

Lessons 24 + 25. The Simplex Method

1 Review

- Given an LP with n decision variables, a solution \mathbf{x} is **basic** if:
 - (a) it satisfies all equality constraints
 - (b) at least n linearly independent constraints are active at \mathbf{x}
- A **basic feasible solution (BFS)** is a basic solution that satisfies all constraints of the LP
- Canonical form LP:

$$\begin{array}{ll}\text{maximize} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}\end{array}$$

- m equality constraints and n decision variables (e.g. A has m rows and n columns).
 - Standard assumptions: $m \leq n$, $\text{rank}(A) = m$
- If \mathbf{x} is a basic solution of a canonical form LP, there exist m **basic variables** of \mathbf{x} such that
 - (a) the columns of A corresponding to these m variables are linearly independent
 - (b) the other $n - m$ **nonbasic variables** are equal to 0
- The set of basic variables is the **basis** of \mathbf{x}

2 Overview

- General improving search algorithm
 - 1 Find an initial feasible solution \mathbf{x}^0
 - 2 Set $t = 0$
 - 3 **while** \mathbf{x}^t is not locally optimal **do**
 - 4 Determine a simultaneously improving and feasible direction \mathbf{d} at \mathbf{x}^t
 - 5 Determine step size λ
 - 6 Compute new feasible solution $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda \mathbf{d}$
 - 7 Set $t = t + 1$
 - 8 **end while**
- The **simplex method** is a specialized version of improving search
 - For canonical form LPs
 - Starts at a BFS in Step 1
 - Considers directions that point towards other BFSes in Step 4
 - Takes the maximum possible step size in Step 5

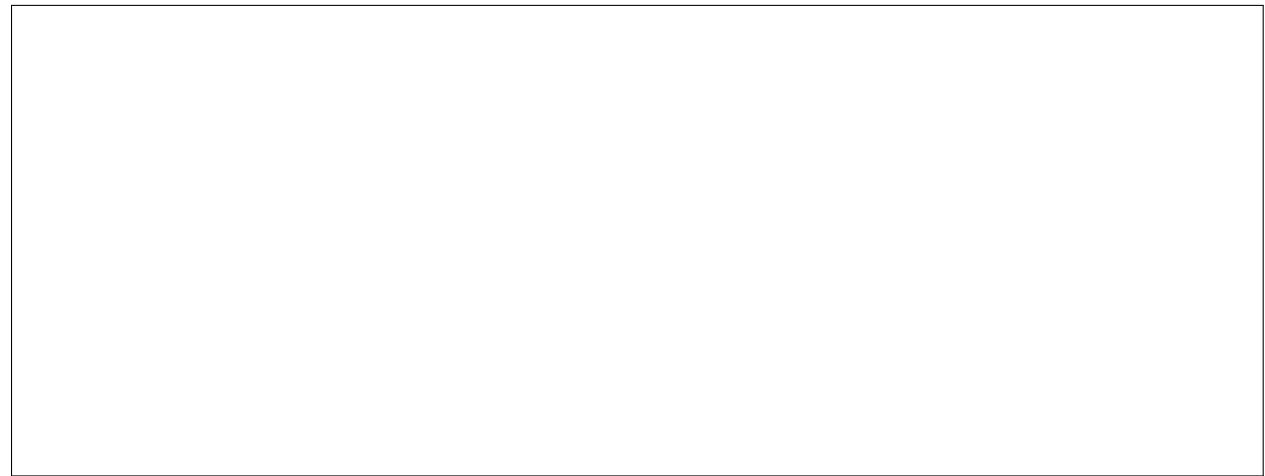
Example 1. Throughout this lesson, we will use the canonical form LP below:

$$\begin{array}{llllll}
 \text{maximize} & 13x + 5y & & & & \\
 \text{subject to} & 4x + y + s_1 & & & = & 24 \\
 & x + 3y & & + s_2 & = & 24 \\
 & 3x + 2y & & & + s_3 & = 23 \\
 & x, & y, & s_1, & s_2, & s_3 \geq 0
 \end{array}$$

3 Initial solutions

- For now, we will start by guessing an initial BFS

Example 2. Verify that $\mathbf{x}^0 = (0, 0, 24, 24, 23)$ is a BFS with basis $\mathcal{B}^0 = \{s_1, s_2, s_3\}$.



4 Finding feasible directions

- Two BFSes are **adjacent** if their bases differ by exactly 1 variable
- Suppose \mathbf{x}^t is the current BFS with basis \mathcal{B}^t
- Approach: consider directions that point towards BFSes adjacent to \mathbf{x}^t
- To get a BFS adjacent to \mathbf{x}^t :
 - Put one nonbasic variable into \mathcal{B}^t
 - Take one basic variable out of \mathcal{B}^t
- Suppose we want to put nonbasic variable y into \mathcal{B}^t
- This corresponds to the **simplex direction** \mathbf{d}^y corresponding to nonbasic variable y
- \mathbf{d}^y has a component for every decision variable
 - e.g. $\mathbf{d}^y = (d_x^y, d_y^y, d_{s_1}^y, d_{s_2}^y, d_{s_3}^y)$ for the LP in Example 1
- The components of the simplex direction \mathbf{d}^y corresponding to nonbasic variable y are:

- $d_y^y = 1$
- $d_z^y = 0$ for all other nonbasic variables z
- d_w^y (uniquely) determined by $A\mathbf{d} = \mathbf{0}$ for all basic variables w
- Why does this work? Remember for LPs, \mathbf{d} is a feasible direction at \mathbf{x} if

$$\mathbf{a}^\top \mathbf{d} \begin{cases} \leq \\ \geq \\ = \end{cases} 0 \quad \text{for each active constraint of the form} \quad \mathbf{a}^\top \mathbf{x} \begin{cases} \leq \\ \geq \\ = \end{cases} b$$

- Each nonbasic variable has a corresponding simplex direction

Example 3. The basis of the BFS $\mathbf{x}^0 = (0, 0, 24, 24, 23)$ is $\mathcal{B}^0 = \{s_1, s_2, s_3\}$. For each nonbasic variable, x and y , we have a corresponding simplex direction. Compute the simplex directions \mathbf{d}^x and \mathbf{d}^y .



5 Finding improving directions

- Once we've computed the simplex direction for each nonbasic variable, which one do we choose?
- We choose a simplex direction \mathbf{d} that is improving
- Recall that if $f(\mathbf{x})$ is the objective function, \mathbf{d} is an improving direction at \mathbf{x} if

$$\nabla f(\mathbf{x})^\top \mathbf{d} \begin{cases} > 0 & \text{when maximizing } f \\ < 0 & \text{when minimizing } f \end{cases}$$

- For LPs, $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$, and so $\nabla f(\mathbf{x}) = \mathbf{c}$ for any \mathbf{x}
- The **reduced cost** associated with nonbasic variable y is

$$\bar{c}_y = \mathbf{c}^\top \mathbf{d}^y$$

where \mathbf{d}^y is the simplex direction associated with y

- The simplex direction \mathbf{d}^y associated with nonbasic variable y is improving if

$$\bar{c}_y \begin{cases} > 0 & \text{for a maximization LP} \\ < 0 & \text{for a minimization LP} \end{cases}$$

Example 4. Consider the BFS $\mathbf{x}^0 = (0, 0, 24, 24, 23)$ with basis $\mathcal{B}^0 = \{s_1, s_2, s_3\}$. Compute the reduced costs \bar{c}_x and \bar{c}_y for nonbasic variables x and y , respectively. Are \mathbf{d}^x and \mathbf{d}^y improving?

- If there is an improving simplex direction, we choose it
- If there is more than 1 improving simplex direction, we can choose any one of them
 - One option – **Dantzig’s rule**: choose the improving simplex direction with the most improving reduced cost (maximization LP – most positive, minimization LP – most negative)
- **If there are no improving simplex directions, then the current BFS is a global optimal solution**

6 Determining the maximum step size

- We’ve picked an improving simplex direction – how far can we go in that direction?
- Suppose \mathbf{x}^t is our current BFS, \mathbf{d} is the improving simplex direction we chose
- Our next solution is $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda \mathbf{d}$ for some value of $\lambda \geq 0$
- How big can we make λ while still remaining feasible?
- Recall that we computed \mathbf{d} so that $A\mathbf{d} = \mathbf{0}$
- \mathbf{x}^{t+1} satisfies the equality constraints $A\mathbf{x} = \mathbf{b}$ no matter how large λ gets, since

$$A\mathbf{x}^{t+1} = A(\mathbf{x}^t + \lambda \mathbf{d}) = A\mathbf{x}^t + \lambda A\mathbf{d} = A\mathbf{x}^t = \mathbf{b}$$
- So, the only thing that can go wrong are the nonnegativity constraints
 - \Rightarrow What is the largest λ such that $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda \mathbf{d} \geq \mathbf{0}$?

Example 5. Suppose we choose the improving simplex direction $\mathbf{d}^x = (1, 0, -4, -1, 3)$. Compute the maximum step size λ for which $\mathbf{x}^1 = \mathbf{x}^0 + \lambda \mathbf{d}^x$ remains feasible.

- Note that only negative components of \mathbf{d} determine maximum step size:

$$(\text{nonnegative number}) + \lambda d \stackrel{?}{\geq} 0$$

- The **minimum ratio test**: starting at the BFS \mathbf{x} , if any component of the improving simplex direction \mathbf{d} is negative, then the maximum step size is

$$\lambda_{\max} = \min \left\{ \frac{x_j}{-d_j} : d_j < 0 \right\}$$

Example 6. Verify that the minimum ratio test yields the same maximum step size you found in Example 5.

- What if \mathbf{d} has no negative components?
- For example:
 - Suppose $\mathbf{x}^0 = (0, 0, 1, 2, 3)$ is a BFS
 - $\mathbf{d} = (1, 0, 2, 4, 3)$ is an improving simplex direction at \mathbf{x}
 - Then the next solution is

$$\mathbf{x}^1 = \mathbf{x}^0 + \lambda \mathbf{d} = (\lambda, 0, 1 + 2\lambda, 2 + 4\lambda, 3 + 3\lambda)$$

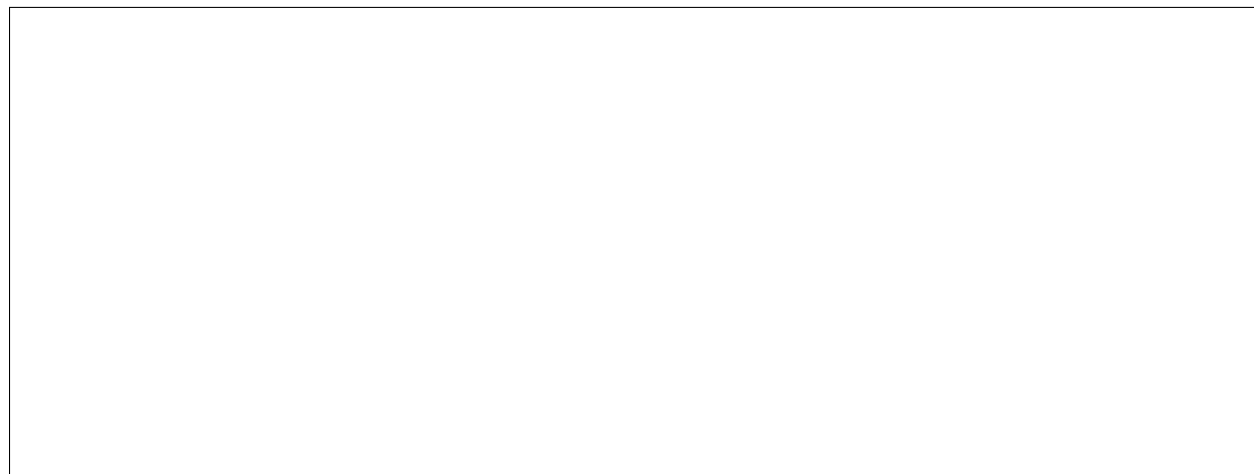
- $\mathbf{x}^1 \geq 0$ for all $\lambda \geq 0$!
 - We can improve our objective function and remain feasible forever!
- \Rightarrow The LP is unbounded

- **Test for unbounded LPs**: if all components of an improving simplex direction are nonnegative, then the LP is unbounded

7 Updating the basis

- We have our improving simplex direction \mathbf{d} and step size λ_{\max}
- We can compute our new solution $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda_{\max}\mathbf{d}$
- We also update the basis: update the set of basic variables

Example 7. Compute \mathbf{x}^1 . What is the basis \mathcal{B}^1 of \mathbf{x}^1 ?



- **Entering and leaving variables**

- The nonbasic variable corresponding to the chosen simplex direction enters the basis and becomes basic: this is the **entering variable**
- Any one of the basic variables that define the maximum step size leaves the basis and becomes nonbasic: this is the **leaving variable**

8 Putting it all together: the simplex method

Step 0: Initialization. Identify a BFS \mathbf{x}^0 . Set solution index $t = 0$.

Step 1: Simplex directions. For each nonbasic variable y , compute the corresponding simplex direction \mathbf{d}^y and its reduced cost \bar{c}_y .

Step 2: Check for optimality. If no simplex direction is improving, stop. The current solution \mathbf{x}^t is optimal. Otherwise, choose any improving simplex direction \mathbf{d} . Let x_e denote the entering variable.

Step 3: Step size. If $\mathbf{d} \geq \mathbf{0}$, stop. The LP is unbounded. Otherwise, choose the leaving variable x_ℓ by computing the maximum step size λ_{\max} according to the minimum ratio test.

Step 4: Update solution and basis. Compute the new solution $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda_{\max}\mathbf{d}$. Replace x_ℓ by x_e in the basis. Set $t = t + 1$. Go to Step 1.