Modelling the motion of a shock absorber

Figure 1: - Few elements of a vehicle's suspension system[1]

[1] "Suspension - Steering System | Advanced Mechanical Services". *Advanced Mechanical Services Coffs Harbour*, https://amscoffs.com.au/services/suspension/. Accessed 5 Jan

Aim: Deriving a mathematical expression for the nature of the damped motion of a shock absorber.

Introduction:
During a drive to the lake, I noticed that the last stretch of the route consisted of an unmetalled road. It was a dirt road with a number of potholes. Intuitively, it made sense that my father would drive slowly over the path. However, he drove over the path without slowing down, and we were able to cross the path without the car jumping too high. In this scenario, I had disregarded the existence of shock absorbers. However, I realized that if a bicycle (with no shock absorbers) had traversed over this path, it would have bounced very high and harmed the cyclist. Although I understood the consequence of the presence and absence of shock absorbers, I did not know much about them. This made me interested to learn more about them and I decided to explore them through a mathematical lens.

Rationale:
Through this investigation, I will derive a mathematical expression which appropriately models the nature of the damped motion of shock absorbers. To carry out this investigation, primary data (values of maximum displacement) obtained from experimentation, whose methodology is outlined ahead, will be used. Furthermore, concepts of logarithms, exponential functions, equation of a straight line, second order homogeneous linear differential equations, and the general form of the cosine function will be employed.

Investigating shock absorbers mathematically, in the form of a function which represents its dampened motion, would be useful for mechanics as it would allow them to understand the time taken for the motion of a bouncing car (due to travelling over an abnormality on the road) or any other object to dampen. This function would allow mechanics to determine the appropriate factors of a shock absorber which could be manipulated to make dampening efficient.

Background Information:
Shock absorbers, springs, tires, and linkages which appropriately connect wheels to a car, allowing relative motion between the vehicle and wheels, constitute the suspension system of a car. As perfectly flat roads do not exist, the job of a car suspension is to maximize the friction between the tires and road surface, to provide steering stability with good handling, and to ensure the comfort of the passengers.[2] In a suspension system, shock absorbers, as seen in *figure 2*, reduce the bumpiness felt by a car traversing on a rough road. This is achieved by dampening the oscillation of the car (up-down motion of the car) and hence lending shock absorbers the name dampers. These dampers damp motion with the help of viscous fluids such as mineral oil, silicone fluids, etc. The viscous fluid, present in shock absorbers, provides resistance to the oscillations generated when a vehicle passes over an abnormality on the road, ultimately damping the up-down motion of the car. Therefore, higher the viscosity (measure of resistance provided by a fluid) of the viscous fluid, more time will be taken by the piston to traverse the fluid region, and higher will be the damping of the motion.

[2] Harris, William. "How Car Suspensions Work". *Howstuffworks*, 2005, https://auto.howstuffworks.com/car-suspension.htm. Accessed 9 Jan 2020.

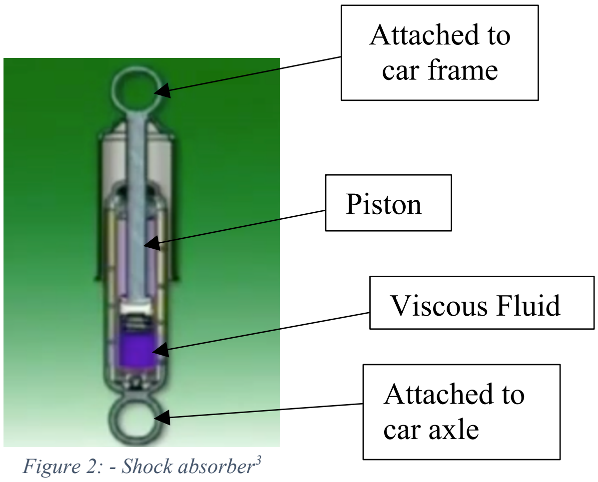

Figure 2: - Shock absorber[3]

Methodology:
To fulfil the aim of the investigation, an experiment was designed to emulate the functioning of a shock absorber. A spring was oriented vertically with one of its ends being fixed, while a block (mass) hung from the other end. This setup was immersed in viscous liquid kept in a beaker. The mass was then dislodged from equilibrium (position of rest), and its maximum displacement was recorded using the Tracker Software. In terms of a shock absorber, the vertical spring can be considered a shock absorber, the fixed end of the spring can be considered to be connected to the car frame, the other end of the spring can be considered to be connected to the car axle, and the viscous liquid can be considered as the viscous fluid of the damper.

Using the experimentally obtained readings, concepts of exponential functions, logarithms, and equation of a straight line will be used to obtain a straight-line relation (linearize) between maximum displacement and time. This relation would be used to determine the decay constant of the exponential relation between the experimentally collected values. Next, using the forces exerted on the object attached to the spring, an equation for the net force (total force), which is in the form of a second order homogeneous linear differential equation is obtained. Finally, on the basis of the nature of the solutions of this equation and using the general form of the cosine function, a mathematical expression for the motion of the shock absorber is obtained.

Data Collection:
Olive oil was used as the viscous fluid as it has properties similar to oils used in shock absorbers. The data collected for the maximum displacement (A) of the block over different times (t), which is represented as $A(t)$, is presented in a table below. The maximum displacement was recorded five times for every oscillation. That is, readings were taken every 0.35 s due to the period (T, time to complete one oscillation[4]) of the spring being 0.35 s.

[3] "Hydraulic Shock Absorbers.OTO-HUI.COM On Make A GIF". *Makeagif*, 2015, https://makeagif.com/gif/hydraulic-shock-absorbersoto-huicom-o9WfeK. Accessed 5 Jan 2020.

[4] "Period And Frequency In Oscillations". *Opentextbc.Ca*, https://opentextbc.ca/physicstestbook2/chapter/period-and-frequency-in-oscillations/. Accessed 6 Jan 2020.

Time t/s	Maximum displacement of block over time $A(t)/\text{cm}$				
	Trial 1	**Trial 2**	**Trial 3**	**Trial 4**	**Trial 5**
0.00	147.96	156.10	152.13	154.84	152.72
0.35	91.89	92.23	91.64	90.99	92.05
0.70	50.23	49.98	50.04	51.01	48.59
1.05	29.42	31.20	32.47	27.79	29.78
1.40	17.45	17.18	19.22	17.76	17.34
1.75	9.71	10.43	10.21	10.91	11.02
2.10	6.12	5.25	5.37	5.93	6.04

Table 1: - Raw data for maximum displacement in olive oil

Data Processing:
Since the values obtained from the Tracker Software are measured from a relative position, the values of maximum displacement collected in *table 1* are arbitrary values. Therefore, data processing involved calculating the average of the arbitrary maximum displacement values to reduce the effect of random error on the investigation and the conversion of these arbitrary average maximum displacement values to real maximum displacement values. Arbitrary average maximum displacement at 0 s has been calculated below

$$\frac{147.96 + 156.10 + 152.13 + 154.84 + 152.72}{5} = 152.75 \text{ cm}$$

To convert the arbitrary average values to real maximum displacement values, the maximum displacement of the block at 0 s was measured using a wooden scale. The obtained real value (10.20 cm) was then compared with the arbitrary average value at 0 s, resulting in a ratio of both values: real maximum displacement value to the arbitrary average value. Since ratios are simply a comparative relation between any two entities, this ratio is then used to calculate the real displacement by multiplying it with every arbitrary average value. Real maximum displacement at 0.35 s has been calculated below

$$\frac{Real\ maximum\ displacement}{Arbitrary\ average\ maximum\ displacement} = \frac{10.20}{152.75}$$

$$\frac{10.20}{152.75} = \frac{Real\ maximum\ displacement}{91.76}$$

(where 91.76 cm is the arbitrary average maximum displacement at 0.35 s)

$$Real\ maximum\ displacement\ value = S(t) = 6.12 \text{ cm} = \frac{6.12}{100}\ m = 0.061 \text{ m}$$

The table below shows the arbitrary average values of the maximum displacement and their corresponding real values.

Time t/s	**Arbitrary Average Maximum Displacement** $A(t)/\mathrm{cm}$	**Real Maximum Displacement** $S(t)/\mathrm{m}$
0.00	152.75	0.102
0.35	91.76	0.061
0.70	49.97	0.033
1.05	30.13	0.020
1.40	17.79	0.012
1.75	10.46	0.007
2.10	5.74	0.004

Table 2: - Processed data for maximum displacement in olive oil

<u>Determining the decay constant</u>:
Using the real maximum displacement values in *table 2*, a graph of $S(t)\ vs\ t$ is plotted

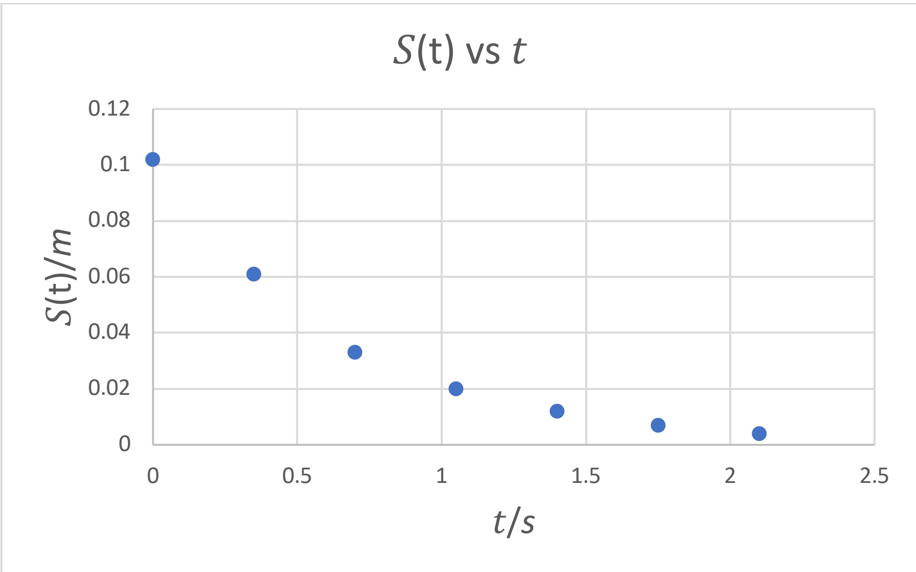

Figure 3: - Graph of real average maximum displacement vs time

Figure 2 shows that an exponential relationship exists between maximum displacement and time. That is, maximum oscillation of the block in olive oil decreases exponentially. Therefore, an exponential function is obtained for the maximum displacement

$$S(t) = S_0 e^{-\lambda t} \ldots\ldots\ldots\ldots\ldots \textit{Equation 1}$$

(where S_0 is the maximum displacement at 0 s and λ is the decay constant)

The value of S_0 and λ can be determined by taking the natural logarithm of both sides of *Equation 1*

$$\ln S(t) = \ln S_0 - \lambda t \ldots\ldots\ldots\ldots\ldots \textit{Equation 2}$$

Equation 2 can then be compared to the equation of a straight line

$$f(t) = mt + b$$
(where $f(t) = \ln S(t)$, $m = \lambda$, and $b = \ln S_0$)

The data obtained in *table 2* can be graphed using *Equation 2*. In the graph, the gradient of the best fit line would equal to λ and the y-intercept of the line of best fit would equal to $\ln S_0$.

Time, t/s	$S(t), m$	$\ln S(t)$
0.00	0.102	-2.283
0.35	0.061	-2.796
0.70	0.033	-3.411
1.05	0.020	-3.912
1.40	0.012	-4.442
1.75	0.007	-4.962
2.10	0.004	-5.521

Table 3: - Data for ln S(t) vs t

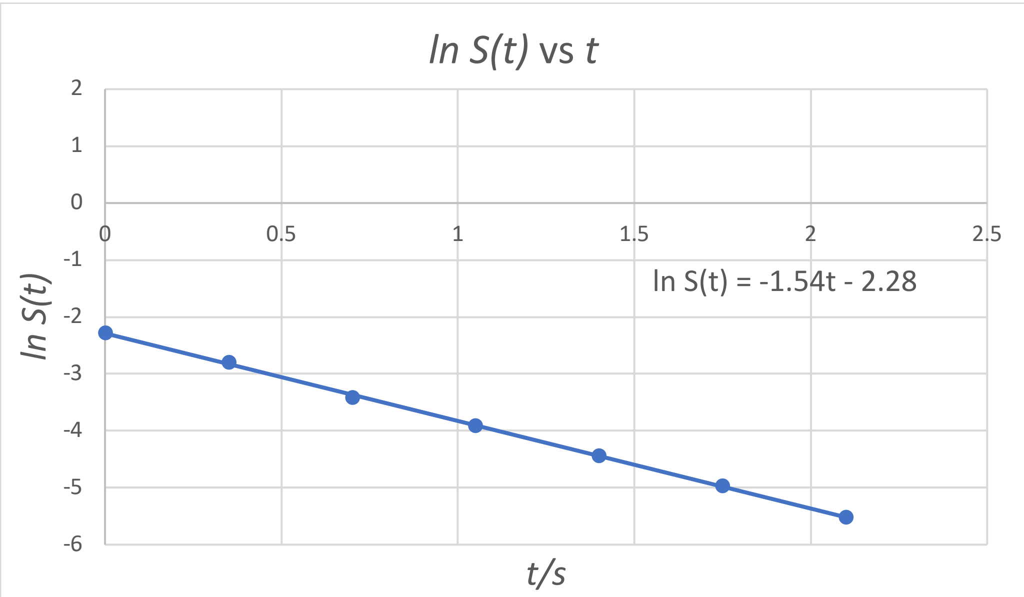

Figure 4: - Graph of ln S(t) vs t

From the equation of the graph displayed in *figure 4*,

$$m = -1.54 \text{ and b} = -2.28$$

Therefore,

$$\ln S_0 = -2.28$$

$$S_0 = e^{-2.28} = 0.10228 \text{ m} \approx 0.102 \text{ m}$$
And
$$\lambda = -1.54$$

By comparing the above values with *equation 1*, we obtain an equation for the maximum displacement of the block at a particular instance of time,

$$S(t) = 0.102e^{-1.54t} \ldots\ldots\ldots\ldots\ldots \textit{Equation 3}$$

Deriving an equation for shock absorber's motion:

Equation 3 indicates maximum displacement with relation to time, not displacement of the block with relation to time. Furthermore, *equation 3* indicates that the graph is a decaying exponential function which is not the oscillating function we set out to obtain. Therefore, we will now determine the equation of displacement with respect to time

$$x = S(t).\, g(t)$$

$$x = 0.102e^{-1.54t}.\, g(t) \ldots\ldots\ldots\ldots\ldots \textit{Equation 4}$$

(where x is the displacement of the block and $g(t)$ is the function we wish to determine)

Till now, it is clear that a sinusoidal function would appropriately model the motion of shock absorbers as dampers exhibit oscillatory motion (up-down motion). Furthermore, since viscous fluids play an important role in shock absorbers, it is clear that the amplitude (maximum displacement) would be continuously decreasing (dampening) due to the resistance provided by the viscous fluid. Hence, *equation 4* should represent a decaying sinusoidal function wherein $S(t)$ is the decaying exponential function and $g(t)$ is a periodic function. We will now go about determining $g(t)$.

Now, as shown in *figure 5*, there are numerous forces acting on the block attached to the spring.

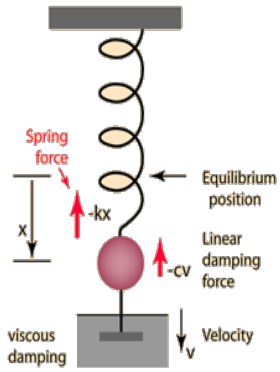

Figure 5: - Forces acting on the block attached to spring[5]

Therefore, the sum of all the forces acting on the block would provide us with the net force (total force) acting on it

$$F = spring\ force + linear\ damping\ force$$

(as seen from *figure 5*)

$$F = -kx - cv \ldots\ldots\ldots\ldots\ldots \textit{Equation 5}$$

(where F is the net force, k is the spring constant, c is a constant proportional to viscosity, and v represents velocity. Both the forces are negative as they are acting in directions opposite to the motion of the block.)

[5] Nave, R. "Damped Harmonic Oscillator". *Hyperphysics.Phy-Astr.Gsu.Edu*, http://hyperphysics.phy-astr.gsu.edu/hbase/oscda.html. Accessed 13 Jan 2020.

Newton's Second Law of Motion can be mathematically expressed as

$$F = ma\text{..............}\textit{Equation 6}$$

(where m represents the mass and a is the acceleration)

Net force on block is known from *equation 5* and therefore, equating *equation 5* and *6*

$$ma = -kx - cv$$
$$ma + cv + kx = 0\text{..............}\textit{Equation 7}$$

In *equation 7*, m, c, and k are constants and a and v can be represented as x:

1. v : Velocity is defined as the rate of change of displacement and the gradient of a $x\ vs\ t$ graph represents the displacement's rate of change

   $$\frac{\Delta x}{\Delta t}$$

   However, this does not provide the velocity at a particular instant of time. Since velocity is the gradient of $x\ vs\ t$, we can find the velocity at a particular instant of time by finding the derivative of $x\ vs\ t$ at that particular instance of time and this can be represented as

   $$x' = \frac{dx}{dt} = v$$

2. a: Similar to the case of velocity, acceleration can be represented as

   $$\frac{\Delta v}{\Delta t}$$

   However, just like with velocity, this form does not provide the acceleration at a particular instant of time. Therefore, acceleration at a particular instant of time would be the derivative of $v\ vs\ t$ at that particular instance of time

   $$v' = \frac{dv}{dt} = a$$

   However, as seen above, since $v = x'$, therefore, $v' = (x')' = x'' = a$

From the above, *equation 7* can be represented as

$$mx'' + bx' + kx = 0\text{..............}\textit{Equation 8}$$

Now, the general form of a second order homogeneous linear differential equation with constant coefficients is

$$Ax'' + Bx' + Cx = 0 \ldots\ldots\ldots\ldots\ldots \textit{Equation 9}$$

(where A, B, and C are constants)[6]

Therefore, comparison of *equation 8* and *9* shows that *equation 8* is actually a second order homogeneous linear differential equation.

For *equation 8*, $e^{-\lambda t}$ from *equation 1* can be assumed as a solution. This is because $e^{-\lambda t} \neq 0$ and $(e^{-\lambda t})' = e^{-\lambda t}$. Therefore, substituting $x = e^{-\lambda t}$ as a solution in *equation 8*

$$m(e^{-\lambda t})'' + c(e^{-\lambda t})' + k(e^{-\lambda t})$$

$$m(-\lambda)^2 e^{-\lambda t} - c\lambda e^{-\lambda t} + ke^{-\lambda t} = 0$$

$$e^{-\lambda t}(m\lambda^2 - c\lambda + k) = 0 \ldots\ldots\ldots\ldots\ldots \textit{Equation 10}$$

Since $e^{-\lambda t} \neq 0$, we can divide *equation 10* by $e^{-\lambda t}$ on both sides to obtain

$$m\lambda^2 - c\lambda + k = 0 \ldots\ldots\ldots\ldots\ldots \textit{Equation 11}$$

For *equation 11*, there are two possible roots: λ_1 and λ_2. These roots can be represented as

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

(as $m\lambda^2 - c\lambda + k$ is a quadratic equation)

The characteristics of the two roots dictate the form taken by the solution

Nature of roots (λ_1 and λ_2)	**Form of general solution**	**Resultant harmonic motion**
Distinct	$x = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_2 t}$	Overdamped motion
Same or repeated	$x = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$	Critically damped motion
Imaginary or complex ($\alpha + \beta i$)	$x = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$	Underdamped motion

Table 4: - General form of solution depending on nature of roots where C_1 and C_2 are constants[7]

In overdamped motion, the mass returns back to equilibrium after a long duration without performing oscillations. On the other hand, in critically damped motion, the mass returns back to equilibrium quickly without performing oscillations. Lastly, in underdamped motion, the maximum displacement decreases periodically with respect to time. As underdamped motion underlies our experiment, the solution to the equation should be of the following form

$$x = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$$

[6] S Tseng, Zachary. *Second Order Linear Differential Equations*. Penn State University, 2008, http://www.personal.psu.edu/sxt104/class/Math251/Notes-2nd%20order%20ODE%20pt1.pdf. Accessed 16 Jan 2020.

[7] Grayling, Michael. "Second Order Differential Equations". *Nrich.Maths.Org*, 2014, https://nrich.maths.org/11054. Accessed 17 Jan 2020.

$$x = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t)) \text{ ....} \textit{Equation 12}$$

Using the addition formula of cosine, *equation 12* could be further simplified

$$x = e^{\alpha t} R cos(\beta(t - t(0)))$$

$$x = e^{\alpha t} R cos(\beta t) \text{ ....} \textit{Equation 13}$$

However, at $t = 0$, $x = S_0$ and hence, $R = S_0$. Therefore, *equation 13* can be written as

$$x = e^{\alpha t} S_0 cos(\beta t) \text{ ....} \textit{Equation 14}$$

Upon comparison of *equation 4* and *14*

$$S(t) = S_0 e^{\alpha t} = S_0 e^{-\lambda t}$$

And

$$g(t) = \cos(\beta t)$$

Using the expression determined for $g(t)$, *equation 4* can now be written as

$$x = 0.102 e^{-1.54t}. \cos(\beta t)$$

Using the general form of a cosine function

$$\beta = \frac{2\pi}{T}$$

Since the period (T) of the spring is 0.35 s

$$\beta = \frac{2\pi}{0.35} = \frac{40\pi}{7}$$

Therefore, we obtain the equation for displacement of the block at a particular instance of time

$$x = 0.102 e^{-1.54t}. \cos\left(\frac{40\pi t}{7}\right) \text{ ....} \textit{Equation 15}$$

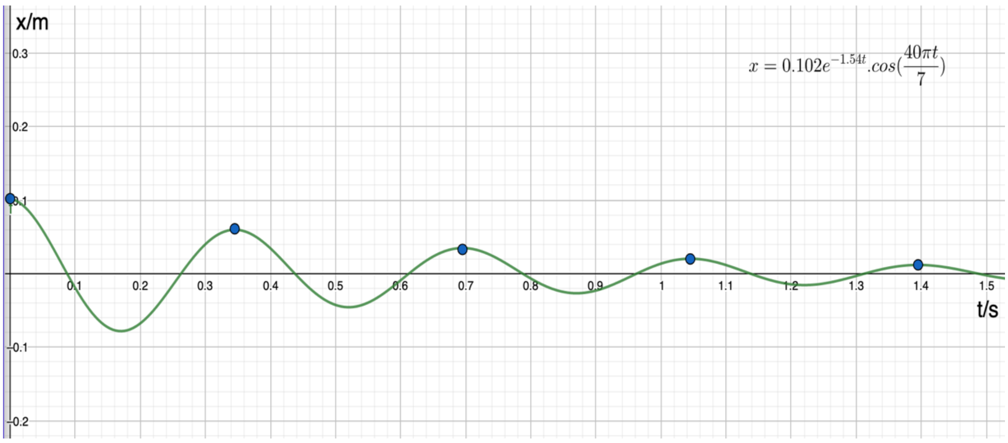

Figure 6: - Graph of equation 14 using GeoGebra

The graph of *equation 15* in *figure 6* models the characteristic damped oscillatory motion of a block attached to a spring which is forced to oscillate in a viscous fluid (olive oil). In other words, *equation 15* models the motion of a shock absorber. The accuracy of *equation 15* in modelling the motion of the block can be seen from its graph wherein we can see the points, which represent maximum displacements ($S(t)$ as seen in *table 2*), fitting in the curve to a large extent.

Analysis of the result:
The experiment was also repeated for glycerol and water as these liquids also resemble the properties of the viscous fluids used in a damper. The equations for the two are given as follows:

$$x = 0.121e^{-1.96t}\cos\left(\frac{40\pi t}{7}\right) \text{ ....} \textit{Equation 16}$$

equation for the damped oscillatory motion of a block in glycerol

$$x = 0.061e^{-0.89t}\cos\left(\frac{40\pi t}{7}\right) \text{ ....} \textit{Equation 17}$$

equation for the damped oscillatory motion of a block in water

Equation 15, *16*, and *17* show that the value of λ is different for the motion of the block in each fluid. This is because as each of these fluids have different values of viscosity, the net force acting on the block differs in each case due to the varying linear damping force (because of changing value of constant (c) in linear damping force (cv)). Therefore, by determining the appropriate fluid to be used in the shock absorber, mechanics can manipulate the value of λ (factor responsible for the damping strength) and ultimately control the damping strength of a shock absorber.

Although the value of λ (damping factor) differed for motion in each fluid, the period of the damped oscillations remain unchanged. This is because the investigation involves Simple Harmonic Motions (SHM) whose isochronous property states that the period of an oscillation is independent of its amplitude. This can be shown mathematically as follows

Displacement of an object at a particular instant of time exhibiting SHM is given as

$$x = Acos(\omega t + \varphi)$$ [8]

(where A represents amplitude, ω represents angular frequency ($\omega = \frac{2\pi}{f}$, where f represents frequency of oscillations), and φ represents phase shift which is defined as the horizontal shift of the function from its usual position[9])

Now, if at $t = 0$ s an object attains a displacement A then,

$$x = Acos(\omega t) \text{ ....} \textit{Equation 18}$$

(as at $t = 0$, $\varphi = 0$)

[8] Tsokos, K. A. *Physics For The IB Diploma*. 6th ed., Cambridge University Press, 2014, pp. 146-149. Accessed 5 Jan 2020.
[9] "Amplitude, Period, Phase Shift And Frequency". *Mathsisfun.Com*, https://www.mathsisfun.com/algebra/amplitude-period-frequency-phase-shift.html. Accessed 6 Jan 2020.

Upon comparing *equation 18* with the equation derived for the motion of the block ($x = S_0 e^{-\lambda t} \cos(\beta t)$), we find that the two equation are of the same form, but *equation 18* lacks the expression $e^{-\lambda t}$. As studied above, the damping factor was not seen to have an effect on the period of the oscillation of the block, and therefore, the period of the oscillatory motion of a mass exhibiting SHM can be shown as

$$S_0 \cos(\beta t) = A cos(\omega t)$$

However, for an object undergoing SHM, the period of the oscillation is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Therefore, mass and the spring constant (elasticity of the spring) are the factors which affect the period and frequency (as $T = \frac{1}{f}$) of the oscillation of an object in SHM.

Conclusion:

Delving into the mathematics of the motion of a shock absorber has ultimately allowed to gain a thorough understanding of its functioning and applications. Through the investigation conducted above, the factors which affect the damping motion of a shock absorber have been identified.

It becomes clear that three factors affect the efficiency of a shock absorber: mass of the car (or object in focus), elasticity of the spring attached to damper (spring constant), and viscosity of the liquid used for the shock absorber. Mechanics ought to take these three factors into consideration to appropriately manipulate shock absorbers and meet their needs.

The damping factor can be manipulated by manipulating the viscosity of the liquid. Intuition would suggest that employing a highly viscous fluid would be an optimum choice for a vehicle. However, this is not viable as there occurs damping at both the compression and decompression, which would result in the wheels of the car hanging when it goes over a bump. Therefore, the viscosity of the fluid ought to be decided by mechanics in accordance with the damping factor.

The viscous fluid aids in damping vibrations and decreasing the amplitude of the oscialltions. Although the vibrations are damped, the vehicle still oscillates with continuously decreasing amplitudes. The frequency of these vibrations can be controlled (aiding in smoother rides for the vehicle after passing over a bump) by manipulating the remaining two factors: mass and the spring constant. From the formulas below

$$T = 2\pi\sqrt{\frac{m}{k}}$$

And

$$f = \frac{1}{T}$$

It is clear that upon increasing the mass (m) and decreasing the spring constant (k), value of the period (T) would increase, resulting in a decrease in the value of frequency (f). Due to this, a greater amount of time would be taken to complete a cycle. That is, the damped motion of the vehicle would slow down and the motion, ultimately, becomes smooth. However, there is a limit to how much the mechanics can increase the mass of the car and decrease the spring constant. This is because if the mass of vehicle is increased and the spring constant is decreased beyond a certain limit, the body of the car may collapse on the wheels, restricting its motion.

The factors outlined above, vary from vehicle to vehicle. For example, for a F1 car, ensuring a smooth ride is not the primary purpose. In such a vehicle, a high spring constant and damping factor of a shock absorber is required to damp the vibrations instantaneously as these cars travel at high speeds any deformation in the road can cause the car to topple and give way to an accident.

Limitations:

Given the outcome of this investigation heavily relies on the values obtained from experimentation, a number of systematic and random errors plague the accuracy of the obtained results.

Due to the paucity of time, only five trials were taken and the maximum displacement was measured up to only $2.10\ s$. The lack of trials conducted and data collected results in the effect of outliers on the outcome of the investigation being amplified. Moreover, the amplified effect of these outliers may also result in the prediction of an incorrect correlation between the two variables (displacement and time).

Furthermore, although, the damping of the oscillations by the viscous fluid was considered, the investigation did not take into consideration the damping effect on the spring created by the air resistance. Assuming that the spring is massless and follows Hooke's Law ($F = -kx$) throughout its range of motion also act as limitations of the investigations. The above-mentioned factors were excluded from the context of the investigation due to their minuscule impact on the accuracy of the investigation and to ensure that focus is not diverted from the investigation's aim by delving into unnecessary complications. To appropriately account for these assumptions in the investigation, the scope of the mathematics used would have to be increased. For example, if air resistance were to be included, an inhomogeneous linear second order differential equation of the form

$$y'' + py' + qy = f(x)$$

(where p and q can are constants but can both real and complex)

would be obtained.

Further scope of the investigation:

The scope of the investigation can be extended if the number of trials taken and data collected is increased. Furthermore, accounting for factors such as air resistance and their appropriate mathematical tools would also provide greater insight into the functioning of shock absorbers in realistic situations.

Lastly, the scope of the investigation can also be extended by investigating different types of shock absorbers. For example, Twin Tube type shock absorbers (contain two tubes) differ from Mono Tube type shock absorbers (contain one tube) in regards to the number of tubes responsible for the generating the damping forces. Therefore, the effect of the number of tubes on a shock absorber's damped motion can be explored.

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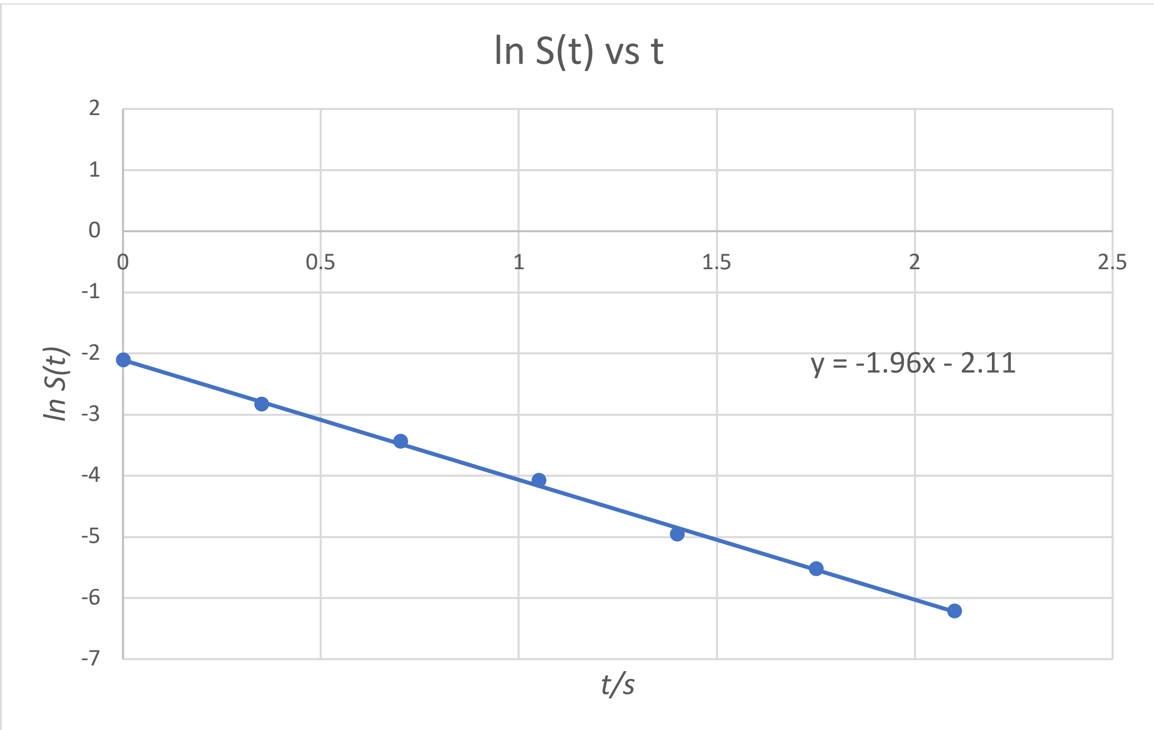
Tsokos, K. A. *Physics For The IB Diploma*. 6th ed., Cambridge University Press, 2014, pp. 146-149. Accessed 5 Jan 2020.

Appendix:
The experiment mentioned in the introduction was replicated for glycerol

Time, t/s	Maximum displacement of block over time, cm						Real Average of Maximum Displacement, $S(t)/m$
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Arbitrary Average	
0.00	171.23	169.76	172.01	171.89	170.97	171.17	0.121
0.35	84.01	81.86	82.21	82.52	82.10	82.54	0.059
0.70	45.22	45.89	45.56	44.99	45.19	45.37	0.032
1.05	24.56	24.02	23.48	23.97	21.77	23.56	0.017
1.40	10.23	9.78	9.25	9.96	9.68	9.78	0.007
1.75	6.02	5.89	6.28	5.98	5.68	5.97	0.004
2.10	3.03	2.87	3.23	3.01	2.76	2.98	0.002

Time, t/s	$S(t), m$	$\ln S(t)$
0.00	0.121	-2.112
0.35	0.059	-2.830
0.70	0.032	-3.442
1.05	0.017	-4.074
1.40	0.007	-4.962
1.75	0.004	-5.521
2.10	0.002	-6.215



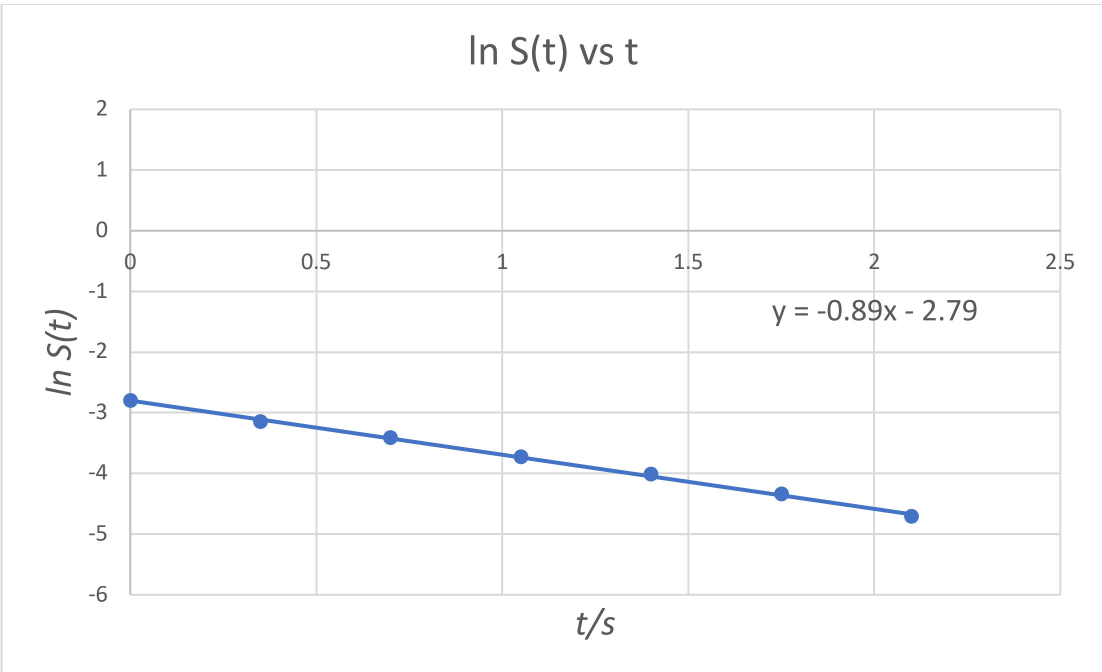
Equation for the damped oscillatory motion of a block in glycerol is given as

$$x = 0.121e^{-1.96t}\cos\left(\frac{40\pi t}{7}\right)$$

The experiment mentioned in the introduction was replicated for water

Time, t/s	**Maximum displacement of block over time, cm**						**Real Average of Maximum Displacement, $S(t)/m$**
	Trial 1	**Trial 2**	**Trial 3**	**Trial 4**	**Trial 5**	**Arbitrary Average**	
0.00	120.11	120.02	120.03	119.45	119.09	119.74	0.061
0.35	84.98	84.68	85.12	84.42	85.15	84.87	0.043
0.70	65.04	65.49	65.96	65.59	65.17	65.45	0.033
1.05	55.56	55.02	54.78	54.97	54.02	54.87	0.024
1.40	35.78	35.28	36.13	36.07	36.19	35.89	0.018
1.75	25.76	25.49	25.28	26.28	26.79	25.92	0.013
2.10	18.92	19.09	18.23	18.73	18.23	18.64	0.009

Time, t/s	**$S(t), m$**	**$\ln S(t)$**
0.00	0.061	-2.796
0.35	0.043	-3.145
0.70	0.033	-3.411
1.05	0.024	-3.730
1.40	0.018	-4.017
1.75	0.013	-4.343
2.10	0.009	-4.710



Equation for the damped oscillatory motion of a block in water is given as

$$x = 0.061e^{-0.89t}\cos\left(\frac{40\pi t}{7}\right)$$