Tab 5:

Fractional Factorials

Tab 5: Fractional Factorials

PURPOSE:

Explore the concept of fractional factorial DOE's, including when to apply them and limitations of the method.

OBJECTIVES:

1. Describe the nomenclature of fractional factorial DOE's.

2. Introduce terms used in fractional factorial DOE's (confounding, design resolution, alias table and screening)

3. Use Minitab to generate and analyze fractional factorial DOE's graphically.

Fractional Factorial DOE...

What is it?

A DOE which allows for more factors (X's) to be included in the experimental design for the same number of runs.

Why use it?

To test a large number of potential X's with a minimum of runs.

When is it used?

- When screening for the Vital Few X's.
- When economic considerations make it difficult to run a full factorial experiment.

Fractional Factorials are a powerful way to perform experiments with a large number of X's

Chemical Reaction Example

In a chemical reaction process, it is believed that the percent material reacted is a result of 5 factors (X's).

- Feedrate of the material into the reactor (Liters / min)
- % Catalyst added
- Agitation Rate (RPM)
- Reactor internal temperature (°C)
- Concentration of the raw compounding material (%)

The purpose of the experiment is to determine which of these factors are Potential 'Vital Few' X's. The levels for each of the five factors is shown below:

	Factor	*Low*	*High*
1.	Feedrate	10	15
2.	Catalyst	1	2
3.	Agitation	100	120
4.	Temperature	140	180
5.	Concentration	3	6

The 2^5 experiment (32 runs) is shown on the next page.

Note: Example from Box, Hunter, Hunter, Statistics for Experimenters, pg. 377

Chemical Reaction 'Full' Factorial design matrix

RunOrde	Feedrate	Catalyst	Agitation	Temperatu	Concentrati	Response
1	10	1	100	140	3	
2	15	1	100	140	3	
3	10	2	100	140	3	
4	15	2	100	140	3	
5	10	1	120	140	3	
6	15	1	120	140	3	
7	10	2	120	140	3	
8	15	2	120	140	3	
9	10	1	100	180	3	
10	15	1	100	180	3	
11	10	2	100	180	3	
12	15	2	100	180	3	
13	10	1	120	180	3	
14	15	1	120	180	3	
15	10	2	120	180	3	
16	15	2	120	180	3	
17	10	1	100	140	6	
18	15	1	100	140	6	
19	10	2	100	140	6	
20	15	2	100	140	6	
21	10	1	120	140	6	
22	15	1	120	140	6	
23	10	2	120	140	6	
24	15	2	120	140	6	
25	10	1	100	180	6	
26	15	1	100	180	6	
27	10	2	100	180	6	
28	15	2	100	180	6	
29	10	1	120	180	6	
30	15	1	120	180	6	
31	10	2	120	180	6	
32	15	2	120	180	6	

When creating a model equation, the 'estimates of the unknown parameters' are the coefficients ('b's):

$$Y = b_0 + b_1X_1 + b_2X_2 + b_{12}X_{12} + \ldots + \text{error}$$

'parameter estimates'

With 32 runs, we can estimate:

1. OVERALL AVERAGE	1	
2. MAIN EFFECTS	5	
3. 2-FACTOR INTERACTIONS	10	(5 x 4)/2
4. 3-FACTOR INTERACTIONS	10	(5 x 4)/2
5. 4-FACTOR INTERACTIONS	5	
6. 5-FACTOR INTERACTIONS	1	
	32	Runs

When planning an experiment, we make a tradeoff between the number of runs and our ability to estimate effects. You must have at least one run for each effect you want to estimate. For example, in the 5-factor experiment above, we need 32 runs to estimate all of the parameters listed.

In most cases, however, we are not interested in high-order interactions. High-order interactions (those with 3 or more factors) are rarely Vital. This fact may allow us to reduce our number of runs.

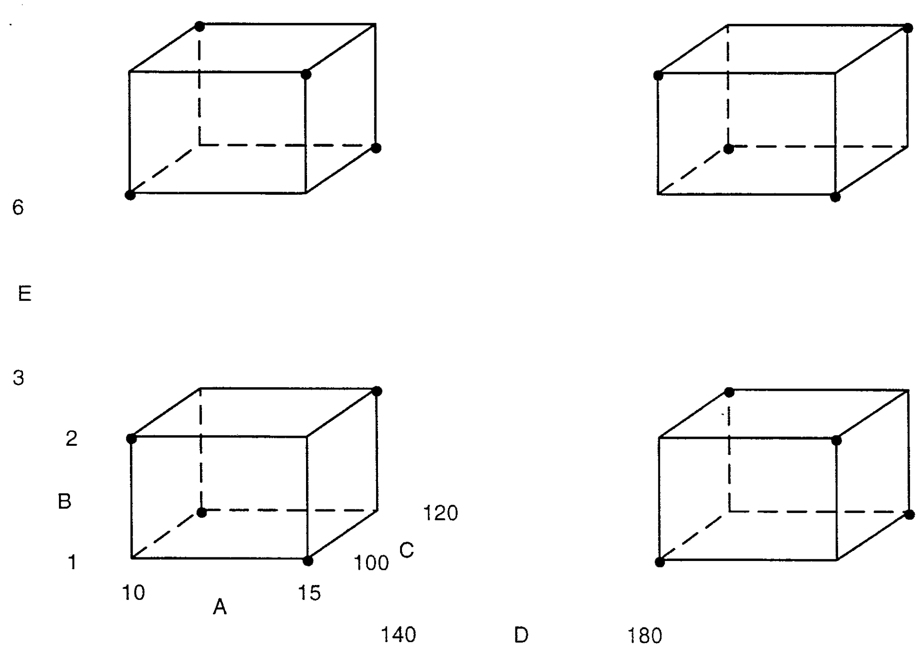
Definition of a Fractional Factorial

Instead of running all 32 runs listed, let's consider a "Fractional Factorial" experiment with only 16 runs.

The following graph shows the 16 points that are used in the 16-run fractional factorial experiment. Note that each cube has:

- 2 runs on the top and 2 runs on the bottom
- 2 runs on the left and 2 runs on the right
- 2 runs on the front and 2 runs on the back

Factorial Design



Orthogonality Test Plans

What is so special about these 16 runs?

They are ORTHOGONAL!

This allows us to estimate the effects of the most important variables . . .

With fewer total runs!!

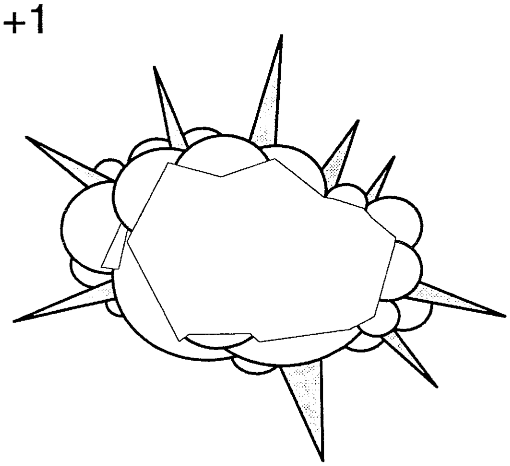
The choice of the 16 runs (combinations of X_1, X_2, X_3, X_4, and X_5) is critical.

We *could* have chosen:

$X_1 \times X_2 \times X_3 \times X_4 \times X_5 = -1$ **OR**

$X_1 \times X_2 \times X_3 \times X_4 \times X_5 = +1$

If we choose the 16 points in another non-orthogonal manner, then we call it a ***fractured*** factorial experiment.



Naming the Fractional Factorial Design

In a **FULL factorial** experiment:

Base Number ➡ Number of Levels

Exponent ➡ Number of factors (X's)

Example: 2^5 ➡ 2 levels, 5 factors; 32 total runs

In a **FRACTIONAL factorial** experiment:

Decide the total number of runs first and "back into" the exponent value.

Base Number = Number of Levels

Exponent = The first number is the number of factors; the second number is derived.

Example: 2^{5-1} = 2 levels, 5 factors; 16 total runs because $2^{5-1} = 2^4 = 16$ runs.

Name fractional factorial designs by starting with the number of runs in the experiment

Creating and Analyzing Fractional Factorial Designs in Minitab

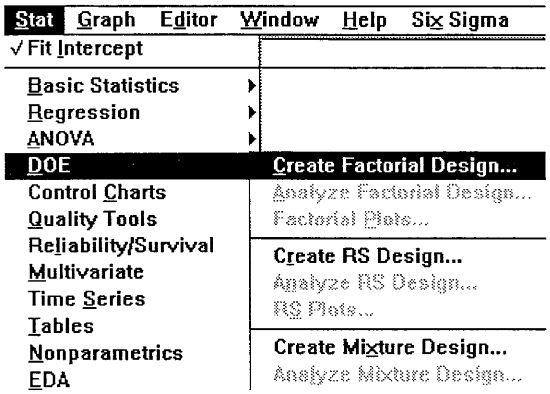
To create a Fractional Factorial design, we will use the same dialog box we used to create Full Factorial designs. As shown earlier in this chapter, we are fitting a 'fraction' of a larger design.

More care has to be taken to ensure "othogonality" and "balance" in the design -- especially when using blocking factors. (Let Minitab take care of this for you!)

Minitab requires that you specify the number of factors, runs, and blocks. Minitab follows the Box, Hunter, and Hunter book, "Statistics for Experimenters" when generating these factorial designs.

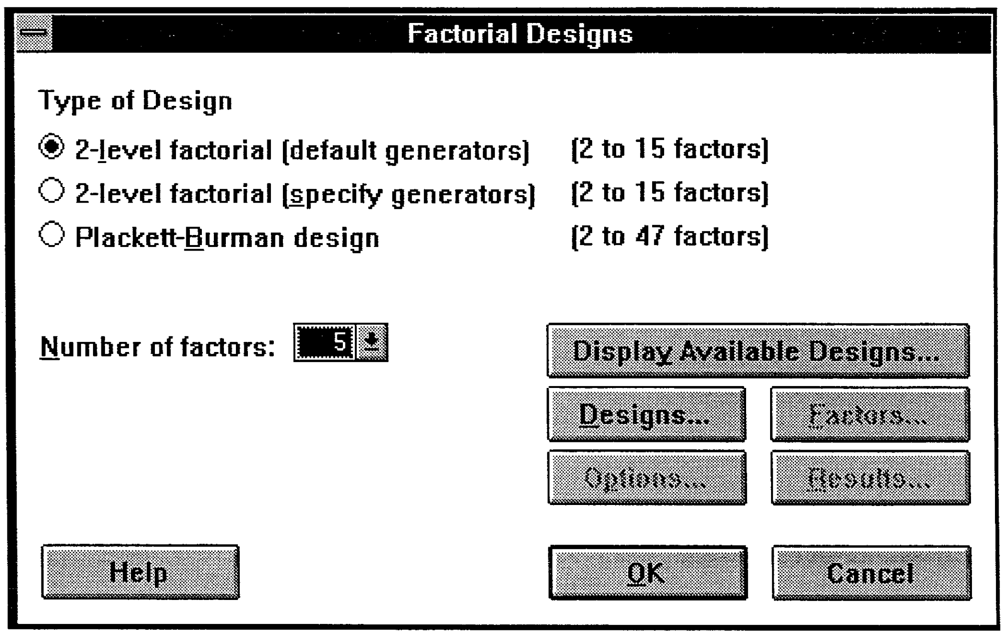
Creating a Fractional Design in Minitab

Stat>DOE>Create Factorial Design...



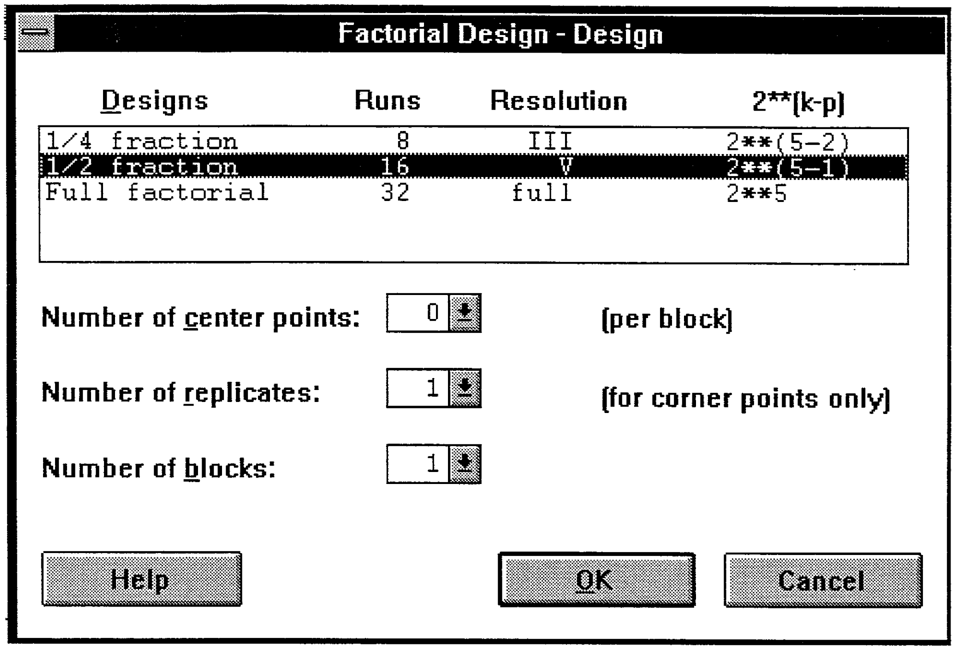
Remember: You can only create one design per worksheet!

In the main dialog box enter 5 for the 'Number of Factors'



We will be entering data into the 'Designs', 'Factors' and 'Options' sub-dialog boxes from this main dialog box

Creating a Fractional Design in Minitab (cont'd)



Highlight the '1/2 fraction' line, indicating a 16-run experiment.

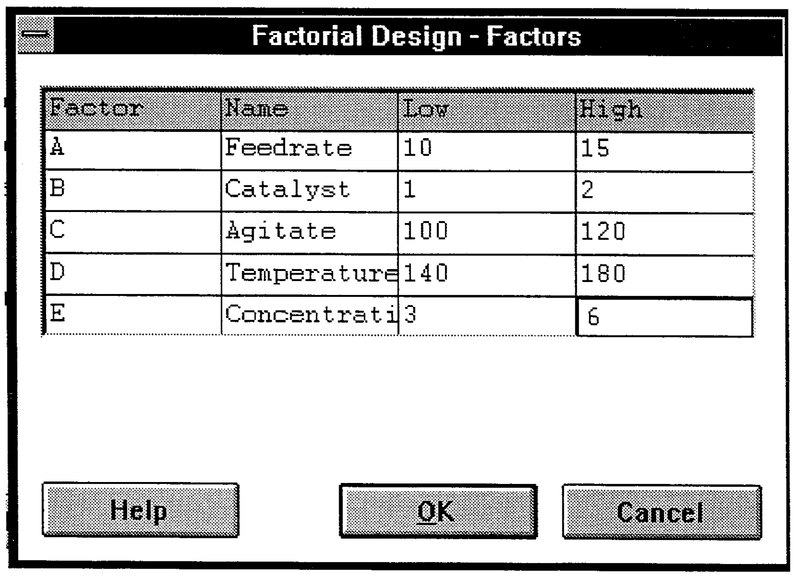
Leave the other boxes at their default settings.

Click 'OK'

From the main dialog box, click on 'Factors...' to bring up the sub-dialog box.

Enter the Factor 'Name' and 'Low' and 'High' values for each factor

Click 'OK'

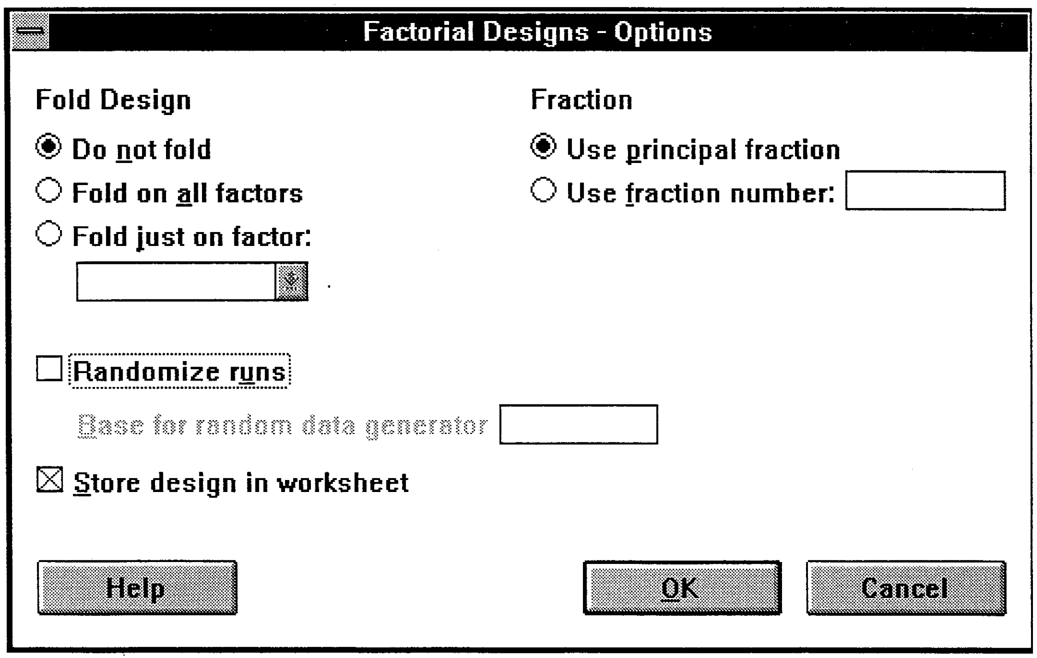


Creating a Fractional Design in Minitab

From the main dialog box, click on 'Options...' to bring up the sub-dialog box.

Un-click 'Randomize runs' for this in-class example ONLY!

Click 'OK' twice



Creating a Fractional Design in Minitab

The worksheet appears in the Data Window:

StdOrder	RunOrde	Blocks	Feedrate	Catalyst	Agitation	Tempera	Concentr	PCReact
·	1	1	10	1	100	140	6	56
2	2	1	15	1	100	140	3	53
3	3	1	10	2	100	140	3	63
4	4	1	15	2	100	140	6	65
5	5	1	10	1	120	140	3	53
6	6	1	15	1	120	140	6	55
7	7	1	10	2	120	140	6	67
8	8	1	15	2	120	140	3	61
9	9	1	10	1	100	180	3	69
10	10	1	15	1	100	180	6	45
11	11	1	10	2	100	180	6	78
12	12	1	15	2	100	180	3	93
13	13	1	10	1	120	180	6	49
14	14	1	15	1	120	180	3	60
15	15	1	10	2	120	180	3	95
16	16	1	15	2	120	180	6	82

Enter this data

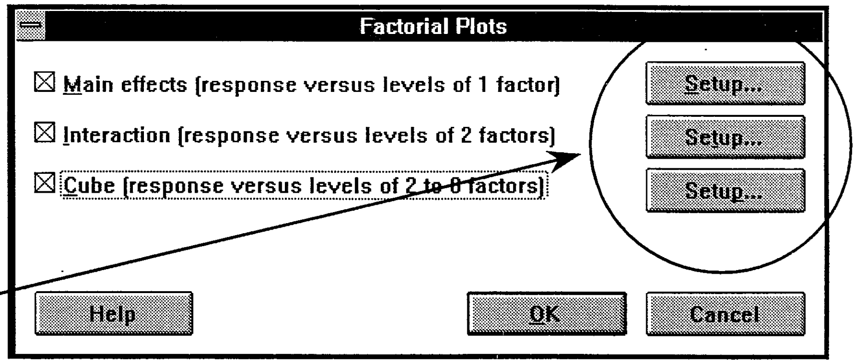
Enter the 'Y' response data gathered from the experiment into a column labeled 'PCReact'.

For a real project, the 'StdOrder' numbers will be scrambled due to the randomization of the runs. When not in class, always randomize runs!

Analyzing a Fractional Design in Minitab

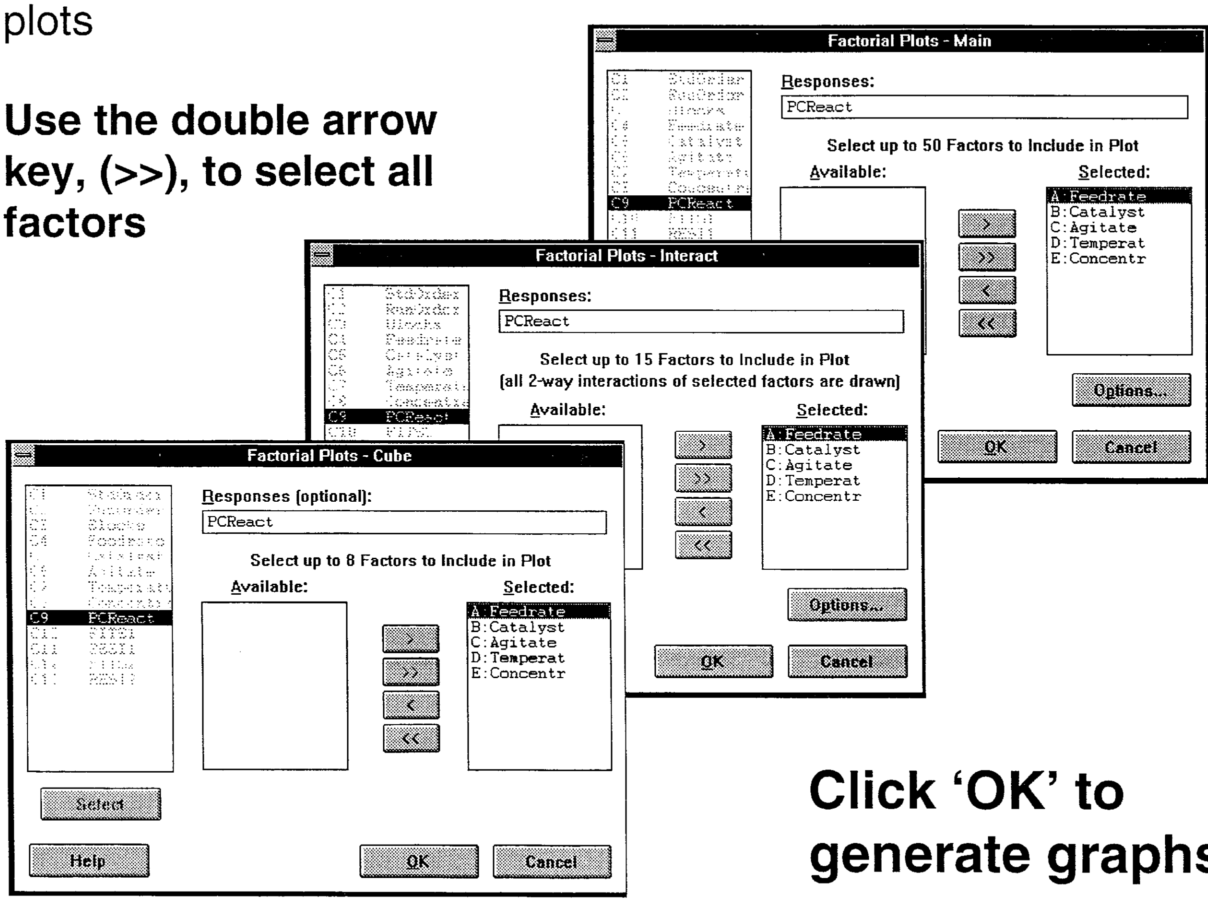
Analyze Fractional Factorial experiments using the same three graphs as all other 2-level experiments. Let's create the Main Effects, Interaction and Cube Plots:

Stat>DOE>
Factorial Plots...



Select and set up all plots

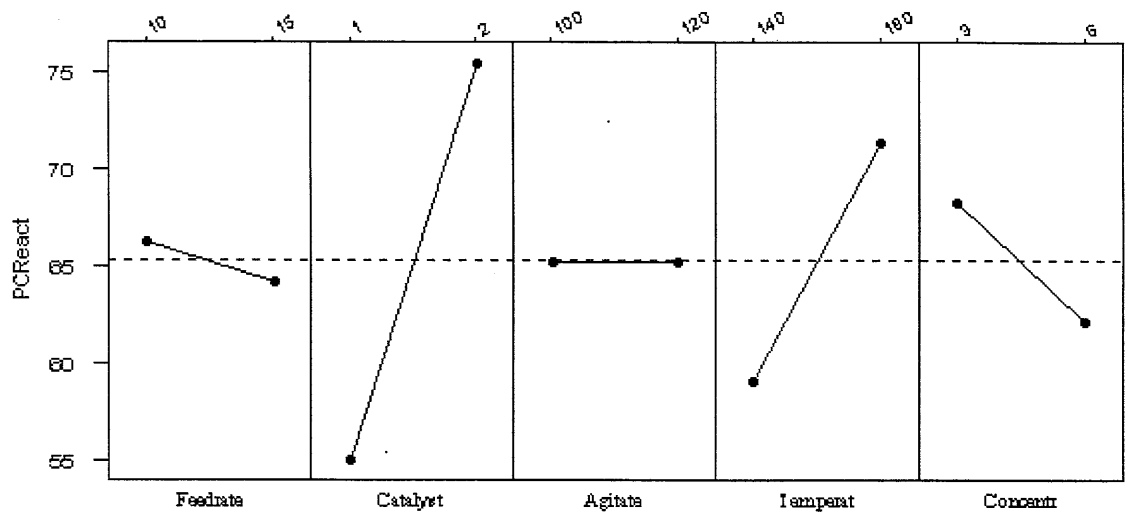
Use the double arrow key, (>>), to select all factors



Click 'OK' to generate graphs

Analyzing a Fractional Design in Minitab

Main Effects for PCReact

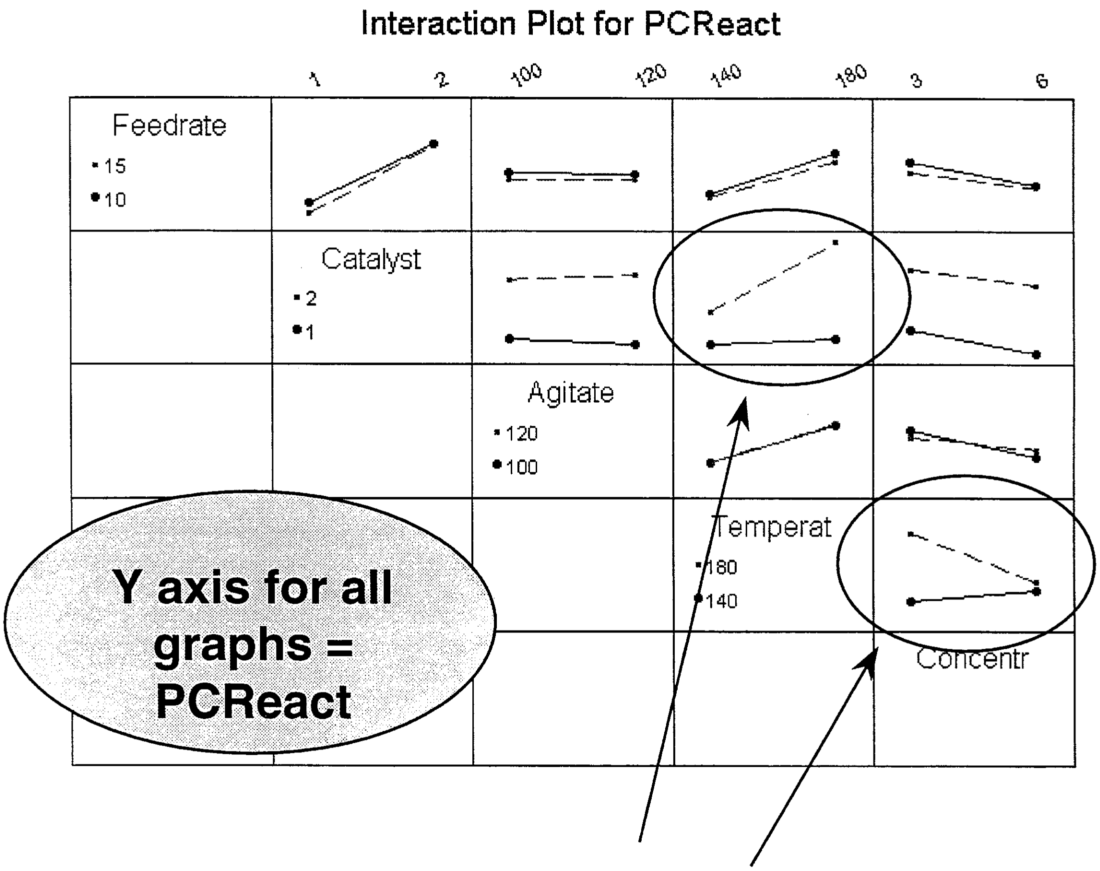


The Main Effects Plot shows:

- A strong influence on PCReact by Catalyst and Temperature, with a minor influence by Concentration.
- Feedrate doesn't have much of an effect.
- Agitation shows no influence on PCReact.
- If agitation has no effect on Y, can the agitator be turned off in this process?

  Why or why not?

Analyzing a Fractional Design in Minitab



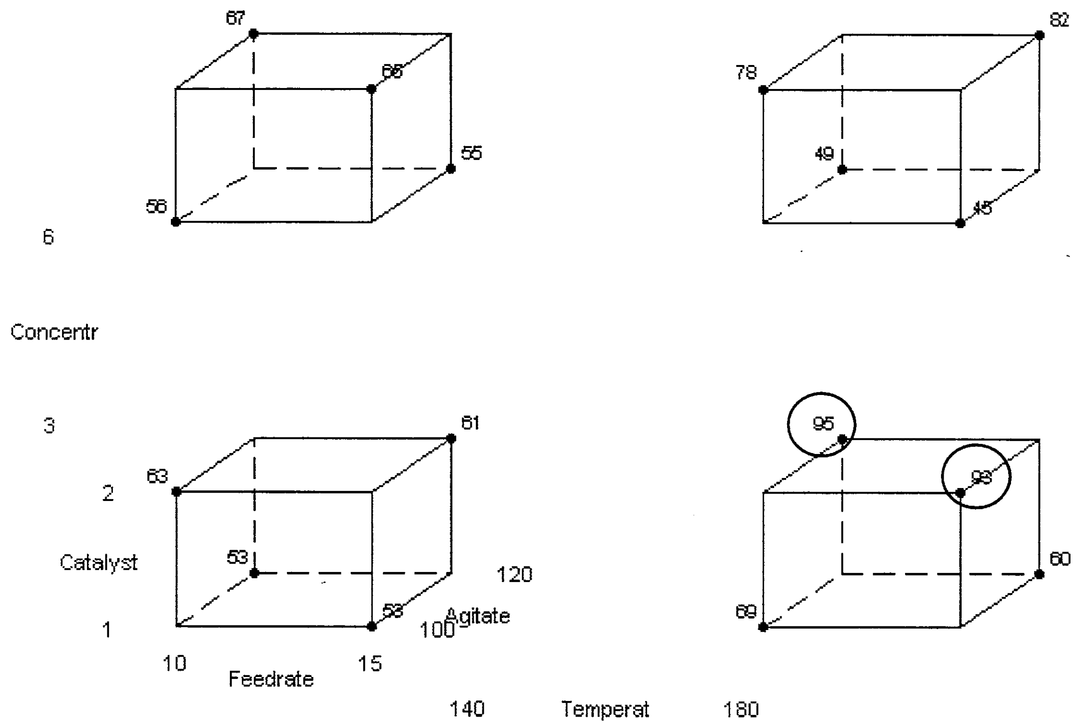
The interaction plot shows significant interaction between 'Temperature and Catalyst' and between 'Concentration and Temperature'.

<u>Interpretation</u>:

- **Temperature has no effect with Catalyst 1, but higher Temperature gives a higher response with Catalyst 2.**
- **Temperature has no effect with Concentration = 6, but higher temperature gives a higher response with Concentration = 3.**

Analyzing a Fractional Design in Minitab

Cube Plot - Means for PCReact

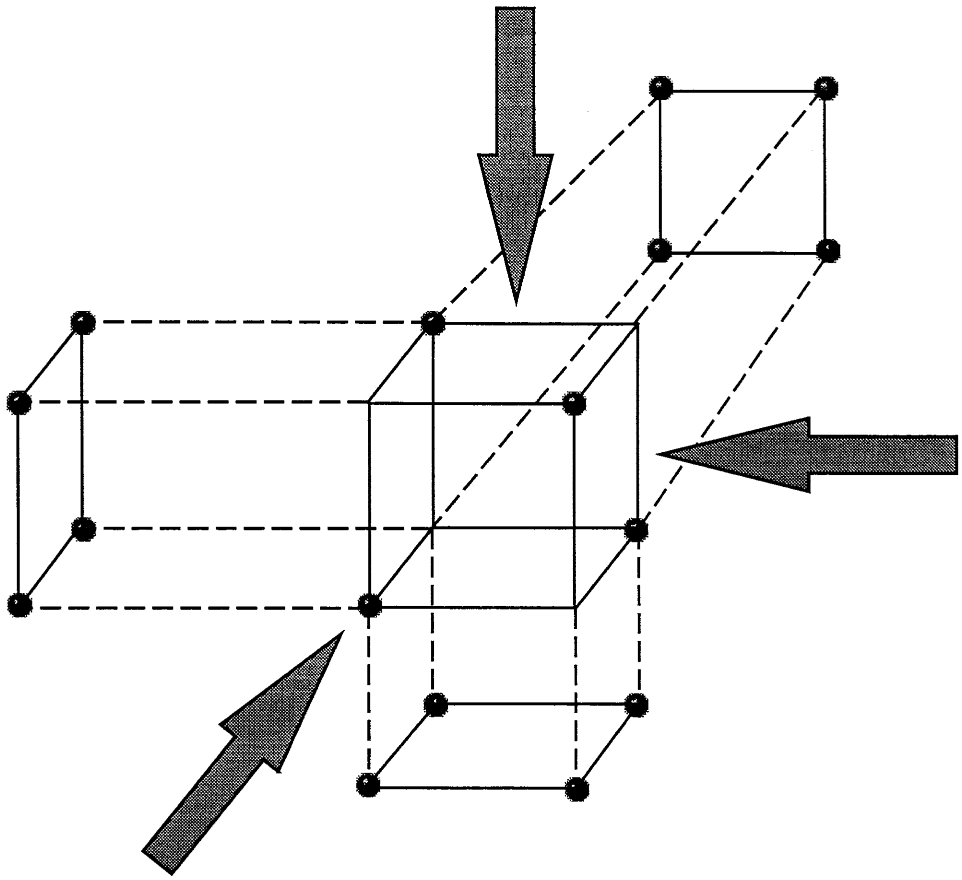


The Cube plot shows the optimum setting (95% and 93% reacted) within the region tested is:

High 'Catalyst' (2%)
High 'Temperature' (180°C)
Low 'Concentration' (3%)

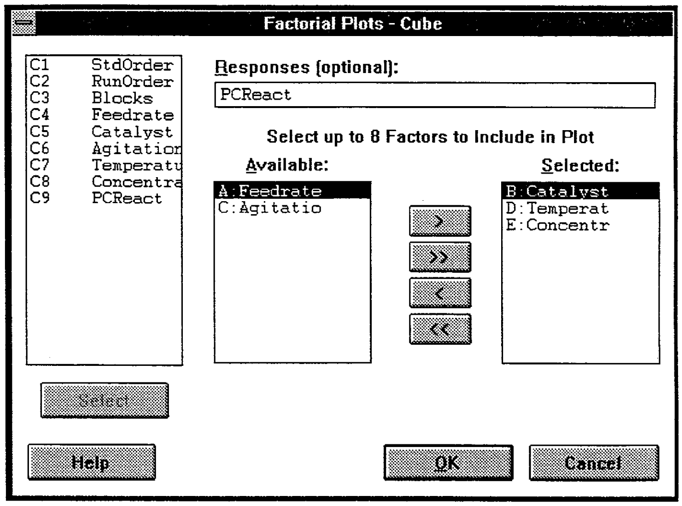
The impact of 'Feedrate' and 'Agitate' is minimal.

Another way to look at the fractional factorial is by looking at a "projection" of the experiment in all 3 directions (X, Y, and Z axes).

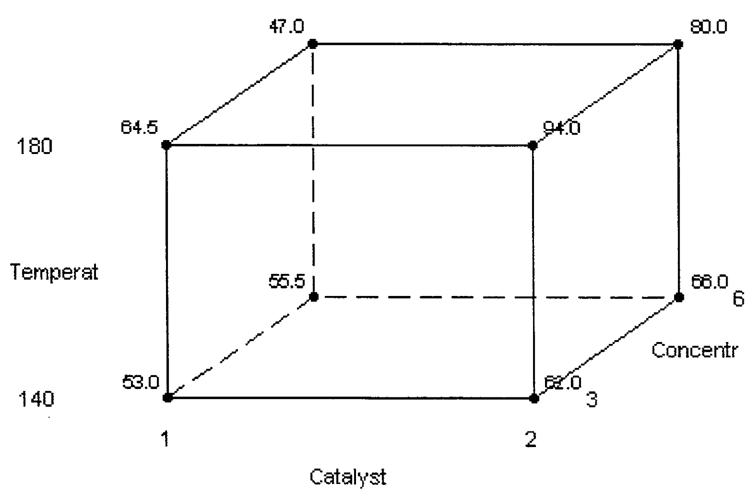


If any of the 3 variables is not important, then the 4 remaining points form a full factorial experiment with the 2 remaining variables.

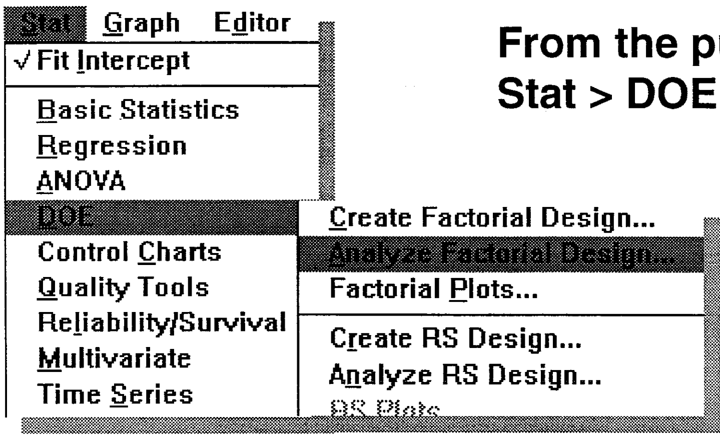
Let's regenerate the Cube plot with only the important variables



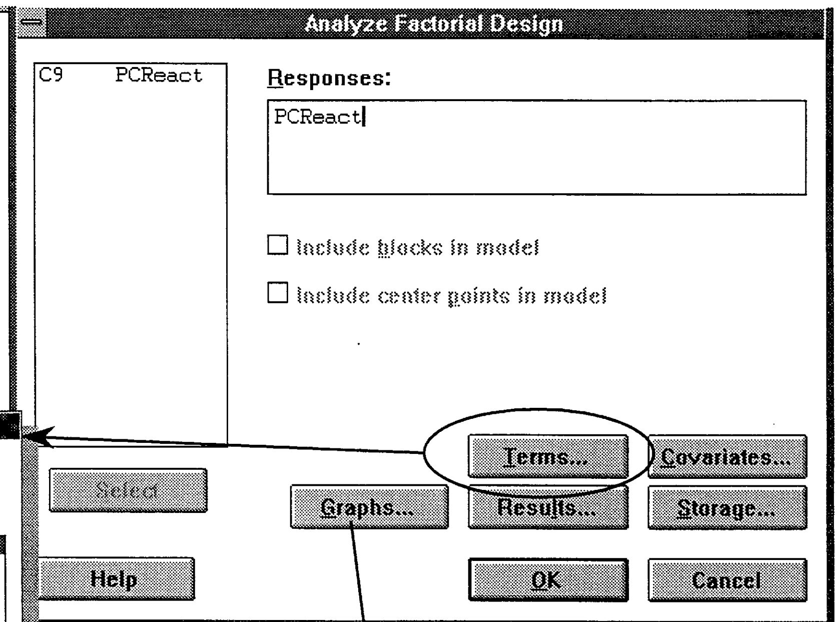
Cube Plot - Means for PCReact



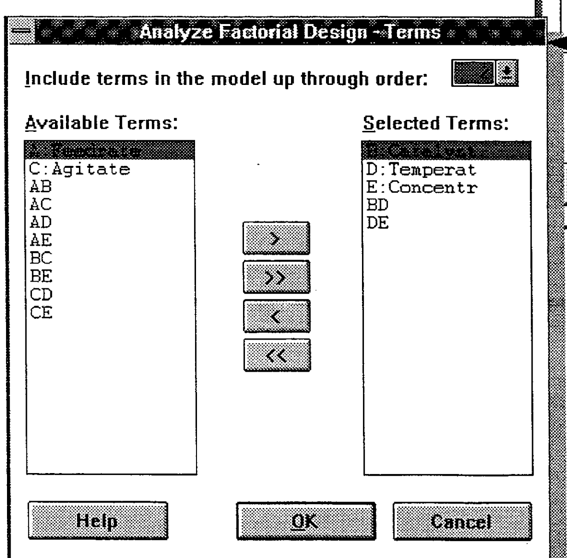
Analyzing a Fractional Design in Minitab for Statistical Significance...



From the pull down menu, select:
Stat > DOE > Analyze Factorial Design



Include only the terms in the model that the factorial plots indicate are the most likely to have a large effect on the response.

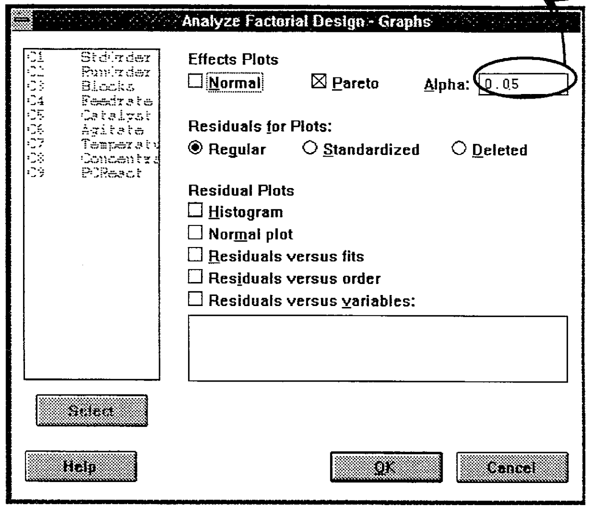


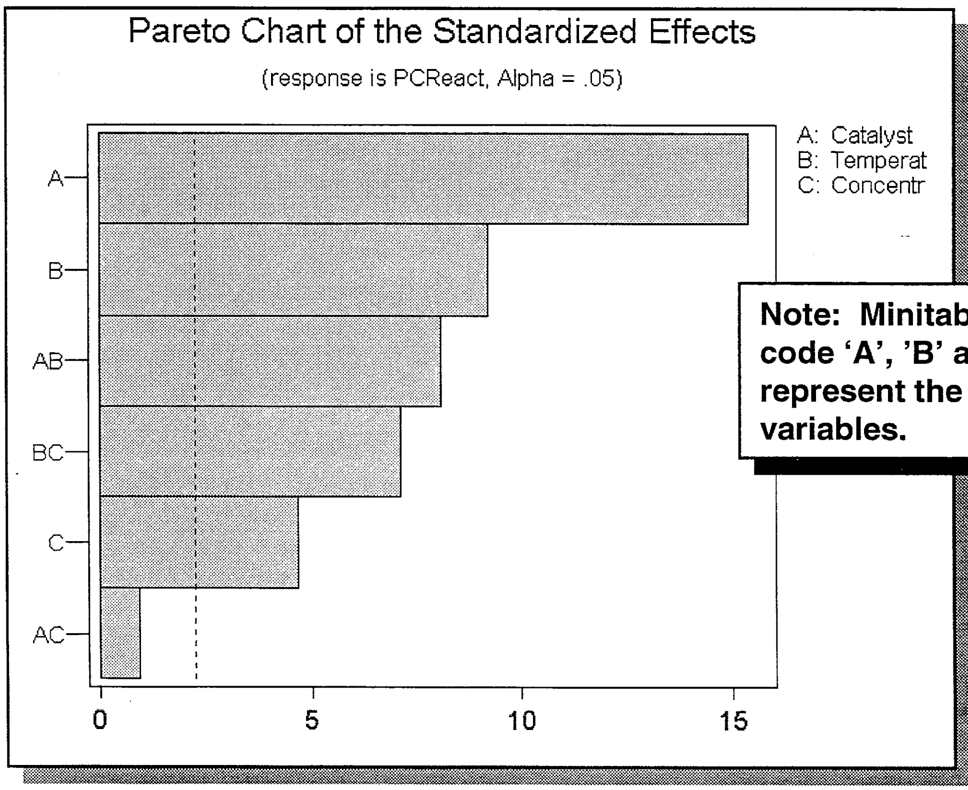
Select 'Graphs . . .', and fill in the dialog box as shown on the next page.

Choosing Graphs in the analyze design dialog box will produce a pareto chart of the model's most significant terms.

A pareto of the ANOVA table's effects is created. All factors extending to the right of the dotted line are significant to $\alpha = 0.05$.

Select Pareto and change α to 0.05





Note: Minitab uses the code 'A', 'B' and 'C' to represent the first 3 variables.

The Pareto graph option makes it easy to see which terms are statistically significant!

Analyzing a Fractional Design in Minitab for Statistical Significance...

Fractional Factorial Fit

Estimated Effects and Coefficients for PCReact

Term	Effect	Coef	StDev Coef	T	P
Constant		65.250	0.6667	97.88	0.000
Catalyst	20.500	10.250	0.6667	15.38	0.000
Temperat	12.250	6.125	0.6667	9.19	0.000
Concentr	-6.250	-3.125	0.6667	-4.69	0.000
Catalyst*Temperat	10.750	5.375	0.6667	8.06	0.000
Catalyst*Concentr	1.250	0.625	0.6667	0.94	0.373
Temperat*Concentr	-9.500	-4.750	0.6667	-7.13	0.000

Analysis of Variance for PCReact

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Main Effects	3	2437.50	2437.50	812.500	114.26	0.000
2-Way Interactions	3	829.50	829.50	276.500	38.88	0.000
Residual Error	9	64.00	64.00	7.111		
Lack of Fit	1	1.00	1.00	1.000	0.13	0.731
Pure Error	8	63.00	63.00	7.875		
Total	15	3331.00				

Unusual Observations for PCReact

Obs	PCReact	Fit	StDev Fit	Residual	St Resid
9	69.0000	64.2500	1.7638	4.7500	2.38R
14	60.0000	64.2500	1.7638	-4.2500	-2.13R

R denotes an observation with a large standardized residual

To get p-values for main effects and interactions, we looked at the factorial plots and pooled the insignificant terms into the error term of the model. Note main effects and specific two-way interactions are statistically significant (p-values < 0.05).

Limitations of Fractional Factorial Experiments - 'Confounding'

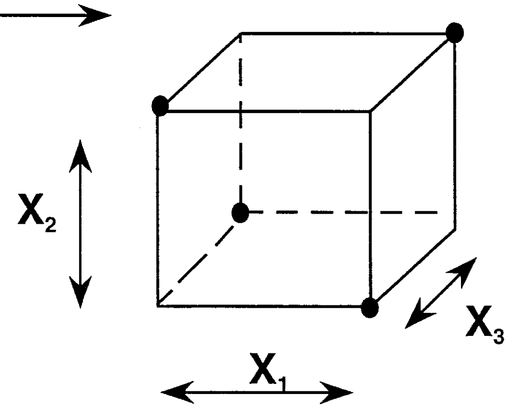
Let's start with a simple example:

Consider a 2^{3-1} fractional factorial with 4 runs. If we wanted to add a third variable, X_3, how could we do it?

Set the X_3 levels at the same settings as the X_1X_2 interaction.

Main Effect Columns					Interaction Columns			Resp
RUN	X_1	X_2	X_1X_2	X_3	X_1X_3	X_2X_3	$X_1X_2X_3$	Y
1	-1	-1	1	1	-1	-1	1	
2	1	-1	-1	-1	-1	1	1	
3	-1	1	-1	-1	1	-1	1	
4	1	1	1	1	1	1	1	

Full Factorial ← → **Interactions**



Note: The X_3 column is equal to the X_1 X_2 column.
the X_1 column is equal to the X_2 X_3 column.
the X_2 column is equal to the X_1 X_3 column.

We estimate the effect of the X_3 by:

X_3 = (Average at the high level) - (Average at the low level)

However, this is also the estimate of the X_1X_2 interaction. These effects are ***CONFOUNDED***, or mixed up. What we call the estimate of the X_3 effect, is actually the estimate of $X_3 + X_1X_2$.

‘Confounding’ (cont’d)

In a 2^{5-1} design, note that:

Main Effects $X_1 = X_2 \times X_3 \times X_4 \times X_5$ **4-way interactions**

$X_2 = X_1 \times X_3 \times X_4 \times X_5$

$X_3 = X_1 \times X_2 \times X_4 \times X_5$

$X_4 = X_1 \times X_2 \times X_3 \times X_5$

$X_5 = X_1 \times X_2 \times X_3 \times X_4$

2-way interactions $X_1 \times X_2 = X_3 \times X_4 \times X_5$ **3-way interactions**

$X_1 \times X_3 = X_2 \times X_4 \times X_5$

$X_1 \times X_2 = X_3 \times X_4 \times X_5$

X_1 is ***confounded*** with $X_2 \times X_3 \times X_4 \times X_5$

$X_1 \times X_2$ is ***confounded*** with $X_3 \times X_4 \times X_5$

'Confounding' (cont'd)

Confounding means 'Mixed up'. We cannot separate the effects of confounded terms.

What we called the estimate of the effect of X_1 was actually the estimate of the effect of X_1, PLUS the effect of $X_2 \times X_3 \times X_4 \times X_5$.

What we called the estimate of the effect of $X_1 \times X_2$ was actually the estimate of the effect of $X_1 \times X_2$, PLUS the effect of $X_3 \times X_4 \times X_5$.

Since higher order interactions are seldom important, and since this is a screening experiment to pick out the most important variables, this loss of information is usually small.

"Alias Structure"

Minitab records the variables which are confounded in the Session window under the heading "Alias Structure". The Alias table lists the factors which are confounded with each other.

For Example:

```
Fractional Factorial Design

Factors:  5   Base Design:    5, 16  Resolution: V
Runs:    16   Replicates:        1   Fraction: 1/2
Blocks: none   Center pts (total):  0

Design Generators:  E = ABCD

Alias Structure

I + ABCDE

A + BCDE
B + ACDE
C + ABDE
D + ABCE
E + ABCD
AB + CDE
AC + BDE
AD + BCE
AE + BCD
BC + ADE
BD + ACE
BE + ACD
CD + ABE
CE + ABD
DE + ABC
```

Resolution number

Shows that the Main effects are confounded with 4-way interactions and the 2-way interactions are confounded with the 3-way interactions.

"Resolution" of a Fractional Factorial

A fractional factorial <u>deliberately</u> confounds factors with interactions. This means that some of the effects of the confounded factors can't be seen as clearly.

The resolution number is an index of the "clarity" of the design - the larger the number, the better the design.

With higher resolution numbers, the Main Effects are confounded only with the higher order interactions (3-way interactions and higher), which are generally less important.

The Notion of Design Resolution

Resolution III:

No main effect is aliased with any other main effect

Main effects are aliased with second order interactions

Second order interactions are aliased with other second order interactions

Resolution IV:

No main effect is aliased with any other main effect

No main effects are aliased with any second order interactions

Second order interactions are aliased with other second order interactions

Resolution V:

No main effect is aliased with any other main effect

No main effect is aliased with any second order interactions

No second order interaction is aliased with any other second order interaction

Second order interactions are aliased with third order interactions

Interpreting the Resolution Index

The Resolution Index helps identify which factors are confounded with which interactions.

First, assign values to the main effects and the various level of interactions as follows:

Main Effects - 1

Two Way Interactions - 2

Three Way Interactions - 3

Four Way Interactions - 4; etc.

Second, subtract the value of the effect you are interested in from the resolution number given in Minitab.

<u>For example</u>, in a resolution "V" design, for Main Effects:

(Resolution) - (Main Effects) = interaction confounded

5 - 1 = 4

Therefore, in a Resolution "V" design, Main effects are confounded with four-way interactions. In that same Resolution V design, Two-way interactions are confounded with three-way interactions.

Resolution "V" designs are desirable, since the loss of information due to confounding is small.

Highly Fractionated Experiments

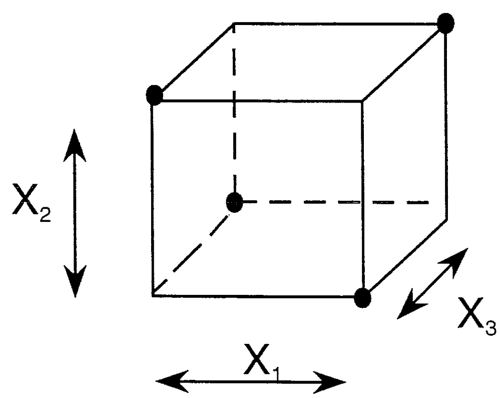
Highly fractionated experiments are experiments where the number of runs is only a little larger than the number of independent variables.

Examples: 2^{3-1}, 4 runs, 3 independent variables
2^{7-4}, 8 runs, 7 independent variables

The highly fractionated experiments are useful ONLY for detecting Main Effects. Caution - You can be misled if interactions are large (but this happens infrequently).

Statistical tests of significance have less meaning, because there is not a good estimate of the error. You typically look for the ***relative magnitude*** of the Main Effects.

Highly fractionated experiments are often used for ***screening***, to find the variables that deserve further study.



OTHER NOTES ON FRACTIONAL Factorial Experiments

- Run confirming experiments on the apparent best set of conditions.
- Remember: you can run the other half-fraction if the results of the first experiment are not clear.
- You can add levels for a more detailed study of the important variables
- 2-level experiments can be used with DISCRETE or CONTINUOUS "X" variables

A Fractional Factorial is a good start for a sequence of experiments because it determines which variables warrant further study (screening).

OK...now you're on your own!

Generate the following design: 7 factors in 16 runs

Look at both the confounding pattern and the design matrix. Use the confounding pattern to determine the resolution.

Note your answers below....you may need to know this in a month or two!

Appendix

SOME FRACTIONAL FACTORIAL TEST PLANS

4 RUNS

Run	1	2	12
1	-	-	+
2	+	-	-
3	-	+	-
4	+	+	+

For a 2^2 full factorial: *Use columns 1 & 2*

For a 2^{3-1} fractional factorial: *Use columns 1, 2, and 12*

Additional Fractional Factorial test plans

8 Runs

Run	1	2	3	12	13	23	123
1	-	-	-	+	+	+	-
2	+	-	-	-	-	+	+
3	-	+	-	-	+	-	+
4	+	+	-	+	-	-	-
5	-	-	+	+	-	-	+
6	+	-	+	-	+	-	-
7	-	+	+	-	-	+	-
8	+	+	+	+	+	+	+

- *For a 2^3 full factorial: Use columns 1, 2, and 3*
- *For a 2^{4-1} fractional factorial: Use columns 1, 2, 3, and 123.*

  Main effects are confounded with 3-factor interactions.

  2-factor interactions are confounded with each other.
- *For a 2^{5-2}, use columns: 1, 2, 3, 12, and 13*
- *For a 2^{6-3}, use columns: 1, 2, 3, 12, 13, and 23*
- *For a 2^{7-4}, use columns: 1, 2, 3, 12, 13, 23, and 123*

The 2^{5-2}, 2^{6-3}, and 2^{7-4} experiments have main effects confounded with 2-factor interactions.

See appendix for additional test plans

Additional Fractional Factorial test plans

16 Runs

Run	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234
1	-	-	-	-	+	+	+	+	+	+	-	-	-	-	+
2	+	-	-	-	-	-	-	+	+	+	+	+	+	-	-
3	-	+	-	-	-	+	+	-	-	+	+	+	-	+	-
4	+	+	-	-	+	-	-	-	-	+	-	-	+	+	+
5	-	-	+	-	+	-	+	-	+	-	+	-	+	+	-
6	+	-	+	-	-	+	-	-	+	-	-	+	-	+	+
7	-	+	+	-	-	-	+	+	-	-	-	+	+	-	+
8	+	+	+	-	+	+	-	+	-	-	+	-	-	-	-
9	-	-	-	+	+	+	-	+	-	-	-	+	+	+	-
10	+	-	-	+	-	-	+	+	-	-	+	-	-	+	+
11	-	+	-	+	-	+	-	-	+	-	+	-	+	-	+
12	+	+	-	+	+	-	+	-	+	-	-	+	-	-	-
13	-	-	+	+	+	-	-	-	-	+	+	+	-	-	+
14	+	-	+	+	-	+	+	-	-	+	-	-	+	-	-
15	-	+	+	+	-	-	-	+	+	+	-	-	-	+	-
16	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

- ***For a 2^4 full factorial, use columns: 1, 2, 3, and 4***
- ***For a 2^{5-1}, use columns: 1, 2, 3, 4, and 1234***

Main effects are confounded with 4-factor interactions. 2-factor interactions are confounded with 3-factor interactions.

- ***For a 2^{6-2}, use columns 1, 2, 3, 4, 1234, and 234***
- ***For a 2^{7-3}, use columns: 1, 2, 3, 4, 123, 234, and 134***
- ***For a 2^{8-4}, use columns: 1, 2, 3, 4, 234, 134, 123, and 124***

The 2^{6-2}, 2^{7-3}, and 2^{8-4} experiments have main effects confounded with 3-factor interactions, and 2-factor interactions confounded with each other.

- ***For a 2^{15-11}, use all 15 columns.***

FRACTIONAL FACTORIAL TEST PLANS

32 RUNS

1	-	-	-	-	-
2	+	-	-	-	-
3	-	+	-	-	-
4	+	+	-	-	-
5	-	-	+	-	-
6	+	-	+	-	-
7	-	+	+	-	-
8	+	+	+	-	-
9	-	-	-	+	-
10	+	-	-	+	-
11	-	+	-	+	-
12	+	+	-	+	-
13	-	-	+	+	-
14	+	-	+	+	-
15	-	+	+	+	-
16	+	+	+	+	-
17	-	-	-	-	+
18	+	-	-	-	+
19	-	+	-	-	+
20	+	+	-	-	+
21	-	-	+	-	+
22	+	-	+	-	+
23	-	+	+	-	+
24	+	+	+	-	+
25	-	-	-	+	+
26	+	-	-	+	+
27	-	+	-	+	+
28	+	+	-	+	+
29	-	-	+	+	+
30	+	-	+	+	+
31	-	+	+	+	+
32	+	+	+	+	+

FRACTIONAL FACTORIAL TEST PLANS

32 RUNS (cont'd)

DESIGN	*COLUMNS*	
2^5:	1, 2, 3, 4, 5	Resolution VI
2^{6-1}:	1, 2, 3, 4, 5, 12345	Resolution VI
2^{7-2}:	1, 2, 3, 4, 5, 1234, 1245	Resolution IV
2^{8-3}:	1, 2, 3, 4, 5, 123, 124, 2345	Resolution IV
2^{9-4}:	1, 2, 3, 4, 5, 2345, 1345, 1245, 1235	Resolution IV
2^{10-5}:	1, 2, 3, 4, 5, 1234, 1235, 1245, 1345, 2345	Resolution IV
2^{11-6}:	1, 2, 3, 4, 5, 123, 234, 345, 134, 145, 245	Resolution IV

To construct a 2^{K-1} Fractional Factorial Design with the highest possible resolution:

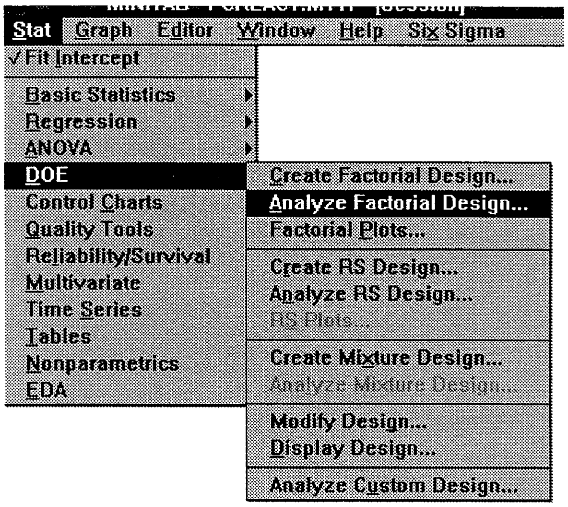
1. **Write a full factorial design for the first K-1 variables.**
2. **Associate the Kth variable with "+" or "-" the interaction column 123...(K-1).**

extras

Pareto Chart of Effects

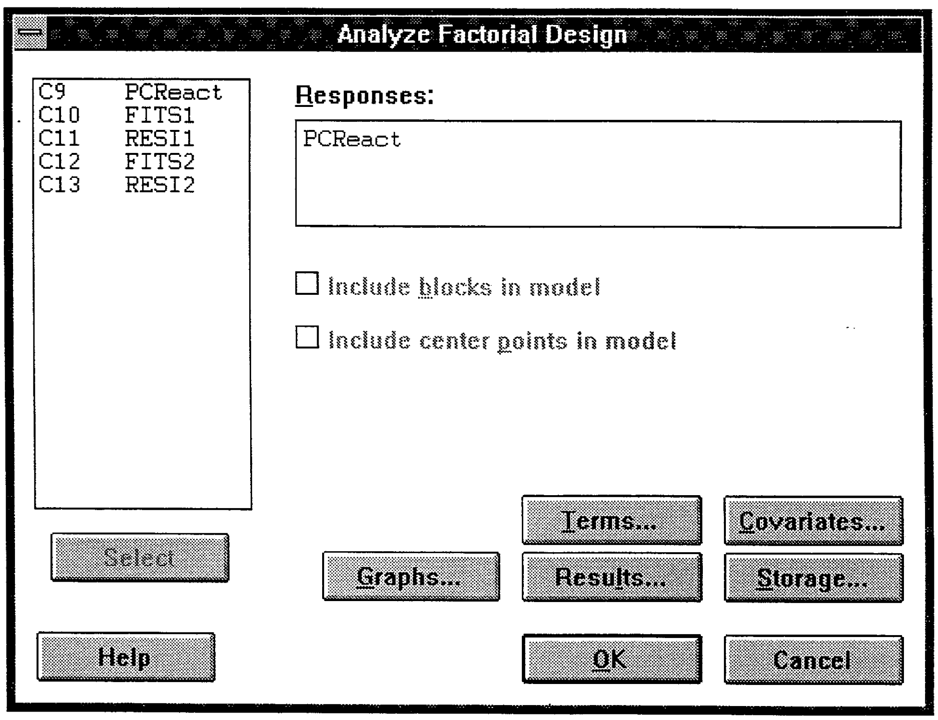
Another way to view the effects is through a Pareto chart. To create a Pareto chart:

Stat>DOE>Analyze Factorial Design...

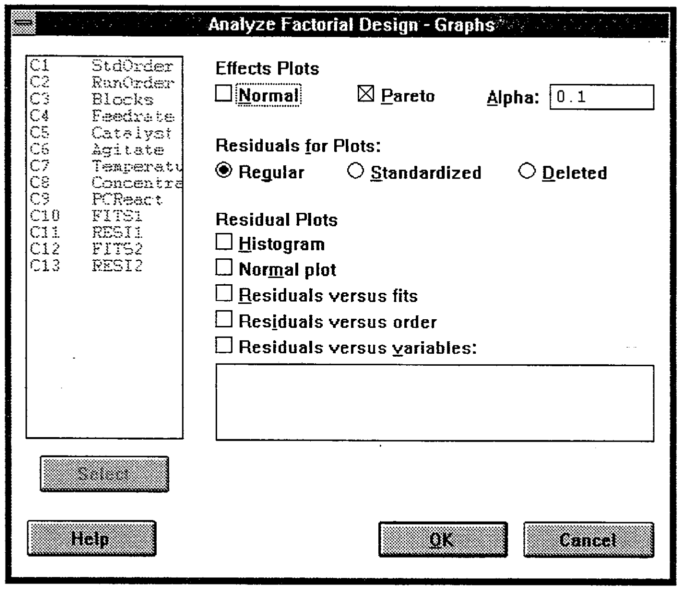


Enter 'PCReact' into Response

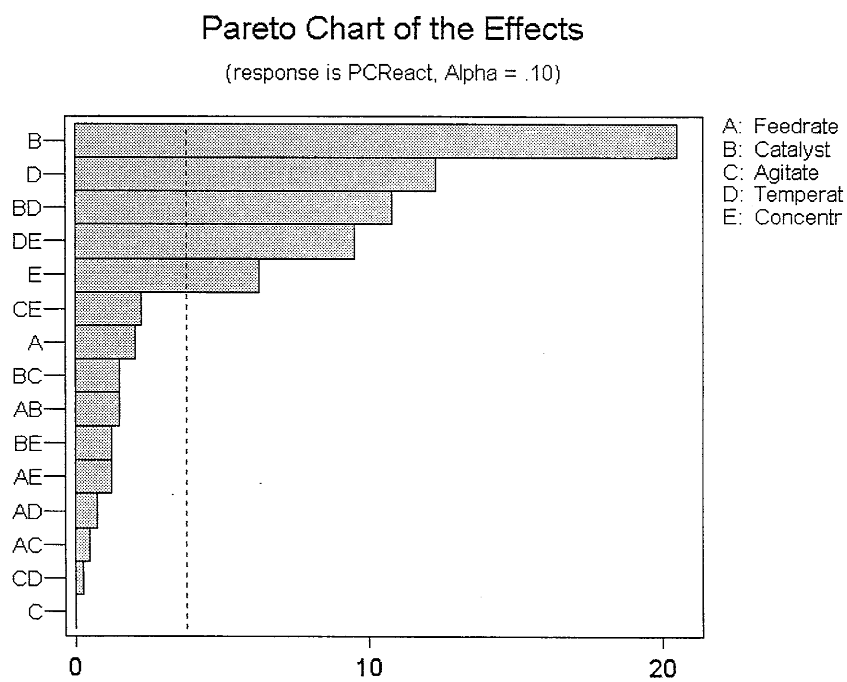
Click on 'Graphs...'



Click on 'Pareto'



Click on 'OK' twice to generate Pareto chart of effects



The Pareto in this example indicates that 'Catalyst', 'Temperature', 'Catalyst x Temperature', 'Temperature x Concentration' and 'Concentration' are vital effects.