The St. Petersburg Game

And Other Paradoxes of Decision Theory

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You can't make an omelet ...

... without decision theory.

	Good	Rotten
Break into Bowl	6-Egg Omelet	No Omelet
Break into Saucer	6-Egg Omelet, extra washing	5-Egg Omelet, extra washing
Throw it Out	5-Egg Omelet, 1 egg wasted	5-Egg Omelet

Decision under Uncertainty

Expected Utility Theory

To make the *rational* decision:

- assign a *probability* to each state of the world;
- assign a number (*utility) to each outcome;*
- *calculate the* expected utility of each act.

Finally, choose the act with the greatest expected utility.

Decision under Uncertainty

Expected Utility Theory

- Fermat, Pascal (1654);
- von Neumann, Morgenstern (1947);
- Savage (1954).

Decision under Uncertainty

	Good: 50%	Rotten: 50%
Break into Bowl (A1)	6-Egg Omelet +10	No Omelet 0
Break into Saucer (A2)	6-Egg Omelet, extra washing +8	5-Egg Omelet, extra washing +4
Throw it Out (A3)	5-Egg Omelet, 1 egg wasted +5	5-Egg Omelet +6

$$\begin{aligned} EU(A1) &= .5*10+.5*0 = & 5 \\ EU(A2) &= .5*8+.5*4 = & 6 \\ EU(A3) &= .5*5+.5*6 = & 5.5 \end{aligned}$$

Decision under Uncertainty

Expected Utility Theory

- Wants to help us make *rational decisions* when faced with uncertainty.
- Is a widely-adopted framework in economics and finance.
- Makes apparently paradoxical recommendations in certain situations!

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The St. Petersburg Game

In 1713, Daniel Bernoulli proposes the following game:

A fair coin is flipped until the first time heads appears. The player wins \$ 2^n where n is the number of times the coin was flipped.

How much should you be willing to pay in order to play this game?

The St. Petersburg Game

What is the probability of each state?

$$
\begin{aligned}
\mathrm{P}(H) &= \tfrac{1}{2} \\
\mathrm{P}(TH) &= \tfrac{1}{2} * \tfrac{1}{2} = \tfrac{1}{4} \\
\mathrm{P}(TTH) &= \tfrac{1}{2} * \tfrac{1}{2} * \tfrac{1}{2} = \tfrac{1}{8} \\
\mathrm{P}(TTTH) &= \tfrac{1}{2} * \tfrac{1}{2} * \tfrac{1}{2} * \tfrac{1}{2} = \tfrac{1}{16} \\
\mathrm{P}(TTTTH) &= \tfrac{1}{2} * \tfrac{1}{2} * \tfrac{1}{2} * \tfrac{1}{2} * \tfrac{1}{2} = \tfrac{1}{32} \\
&\dots
\end{aligned}
$$

The St. Petersburg Game

What is the payout for each state?

$$
\begin{aligned}
\$(H) &= \$2 \\
\$(TH) &= 2*2 = \$4 \\
\$(TTH) &= 2*2*2 = \$8 \\
\$(TTTH) &= 2*2*2*2 = \$16 \\
\$(TTTTH) &= 2*2*2*2*2 = \$32 \\
&\dots
\end{aligned}
$$

The St. Petersburg Game

	H 1/2	TH 1/4	TTH 1/8	$TTTH$ 1/16	$TTTTH$ 1/32	...
Play	\$2	\$4	\$8	\$16	\$32	...
Don't Play	\$0	\$0	\$0	\$0	\$0	...

The St. Petersburg Game

	H 1/2	TH 1/4	TTH 1/8	$TTTH$ 1/16	$TTTTH$ 1/32	...
Play	\$2	\$4	\$8	\$16	\$32	...

$$
\begin{aligned}
EU(Play) &= \frac{1}{2} * 2 + \frac{1}{4} * 4 + \frac{1}{8} * 8 + \frac{1}{16} * 16 + \frac{1}{32} * 32 + \cdots \\
&= 1 \quad + 1 \quad + 1 \quad + 1 \quad + 1 \quad + \cdots \\
&= \infty
\end{aligned}
$$

The St. Petersburg Game

If the expected utility of the game is infinite, a rational person ought to be willing to pay *any price* for a chance to play the game.

The St. Petersburg Game

But, intuitively, this is totally wrong! People *aren't* willing to pay an arbitrary amount for such a game. And it doesn't seem like they are *irrational*.

Bernoulli's Resolution

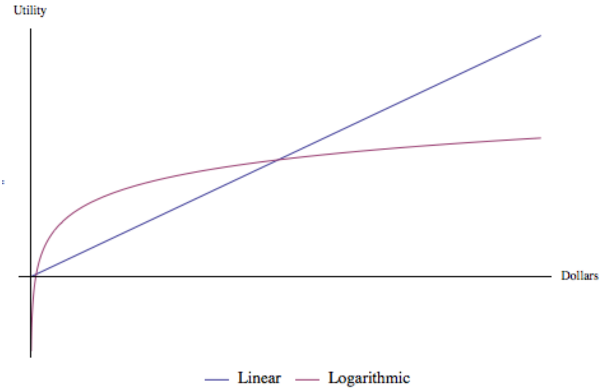
...there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount. ...any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversely proportionate to the quantity of goods already possessed (Daniel Bernoulli, 1738).

Bernoulli's Resolution

We assumed that utility is a linear function of the amount of money you have. Bernoulli argues that the more money you have, the less you are interested in an extra dollar. He suggests that $U(\$) = \log(\$)$.

We now call this phenomenon *decreasing marginal utility*.

Bernoulli's Resolution



The St. Petersburg Game

	H 1/2	TH 1/4	TTH 1/8	$TTTH$ 1/16	$TTTTH$ 1/32	...
Play	\$2 .3	\$4 .6	\$8 .9	\$16 1.2	\$32 1.5	...

$$EU(Play) = \frac{1}{2} * \log(2) + \frac{1}{4} * \log(4) + \frac{1}{8} * \log(8) + \frac{1}{16} * \log(16) + \cdots$$
$$= \$4$$

Bernoulli's Resolution

But the paradox returns if we just change the payoffs. Suppose the player wins \$ $\exp(2^n)$ where n is the number of times the coin was flipped.

Bernoulli's Resolution

	H 1/2	TH 1/4	TTH 1/8	$TTTH$ 1/16	$TTTTH$ 1/32	...
Play	$\$e^2$	$\$e^4$	$\$e^8$	$\$e^{16}$	$\$e^{32}$	...

$$
\begin{aligned}
EU(Play) &= \frac{1}{2} * \log(e^2) + \frac{1}{4} * \log(e^4) + \frac{1}{8} * \log(e^8) + \frac{1}{16} * \log(e^{16}) + \cdots \\
&= 1 \quad + 1 \quad + 1 \quad + 1 \quad + 1 \quad + \cdots \\
&= \infty
\end{aligned}
$$

Risk Aversion

Different people have different attitudes toward risk.

Offer 1: A guaranteed \$50.
Offer 2: A 50% chance of \$100 and a 50% chance of \$0.

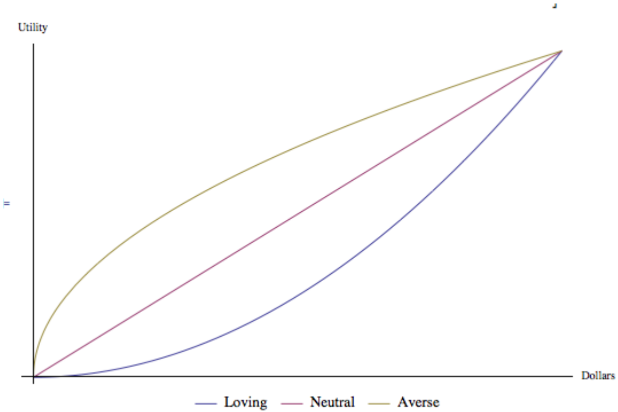
The mathematical expectation is the same for both offers. If you are indifferent between 1 and 2, then you are *risk neutral*. If you prefer 1, you are *risk averse*. If you prefer 2, you are *risk loving*.

Risk Aversion

The St. Petersburg game pays out large rewards with very small probability. But the probability of getting \$2 is 50%.

Does risk aversion explain away the paradox?

Risk Aversion



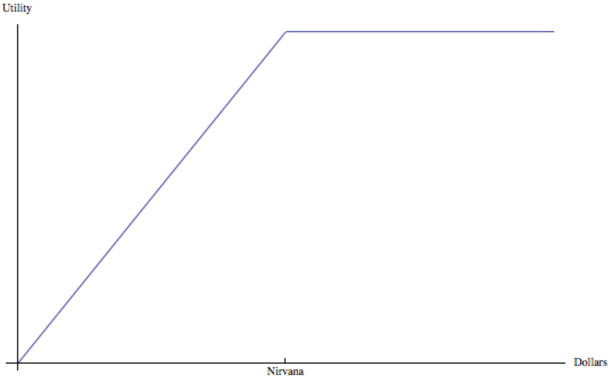
Risk affinity is modeled by curving the utility function.

Risk Aversion

But we can always compensate for risk aversion by increasing the payoffs.

In fact, Karl Menger (1923) showed that a St. Petersburg game exists for any *unbounded* utility function.

Bounded Utility



Maybe everyone has a finite upper bound for utility.

The Pasadena Game

In 2004, Harris Nover and Alan Hájek introduced a variant of the St. Petersburg Game.

The Pasadena Game

	H 1/2	TH 1/4	TTH 1/8	$TTTH$ 1/16	$TTTTH$ 1/32	...
Play	\$2	-\$4/2	\$8/3	-\$16/4	\$32/5	...

$$EU(Play) = \frac{1}{2} * 2 - \frac{1}{4} * \frac{4}{2} + \frac{1}{8} * \frac{8}{3} - \frac{1}{16} * \frac{16}{4} + \frac{1}{32} * \frac{32}{5} + \cdots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

$$= \log(2)$$

The Pasadena Game

But rearranging the terms of the series yields different results!

The Pasadena Game

$$EU(Play) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \ldots \qquad = \log(2)$$

$$EU(Play) = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} + \ldots \qquad = \frac{\log(2)}{2}$$

$$EU(Play) = (1) + \left(\frac{1}{3} + \frac{1}{5} + \ldots - \frac{1}{2}\right) + \left(\frac{1}{25} + \ldots - \frac{1}{4}\right) + \ldots = \infty$$

The Pasadena Game

Riemann series theorem (1853): the terms of a conditionally convergent series can be rearranged so that the new series converges to any given value, $+\infty$, or $-\infty$.

The expectation value of the Pasadena game is *undefined.*

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Allais Paradox (1953)

	89%	1%	10%
A	\$1mn	\$1mn	\$1mn
B	\$1mn	\$0	\$5mn
C	\$0	\$1mn	\$1mn
D	\$0	\$0	\$5mn

EU Theory: if you prefer A to B, then you ought to prefer C to D. People often do not reason this way!

Allais Paradox (1953)

	89%	1%	10%
A	\$1mn	\$1mn	\$1mn
B	\$1mn	\$0	\$5mn
C	\$0	\$1mn	\$1mn
D	\$0	\$0	\$5mn

If you prefer A to B, then

$$1.00 * U(\$1\text{mn}) > .89 * U(\$1\text{mn}) + .01 * U(\$0) + .1 * U(\$5\text{mn})$$

Allais Paradox (1953)

	89%	1%	10%
A	\$1mn	\$1mn	\$1mn
B	\$1mn	\$0	\$5mn
C	\$0	\$1mn	\$1mn
D	\$0	\$0	\$5mn

If you prefer D to C, then

$$.11 * U(\$1\text{mn}) + .89 * U(\$0) < .1 * U(\$5\text{mn}) + .9 * U(\$0)$$

$$1.00 * U(\$1\text{mn}) - .89 * U(\$1\text{mn}) < .1 * U(\$5\text{mn}) + .01 * U(\$0)$$

$$1.00 * U(\$1\text{mn}) < .89 * U(\$1\text{mn}) + .1 * U(\$5\text{mn}) + .01 * U(\$0)$$

Allais Paradox (1953)

So both

$$1.00 * U(\$1\text{mn}) > .89 * U(\$1\text{mn}) + .01 * U(\$0) + .1 * U(\$5\text{mn})$$

$$1.00 * U(\$1\text{mn}) < .89 * U(\$1\text{mn}) + .01 * U(\$0) + .1 * U(\$5\text{mn})$$

Contradiction!

Ellsberg Paradox (1961)

	Red: 1/3	Black: ?	Yellow: ?
A	\$100	\$0	\$0
B	\$0	\$100	\$0
C	\$100	\$0	\$100
D	\$0	\$100	\$100

EU Theory: if you prefer A to B, then you ought to prefer C to D. People often do not reason this way!

Questions?

Thank you!