Pricing Models

ARIMA Models

An ARIMA model is written at ARIMA(p, d, q), where:

- p: Number of Autoregressive (AR) terms
- d: Number of differences (how many times we subtract past values to make the data stationary)
- q: Number of moving average (MA) terms

After differencing d times, the ARIMA model is:

$$y_t' = c + \sum_{i=1}^{p} \phi_i y_{t-i}' + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \epsilon_t$$

- y_t - original time series
- d - Differncing order

  If a time series has a trend or isn't stationary, you difference it, and you keep on differencing it until its mean and variance stays stable.

  - If $d = 0$, $y_t' = y_t$
  - If $d = 1$, $y_t' = y_t - y_{t-1}$
  - If $d = 2$, $y_t' = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$
- c - Constant term, helps with differencing less times
- ϕ_i: AR coefficients (AutoRegressive Terms) - this captures momentum or trend following behavior
- $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ - random noise
- θ_j - MA coefficient

One-Period Binomial Model

Let S_0 denote the initial value of a stock. After 1 period, we have two possibilities:

1. Up: $S_U = uS_0$
2. Down: $S_d = dS_0$
3. Risk-free rate: $r, B_1 = (1 + r)$

Risk-Neutral Probability:

$$\bar{p} = \frac{(1+r)-d}{u-d} \qquad \bar{q} = 1 - \bar{p} \tag{1}$$

Derivative Pricing
For a derivative with payoff V_u and V_d in the up/down states:

$$V_0 = \frac{1}{1+r}(\bar{p}V_u + \bar{q}V_d) \tag{2}$$

Multi-Period Binomial Model

Let $S_n^i = S_0 \cdot u^i d^{m-i}$ denote the stock price at node (n,i). Let $V_N^i = f(S_N^i)$ be the final payoff of the **european** option:

$$V_n^i = \frac{1}{1+r}(\bar{p}V_{n+1}^{i+1} + \bar{q}V_{n+1}^i) \tag{3}$$

Hedging and Replication

Replication means constructing a portfolio of traded assets that mimics the payoff of some derivative security. If your portfolio always matches the option's value at every step/node in the pricing tree, it is a replicating portfolio. By the no-arbitrage (there is not way to make a profit without risk/investment).
Hedging is about reducing or eliminating risk. Replication is the tool to build a hedge.
Our goal is to replicate the value of an option at that point using:

1. Δ_n^i shares of stock
2. B_n^i in the risk-free bank account

Assuming you know the option values in the next time step, both in the up (V_{n+1}^{i+1}) and down (V_{n+1}^i) states, you can solve for stock holding:

$$\Delta_n^i = \frac{V_{n+1}^{i+1} - V_{n+1}^i}{S_{n+1}^{i+1} - S_{n+1}^i} \tag{4}$$

Then, we can solve for bond (any risk free asset whose value is $B_n = (1+r)^n$ in discrete situations), B_n^i:

$$B_n^i = \frac{1}{1+r}(V_{n+1}^i - \Delta_n^i S_{n+1}^i) \tag{5}$$

Then, your portfolio has value:

$$X_n^i = \Delta_n^i S_n^i + B_n^i$$

Essentially, Δ tells you how many shares of stock to hold to hedge one unit of the derivative.

American Derivative Securities

American options differ from European options in that a **European Option** can only be exercised at maturity, while **American Options** gives the holder the right to exercise at any time up to and including expiration.

- The value of an American option is always at least its intrinsic value.
- Its discounted price process is a super-martingale (type of stochastic process where it is expected to decrease of stay the same) under the risk-neutral measure.
- American options are priced as the maximum expected discounted payoff over all stopping times.

Definition:
A **call option** gives the holder (buyer) the right but not the obligation to buy an asset at a specified price (the **strike** price K) on or before a certain expiration date. The payoff at expiration is:

$$\max(S_T - K, 0)$$

Definition:
A **put option** gives the holder the right but not the obligation to sell the asset at the strike price K on or before expiration. The payoff at expiration is:

$$\max(K - S_T, 0)$$

Basically, if you think it's going to go up buy a call, if you think it's going to go down buy a put.
The value at time n of an American option is given by:

$$V_n = \max\left\{G_n, \frac{1}{1+r}(pV_{n+1}^{\text{up}}, qV_{n+1}^{\text{down}})\right\}$$

where G_n is the intrinsic value at time n.