The 2k-p Fractional Factorial Designs Part II
Author(s): G. E. P. Box and J. S. Hunter
Source: *Technometrics*, Vol. 3, No. 4 (Nov., 1961), pp. 449-458
Published by: American Statistical Association and American Society for Quality
Stable URL: http://www.jstor.org/stable/1266553
Accessed: 09/03/2010 11:15

The 2^{k-p} Fractional Factorial Designs Part II.

G. E. P. BOX AND J. S. HUNTER†

*Statistics Department, University of Wisconsin and Mathematics Research Center, University of Wisconsin**

"......... but that there turtle is an insect."

6. RESOLUTION V DESIGNS

In Part I of this paper the construction of two version factorials of resolution III and IV was discussed. In some situations we need experimental designs in which all main effects and two factor interactions are unconfounded with all other main effects and two factor interactions. Such designs are called designs of resolution V since each word in the defining relation has five or more characters. The one-half replicate of the 2^5 factorial with defining relation $\mathbf{I} = \mathbf{1\ 2\ 3\ 4\ 5}$ mentioned in Part I of this paper is a resolution V design, that is, a 2_{V}^{5-1} fractional.

Since there are $k(k + 1)/2$ main effects and two factor interactions to be estimated, these designs require considerably larger numbers of experimental runs than designs of resolution III or IV. No simple fractional of resolution V exist for two, three or four variables. In fact, the largest number of factors which can be included in a fractional of resolution V is as follows:

Resolution V Designs

Number of Runs	Number of Factors
16	5
32	6
64	8
128	11

Construction of Resolution V Designs

Table 23 lists the resolution V designs for $k = 5, 6, \cdots, 11$. The use of this table for constructing the appropriate design and arranging it in blocks may be illustrated in the case of the one-sixteenth replicate of the 2^{11} design, i.e., the 2_{V}^{11-4} design. In this design there are $2^{11-4} = 2^7 = 128$ runs so that we first write down in standard order the seven columns of the full 2^7 design. The first column

* Research Sponsored by the United States Army under Contract No. DA-11-022-ORD-2059.

† Now with the Chemical Engineering Department, Princeton University.

TABLE 23

Number of variabies	Number of Runs	Degree of Replication	Type of Design	Method of Introducing "new" factors	Blocking	Method of Introducing Blocks
5	16	$\frac{1}{2}$	2_V^{5-1}	$\pm$ 5 = 1 2 3 4	Not Available	
6†	32	$\frac{1}{2}$	2_V^{6-1}	$\pm$ 6 = 1 2 3 4 5	Two blocks of sixteen runs	B_1 = 1 2 3
7†	64	$\frac{1}{2}$	2_V^{7-1}	$\pm$ 7 = 1 2 3 4 5 6	Eight blocks of eight runs	B_1 = 1 3 5 7 B_2 = 1 2 5 6 B_3 = 1 2 3 4
8	64	$\frac{1}{4}$	2_V^{8-2}	$\pm$ 7 = 1 2 3 4 $\pm$ 8 = 1 2 5 6	Four blocks of sixteen runs	B_1 = 1 3 5 B_2 = 3 4 8
9*†	128	$\frac{1}{4}$	2_V^{9-2}	$\pm$ 9 = 1 4 5 7 8 $\pm$ 10 = 2 4 6 7 8	Eight blocks of sixteen runs	B_1 = 1 4 9 B_2 = 1 2 10 B_3 = 8 9 10
10	128	$\frac{1}{8}$	2_V^{10-3}	$\pm$ 8 = 1 2 3 7 $\pm$ 9 = 2 3 4 5 $\pm$ 10 = 1 3 4 6	Eight blocks of sixteen runs	B_1 = 1 4 9 B_2 = 1 2 10 B_3 = 8 9 10
11	128	$\frac{1}{16}$	2_V^{11-4}	$\pm$ 8 = 1 2 3 7 $\pm$ 9 = 2 3 4 5 $\pm$ 10 = 1 3 4 6 $\pm$ 11 = 1 2 3 4 5 6 7	Eight blocks of sixteen runs	B_1 = 1 4 9 B_2 = 1 2 10 B_3 = 8 9 10

Summary of Resolution V designs and their blocks. (All main effects and two-factor interactions clear of one another.)

*The nine factors in this design are **1**, **2**, **4**, **5**, **6**, **7**, **8**, **9**, **10**. In the text this design is derived from the 2^{11-4} and, as is there explained, it is convenient to drop factors **3** and **11**.

† The designs for 6, 7 and 9 variables are of resolution VI, VII and VI respectively *before* blocking.

corresponding to variable **1** consists of alternating minus and plus signs, the second column alternating pairs of minus and plus signs and so on. We now add four further columns using the relations found in column five of Table 23. The new vector associated with the variable **8** is generated by multiplying together the minus and plus signs of columns **1**, **2**, **3** and **7** to obtain the elements for the interaction column **1 2 3 7**. These are then used to identify the minus and plus signs of variable **8**, that is, **8** = **1 2 3 7**. A second new column is written down in a similar way for variable **9** using **9** = **2 3 4 5**. Similarly **10** = **1 3 4 6** and **11** = **1 2 3 4 5 6 7**. We now have a complete one-sixteenth replicate of the 2^{11} design with generators

1 2 3 7 8,   2 3 4 5 9,   1 3 4 6 10,   1 2 3 4 5 6 7 11,

The full defining relation of this design is therefore made up of the words:

I	**1 2 5 6 9 10**
1 2 3 7 8	**1 6 7 9 11**
2 3 4 5 9	**2 5 7 10 11**
1 3 4 6 10	**3 5 6 7 8 9 10**
1 2 3 4 5 6 7 11	**1 3 5 8 10 11**
1 4 5 7 8 9	**2 3 6 8 9 11**
2 4 6 7 8 10	**3 4 7 9 10 11**
4 5 6 8 11	**1 2 4 8 9 10 11**

We observe that all of the words in this defining relation have five or more characters identifying the design as one of resolution V. The generators given above are for the principal fraction. The generators for all sixteen fractions are given by

±1 2 3 7 8,   ±2 3 4 5 9,   ±1 3 4 6 10,   ±1 2 3 4 5 6 7 11

To block this design in eight blocks of 16 runs we write down three further columns for $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ using the relations in column seven of Table 23. The two versions of $\mathbf{B}_1$ are obtained by multiplying together the minus and plus signs of columns **1**, **4** and **9**, the versions of $\mathbf{B}_2$ using columns **1 2** and **10**, and the versions of $\mathbf{B}_2$ using columns **8 9** and **10**. The eight blocks are then identified by those runs having the signs of $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ equal to $(-,-,-)$, $(+,-,-)$, $(-,+,-)$, $(+,+,-)$, $(-,-,+)$, $(+,-,+)$, $(-,+,+)$, and $(+,+,+)$.

Designs with 16 Runs: The 2_{V}^{5-1} Design

With five variables, a 2_{V}^{5-1} design is obtained using the defining relation

$$\mathbf{I} = \pm \mathbf{1\ 2\ 3\ 4\ 5}.$$

The 16 version combinations for such a design are written down by first writing out the design matrix for the 2^4 complete factorial and then identifying the versions of the fifth variable with those of the interaction **1 2 3 4**. Clearly all main effects are here confounded with four-factor interactions since the alias relationships of main effects are of the type

$$\pm \mathbf{1} = \mathbf{2\ 3\ 4\ 5}$$

while those for the two-factor interactions are of the type

$$\pm\mathbf{1\ 2} = \mathbf{3\ 4\ 5}.$$

It follows that if we can ignore three and four-factor interactions we can estimate with these sixteen runs the average, the five main effects and the ten two-factor interactions. This design is remarkable in that every degree of freedom is used to estimate effects of interest. However none are available for confounding so that it is not possible for this design to be run in blocks without associating one or more main effects or two-factor interactions with the block variables.

*Designs with 32 Runs: The 2_{V}^{6-1} Design**

For six variables there are 21 effects to estimate (the six main effects and the 15 two-factor interactions). It is clear that a quarter replicate of a 2^6 factorial involving 16 points would not be large enough to estimate all the quantities of interest. The half replicate has as its defining relation

$$\mathbf{I} = \pm\mathbf{1\ 2\ 3\ 4\ 5\ 6}.$$

Using this design main effects are associated with five-factor interactions, for example, $\pm\mathbf{1} = \mathbf{2\ 3\ 4\ 5\ 6}$ and two-factor interactions are associated with four-factor interactions, for example, $\pm\mathbf{1\ 2} = \mathbf{3\ 4\ 5\ 6}$.

32 Runs, The 2_{V}^{6-1} Design in Two Blocks of 16

In blocking this and other fractional factorial designs it is important to bear in mind that any factor associated with blocks will have its aliases also associated with blocks. For the 2_{V}^{6-1} design for instance, where the alias relation $\mathbf{I} = \mathbf{1\ 2\ 3\ 4\ 5\ 6}$ already exists, if we adopt for the block arrangement $\mathbf{B}_1 = \mathbf{1\ 2\ 3\ 4\ 5}$ then we should also have $\mathbf{B}_1 = \mathbf{6}$. Thus we would confound the main effect of variable **6** with blocks. In general, when choosing a suitable factor to be associated with the block variables we must be careful to ensure that not only the chosen factor but also its aliases correspond to at least three-factor interactions. The present design can be run in two blocks of 16 by using any three-factor interaction to define the block. For example, for block variable $\mathbf{B}_1$ we can set

$$\mathbf{B}_1 = \mathbf{1\ 2\ 3}$$

whence, because of the alias relationship $\mathbf{I} = \mathbf{1\ 2\ 3\ 4\ 5\ 6}$, we find that $\mathbf{B}_1 = \mathbf{1\ 2\ 3} = \mathbf{4\ 5\ 6}$, and no two-factor interaction is confounded.

Designs with 64 Runs: The 2_{V}^{7-1} Design

With seven variables there are 28 effects to be determined (the seven main effects and twenty-one two-factor interactions) so that in principle it might be possible to use a quarter replicate of the full 2^7 design which would involve 32 experimental runs. In practice however this is not possible for the type of fractional factorial designs discussed here. Other designs, such as the three quarter replicate designs discussed elsewhere in this issue, can however be em-

* The 2^{6-1} and 2^{7-1} designs are properly of resolution VI and VII respectively. The block arrangements however insure that first and second order effects will not be associated with blocks or with each other.

ployed. The best that can be done for this series of conventional fractional factorials is to use of half replicate and once again it is desirable to use for the defining contrast the highest possible order interaction, i.e.,

$$\mathbf{I} = \mathbf{1\ 2\ 3\ 4\ 5\ 6\ 7}.$$

This seven variable design can be run in eight blocks of eight experimental runs each using the following four-factor interactions to define the block variables: $\mathbf{B}_1 = \mathbf{1\ 3\ 5\ 7}$, $\mathbf{B}_2 = \mathbf{1\ 2\ 5\ 6}$ and $\mathbf{B}_3 = \mathbf{1\ 2\ 3\ 4}$. To write down the complete design, first write down all the minus and plus signs for the 2^6 factorial in standard order. Introduce the seventh variable using $\mathbf{7} = \pm\mathbf{1\ 2\ 3\ 4\ 5\ 6}$. The eight blocks can then be obtained by writing down the three additional vectors $\mathbf{B}_1 = \mathbf{1\ 3\ 5\ 7}$, $\mathbf{B}_2 = \mathbf{1\ 2\ 5\ 6}$ and $\mathbf{B}_3 = \mathbf{1\ 2\ 3\ 4}$ and allocating the experimental points to the first block, second block, $\cdots$, eighth block in accordance as the signs of $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ are $(-, -, -)$, $(+, -, -)$, $(-, +, -)$, $(+, +, -)$, $(-, -, +)$, $(+, -, +)$, $(-, +, +)$ and $(+, +, +)$. The derivation of this design is given in the Appendix.

64 Runs: The 2_{V}^{8-2} Design

For eight variables there are 36 effects to be estimated, the eight main effects and twenty-eight two-factor interactions. These estimates can be obtained with a one-quarter replicate of the 2^8 design involving 64 experimental points, that is, a 2_{V}^{8-2} design. The defining relation will include three words in addition to the identity $\mathbf{I}$ and if all main effects and two-factor interactions are not to be confounded then all the words in the defining contrast must be at least five-factor interactions. Bearing in mind that the three words in the defining contrast will be such that any two of them multiply to give the third, the best arrangement will be of the type indicated below, in which the interactions involved are $\mathbf{1\ 2\ 3\ 4\ 7}$, $\mathbf{1\ 2\ 5\ 6\ 8}$ and their product $\mathbf{3\ 4\ 5\ 6\ 7\ 8}$ giving:

$$\begin{array}{cccccccc}
\mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & & & \mathbf{7} & \\
\mathbf{1} & \mathbf{2} & & & \mathbf{5} & \mathbf{6} & & \mathbf{8} \\
 & & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8}
\end{array}$$

From the above array it will be seen that we are trying to pick two interactions containing five factors or more which have "minimum overlap", so that their product will also be an interaction of five factors or more. We therefore take for the generating relations

$$\mathbf{I} = \pm\mathbf{1\ 2\ 3\ 4\ 7}; \qquad \mathbf{I} = \pm\mathbf{1\ 2\ 5\ 6\ 8}$$

and write down the design by first setting out the full 2^6 design for the variables **1**, **2**, **3**, **4**, **5** and **6**. The versions of the variables **7** and **8** then follow the minus and plus elements of the interactions **1 2 3 4** and **1 2 5 6**.

It is possible to arrange this 2_{V}^{8-2} design in four blocks of 16 runs each without confounding block variables with any main effect or two-factor interaction. To see how this can be done, we remember that although it would be possible to write down 63 columns associated with all the main effects and interactions of the basic 2^6 design only 36 of these will be associated with main effects and two-factor interactions for the eight variables of interest. There remain therefore $63 - 36 = 27$ columns containing + and − signs which can be utilized to accommodate block effects. If the experiment is run in *two* blocks of 32 *any* one of these 27 columns can be used to accommodate the blocks. In practice it is not necessary to write down all the 63 columns. In fact we select the columns for blocking by inspection of the generators. Some care is needed in this selection. For example, the **1 2 3 4** interaction contains four symbols and so on first sight appears to provide as a suitable column to be used in blocking. However, it cannot be used since **1 2 3 4** is an alias of **7** and therefore the main effect of **7** would be confounded with blocks. This is so since one of the generators for the quarter replicate design is **1 2 3 4 7** and consequently **1 2 3 4** = **7**.

Clearly it is necessary to check not only that the interaction employed involves three factors or more but also that all its aliases involve three factors or more. The simplest way to pick out a suitable interaction is to write out the defining relation for the fractional factorial

$$\mathbf{I} = \mathbf{1\,2\,3\,4\,7} = \mathbf{1\,2\,5\,6\,8} = \mathbf{3\,4\,5\,6\,7\,8}$$

and then by inspection select interactions whose multiples with the words of the defining relation contain three or more characters. For instance, suppose **1 3 5** is selected as a possible interaction to confound. Then

$$\mathbf{1\,3\,5} = \mathbf{2\,4\,5\,7} = \mathbf{2\,3\,6\,8} = \mathbf{1\,4\,6\,7\,8}.$$

Thus the three-factor interaction **1 3 5** is confounded with no interaction containing fewer than three factors, and hence, **1 3 5** is a suitable contrast to use for blocking.

If we wish to run the experiment in four blocks of 16 then another interaction of three factors or more having no aliases with less than three factors must be chosen with the additional requirement that its interaction with the first interaction used to identify blocks has aliases all of three factors or more. For example, take **1 3 5** and **3 4 8**. Using these, the three degrees of freedom among the four blocks will be associated with **1 3 5** and **3 4 8** and with their interaction **1 4 5 8**. That these give satisfactory aliases is seen by obtaining their multiples with the defining relation of the fractional factorial giving:

$$\begin{aligned}
\mathbf{1\,3\,5} &= \mathbf{2\,4\,5\,7} = \mathbf{2\,3\,6\,8} = \mathbf{1\,4\,6\,7\,8}\\
\mathbf{3\,4\,8} &= \mathbf{1\,2\,7\,8} = \mathbf{1\,2\,3\,4\,5\,6} = \mathbf{5\,6\,7}\\
\mathbf{1\,4\,5\,8} &= \mathbf{2\,3\,5\,7\,8} = \mathbf{2\,4\,6} = \mathbf{1\,3\,6\,7}
\end{aligned}$$

Thus as indicated in column six of Table 23, the + and − signs associated with the interaction vectors **1 3 5** and **3 4 8** can be used to identify four blocks of sixteen runs each.

Designs with 128 Runs: The 2_{V}^{9-2}, 2_{V}^{10-3} and 2_{V}^{11-4} Designs

The designs for nine and ten variables using 128 runs can be regarded as a special case of the 2_{V}^{11-4} design from which one or two variables have been dropped. In many cases therefore where nine or ten variables are to be studied and where there are one or two further variables of possible importance, it would be worthwhile to include these and in fact to run the full eleven variable design. We shall first discuss the 2_{V}^{11-4} design and then discuss the designs for nine and ten variables as special cases. To construct the 2_{V}^{11-4} design we begin by writing down the $-$ and $+$ signs comprising the seven columns for the complete 2^7 factorial. The sixteenth fraction of the 2^{11} design is then obtained by associating certain interactions between the seven variables with the new variables **8**, **9**, **10** and **11**. Let us call these interactions **W**, **X**, **Y** and **Z**, so that the column headings for our final design matrix will be as follows:

$$\begin{array}{ccccccccccc} & & & & & & & \mathbf{W} & \mathbf{X} & \mathbf{Y} & \mathbf{Z} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} & \mathbf{10} & \mathbf{11} \end{array}$$

It will be understood that each of the symbols **W**, **X**, **Y**, **Z**, corresponds to an interaction of the first set of seven variables. For example, if **W** is the interaction **1 2 3 4**, the generators produced by associating **W** with the variable **8** would be **1 2 3 4 8** or **W8**. In general the generating relation for the complete design are

$$\mathbf{I} = \mathbf{W\,8}, \qquad \mathbf{I} = \mathbf{X\,9}, \qquad \mathbf{I} = \mathbf{Y\,10}, \qquad \mathbf{I} = \mathbf{Z\,11}$$

and hence the defining relation is $\mathbf{I} = \mathbf{W\,8} = \mathbf{X\,9} = \mathbf{Y\,10} = \mathbf{Z\,11} = \mathbf{W\,X\,8\,9} = \mathbf{W\,Y\,8\,10} = \mathbf{W\,Z\,8\,11} = \mathbf{X\,Y\,9\,10} = \mathbf{X\,Z\,9\,11} = \mathbf{Y\,Z\,10\,11} = \mathbf{W\,X\,Y\,8\,9\,10} = \mathbf{W\,X\,Z\,8\,9\,11} = \mathbf{W\,Y\,Z\,8\,10\,11} = \mathbf{X\,Y\,Z\,9\,10\,11} = \mathbf{W\,X\,Y\,Z\,8\,9\,10\,11}$. Now if this design is to be of resolution V all the words in the defining relation must contain at least five symbols; it follows that:

(i) **W**, **X**, **Y** and **Z** must themselves each contain at least four symbols.
(ii) The products **W X**, **W Y**, **W Z** etc., must contain at least three symbols since all words in the defining relation containing these pairs will also contain two further symbols from the group **8**, **9**, **10**, **11**.
(iii) Similarly the products in threes **W X Y**, **W X Z**, etc., must contain two symbols and
(iv) the product **W X Y Z** must contain at least one symbol.

The problem is therefore to find four interactions each containing at least four symbols with all the above properties. A set of interactions having these properties has in fact already been derived. It was shown earlier that the defining relation for the 2_{III}^{7-3} design is:

$$\mathbf{I} = \mathbf{1\,2\,4} = \mathbf{1\,3\,5} = \mathbf{2\,3\,6} = \mathbf{1\,2\,3\,7} = \mathbf{2\,3\,4\,5} = \mathbf{1\,3\,4\,6} = \mathbf{3\,4\,7} = \mathbf{1\,2\,5\,6}$$
$$= \mathbf{2\,5\,7} = \mathbf{1\,6\,7} = \mathbf{4\,5\,6} = \mathbf{1\,4\,5\,7} = \mathbf{2\,4\,6\,7} = \mathbf{3\,5\,6\,7} = \mathbf{1\,2\,3\,4\,5\,6\,7}.$$

A set of generators for this defining relation are **1 2 3 7**, **2 3 4 5**, **1 3 4 6** and **1 2 3 4 5 6 7**. Suppose then we choose:

$$\mathbf{W} = \mathbf{1\,2\,3\,7}$$

$$\mathbf{X} = \mathbf{2\,3\,4\,5}$$

$$\mathbf{Y} = \mathbf{1\,3\,4\,6}$$

$$\mathbf{Z} = \mathbf{1\,2\,3\,4\,5\,6\,7}$$

Then clearly all conditions (i), (ii), (iii) and (iv) mentioned above will be satisfied. A one-sixteenth replicate of the 2^{11} for which no main effect or two-factor interaction is confounded with any other main effect or interaction, that is, the 2_{V}^{11-4} design can be obtained by setting

$$\pm\ \mathbf{8} = \mathbf{1\ 2\ 3\ 7}$$

$$\pm\ \mathbf{9} = \mathbf{2\ 3\ 4\ 5}$$

$$\pm\mathbf{10} = \mathbf{1\ 3\ 4\ 6}$$

$$\pm\mathbf{11} = \mathbf{1\ 2\ 3\ 4\ 5\ 6\ 7}$$

to give the generators

$$\pm\mathbf{1\ 2\ 3\ 7\ 8};\ \pm\mathbf{2\ 3\ 4\ 5\ 9},\ \pm\mathbf{1\ 3\ 4\ 6\ 10},\ \pm\mathbf{1\ 2\ 3\ 4\ 5\ 6\ 7\ 11}$$

Given that three-factor and higher order interaction effects are negligible the resulting 2_{V}^{11-4} design provides separate estimates of the eleven main effects and fifty-five two-factor interactions.

The 2_{V}^{10-3} and 2_{V}^{9-2} designs may be obtained from the 2_{V}^{11-4} design by dropping out respectively one or two of the variables. The defining relations for these designs omit all words containing the dropped variables. All designs obtained by dropping variables will have the properties of the parent design. However, selections can be made which improve the reduced design. For example, if we drop variable **11** the generators for the resultant 2_{V}^{10-3} design are

$$\pm\mathbf{1\ 2\ 3\ 7\ 8};\qquad \pm\mathbf{2\ 3\ 4\ 5\ 9};\qquad \pm\mathbf{1\ 3\ 4\ 6\ 10}$$

The corresponding defining relation contains three words with five characters, three with six and one word with seven characters. However, if variable **10** is dropped the generators are

$$\pm\mathbf{1\ 2\ 3\ 7\ 8};\qquad \pm\mathbf{2\ 3\ 4\ 5\ 9};\qquad \pm\mathbf{1\ 2\ 3\ 4\ 5\ 6\ 7\ 11}$$

and the defining relation now contains four words with five characters two with six and one word with eight. The first arrangement is slightly preferable since, with fewer five character words, fewer two-factor interactions will beconfounded with three-factor interactions.

Similarly for the 2_{V}^{9-7} design it is best to drop variable **3** in addition to variable **11**. The generators for this design are

$$\pm\mathbf{1\ 4\ 5\ 7\ 8\ 9};\qquad \pm\mathbf{2\ 4\ 6\ 7\ 8\ 10}$$

from which we see that all main effects will be confounded with only five-factor interactions and two-factor interactions only with four-factor interactions. (This design is, in fact, of resolution VI).

128 Runs: The 2_{V}^{11-4} in 8 Blocks of 16 Runs

It is a somewhat remarkable fact that the 2_{V}^{11-4} design can be blocked into eight groups of sixteen runs. Using the generators for the principal fraction of the 2_{V}^{11-4} given in the previous section, we obtain the defining relation:

$$\mathbf{I} = \mathbf{1\,2\,3\,7\,8} = \mathbf{2\,3\,4\,5\,9} = \mathbf{1\,3\,4\,6\,10} = \mathbf{1\,2\,3\,4\,5\,6\,7\,11} = \mathbf{1\,4\,5\,7\,8\,9}$$
$$= \mathbf{2\,4\,6\,7\,8\,10} = \mathbf{4\,5\,6\,8\,11} = \mathbf{1\,2\,5\,6\,9\,10} = \mathbf{1\,6\,7\,9\,11} = \mathbf{2\,5\,7\,10\,11}$$
$$= \mathbf{3\,5\,6\,7\,8\,9\,10} = \mathbf{2\,3\,6\,8\,9\,11} = \mathbf{1\,3\,5\,8\,10\,11}$$
$$= \mathbf{3\,4\,7\,9\,10\,11} = \mathbf{1\,2\,4\,8\,9\,10\,11}.$$

To obtain eight blocks we require seven interactions produced by three block generators which, when multiplied together and by the words in the above defining relation produce words with three or more characters. It can be confirmed by actual multiplication that a suitable group of block generators is

$$\mathbf{B}_1 = \mathbf{1\,4\,9}, \qquad \mathbf{B}_2 = \mathbf{1\,2\,10}, \qquad \mathbf{B}_3 = \mathbf{8\,9\,10}.$$

We find that the following seven interactions will be confounded with block effects:

$\mathbf{B}_1 = \mathbf{1\,4\,9}$	$\mathbf{B}_1\mathbf{B}_2 = \mathbf{2\,4\,9\,10}$
$\mathbf{B}_2 = \mathbf{1\,2\,10}$	$\mathbf{B}_1\mathbf{B}_3 = \mathbf{1\,4\,8\,10}$
$\mathbf{B}_3 = \mathbf{8\,9\,10}$	$\mathbf{B}_2\mathbf{B}_3 = \mathbf{1\,2\,8\,9}$
	$\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3 = \mathbf{2\,4\,8}$

When variables **3** and **11** are dropped from the eleven variable design these same block generators may also be used to generate the blocks for the ten and nine factor resolution V designs.

Appendix

The Derivation of the 2_V^{7-1} in Blocks of Eight

The 2_V^{7-1} contains sixty-four experimental runs and is constructed by first writing down a full 2^6 factorial and then setting $\mathbf{7} = \mathbf{1\,2\,3\,4\,5\,6}$. To divide the design into eight blocks of eight runs such that no main effect or two-factor interaction is confounded with any block effect we must be able to find seven interactions to associate with the seven degrees of freedom between the eight blocks and each of these interactions and their aliases must be at least three-factor interactions. To see how this can be done we refer once again to the table of minus and plus signs appropriate to the 2^3 factorial. This table, shown below, has been used to produce a second table in which a number appears if there is a minus sign in the first table, and does not appear if there is a plus sign. If we associate the first three columns of the first table with B_1, B_2 and B_3 we see, that the interactions B_1B_2, B_1B_3, B_2B_3 and $B_1B_2B_3$ will identify the remaining columns. We also note that all of the columns contain four minus and four plus signs. Referring now to the second table, when two columns are multiplied together only those numerals which appear once, or an odd number of times, will appear in the product, and those numerals appearing an even number of times will not appear in the product. This is exactly the same rule adopted earlier to assist us in identifying the alias terms. It is clear therefore that setting $\mathbf{B}_1$ equal to the $\mathbf{1\,3\,5\,7}$ interaction effect, $\mathbf{B}_2 = \mathbf{1\,2\,5\,6}$ and $\mathbf{B}_3 =$

TABLE

	B_1	B_2	B_3	B_1B_2	B_1B_3	B_2B_3	$B_1B_2B_3$	
(1)	−1	−1	−1	1	1	1	−1	
(2)	1	−1	−1	−1	−1	1	1	
(3)	−1	1	−1	−1	1	−1	1	
(4)	1	1	−1	1	−1	−1	−1	± sign in
(5)	−1	−1	1	1	−1	−1	1	2^3 factorial
(6)	1	−1	1	−1	1	−1	−1	
(7)	−1	1	1	−1	−1	1	−1	
(8)	1	1	1	1	1	1	1	
	1	1	1				1	
		2	2	2	2			confounded inter-
	3		3	3		3		actions for the
			4		4	4	4	$\frac{1}{2}$th replicate of
	5	5			5	5		the 2^7 design
		6		6		6	6	
	7			7	7		7	

1 2 3 4 that all seven of the block effects will be confounded with four-factor interactions. Since the 2_{V}^{7-1} has the defining relation $\mathbf{I} = \mathbf{1\ 2\ 3\ 4\ 5\ 6\ 7}$, each of the block defining contrasts will therefore be aliased with those three of the seven numerals which do not appear in it. Thus, the 2_{V}^{7-1} is blocked such that all block effects are aliased with at least a three factor interaction and hence clear of both main effects and two factor interactions as required. Of course, *before blocking,* the design with defining relation $\mathbf{I} = \mathbf{1\ 2\ 3\ 4\ 5\ 6\ 7}$ is of resolution VII.