}{x^2}$

$\lim_{x\to\infty} \frac{\sin x}{x^2} = 0$

$\lim_{x\to\infty} \frac{\cos x}{x^2} = 0$

$\rightarrow 0 + 0 = 0$

Because of the squeeze theorem



Problem 1 (12 points)

Suppose that $S(t) = \sin^2(t) + t$ is the position of a particle at time t (in seconds) on a line.
Find:
(a) the velocity at time t
(b) the displacement from $t = 0$ to $t = \pi/2$
(c) the displacement from $t = \pi/2$ to $t = \pi$
(d) the displacement from $t = 0$ to $t = \pi$.
Recall that the displacement could be positive or negative depending on the direction of movement.

$s''(x) = v'(x) = a(x)$

d'' of position — d' of velocity — acceleration

a) $s(t) = \sin^2(t) + t \longrightarrow v(x) = s'(x) = \cos^2(t) + 1$

b) Final − Initial

$\sin^2(\pi/2) + (\pi/2) - \sin^2(0) + 0 =$

$1.57 - 0 = 1.57$

c) $\sin^2(\pi) + \pi - \sin^2(\pi/2) + \pi/2$

$3.145 - 1.57 = 1.58$

d) $\sin^2(\pi) + \pi - \sin^2(0) + 0$

$3.145 - 0 = 3.145$

Question 4

Find the derivative of $f(x) = \sqrt[4]{x}(x^2 + x - 1)$. ($f(x)$ — $g(x)$)

Product Rule

$f(x)g(x) = f'(x)g + f(x)g'(x)$

$x^{1/4}(x^2 + x + 1)$

$x^{(1/4)-1} = \frac{1}{4} x^{-\frac{3}{4}} \rightarrow \sqrt[4]{x^{-3}} \rightarrow \frac{1}{\sqrt[4]{x^3}}$

$= \frac{1}{4}\left(\frac{1}{\sqrt[4]{x^3}}\right) = \frac{1}{4\sqrt[4]{x^3}} = f'(x)$

$g(x) = x^2 + x + 1$

$g'(x) = 2x + 1$

$f'(x)g(x) + f(x)g'(x)$

$\frac{1}{4\sqrt[4]{x^3}}(x^2 + x + 1) + \sqrt[4]{x}(2x + 1)$