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A Review of Response Surface Methodology: A Literature Survey*

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Response surface methodology, an experimental strategy initially developed and described by Box and Wilson, has been employed with considerable success in a wide variety of situations, especially in the fields of chemistry and chemical engineering. It is the purpose of this paper to review the literature of response surface methodology, emphasizing especially the practical applications of the method. A comprehensive bibliography is included.

INTRODUCTION

Response surface methodology (RSM) was initially developed and described by Box and Wilson (1951). The fundamentals of RSM and its underlying philosophy are discussed in many papers and a number of textbooks. The most comprehensive discussion is that given in a book written by a group of chemists, engineers, and statisticians, and edited by Davies (1954). Anyone beginning a study of RSM is strongly encouraged to read this account. Box (1954a) emphasizes the fundamentals of RSM and illustrates them by means of three different chemical examples. Bradley (1958) and Hunter (1958, 1959a, 1959b) offer a most readable and educational series of articles. Furthermore, there are articles by Box (1960, 1964) and Box and Hunter (1958). In the second edition of their book, Cochran and Cox (1957) have included a discussion of RSM. A number of expository articles have also appeared; Andersen (1959), Baasel (1965), Berg (1960), Carr and McCracken (1960), Grohskopf (1960), Hunter (1954, 1956, 1960), Koehler (1960), Li (1958), Neuwirth and Naphtali (1957), and Read (1954).

This review article is divided into eight sections, the first three dealing with the basic principles and certain theoretical aspects of RSM and the remaining five dealing with practical application.

THEORY

1. *Postulation of a Mathematical Model*

An experimenter is about to study a system for which there is a mathematical equation relating the expected value of a response $\eta = E(y)$ to the experimental

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variables $x_1, x_2, \cdots, x_k$,

$$\eta = f(\theta_1, \theta_2, \cdots, \theta_p; x_1, x_2, \cdots, x_k), \tag{1}$$

where $\theta_1, \theta_2, \cdots, \theta_p$ are the parameters of the system. For example, in a chemical reaction the response could be the yield; the variables might include temperature, pressure, and pH; and the parameters might include the reaction rate constant and the heat tranfer coefficient. Of course, for each experimental run, more than one response may be measured; for example, cost may be measured in addition to yield. Such cases will be discussed later, in particular in Section 5.

It is likely, especially in complex situations, that the *exact* form of the response function f in Equation 1 will be unknown. In fact, a good case can be made for the claim that it is never known exactly. In any event, it is true for a vast majority of industrial processes that complete theoretical mathematical models are not available. Furthermore, in many circumstances, any attempt to develop one could not be justified from an economic point of view. It would be interesting perhaps to understand the mechanism of the process but of more immediate concern are questions such as: what operating conditions should be maintained so that a maximum yield is achieved? In development work the main question might well be: Are there *any* settings of the variables that will give a product satisfying all desirable specifications? RSM has been used primarily to answer questions such as these.

Fortunately, for many purposes, consideration of the possible forms of the true function f is unnecessary. A flexible graduating function g (for example, a polynomial) will often be satisfactory to express the relationship between the response η and the k important variables $x_1, x_2, \cdots, x_k$. In other words, g is an adequate approximation of f over the region of experimentation. The two most common forms of g are the first-order polynomial

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k \tag{2}$$

and the second-order polynomial

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \cdots + \beta_{k-1,k} x_{k-1} x_k + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \cdots + \beta_{kk} x_k^2 . \tag{3}$$

The coefficients $\beta_0, \beta_1, \beta_2, \cdots$ are parameters to be estimated from the data. If transformations of the x's and the y's are considered (see Box and Tidwell (1962) and Box and Cox (1964)), the flexibility of such first-order and second-order models is increased substantially. For $k = 2$ experimental variables, these general polynomials reduce to

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \tag{4}$$

and

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 . \tag{5}$$

Strictly speaking, Equations 2, 3, 4 and 5 should not be written as equalities but rather as approximations. But it is usually assumed that the approximation is so close that lack of fit will remain undetected with a normal amount of experimentation so that for practical purposes it is reasonable to write them as equalities and this is customarily done.

Although polynomials have been most commonly used, other graduating functions which similarly act as mathematical French curves can be employed. Graduating functions which are not based directly on an understanding of the underlying mechanism are sometimes referred to as *empirical* models to distinguish them from *theoretical* models, which are developed from a more fundamental approach. Between these two extremes there exist models which combine elements of both empiricism and theory. Actually there is an entire spectrum of models from the purely empirical on the one hand to the purely theoretical on the other.

Models all along this spectrum can be used in response surface studies. As long as the models are not too cumbersome, those which incorporate as much of the currently available theoretical knowledge of the system are preferable to those which ignore this information. All other things being comparable a simple model is better than a complicated one. Consequently, in constructing a mathematical model an experimenter should try to blend in as much theoretical knowledge as possible while maintaining a desirable level of simplicity.

It is of utmost importance that the experimenter be able to appreciate the results of an RSM study. This point cannot be overemphasized. There have been cases where rather elaborate mathematical models have been proposed, the appropriate data collected, the models fitted, and the results presented in the form of mathematical equations, only to have the practical implications of these results remain unappreciated and therefore unexploited.

Considerable thought, therefore, should be given to the problem of how to most effectively present the results so that the experimenter concerned can appreciate their significance. Graphical summaries such as contour plots have proven highly effective for this purpose. Also, a canonical analysis, which is discussed in Section 6, can often provide valuable insight into the nature of the response surface. The ultimate problem is not one of mathematics but of communication.

2. *Selection of an Experimental Design*

The principal papers dealing with experimental designs for RSM are Box and Wilson (1951), which introduced central composite designs; Box and Hunter (1957), which introduced the concept of rotatability; and Box and Draper (1959), which introduced the bias criterion for design purposes. In addition, the following are important references. Note that designs which allow estimation of the parameters in first-order models (see Equation 2) are called first-order designs and designs which allow estimation of the parameters in second-order models (see Equation 3) are called second-order designs, etc.

First-Order Designs

Box and Hunter (1961a, 1961b) are concerned with factorial and fractional factorial designs, and ideas contained in these papers are often useful at the outset of an investigation. First-order designs which are appropriate when time trends exist are considered by Box (1952) and Hill (1960).

Second-Order Designs

In addition to the principal papers mentioned above — Box and Wilson (1951), Box and Hunter (1957), and Box and Draper (1959) — second-order designs are also discussed in the following papers: Bose and Carter (1959), Bose and Draper (1959), Box and Behnken (1960a, 1960b), Box and Draper (1959, 1963), Das and Narasimham (1962), and Das (1963), DeBaun (1959), Draper (1960a), Dykstra (1959, 1960), and Hartley (1959).

Third-Order Designs

Third-order designs are discussed in the following papers: Draper (1960b, 1960c, 1961b, 1962), Gardiner, Grandage, and Hader (1959), and Herzberg (1964). Third-order models have been employed in practice (see, for example, Pike et al (1954)) when second-order models were found to be inadequate. This is not to say, however, that one would automatically use a higher-order model in such circumstances, that is, when a given model has been found to be inadequate. This procedure may involve an unreasonably large increase in the number of parameters; as an alternative, a simple transformation analysis involving fewer parameters may suffice (see, in particular, Box and Tidwell (1962) and Box and Cox (1964)).

Blocking

Box and Hunter (1957, 1961a, 1961b) develop appropriate design criteria for the elimination of extraneous factors (block effects) and describe the corresponding analyses. DeBaun (1956) discusses the influence of block effects in the determination of optimum conditions. Box (1959) discusses replication and blocking with respect to central composite designs.

3. *Analysis of the Data*

A basic description of RSM analysis is contained in Davies (1954). Let us briefly summarize the underlying theory. If a graduating function which is linear in the parameters is used (for example, a polynomial), the mathematical model can be written in matrix form simply as

$$\eta = \mathbf{x}\boldsymbol{\beta}, \tag{6}$$

where η denotes the true response value, $\mathbf{x}$ the $1 \times r$ vector of independent variables, and $\boldsymbol{\beta}$ the $r \times 1$ vector of unknown parameter values to be estimated. That is, there are r parameters to be estimated. (Equation 6 is the matrix generalization of equations such as Equation 2, where $r = k + 1$, and Equation 3, where $r = (k + 1)(k + 2)/2$. Thus, $r = 3$ for Equation 4 and $r = 6$ for Equation 5.) Let $\mathbf{y}$ be the $N \times 1$ vector of observations and $\mathbf{X}$ the $N \times r$

matrix of independent variables. If the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular, least squares estimates $\hat{\boldsymbol{\beta}}$ are readily obtained from the equation

$$\hat{\boldsymbol{\beta}} = [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{y}. \tag{7}$$

It is then possible to construct a response surface of the predicted value

$$\hat{y} = \mathbf{x}\hat{\boldsymbol{\beta}} \tag{8}$$

as a function of the variables $\mathbf{x}$.

This surface in the form of contours (loci of constant $\hat{y}$ values) can be studied visually to gain an appreciation of the relationship between the variables and the response. Three-dimensional devices for the representation of response surfaces have been described by Box (1954a) and by Lind and Young (1965). If more than one response is being followed, a surface can be constructed for each response and these surfaces can be studied in juxtaposition to one another. This can often best be done by means of superimposing the various contour plots. But it should be remembered that these contour plots are, of course, not exact but only estimated representations of the true surface. Allowance for this fact should be made when evaluating the results.

A method using Lagrangian multipliers for optimizing a response subject to constraints is suggested in Umland and Smith (1959) and Chow (1962). Hoerl (1959, 1960) and Draper (1963) have discussed ridge analysis as a technique for the study of response surfaces.

When maximization (minimization) of a response is an important part of an experimental program, the method of steepest ascent (descent) is a useful procedure. As Box and Wilson (1951) pointed out in their original paper, like most other experimental procedures, this method is not scale-invariant. It finds the path in any units of measurement preferred by the experimenter, and it has been found valuable in practical applications when searching for optimum conditions. Box (1954b) derived an expression for the confidence cone about the direction of steepest ascent. Brooks and Mickey (1961) explored some theoretical considerations of the best strategy in steepest ascent experiments. (The method of steepest descent in conjunction with the principles of RSM has been applied by Box and Coutie (1956) for nonlinear estimation in a setting rather different from those previously described. Instead of examining the surface of $\hat{y}$ in the $\mathbf{x}$-space, they explore the sum of squares surface in the $\boldsymbol{\theta}$-space, where $\boldsymbol{\theta}$ is the set of parameters to be estimated.)

As the region of the optimum is approached it is often necessary to extend a first-order polynomial to a second-order polynomial that will better account for curvature in the response surface. This transition may be prompted by a significant lack of fit in the first-order model. Lack of fit in RSM studies has recently been explored by Wetz (1964). Methods for augmenting first-order designs are discussed in papers mentioned in Section 2.

Box and Hunter (1954) and Wallace (1958) have studied confidence regions for stationary points of fitted second-order surfaces.

If there are more than three independent variables involved in the analysis, it is often difficult to gain an understanding of the surface by either an inspection of Equation 8 or by a visual study of the contour surface. A useful aid in this situation is canonical analysis as described in Box and Wilson (1951) and Davies (1954). Van der Vaart (1960, 1961) has discussed certain theoretical aspects of canonical analysis.

Draper (1961a) describes analysis procedures in RSM when there are missing values.

The previous three sections (postulation of a mathematical model, selection of an experimental design, and analysis of the data) serve as a skeletal outline of RSM. It must be remembered, however, that experimentation is in general an iterative process, and these three steps executed only once will rarely suffice in practice. Very possibly the analysis of an initial set of data will suggest the need for either further experimentation or a modification of the current model or both. Thus a cycle is initiated which is repeated as often as is necessary to reach a satisfactory conclusion.

Applications

A typical RSM study begins with a definition of the problem (which responses is to be measured, how it is to be measured, which variables are to be studied, over what ranges they are to be explored, etc.) and includes, in particular, (1) performing statistically designed experiments, (2) estimating the coefficients in the response surface equation, (3) checking on the adequacy of the equation, and (4) studying the response surface in the region of interest. Many experimenters, however, have gone beyond these four steps. For instance, using RSM, some have studied multiple-response problems while others have attempted to elucidate the underlying physical mechanisms of systems.

4. *Three Selected Applications*

The imaginative use of RSM has brought dividends in a wide variety of fields. To illustrate this point, three successful applications are presented in this section, the fields represented being tool-life testing, chemistry, and foodstuffs. In each case the principles of RSM were utilized effectively and as a consequence useful information was obtained.

A Tool-life Application

The first of these applications concerns research on tool-life testing by Wu (1964a, 1964b). For about sixty years experimenters had been plagued by the problem of how to collect data on tool life and subsequently estimate the parameters a, b, c, and K in the tool-life equation

$$T = KV^a f^b d^c, \tag{9}$$

where the variables are the cutting speed V, the feed f and the depth of cut d, and the response is the tool life T. A typical tool-life testing investigation frequently included a large number of undesigned experimental trials the data from which were analyzed by inefficient methods. The situation was clearly unsatisfactory. By the proper use of RSM principles efficiency was brought to both aspects of the problem, data generation and data analysis.

A logarithmic transformation reduces Equation 9 to a first-order model linear in the parameters β_0, β_1, β_2, and β_3,

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 . \tag{10}$$

The variables x_1, x_2, and x_3 are coded values of the original variables V, f, and d. In terms of these x's a 2^3-factorial design augmented with four center points was constructed. From the resulting data it was possible to calculate estimates for the parameters $\boldsymbol{\beta}$, estimate the experimental error, test for lack of fit, and illustrate the results in terms of a contour diagram which is presented in Figure 1. Furthermore, a comparison between the tool life and a second response, rate of metal removal, was made by plotting contour surfaces for both these responses in one picture. In summary, RSM was utilized to clarify in only twelve runs a tool-life testing problem which, had conventional methods been used, could not have been handled so satisfactorily even if many more runs had been made.

A Chemical Application

The second selected application of RSM is in the field of chemistry. Tidwell (1960) describes how improvement was achieved in the processing of a product D in the reaction

$$A + B + C \rightarrow D + B + C + \text{other products.}$$

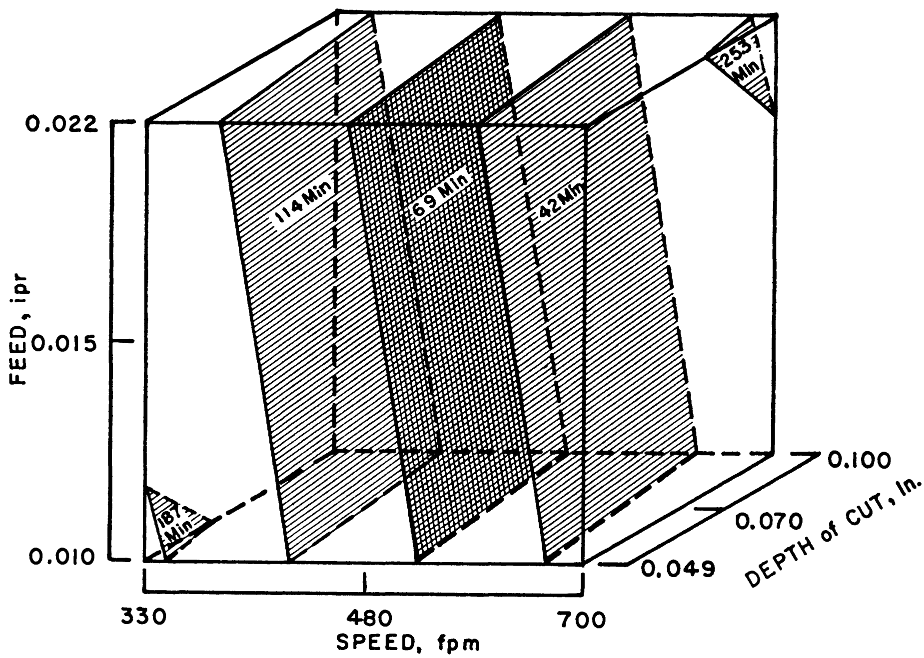

FIGURE 1.—Tool-life response surface contour planes for $\hat{y} = 4.231 - 0.846x_1 - 0.164x_2 - 0.089x_3$, where $\hat{y}$ = predicted tool life, x_1 = speed, x_2 = feed, and x_3 = depth of cut. (Reproduced from Wu (1964a) with permission of the author and the published, The American Society of Mechanical Engineers.)

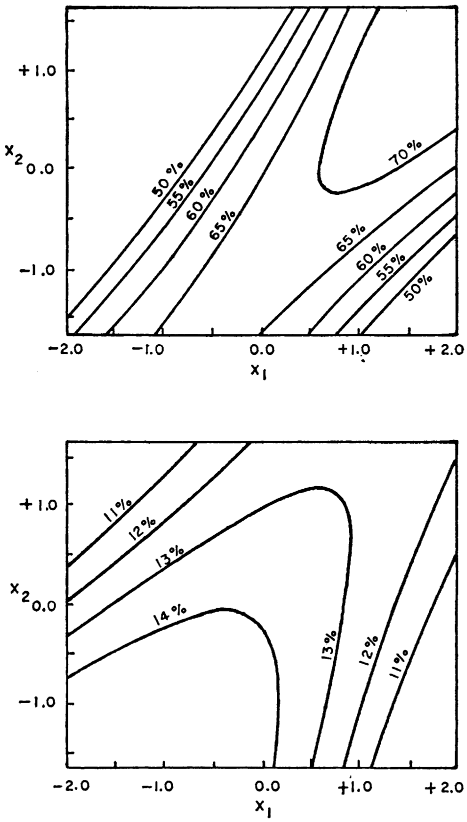

FIGURE 2.—Effects of $x_1(\log B/C)$ and $x_2(\log A/(B + C))$ on yield (upper) and % D, (lower) in the product stream for the reaction $A + B + C \rightarrow D + B + C +$ other products. (Reproduced from Tidwell (1960) with the permission of the author and the publisher, The American Chemical Society.)

A composite design was employed and the analysis revealed (1) the presence of rising ridge systems (see Figure 2) for both total yield and the yield of D, (2) significant interactions among the variables, (3) a decrease in D with an increase in over all product yield, and (4) the advantage of one catalyst over another.

A Foodstuffs Application

A third noteworthy application concerns foodstuffs. Smith and Rose (1963) applied RSM to the improvement of a pie crust. Choosing water, flour and shortening as their independent variables, they used a replicated central composite design to study two qualitative responses (flakiness and gumminess) and one quantitative response (specific volume). By fitting second-order models for all three responses and by superimposing the resulting contours, they were able to determine the best settings of the variables.

These three examples have been singled out because of their sound application of RSM principles as well as their diversified areas of interest. Many other articles on applications of RSM have appeared and these are included in the following discussion where papers are in general grouped according to their outstanding characteristics.

5. *Multiple Response Applications*

The three applications just discussed–Wu (1964a), Tidwell (1960), and Smith and Rose (1963)–each consider more than one response. Many other papers also deal with multiple-response situations where, for instance, the cost response as well as the yield response is studied.

Lind, Goldin, and Hickman (1960) studied the effect of complexing agents on the yield of a certain antibiotic. Fitting second-order models for both cost and yield responses and superimposing the corresponding contour diagrams (see Figure 2), they sucessfully found new operating conditions that increased yield by 5% and reduced cost by $5 per kilogram of product. Using a similar

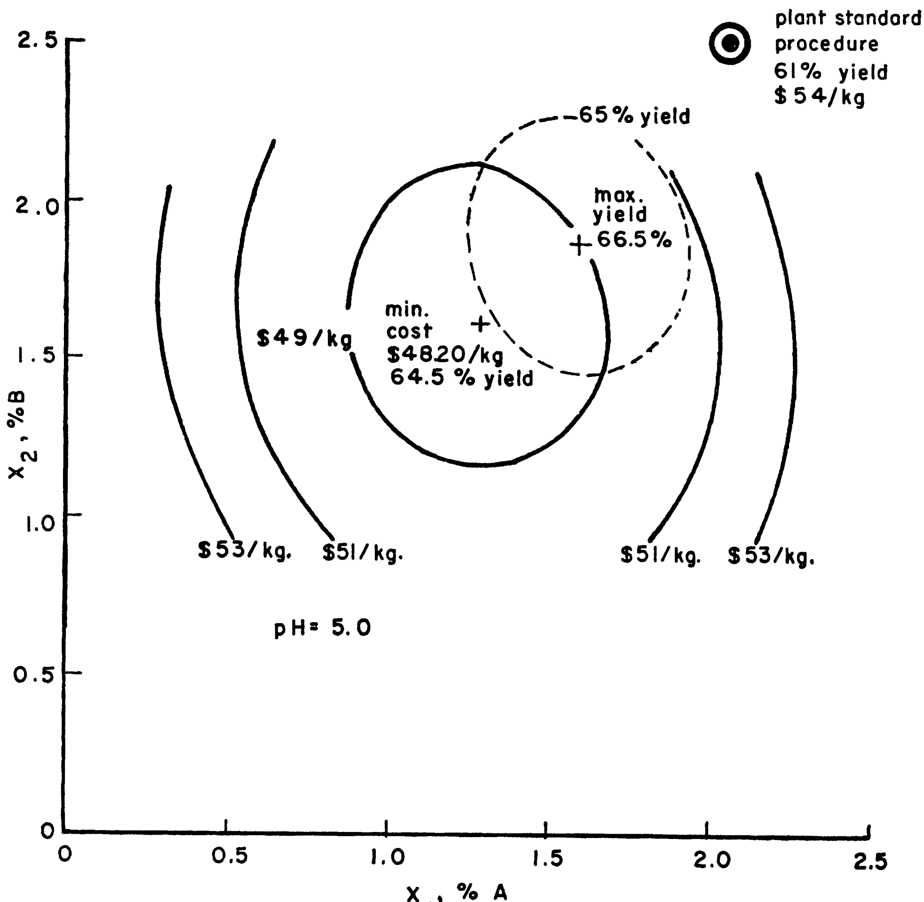

FIGURE 3.—Cost contours and yield contours as a function of two complexing agents A and B for a certain antibiotic. (Reproduced from Lind, Goldin, and Hickman (1960) with permission of the authors and the publisher, The American Institute of Chemical Engineers.)

analysis, Remmers and Dunn (1961), for the initial stages, used the method of steepest ascent to guide them to better operating conditions and, at the final stage, superimposed the resulting contours to find the best compromise conditions for both the cost and yield responses of a fermentation process involving Aspergillus Niger NRRL 337. In a paper by Weissert and Cundiff (1963) six responses were compared where the effect of five variables on the manufacture of rubber for truck tires was studied. Sixty contour diagrams of different combinations of two-variable spaces were compared in an attempt to arrive at the best compromise conditions.

Important applications of RSM have occurred in the dyestuff industry. Gaido and Terhune (1961) studied the effect of four variables on three responses (whiteness, fluidity, and absorbency) in the single-stage pressure kier bleaching of cotton. In a second paper, Terhune (1963) studied the effect of four variables on reflectance and fluidity in a bleaching operation. The contours presented in this paper very effectively summarize the main features of the system (for example, see Figure 3). This paper offers an excellent example of the value of contour diagrams for presenting results in a readily appreciated manner. Also concerned with dyestuffs, Ferrante (1962) studied the effect of four variables on tensile strength, crease recovery angles, and whiteness of the cotton fabric.

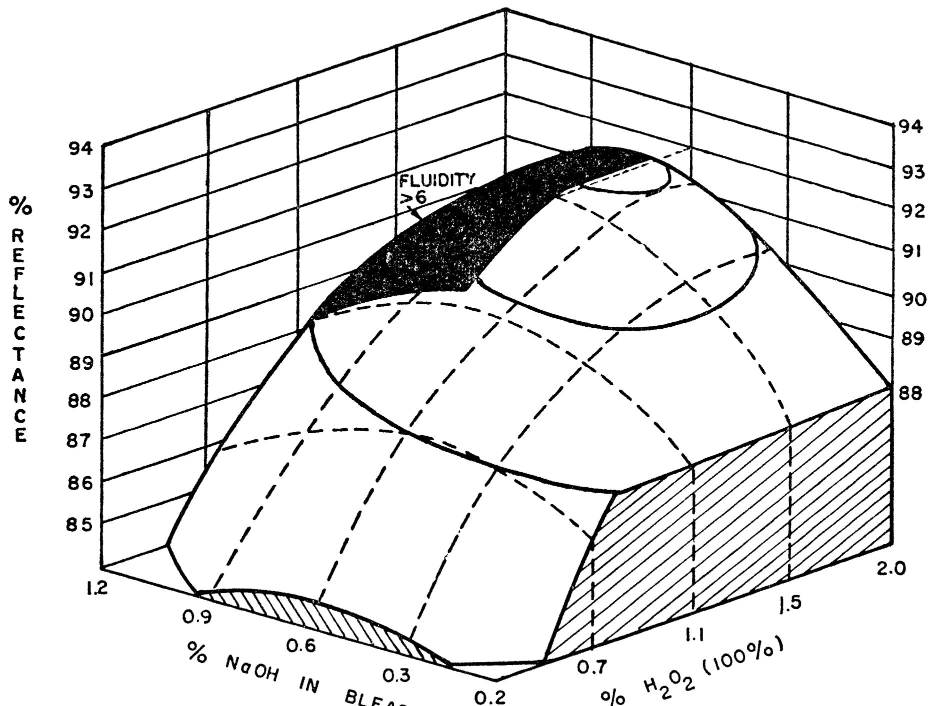

FIGURE 4.—Effect of H_2O_2 and NaOH concentration in bleach on reflectance and fluidity. (Reproduced from Terhune (1963) with permission of the author, the publisher, Howes Publishing Co., Inc., and the copyright holder, The American Association of Textile Chemists and Colorists.)

Other papers where multiple responses were studied include those by Bolker (1965) in an investigation on the delignification by nitrogen compounds; Swanson, Naffzigor, Russel, Hofreiter and Rist (1964), on the xanthation of starch by a continuous process where contours were drawn on triangular coordinates; Ellis, Jeffreys and Wharton (1964), on the production of chloramine from the Raschig synthesis of hydrazine; Norton and Moss (1963, 1964), on the oxidation of 1-methyl-naphthalene to naphthalene; Novak, Lynn, and Harrington (1962), on process scale-up of a polymerization reaction; Underwood (1962), on the designing of extrusion screws; Aia, Goldsmith, and Mooney (1961), on the precipitation of $CaHPO_4 \cdot 2H_2O$; and Dechman and Van Winkle (1959), on the performance of a perforated plate column. The paper by Aia, Goldsmith, and Mooney is one of the very few papers on the application of RSM which includes a canonical analysis. The next section includes a further mention of this paper.

6. *Applications of Canonical Analysis*

The main reason for performing a canonical analysis is to gain insight into the nature of the response surface, to discover for example whether one has simply a true maximum or minimum or, on the contrary, a minimax or a rising ridge. A complete account of this topic is to be found in Davies (1954).

Chang, Kononenko, and Franklin (1960) performed a canonical reduction. They studied the yield surface of 2, 5-dimethyl-piperazine and obtained the response equation

$$\begin{aligned}\hat{y} = 39.846 &- 1.511x_1 + 1.284x_2 - 8.739x_3 + 4.955x_4 - 6.332x_1^2 \\ &- 4.391x_2^2 + 0.021x_3^2 - 2.505x_4^2 + 2.194x_1x_2 - 0.144x_1x_3 \\ &+ 1.581x_1x_4 + 8.006x_2x_3 + 2.806x_2x_4 + 0.294x_3x_4 \qquad (11)\end{aligned}$$

where, in coded units,

x_1 = concentration of NH_3 ,
x_2 = temperature,
x_3 = amount of water,
x_4 = hydrogen pressure,

which, in canonical form, is

$$\hat{y} - 43.07 = -7.59X_1^2 - 6.03X_2^2 - 2.16X_3^2 + 2.58X_4^2 \qquad (12)$$

This equation represents a minimax surface where a decrease in yield is predicted when one moves away from the center of the system in either the positive or negative directions of X_1 , X_2 , and X_3 , and correspondingly an increase in yield is predicted in either the positive or negative directions of X_4 .

Aia, Goldsmith, and Mooney (1961) performed a canonical analysis in their study of the precipitation of $CaHPO_4 \cdot 2H_2O$ and reduced the response equation

$$\hat{y} = 76.39 + 5.49x_1 + 10.18x_3 + 1.46x_1x_3 + .64x_1^2 - 7.22x_3^2 \qquad (13)$$

where, in coded units,

x_1 = ratio of concentrations of NH_3 and $CaCl_2$
$x_3 = p\mathrm{H}$

to the equivalent but simpler canonical equation

$$\hat{y} = 0.71Z^2 + 4.42Z - 7.29X_3^2 + 79.71 \tag{14}$$

with origin at $X_3 = 0$, $Z = X_1 - 3.11 = 0$. This equation reveals that the surface is a rising ridge.

A most interesting application of canonical analysis is that by Comp (1963) in a patent on a process for the preparation of cyanoacetylenes. Moorhead and Himmelblau (1962), a study on the operating conditions in a packed liquid-liquid extraction column, is another paper utilizing canonical analysis. Box and Youle (1955) and Wu (1964b) also contain canonical analyses.

In summary, however, it is somewhat surprising that so few of the papers reviewed here in the sections on applications of RSM contain a canonical analysis. A study of the papers in this survey reveals a definite need for more extensive use of canonical analysis. In some cases ambiguous and contradictory conclusions were reached, a situation which could have been avoided if a canonical analysis had been carried out. For example, Berry, Tucker, and Deutschman (1963) did not include a canonical analysis in their study on starch vinylation. They based some of their conclusions on an incorrect diagram in the space of two of the variables. This mistake was later noticed by Klein and Marshall (1964). A canonical analysis would have reduced the possibility of making these erroneous conclusions in this particular case. There are other examples where a canonical analysis would have been beneficial, especially in determining the shape of the contour systems from the prediction equation. We feel that more emphasis should be put on this phase of RSM in future applications because it provides the experimenter with a deeper, more incisive analysis and consequently a better appreciation of the system under investigation.

7. *Elucidation of Underlying Mechanisms*

RSM has been used effectively by some researchers in elucidating the basic mechanism of the particular physical system under study. Box and Youle (1955) indicate how information from an empirically fitted surface can be used to cast light on the actual mechanism of the system involved. An example is first treated from an empirical point of view and then a theoretical surface based on reaction kinetics is derived. It is further illustrated how canonical variables of an empirical surface may serve as guides to the basic underlying mechanism.

An outstanding example of the application of RSM in the elucidation of a basic mechanism appears in the papers of Franklin, Pinchbeck, and Popper (1956, 1958) and Pinchbeck (1957) where the underlying mechanism of the vapor-phase oxidation of naphthalene to phthalic anhydride is studied. Pinchbeck compared the previously developed empirical model from Franklin, Pinchbeck, and Popper (1956) with three theoretical surfaces derived from three different

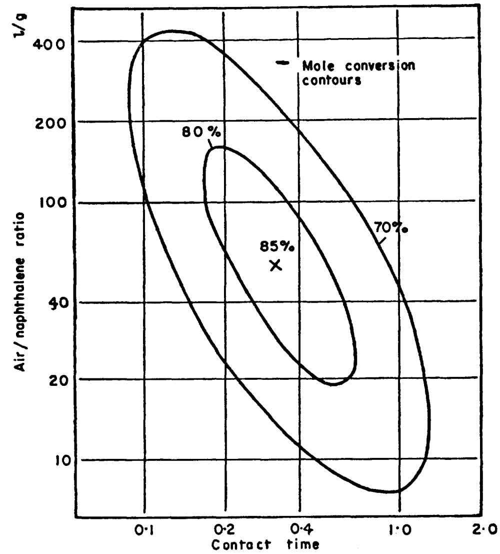

FIGURE 5.—Experimental response surface of mole conversion as a function of air-to-naphthalene ratio and contact time. (Reproduced from Pinchbeck (1957) with permission of the author and the publisher, Pergamon Press, Inc.)

possible mechanisms. This comparison indicated that the theoretical surface (see Figure 6) based on a half-order reaction step for the disappearance of naphthalene closely resembled, in shape, the empirical surface (see Figure 5). The conclusion reached was that the rate-determining step was between half and first-order rather than the alternative first or second-order cases.

Another article dealing with the elucidation of an underlying mechanism is that by Carr (1960) concerning the conversion of n-pentane to isopentane. By linearizing four theoretically-based nonlinear reaction rate models and by fitting them to the data, he arrived at his choice of a dual site mechanism. This paper indicates a possible approach to further work in the area of discovering the underlying mechanism through response surface methods.

Aia, Goldsmith, and Mooney (1961) should also be mentioned here since this paper includes a study of the basic mechanism.

8. *Further Applications*

DeBaun and Schneider (1958) illustrated with examples three different applications of RSM: (1) the use of an orthogonally blocked design for a spray-drying problem with large batch-to-batch variations, (2) the use of a non-orthogonally blocked design for a batch reaction involving an organic intermediate, and (3) the use of a noncentral composite design for a staged chemical process.

In an excellent paper Frankel (1961) described the application of response methods to the production of mercaptobenzathiazole with the use of an octogonal rotatable design in two variables, temperature and time. The resulting contour diagram was saddle-shaped. Ross (1961) effectively used response surface

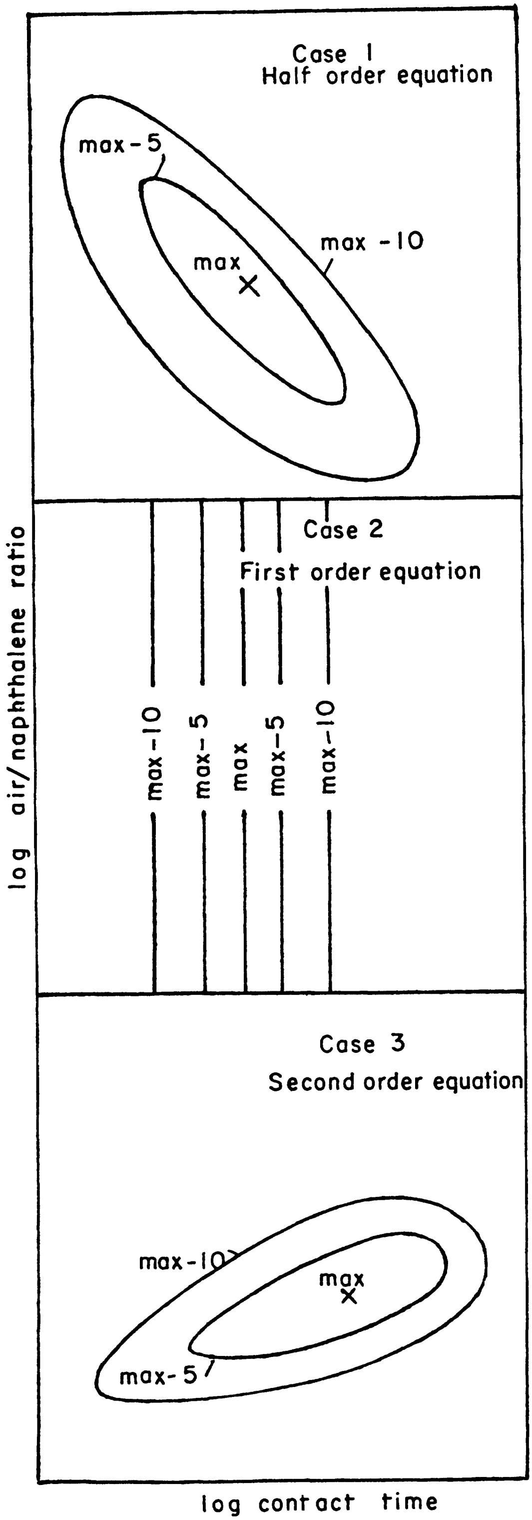

FIGURE 6.—Theoretical response surfaces of mole conversion as a function of air-to-naphthalene ratio and contact time. (Reproduced from Pinchbeck (1957) with permission of the author and the publisher, Pergamon Press, Inc.)

techniques to study ultrasonic welding. Another interesting paper is that by Schneider and Stockett (1963) on the effects of gas scrubbing on the "odor acceptability" of an unpleasant smelling product. Gardiner and Cowser (1961) used the method of steepest ascent in a paper on the optimization of radionuclide removal from process wastes. Similarly Homme and Othmer (1961) used the method of steepest ascent when optimizing a profit index of the production of sulphuric acid by the contact process.

Further applications are those by Grové (1965) on the influence of temperature on the scouring of raw wool; Hermanson (1965) on the maximization of potato yield; Stone, Wu, and Tiemann (1965) on the extraction of silica from quartz in sodium hydrozide solutions; Hermanson, Gates, Chapman, and Farnham (1964) on a study of corn yield; Wu and Meyer on the effect of speed, feed, depth of cut (1964a), side-cutting-edge angle, and nose radius (1964b) on cutting-tool temperature; Meyer (1963) on the application of RSM in education and psychology; Michaels and Pengily (1963) on the maximization of yield for a specified cost; Welch, Adams, and Carmon (1963) on a study of the yield of Bermudagrass; Miller and Ashton (1960) on the absorption of fertilizer phosphorus by oats; Mooney, Comstock, Goldsmith, and Meisenhelter (1960) on the precipitation of calcium hydrogen orthophosphate; Phifer and Maginnis (1960) on the factors which influence dry ashing of pulps; Robinson and Nielsen (1960) on the effect of nitrogen, phosphorus, and potassium on the growth of tomato plants; Sheldon (1960) on the effect of water, polyphosphate, and ethylenediaminetetraacetic acid on the brightness of kraft pulp; Baird and Mason (1959) on a study of corn yield; Hader, Harward, Mason, and Moore (1957) and Moore, Harward, Mason, Hader, Lott, and Jackson (1957) on the effect of copper, iron, and molybdenun on the growth and nutrition of lettuce; Roth and Switlyk (1957) on a study of textile resin finishes; Sanderson (1957) on a study of a titania pigment process; Hackler, Kreigel, and Hader (1956) on the effect of raw-material ratios on absorption of Whiteware compositions; Shewell (1956) on an investigation of catalytic cracking; Whidden (1956) on a study of metals processing; and Cragle, Myers, Waugh, Hunter, and Anderson (1955) on the effects of various levels of sodium citrate, glycerol, and equilibration time on survival of bovine spermatozoa after storage at −79°C.

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