



# Simplicity and Scientific Progress

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# The Synchronic and Diachronic Schools

**Synchronic School:** focused on the finished products of science, esp. characterizing which beliefs (or systems of belief) constitute **rational** responses to evidence.

**Diachronic School:** characterize which methods are conducive to scientific progress.

Ilkka Niiniluoto, *Scientific Progress* (2015)

# Diachronic School

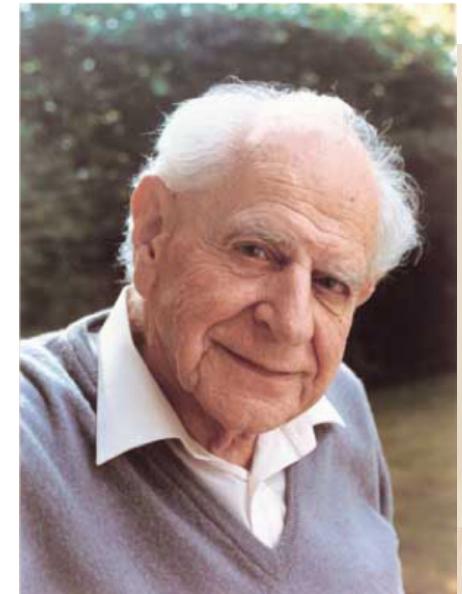
“... progress necessarily involves the idea of a **process through time**. Rationality, on the other hand, has tended to be viewed as an **atemporal** concept ... most writers see progress as nothing more than the temporal projection of a series of individual rational choices .... we may be able to learn something **by inverting** the presumed dependence of progress on rationality.”



Laudan, *Progress and its Problems* (1978).

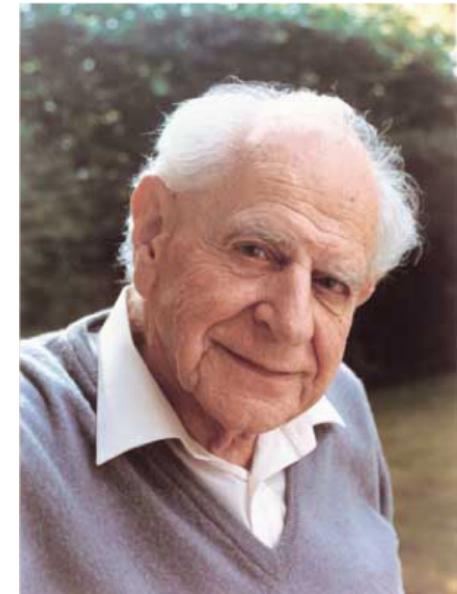
# Popper's Critical Rationalism

**Popper:** Science progresses through a series of highly testable conjectures, followed by dogged attempts at refutation.



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But why think this is anything more than a series of **bold mistakes**, yielding to new, and bolder, mistakes?

# Lakatos Objects

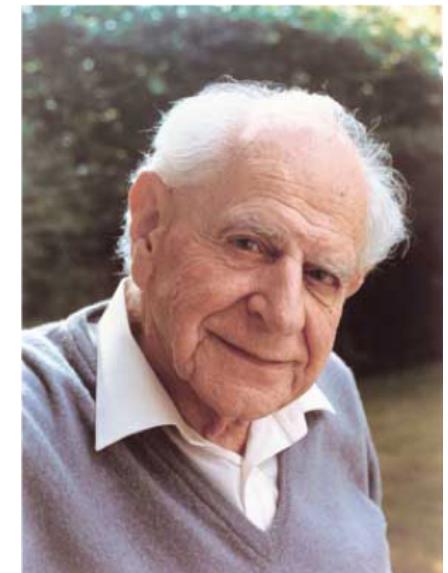
Popper “offers a methodology without an epistemology or a learning theory, and confesses explicitly that his methodology may lead us epistemologically astray, and implicitly, that *ad hoc* stratagems might lead us to Truth.”



Imre Lakatos, *The Role of Crucial Experiments in Science* (1971).

# Truthlikeness

Popper developed a theory of verisimilitude, hoping to show that the process of conjectures and refutations leads to theories of increasing truthlikeness (1963, 1970).



Popper's idea was famously trivialized (independently) by Pavel Tichy and David Miller (1974). On Popper's account, no false theory is more truthlike than any other!

# Truthlikeness Redux

Oddie (1986) and Niiniluoto (1987, 1999) make more sophisticated attempts at a definition of truthlikeness.

# Truthlikeness Redux

But there is no demonstration that any method is guaranteed to produce increasingly truthlike theories!

# Truthlikeness Redux

“appraisals of the relative distances from the truth presuppose that an epistemic probability distribution . . . is available. In this sense ... the problem of estimating verisimilitude is neither more nor less difficult than the traditional problem of induction.”



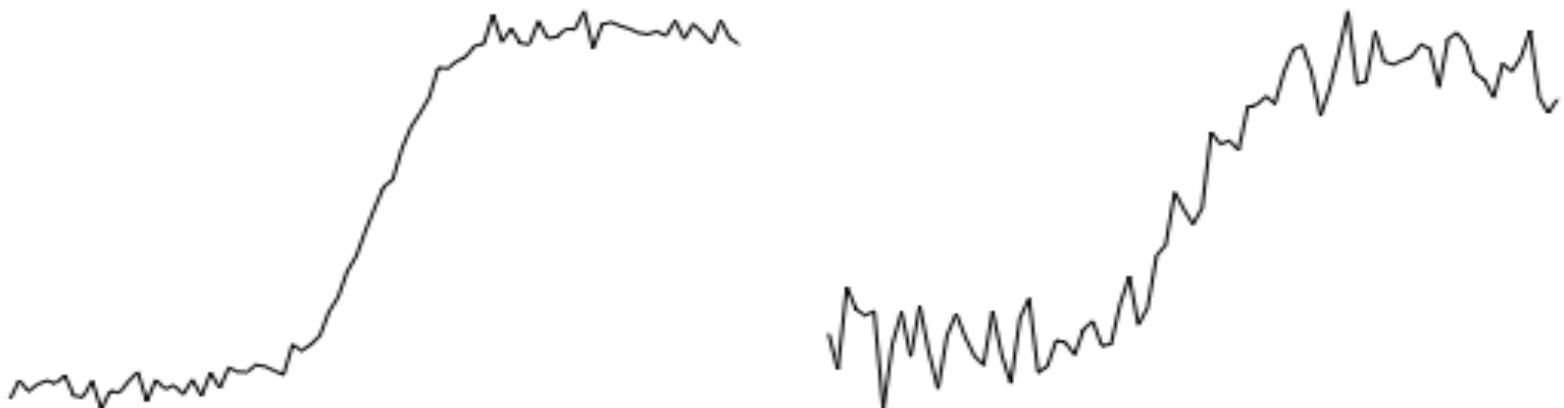
Illka Niiniluoto, *Truthlikeness* (1987).

# Progressive Methods

- Say that a method for answering a question is **progressive** if the **chance** that it outputs the **true answer** is strictly **increasing** with sample size.
- That notion makes sense, even if it does not make sense to ask which of two false theories is closer to the truth!

# Progressive Methods

- A method is  $\alpha$ -progressive if the chance that it outputs the true answer never decreases by more than  $\alpha$ .



# Progressive Methods

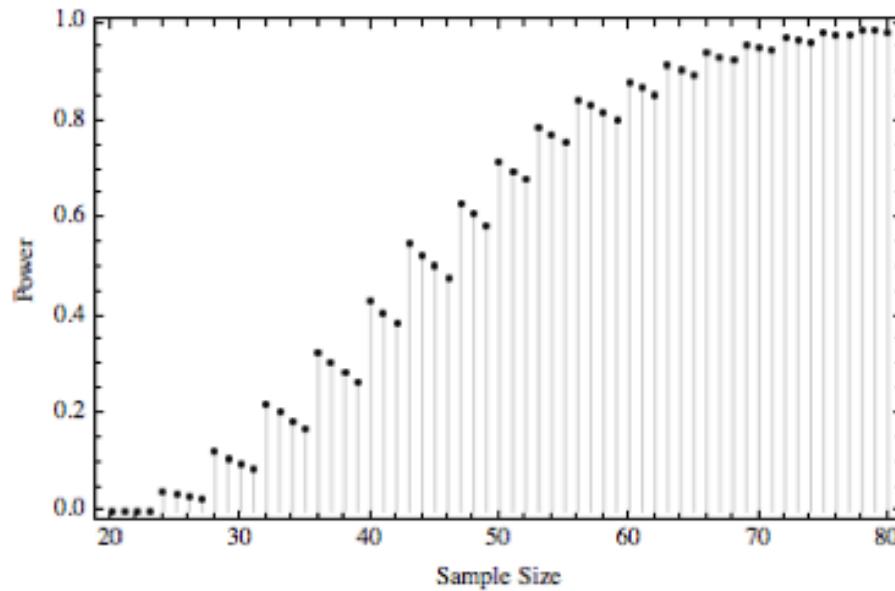
Researchers propose recruiting 100 patients to investigate whether a new drug is better at treating migraine than placebo. In their grant, they analyze their statistical method and conclude the following: if the new drug is significantly better than placebo, the chance that their method detects the improvement is greater than 50%. The funding agency is satisfied. Soon after, the researchers publish a paper claiming to have discovered a promising new treatment!

# Progressive Methods

Now, suppose that a replication study is proposed with 150 patients. However, the *ex ante* analysis reveals that the objective chance of detecting an improvement over placebo, if one exists, has decreased to 40%. The **chance of replicating successfully has gone down**, even though the first study may well be correct, and yet the investigators propose performing a larger study!

# Progressive Methods

Surprisingly, many textbook methods in frequent hypothesis testing exhibit this perverse behavior.



Chernick and Liu, *The Saw-toothed behavior of power vs. sample size and software solutions.* (2012)

# Progressive Methods

**Theorem (Genin):** For typical problems, there exists an  $\alpha$ -progressive method for every  $\alpha > 0$ .

# A Vindication of Neo-Popperian Method

**Theorem (Genin):** All progressive methods must systematically prefer simpler (more falsifiable) theories.

# The Plan

1. Prove this result in the simplified setting of propositional information.
2. Port this result to the setting of statistical information.

# The Topological Bridge

- Start with **logical** insights.
- Allow methods a small chance  $\alpha$  of error.
- Obtain corresponding **statistical** insights



# The Topology of Information

I ● topology



# Possible Worlds

$W$

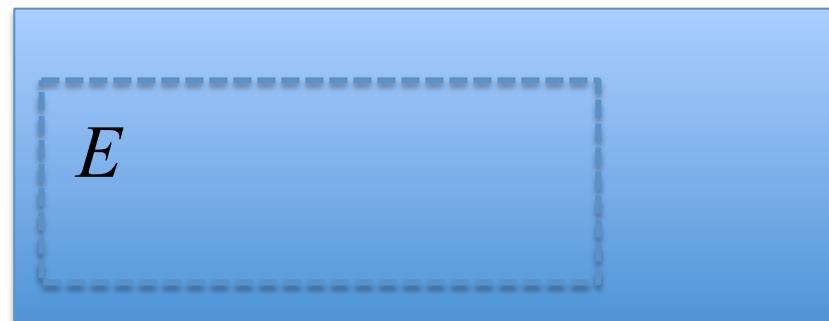
$w$



# Propositional Information State

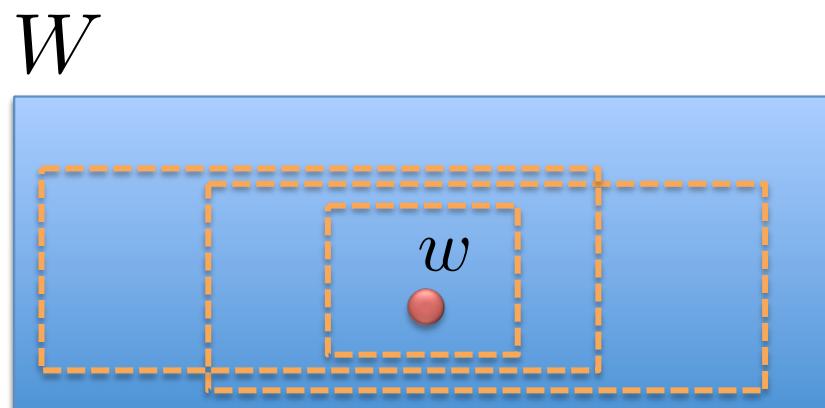
The logically strongest proposition you are informed of.

$W$



# Propositional Information State

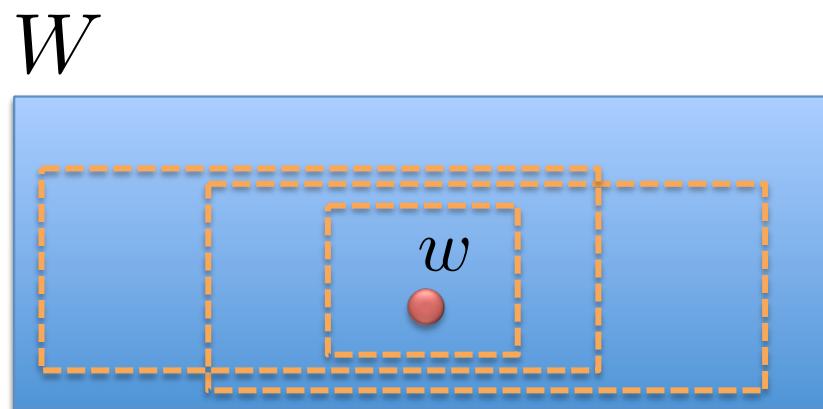
- $\mathcal{I}$  is the set of **all** possible information states.
- $\mathcal{I}(w)$  is the set of all information states **true** in  $w$ .
- $\mathcal{I}(w \mid E) = \{ F \text{ in } \mathcal{I}(w) : F \subseteq E \}$



# Propositional Information State

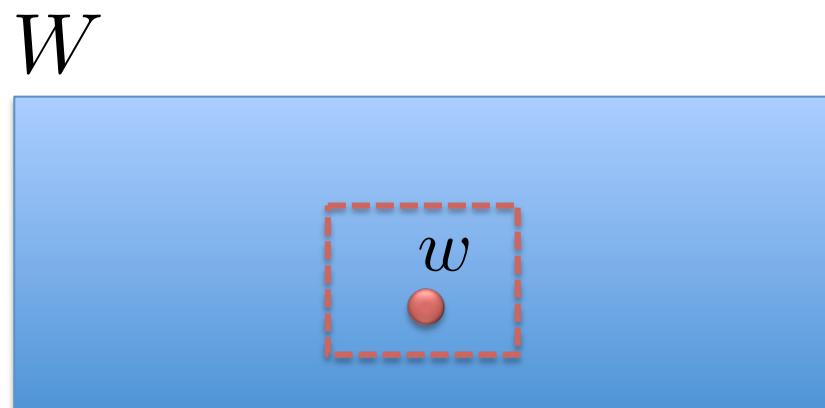
**Intended Interpretation:**  $E$  is in  $\mathcal{I}(w)$  iff

a diligent inquirer in  $w$  will eventually be afforded information at least as strong as  $E$ .



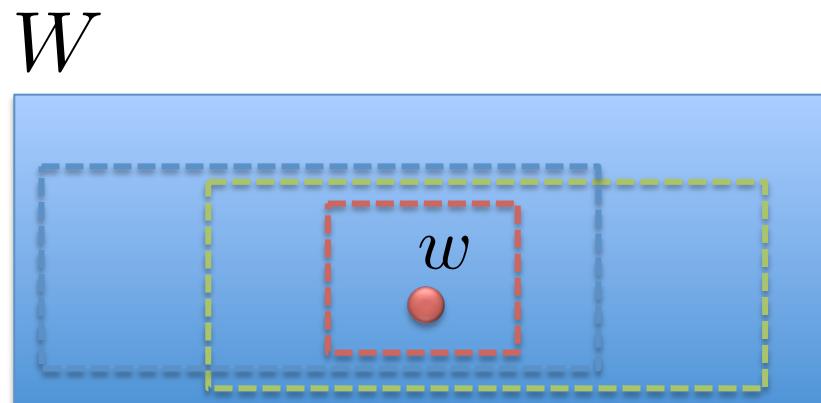
# Three Axioms

1. Some information state is true in  $w$ .



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2. Each pair of information states **true** in  $w$  is **entailed** by an information state **true** in  $w$ .

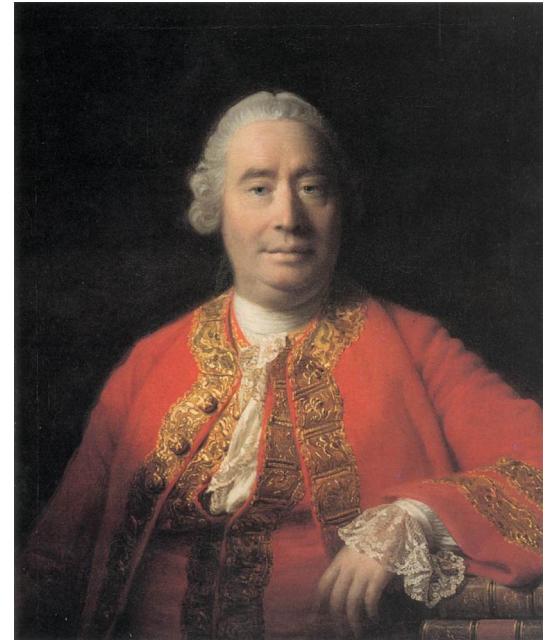


# Three Axioms

1. Some information state is true in  $w$ .
2. Each pair of information states true in  $w$  is entailed by an information state true in  $w$ .
3. There are at most **countably many** information states.

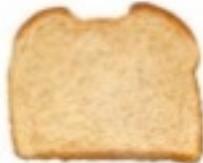
# Hume's Problem

“The bread, which I formerly ate,  
nourished me ... but does it  
follow, that **other** bread must **also**  
nourish me at another time ... ?  
The consequence seems nowise  
necessary.”

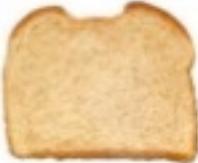


Hume, *Enquiry*.

# Hume's Problem, Topologized.



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# Hume's Problem, Topologized.



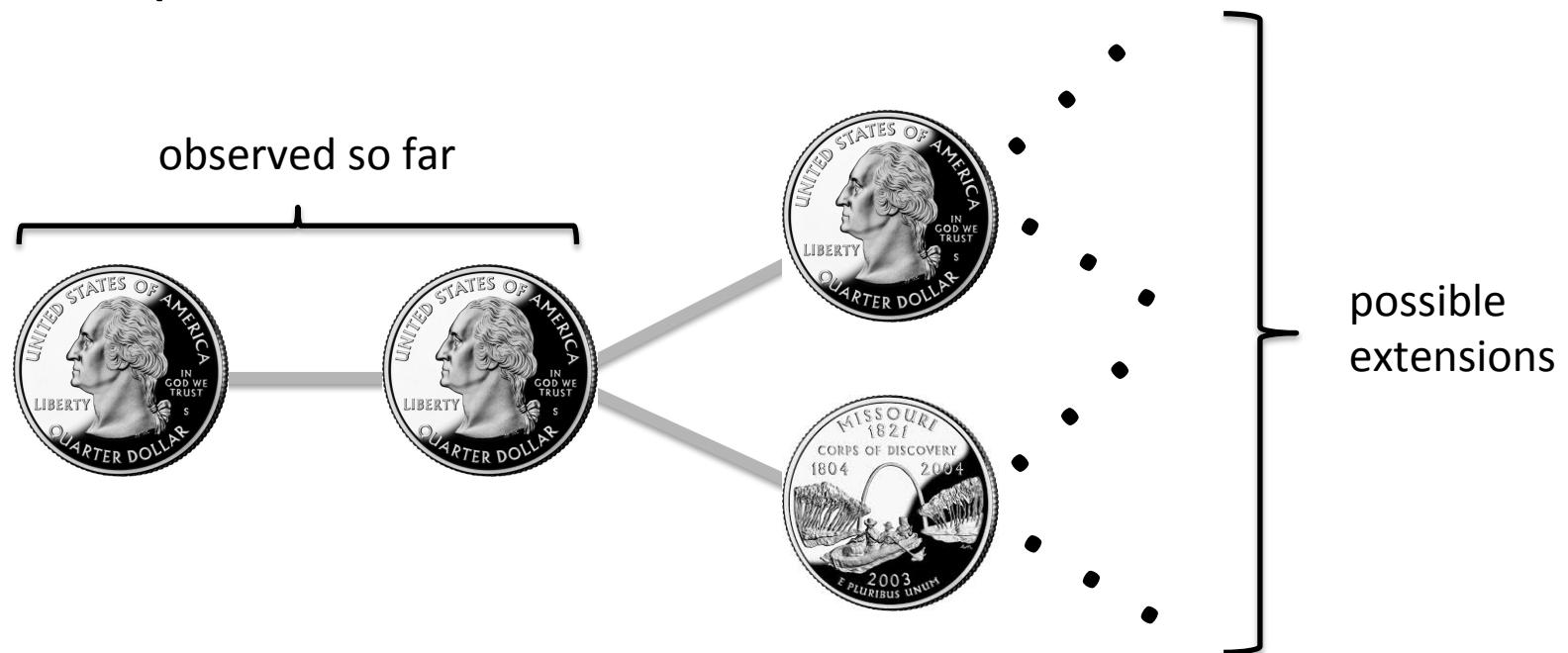
# Hume's Problem, Topologized.



# Example: Sequential Binary Experiment

**Worlds** = infinite sequences of coin flips.

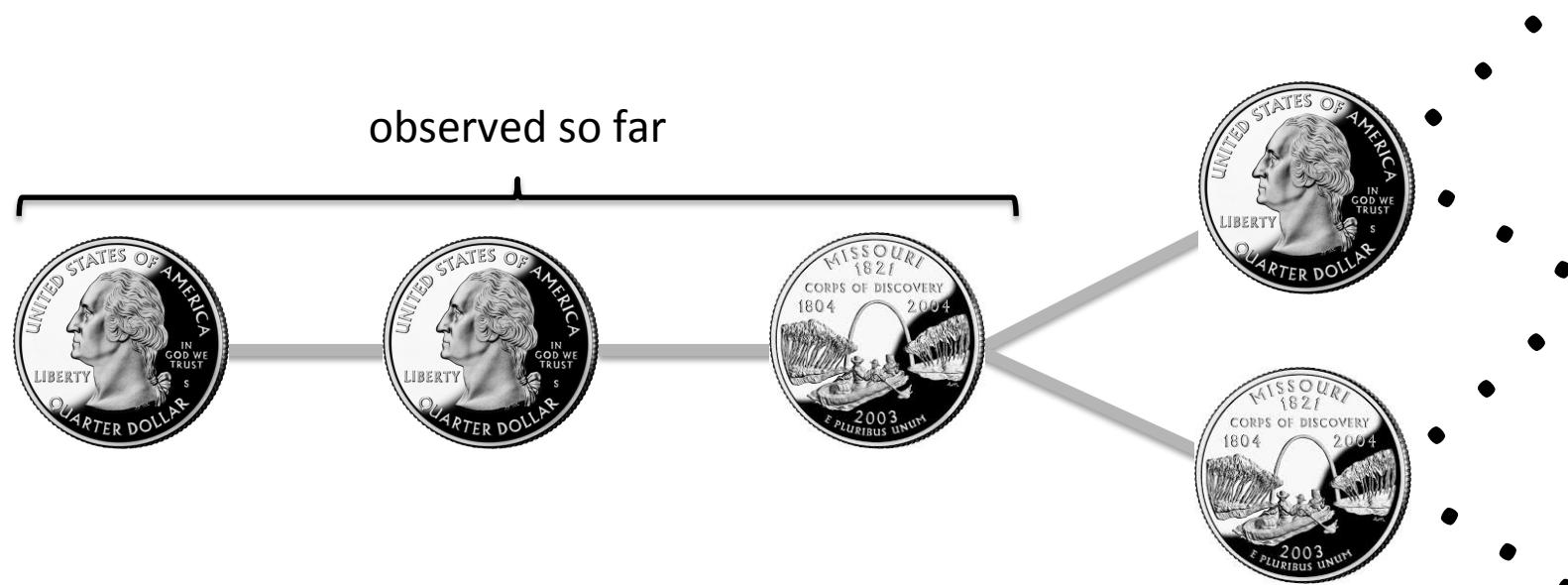
**Evidential states** = cones of possible extensions of finite sequences:



# Example: Sequential Binary Experiment

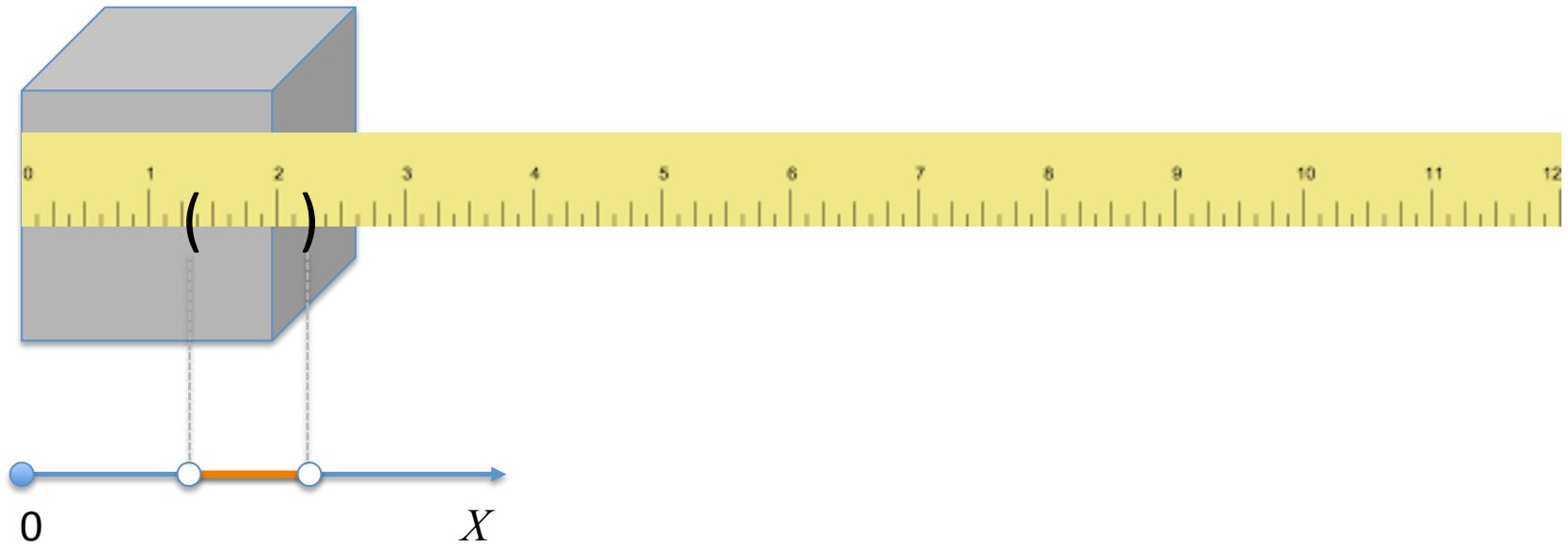
**Worlds** = infinite sequences of coin flips.

**Evidential states** = cones of possible extensions of finite sequences:



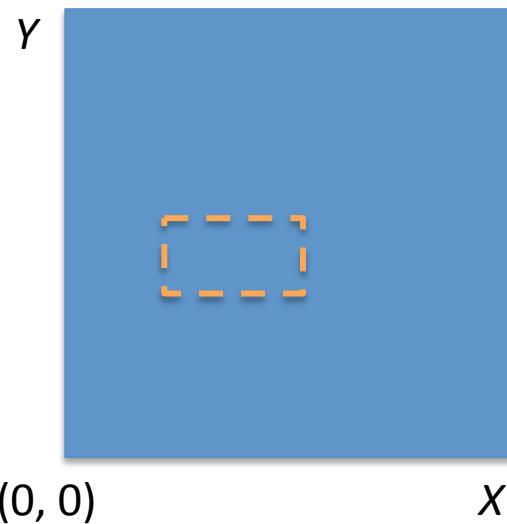
# Example: Measurement of $X$

- **Worlds** = real numbers.
- **Information states** = open intervals.



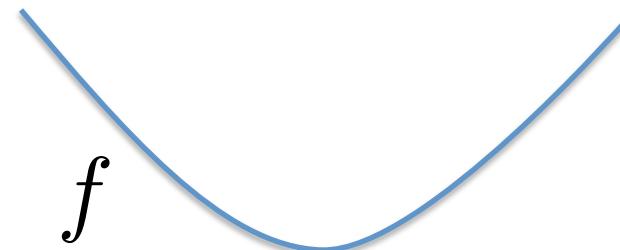
# Example: Joint Measurement

- **Worlds** = points in real plane.
- **Information states** = open rectangles.



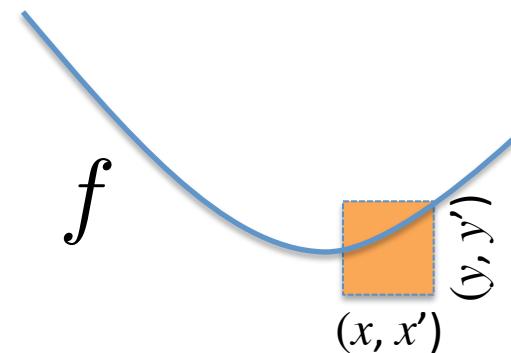
# Example: Functions

- **Worlds** = functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ .



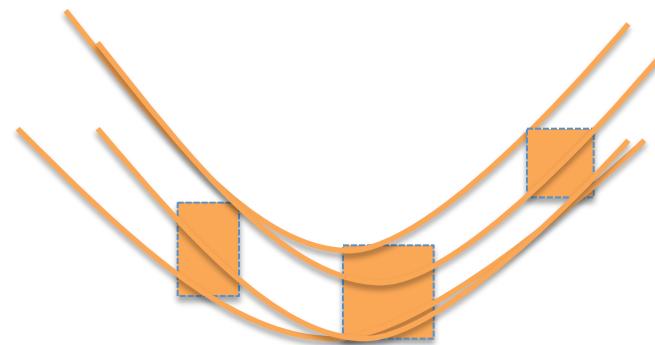
# Example: Functions

- An **observation** is a joint measurement.



# Example: Functions

- The **information state** is the set of all worlds that touch each observation.

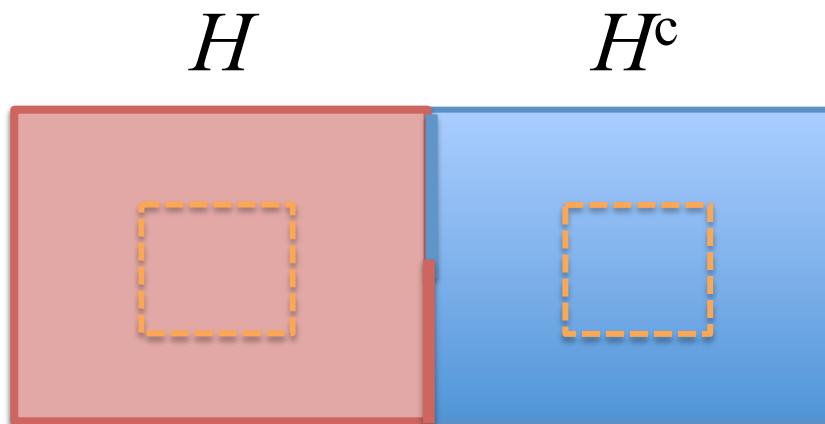


# Deductive Verification and Refutation

$H$  is **verified** by  $E$  iff  $E \subseteq H$ .

$H$  is **refuted** by  $E$  iff  $E \subseteq H^c$ .

$H$  is **decided** by  $E$  iff  $H$  is either verified or refuted by  $E$ .

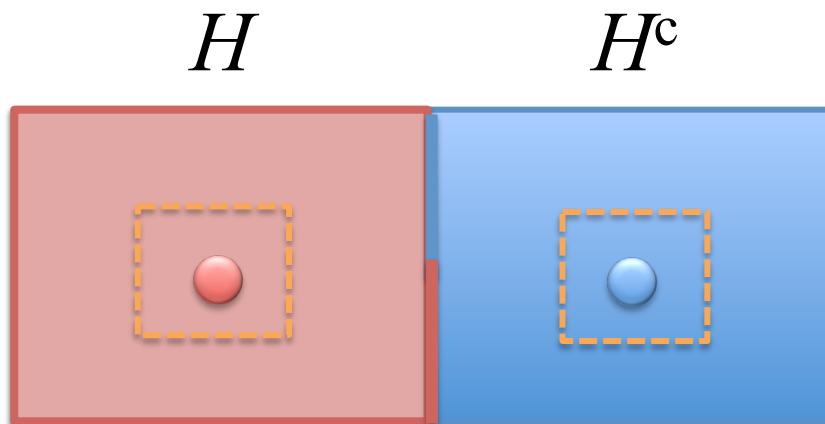


# Will be Verified

$w$  is an **interior point** of  $H$  iff

iff  $H$  **will be verified** in  $w$ ;

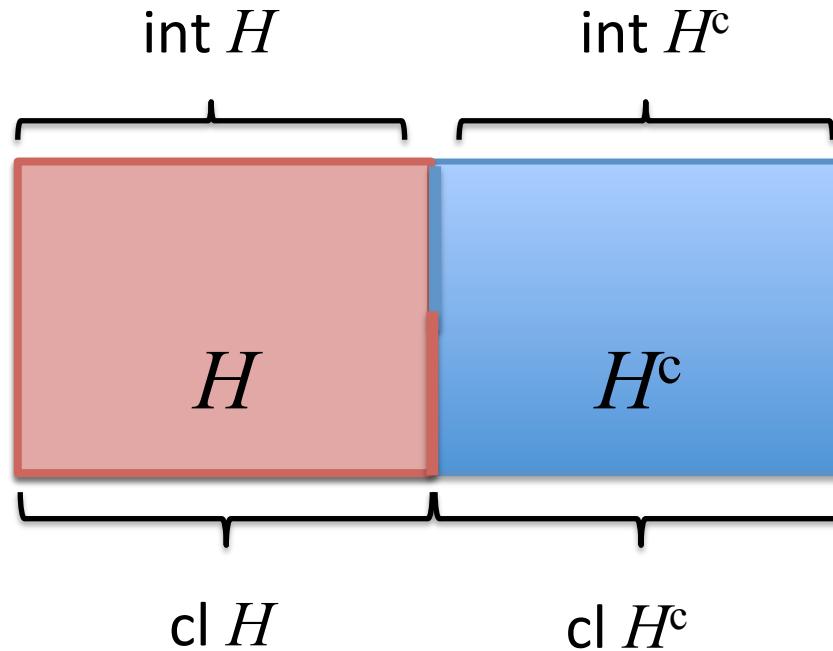
iff there is  $E$  in  $\mathcal{I}(w)$  s.t.  $H$  is **verified** by  $E$ .



# Topological Operators as Modal Operators

$\text{int } H$  := the proposition that  $H$  **will be verified**.

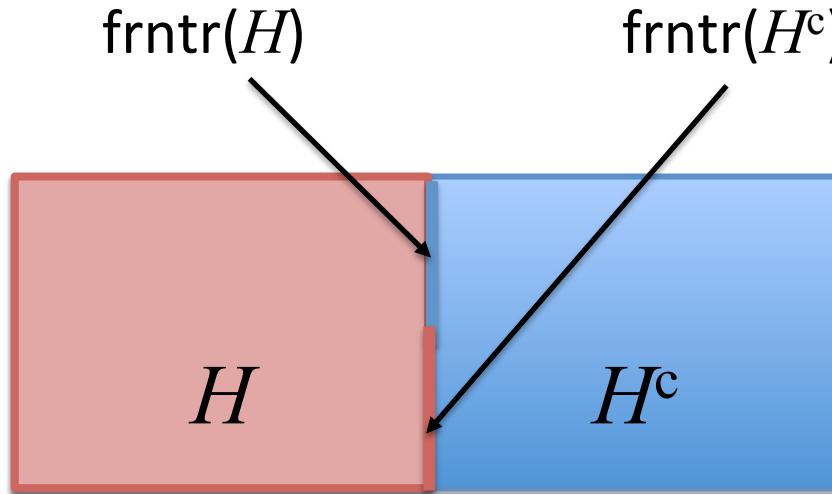
$\text{cl } H$  := the proposition that  $H$  **will never be refuted**.



# Topological Operators

**frntr**  $H$  := the proposition that  $H$  is **false** but will never be **refuted**.

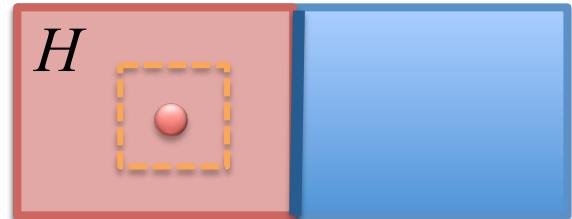
**frntr**  $H^c$  := the proposition that  $H$  is **true** but will never be **verified**.



# Verifiability, Refutability, Decidability

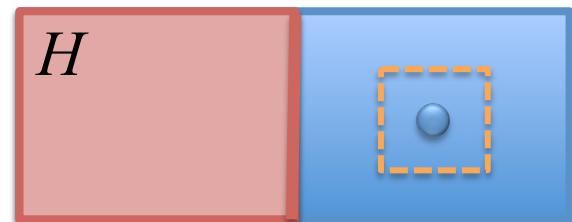
$H$  is **verifiable (open)** iff  $H \subseteq \text{int}(H)$ .

i.e., iff  $H$  will be **verified** however  $H$  is **true**.

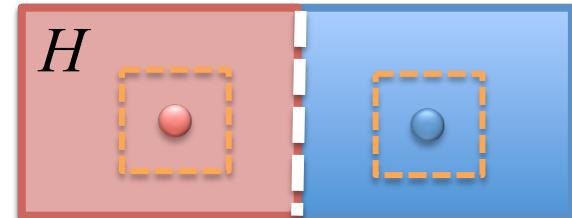


$H$  is **refutable (closed)** iff  $\text{cl}(H) \subseteq H$ .

i.e., iff  $H$  will be **refuted** however  $H$  is **false**.

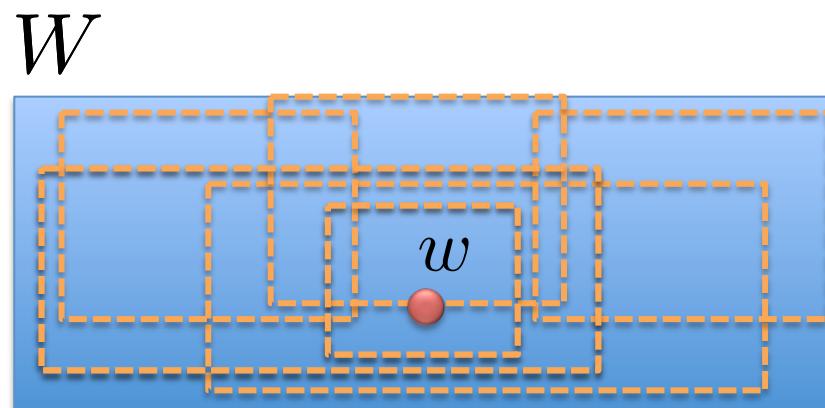


$H$  is **decidable (clopen)** iff  $H$  is both  
verifiable and refutable.



# The Topology of Information

- A **topology** on  $W$  is determined by its **open** (verifiable) propositions.
- Every verifiable proposition is a **disjunction** of information states in  $\mathcal{I}$ .



# Interior

$\text{int } H$  = the proposition that  $H$  **will be verified**.



$$\text{Int} \{ \text{skull} \} = \{ \text{skull} \}$$

$$\text{Int} \{ \text{bread} \} = \cancel{\emptyset}$$

# Open = Verifiable

$H$  is open (verifiable) iff  $H$  entails **int**  $H$ .



$$\text{Int} \{ \text{skull} \} = \{ \text{skull} \}$$

$$\text{Int} \{ \text{bread} \} = \emptyset$$

# Closure

$\text{cl } H$  = the proposition that  $H$  **will never be refuted**.



$$\text{Cl } \{\text{skull}\} = \{\text{bread}, \text{skull}\}$$

$$\text{Cl } \{\text{bread}\} = \{\text{bread}\}$$

# Closed = Refutable

$H$  is closed (refutable) iff  $\text{cl } H$  entails  $H$ .



$$\text{Cl } \{ \text{skull} \} = \{ \text{bread}, \text{skull} \}$$

$$\text{Cl } \{ \text{bread} \} = \{ \text{bread} \}$$

# Frontier

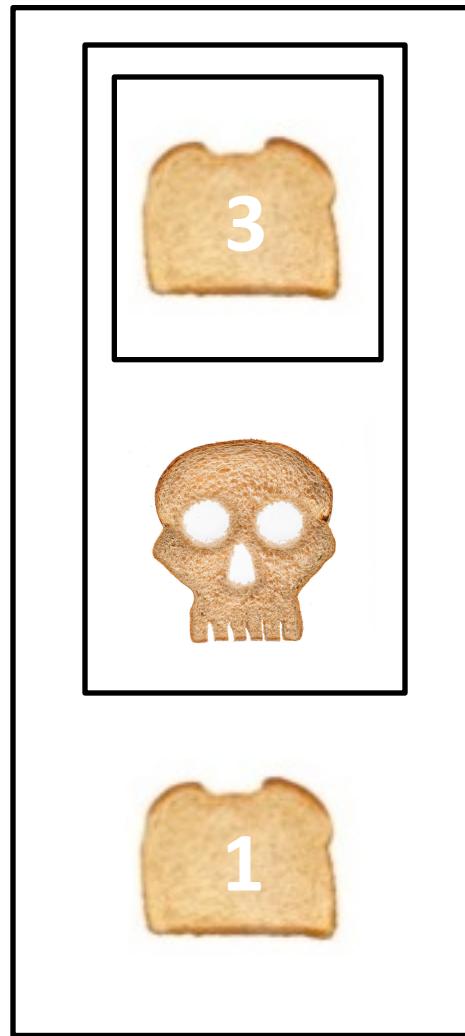
**frntr**  $H = H$  is false, but will never be refuted.



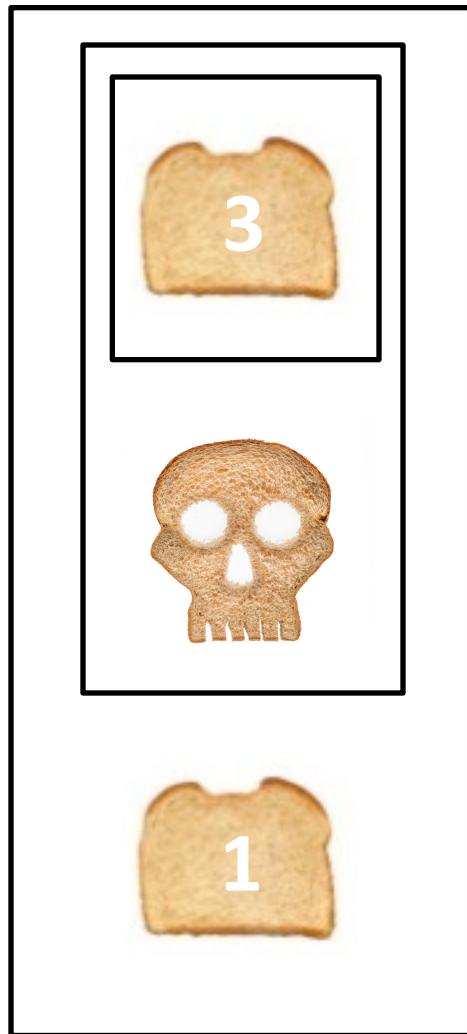
Frntr  $\{\ skull \} = \{ \text{bread} \}$

Frntr  $\{ \text{bread} \} = \emptyset$

# Hume's Problem, Enhanced.

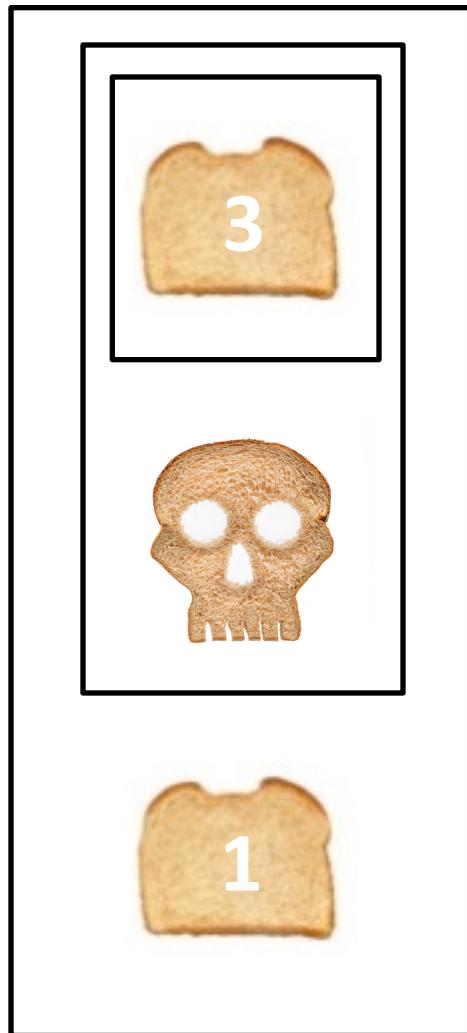


# Hume's Problem, Enhanced.



Frntr { } = { }

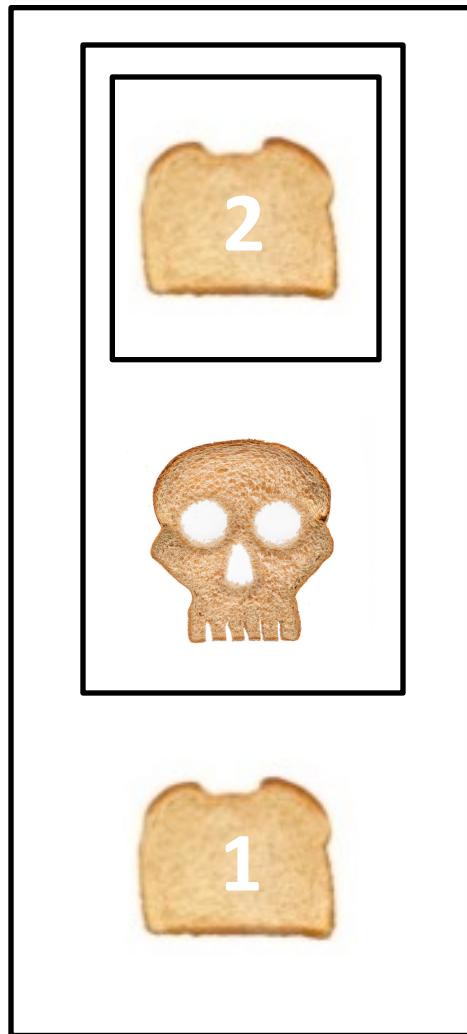
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Frntr { } = { }

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# Hume's Problem, Enhanced.



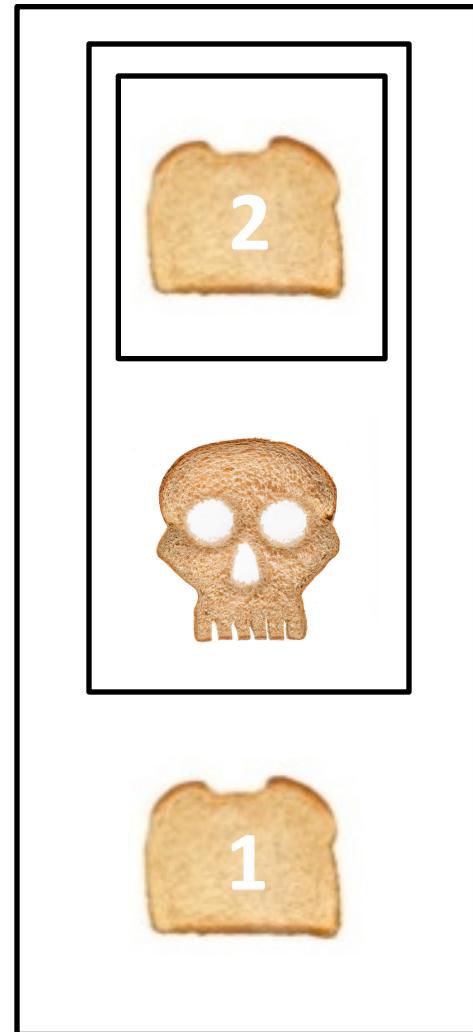
Frntr { } = { }

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# Locally Closed

$H$  is locally closed iff **frntr**  $H$  is closed.



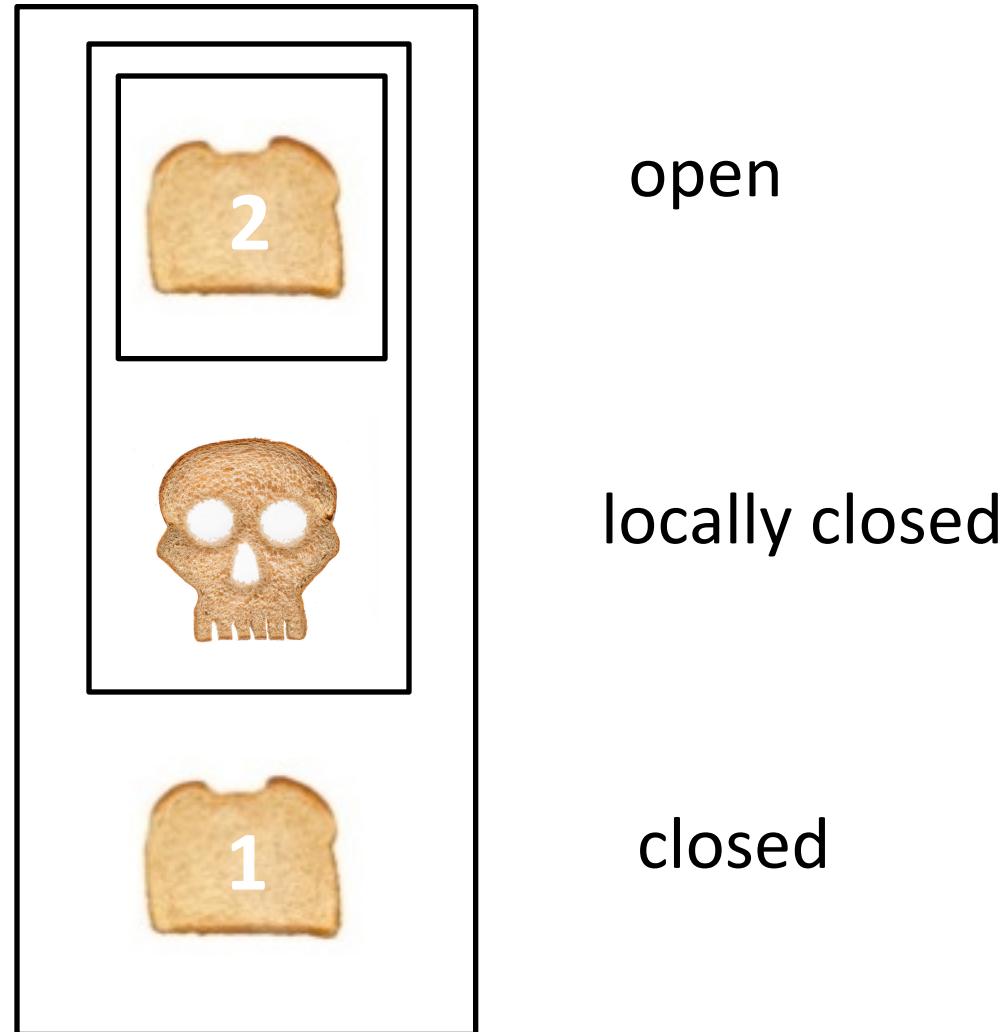
open

locally closed

closed

# Locally Closed

$H$  is **locally closed** iff  $H$  entails that  $H$  will be refutable (closed).



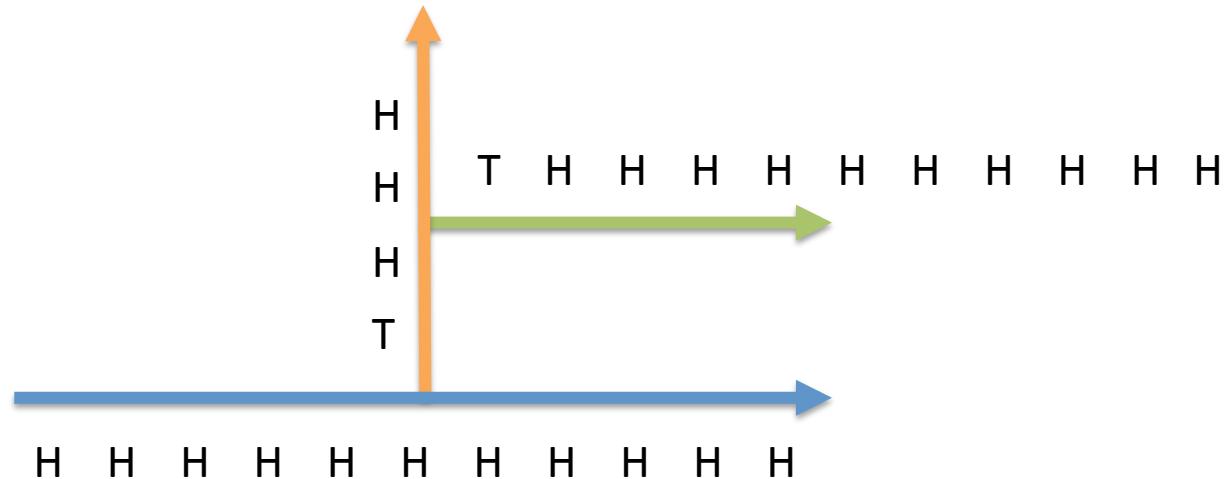
# Sequential Example

etc.

$H_2$  = “You will see T exactly twice” is locally closed.

$H_1$  = “You will see T exactly once” is locally closed.

$H_0$  = “You will never see T” is closed.



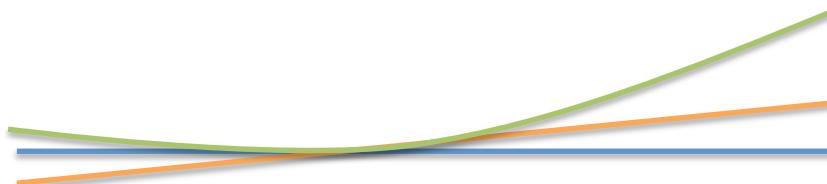
# Equation Example

*etc.*

$H_2$  = “quadratic” is locally closed.

$H_1$  = “linear” is locally closed.

$H_0$  = “constant” is closed.

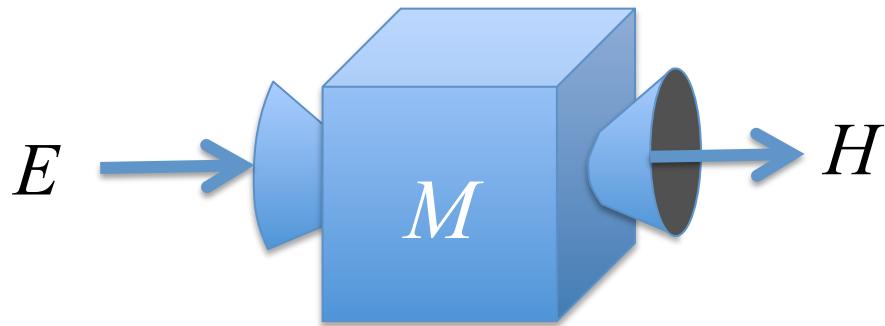


# Topology

- $H$  is **limiting open** iff  $H$  is a countable union of locally closed sets.
- $H$  is **limiting closed** iff  $H^c$  is limiting open.
- $H$  is **limiting clopen** iff  $H$  is both limiting open and limiting closed.

# Propositional Methods

- **Propositional methods** produce **propositional conclusions** in response to **propositional information**.



# Propositional Methods

- $M$  is **infallible** iff  $w \in M(E)$ , whenever  $E \in \mathcal{I}(w)$ .
- $M$  is **monotonic** iff  $M(F) \subseteq M(E)$ , whenever  $F \subseteq E$ .

# Convergence

$M$  converges to  $H$  in  $w$  iff

there is  $E$  in  $\mathcal{I}(w)$  such that

for all  $F$  in  $\mathcal{I}(w \mid E)$ ,

$$M(F) \subseteq H.$$

# Deductive Methods

- A **verification method** for  $H$  is an infallible, monotonic method  $V$  such that:
  1.  $w \in H^c$  implies  $V(E) = W$  for  $E$  in  $\mathcal{I}(w)$ ;
  2.  $w \in H$  implies  $V$  converges to  $H$  in  $w$ .



# Deductive Methods

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- A **refutation method** for  $H$  is just a verification method for  $H^c$ .
- A **decision method** for  $H$  converges to  $H$  or to  $H^c$  without error.

# Deductive Methods

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- A **refutation method** for  $H$  is just a verification method for  $H^c$ .
- A **decision method** for  $H$  converges to  $H$  or to  $H^c$  without error.
- $H$  is **methodologically verifiable [refutable, decidable, etc.]** iff  $H$  has a method of the corresponding kind.

# Inductive Methods

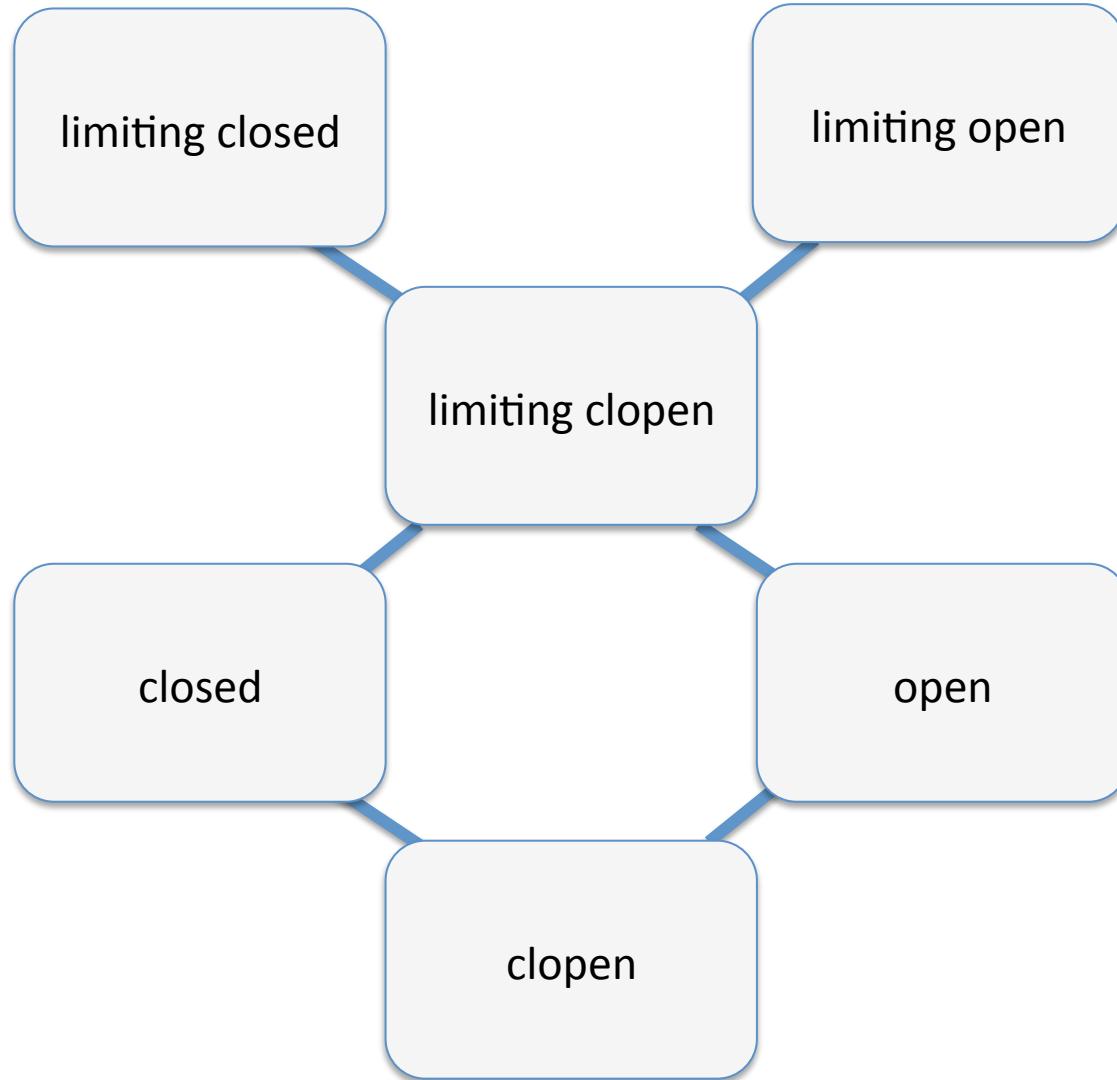
- A **limiting verification method** for  $H$  is a method  $V$  such that:  
 $w \in H$  iff  $V$  converges in  $w$  to some true  $H'$  that entails  $H$ .



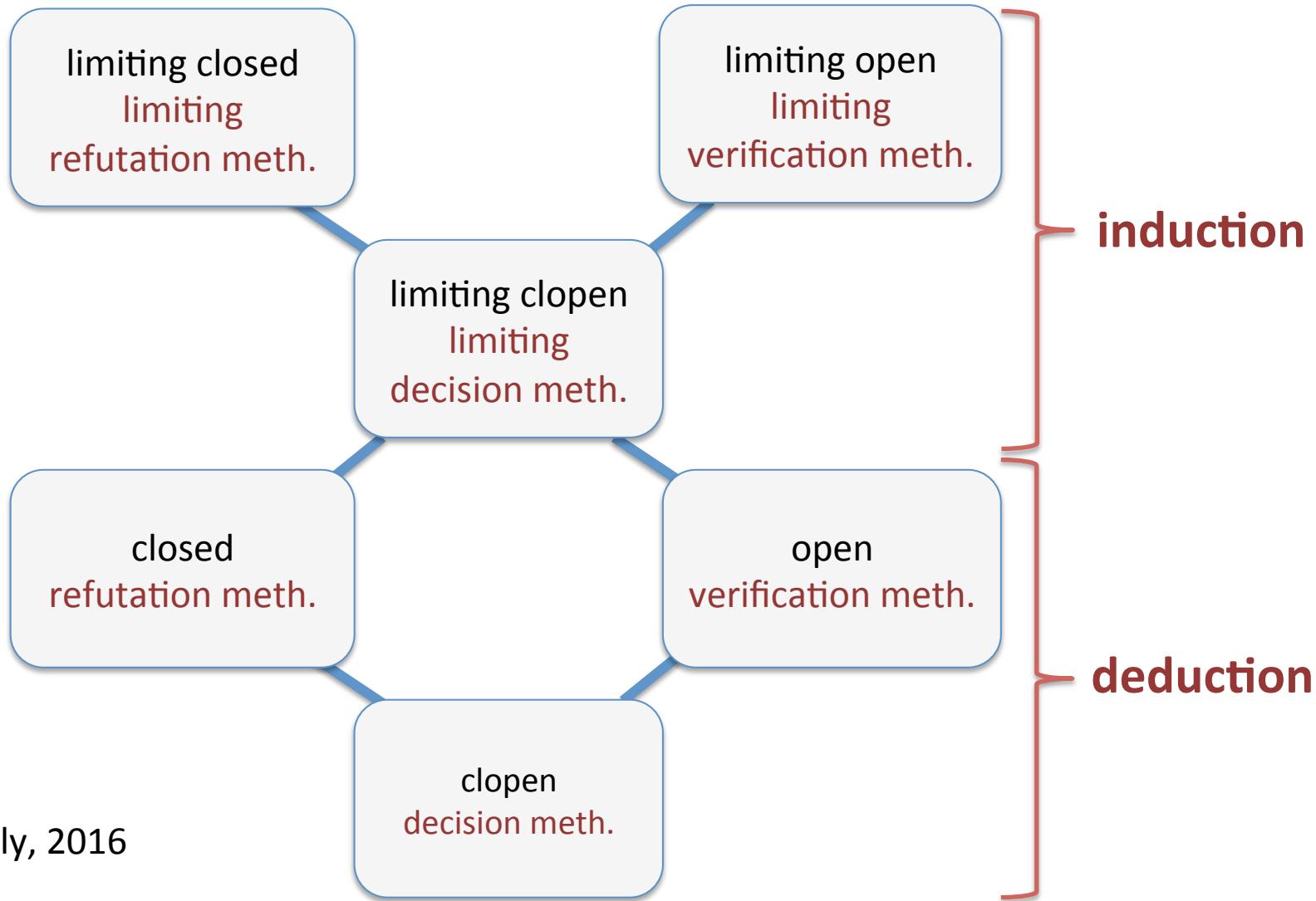
# Inductive Methods

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 $w \in H$  iff  $V$  converges in  $w$  to some true  $H'$  that entails  $H$ .
- A **limiting refutation method** for  $H$  is a limiting verification method for  $H^c$ .
- A **limiting decision method** for  $H$  is a limiting verification method and a limiting refutation for  $H$ .

# Topological Complexity



# Characterization Theorem



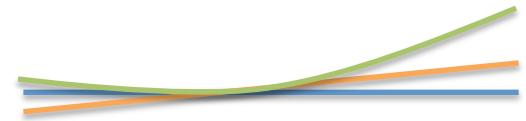
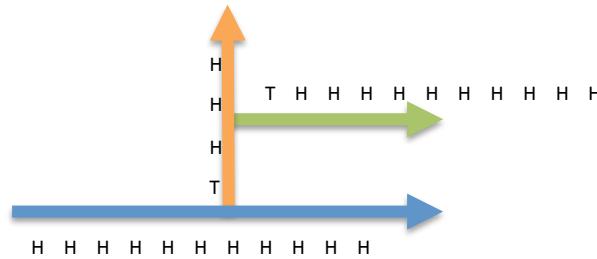


# OCKHAM'S TOPOLOGICAL RAZOR

# Popper's Simplicity Order

- “More falsifiable hypotheses are simpler”.

$$A \preceq B \Leftrightarrow A \subseteq \text{cl}B.$$



$$H_1 \prec H_2 \prec H_3.$$

# A Big Mistake

$$A \preceq B \Leftrightarrow A \subseteq \text{cl}B.$$

1. Weaker hypotheses are **less falsifiable**.

$A \subseteq B$  implies  $A \preceq B$ .

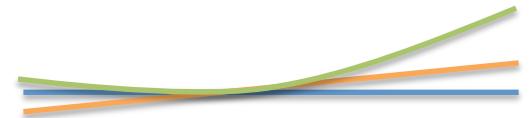
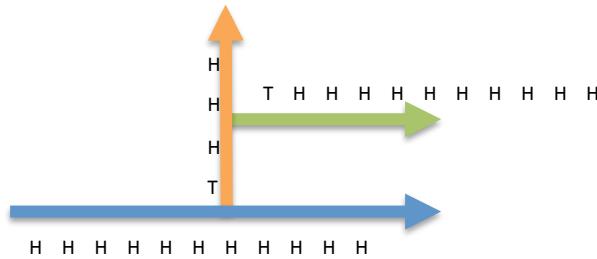
2. So **suspending judgment** violates **Ockham's razor!**

$A \preceq W$ .

# Easy and Natural Fix

Lack of falsifiers is **bad** only if  $A$  is **false**!

$$A \preceq B \Leftrightarrow A \subseteq \text{frntr}B$$



$$H_1 \prec H_2 \prec H_3.$$

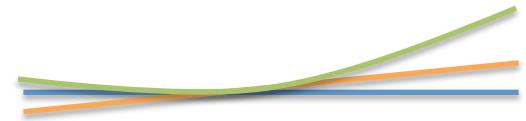
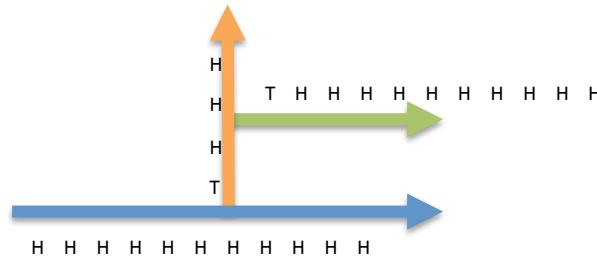
# A Smaller Issue

- Gerrymandered hypotheses can obscure simplicity relations.
- E.g., “The true law is linear, or the cat is on the mat” is not simpler than “The true law is quadratic”.

# A Response

Simpler theories have simpler ways of being true.

$$A \triangleleft B \Leftrightarrow A \cap \text{frntr} B \neq \emptyset$$



$$H_1 \triangleleft H_2 \triangleleft H_3.$$

# Example: Competing Paradigms

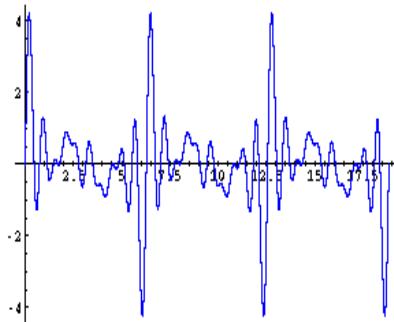
Polynomial paradigm

$$Y = \sum_{i=0}^N a_i X^i.$$



Trigonometric polynomial paradigm

$$Y = \sum_{i=0}^N a_i \sin(iX) + b_i \cos(iX).$$



# Example: Competing Paradigms

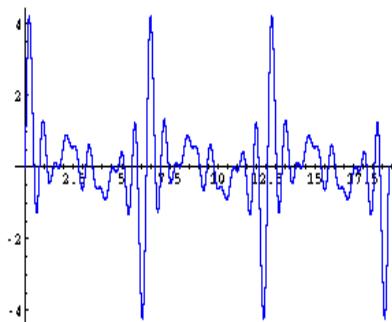
Polynomial paradigm

$$Y = \sum_{i=0}^N a_i X^i.$$

degree

Trigonometric polynomial paradigm

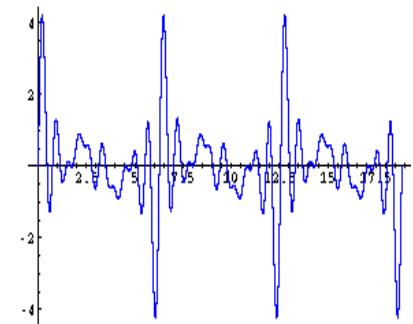
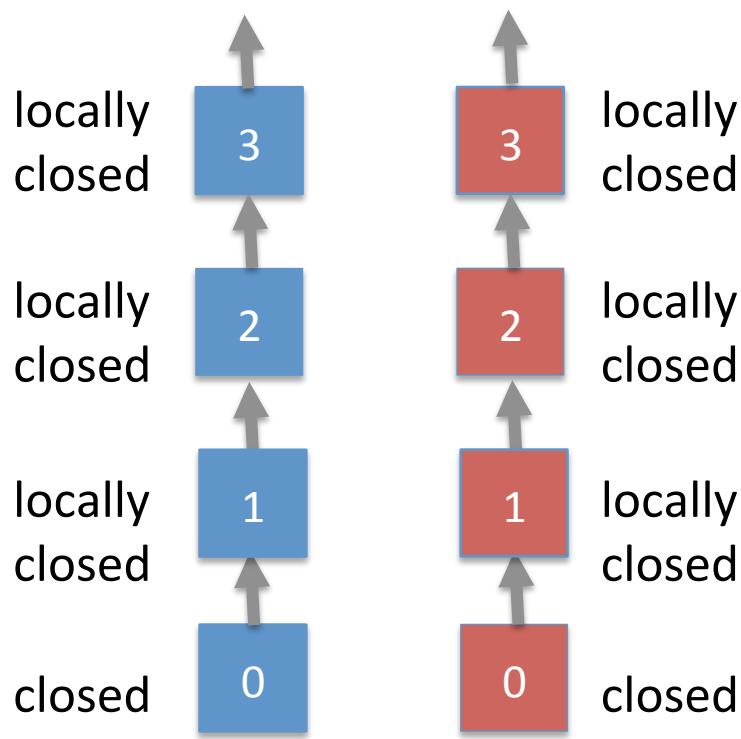
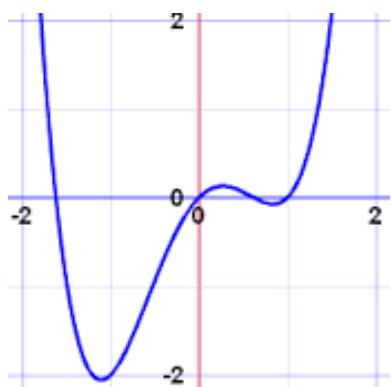
$$Y = \sum_{i=0}^N a_i \sin(iX) + b_i \cos(iX).$$



# Example: Competing Paradigms

$\mathcal{Q}$  = which degree and which paradigm is true?

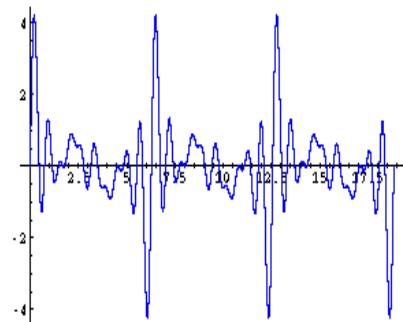
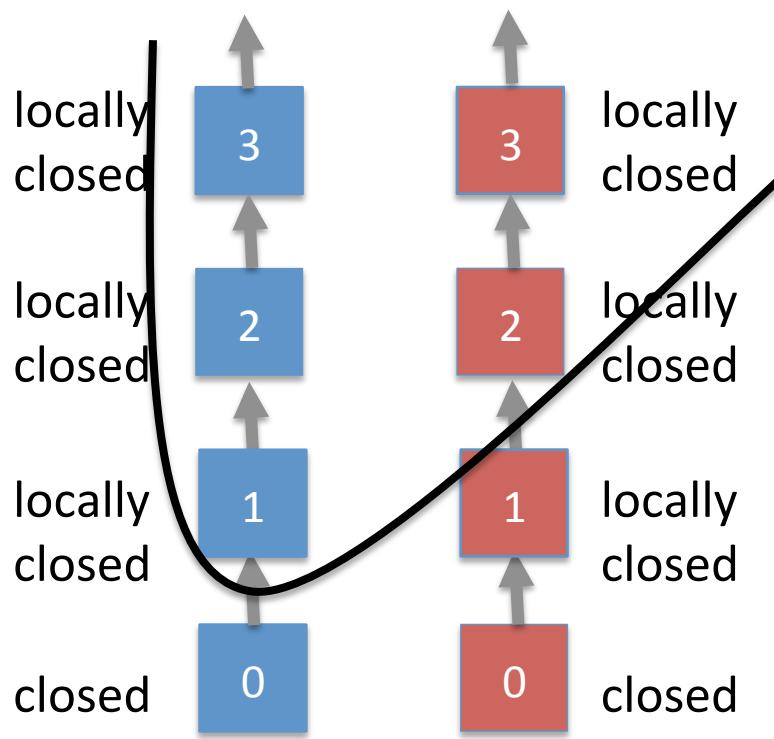
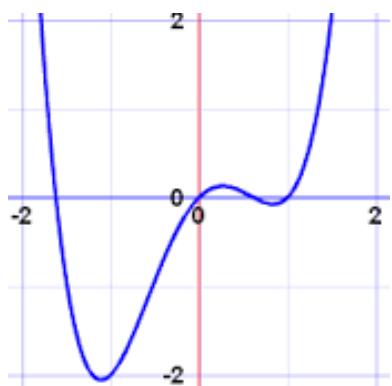
$\mathcal{I}$  = finitely many inexact measurements.



# Example: Competing Paradigms

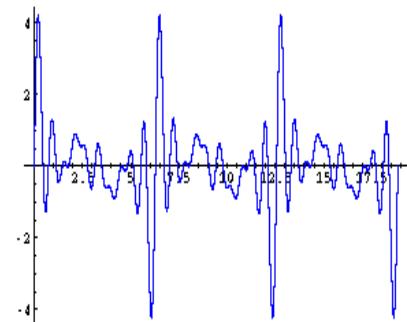
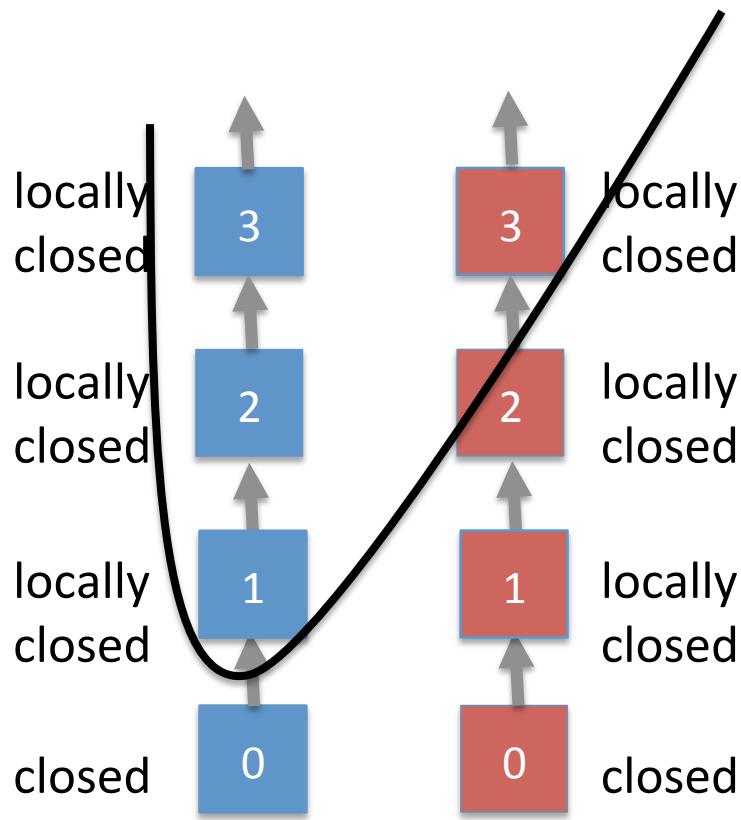
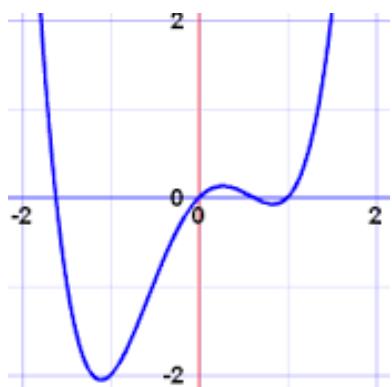
$\mathcal{Q}$  = which degree and which paradigm is true?

$\mathcal{I}$  = finitely many inexact measurements.



# Example: Competing Paradigms

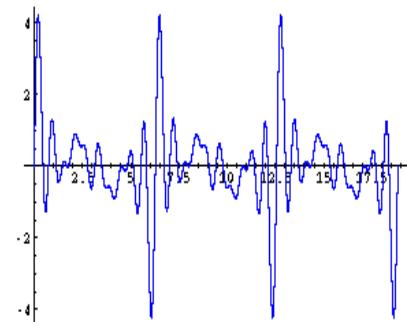
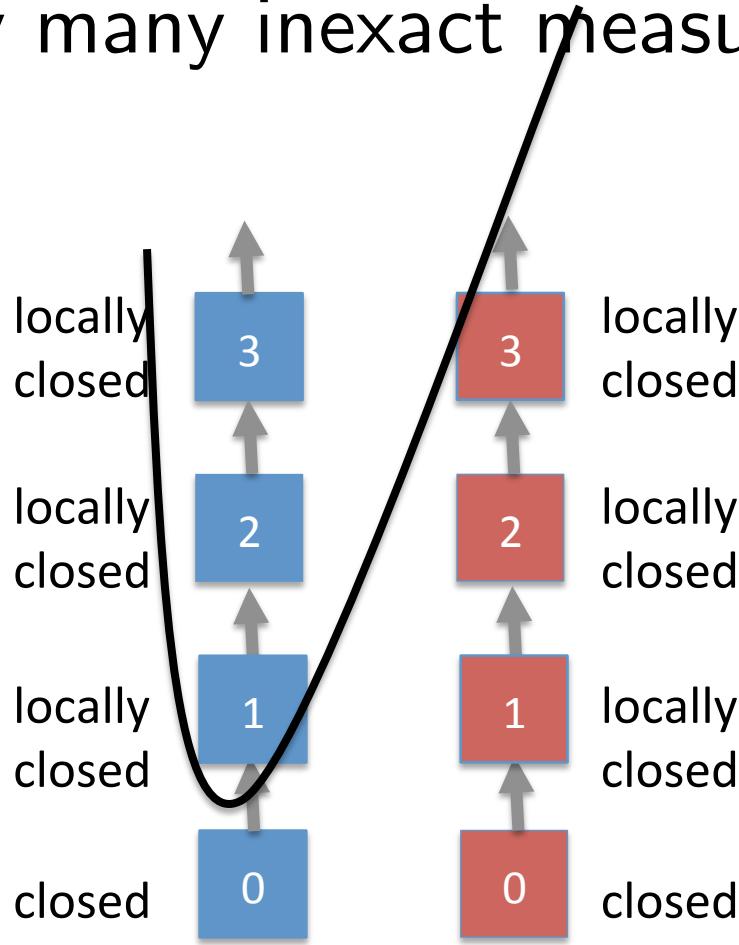
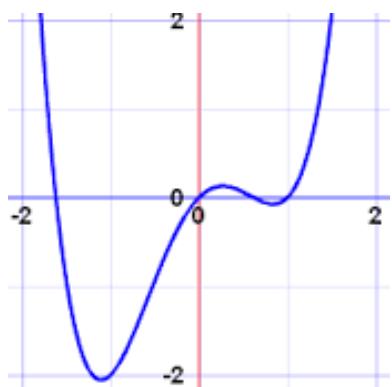
$\mathcal{Q}$  = which degree and which paradigm is true?  
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# Example: Competing Paradigms

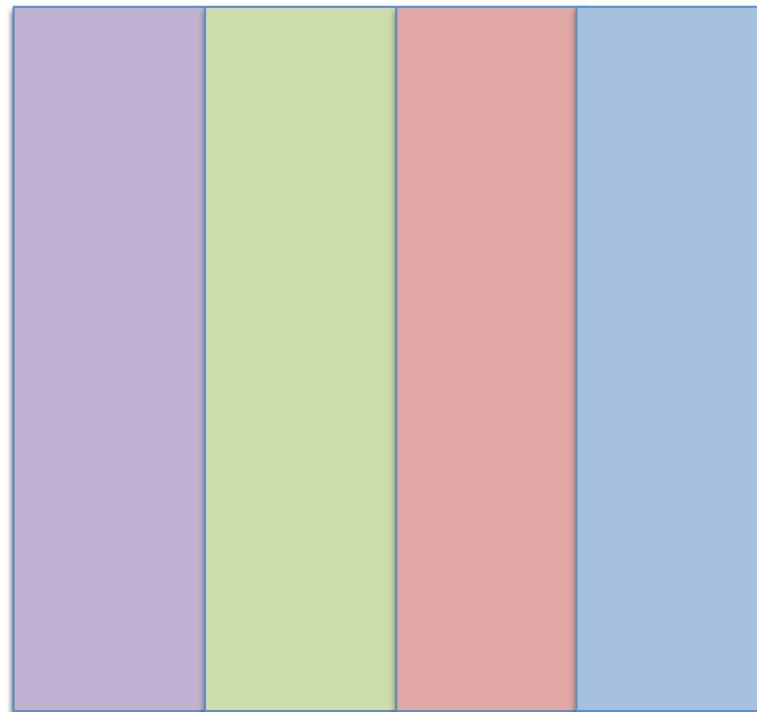
$\mathcal{Q}$  = which degree and which paradigm is true?

$\mathcal{I}$  = finitely many inexact measurements.



# Questions

- A **question** partitions  $W$  into countably many possible answers (Hamblin 1958)
- **Relevant responses** are disjunctions of answers.





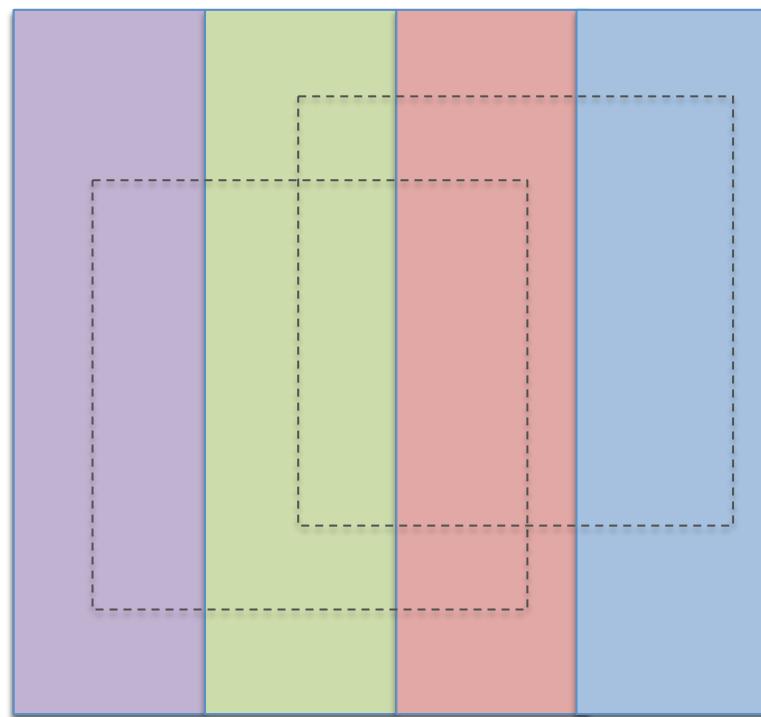
# Ockham's Razor

**Proposition (Genin and Kelly, 2016).** The following principles are **equivalent**.

1. Infer a **simplest** relevant response in light of  $E$ .
2. Infer a **refutable** relevant response compatible with  $E$ .
3. Infer a relevant response that is **not more complex than the true answer**.

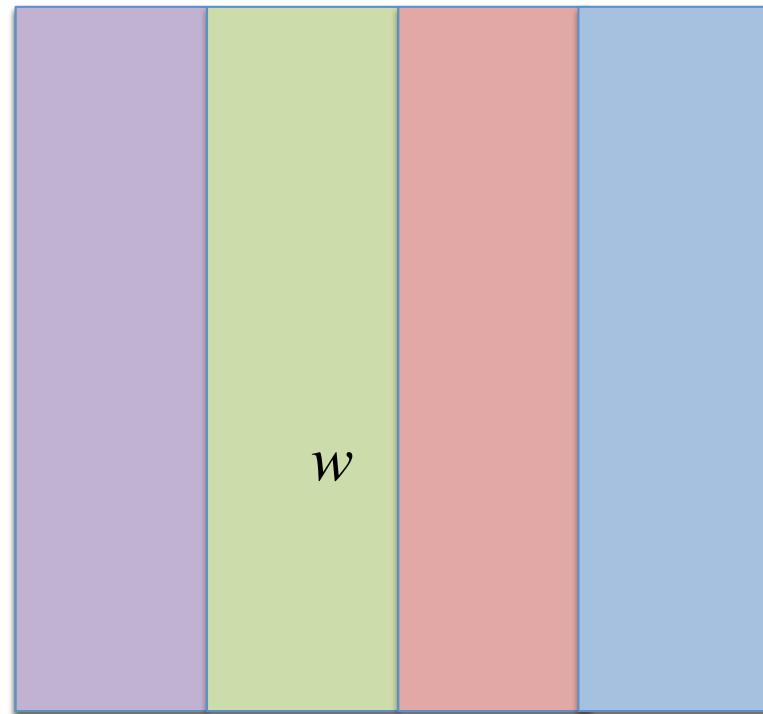
# Empirical Problem

$$\mathfrak{P} = (W, \mathcal{I}, \mathcal{Q}).$$



# Empirical Problem

$\mathcal{Q}(w)$  is the answer true in  $w$ .



# Solutions

A **solution** for  $\mathfrak{P} = (W, \mathcal{I}, \mathcal{Q})$  is a propositional method  $V$  such that

$w \in H$  iff  $V$  converges in  $w$  to some true  $H'$  that entails  $\mathcal{Q}(w)$ .

A problem is **solvable** iff it has a solution.

# Solvability, Characterized.

**Proposition.** A problem  $\mathfrak{P} = (W, \mathcal{I}, \mathcal{Q})$  is solvable iff every answer is limiting open.

de Brecht and Yamamoto (2009)

Baltag, Gerasimczuk, and Smets (2015)

Genin and Kelly (2015)

# Progressive Solutions

A solution for  $\mathfrak{P} = (W, \mathcal{I}, \mathcal{Q})$  is **progressive** iff for all  $E$  in  $\mathcal{I}(w)$  and  $F$  in  $\mathcal{I}(w \mid E)$  :

if  $V(E)$  entails  $Q(w)$ , then  $V(F)$  entails  $Q(w)$ .

That is: the true answer is a **fixed point** of inquiry.

# Progressive Solutions

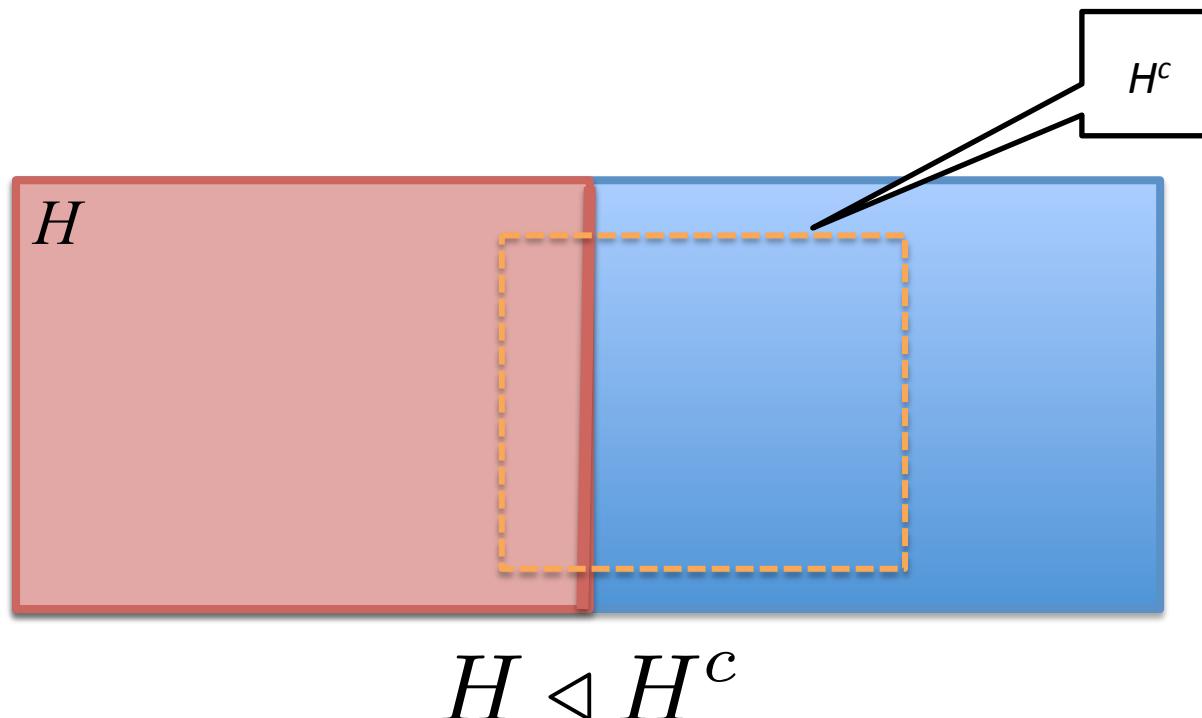
**Proposition.** If there exists an enumeration  $A_1, A_2, \dots$  of the answers to  $Q$ , agreeing with the **simplicity order**, then  $Q$  is progressively solvable.

# Epistemic Mandate for Ockham's Razor

**Proposition** (Genin and Kelly, 2016). Every progressive solution obeys Ockham's razor.

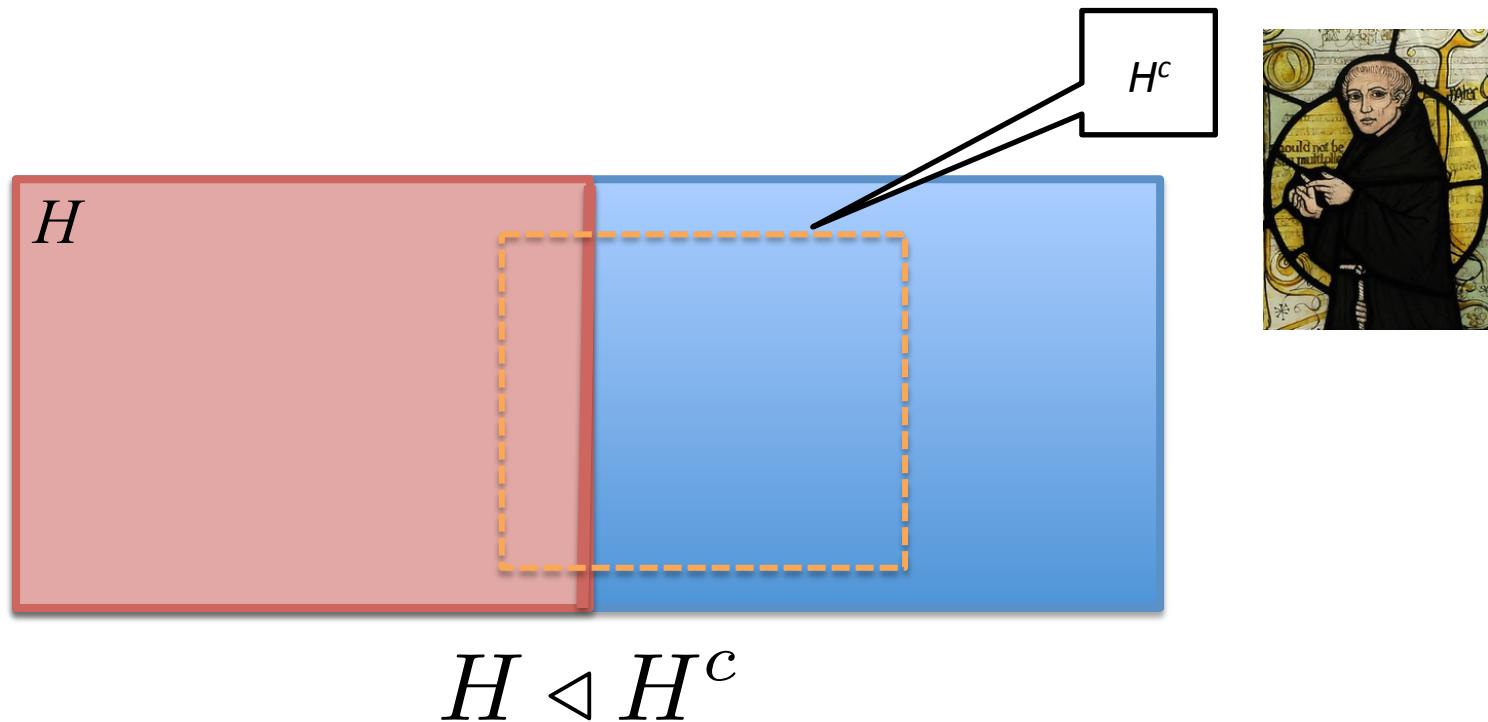
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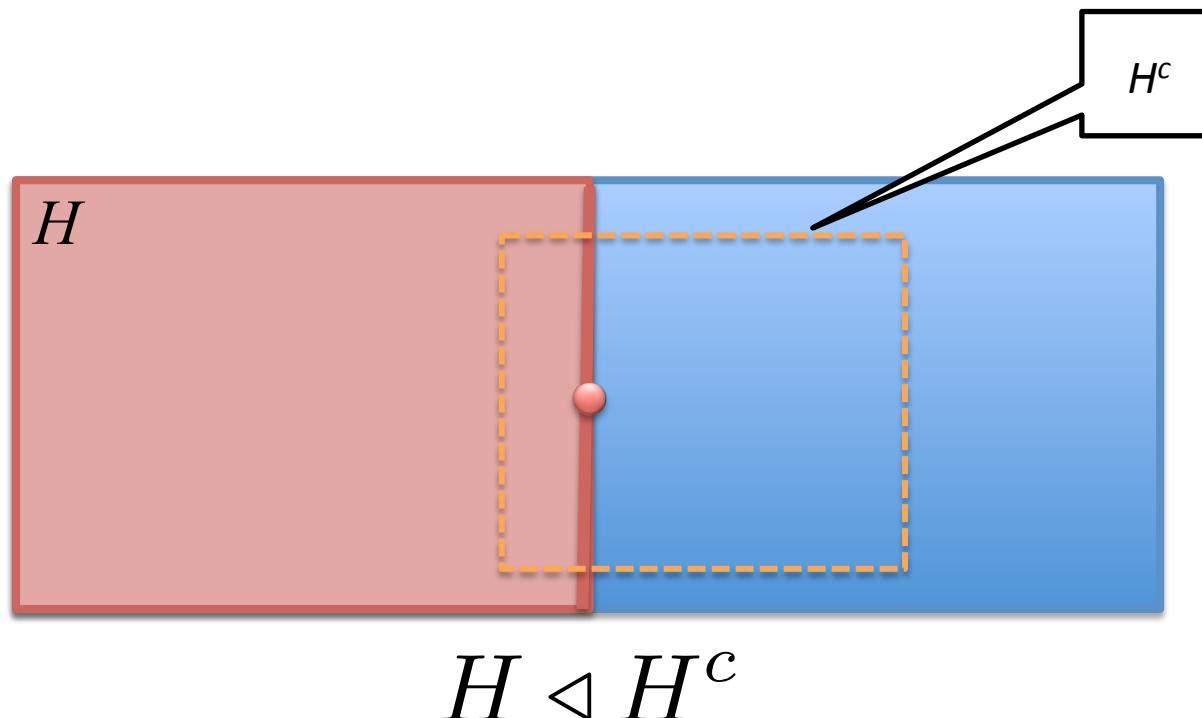
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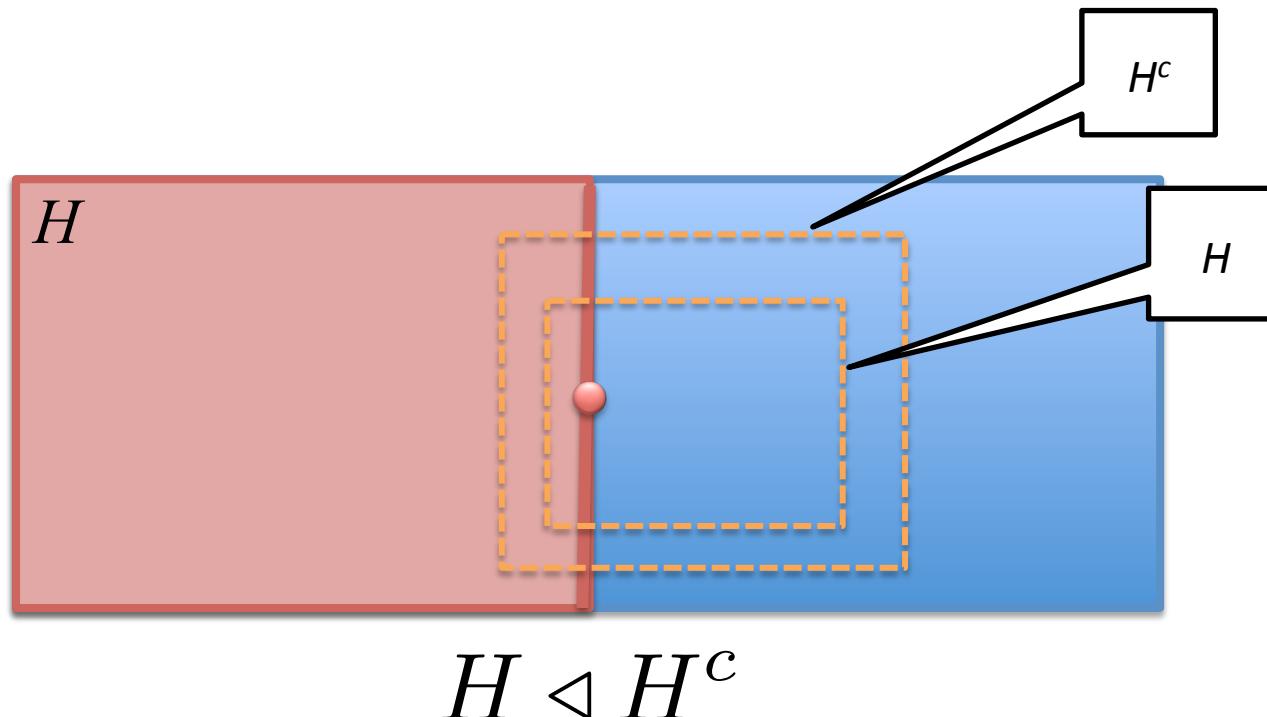
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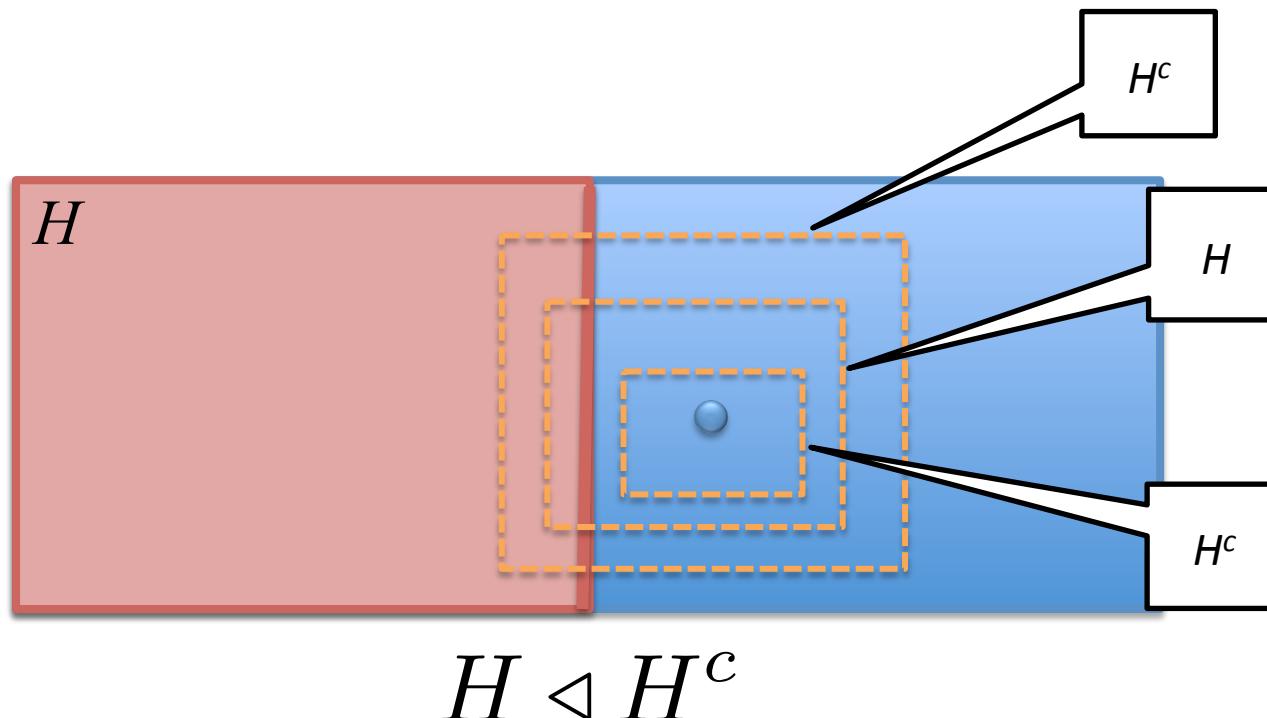
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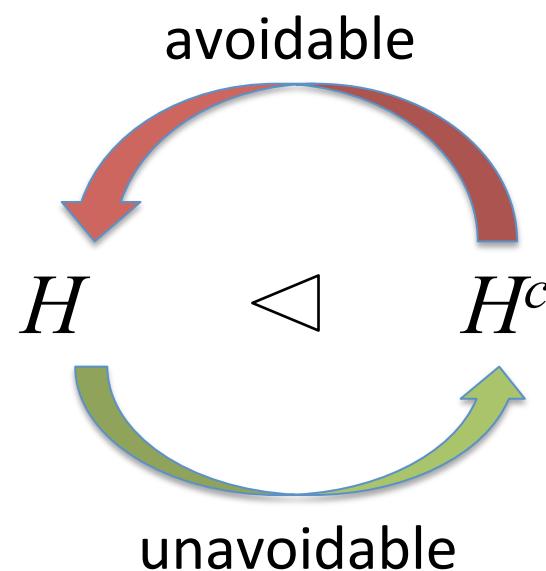
# Epistemic Mandate for Ockham's Razor

**Proposition** (Genin and Kelly, 2016). Every progressive solution obeys Ockham's razor.



# Non-Circular

By **favoring** a **complex** hypothesis, you lose in a **complex** world!



# Skepticism

That story

“... may be okay if the candidate theories are **deductively** related to observations, but when the relationship is **probabilistic**, I am skeptical ...”

Elliot Sober (2015).



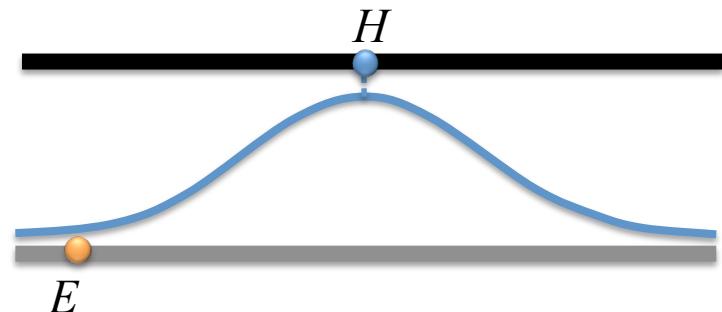
# A Worry

- Propositional information refutes logically incompatible possibilities.



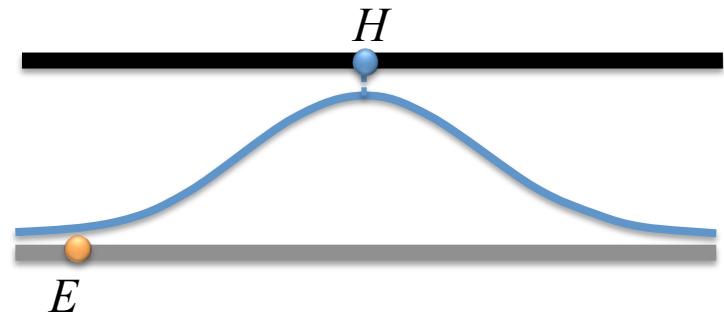
# A Worry

- Propositional information refutes logically incompatible possibilities.
- Typically, statistical samples are logically compatible with **every** possibility.



# Response

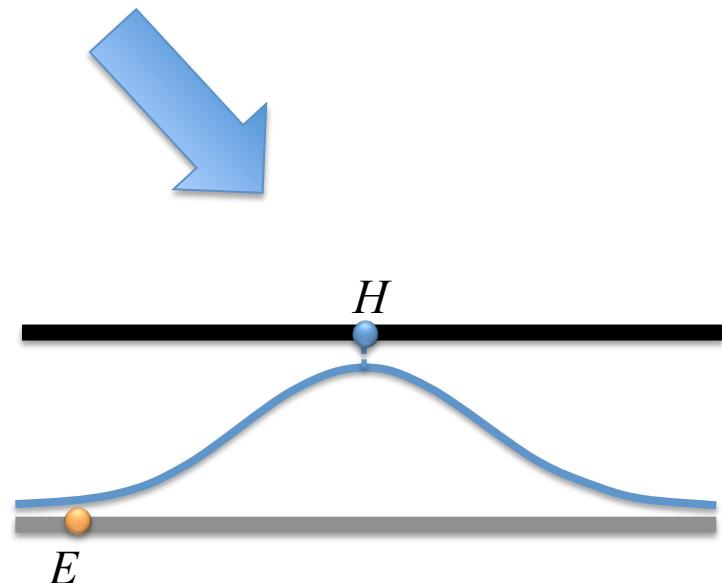
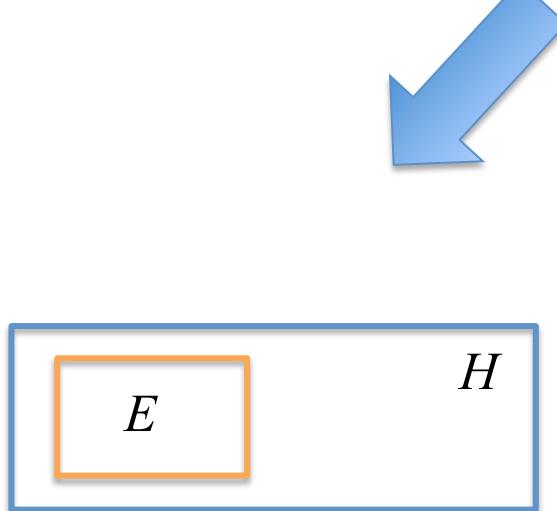
Don't worry!



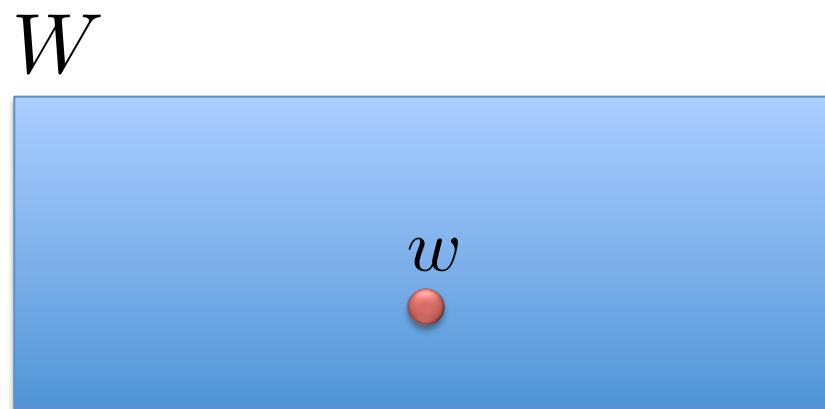
# Response

Don't worry!

Common **topological** structure

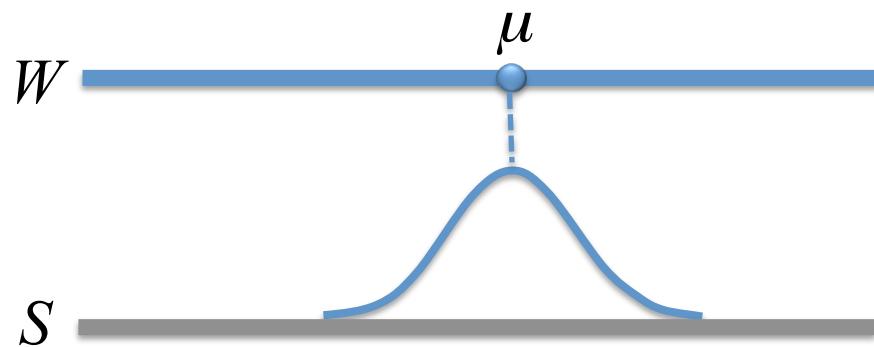


# Recall: Possible Worlds



# Statistical Worlds

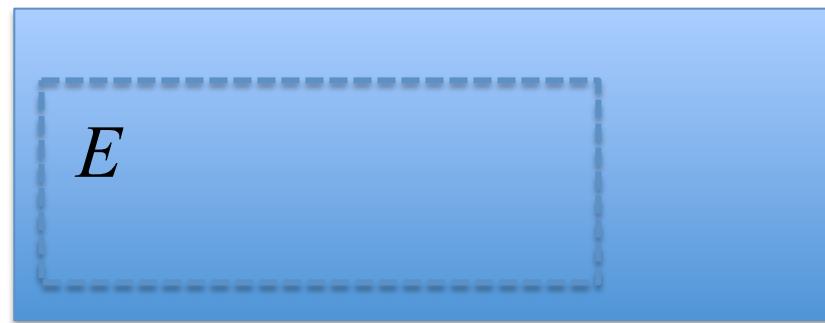
- Probability measures over a sample space.



# Recall: Information States

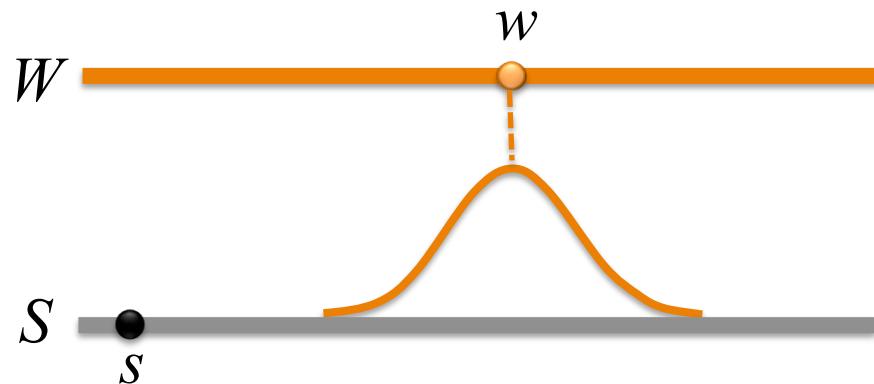
The logically strongest proposition you are informed of.

$W$

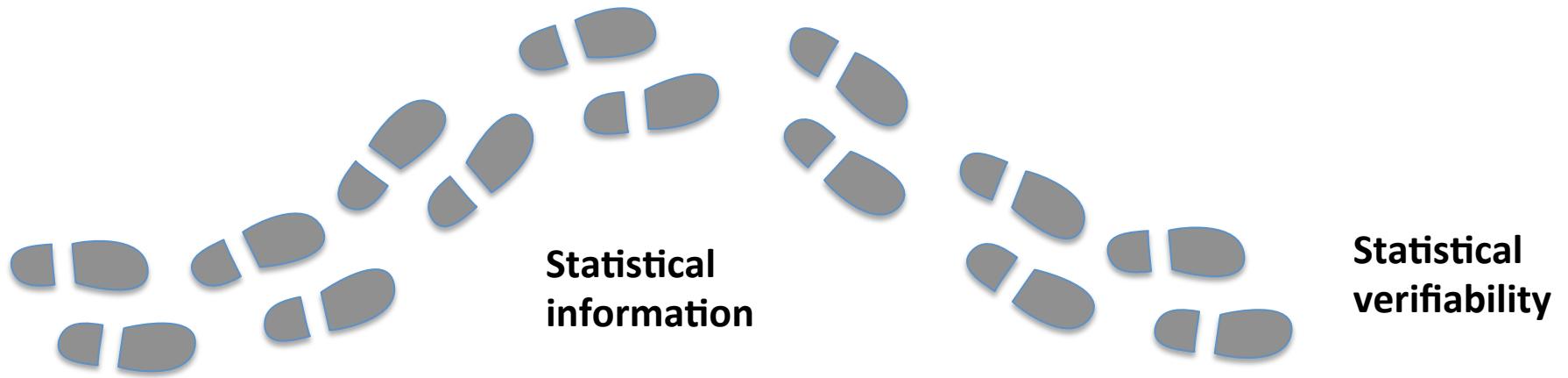


# Statistical Information?

- It seems that the only statistical information state is  $W$ .

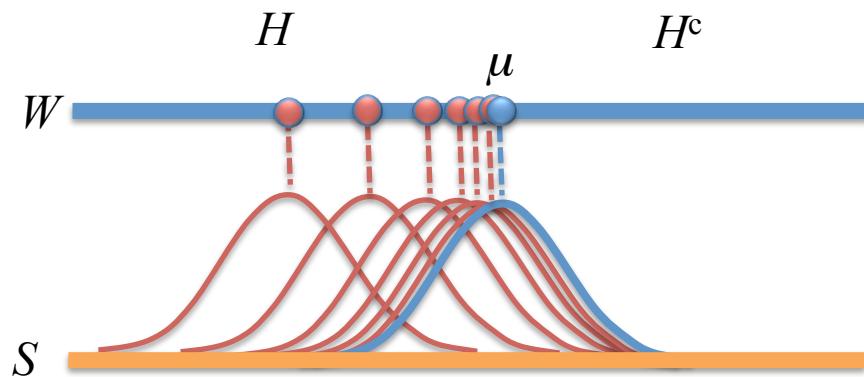


# Side-step the Worry



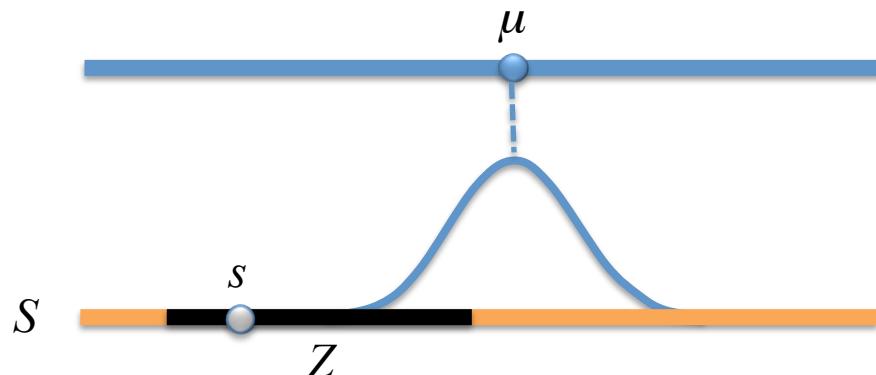
# Statistical Information Topology

Possibilities **nearer** to the truth should be **harder** to rule out by **statistical** methods.



# Gathering Statistical Information

1. The sample space  $S$  has its own topology.
2. Choose a sample event  $Z$  over  $S$ .
3. Obtain sample  $s$ .
4. Observe whether  $Z$  occurs.

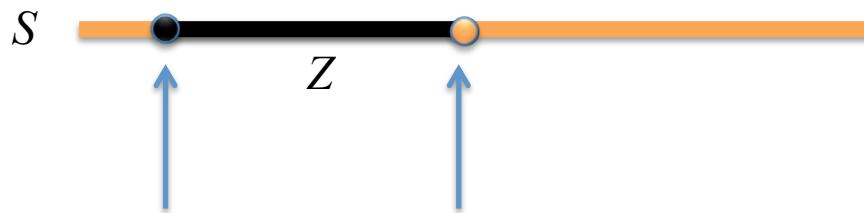
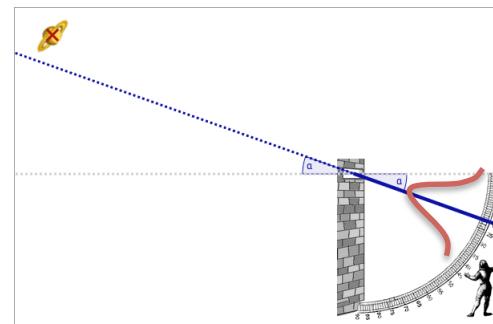


# Feasible Sample Events

- You can't **decide** whether a sample is rational-valued.

# Feasible Sample Events

- You can't **determine** whether a sample hits **exactly** on the **boundary** of an open interval.



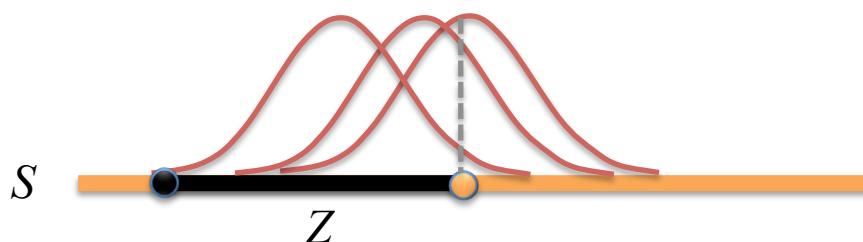
# Feasible Sample Events

- But **every** non-trivial  $Z$  on the real line has **boundary points**.



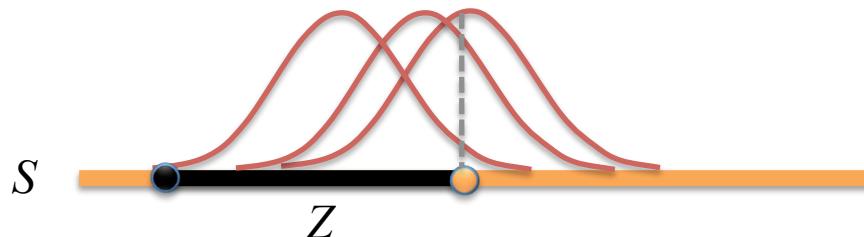
# Feasible Sample Events

- That doesn't matter **statistically** as long as the boundary carries 0 probability.
- So  $Z$  is a **feasible** sample event iff
$$p(\text{bdry } Z) = 0, \text{ for each } p \text{ in } W.$$
- I.e, **feasible**  $Z$  is **almost surely clopen** (decidable) in  $S$ .



# Feasible Statistical Models

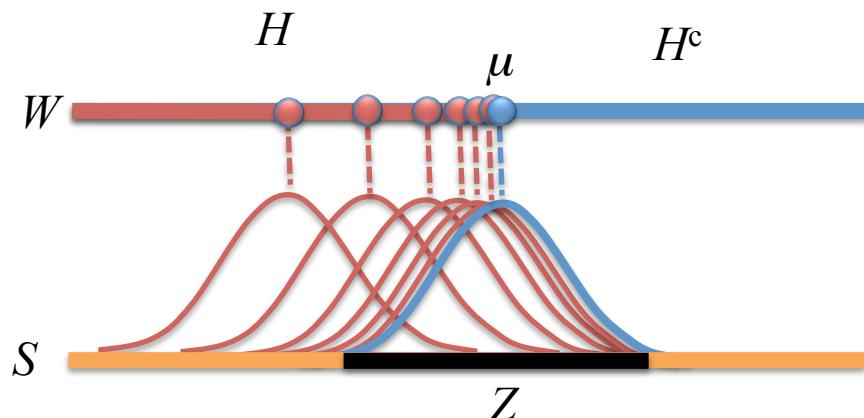
- $S$  is **feasible** for  $W$  iff  
 $S$  has a **countable topological basis** of **feasible zones**.



# Statistical Information Topology

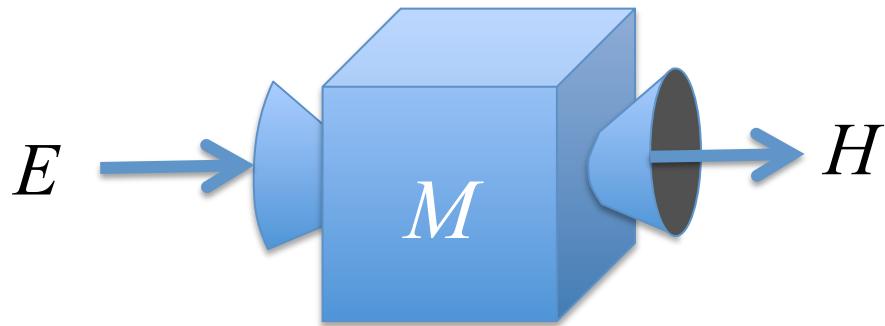
$w \in \text{cl}(H)$  iff  $H$  contains a sequence of worlds  $\mu_1, \dots, \mu_n, \dots$  such that for **every feasible** sample event  $Z \subseteq S$ :

$$\lim_{n \rightarrow \infty} \mu_n(Z) \rightarrow \mu(Z).$$



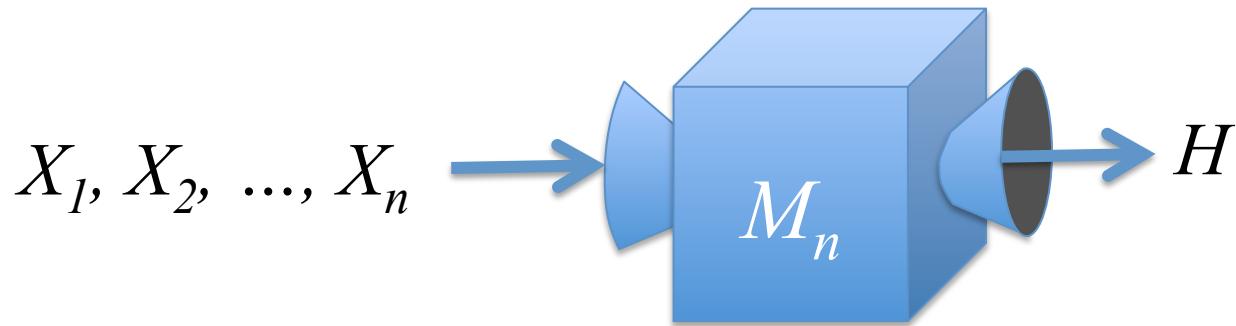
# Recall: Propositional Methods

- **Propositional methods** produce **propositional conclusions** in response to **propositional information**.



# Statistical Methods

- **Statistical methods** produce **propositional conclusions** in response to **statistical samples**.



# Feasible Statistical Methods

A **feasible statistical method** at sample size  $n$  is a function  $M_n$  from sample events in  $S^n$  to **propositions** over  $W$  such that:

$(M_n)^{-1}(H)$  is **feasible**.

A **feasible statistical method** is a collection

$(M_n : n \in \mathbb{N})$

of feasible statistical methods at each sample size.

# Recall: Verification Methods

- A **verification method** for  $H$  is an infallible, monotonic method  $V$  such that:
  1.  $w \in H^c$  implies  $V$  always concludes  $W$ .
  2.  $w \in H$  implies  $V$  converges to  $H$ .



# Statistical Verification

- A **statistical verification method** for  $H$  at **significance level**  $\alpha > 0$  is a feasible method  $(V_n : n \geq 1)$ , such that:
  1. at each sample size, outputs  $W$  with probability at least  $1-\alpha$ , if  $H$  is **false**.
  2. converges in probability to  $H$ , if  $H$  is **true**.
- $H$  is **statistically verifiable** iff  $H$  has a **statistical verification method** at **each**  $\alpha > 0$ .

# Statistical Verification

- A **statistical verification method** for  $H$  at **significance level**  $\alpha > 0$  is a feasible method  $(V_n : n \geq 1)$ , such that:
  1.  $\mu^n [V_n^{-1}(W)] \geq 1 - \alpha$ , if  $H$  is false in  $\mu$ ;
  2.  $\mu^n [V_n^{-1}(H)] \rightarrow 1$ , if  $H$  is true in  $\mu$ .
- $H$  is **statistically verifiable** iff  $H$  has a **statistical verification method** at **each**  $\alpha > 0$ .

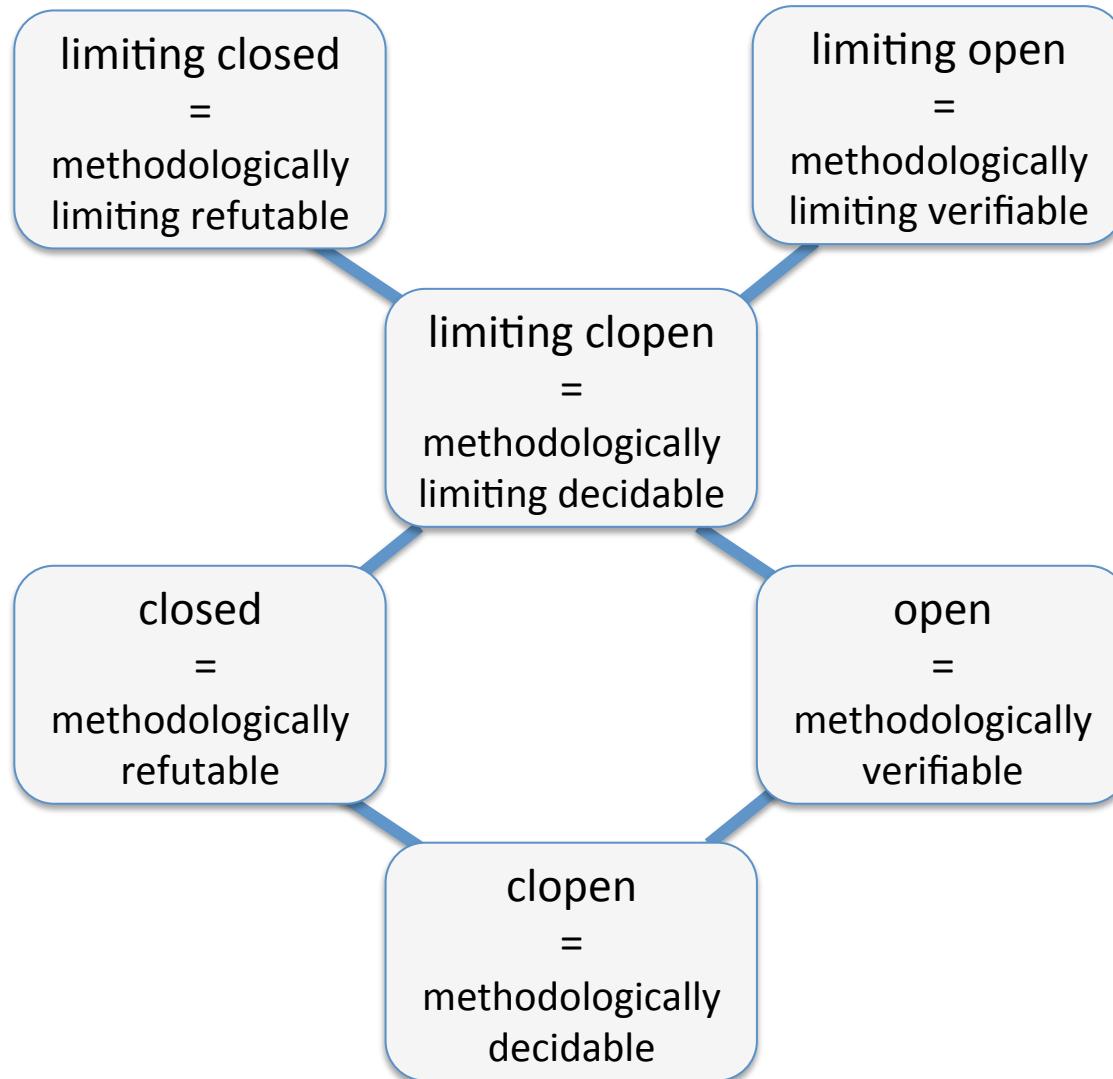
# Recall: Verification in the Limit

- A **limiting verification method** for  $H$  is a method  $M$  such that in **every** world  $w$ :  
 $H$  is true in  $w$  iff  $M$  converges to some true  $H'$  that entails  $H$ .
- $H$  is **verifiable in the limit** iff  $H$  has a limiting verifier.

# Statistical Verification in the Limit

- A **limiting statistical verification method** for  $H$ 
  - converges in probability to some  $H'$  entailing  $H$  iff  $H$  is true.
- $H$  is **statistically verifiable in the limit** iff  $H$  has a limiting **statistical verifier**.

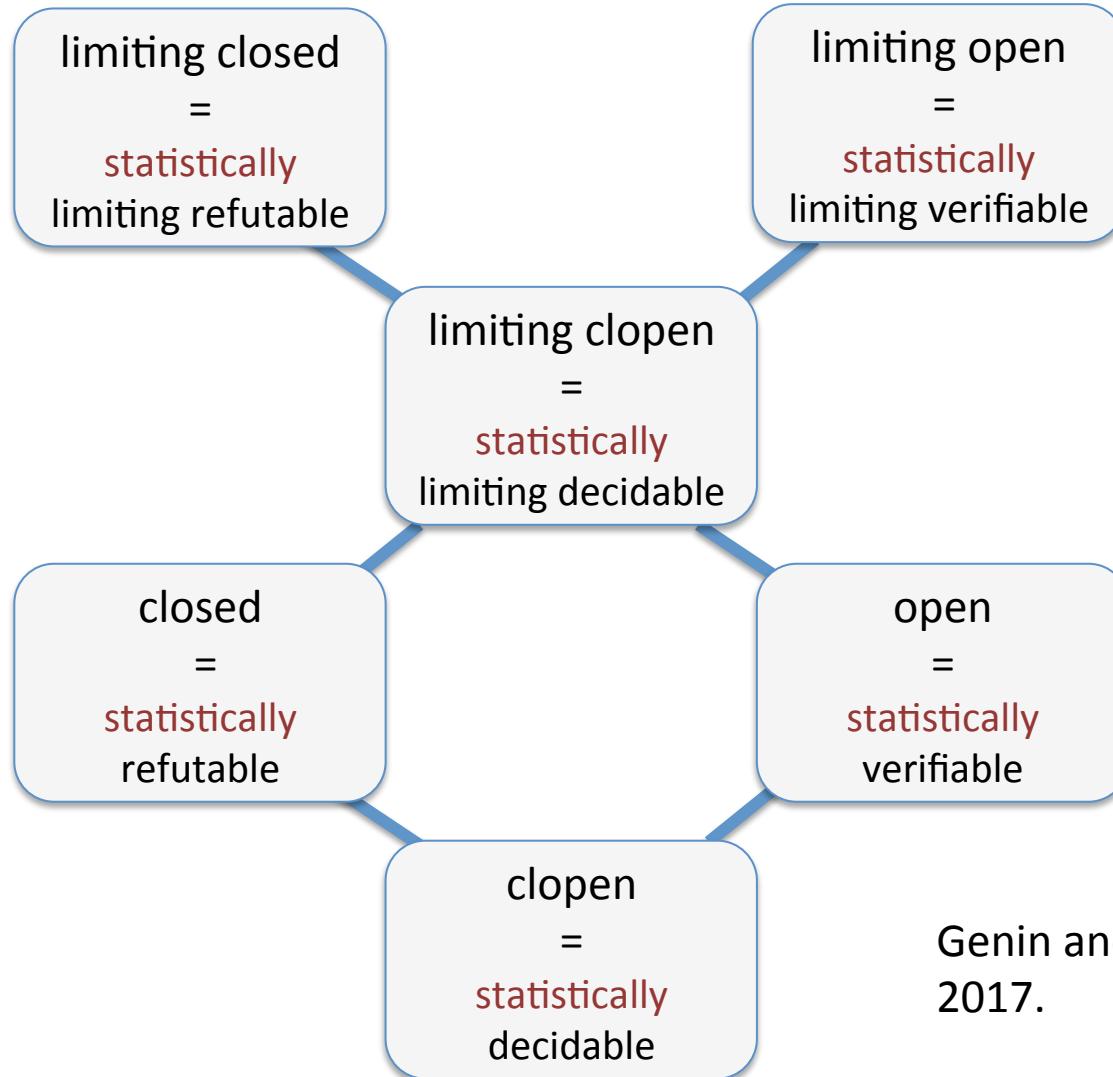
# The Propositional Hierarchy



# The Main Result

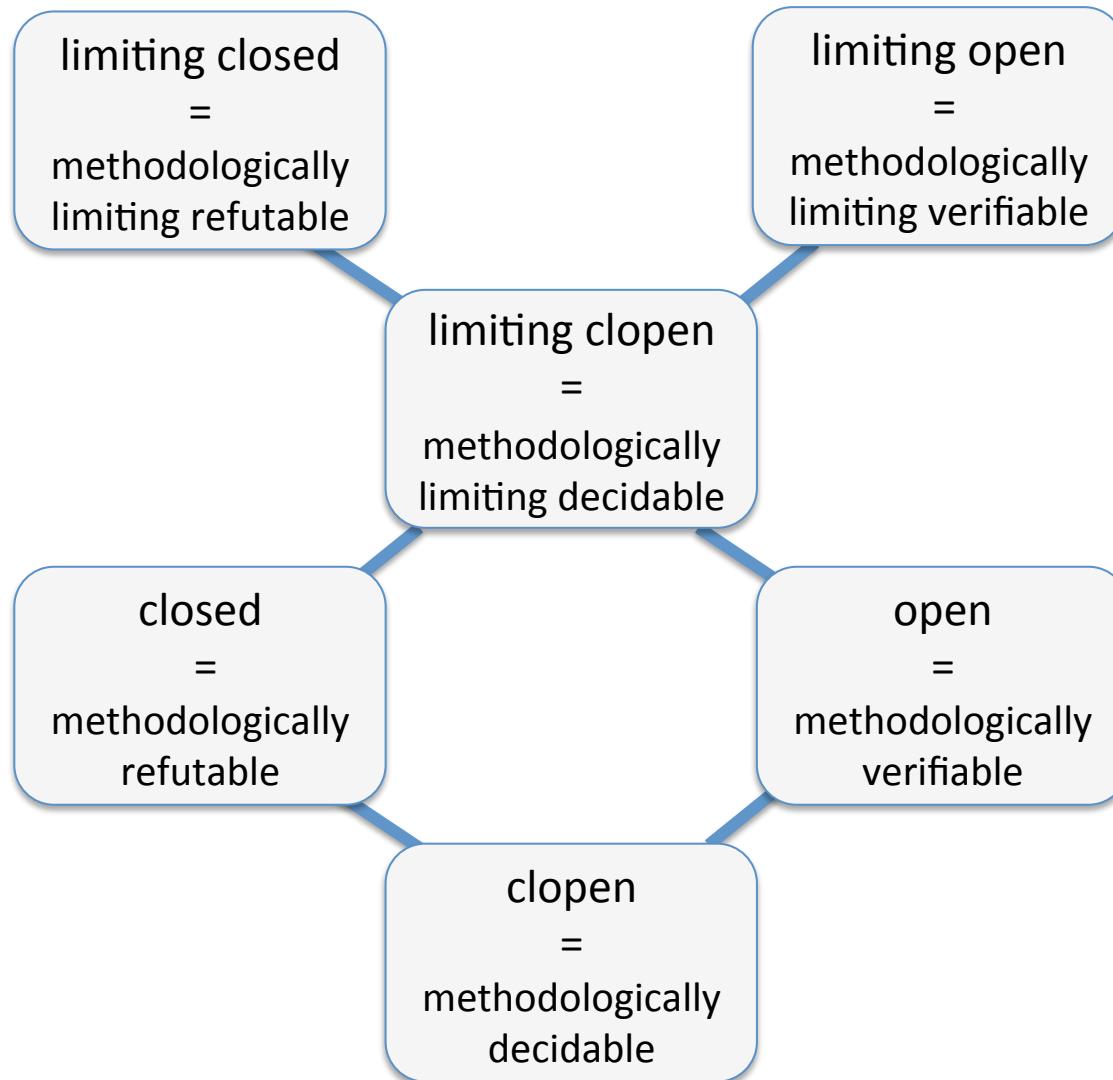
**Proposition.** (Genin, Kelly 2017) Suppose that  $S$  is feasible for  $W$ . Then, the open sets in the weak topology are exactly the statistically verifiable hypotheses.

# The Statistical Hierarchy



Genin and Kelly,  
2017.

# So in Both Logic and Statistics:



# The Topological Bridge



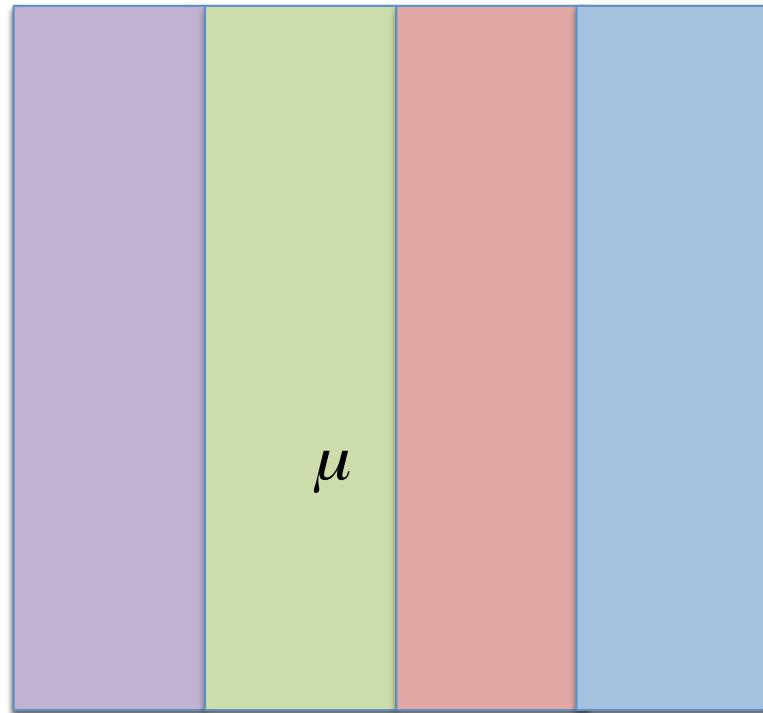
# The Topological Bridge

- Start with **logical** insights.
- Allow methods a small chance  $\alpha$  of error.
- Obtain corresponding **statistical** insights



# Statistical Problem

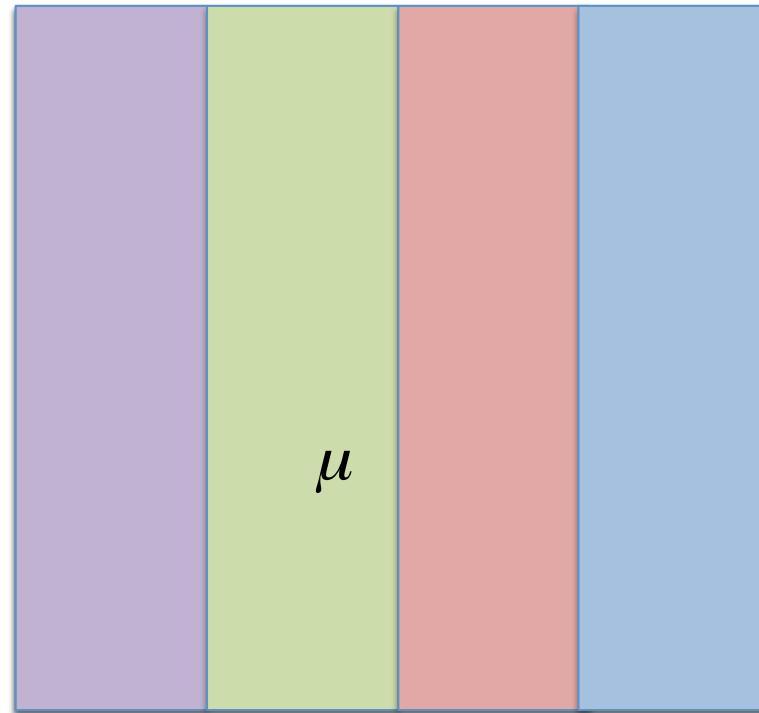
A statistical **question** partitions a set of probability measures into countably many **answers**.



# Statistical Solutions

A statistical method ( $M_n$ ) is a solution to  $Q$  iff for all  $\mu$

$$\mu^n[M_n^{-1}(Q(\mu))] \xrightarrow{n} 1.$$



# Recall: Ockham's Razor

**Proposition (Genin and Kelly, 2016).** The following principles are **equivalent**.

1. Infer a **simplest** relevant response in light of  $E$ .
2. Infer a **refutable** relevant response compatible with  $E$ .
3. Infer a relevant response that is **not more complex than the true answer**.

# Ockham's Statistical Razor

**Concern:** “consistency with  $E$ ” is **trivial** in statistics.



**Response:** the “err on the side of simplicity” version of Ockham’s razor **does not mention consistency with  $E$ .**

3. Infer a relevant response that is more complex than the true answer **with chance  $< \alpha$** .

# Ockham's Statistical Razor

A solution  $(M_n)$  to  $\mathcal{Q}$  satisfies **Ockham's  $\alpha$ -razor** iff

if  $A \in \mathcal{Q}$  and  $\mathcal{Q}(\mu) \triangleleft A$ , then  $\mu^n[M_n^{-1}(A)] < \alpha$ .

# Progressive Methods

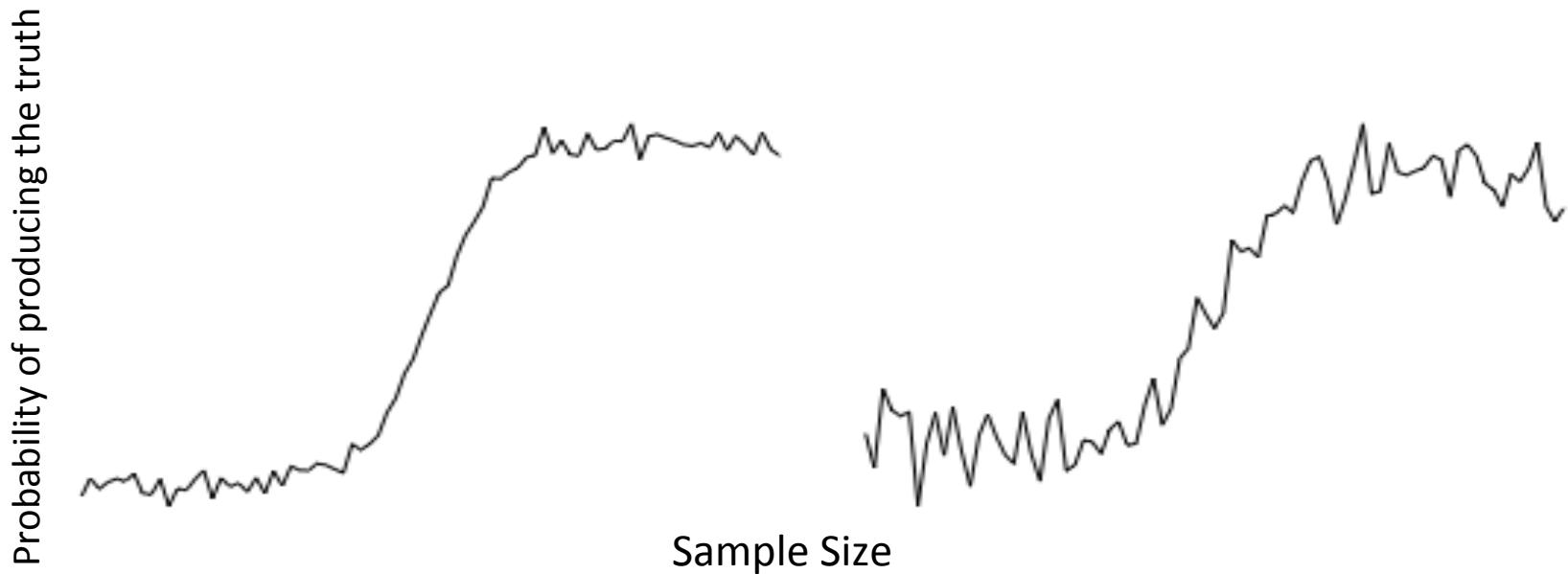
A solution ( $M_n$ ) to question  $Q$  is **progressive** if the chance that it outputs the **true answer** is strictly **increasing** with sample size, i.e. for all  $n_1 < n_2$ :

$$\mu^{n_2} [M_{n_2}^{-1}(\mathcal{Q}(\mu))] > \mu^{n_1} [M_{n_1}^{-1}(\mathcal{Q}(\mu))].$$

# $\alpha$ -Progressive Methods

- $(M_n)$  is  $\alpha$ -progressive if the chance that it outputs the true answer **never decreases** by more than  $\alpha$ , i.e. for  $n_1 < n_2$ :

$$\mu^{n_2} [M_{n_2}^{-1}(\mathcal{Q}(\mu))] + \alpha > \mu^{n_1} [M_{n_1}^{-1}(\mathcal{Q}(\mu))].$$



# Progressive Methods

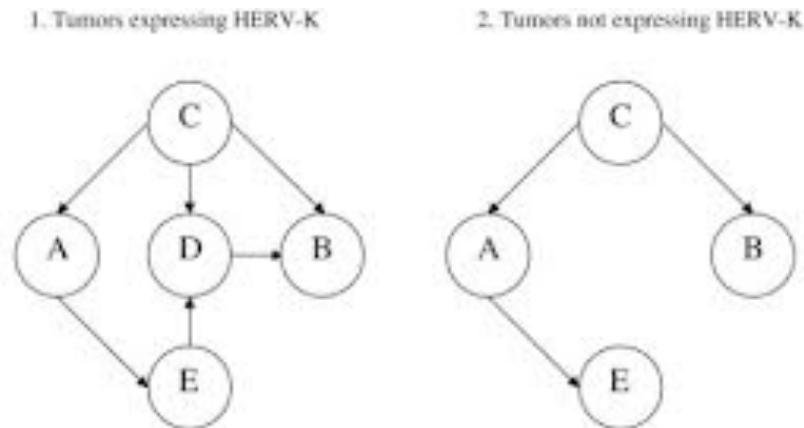
**Theorem** (Genin, 2017): If there exists an enumeration  $A_1, A_2, \dots$  of the answers to  $Q$  that agrees with the simplicity order, then there exists an  $\alpha$ -progressive method for every  $\alpha > 0$ .

# Ockham and Progress

**Theorem** (Genin, 2017): Every  $\alpha$ -progressive solution satisfies Ockham's  $\alpha$ -razor.

# Application: Causal Inference from Non-experimental Data

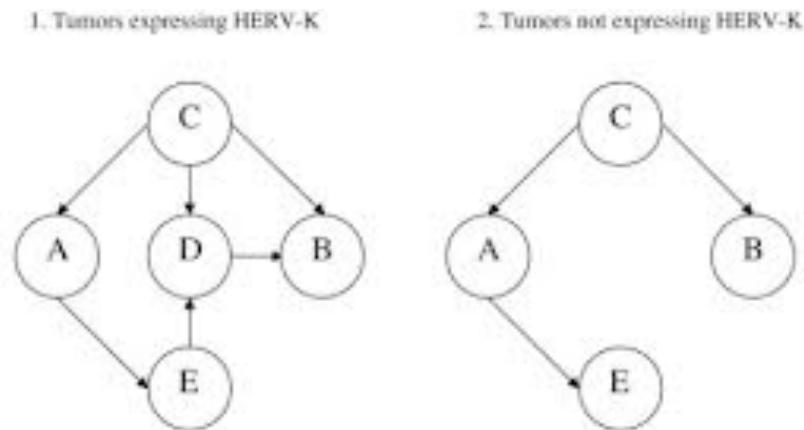
- Causal inference from observational data.
- The search is strongly guided by Ockham's razor.
- Previously, methods were only proven to be point-wise-consistent.



Symbols: "A" = cause (YFV); "B" = outcome (cancer); "C" = confounders (recreational solar exposure and high social class); "D" and "E" = mediators (HERV-K antigen and immune response).

# Application: Causal Inference from Non-experimental Data

**Proposition** (Genin, 2018). For the problem of inferring Markov equivalence classes, there exist  $\alpha$ -progressive solution for every  $\alpha > 0$ .



Symbols: "A" = cause (YFV); "B" = outcome (cancer); "C" = confounders (recreational solar exposure and high social class); "D" and "E" = mediators (HERV-K antigen and immune response).

**Thank you!**