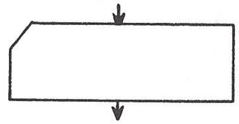
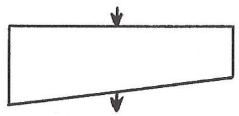
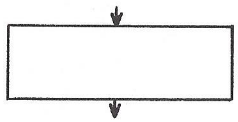
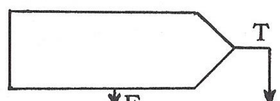
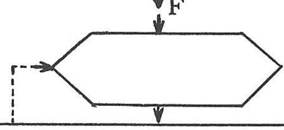
A statement may be preceded by a *label*. Statement labels serve to identify particular statements and are useful for the purpose of proceeding from one statement to another in a sequence other than that in which they are listed.

The types of statements used in the programs in this book are summarized in Table 2-2. The flow chart symbols denoting these statements also appear in the table. Each of the statements listed in Table 2-2 is described briefly in the following discussion.

TABLE 2-2

Statements Used in Computer Programs

Type of Statement	Flow Chart Symbol
(a) Input	
(b) Output	
(c) Assignment	
(d) Unconditional Control	
(e) Conditional Control	
(f) Iterative Control	

The *input statement* causes information to be transferred into the memory unit of the computer from some external source, such as data cards or tape. An input statement is represented in a flow chart by a box in the shape of a data card (see Table 2-2). As an example, suppose that the numbers A, B, and C are to be brought into the computer as input data. This may be represented in the flow chart as shown in Fig. 2-1a. (The arrows denote the flow of steps in a chart.)

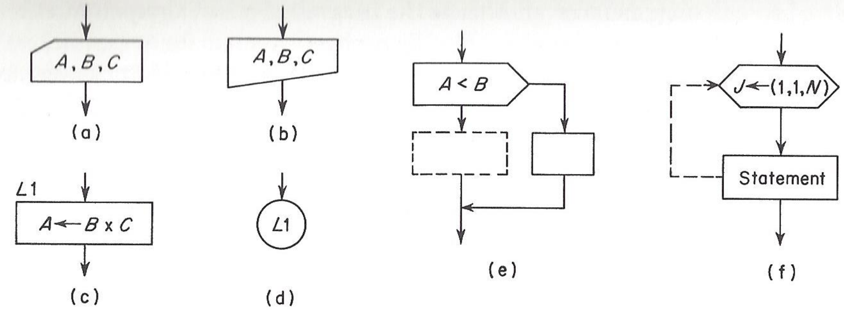


FIG. 2-1. Flow chart symbols for statements: (a) input statement; (b) output statement; (c) assignment statement; (d) unconditional control statement; (e) conditional control statement; (f) iterative control statement

The *output statement* causes information in the computer to be transferred outside and communicated to the programmer by means of a printer, a card punch, or other device. An output statement appears in a flow chart as a box with a diagonal line at the base, as shown in Table 2-2. Figure 2-1b demonstrates the appearance in a flow chart of an output statement that causes the numbers A, B, and C to be printed (or otherwise communicated to the programmer).

The *assignment statement* causes the current value of a variable to be replaced with a new value. For example, the statement

$$A \leftarrow B \times C$$

means that $B \times C$ is to be evaluated and put into the storage location for the variable A. The current value of A is, therefore, replaced by the product of B times C. The replacement operator (a left arrow) is an integral part of the assignment statement. In a flow chart the assignment statement is enclosed in a rectangle (see Table 2-2). The statement described above is shown in Fig. 2-1c as it would appear in a flow chart. The manner of showing a statement label in a flow chart is also illustrated in Fig. 2-1c. The label is written at the upper left-hand corner of the rectangle, and in this example the label is assumed to be the identifier $L1$.

The *unconditional control statement* causes a change in the order of execution of statements in a program. By means of this statement, control may be transferred to any other statement in the program by referring to its statement label. The unconditional control statement is represented in a flow chart by a circle, as shown in Table 2-2. The label of the statement to which control is transferred is indicated inside the circle. For example, in Fig. 2-1d the statement label $L1$ in the circle indicates that control is to be transferred to the statement in the program having the label $L1$.

The *conditional control statement* provides the ability to make decisions based upon a test, which may be a relational expression. Control is trans-

2.3 Method of Decomposition for Symmetric Matrices. Equation (1-65) in Art. 1.9 represents a set of n simultaneous linear algebraic equations containing n unknowns. There are many methods for computing the unknowns in such equations.* An approach which is particularly well suited for structural analysis programs is called the *method of decomposition* (also referred to as the *factorization method*). Since the stiffness matrices of linearly elastic structures are always symmetric, an especially efficient form of the method of decomposition called *Cholesky's square root method*† may be applied to such problems. Details of this special form of the method are developed in the following discussion. The algorithms and notation adopted in this article are incorporated into the procedures presented in the remainder of the chapter.

Let the symbol **A** represent a symmetric, nonsingular matrix. Such a matrix can be decomposed (or factored) into the product of a lower triangular matrix and an upper triangular matrix, each of which is the transpose of the other. Thus, the decomposition of **A** may be stated as:

$$\mathbf{A} = \mathbf{U}'\mathbf{U} \tag{2-1}$$

The symbol **U** in this expression denotes the upper triangular matrix, and **U**′ is its transpose. Equation (2-1) in expanded form becomes:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{bmatrix} = \begin{bmatrix} U_{11} & 0 & 0 & \cdots & 0 \\ U_{12} & U_{22} & 0 & \cdots & 0 \\ U_{13} & U_{23} & U_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ U_{1n} & U_{2n} & U_{3n} & \cdots & U_{nn} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ 0 & 0 & U_{33} & \cdots & U_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & U_{nn} \end{bmatrix} \tag{2-2}$$

It can be seen from Eq. (2-2) that elements of the matrix **A** consist of inner products of the rows of **U**′ and the columns of **U**. Equivalently, the elements of **A** may be calculated as inner products among the columns of **U**. Thus,

$$A_{ii} = U_{1i}^2 + U_{2i}^2 + \cdots + U_{ii}^2$$

Or,

$$A_{ii} = \sum_{k=1}^{i} U_{ki}^2 \qquad (i = j) \tag{a}$$

And,

$$A_{ij} = U_{1i}U_{1j} + U_{2i}U_{2j} + \cdots + U_{ii}U_{ij}$$

Or,

$$A_{ij} = \sum_{k=1}^{i} U_{ki}U_{kj} \qquad (i < j) \tag{b}$$

*For detailed descriptions of various calculation techniques, see Faddeeva, V. N., *Computational Methods of Linear Algebra*, Dover Publications, Inc., New York, 1959.

†Salvadori, M. C., and Baron, M. L., *Numerical Methods in Engineering*, 2nd ed., Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961, pp. 27–32.

The elements of **U** may be determined by rearranging Eqs. (a) and (b) as follows:

$$U_{ii} = \sqrt{A_{ii} - \sum_{k=1}^{i-1} U_{ki}^2} \qquad (1 < i = j) \tag{2-3}$$

$$U_{ij} = \frac{A_{ij} - \sum_{k=1}^{i-1} U_{ki}U_{kj}}{U_{ii}} \qquad (1 < i < j) \tag{2-4}$$

$$U_{ij} = 0 \qquad (i > j) \tag{2-5}$$

Equations (2-3), (2-4), and (2-5) represent steps in an algorithm for decomposing the matrix **A**. The square root term in Eq. (2-3) leads to the name *square root method* for symmetric matrices.

Assume that the objective in an analysis is to solve the following system of equations:

$$\mathbf{AX} = \mathbf{B} \tag{2-6}$$

in which **X** is a column vector of n unknowns and **B** is a column vector of constant terms. As a preliminary step, substitute Eq. (2-1) into Eq. (2-6):

$$\mathbf{U}'\mathbf{UX} = \mathbf{B} \tag{2-7}$$

Then define the vector $\mathbf{X}^*$ to be:

$$\mathbf{UX} = \mathbf{X}^* \tag{2-8}$$

In expanded form this expression is:

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ 0 & 0 & U_{33} & \cdots & U_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & U_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ X_n \end{bmatrix} = \begin{bmatrix} X_1^* \\ X_2^* \\ X_3^* \\ \cdots \\ X_n^* \end{bmatrix} \tag{2-9}$$

Substitution of Eq. (2-8) into Eq. (2-7) yields:

$$\mathbf{U}'\mathbf{X}^* = \mathbf{B} \tag{2-10}$$

Or, in expanded form:

$$\begin{bmatrix} U_{11} & 0 & 0 & \cdots & 0 \\ U_{12} & U_{22} & 0 & \cdots & 0 \\ U_{13} & U_{23} & U_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ U_{1n} & U_{2n} & U_{3n} & \cdots & U_{nn} \end{bmatrix} \begin{bmatrix} X_1^* \\ X_2^* \\ X_3^* \\ \cdots \\ X_n^* \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \cdots \\ B_n \end{bmatrix} \tag{2-11}$$

The vector of unknowns **X** may now be obtained in two steps using Eqs. (2-10) and (2-8). In the first step Eq. (2-10) is solved for the vector $\mathbf{X}^*$. Since **U**′ is a lower triangular matrix (see Eq. 2-11), the elements of

$\mathbf{X}^*$ can be obtained in a series of forward substitutions. For example, the first element of $\mathbf{X}^*$ is:

$$X_1^* = \frac{B_1}{U_{11}} \tag{c}$$

In general, the recurrence formula for elements of $\mathbf{X}^*$ becomes:

$$X_i^* = \frac{B_i - \sum_{k=1}^{i-1} U_{ki}X_k^*}{U_{ii}} \qquad (1 < i) \tag{2-12}$$

The second step consists of solving for the vector $\mathbf{X}$ in Eq. (2-8). Since $\mathbf{U}$ is an upper triangular matrix (see Eq. 2-9), the elements of $\mathbf{X}$ are determined in a backward substitution procedure. The last element of $\mathbf{X}$ is:

$$X_n = \frac{X_n^*}{U_{nn}} \tag{d}$$

And all other elements of $\mathbf{X}$ may be calculated from the recurrence formula:

$$X_i = \frac{X_i^* - \sum_{k=i+1}^{n} U_{ik}X_k}{U_{ii}} \qquad (i < n) \tag{2-13}$$

This step completes the solution of the original equations (Eq. 2-6) for the unknown quantities.

The decomposition and solution procedures described above for a general symmetric matrix apply also to a band symmetric matrix. For the latter type of matrix the upper triangular component $\mathbf{U}$ has the same band width, and fewer calculations are required in the recurrence formulas because all elements outside the band are zero.

Assume next that the immediate objective in an analysis is to calculate the inverse of the matrix $\mathbf{A}$. Referring again to the factored form of $\mathbf{A}$ in Eq. (2-1), the inverse matrix is seen to be:

$$\mathbf{A}^{-1} = (\mathbf{U}'\mathbf{U})^{-1} = \mathbf{U}^{-1}(\mathbf{U}')^{-1} = \mathbf{U}^{-1}(\mathbf{U}^{-1})' \tag{2-14}$$

Let $\mathbf{V} = \mathbf{U}^{-1}$; then Eq. (2-14) becomes:

$$\mathbf{A}^{-1} = \mathbf{V}\mathbf{V}' \tag{2-15}$$

Thus, the primary task is to generate the upper triangular matrix $\mathbf{V}$ (inverse of $\mathbf{U}$), after which the inverse matrix $\mathbf{A}^{-1}$ may be calculated by postmultiplying $\mathbf{V}$ by its transpose.

In order to develop a method for calculating the matrix $\mathbf{V}$, consider the following expression:

$$\mathbf{V}\mathbf{U} = \mathbf{I} \tag{2-16}$$

in which the symbol $\mathbf{I}$ denotes the identity matrix. This statement in expanded form is:

$$\begin{bmatrix} V_{11} & V_{12} & V_{13} & \cdots & V_{1n} \\ 0 & V_{22} & V_{23} & \cdots & V_{2n} \\ 0 & 0 & V_{33} & \cdots & V_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & V_{nn} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ 0 & 0 & U_{33} & \cdots & U_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & U_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{2-17}$$

Elements of the identity matrix consist of the inner products of the rows of $\mathbf{V}$ and the columns of $\mathbf{U}$. For example, multiplication of the first row of $\mathbf{V}$ and the first column of $\mathbf{U}$ produces:

$$V_{11}U_{11} = 1 \tag{e}$$

from which the element V_{11} is found to be:

$$V_{11} = \frac{1}{U_{11}} \tag{f}$$

As a second example, multiplication of the first row of $\mathbf{V}$ and the second column of $\mathbf{U}$ gives:

$$V_{11}U_{12} + V_{12}U_{22} = 0 \tag{g}$$

Solving for V_{12} in this expression,

$$V_{12} = -\frac{V_{11}U_{12}}{U_{22}} \tag{h}$$

In general, the elements of the matrix $\mathbf{V}$ are obtained by the following formulas:

$$V_{ii} = \frac{1}{U_{ii}} \qquad (i = j) \tag{2-18}$$

$$V_{ij} = -\frac{\sum_{k=1}^{j-1} V_{ik}U_{kj}}{U_{jj}} \qquad (i < j) \tag{2-19}$$

Subsequently, the inverse matrix $\mathbf{A}^{-1}$ is calculated from Eq. (2-15) by taking inner products among the rows of $\mathbf{V}$, omitting the multiplication of zero terms. In order to show the sequence of operations, let $\mathbf{C} = \mathbf{A}^{-1}$. Then the recurrence formula for the elements of the upper triangular part of $\mathbf{C}$ becomes:

$$C_{ij} = \sum_{k=j}^{n} V_{ik}V_{jk} \qquad (i \leq j) \tag{2-20}$$

Elements in C below the main diagonal may then be set equal to those above.

$$C_{ji} = C_{ij} \qquad (i < j) \tag{2-21}$$

This step completes the calculations for the inverse matrix $\mathbf{A}^{-1}$.

In summary, the method of decomposition proves to be useful for the purpose of either solving simultaneous equations or determining the inverse of a symmetric coefficient matrix. In both cases the first phase of the analysis consists of decomposing the coefficient matrix into the product of lower and upper triangular arrays. Equations (2-3), (2-4), and (2-5) serve to compute the elements of the upper triangular matrix. The equations may then be solved simultaneously in a two-step procedure consisting of the forward substitutions represented by Eq. (2-12) and the backward substitutions accomplished with Eq. (2-13). Alternatively, the inverse of the coefficient matrix may be obtained by first determining the inverse of the upper triangular matrix using Eqs. (2-18) and (2-19), and then by calculating elements of the final matrix according to Eqs. (2-20) and (2-21). Formal computer procedures for these operations are presented in the remaining articles of this chapter for both filled and banded matrices.

2.4 Procedure *DECOMPOSE*. In this article the decomposition of a symmetric matrix, as described in Art. 2.3, is cast into the form of a computer procedure. The procedure name is:

DECOMPOSE(N,A,EXIT)

The first argument in the parentheses is the integer number N, which denotes the size of the matrix to be decomposed. The second symbol represents a symmetric matrix **A** of real numbers, and the third identifier *EXIT* is a label for an error message in the main program. In addition to this notation, the integer numbers I, J, and K serve as local indexes in the body of the procedure, and the real variables *SUM* and *TEMP* are used for temporary storage.

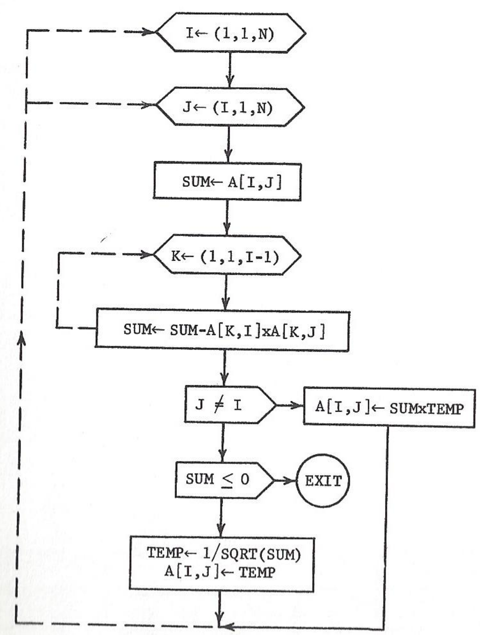
Table 2-3 shows the flow chart for the procedure *DECOMPOSE*. This procedure implements Eqs. (2-3) and (2-4) from the previous article to produce elements of the upper triangular matrix **U** in the storage locations originally occupied by the matrix **A**. However, the identifier A remains in use in order to conserve storage. For convenience in later calculations, the reciprocals of diagonal elements U_{ii} are stored in the diagonal positions. Elements of **A** below the main diagonal are left undisturbed by this procedure. If a zero or imaginary value of U_{ii} is detected, control is transferred to an error message in the main program.

Since the matrix **A** is symmetric and the matrix **U** is upper triangular, there is no need in the decomposition procedure to deal with the elements below the main diagonal. Thus, the upper triangular part of the matrix could be stored as a vector and manipulated in that form. However, the procedure *INVERT* (given subsequently in Art. 2.6) is devised as a tandem subroutine which utilizes the lower triangular part of the matrix as temporary storage for the matrix **V** (defined previously in Art. 2.3).

TABLE 2-3*

Flow Chart for Procedure *DECOMPOSE*

Procedure Name: DECOMPOSE(N,A,EXIT)



*The flow charts in this chapter are formulated under the assumption that the user's compiler will automatically test the validity of the indexes in an iterative control statement before execution. For example, when I = 1 in procedure DECOMPOSE, the initial value for K exceeds the final value; and in that instance the third iterative control statement would not be executed.

2.5 Procedure *SOLVE*. The next procedure in this series accepts the upper triangular matrix from procedure *DECOMPOSE* and solves for the unknowns in the original system of equations. The name of this procedure is:

SOLVE(N,U,B,X)

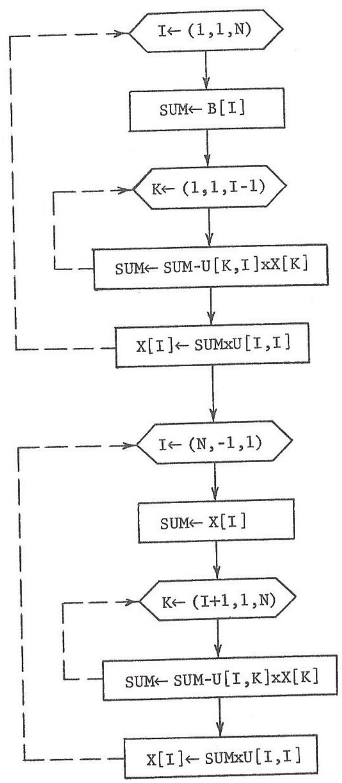
The argument N has the same meaning as previously, and the symbol U denotes the matrix from procedure *DECOMPOSE*. The identifiers B and X represent real vectors of constant terms and unknowns, respectively (see Eq. 2-6). Other symbols in this procedure are unchanged.

The flow chart for the procedure *SOLVE* appears in Table 2-4. In the first half of the flow chart the intermediate vector **X*** is computed by forward substitutions according to Eq. (2-12). Note that the vector **X** is used as temporary storage for **X*** in this part of the procedure. The second

TABLE 2-4

Flow Chart for Procedure *SOLVE*

Procedure Name: SOLVE(N,U,B,X)



half of the flow chart involves the backward substitutions of Eq. (2-13). This part of the procedure produces the final values of the elements in the vector **X**, and the elements of **U** and **B** are left unaltered.

The procedure *SOLVE* is not actually called upon in the structural analysis programs in later chapters, but is included in the series because of its general usefulness. In addition, it serves as a guide to understanding the more complicated procedure *SOLVEBAND*, which is given in Art. 2.8 and is used in the structural analysis program in Chapter 4.

2.6 Procedure *INVERT*. The procedure presented in this article accepts the upper triangular matrix from procedure *DECOMPOSE*, inverts it, and then generates the inverse of the original coefficient matrix. This procedure is assigned the name:

$$INVERT(N,U)$$

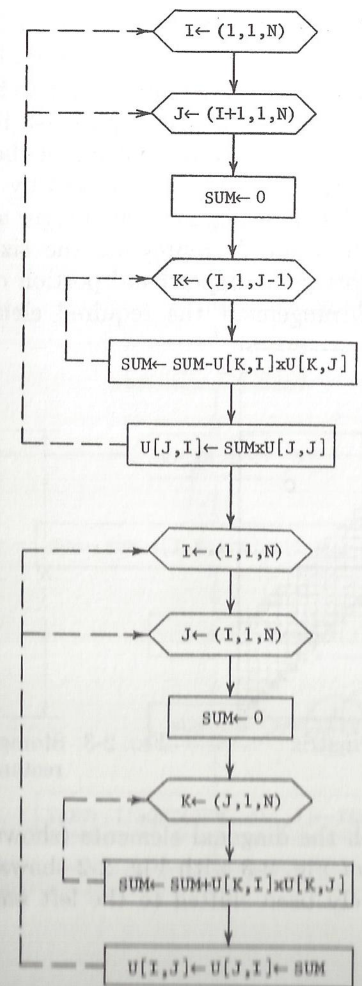
The arguments N and U carry the same meanings as in the procedure *SOLVE*, and other identifiers used in the body of the procedure also remain the same.

Table 2-5 contains the flow chart for the procedure *INVERT*. The first half of this procedure is devoted to the task of determining the matrix **V**, which was defined to be the inverse of the upper triangular matrix **U**. Elements of **V** are calculated according to Eq. (2-19), but they are placed in the lower triangular portion of **U** in order to conserve storage. Note that the elements on the main diagonal are already in the correct form (see

TABLE 2-5

Flow Chart for Procedure *INVERT*

Procedure Name: INVERT(N,U)



Eq. 2-18) as a result of previous calculations in the procedure *DECOMPOSE*.

The second half of procedure *INVERT* deals with obtaining the inverse of the original coefficient matrix by executing the operation indicated in Eq. (2-15). Since the matrix **V** has been stored in the lower triangular part of **U**, however, the necessary operations cannot be carried out in the manner indicated by Eq. (2-20). The subscripts of both terms inside the summation sign in that equation must be reversed to correspond to the fact that inner products among columns instead of rows are now desired. The last statement in the procedure assigns the results of these inner products to the appropriate locations above and below the main diagonal. Thus, the procedure produces the desired symmetric inverse $\mathbf{A}^{-1}$ in the storage locations previously occupied by the matrix **U**, and originally occupied by the symmetric coefficient matrix **A**.

2.7 Procedure *DECOMPOSEBAND*. The method of decomposition is particularly efficient when applied to a symmetric band matrix. For this type of matrix fewer calculations are required due to the fact that elements outside the band are all equal to zero. Figure 2-2 illustrates the general form of a band matrix. Only the upper portion of the band (including the diagonal elements) has to be stored, as indicated by the small squares in the figure. The symbol *UBW* in Fig. 2-2 is an integer number which denotes the upper band width, while *N* represents the size of the matrix. An efficient pattern for storing the upper band portion of the matrix appears in Fig. 2-3. In this arrangement the required elements are stored as a

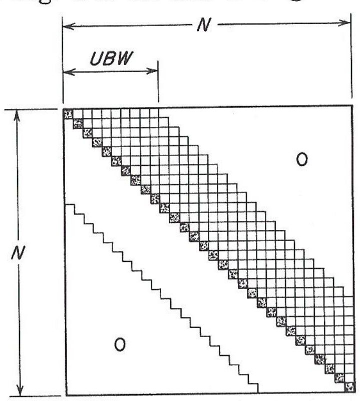


FIG. 2-2. Band matrix

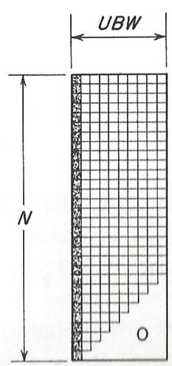


FIG. 2-3. Storage of upper band as a rectangular array

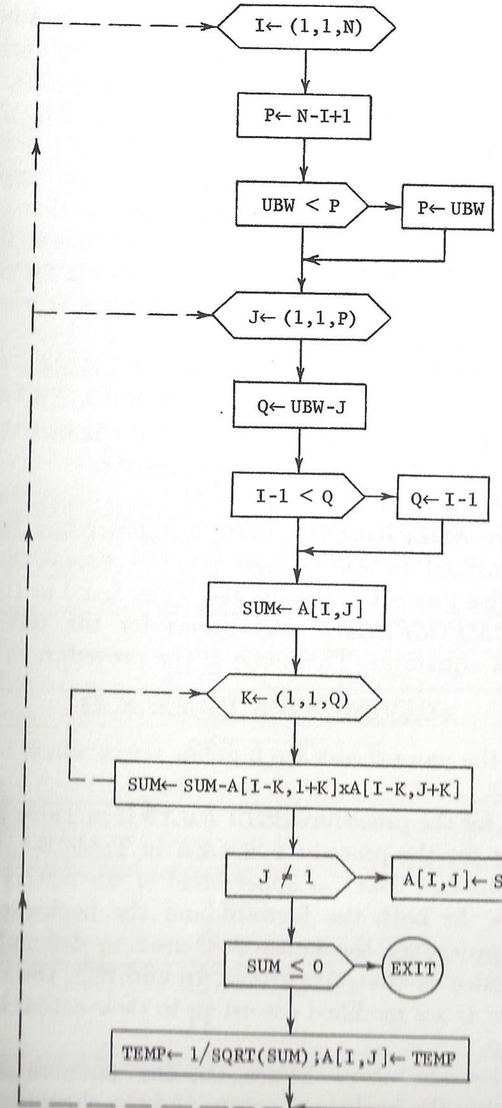
rectangular array with the diagonal elements (shown shaded) in the first column. Comparison of Fig. 2-3 with Fig. 2-2 shows that the rows of the matrix have figuratively been shifted to the left and the excess columns removed.

Table 2-6 shows the flow chart for a procedure which decomposes the upper band of a symmetric band matrix. The name of the procedure is:

DECOMPOSEBAND(*N*,*UBW*,*A*,*EXIT*)

TABLE 2-6

Flow Chart for Procedure *DECOMPOSEBAND*



The arguments of this procedure name have all been defined, and most of the other identifiers in the body of the procedure were used before. However, two new integer numbers P and Q are introduced for the purpose of limiting calculations to nonzero elements. The sequence of operations in procedure *DECOMPOSEBAND* follows that in procedure *DECOMPOSE* (see Table 2-3), except that additional statements are required to determine the values of the indexes P and Q. Furthermore, the summations of products included in Eqs. (2-3) and (2-4) are complicated by the fact that the matrix is stored as a rectangular array. Whereas such summations of products involve the columns of the matrix in Fig. 2-2, they involve staircase patterns of elements in the matrix of Fig. 2-3. Therefore, the subscripts of such elements are modified accordingly.

As in the earlier procedure for decomposition, the upper triangular matrix **U** is generated and placed in the storage locations originally occupied by the matrix **A**, but the identifier A remains in use. The reciprocals of diagonal elements U_{ii} are stored in the first column for convenience in later calculations. If U_{ii} is zero or imaginary, control is transferred to an error message in the main program.

The procedure *SOLVEBAND* given in the next article is intended to be a tandem subroutine for the procedure *DECOMPOSEBAND*. They are both used in the structural analysis program in Chapter 4, which takes advantage of the band width of the stiffness matrix.

2.8 Procedure *SOLVEBAND*. The last procedure in the series is analogous to the procedure *SOLVE* (see Art. 2.5), except that it applies to a band matrix. This procedure accepts the upper band of the matrix from procedure *DECOMPOSEBAND* and solves for the unknowns in the original system of equations. The name of the procedure is:

SOLVEBAND(N,UBW,U,B,X)

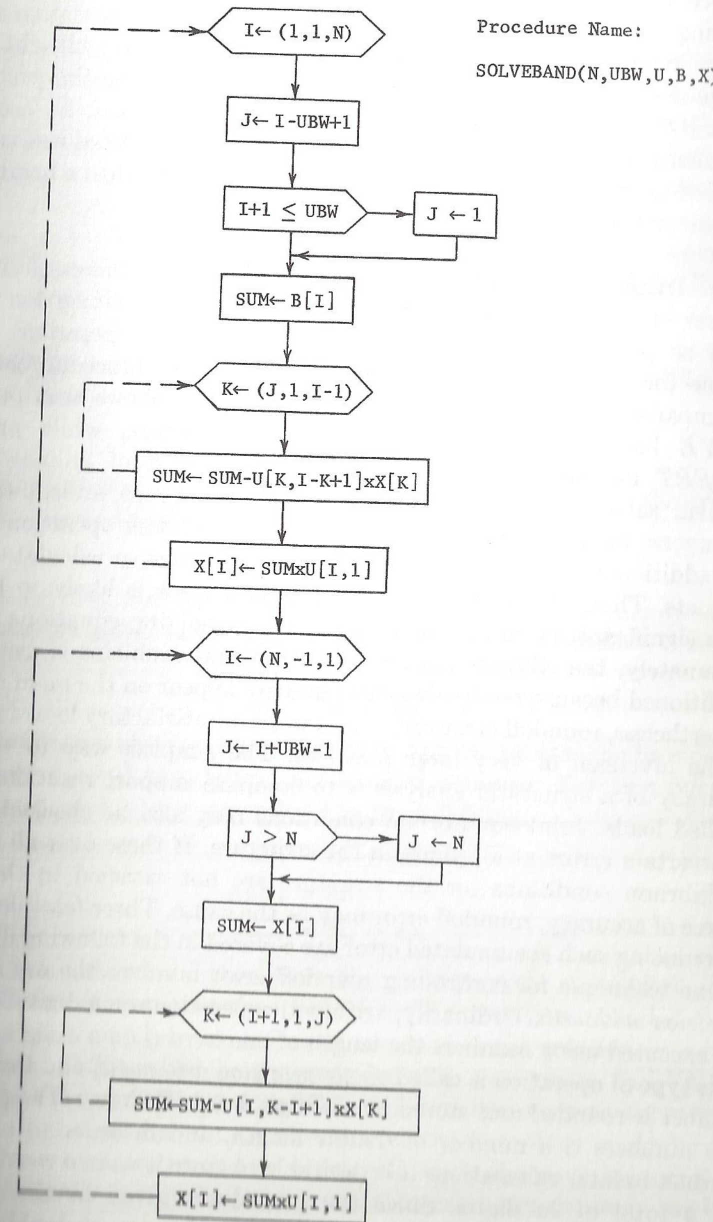
All identifiers in the parentheses are familiar terms which have been used before.

The flow chart for the procedure *SOLVEBAND* in Table 2-7 bears much similarity to that for the procedure *SOLVE* in Table 2-4. However, it is complicated by the fact that the upper band of the matrix **U** is stored in rectangular form. In both the forward and the backward substitution portions of the procedure the index J is used to delineate the nonzero terms to be included in the calculations. In addition, the column indexes of the elements of **U** are modified according to their actual locations in the rectangular array.

As in the earlier solution procedure, the final answers are generated in the vector **X** during the backward sweep; and the elements of **U** and **B** are left unaltered. Thus, the procedure for solution can be used repeatedly for the same matrix **U** but for different vectors of constant terms. If the

TABLE 2-7

Flow Chart for Procedure *SOLVEBAND*



columns of the identity matrix **I** were to be assigned in succession to the vector **B**, each solution vector **X** would represent a column of the inverse matrix $\mathbf{A}^{-1}$. This technique constitutes a possible means of obtaining the inverse of a matrix by repeated applications of a solution routine.

There is little benefit to be derived from composing a procedure for inverting a band matrix, because the inverse is not necessarily a band matrix. When the objective is to obtain the inverse, the efficient use of storage depicted by Fig. 2-3 is lost; and for this purpose the procedures *DECOMPOSE* and *INVERT* (see Arts. 2.4 and 2.6) can be used with little sacrifice of computer time. Alternatively, these procedures could be modified in order to omit operations with zero elements when a band matrix is encountered.

2.9 Improvement of Accuracy. *Roundoff error* increases with the number of arithmetic operations to be performed. An inversion routine (such as procedure *INVERT*) requires many more operations than a routine for solving simultaneous equations (such as procedure *SOLVE*). A comparison of Tables 2-4 and 2-5, for example, shows that procedure *SOLVE* has only four iterative control statements, while procedure *INVERT* has six. Furthermore, the determination of unknowns in a problem solved by inversion involves the subsequent multiplication of the inverse matrix by a vector of constants. This latter operation requires two additional iterative control statements in order to calculate n inner products. Thus, the accumulation of roundoff error is likely to be much more significant for matrix inversion than for solving equations directly. Fortunately, the stiffness matrix $\mathbf{S}$ in structural problems is usually well conditioned because relatively large numbers appear on the main diagonal. Nevertheless, roundoff error may result in an unsatisfactory loss of accuracy in the inversion of very large matrices. The simplest way to check the accuracy of a structural analysis is to compare support reactions against applied loads. Joint equilibrium conditions may also be checked in order to ascertain errors at all joints in the structure. If these over-all and local equilibrium conditions for the structure are not satisfied to the desired degree of accuracy, roundoff error may be the cause. Three feasible methods for reducing such accumulated error are covered in the following discussion.

One technique for controlling roundoff error involves the use of *double-precision arithmetic*. Ordinarily, arithmetic operations on a digital computer are executed using numbers the length of one *word* (i.e., a d-digit number). This type of operation is called *single-precision arithmetic*, and the resulting number is rounded and stored as a d-digit word. However, the product of two numbers is a number of double length, and in order to use such a product in later calculations it is desirable to store it as two words containing a total of $2d$ digits. Since the procedures given in the preceding articles consist primarily of inner products, the use of double-precision arithmetic for these operations is particularly effective. Most digital computers have library routines for double-precision arithmetic, and such

routines may be called upon to perform the operations indicated in the flow charts of this book. The technique may be applied to equation solving, matrix inversion, or any other step in a computer program where roundoff error could be significant.

The second method for improving accuracy relates to the solution of simultaneous equations. Regardless of whether the numerical calculations in the solution are performed with single- or double-precision arithmetic, there will always be some errors in the results. In order to develop a technique for correcting these errors, let the identifier $\mathbf{X}_1$ represent the calculated solution vector. The errors in the solution may be assessed by substituting the vector $\mathbf{X}_1$ into the original set of simultaneous equations (see Eq. 2-6), evaluating the product $\mathbf{AX}_1$, and comparing the results with the elements of the vector $\mathbf{B}$. (In the stiffness method of analysis this comparison represents a check of joint equilibrium conditions.) Let the symbol $\delta\mathbf{B}_1$ denote a *residual vector* of elements which are the differences between elements of the vector $\mathbf{B}$ and the product $\mathbf{AX}_1$.

$$\delta\mathbf{B}_1 = \mathbf{B} - \mathbf{AX}_1 \tag{a}$$

This residual vector can be treated as a new right-hand side in the original matrix equation. Thus,

$$\mathbf{A}\delta\mathbf{X}_1 = \delta\mathbf{B}_1 \tag{b}$$

Equation (b) may then be solved for the *incremental vector* of unknowns $\delta\mathbf{X}_1$, which must be added to the first solution vector $\mathbf{X}_1$ to obtain a corrected vector $\mathbf{X}_2$

$$\mathbf{X}_2 = \mathbf{X}_1 + \delta\mathbf{X}_1 \tag{c}$$

The vector $\mathbf{X}_2$ will also contain small errors, and the process can be repeated (if necessary) until the desired accuracy is attained. For this purpose, Eqs. (a), (b), and (c) are generalized to the following:

$$\begin{aligned} \delta\mathbf{B}_i &= \mathbf{B} - \mathbf{AX}_i \\ \mathbf{A}\delta\mathbf{X}_i &= \delta\mathbf{B}_i \\ \mathbf{X}_{i+1} &= \mathbf{X}_i + \delta\mathbf{X}_i \end{aligned} \tag{2-22}$$

in which the subscript i represents the current cycle of solution. This method for *successive corrections* causes rapid convergence, and a single correction cycle may provide sufficient accuracy in the solution.

Equations (2-22) could be incorporated into procedures *SOLVE* and *SOLVEBAND* (see Tables 2-4 and 2-7) if it were necessary to improve the accuracy of the results. As an alternative, the equations could be programmed in a separate procedure which calls upon the procedures *SOLVE* or *SOLVEBAND* in a cyclic fashion until the necessary accuracy has been achieved.

The third method for reducing roundoff error applies to the task of matrix inversion. Because of numerous arithmetic operations, the elements of the calculated inverse matrix will always be inaccurate to a certain extent. Let the symbol $\delta\mathbf{C}_1$ represent a matrix of elements which are the differences between the elements of the true inverse $\mathbf{A}^{-1}$ and the first calculated inverse matrix $\mathbf{C}_1$.

$$\delta\mathbf{C}_1 = \mathbf{A}^{-1} - \mathbf{C}_1 \tag{d}$$

Then the true inverse may be expressed as the sum of the other two matrices.

$$\mathbf{A}^{-1} = \mathbf{C}_1 + \delta\mathbf{C}_1 \tag{e}$$

An approximate expression for $\delta\mathbf{C}_1$ in terms of $\mathbf{A}$ and $\mathbf{C}_1$ only may be obtained by rewriting Eq. (d) as:

$$\delta\mathbf{C}_1 = \mathbf{A}^{-1}(\mathbf{I} - \mathbf{A}\mathbf{C}_1) \approx \mathbf{C}_1(\mathbf{I} - \mathbf{A}\mathbf{C}_1) \tag{f}$$

Substitution of this expression for the *error matrix* into Eq. (e) produces:

$$\mathbf{A}^{-1} \approx \mathbf{C}_2 = \mathbf{C}_1(2\mathbf{I} - \mathbf{A}\mathbf{C}_1) \tag{g}$$

Equation (g) provides a means of improving the accuracy of the inverse matrix by an iterative procedure represented by the following formula:

$$\mathbf{C}_{i+1} = \mathbf{C}_i(2\mathbf{I} - \mathbf{A}\mathbf{C}_i) \tag{2-23}$$

in which the subscript i denotes the current cycle of calculations. This technique for *successive approximations* results in quick convergence to the desired degree of accuracy, and a single application of Eq. (2-23) may suffice in many instances.

The procedure *INVERT* in Table 2-5 may be revised to include one or more applications of Eq. (2-23). Alternatively, this algorithm for successive approximations could be put into a tandem procedure to be executed at the option of the programmer. In either case the original matrix $\mathbf{A}$ must be retained for use in Eq. (2-23). Therefore, the procedure *INVERT* would have to be revised to save $\mathbf{A}$ in the lower triangular part of the array and to generate $\mathbf{A}^{-1}$ in the upper triangular part. The original diagonal elements of $\mathbf{A}$ must also be saved by introducing an auxiliary storage vector (of length n) for this purpose.

In summary, roundoff error is not likely to be significant in the stiffness method of analysis except for structures with large or ill-conditioned stiffness matrices. The degree of inaccuracy may always be ascertained by checking equilibrium in the structure, and errors may be reduced by using double-precision arithmetic. If necessary, the results of a solution procedure may be corrected as indicated by Eqs. (2-22), and those for a matrix inversion routine may be improved by one or more applications of Eq. (2-23). These methods for controlling accuracy do not appear explicitly in the structural analysis programs of succeeding chapters, but their use on an optional basis is implied.

Chapter 3

ANALYSIS BY INVERSION OF STIFFNESS MATRIX

3.1 Introduction. In this chapter a flow chart is presented for a general-purpose program named *FR*1, which is capable of analyzing any of the six basic types of framed structures described in Art. 1.2. The program implements the theory of the stiffness method given in Chapter 1, including the inversion of the stiffness matrix $\mathbf{S}$ as indicated in Eq. (1-66). For this purpose the tandem procedures *DECOMPOSE* and *INVERT* (see Arts. 2.4 and 2.6) are incorporated into the program.

The reader will appreciate the fact that there are many ways of carrying out the detailed steps involved in structural calculations. Also, there are many possible forms into which a computer program can be organized, including variations in the manner of reading data into the program, printing the results, etc. Thus, it is apparent that the program in this chapter, while representing a satisfactory method of calculation, does not represent the only method. The advantage of combining the analyses of all types of framed structures into one program is that only one deck of cards (or block of storage on tape) is required instead of six, and duplication of programming effort is thereby avoided. Of greater significance is the fact that many methods of solution are available, and inversion of the stiffness matrix represents only one possibility. This approach is pedagogically appealing, however, because it parallels the matrix formulation of the stiffness method in every step. Two other methods are implemented in Programs *FR*2 and *FR*3, which appear in Chapters 4 and 5, respectively.

All of the structures to be analyzed by Programs *FR*1, *FR*2, and *FR*3 are assumed to consist of straight, prismatic members. The material properties for a given structure are taken to be constant throughout the structure. Only the effects of loads are considered, and no other influences, such as temperature change, are taken into account. The programs are designed to handle in a single computer run any number of each type of framed structure as well as any number of load systems for the same structure.

Articles 3.2 through 3.4 contain a list of notation, a detailed description of the required input data, and an outline of Program *FR*1. The flow chart for the program itself appears in Art. 3.5, and this is followed in Art. 3.6 by a series of explanatory notes for the steps in the flow chart. Finally, Art. 3.7 contains a set of examples with computer solutions obtained using the program.

3.2 Notation. The notation used in Program *FR*1 (and later in Programs *FR*2 and *FR*3) is summarized in this article. The items are listed approximately in the order of their appearance in the program. A symbol in the list which is followed by square brackets [ ] denotes a subscripted variable with a single subscript; that is, a vector. A symbol followed by square brackets enclosing a comma [,] denotes a subscripted variable (or a matrix) with two subscripts. The type (integer or decimal) of each identifier is also given; the letter *I* indicates an integer identifier, and *D* indicates that an identifier is of type decimal.

In conjunction with a computer program, symbols such as $\mathbf{S}_{MD}$ and *SMD* may be used interchangeably. It is understood that the symbol $\mathbf{S}_{MD}$ represents a matrix and that the identifier *SMD* stands for its counterpart in the program.

Identifiers Used in Computer Programs

Identifier	*Definition*	*Type*
SN	Structure number	*I*
TS	Type of structure	*I*
NLS	Number of loading systems	*I*
NDJ	Number of displacements per joint	*I*
M	Number of members	*I*
N	Number of degrees of freedom	*I*
NJ	Number of joints	*I*
NR	Number of support restraints	*I*
NRJ	Number of restrained joints	*I*
E	Elastic modulus for tension or compression	*D*
G	Elastic modulus for shear	*D*
I	Member index	*I*
J, *K*	Joint indexes	*I*
X[ ], *Y*[ ], *Z*[ ]	x, y, and z coordinates of joint	*D*
JJ[ ]	Designation for j end of member (joint j)	*I*
JK[ ]	Designation for k end of member (joint k)	*I*
L[ ]	Length of member	*D*
AX[ ]	Cross-sectional area of member	*D*
IX[ ]	Torsion constant of member	*D*
IY[ ], *IZ*[ ]	Moments of inertia about the y_M and z_M axes	*D*
AA	Identifier used to indicate whether or not the angle α is zero	*I*
XCL, *YCL*, *ZCL*	x, y, and z components of length of member	*D*
CX, *CY*, *CZ*	x, y, and z direction cosines of member	*D*
R[,]	Rotation matrix	*D*
XP, *YP*, *ZP*	x, y, and z coordinates of point p	*D*
XPS, *YPS*, *ZPS*	x_S, y_S, and z_S coordinates of point p	*D*
YPG, *ZPG*	y_γ and z_γ coordinates of point p	*D*
Q, *SQ*	Identifiers used for temporary storage of square root expressions	*D*

Identifier	*Definition*	*Type*
SINA, *COSA*	Sine and cosine of the angle α	*D*
RL[ ]	Joint restraint list	*I*
CRL[ ]	Cumulative restraint list	*I*
*J*1, . . ., *J*6	Indexes for displacements at j end of member	*I*
*K*1, . . ., *K*6	Indexes for displacements at k end of member	*I*
SCM	Stiffness constant of member	*D*
SM[,]	Member stiffness matrix for member-oriented axes	*D*
SMR[,]	Identifier used for temporary storage of the product $\mathbf{S}_M\mathbf{R}_T$	*D*
SMD[,]	Member stiffness matrix for structure-oriented axes	*D*
S[,]	Joint stiffness matrix ($\mathbf{S}_J$)	*D*
LN	Loading number	*I*
NLJ	Number of loaded joints	*I*
NLM	Number of loaded members	*I*
A[ ]	Actions (loads) applied at joints (in directions of structure axes)	*D*
AML[,]	Actions at ends of restrained members (in directions of member axes) due to loads	*D*
LML[ ]	Loaded member list	*I*
AE[ ]	Equivalent joint loads (in directions of structure axes)	*D*
AC[ ]	Combined joint loads (in directions of structure axes)	*D*
D[ ]	Joint displacements (in directions of structure axes)	*D*
DJ[ ]	Expanded vector of joint displacements	*D*
AR[ ]	Support reactions (in directions of structure axes)	*D*
JE, *KE*	Indexes for expanded vectors	*I*

3.3 Preparation of Data. The manner in which the program is written depends to some extent upon the form in which the data is supplied. Therefore, the required input data for Program *FR*1 is described in this article as a preliminary matter.

The data cards for a particular structure are divided into the three categories of *control data, structure data,* and *load data.* Only one set of control and structure data is required for each structure, but this may be followed by any number of sets of load data. Since the program is designed to accommodate an indefinite number of six different types of framed structures, the first data card for a particular structure must contain certain control parameters. These are the structure number *SN*, the type of structure *TS*, and the number of loading systems *NLS*. The structure number *SN* will be 1 for the first structure, 2 for the second structure, etc. On the other hand, the type of structure *TS* is represented by one of the numbers 1 through 6 as follows:

$TS = 1$: Continuous beam	$TS = 4$: Grid
$TS = 2$: Plane truss	$TS = 5$: Space truss
$TS = 3$: Plane frame	$TS = 6$: Space frame

The number of loading systems *NLS* indicates the number of sets of load data which accompany a given set of structure data.

The input data required for continuous beams are given in Table 3-1. This table specifies the number of data cards required for each category of information as well as the items which are to be punched on the card. The

TABLE 3-1

Preparation of Data for Continuous Beams

	Data	Number of Cards	Items on Data Cards
	Control Data	1	SN TS NLS
Structure Data	a. Structure parameters and elastic modulus	1	M NR NRJ E
	b. Member designations and properties	M	I L[I] IZ[I]
	c. Joint restraint list	NRJ	K RL[2K–1] RL[2K]
Load Data	a. Numbers of loaded joints and members	1	NLJ NLM
	b. Actions applied at joints	NLJ	K A[2K–1] A[2K]
	c. Actions at ends of restrained members due to loads	NLM	I AML[I,1] AML[I,2] AML[I,3] AML[I,4]

first card listed in the table under the category of structure data contains the number of members *M*, the number of restraints *NR*, the number of restrained joints *NRJ*, and the elastic modulus *E* of the material. The reason for including *NRJ* is that the number of data cards required to fill the joint restraint list (see line 4 of the table) is minimized if the number of restrained joints is known.

Each data card containing member information (line 3 of Table 3-1) includes for one member the member number *I*, the length *L*[*I*], and the moment of inertia *IZ*[*I*] about the *z* axis. A total of *M* cards is required for the member data.

Each of the cards in the next series (a total of *NRJ* cards) contains a joint number *K* and two code numbers which indicate the conditions of restraint at that joint. The term *RL*[2*K*-1] denotes the restraint against translation in the *y* direction at joint *k*, and the term *RL*[2*K*] denotes the restraint against rotation in the *z* sense at joint *k*. The convention adopted in this program is the following. If the restraint exists, the integer 1 is assigned as the value of *RL*; and if there is no restraint a value of zero is assigned.

The first card in the load data contains two items. These are the number of loaded joints *NLJ* and the number of loaded members *NLM*. One reason for introducing these numbers is that they serve to minimize the number of subsequent load data cards required. Another reason is that certain calculations in the program can be bypassed if either *NLJ* or *NLM* is equal to zero; for example, if *NLM* is equal to zero, the calculation of the vector $\mathbf{A}_E$ can be omitted.

Each joint load card (a total of *NLJ* cards) contains a joint number *K* and the two actions applied at that joint. These actions are the applied force and couple in the *y* direction and the *z* sense. Finally, each card for member loads (*NLM* cards total) contains a member number *I* and the four fixed-end actions for that member. The fixed-end actions consist of a force in the *y* direction and a couple in the *z* sense at each end of the member.

Table 3-2 shows the input data required for plane trusses. Since both beams and plane trusses have two possible displacements at each joint, Table 3-1 and 3-2 are very similar. The structure parameters required for plane trusses are similar to those for beams except that the number of joints *NJ* must be specified in addition to the other items. Moreover, a set of cards (*NJ* cards total) is required to specify the coordinates of the joints of a truss. Each card in this set contains a joint number *J*, the *x* coordinate *X*[*J*] of the joint, and the *y* coordinate *Y*[*J*] of the joint.

On each of the cards pertaining to the member designations and properties, the member number *I* is listed first, followed by the joint *j* number *JJ*[*I*] and the joint *k* number *JK*[*I*] for the two ends of the member. The choice of which end of the member is to be the *j* end and which is to be the *k* end is made arbitrarily by the programmer. The last item on each card is the cross-sectional area *AX*[*I*] of the member.

The data cards for the joint restraint list are similar to those for continuous beams, except that the types of restraints differ. For the plane truss, the terms *RL*[2*K*-1] and *RL*[2*K*] denote the restraints against translations in the *x* and *y* directions, respectively, at joint *k*. Similarly, the cards

TABLE 3-2

Preparation of Data for Plane Trusses

	Data	Number of Cards	Items on Data Cards
	Control Data	1	SN TS NLS
Structure Data	a. Structure parameters and elastic modulus	1	M NJ NR NRJ E
	b. Joint coordinates	NJ	J X[J] Y[J]
	c. Member designations and properties	M	I JJ[I] JK[I] AX[I]
	d. Joint restraint list	NRJ	K RL[2K−1] RL[2K]
Load Data	a. Numbers of loaded joints and members	1	NLJ NLM
	b. Actions applied at joints	NLJ	K A[2K−1] A[2K]
	c. Actions at ends of restrained members due to loads	NLM	I AML[I,1] AML[I,2] AML[I,3] AML[I,4]

for load data are symbolically the same as for continuous beams, but the applied actions A consist of forces in the x and y directions, and the actions AML consist of forces in the x_M and y_M directions at the ends of the members.

Table 3-3 lists the input data required for plane frames. This table contains information which is very similar to that in Table 3-2 for plane trusses, but there are certain differences caused by the fact that plane frames have three possible displacements at each joint. One dissimilarity is that an additional member property IZ (moment of inertia of the cross-section) is required on each of the cards pertaining to member designations and properties. Another distinction is that each of the cards in the joint restraint series contains (in addition to the joint number K) three code numbers instead of two. The terms $RL[3K\text{-}2]$, $RL[3K\text{-}1]$, and $RL[3K]$ denote the restraints at joint k against translations in the x and y directions and rotation in the z sense, respectively.

Each joint load card contains a joint number K and the three actions applied at that joint. These actions are the applied forces in the x and y directions and the couple in the z sense. In addition, each card for member loads contains a member number I and the six actions at the ends of the restrained member. These actions consist of forces in the x_M and y_M directions and a couple in the z_M (or z) sense at the ends j and k, respectively.

TABLE 3-3

Preparation of Data for Plane Frames

	Data	Number of Cards	Items on Data Cards
	Control Data	1	SN TS NLS
Structure Data	a. Structure parameters and elastic modulus	1	M NJ NR NRJ E
	b. Joint coordinates	NJ	J X[J] Y[J]
	c. Member designations and properties	M	I JJ[I] JK[I] AX[I] IZ[I]
	d. Joint restraint list	NRJ	K RL[3K−2] RL[3K−1] RL[3K]
Load Data	a. Numbers of loaded joints and members	1	NLJ NLM
	b. Actions applied at joints	NLJ	K A[3K−2] A[3K−1] A[3K]
	c. Actions at ends of restrained members due to loads	NLM	I AML[I,1] AML[I,2] AML[I,3] AML[I,4] AML[I,5] AML[I,6]

The input data required for grids, shown in Table 3-4, are nearly the same as for plane frames (see Table 3-3). However, an additional material property (the shear modulus of elasticity G) is required on the first structure data card in Table 3-4. Also, on the cards containing member properties appear the torsion constant IX and moment of inertia IY instead of the cross-sectional area AX and moment of inertia IZ.

The data cards for the joint restraint list are similar to those for plane frames, except that the nature of the restraints is different. For grids, the terms $RL[3K\text{-}2]$, $RL[3K\text{-}1]$, and $RL[3K]$ denote the restraints against rotations in the x and y senses and translation in the z direction, respectively, at joint k.

Similarly, the cards for load data are symbolically the same as for plane frames, but the meanings are different. For a grid structure, the actions applied at joints consist of couples in the x and y senses and a force in the z direction. Finally, the actions AML at the ends of restrained members due to loads are couples in the x_M and y_M senses and a force in the z_M (or z) direction at the ends j and k, respectively.

TABLE 3-4

Preparation of Data for Grids

	Data	Number of Cards	Items on Data Cards
	Control Data	1	SN TS NLS
Structure Data	a. Structure parameters and elastic moduli	1	M NJ NR NRJ E G
	b. Joint coordinates	NJ	J X[J] Y[J]
	c. Member designations and properties	M	I JJ[I] JK[I] IX[I] IY[I]
	d. Joint restraint list	NRJ	K RL[3K–2] RL[3K–1] RL[3K]
Load Data	a. Numbers of loaded joints and members	1	NLJ NLM
	b. Actions applied at joints	NLJ	K A[3K–2] A[3K–1] A[3K]
	c. Actions at ends of restrained members due to loads	NLM	I AML[I,1] AML[I,2] AML[I,3] AML[I,4] AML[I,5] AML[I,6]

Table 3-5 shows the input data required for space trusses. This table is similar to Table 3-3 for plane frames, with some exceptions. For example, each of the cards in Table 3-5 containing joint coordinates provides not only the x and y coordinates but also the z coordinate for a joint. On the cards containing member properties, only the cross-sectional area AX is required.

The data cards for the joint restraint list are similar to those for plane frames, except that the types of restraints are different. For space trusses, the terms $RL[3K\text{-}2]$, $RL[3K\text{-}1]$, and $RL[3K]$ denote the restraints at joint k against translations in the x, y, and z directions, respectively.

Similarly, the cards for load data are symbolically the same as for plane frames, but the meanings are different. Actions applied at joints consist of forces in the x, y, and z directions, and end-actions AML consist of forces in the x_M, y_M, and z_M directions at each end of a loaded member.

The input data required for space frames are summarized in Table 3-6, which is also very similar to Table 3-3 for plane frames. However, since the space frame is three-dimensional and has six possible displacements per

TABLE 3-5

Preparation of Data for Space Trusses

	Data	Number of Cards	Items on Data Cards
	Control Data	1	SN TS NLS
Structure Data	a. Structure parameters and elastic modulus	1	M NJ NR NRJ E
	b. Joint coordinates	NJ	J X[J] Y[J] Z[J]
	c. Member designations and properties	M	I JJ[I] JK[I] AX[I]
	d. Joint restraint list	NRJ	K RL[3K–2] RL[3K–1] RL[3K]
Load Data	a. Numbers of loaded joints and members	1	NLJ NLM
	b. Actions applied at joints	NLJ	K A[3K–2] A[3K–1] A[3K]
	c. Actions at ends of restrained members due to loads	NLM	I AML[I,1] AML[I,2] AML[I,3] AML[I,4] AML[I,5] AML[I,6]

joint, the data for this type of structure is more extensive than that for plane frames. The first structure data card in Table 3-6 includes the shear modulus G as well as the modulus E, and each of the cards containing joint coordinates must provide the z coordinate as well as the x and y coordinates of a joint.

The cards pertaining to member designations, properties, and orientations must include the cross-sectional properties AX, IX, IY, and IZ. In addition, an identifier AA is provided in order to indicate whether the rotation angle α is zero or not (see Art. 1.6 for the meaning of the angle α). If the angle α is zero for a given member, the value of zero is assigned to AA. On the other hand, if the angle α is not zero, the integer 1 is assigned. This is an arbitrary convention which serves to indicate that the member has its principal axes in skew directions. Under these conditions the principal axes are located by means of the coordinates of a point p in the x_M-y_M plane (see Figs. 1-16 and 1-19), but not on the axis of the member. Therefore, each data card for which AA is equal to 1 must be followed by an additional data card on which the member index I and the three coor-

Table 3-6

Preparation of Data for Space Frames

	Data	Number of Cards	Items on Data Cards
	Control Data	1	SN TS NLS
Structure Data	a. Structure parameters and elastic moduli	1	M NJ NR NRJ E G
Structure Data	b. Joint coordinates	NJ	J X[J] U[J] Z[J]
Structure Data	c. Member designations, properties, and orientations	M	I JJ[I] JK[I] AX[I] IX[I] IY[I] IZ[I] AA
Structure Data	Coordinates of point *p* (required when AA = 1)		I XP YP ZP
Structure Data	d. Joint restraint list	NRJ	K RL[6K–5] RL[6K–4] RL[6K–3] RL[6K–2] RL[6K–1] RL[6K]
Load Data	a. Numbers of loaded joints and members	1	NLJ NLM
Load Data	b. Actions applied at joints	NLJ	K A[6K–5] A[6K–4] A[6K–3] A[6K–2] A[6K–1] A[6K]
Load Data	c. Actions at ends of restrained members due to loads	2NLM	I AML[I,1] AML[I,2] AML[I,3] AML[I,4] AML[I,5] AML[I,6] AML[I,7] AML[I,8] AML[I,9] AML[I,10] AML[I,11] AML[I,12]

dinates *XP*, *YP*, and *ZP* are given. These coordinates are used to obtain the rotation matrix as described in Art. 1.6.

An alternate method of programming would be to provide for the contingency that the angle α is known, in which case it would not be necessary to give the coordinates of a point p. The actual value of α could be assigned to the identifier *AA*, after which the rotation matrix can be calculated directly from Eq. (1-29) or Eq. (1-33).

Each of the cards in the joint restraint series contains a joint number K and six code numbers which indicate the conditions of restraint at that joint. The terms *RL*[6*K*-5] through *RL*[6*K*] denote the restraints against translations and rotations in the x, y, and z directions at joint k.

Each joint load card contains a joint number K and the six actions applied at that joint. These actions are the components of the applied force vector and the applied moment vector in the x, y, and z directions.

In the last series, two cards must be provided for each member to which loads are applied because the information may not fit on one card. The first card contains a member number I and the six actions at the j end of the restrained member. The second card provides the six actions at the k end of the member. The actions at each end are the forces and couples in the x_M, y_M and z_M directions.

A consistent system of units must be used for the input data, which involves units of force and length only. The results of the calculations will have the same units as the input data, with the addition that all angles will automatically be in radians. One example of a consistent system of units is to give loads in kips, lengths in inches, areas in square inches, modulus of elasticity in kips per square inch, etc. In such a case all of the final results will be in units of kips, inches, and radians. On the other hand, if units of feet and pounds are used for the input data, the final results will be in terms of feet, pounds and radians. In all of the examples given at the end of this chapter units of kips and inches are used.

3.4 Outline of Program *FR*1. The general outline of Program *FR*1 follows directly from the theory of the stiffness method described in Chapter 1. However, only a part of the over-all joint stiffness matrix $\mathbf{S}_J$ is generated for a given structure. Since only the effects of loads are taken into account, the required portions of $\mathbf{S}_J$ are the matrices $\mathbf{S}$ and $\mathbf{S}_{RD}$ (see Eq. 1-2). The other two portions ($\mathbf{S}_{DR}$ and $\mathbf{S}_{RR}$) are not generated in this program. However, only a short additional step is required to generate them, which could be accomplished in a more extensive program that included the effects of support displacements.

The outline of the program is shown in the following five steps, each of which is explained in more detail in connection with the flow chart:

1. Input and Print Control and Structure Data
   a. Control data, structure parameters, and elastic moduli
   b. Joint coordinates (except continuous beams)
   c. Member information; rotation matrix
   d. Joint restraint list; cumulative restraint list
2. Structure Stiffness Matrix
   a. Generation of stiffness matrix
   b. Decomposition and inversion of stiffness matrix
3. Input and Print Load Data
   a. Load parameters
   b. Actions applied at joints
   c. Actions at ends of restrained members due to loads

4. Construction of Vectors Associated with Loads
   a. Equivalent joint loads
   b. Combined joint loads
5. Calculation and Output of Results
   a. Joint displacements
   b. Member end-actions
   c. Support reactions.

In the outline above and in the program itself, it is assumed that the input data and certain descriptive headings will be printed, as well as the final results. Such information is usually needed to identify the structure being analyzed and to provide the programmer with the results which constitute the goal of the analysis. Of course, it is always possible for a programmer to add additional output statements at any intermediate stage of the calculations if it is desired to print some of the intermediate results. For example, under certain conditions it might be desirable to print the stiffness matrix for the structure.

At the beginning of a computer program there are various preliminary matters which should be taken into account. It is good practice to introduce a program with a description of its content. This includes the scope of the program, an outline of the steps, the notation used, and the data required. Since this is done primarily for the convenience of the programmer, it is not included in the flow chart.

Another preliminary matter is to declare those identifiers in a program which are to be integer numbers as distinct from those which are to be decimal numbers. The identifiers of type integer in the program may be determined from the list of notation given in Art. 3.2. In addition, storage in the memory of the computer must be reserved for the subscripted variables in the program,* and these variables are also indicated in Art. 3.2. Finally, any procedures to be used in the program (such as the procedures for decomposition and inversion) must be inserted at the beginning so that they will be available for later use.

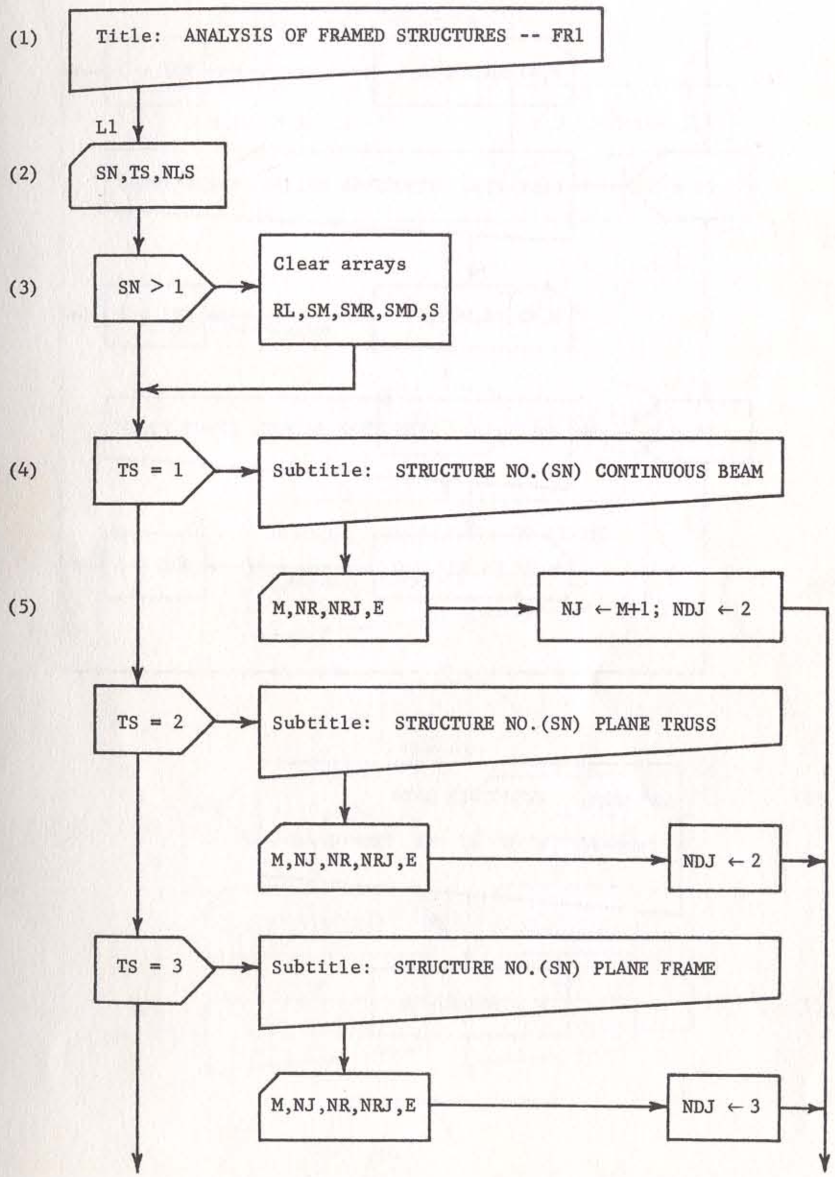
The flow chart for Program *FR*1 is given in the next article. The outline of the program is interspersed in the chart, and comments noted by the numbers in parentheses are given in Art. 3.6.

*The flow charts in this book are formulated under the assumption that arrays are cleared initially by the compiler upon encountering array declarations.

3.5 Flow Chart for Program *FR*1.

1. <u>Input and Print Control and Structure Data</u>

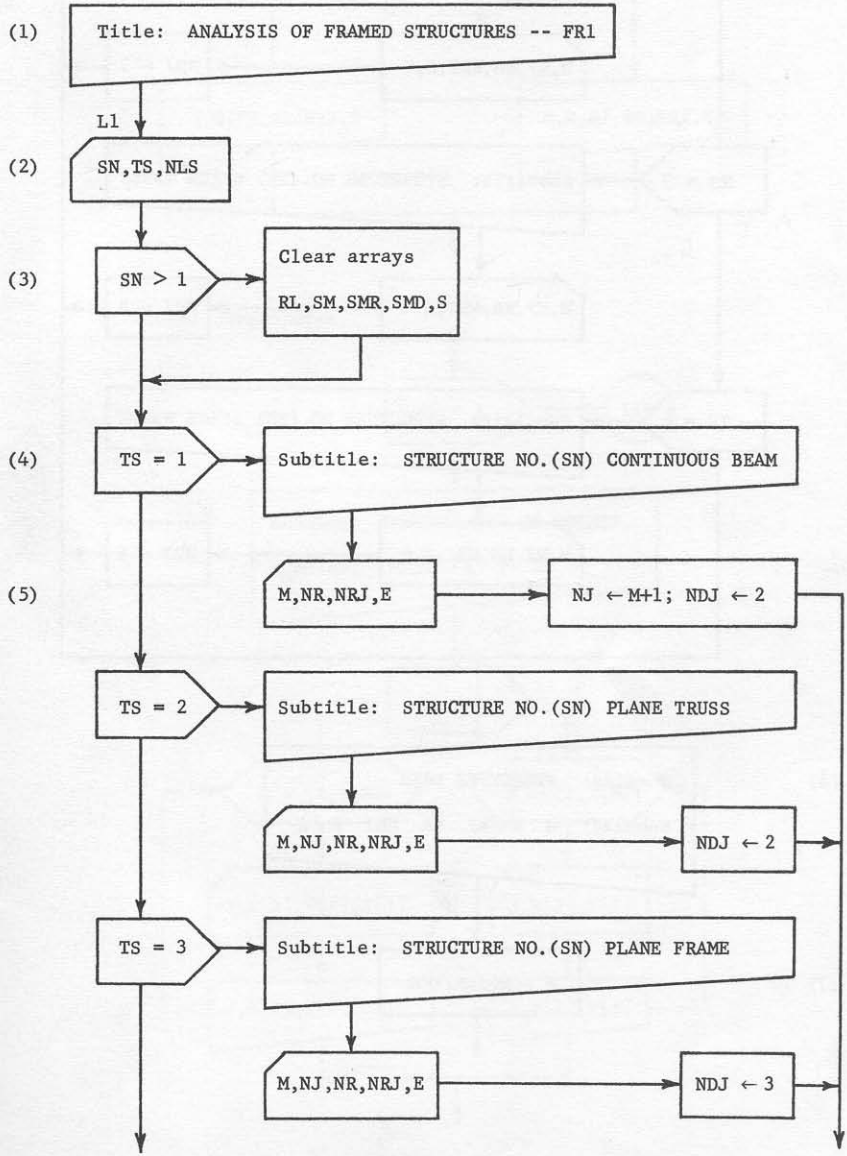
   a. Control data, structure parameters, and elastic moduli

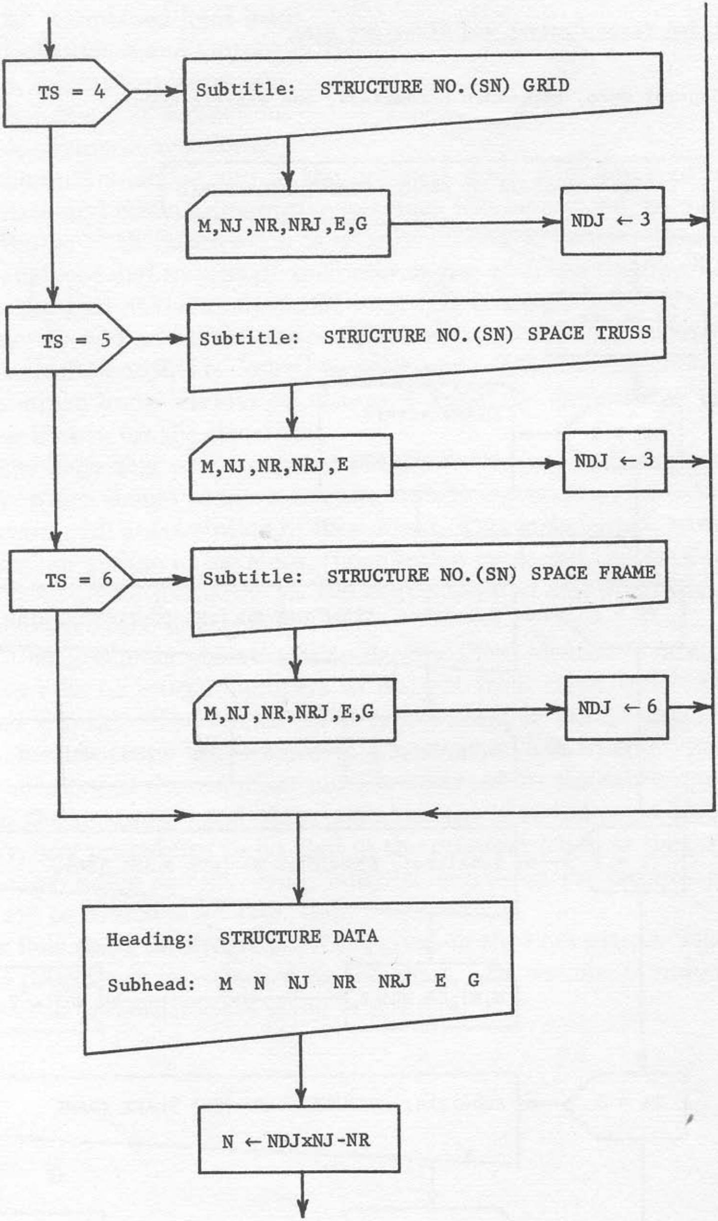


3.5 Flow Chart for Program *FR*1.

1. <u>Input and Print Control and Structure Data</u>

a. Control data, structure parameters, and elastic moduli

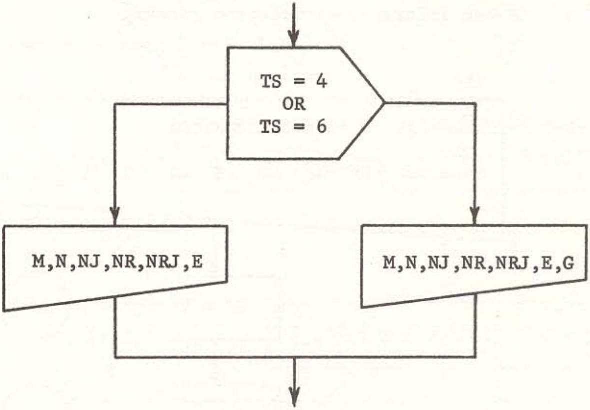



TS = 4
Subtitle: STRUCTURE NO.(SN) GRID
M,NJ,NR,NRJ,E,G
NDJ ← 3
TS = 5
Subtitle: STRUCTURE NO.(SN) SPACE TRUSS
M,NJ,NR,NRJ,E
NDJ ← 3
TS = 6
Subtitle: STRUCTURE NO.(SN) SPACE FRAME
M,NJ,NR,NRJ,E,G
NDJ ← 6
Heading: STRUCTURE DATA
Subhead: M N NJ NR NRJ E G
N ← NDJxNJ-NR

(6)

(7)

(8)

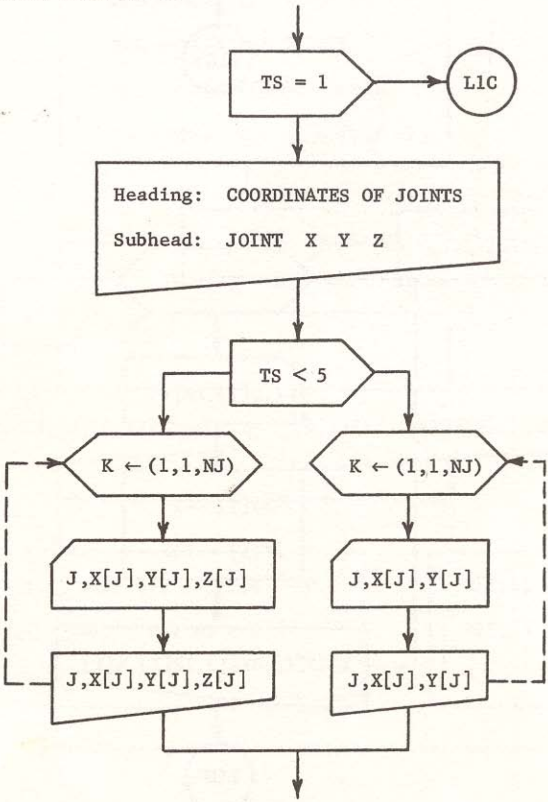


b. Joint coordinates

(1)

(2)

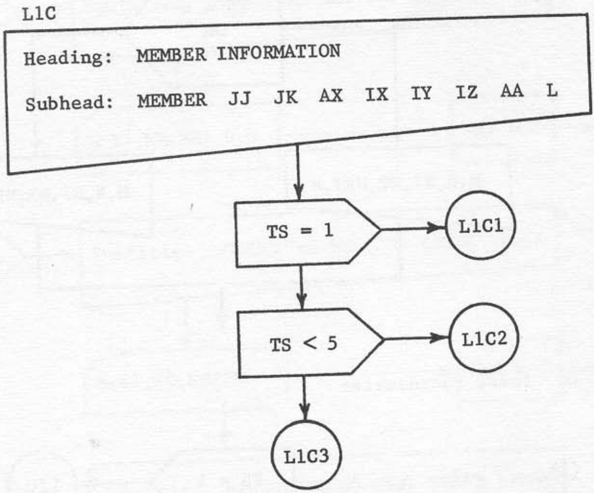
(3)



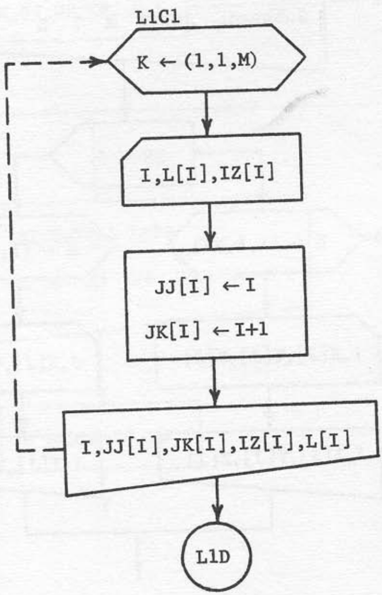
c. Member information; rotation matrix

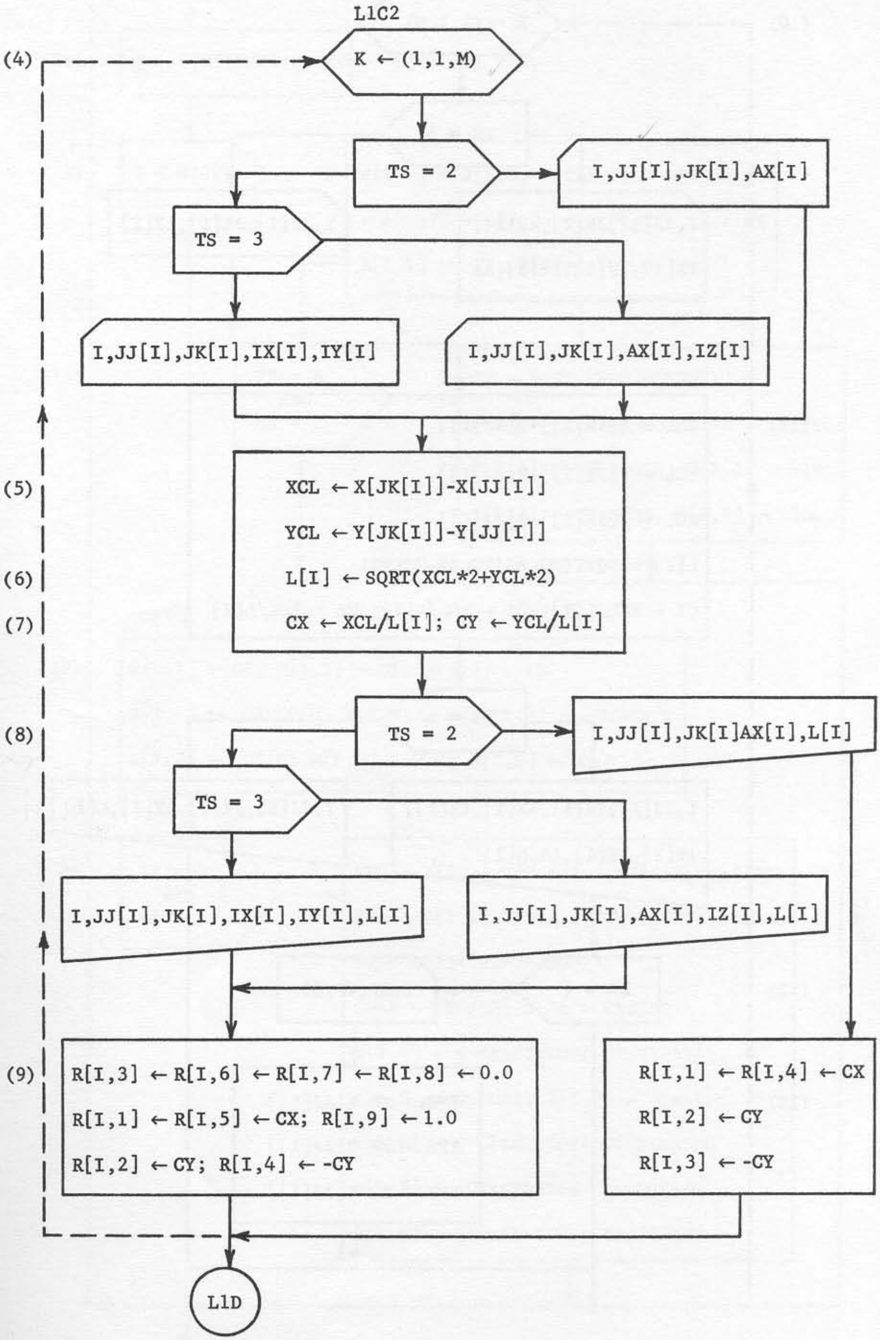
(1)

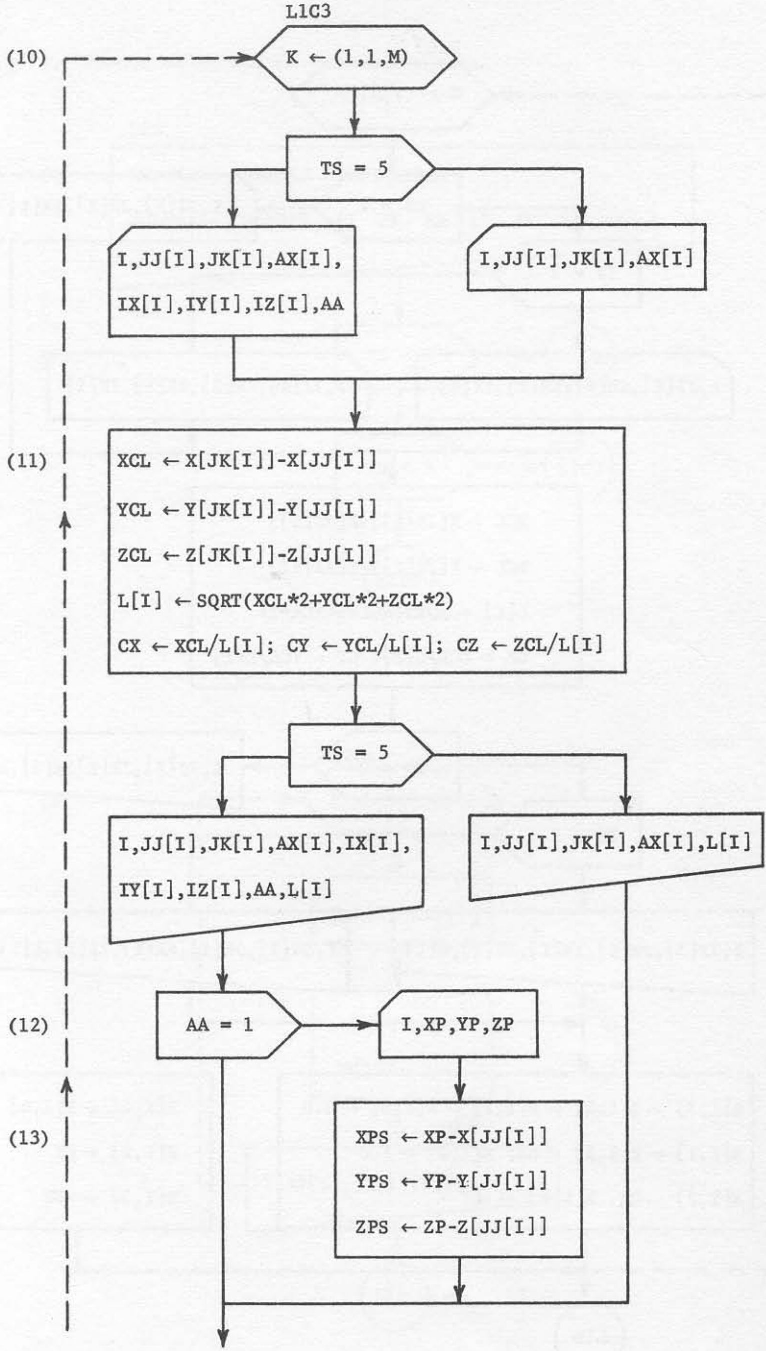
(2)

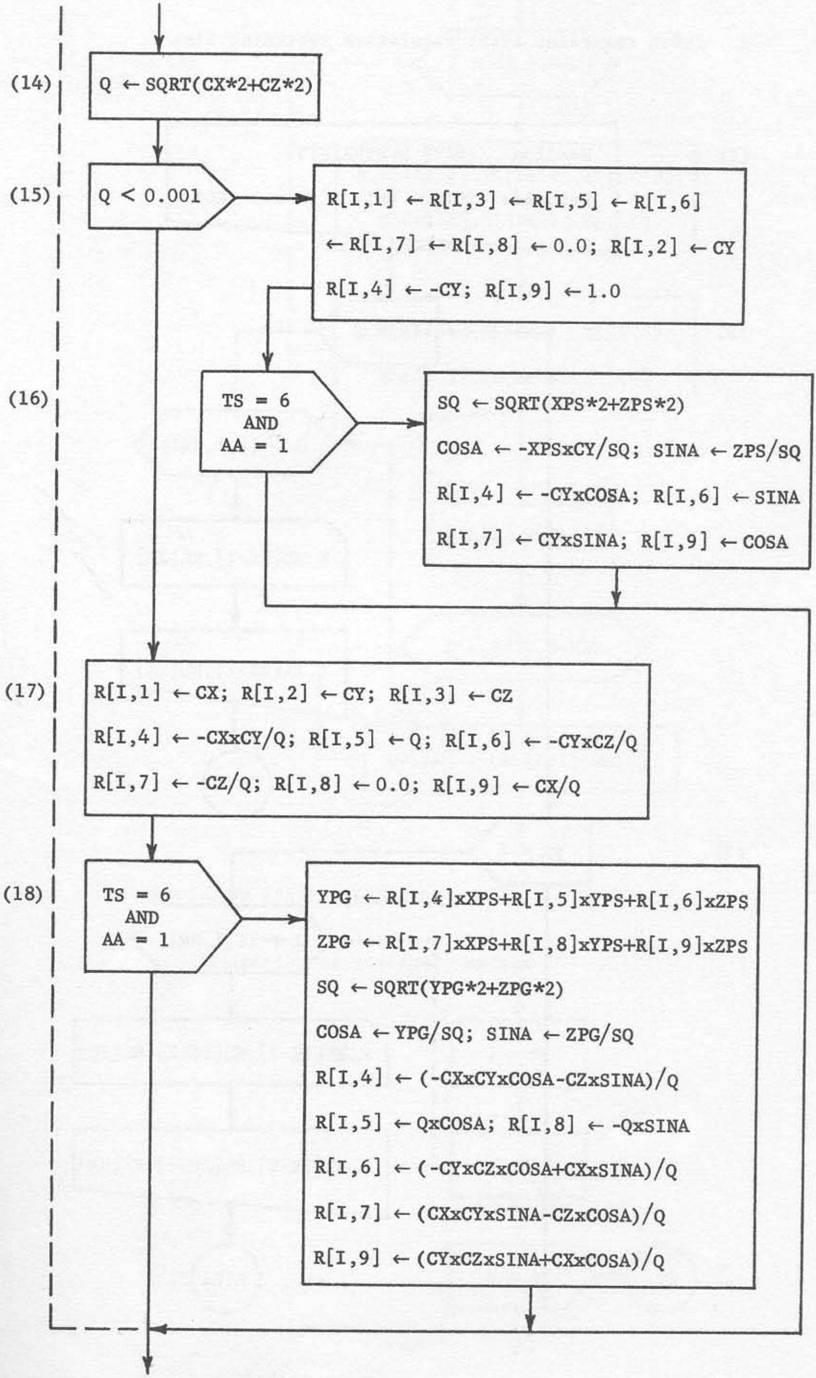


(3)

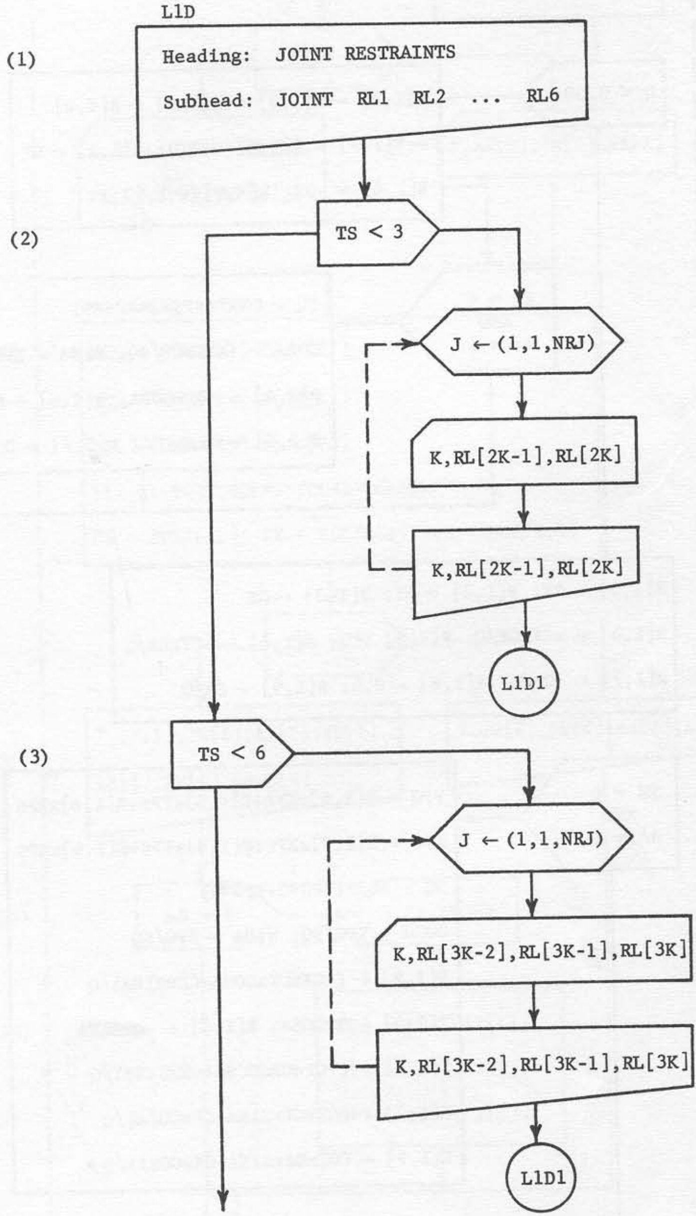


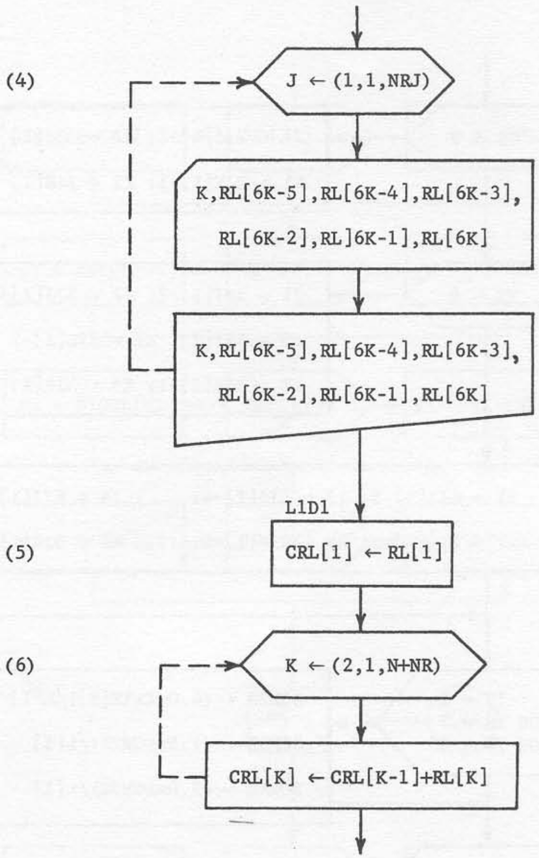

L1C2
(4)
K ← (1,1,M)
TS = 2
I,JJ[I],JK[I],AX[I]
TS = 3
I,JJ[I],JK[I],IX[I],IY[I]
I,JJ[I],JK[I],AX[I],IZ[I]
(5)
XCL ← X[JK[I]]-X[JJ[I]]
YCL ← Y[JK[I]]-Y[JJ[I]]
(6)
L[I] ← SQRT(XCL*2+YCL*2)
(7)
CX ← XCL/L[I]; CY ← YCL/L[I]
(8)
TS = 2
I,JJ[I],JK[I]AX[I],L[I]
TS = 3
I,JJ[I],JK[I],IX[I],IY[I],L[I]
I,JJ[I],JK[I],AX[I],IZ[I],L[I]
(9)
R[I,3] ← R[I,6] ← R[I,7] ← R[I,8] ← 0.0
R[I,1] ← R[I,5] ← CX; R[I,9] ← 1.0
R[I,2] ← CY; R[I,4] ← -CY
R[I,1] ← R[I,4] ← CX
R[I,2] ← CY
R[I,3] ← -CY
L1D


L1C3
(10)
K ← (1,1,M)
TS = 5
I,JJ[I],JK[I],AX[I],
IX[I],IY[I],IZ[I],AA
I,JJ[I],JK[I],AX[I]
(11)
XCL ← X[JK[I]]-X[JJ[I]]
YCL ← Y[JK[I]]-Y[JJ[I]]
ZCL ← Z[JK[I]]-Z[JJ[I]]
L[I] ← SQRT(XCL*2+YCL*2+ZCL*2)
CX ← XCL/L[I]; CY ← YCL/L[I]; CZ ← ZCL/L[I]
TS = 5
I,JJ[I],JK[I],AX[I],IX[I],
IY[I],IZ[I],AA,L[I]
I,JJ[I],JK[I],AX[I],L[I]
(12)
AA = 1
I,XP,YP,ZP
(13)
XPS ← XP-X[JJ[I]]
YPS ← YP-Y[JJ[I]]
ZPS ← ZP-Z[JJ[I]]


(14)
Q ← SQRT(CX*2+CZ*2)
(15)
Q < 0.001
R[I,1] ← R[I,3] ← R[I,5] ← R[I,6]
← R[I,7] ← R[I,8] ← 0.0; R[I,2] ← CY
R[I,4] ← -CY; R[I,9] ← 1.0
(16)
TS = 6
AND
AA = 1
SQ ← SQRT(XPS*2+ZPS*2)
COSA ← -XPSxCY/SQ; SINA ← ZPS/SQ
R[I,4] ← -CYxCOSA; R[I,6] ← SINA
R[I,7] ← CYxSINA; R[I,9] ← COSA
(17)
R[I,1] ← CX; R[I,2] ← CY; R[I,3] ← CZ
R[I,4] ← -CXxCY/Q; R[I,5] ← Q; R[I,6] ← -CYxCZ/Q
R[I,7] ← -CZ/Q; R[I,8] ← 0.0; R[I,9] ← CX/Q
(18)
TS = 6
AND
AA = 1
YPG ← R[I,4]xXPS+R[I,5]xYPS+R[I,6]xZPS
ZPG ← R[I,7]xXPS+R[I,8]xYPS+R[I,9]xZPS
SQ ← SQRT(YPG*2+ZPG*2)
COSA ← YPG/SQ; SINA ← ZPG/SQ
R[I,4] ← (-CXxCYxCOSA-CZxSINA)/Q
R[I,5] ← QxCOSA; R[I,8] ← -QxSINA
R[I,6] ← (-CYxCZxCOSA+CXxSINA)/Q
R[I,7] ← (CXxCYxSINA-CZxCOSA)/Q
R[I,9] ← (CYxCZxSINA+CXxCOSA)/Q

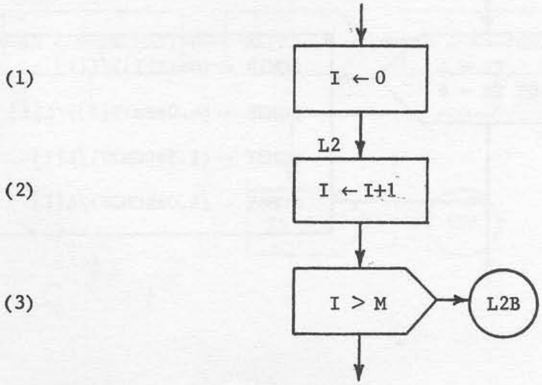
d. Joint restraint list; cumulative restraint list

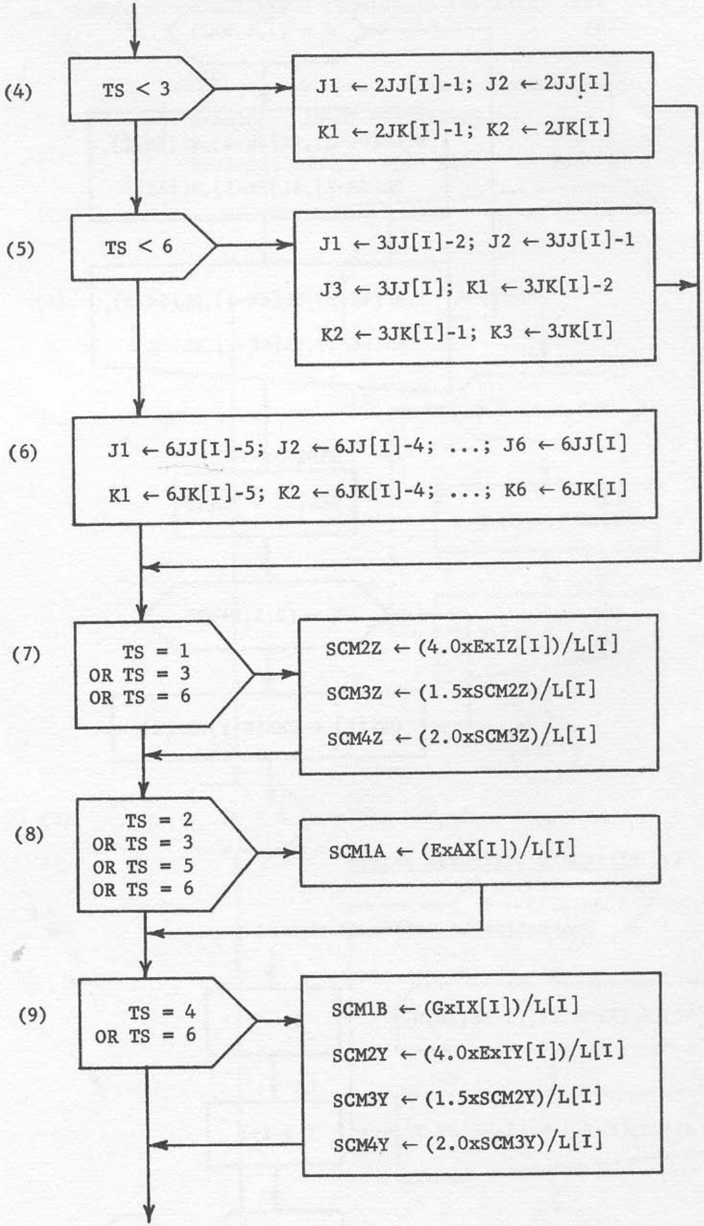




2. Structure Stiffness Matrix

a. Generation of stiffness matrix



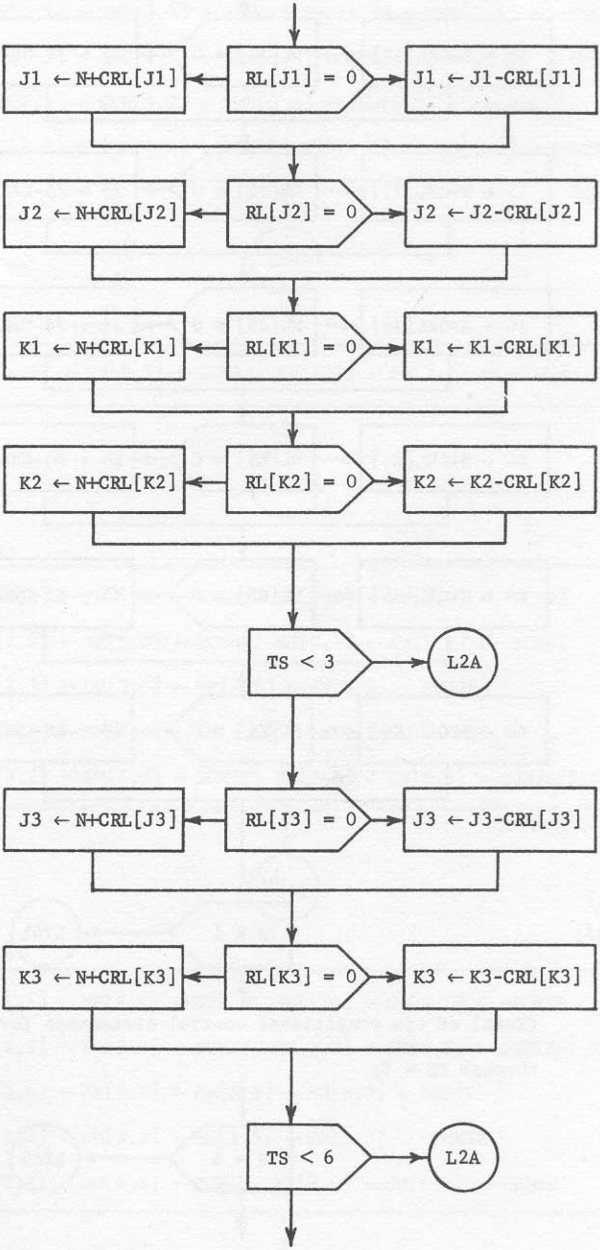

(4)
TS < 3
J1 ← 2JJ[I]-1; J2 ← 2JJ[I]
K1 ← 2JK[I]-1; K2 ← 2JK[I]
(5)
TS < 6
J1 ← 3JJ[I]-2; J2 ← 3JJ[I]-1
J3 ← 3JJ[I]; K1 ← 3JK[I]-2
K2 ← 3JK[I]-1; K3 ← 3JK[I]
(6)
J1 ← 6JJ[I]-5; J2 ← 6JJ[I]-4; ...; J6 ← 6JJ[I]
K1 ← 6JK[I]-5; K2 ← 6JK[I]-4; ...; K6 ← 6JK[I]
(7)
TS = 1 OR TS = 3 OR TS = 6
SCM2Z ← (4.0xExIZ[I])/L[I]
SCM3Z ← (1.5xSCM2Z)/L[I]
SCM4Z ← (2.0xSCM3Z)/L[I]
(8)
TS = 2 OR TS = 3 OR TS = 5 OR TS = 6
SCM1A ← (ExAX[I])/L[I]
(9)
TS = 4 OR TS = 6
SCM1B ← (GxIX[I])/L[I]
SCM2Y ← (4.0xExIY[I])/L[I]
SCM3Y ← (1.5xSCM2Y)/L[I]
SCM4Y ← (2.0xSCM3Y)/L[I]

(10)

(11)

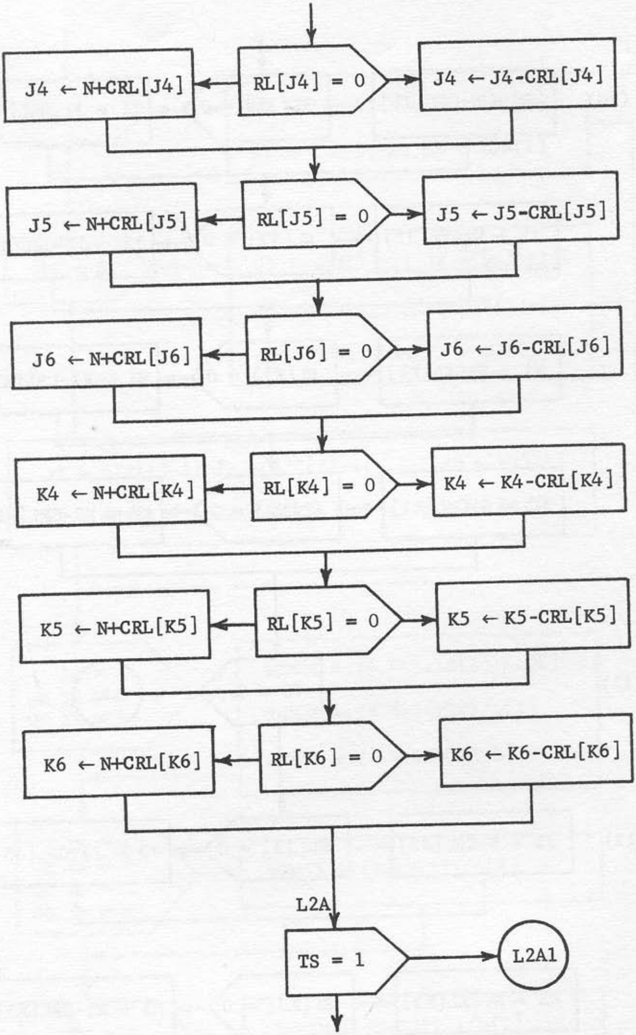
(12)

(13)

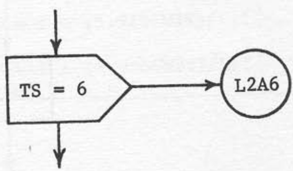

J1 ← N+CRL[J1]
RL[J1] = 0
J1 ← J1-CRL[J1]
J2 ← N+CRL[J2]
RL[J2] = 0
J2 ← J2-CRL[J2]
K1 ← N+CRL[K1]
RL[K1] = 0
K1 ← K1-CRL[K1]
K2 ← N+CRL[K2]
RL[K2] = 0
K2 ← K2-CRL[K2]
TS < 3
L2A
J3 ← N+CRL[J3]
RL[J3] = 0
J3 ← J3-CRL[J3]
K3 ← N+CRL[K3]
RL[K3] = 0
K3 ← K3-CRL[K3]
TS < 6
L2A

(14)

(15)



(Total of six conditional control statements for TS = 1 through TS = 6)



L2A1

```
(16)   SMD[1,1] ← SMD[3,3] ← SCM4Z; SMD[1,3] ← SMD[3,1] ← -SCM4Z
       SMD[1,2] ← SMD[2,1] ← SMD[1,4] ← SMD[4,1] ← SCM3Z
       SMD[2,3] ← SMD[3,2] ← SMD[3,4] ← SMD[4,3] ← -SCM3Z
       SMD[2,2] ← SMD[4,4] ← SCM2Z; SMD[2,4] ← SMD[4,2] ← SCM2Z/2.0
```

L2A9

L2A2

```
(17)   SM[1,1] ← SM[3,3] ← SCM1A; SM[1,3] ← SM[3,1] ← -SCM1A
```

L2A7

L2A3

```
(18)   SM[1,1] ← SM[4,4] ← SCM1A; SM[1,4] ← SM[4,1] ← -SCM1A
       SM[2,2] ← SM[5,5] ← SCM4Z; SM[2,5] ← SM[5,2] ← -SCM4Z
       SM[2,3] ← SM[3,2] ← SM[2,6] ← SM[6,2] ← SCM3Z
       SM[3,5] ← SM[5,3] ← SM[5,6] ← SM[6,5] ← -SCM3Z
       SM[3,3] ← SM[6,6] ← SCM2Z; SM[3,6] ← SM[6,3] ← SCM2Z/2.0
```

L2A8

L2A4

```
(19)   SM[1,1] ← SM[4,4] ← SCM1B; SM[1,4] ← SM[4,1] ← -SCM1B
       SM[2,2] ← SM[5,5] ← SCM2Y; SM[2,5] ← SM[5,2] ← SCM2Y/2.0
       SM[2,6] ← SM[6,2] ← SM[5,6] ← SM[6,5] ← SCM3Y
       SM[2,3] ← SM[3,2] ← SM[3,5] ← SM[5,3] ← -SCM3Y
       SM[3,3] ← SM[6,6] ← SCM4Y; SM[3,6] ← SM[6,3] ← -SCM4Y
```

L2A8

L2A5

(20) SM[1,1] ← SM[4,4] ← SCM1A; SM[1,4] ← SM[4,1] ← -SCM1A

↓

L2A8

L2A6

(21) SM[1,1] ← SM[7,7] ← SCM1A; SM[1,7] ← SM[7,1] ← -SCM1A

SM[2,2] ← SM[8,8] ← SCM4Z; SM[2,8] ← SM[8,2] ← -SCM4Z

SM[2,6] ← SM[6,2] ← SM[2,12] ← SM[12,2] ← SCM3Z

SM[6,8] ← SM[8,6] ← SM[8,12] ← SM[12,8] ← -SCM3Z

SM[3,3] ← SM[9,9] ← SCM4Y; SM[3,9] ← SM[9,3] ← -SCM4Y

SM[3,5] ← SM[5,3] ← SM[3,11] ← SM[11,3] ← -SCM3Y

SM[4,4] ← SM[10,10] ← SCM1B; SM[4,10] ← SM[10,4] ← -SCM1B

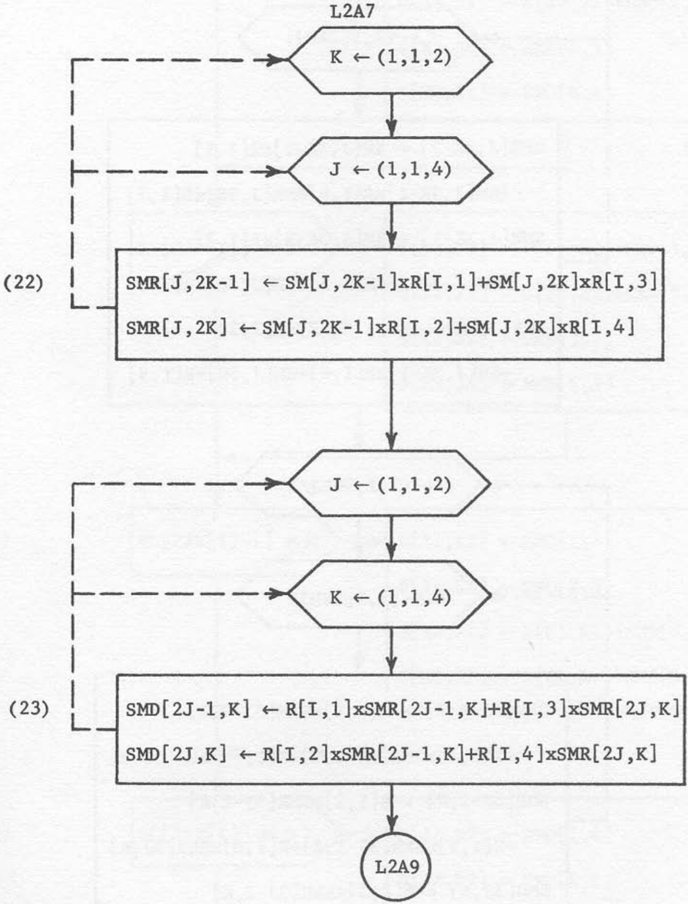
SM[5,5] ← SM[11,11] ← SCM2Y; SM[5,11] ← SM[11,5] ← SCM2Y/2.0

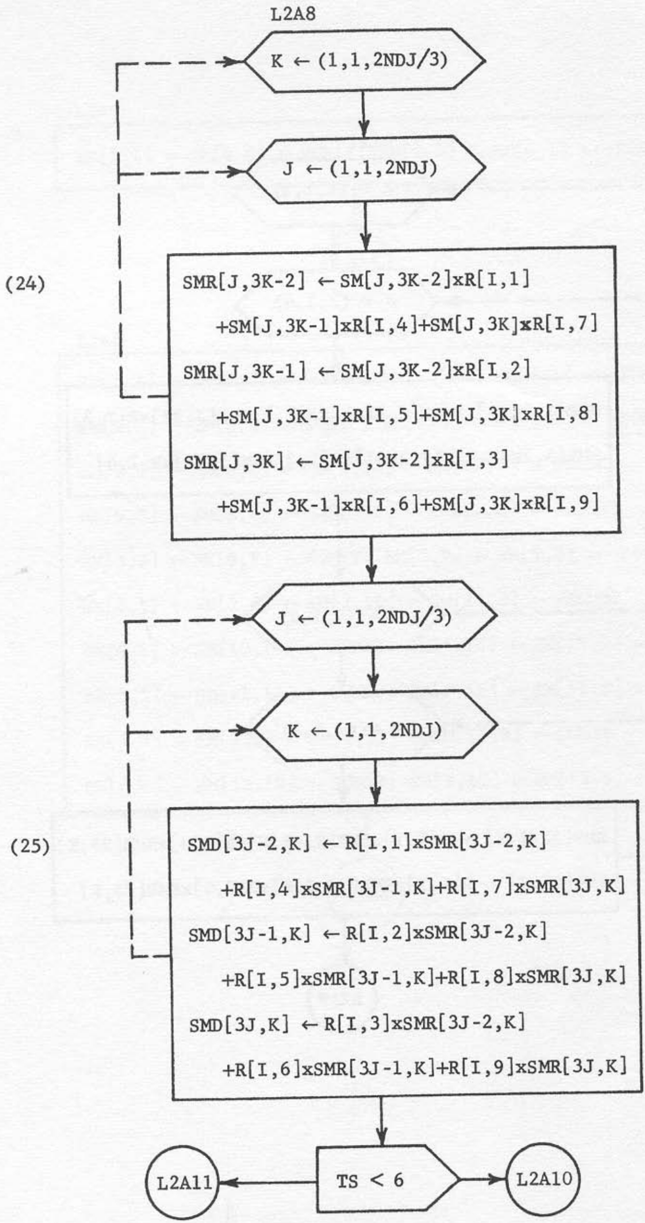
SM[5,9] ← SM[9,5] ← SM[9,11] ← SM[11,9] ← SCM3Y

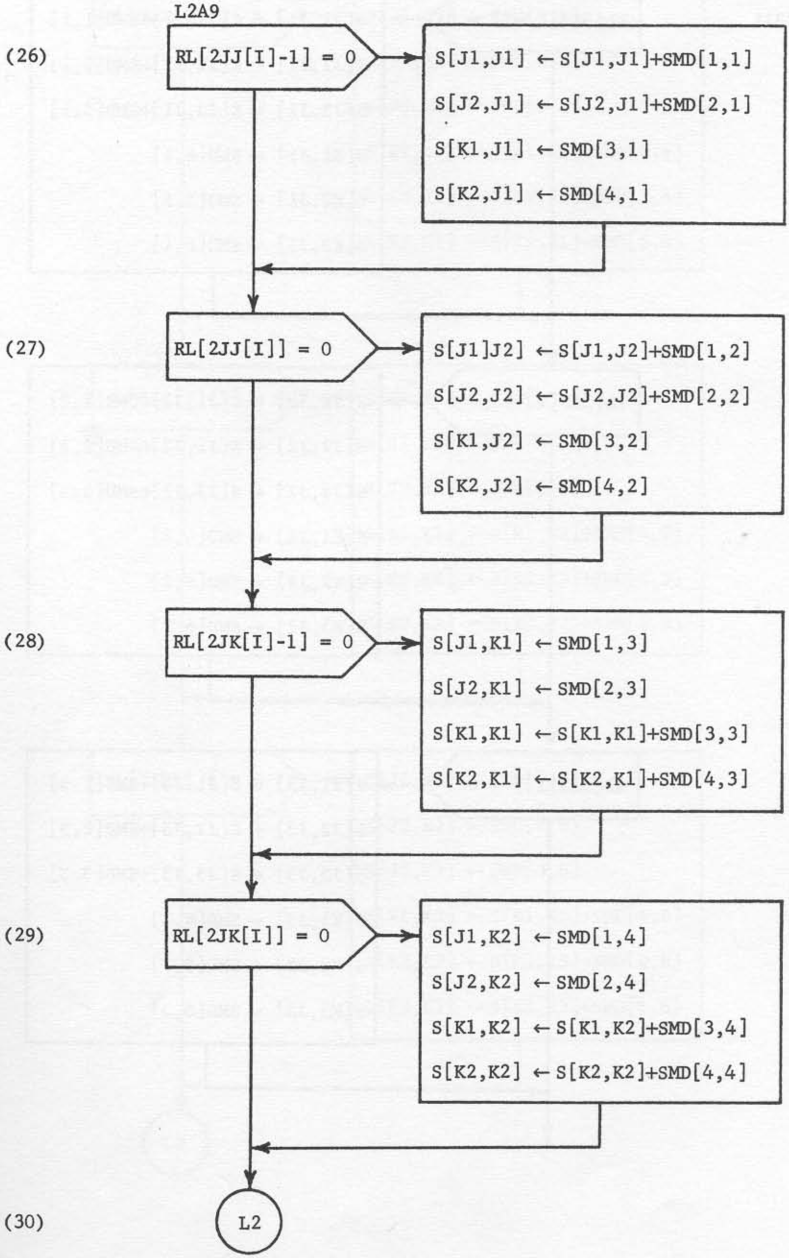
SM[6,6] ← SM[12,12] ← SCM2Z; SM[6,12] ← SM[12,6] ← SCM2Z/2.0

↓

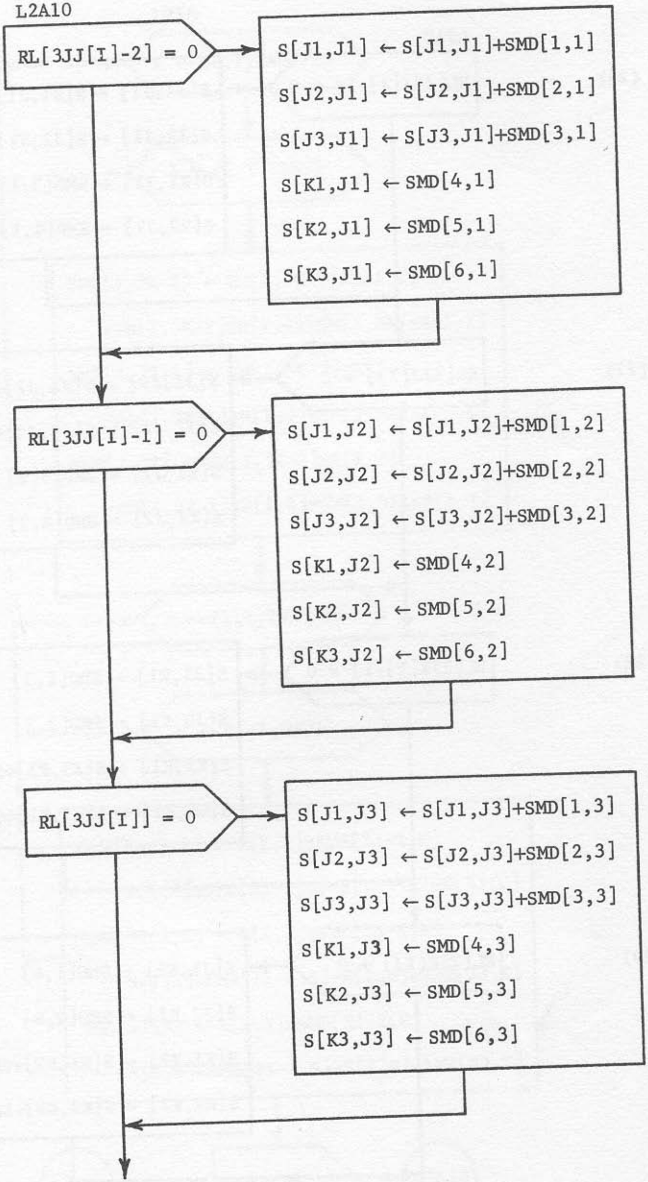
L2A8

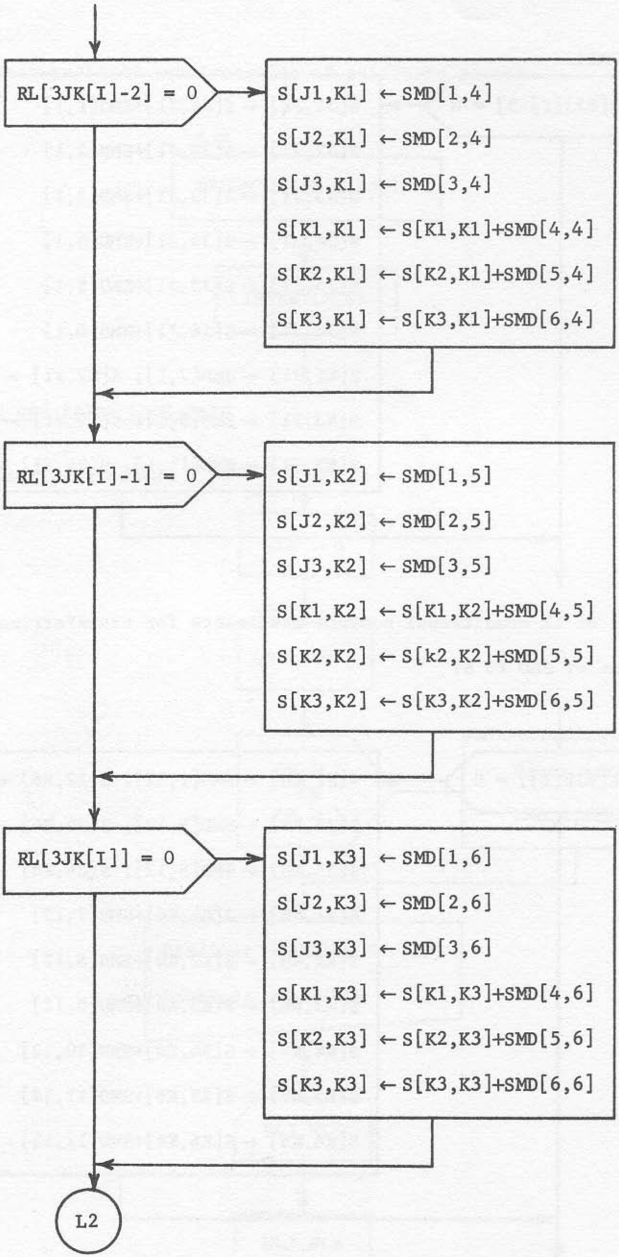

L2A7
K ← (1,1,2)
J ← (1,1,4)
(22)
SMR[J,2K-1] ← SM[J,2K-1]xR[I,1]+SM[J,2K]xR[I,3]
SMR[J,2K] ← SM[J,2K-1]xR[I,2]+SM[J,2K]xR[I,4]
J ← (1,1,2)
K ← (1,1,4)
(23)
SMD[2J-1,K] ← R[I,1]xSMR[2J-1,K]+R[I,3]xSMR[2J,K]
SMD[2J,K] ← R[I,2]xSMR[2J-1,K]+R[I,4]xSMR[2J,K]
L2A9


L2A8
K ← (1,1,2NDJ/3)
J ← (1,1,2NDJ)
(24)
SMR[J,3K-2] ← SM[J,3K-2]xR[I,1]
+SM[J,3K-1]xR[I,4]+SM[J,3K]xR[I,7]
SMR[J,3K-1] ← SM[J,3K-2]xR[I,2]
+SM[J,3K-1]xR[I,5]+SM[J,3K]xR[I,8]
SMR[J,3K] ← SM[J,3K-2]xR[I,3]
+SM[J,3K-1]xR[I,6]+SM[J,3K]xR[I,9]
J ← (1,1,2NDJ/3)
K ← (1,1,2NDJ)
(25)
SMD[3J-2,K] ← R[I,1]xSMR[3J-2,K]
+R[I,4]xSMR[3J-1,K]+R[I,7]xSMR[3J,K]
SMD[3J-1,K] ← R[I,2]xSMR[3J-2,K]
+R[I,5]xSMR[3J-1,K]+R[I,8]xSMR[3J,K]
SMD[3J,K] ← R[I,3]xSMR[3J-2,K]
+R[I,6]xSMR[3J-1,K]+R[I,9]xSMR[3J,K]
TS < 6
L2A11
L2A10


L2A9
(26)
RL[2JJ[I]-1] = 0
S[J1,J1] ← S[J1,J1]+SMD[1,1]
S[J2,J1] ← S[J2,J1]+SMD[2,1]
S[K1,J1] ← SMD[3,1]
S[K2,J1] ← SMD[4,1]
(27)
RL[2JJ[I]] = 0
S[J1]J2] ← S[J1,J2]+SMD[1,2]
S[J2,J2] ← S[J2,J2]+SMD[2,2]
S[K1,J2] ← SMD[3,2]
S[K2,J2] ← SMD[4,2]
(28)
RL[2JK[I]-1] = 0
S[J1,K1] ← SMD[1,3]
S[J2,K1] ← SMD[2,3]
S[K1,K1] ← S[K1,K1]+SMD[3,3]
S[K2,K1] ← S[K2,K1]+SMD[4,3]
(29)
RL[2JK[I]] = 0
S[J1,K2] ← SMD[1,4]
S[J2,K2] ← SMD[2,4]
S[K1,K2] ← S[K1,K2]+SMD[3,4]
S[K2,K2] ← S[K2,K2]+SMD[4,4]
(30)
L2

(31)


L2A10
RL[3JJ[I]-2] = 0
S[J1,J1] ← S[J1,J1]+SMD[1,1]
S[J2,J1] ← S[J2,J1]+SMD[2,1]
S[J3,J1] ← S[J3,J1]+SMD[3,1]
S[K1,J1] ← SMD[4,1]
S[K2,J1] ← SMD[5,1]
S[K3,J1] ← SMD[6,1]
RL[3JJ[I]-1] = 0
S[J1,J2] ← S[J1,J2]+SMD[1,2]
S[J2,J2] ← S[J2,J2]+SMD[2,2]
S[J3,J2] ← S[J3,J2]+SMD[3,2]
S[K1,J2] ← SMD[4,2]
S[K2,J2] ← SMD[5,2]
S[K3,J2] ← SMD[6,2]
RL[3JJ[I]] = 0
S[J1,J3] ← S[J1,J3]+SMD[1,3]
S[J2,J3] ← S[J2,J3]+SMD[2,3]
S[J3,J3] ← S[J3,J3]+SMD[3,3]
S[K1,J3] ← SMD[4,3]
S[K2,J3] ← SMD[5,3]
S[K3,J3] ← SMD[6,3]



RL[3JK[I]-2] = 0
S[J1,K1] ← SMD[1,4]
S[J2,K1] ← SMD[2,4]
S[J3,K1] ← SMD[3,4]
S[K1,K1] ← S[K1,K1]+SMD[4,4]
S[K2,K1] ← S[K2,K1]+SMD[5,4]
S[K3,K1] ← S[K3,K1]+SMD[6,4]
RL[3JK[I]-1] = 0
S[J1,K2] ← SMD[1,5]
S[J2,K2] ← SMD[2,5]
S[J3,K2] ← SMD[3,5]
S[K1,K2] ← S[K1,K2]+SMD[4,5]
S[K2,K2] ← S[k2,K2]+SMD[5,5]
S[K3,K2] ← S[K3,K2]+SMD[6,5]
RL[3JK[I]] = 0
S[J1,K3] ← SMD[1,6]
S[J2,K3] ← SMD[2,6]
S[J3,K3] ← SMD[3,6]
S[K1,K3] ← S[K1,K3]+SMD[4,6]
S[K2,K3] ← S[K2,K3]+SMD[5,6]
S[K3,K3] ← S[K3,K3]+SMD[6,6]
L2

(32)

L2A11

```
RL[6JJ[I]-5] = 0  →
    S[J1,J1] ← S[J1,J1]+SMD[1,1]
    S[J2,J1] ← S[J2,J1]+SMD[2,1]
    S[J3,J1] ← S[J3,J1]+SMD[3,1]
    S[J4,J1] ← S[J4,J1]+SMD[4,1]
    S[J5,J1] ← S[J5,J1]+SMD[5,1]
    S[J6,J1] ← S[J6,J1]+SMD[6,1]
    S[K1,J1] ← SMD[7,1]; S[K2,J1] ← SMD[8,1]
    S[K3,J1] ← SMD[9,1]; S[K4,J1] ← SMD[10,1]
    S[K5,J1] ← SMD[11,1]; S[K6,J1] ← SMD[12,1]
```

(Total of 12 conditional control statements for transferring 12 columns of SMD to S)

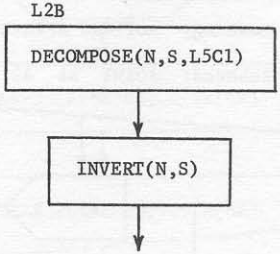
```
RL[6JK[I]] = 0  →
    S[J1,K6] ← SMD[1,12]; S[J2,K6] ← SMD[2,12]
    S[J3,K6] ← SMD[3,12]; S[J4,K6] ← SMD[4,12]
    S[J5,K6] ← SMD[5,12]; S[J6,K6] ← SMD[6,12]
    S[K1,K6] ← S[K1,K6]+SMD[7,12]
    S[K2,K6] ← S[K2,K6]+SMD[8,12]
    S[K3,K6] ← S[K3,K6]+SMD[9,12]
    S[K4,K6] ← S[K4,K6]+SMD[10,12]
    S[K5,K6] ← S[K5,K6]+SMD[11,12]
    S[K6,K6] ← S[K6,K6]+SMD[12,12]
```

L2

b. Decomposition and inversion of stiffness matrix

(1)

(2)



3. Input and Print Load Data

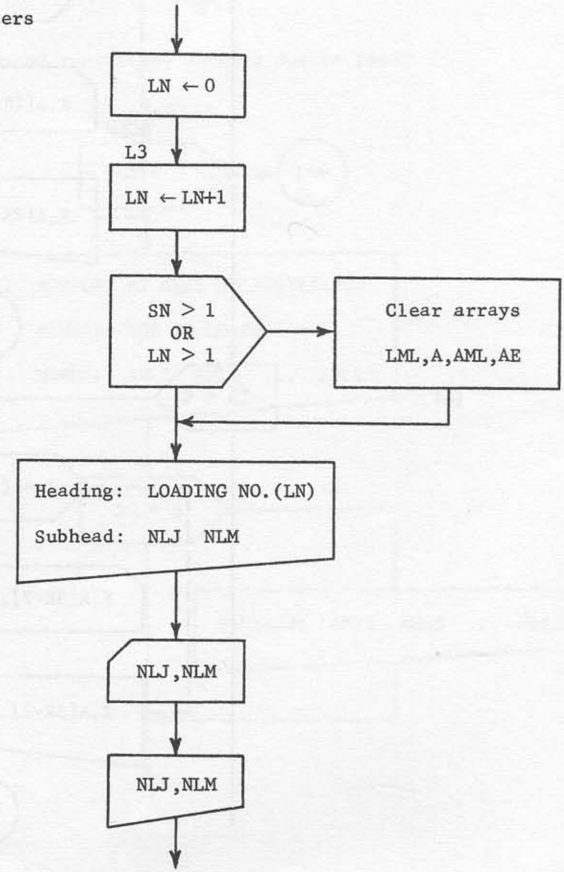
a. Load parameters

(1)

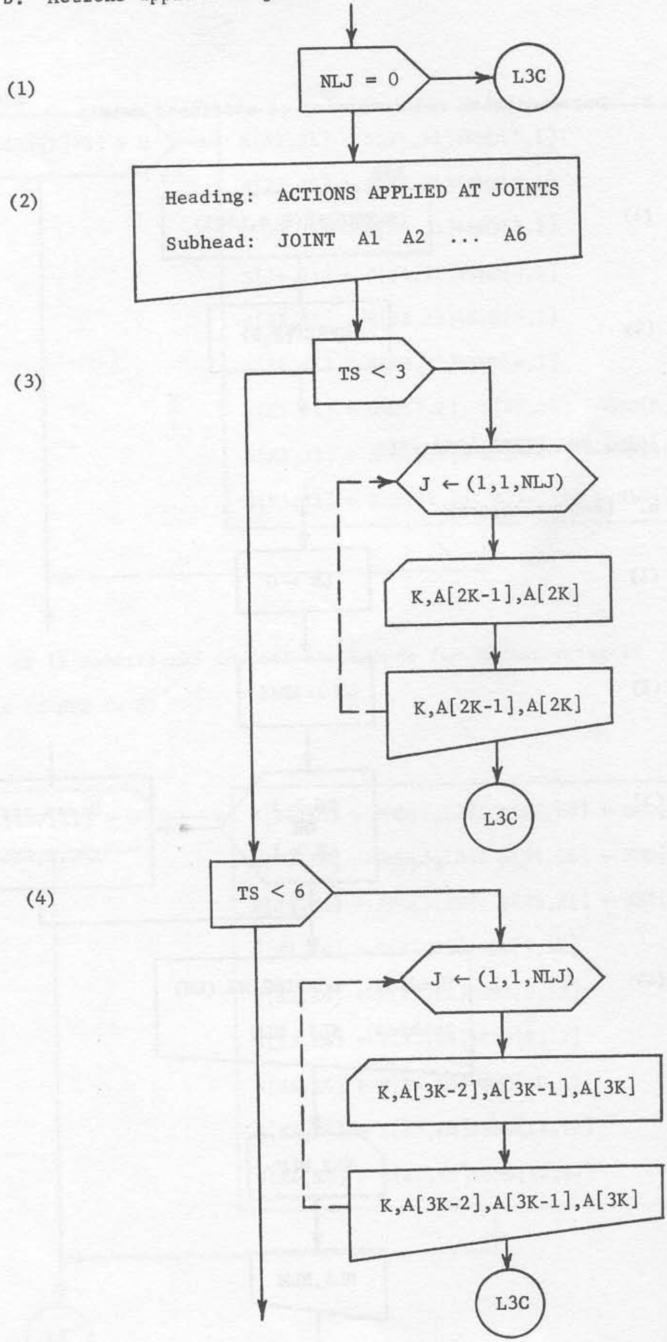
(2)

(3)

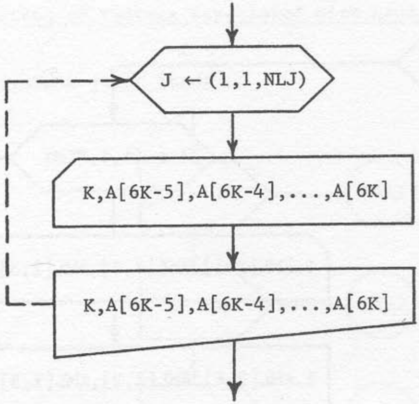
(4)



b. Actions applied at joints



(5)

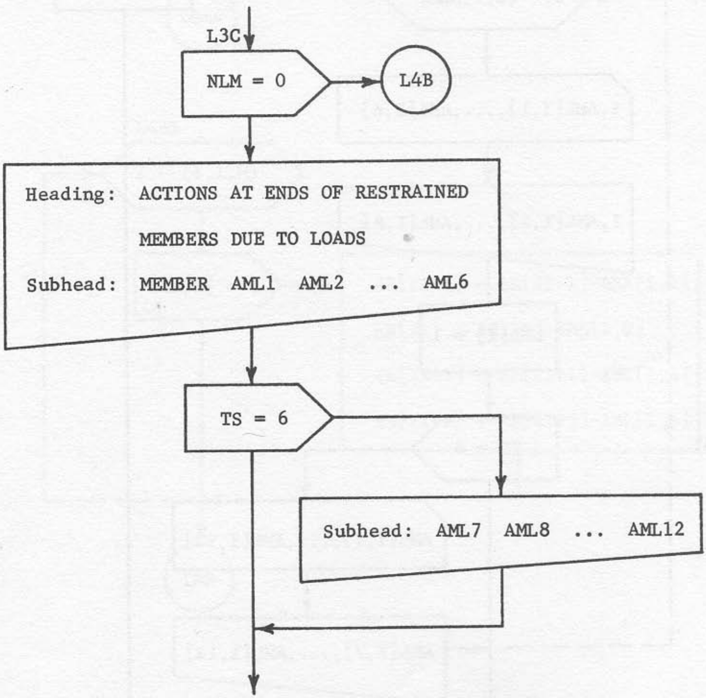


c. Actions at ends of restrained members due to loads

(1)

(2)

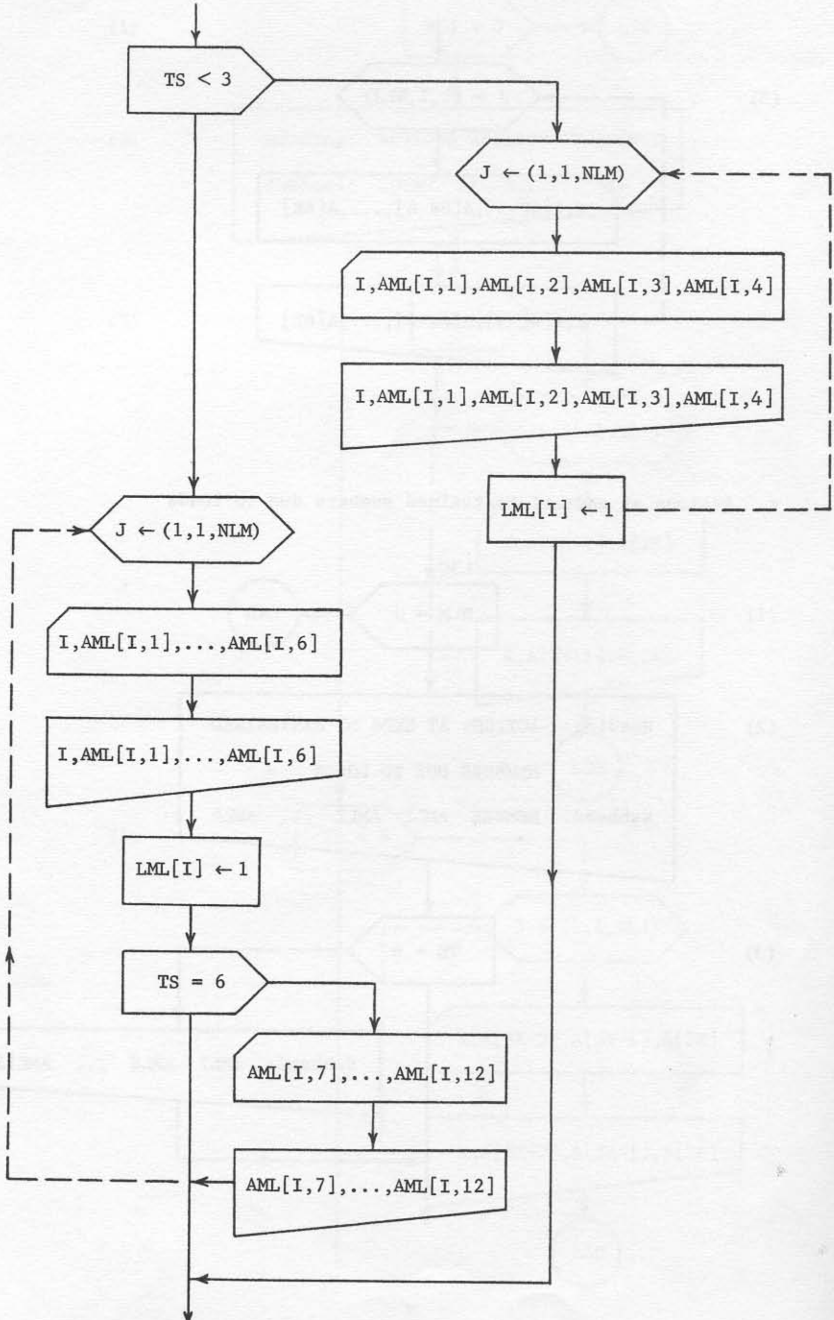
(3)



(4)

(5)

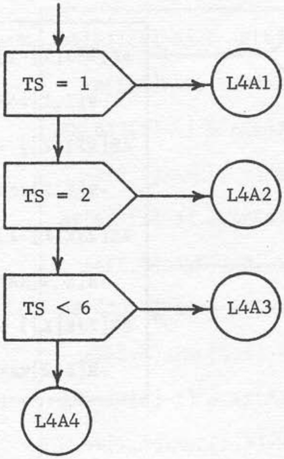
(6)


TS < 3
J ← (1,1,NLM)
I,AML[I,1],AML[I,2],AML[I,3],AML[I,4]
I,AML[I,1],AML[I,2],AML[I,3],AML[I,4]
LML[I] ← 1
J ← (1,1,NLM)
I,AML[I,1],...,AML[I,6]
I,AML[I,1],...,AML[I,6]
LML[I] ← 1
TS = 6
AML[I,7],...,AML[I,12]
AML[I,7],...,AML[I,12]

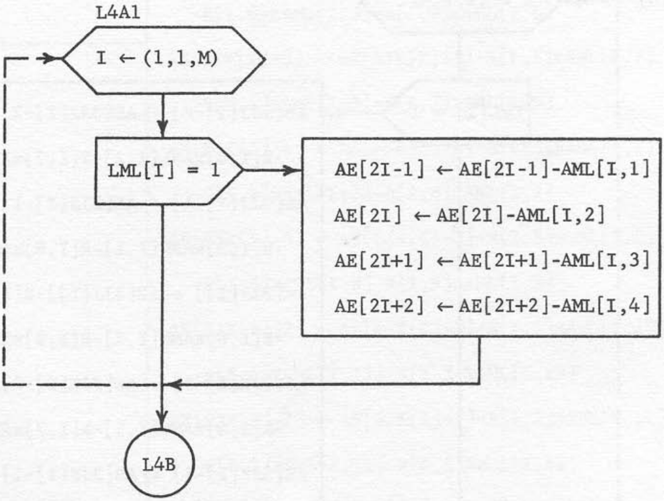
4. Construction of Vectors Associated with Loads

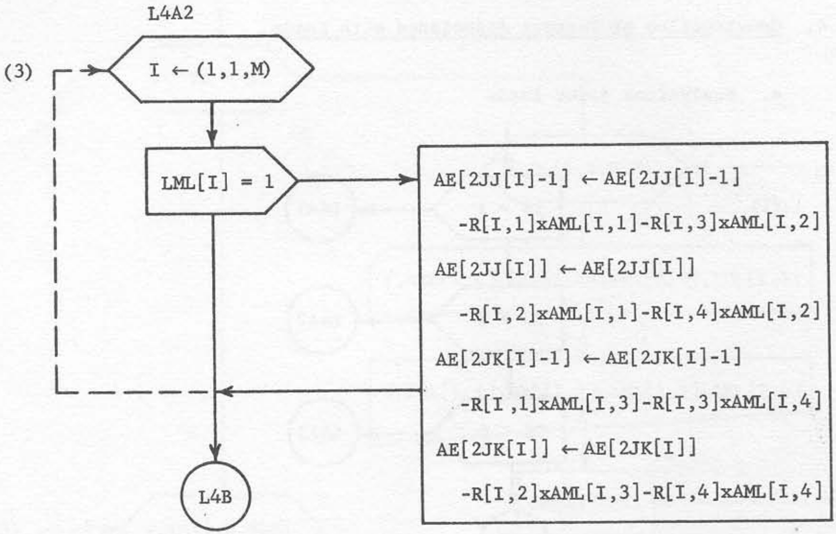
a. Equivalent joint loads

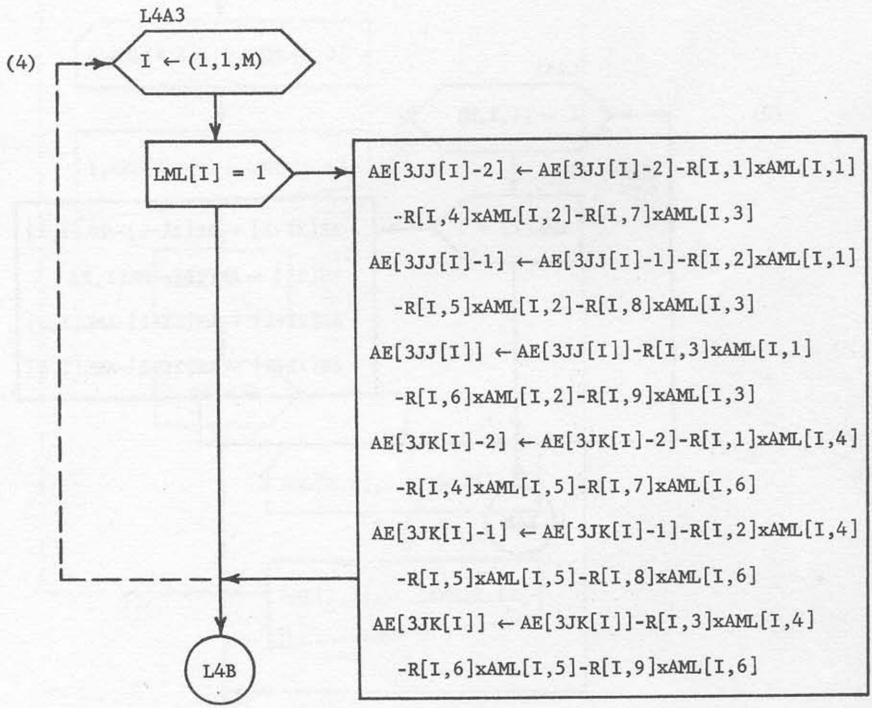
(1)



(2)

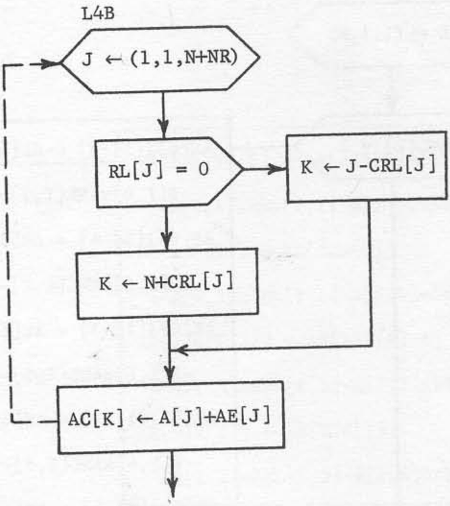



L4A2
(3)
I ← (1,1,M)
LML[I] = 1
AE[2JJ[I]-1] ← AE[2JJ[I]-1]
-R[I,1]xAML[I,1]-R[I,3]xAML[I,2]
AE[2JJ[I]] ← AE[2JJ[I]]
-R[I,2]xAML[I,1]-R[I,4]xAML[I,2]
AE[2JK[I]-1] ← AE[2JK[I]-1]
-R[I ,1]xAML[I,3]-R[I,3]xAML[I,4]
AE[2JK[I]] ← AE[2JK[I]]
-R[I,2]xAML[I,3]-R[I,4]xAML[I,4]
L4B


L4A3
(4)
I ← (1,1,M)
LML[I] = 1
AE[3JJ[I]-2] ← AE[3JJ[I]-2]-R[I,1]xAML[I,1]
-R[I,4]xAML[I,2]-R[I,7]xAML[I,3]
AE[3JJ[I]-1] ← AE[3JJ[I]-1]-R[I,2]xAML[I,1]
-R[I,5]xAML[I,2]-R[I,8]xAML[I,3]
AE[3JJ[I]] ← AE[3JJ[I]]-R[I,3]xAML[I,1]
-R[I,6]xAML[I,2]-R[I,9]xAML[I,3]
AE[3JK[I]-2] ← AE[3JK[I]-2]-R[I,1]xAML[I,4]
-R[I,4]xAML[I,5]-R[I,7]xAML[I,6]
AE[3JK[I]-1] ← AE[3JK[I]-1]-R[I,2]xAML[I,4]
-R[I,5]xAML[I,5]-R[I,8]xAML[I,6]
AE[3JK[I]] ← AE[3JK[I]]-R[I,3]xAML[I,4]
-R[I,6]xAML[I,5]-R[I,9]xAML[I,6]
L4B

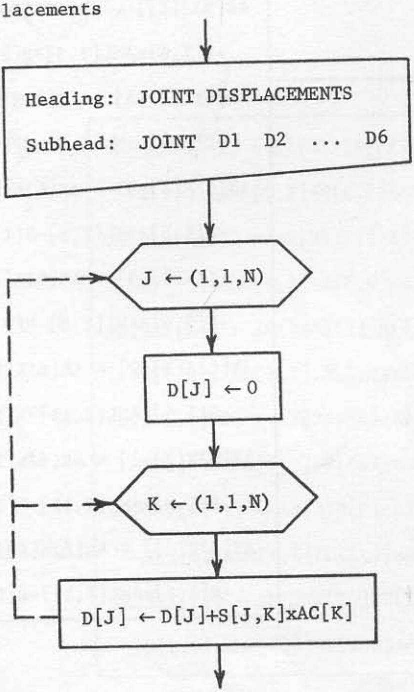
```
L4A4
(5)  I ← (1,1,M)

LML[I] = 1

AE[6JJ[I]-5] ← AE[6JJ[I]-5]-R[I,1]xAML[I,1]
   -R[I,4]xAML[I,2]-R[I,7]xAML[I,3]
AE[6JJ[I]-4] ← AE[6JJ[I]-4]-R[I,2]xAML[I,1]
   -R[I,5]xAML[I,2]-R[I,8]xAML[I,3]
AE[6JJ[I]-3] ← AE[6JJ[I]-3]-R[I,3]xAML[I,1]
   -R[I,6]xAML[I,2]-R[I,9]xAML[I,3]
AE[6JJ[I]-2] ← AE[6JJ[I]-2]-R[I,1]xAML[I,4]
   -R[I,4]xAML[I,5]-R[I,7]xAML[I,6]
AE[6JJ[I]-1] ← AE[6JJ[I]-1]-R[I,2]xAML[I,4]
   -R[I,5]xAML[I,5]-R[I,8]xAML[I,6]
AE[6JJ[I]] ← AE[6JJ[I]]-R[I,3]xAML[I,4]
   -R[I,6]xAML[I,5]-R[I,9]xAML[I,6]
AE[6JK[I]-5] ← AE[6JK[I]-5]-R[I,1]xAML[I,7]
   -R[I,4]xAML[I,8]-R[I,7]xAML[I,9]
AE[6JK[I]-4] ← AE[6JK[I]-4]-R[I,2]xAML[I,7]
   -R[I,5]xAML[I,8]-R[I,8]xAML[I,9]
AE[6JK[I]-3] ← AE[6JK[I]-3]-R[I,3]xAML[I,7]
   -R[I,6]xAML[I,8]-R[I,9]xAML[I,9]
AE[6JK[I]-2] ← AE[6JK[I]-2]-R[I,1]xAML[I,10]
   -R[I,4]xAML[I,11]-R[I,7]xAML[I,12]
AE[6JK[I]-1] ← AE[6JK[I]-1]-R[I,2]xAML[I,10]
   -R[I,5]xAML[I,11]-R[I,8]xAML[I,12]
AE[6JK[I]] ← AE[6JK[I]]-R[I,3]xAML[I,10]
   -R[I,6]xAML[I,11]-R[I,9]xAML[I,12]
```

b. Combined joint loads



5. <u>Calculation and Output of Results</u>

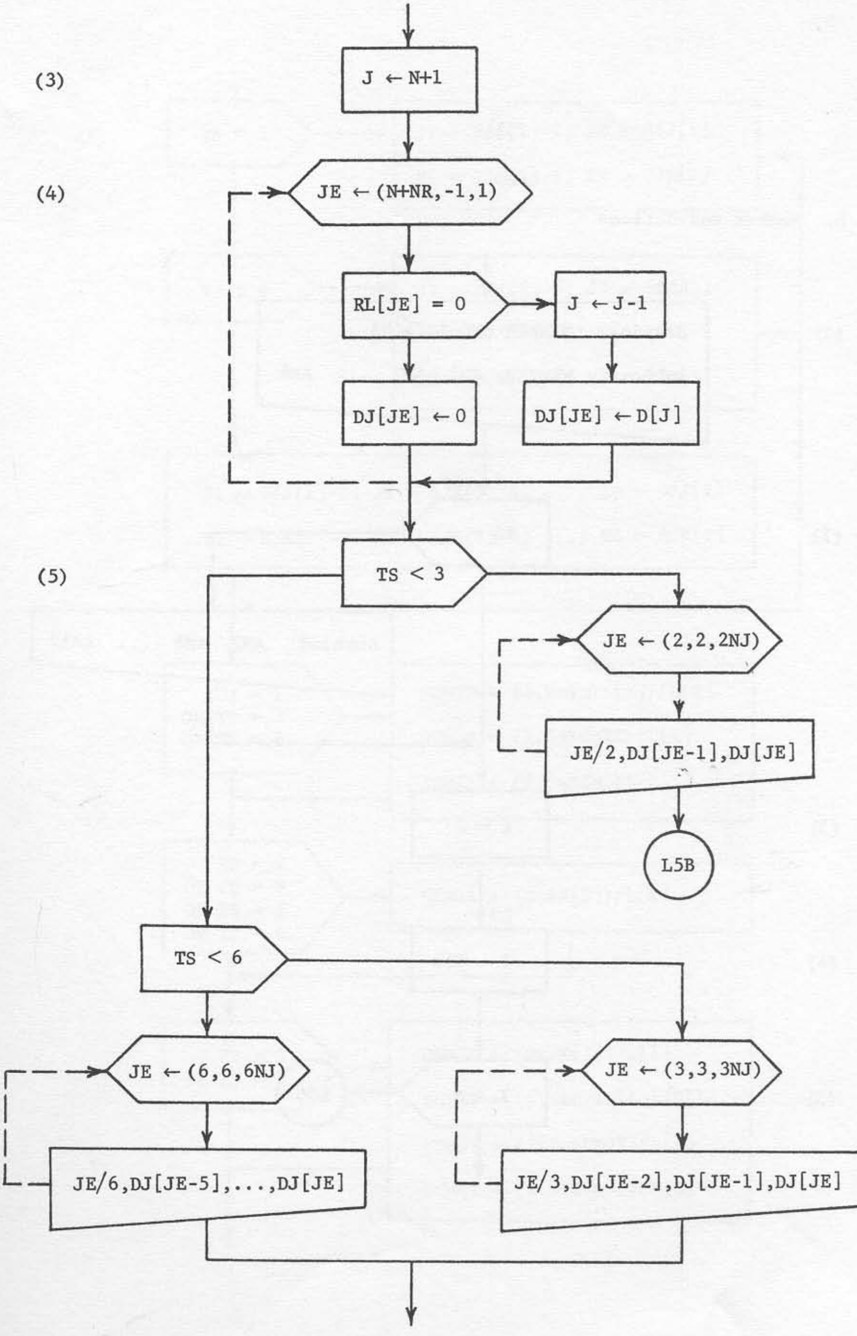
a. Joint displacements



(3)

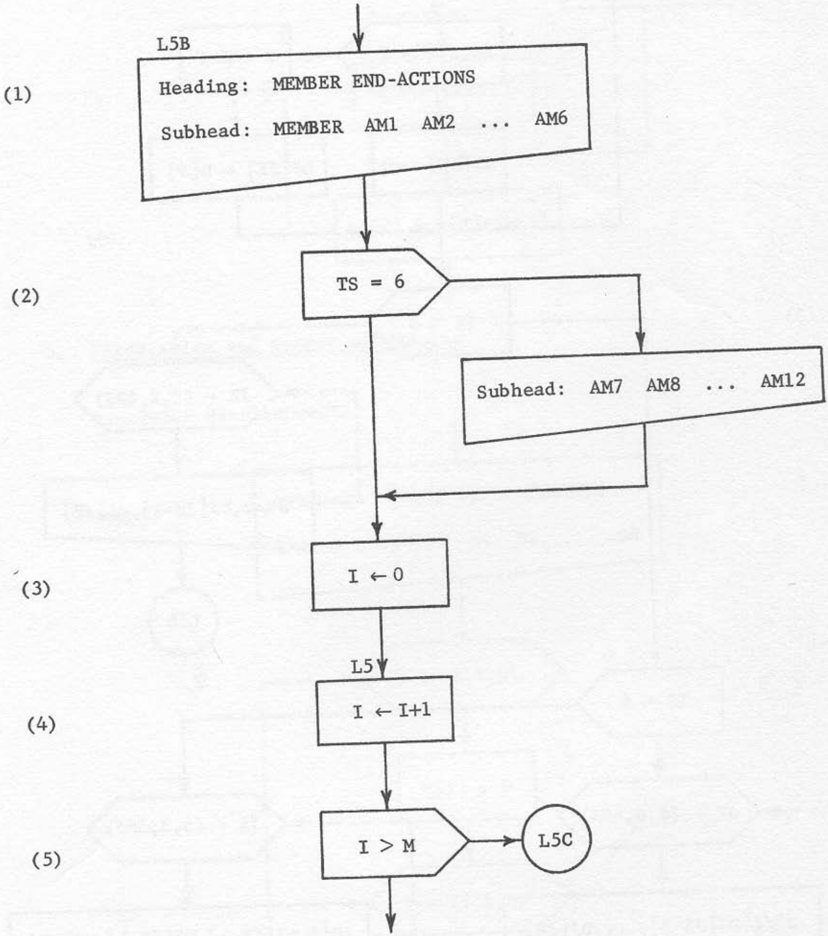
(4)

(5)

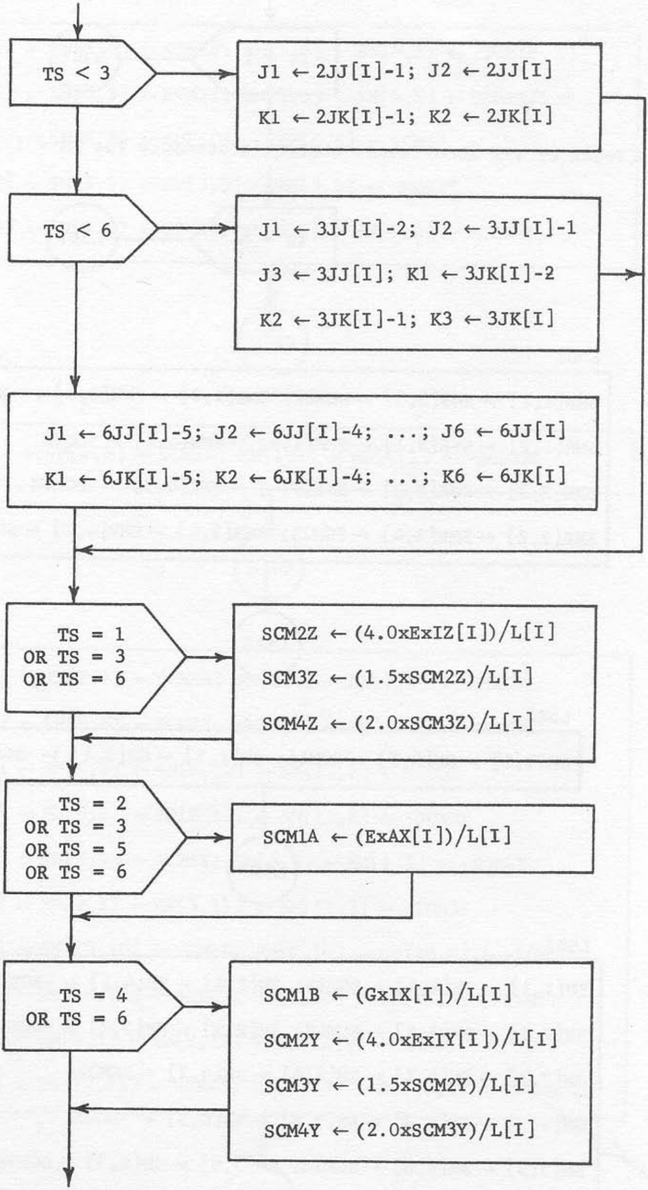


J ← N+1
JE ← (N+NR,-1,1)
RL[JE] = 0
J ← J-1
DJ[JE] ← 0
DJ[JE] ← D[J]
TS < 3
JE ← (2,2,2NJ)
JE/2,DJ[JE-1],DJ[JE]
L5B
TS < 6
JE ← (6,6,6NJ)
JE ← (3,3,3NJ)
JE/6,DJ[JE-5],...,DJ[JE]
JE/3,DJ[JE-2],DJ[JE-1],DJ[JE]

b. Member end-actions



(6)


TS < 3
J1 ← 2JJ[I]-1; J2 ← 2JJ[I]
K1 ← 2JK[I]-1; K2 ← 2JK[I]
TS < 6
J1 ← 3JJ[I]-2; J2 ← 3JJ[I]-1
J3 ← 3JJ[I]; K1 ← 3JK[I]-2
K2 ← 3JK[I]-1; K3 ← 3JK[I]
J1 ← 6JJ[I]-5; J2 ← 6JJ[I]-4; ...; J6 ← 6JJ[I]
K1 ← 6JK[I]-5; K2 ← 6JK[I]-4; ...; K6 ← 6JK[I]
TS = 1
OR TS = 3
OR TS = 6
SCM2Z ← (4.0xExIZ[I])/L[I]
SCM3Z ← (1.5xSCM2Z)/L[I]
SCM4Z ← (2.0xSCM3Z)/L[I]
TS = 2
OR TS = 3
OR TS = 5
OR TS = 6
SCM1A ← (ExAX[I])/L[I]
TS = 4
OR TS = 6
SCM1B ← (GxIX[I])/L[I]
SCM2Y ← (4.0xExIY[I])/L[I]
SCM3Y ← (1.5xSCM2Y)/L[I]
SCM4Y ← (2.0xSCM3Y)/L[I]

(7)

TS = 1 → L5B1

(Total of six conditional control statements for TS = 1 through TS = 6)

TS = 6 → L5B6

L5B1

(8)

```
SMR[1,1] ← SMR[3,3] ← SCM4Z; SMR[1,3] ← SMR[3,1] ← -SCM4Z
SMR[1,2] ← SMR[2,1] ← SMR[1,4] ← SMR[4,1] ← SCM3Z
SMR[2,3] ← SMR[3,2] ← SMR[3,4] ← SMR[4,3] ← -SCM3Z
SMR[2,2] ← SMR[4,4] ← SCM2Z; SMR[2,4] ← SMR[4,2] ← SCM2Z/2.0
```

L5B8

L5B2

```
SM[1,1] ← SM[3,3] ← SCM1A; SM[1,3] ← SM[3,1] ← -SCM1A
```

L5B7

L5B3

```
SM[1,1] ← SM[4,4] ← SCM1A; SM[1,4] ← SM[4,1] ← -SCM1A
SM[2,2] ← SM[5,5] ← SCM4Z; SM[2,5] ← SM[5,2] ← -SCM4Z
SM[2,3] ← SM[3,2] ← SM[2,6] ← SM[6,2] ← SCM3Z
SM[3,5] ← SM[5,3] ← SM[5,6] ← SM[6,5] ← -SCM3Z
SM[3,3] ← SM[6,6] ← SCM2Z; SM[3,6] ← SM[6,3] ← SCM2Z/2.0
```

L5B9

L5B4

```
SM[1,1] ← SM[4,4] ← SCM1B; SM[1,4] ← SM[4,1] ← -SCM1B
SM[2,2] ← SM[5,5] ← SCM2Y; SM[2,5] ← SM[5,2] ← SCM2Y/2.0
SM[2,6] ← SM[6,2] ← SM[5,6] ← SM[6,5] ← SCM3Y
SM[2,3] ← SM[3,2] ← SM[3,5] ← SM[5,3] ← -SCM3Y
SM[3,3] ← SM[6,6] ← SCM4Y; SM[3,6] ← SM[6,3] ← -SCM4Y
```

↓

L5B9

L5B5

```
SM[1,1] ← SM[4,4] ← SCM1A; SM[1,4] ← SM[4,1] ← -SCM1A
```

↓

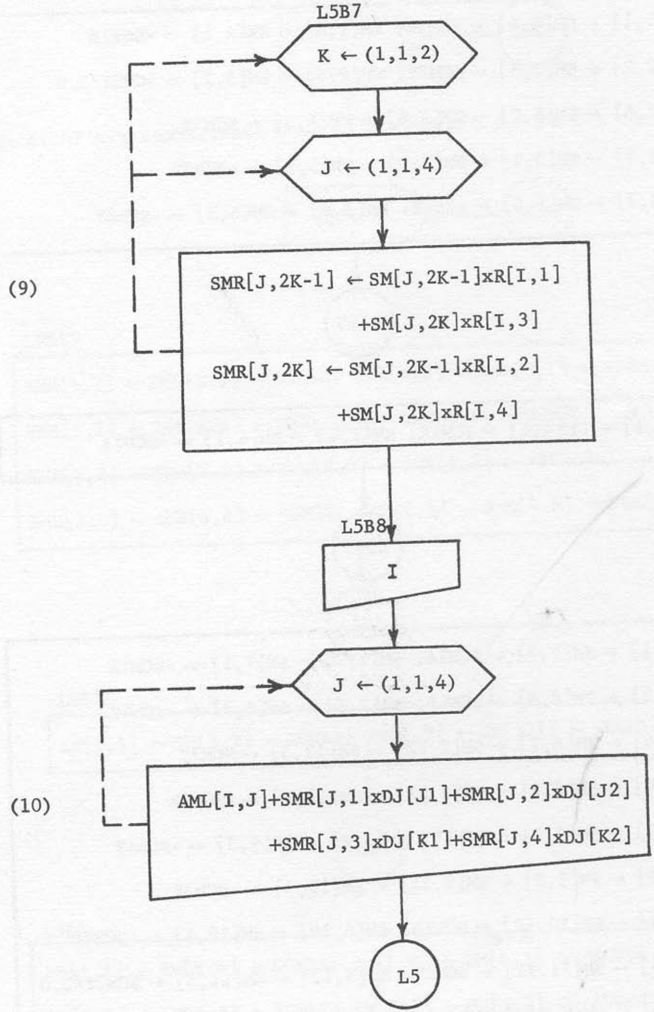
L5B9

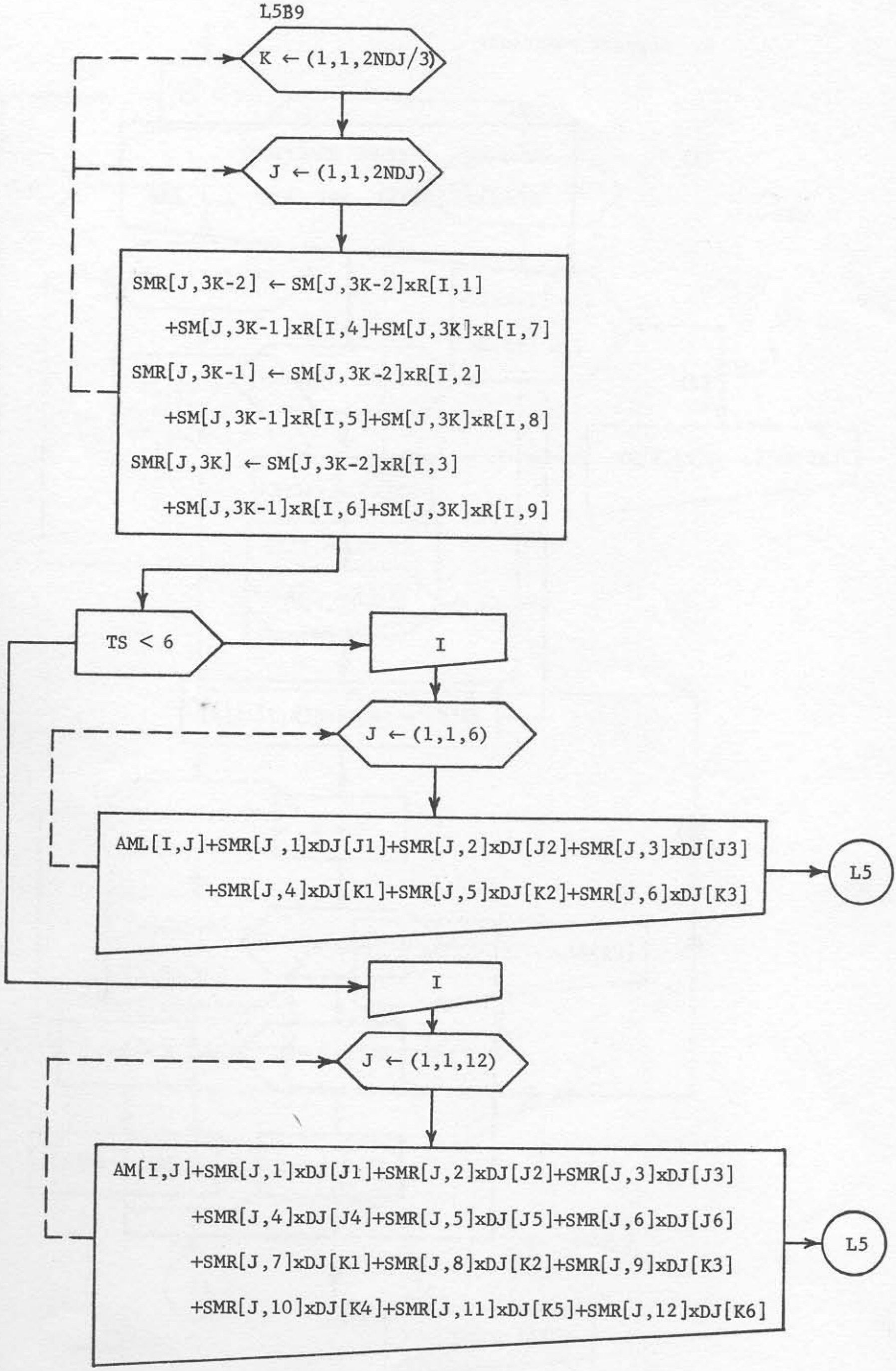
L5B6

```
SM[1,1] ← SM[7,7] ← SCM1A; SM[1,7] ← SM[7,1] ← -SCM1A
SM[2,2] ← SM[8,8] ← SCM4Z; SM[2,8] ← SM[8,2] ← -SCM4Z
SM[2,6] ← SM[6,2] ← SM[2,12] ← SM[12,2] ← SCM3Z
SM[6,8] ← SM[8,6] ← SM[8,12] ← SM[12,8] ← -SCM3Z
SM[3,3] ← SM[9,9] ← SCM4Y; SM[3,9] ← SM[9,3] ← -SCM4Y
SM[3,5] ← SM[5,3] ← SM[3,11] ← SM[11,3] ← -SCM3Y
SM[4,4] ← SM[10,10] ← SCM1B; SM[4,10] ← SM[10,4] ← -SCM1B
SM[5,5] ← SM[11,11] ← SCM2Y; SM[5,11] ← SM[11,5] ← SCM2Y/2.0
SM[5,9] ← SM[9,5] ← SM[9,11] ← SM[11,9] ← SCM3Y
SM[6,6] ← SM[12,12] ← SCM2Z; SM[6,12] ← SM[12,6] ← SCM2Z/2.0
```

↓

L5B9


L5B7
K ← (1,1,2)
J ← (1,1,4)
(9)
SMR[J,2K-1] ← SM[J,2K-1]xR[I,1]
+SM[J,2K]xR[I,3]
SMR[J,2K] ← SM[J,2K-1]xR[I,2]
+SM[J,2K]xR[I,4]
L5B8
I
J ← (1,1,4)
(10)
AML[I,J]+SMR[J,1]xDJ[J1]+SMR[J,2]xDJ[J2]
+SMR[J,3]xDJ[K1]+SMR[J,4]xDJ[K2]
L5



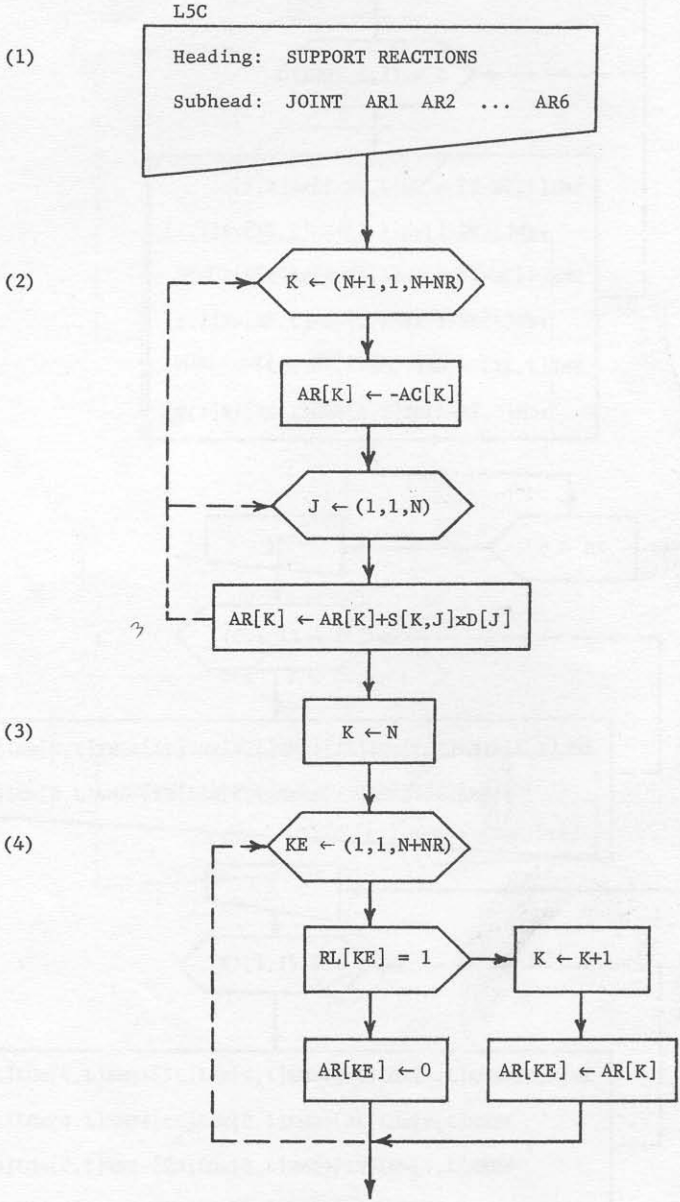
L5B9
K ← (1,1,2NDJ/3)
J ← (1,1,2NDJ)
SMR[J,3K-2] ← SM[J,3K-2]xR[I,1]
+SM[J,3K-1]xR[I,4]+SM[J,3K]xR[I,7]
SMR[J,3K-1] ← SM[J,3K-2]xR[I,2]
+SM[J,3K-1]xR[I,5]+SM[J,3K]xR[I,8]
SMR[J,3K] ← SM[J,3K-2]xR[I,3]
+SM[J,3K-1]xR[I,6]+SM[J,3K]xR[I,9]
TS < 6
I
J ← (1,1,6)
AML[I,J]+SMR[J,1]xDJ[J1]+SMR[J,2]xDJ[J2]+SMR[J,3]xDJ[J3]
+SMR[J,4]xDJ[K1]+SMR[J,5]xDJ[K2]+SMR[J,6]xDJ[K3]
L5
I
J ← (1,1,12)
AM[I,J]+SMR[J,1]xDJ[J1]+SMR[J,2]xDJ[J2]+SMR[J,3]xDJ[J3]
+SMR[J,4]xDJ[J4]+SMR[J,5]xDJ[J5]+SMR[J,6]xDJ[J6]
+SMR[J,7]xDJ[K1]+SMR[J,8]xDJ[K2]+SMR[J,9]xDJ[K3]
+SMR[J,10]xDJ[K4]+SMR[J,11]xDJ[K5]+SMR[J,12]xDJ[K6]
L5

(11)

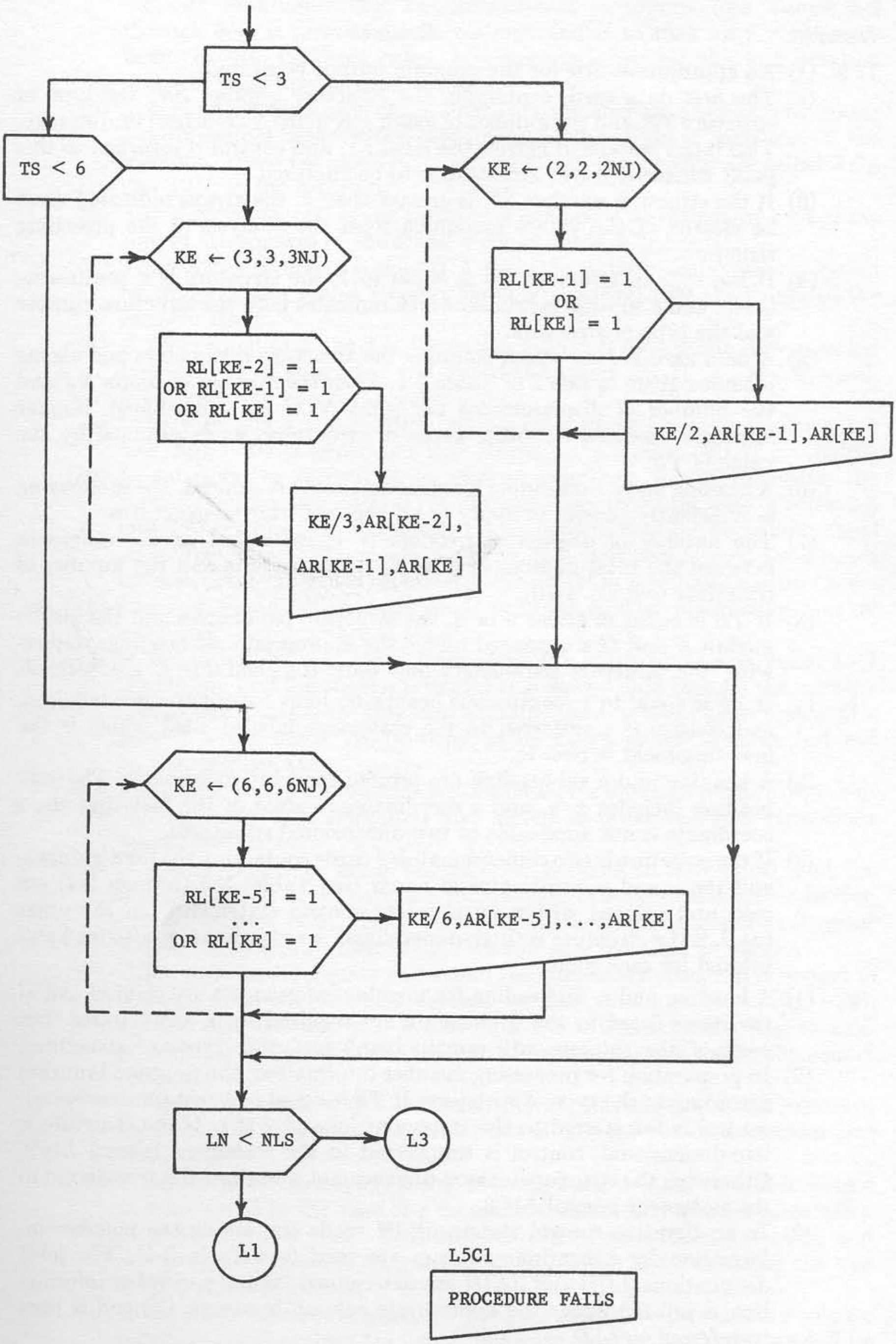
(12)

(13)

c. Support reactions



(5)


TS < 3
TS < 6
KE ← (2,2,2NJ)
KE ← (3,3,3NJ)
RL[KE-1] = 1
OR
RL[KE] = 1
RL[KE-2] = 1
OR RL[KE-1] = 1
OR RL[KE] = 1
KE/2,AR[KE-1],AR[KE]
KE/3,AR[KE-2],
AR[KE-1],AR[KE]
KE ← (6,6,6NJ)
RL[KE-5] = 1
...
OR RL[KE] = 1
KE/6,AR[KE-5],...,AR[KE]
LN < NLS
L3
L1
L5C1
PROCEDURE FAILS

(6)

(7)

3.6 Explanation of Flow Chart for Program *FR*1.

Item No. *Remarks*

1. a. (1) An appropriate title for the program output is printed.
(2) The first data card, containing the structure number *SN*, the type of structure *TS*, and the number of loading systems *NLS* is read into storage. This input statement carries the label *L*1, and control is returned to this point whenever a new structure is to be analyzed.
(3) If the structure number *SN* is greater than 1, the arrays indicated must be cleared of the values remaining from the analysis of the preceding structure.
(4) If the type of structure *TS* is equal to 1, the structure is a continuous beam, and a subtitle is printed which indicates both the structure number and the type of structure.
(5) A data card is read which contains the structure parameters and elastic modulus given in line 2 of Table 3-1. Then the number of joints *NJ* and the number of displacements per joint *NDJ* are determined. Similar steps are executed for other types of structures, as determined by the value of *TS*.
(6) A heading and a subheading for structure data are printed. The subheading is sufficiently general to apply to all types of framed structures.
(7) The number of degrees of freedom *N* is calculated as the difference between the total number of possible displacements and the number of restraints (see Eq. 1-70).
(8) If *TS* is equal to either 4 or 6, the structure parameters and the elastic moduli *E* and *G* are printed under the appropriate subheadings. Otherwise, the structure parameters and only the modulus *E* are printed.

b. (1) If *TS* is equal to 1 (continuous beam), no joint coordinates are required, and control is transferred to the statement labeled *L*1*C*, which is the first statement in Sec. 1c.
(2) A heading and a subheading are printed for joint coordinates. The subheading includes x, y, and z coordinates in spite of the fact that the z coordinate is not applicable to two-dimensional structures.
(3) If the structure is two-dimensional, *NJ* cards containing the joint numbers and the x and y coordinates of joints (see Tables 3-2 through 3-4) are read and printed within an iterative control statement. On the other hand, if the structure is three-dimensional, a z coordinate is also read and printed for each joint.

c. (1) A heading and a subheading for member information are printed. All of the items listed in the subheading are required for a space frame, but some of the columns will remain blank for other types of structures.
(2) In preparation for processing member information, the program branches according to the type of structure. If *TS* is equal to 1 (continuous beam), control is transferred to the statement labeled *L*1*C*1. If the structure is two-dimensional, control is transferred to the statement labeled *L*1*C*2. Otherwise, the structure is three-dimensional, and control is transferred to the statement labeled *L*1*C*3.
(3) In an iterative control statement, *M* cards containing the member information for a continuous beam are read (see Table 3-1). The joint designations *JJ*[*I*] and *JK*[*I*] are determined, and the member information is printed under the appropriate column headings. Control is then transferred to *L*1*D*.

(4) In an iterative control statement on members, for *M* members, the member information for two-dimensional structures (see Tables 3-2 through 3-4) is processed in a manner similar to that for a continuous beam in (3). In these cases, however, member lengths and rotation matrices must also be determined.
(5) The x and y components of the length of a member are computed by taking the differences between the coordinates of the joints at each end of the member. These components of the length of the member are called *XCL* and *YCL* in the program.
(6) The true length *L*[*I*] of the member is computed as the square root of the sum of the squares of the x and y components of the length.
(7) The x and y direction cosines (*CX* and *CY*) of the member are computed by dividing the x and y components of the length by the length of the member itself.
(8) If *TS* is equal to 2, the 2×2 rotation matrix **R** for a plane truss member (see Eq. 1-19) is generated and stored as a vector *R*[*I*,1], . . . , *R*[*I*,4], which is the *I*-th row of a rectangular matrix.
(9) Otherwise, the 3×3 rotation matrix for a plane frame or grid member (see Eq. 1-20) is generated and stored as a vector *R*[*I*,1], . . . ,*R*[*I*,9].
(10) In an iterative control statement on members, for *M* members, the member information for three-dimensional structures (see Tables 3-5 and 3-6) is processed in a manner analogous to that for two-dimensional structures above. The rotation matrix **R** is first determined for a space truss structure and then modified if the structure is actually a space frame.
(11) The x, y, and z components, the true length, and the direction cosines are computed for a member in space.
(12) If the structure is a space frame and the identifier *AA* is equal to 1 (signifying that the angle α is nonzero), a second data card for the member is read (see Table 3-6). This card contains the x, y, and z coordinates for a point p which lies in the principal plane x_M-y_M of the member but not on the member axis.
(13) The x_S, y_S, and z_S coordinates are computed for the point p (see Eqs. 1–30).
(14) The value of Q is computed and stored in order to avoid repetitious calculations later.
(15) If Q is zero or very small, the member is taken to be vertical, and the rotation matrix **R** is generated for the case of a vertical member having $\alpha = 0$ (see Eq. 1-26). The 3×3 rotation matrix **R** is stored as a vector *R*[*I*,1], . . . , *R*[*I*,9], which is the *I*-th row of a rectangular matrix.
(16) If the structure is a space frame and the angle α is nonzero, **R** is revised to correspond to the case of a vertical member with $\alpha \neq 0$ (see Eq. 1-33). In this case the sine and cosine of α are evaluated according to Eqs. (1-34), and the identifier *SQ* is used as temporary storage for the square root term.
(17) On the other hand, if Q is not small (hence, the member is not vertical), the matrix **R** is generated on the basis of $\alpha = 0$ for a skew member (see Eq. 1-25).
(18) If the structure is a space frame and the angle α is nonzero, **R** is revised to correspond to the case of a member of completely general orientation in space (see Eq. 1-29). This operation involves calculating the y_γ and z_γ coordinates of point p according to Eq. (1-31) and evaluating the sine and cosine of α by Eqs. (1-32).

d. (1) A heading and a subheading for joint restraints are printed. Six column headings for restraints are listed, but only the first two columns will be used for structures having two possible displacements per joint. Similarly, only the first three columns will be used for structures having three possible displacements per joint.

(2) If TS is less than 3, the structure is either a continuous beam or a plane truss, and NRJ restraint list cards for two possible displacements per joint are read and printed (see Tables 3-1 and 3-2).

(3) If TS is equal to 3, 4, or 5, the structure is a plane frame, a grid, or a space truss, and restraint list data for three possible displacements per joint are read and printed (see Tables 3-3, 3-4, and 3-5).

(4) Alternatively, TS is equal to 6, and the restraint list data for a space frame (see Table 3-6) are read and printed.

(5) In this portion of the program the cumulative restraint list CRL is computed. This list consists of the cumulative sums of the numbers in the restraint list RL. As an example, the restraint list for the first problem in the next article is:

$$RL = \{1, 1, 0, 0, 1, 0, 1, 1\}$$

The corresponding cumulative restraint list consists of the following:

$$CRL = \{1, 2, 2, 2, 3, 3, 4, 5\}$$

Note that the first element of CRL is always equal to the first element of RL, and the last element of CRL is numerically equal to the total number of restraints NR. The first step in generating CRL is to assign $RL[1]$ to $CRL[1]$.

(6) In an iterative control statement, all of the other elements of CRL are calculated in sequence.

2. a. (1) The member index I is initialized to zero in preparation for generating the structure stiffness matrix from member stiffness matrices.

(2) The index I is incremented by 1 each time that control is transferred to the statement labeled $L2$.

(3) If the member index I exceeds the number of members M, all members have been processed, and control is transferred to the statement labeled $L2B$ (Sec. 2b).

(4) If TS is either 1 or 2, the two possible displacements are indexed for the joint j ($J1$ and $J2$) and the joint k ($K1$ and $K2$) according to Eqs. (1-47).

(5) If TS is 3, 4, or 5, the three possible displacements are indexed for the joints j ($J1$, $J2$, and $J3$) and k ($K1$, $K2$, and $K3$), as indicated in Eqs. (1-48).

(6) Alternatively, TS is equal to 6, and the six possible displacements are indexed for the joints j ($J1$ through $J6$) and k ($K1$ through $K6$) as in Eqs. (1-49).

(7) If TS is equal to 1, 3, or 6, three flexural stiffness constants for the member are calculated and stored in order to avoid repetition of these expressions when generating the member stiffness matrix. These constants are:

$$SCM2Z = \frac{4EI_Z}{L} \qquad SCM3Z = \frac{6EI_Z}{L^2} \qquad SCM4Z = \frac{12EI_Z}{L^3}$$

(8) If TS is equal to 2, 3, 5, or 6, the following axial stiffness constant is determined:

$$SCM1A = \frac{EA_X}{L}$$

(9) If TS is equal to 4 or 6, the torsional stiffness constant and three additional flexural stiffness constants are computed as follows:

$$SCM1B = \frac{GI_X}{L} \qquad SCM2Y = \frac{4EI_Y}{L}$$

$$SCM3Y = \frac{6EI_Y}{L^2} \qquad SCM4Y = \frac{12EI_Y}{L^3}$$

(10) Using the joint restraint list as a guide, the first two possible displacements for the joints at each end of the member are re-indexed according to whether the joints are actually free to move or not. That is, if a certain displacement is free to occur, the index is decreased by the corresponding element of CRL, placing it in the proper order among the degrees of freedom (see Eq. 1-71). On the other hand, if a displacement is not free to occur, the index is reset to N plus the corresponding element of CRL, placing it in the proper order among the support restraints (see Eq. 1-72).

(11) If TS is less than 3, control is transferred to $L2A$.

(12) Otherwise, the third possible displacement for the joint at each end is re-indexed as in (10).

(13) If TS is less than 6, control is transferred to $L2A$.

(14) Otherwise, the fourth through the sixth possible displacements for the joint at each end of the member are re-indexed as in (10).

(15) In preparation for generating the member stiffness matrix, the program branches according to the type of structure.

(16) If TS is equal to 1, the 4×4 member stiffness matrix $\mathbf{S}_{\mathrm{MD}}$ (equal to $\mathbf{S}_{\mathrm{M}}$ for a beam) is generated according to Table 1-1, using the stiffness constants previously evaluated. Then control is transferred to $L2A9$.

(17) If TS is equal to 2, the nonzero elements of the 4×4 member stiffness matrix $\mathbf{S}_{\mathrm{M}}$ for a plane truss member (see Table 1-2) are calculated, and control is then transferred to $L2A7$.

(18) If TS is equal to 3, the nonzero elements of the 6×6 member stiffness matrix $\mathbf{S}_{\mathrm{M}}$ for a plane frame member (see Table 1-3) are calculated, and control is transferred to $L2A8$.

(19) Similar to (18), but for a grid member (see Table 1-4).

(20) Similar to (18), but for a space truss member (see Table 1-5).

(21) If TS is equal to 6, the nonzero elements of the 12×12 member stiffness matrix $\mathbf{S}_{\mathrm{M}}$ for a space frame member (see Table 1-6) are calculated, and control is transferred to $L2A8$.

(22) A matrix $\mathbf{S}_{\mathrm{MR}}$ is calculated as an intermediate step in the transformation of the matrix $\mathbf{S}_{\mathrm{M}}$ to the matrix $\mathbf{S}_{\mathrm{MD}}$ for a plane truss, using Eq. (1-45). This intermediate matrix is calculated by postmultiplying the matrix $\mathbf{S}_{\mathrm{M}}$ by the rotation transformation matrix $\mathbf{R}_{\mathrm{T}}$ as follows:

$$\mathbf{S}_{\mathrm{MR}} = \mathbf{S}_{\mathrm{M}}\mathbf{R}_{\mathrm{T}}$$

In the program, the matrix $\mathbf{R}_{\mathrm{T}}$ is not actually formed, but the rotation matrix $\mathbf{R}$ (which makes up $\mathbf{R}_{\mathrm{T}}$) is used.

(23) Next, the matrix $\mathbf{S}_{\mathrm{MD}}$ is calculated by premultiplication of the matrix $\mathbf{S}_{\mathrm{MR}}$ by the transpose of $\mathbf{R}_{\mathrm{T}}$:

$$\mathbf{S}_{\mathrm{MD}} = \mathbf{R}_{\mathrm{T}}'\mathbf{S}_{\mathrm{MR}}$$

(24) Similar to (22), but for TS = 3, 4, 5, or 6.

(25) Similar to (23), but for $TS = 3, 4, 5$, or 6.

(26) If the index $2JJ[I]$-1 (the original value of $J1$) corresponds to a degree of freedom, the first column of a 4×4 member stiffness matrix $\mathbf{S}_{MD}$ is transferred to the joint stiffness matrix for the structure (see Eqs. 1-50). However, if $2JJ[I]$-1 corresponds to a restraint, the first column of $\mathbf{S}_{MD}$ is bypassed. By this device, only the required portions of the overall joint stiffness matrix are generated. (The result is a rectangular array consisting of the matrix $\mathbf{S}$ in the upper portion and the matrix $\mathbf{S}_{RD}$ in the lower portion. Nevertheless, the whole array is identified as S in the program.) Elements of S involving the coupling of $J1$ and $J2$ are cumulative because, in general, they receive contributions from more than one member. On the other hand, elements of S involving the coupling of $J1$ with $K1$ and $K2$ are single values.

(27) Similar to (26), but for $2JJ[I]$, which is the original value of $J2$ (see Eqs. 1-51). The second column of SMD is transferred to S.

(28) Similar to (26), but for $2JK[I]$-1, which is the original value of $K1$ (see Eqs. 1-52). The third column of SMD is transferred to S.

(29) Similar to (26), but for $2JK[I]$, which is the original value of $K2$ (see Eqs. 1-53). The fourth column of SMD is transferred to S.

(30) Control is returned to the statement labeled $L2$ for the purpose of processing the next member in the structure.

(31) If the index $3JJ[I]$-2 (the original value of $J1$) corresponds to a degree of freedom, the first column of a 6×6 member stiffness matrix is transferred to the joint stiffness matrix for the structure (see Eqs. 1-54). This transfer process is repeated for all six columns of SMD, and control is then returned to $L2$.

(32) If the index $6JJ[I]$-5 (the original value of $J1$) corresponds to a degree of freedom, the first column of a 12×12 member stiffness matrix is transferred to the joint stiffness matrix (see Eqs. 1-55). This transfer process is repeated for all twelve columns of SMD, and control is then returned to $L2$.

b. (1) The upper part of the rectangular array S is decomposed using the procedure *DECOMPOSE* from Art. 2.4. If the procedure fails, control is transferred to the last statement in the program (labeled $L5C1$).

(2) Then the procedure *INVERT* (see Art. 2.6) is called, and the inverse matrix $\mathbf{S}^{-1}$ is produced in the upper part of the array.

3. a. (1) The loading number LN is initialized to zero in preparation for handling the first load system.

(2) The loading number is incremented by 1 each time that control is returned to the statement labeled $L3$. This occurs whenever a new load system is encountered for the same structure.

(3) If either SN or LN is greater than 1, the arrays indicated must be cleared of the values remaining from the previous analysis.

(4) An appropriate heading and subheading are printed, and the numbers of loaded joints and loaded members are read (see Tables 3-1 through 3-6) and printed.

b. (1) If the number of loaded joints is not equal to zero, actions applied at joints must be read and printed.

(2) A heading and a subheading for actions applied at joints are printed in a form sufficiently general to apply to all types of framed structures.

(3) If TS is less than 3, NLJ cards containing the joint designations and two actions applied at each loaded joint (see Tables 3-1 and 3-2) are read and printed.

(4) If TS is equal to 3, 4, or 5, NLJ cards containing the joint designations and three actions applied at each loaded joint (seee Tables 3-3, 3-4, and 3-5) are read and printed.

(5) Alternatively, TS is equal to 6, and the joint load data for a space frame (see Table 3-6) are read and printed.

c. (1) If the number of loaded members is not equal to zero, data concerning loads on members must be read and printed. Subsequently, in Sec. 4a, equivalent joint loads must be calculated from the actions at the ends of restrained members.

(2) A heading and a subheading sufficiently general to apply to structures of types 1 through 5 are printed.

(3) If TS is equal to 6, a second set of column headings is also printed for a space frame.

(4) If TS is less than 3, NLM cards containing the member designations and four actions at the ends of each loaded member (see Tables 3-1 and 3-2) are read and printed. In addition, the I-th element of the loaded member list LML is given the value 1. This list is similar to the restraint list RL, and it indicates which members in the structure are loaded. The loaded member list LML is used subsequently in Sec. 4a to determine which members contribute values to the equivalent joint loads.

(5) Otherwise, NLM cards containing the member designations and six end-actions for each loaded member (see Tables 3-3 through 3-6) are read and printed, and $LML[I]$ is given the value 1.

(6) If TS is equal to 6, a card containing six additional end-actions for each loaded space frame member (see Table 3-6) is read and printed.

4. a. (1) In preparation for computing equivalent joint loads, the program branches according to the type of structure.

(2) In an iterative control statement for M members, the contributions of AML to AE for each loaded member of a beam are identified and transferred. The loaded member list LML is used as a guide for this purpose. Note that the calculation of equivalent joint loads involves taking the negatives of fixed-end actions for members. Each of the statements for generating an element of AE consists of a summation of contributions from more than one member, because only the end points of the continuous beam receive contributions from a single member.

(3) Contributions to AE are computed for each loaded member of a plane truss using the product of the matrix $[R_T']_i$ and the vector $\{A_{ML}\}_i$, as indicated by the expression numbered (1-62) in Chapter 1.

(4) Similar to (3), but for a plane frame, grid, or space truss.

(5) Similar to (3), but for a space frame.

b. (1) Iterative control statement on all possible joint displacements ($N + NR$ displacements).

(2) If the J-th possible displacement is unrestrained, the index K for an element of AC is set to the proper position in the first part of the vector: $K \leftarrow J - CRL[J]$ (see Eq. 1-71).

(3) Otherwise, the J-th possible displacement is restrained; so the index K is set to the proper position in the latter part of the vector: $K \leftarrow N + CRL[J]$ (see Eq. 1-72).

(4) Elements of AC are calculated as the sums of elements of A and AE, as denoted by Eq. (1-63). Because of the fact that the indexes have been reset as described above, the vector AC is generated in the form given by Eq. (1-64).

5. a. (1) A heading and a subheading for joint displacements are printed in a form sufficiently general to accommodate all types of framed structures.
(2) Within two iterative control statements, the joint displacements are calculated by matrix multiplication, using Eq. (1-66).
(3) In preparation for expanding the displacement vector, J is initialized to $N+1$.
(4) In an iterative control statement with descending values of the variable JE, the displacement vector is expanded in accordance with the degrees of freedom denoted by the joint restraint list. J is the index for the vector D, and JE is the index for the expanded vector DJ.
(5) The joint number and the displacements of each joint are printed for each joint in the structure. The number of displacements printed per joint is 2, 3, or 6, depending upon the type of structure.

b. (1) A heading and a subheading for member end-actions are printed in a form suitable for structures of types 1 through 5.
(2) If TS is equal to 6, a second set of column headings is also printed for a space frame.
(3) The member index I is initialized to zero in preparation for processing the members and computing end-actions.
(4) The index I is incremented by 1 each time that control is returned to the statement labeled $L5$.
(5) If the member index I exceeds the number of members M, all members have been processed, and control is transferred to the statement labeled $L5C$ (Sec. 5c).
(6) Joint displacement indexes and member stiffness constants are calculated as in Sec. 2a, parts (4) through (9).
(7) In preparation for computing the member stiffness matrix, the program branches according to the type of structure.
(8) Member stiffness matrices are generated as in Sec. 2a, parts (16) through (21), except that the beam matrix is stored in SMR instead of in SMD.
(9) The matrix SMR for a plane truss is calculated as in Sec. 2a(22).
(10) The member number I is printed, and the four end-actions for a beam or plane truss member are computed and printed on the same line, using Eq. (1-69). Control is then returned to $L5$.
(11) The matrix SMR is computed for other types of structures.
(12) If TS is equal to 3, 4, or 5, the member number is printed; and six member end-actions are computed and printed on the same line, after which control is returned to $L5$.
(13) Otherwise, if TS is equal to 6, the member number is printed, and the twelve member end-actions for a space frame member are computed and printed (in two lines, six actions per line). Control is then returned to $L5$.

c. (1) A heading and a subheading for support reactions are printed in a form sufficiently general to accommodate all types of framed structures.
(2) Within two iterative control statements, the support reactions are calculated by matrix multiplication, using Eq. (1-6).
(3) In preparation for expanding the vector of support reactions, K is initialized to N.
(4) In an iterative control statement, the vector of support reactions is expanded in accordance with the restraints denoted by the joint restraint list. K is the index for the condensed form of AR, and KE is the index for the expanded form of AR.

(5) The joint number and the reactions are printed for each joint in the structure at which a restraint exists. The restraint list RL is used as an indicator for this purpose. The number of reactions printed per joint is 2, 3, or 6, depending upon the type of structure.
(6) If the loading number LN is less than the number of loading systems NLS, control is transferred to $L3$ for the purpose of reading data for another load system. On the other hand, if all load systems have been processed, control is transferred to $L1$ for the purpose of reading data for another structure.
(7) In the event that the procedure *DECOMPOSE* (see Sec. 2b) detects a singular matrix, control is transferred to this statement, and the message *PROCEDURE FAILS* is printed. No further calculations are made for such a structure.

3.7 Examples. The examples of this article were analyzed previously,* using a special program for each type of structure. In this presentation, however, all of the structures are analyzed in one run of the general-purpose program $FR1$, and the majority of them are analyzed for more than one load system.

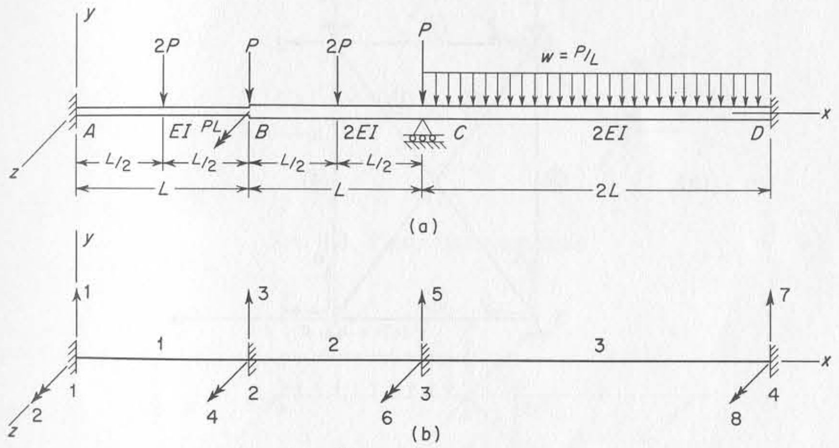


FIG. 3-1. Continuous beam example

Figures 3-1 through 3-9 illustrate the structures to be analyzed. Part (a) of each figure shows the dimensions, the applied loads, and the support restraints, whereas part (b) contains the restrained structure with the numbering systems for members, joints, and displacements. Numerical values of the parameters in each problem are summarized in Table 3-7. The first three columns of the table contain the structure number SN, the type of structure TS, and the number of loading systems NLS. Structures for which $NLS = 1$ are to be analyzed for all of the loads shown in the figures, but those with $NLS = 2$ are to be analyzed for joint loads and member loads separately. Elastic moduli, loads, lengths, and member properties constitute the remainder of Table 3-7. For simplicity, each structure is assumed to be composed of members which all have the same cross-sectional properties.

*See Chapter 5 of *Analysis of Framed Structures*, by J. M. Gere and W. Weaver, Jr., D. Van Nostrand Company, Inc., Princeton, N. J., 1965.

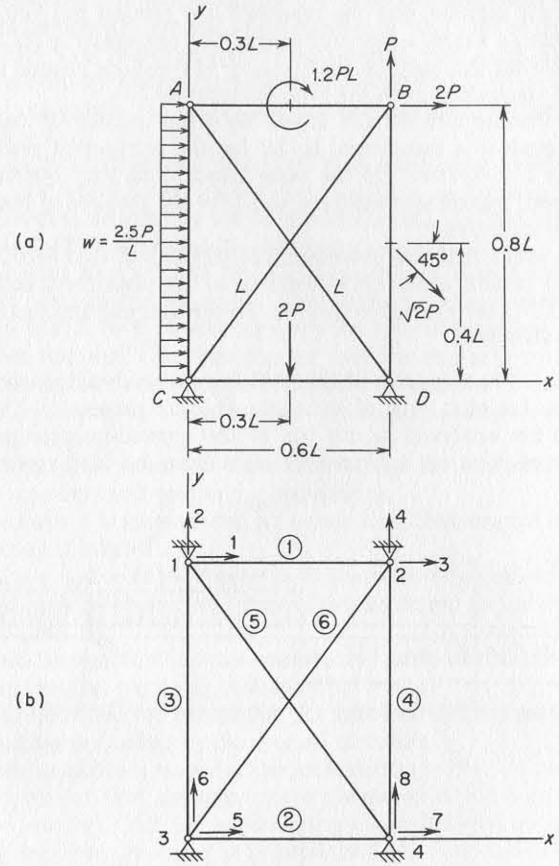


Fig. 3-2. Plane truss example

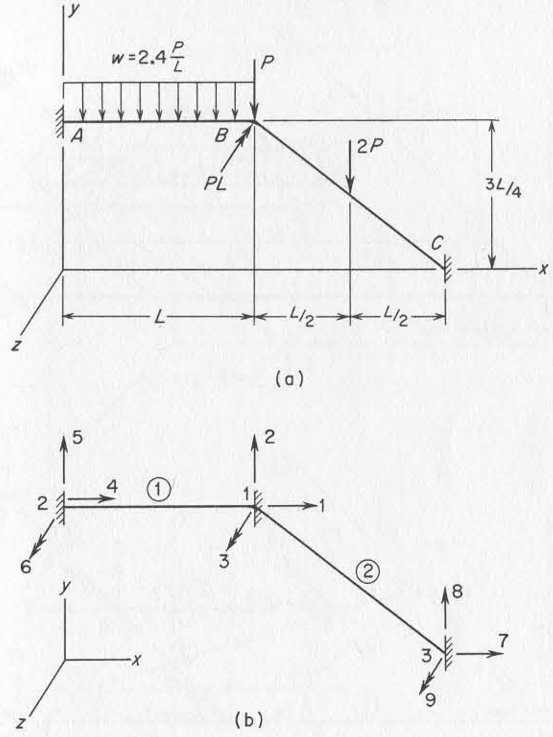


Fig. 3-3. Plane frame example

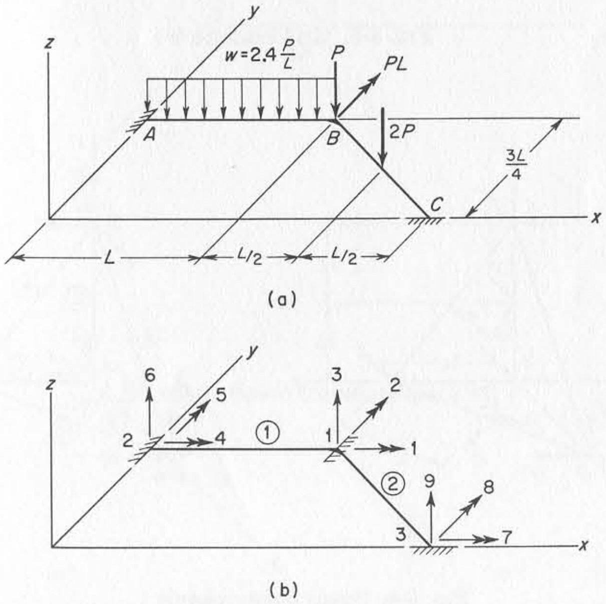


Fig. 3-4. Grid example 1

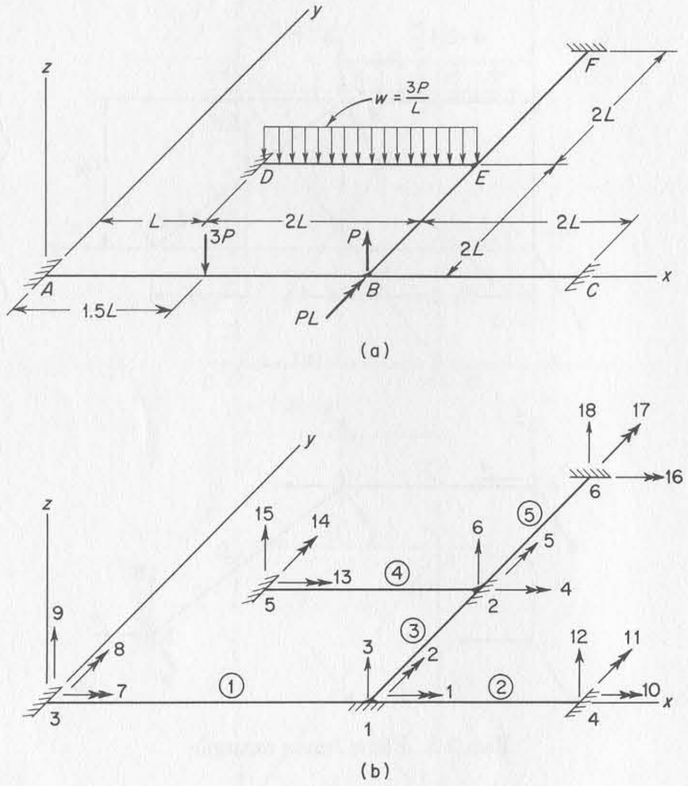


FIG. 3-5. Grid example 2

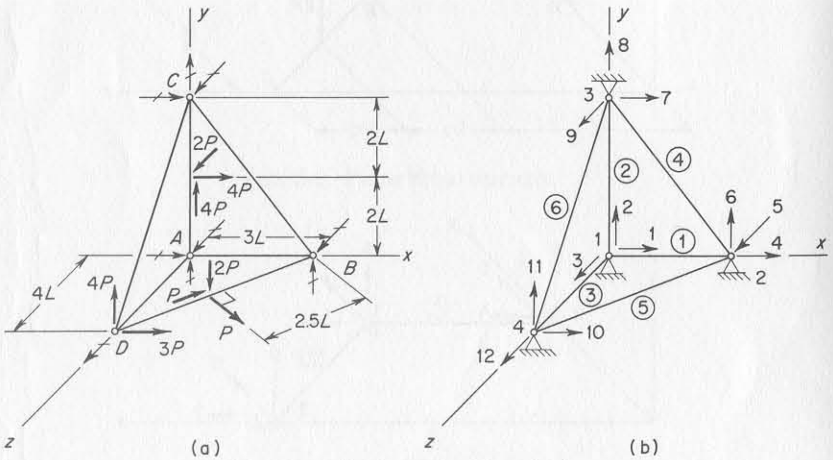


FIG. 3-6. Space truss example 1

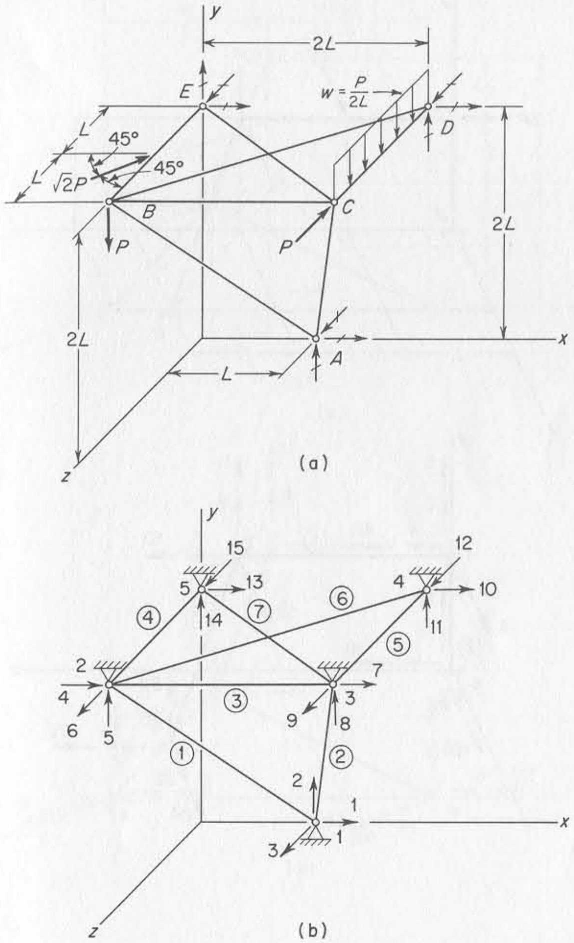


FIG. 3-7. Space truss example 2

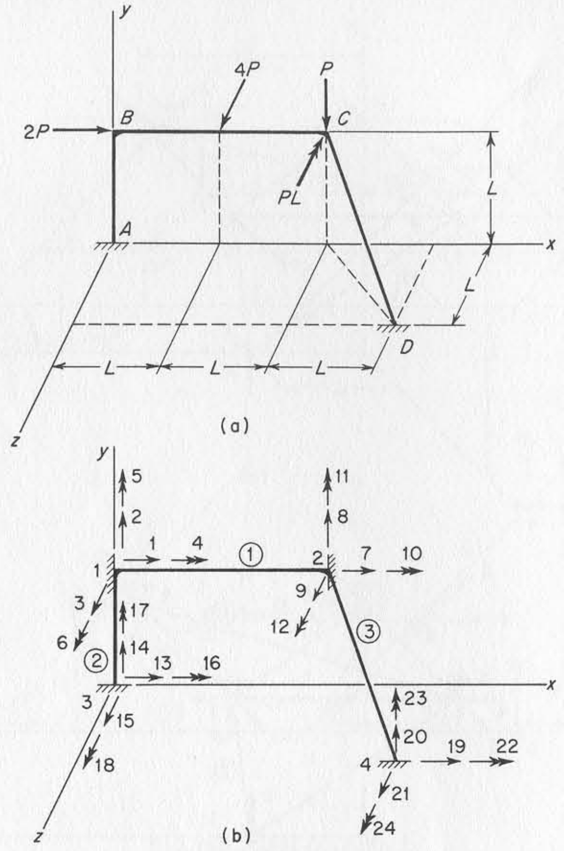


Fig. 3-8. Space frame example 1

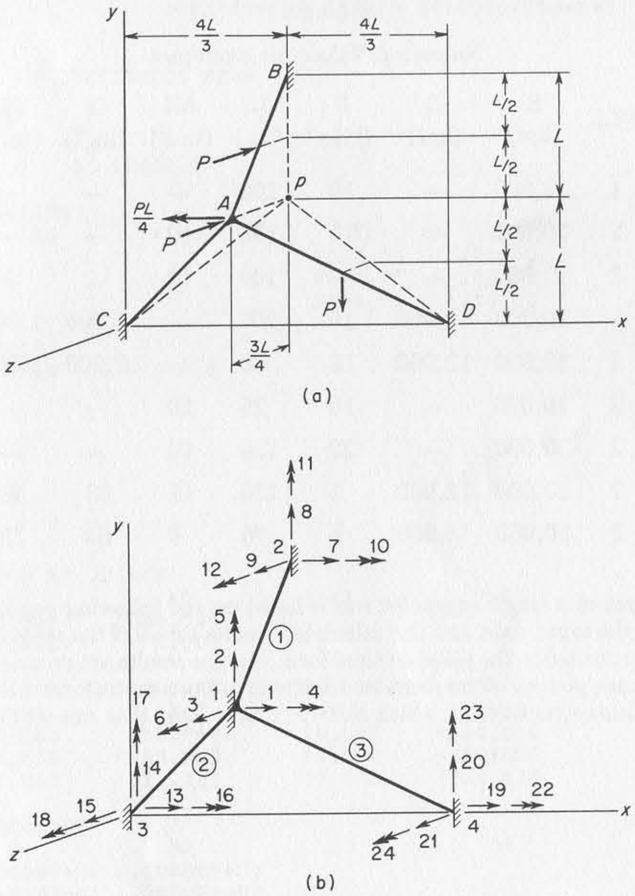


Fig. 3-9. Space frame example 2

TABLE 3-7

Numerical Values for Examples

SN	TS	NLS	E (ksi)	G (ksi)	P (kips)	L (in.)	AX (in.^{2})	IX (in.^{4})	IY (in.^{4})	IZ (in.^{4})
1	1	1	10,000	—	10	100	—	—	—	1,000
2	2	2	10,000	—	10	100	10	—	—	—
3	3	1	10,000	—	10	100	10	—	—	1,000
4	4	1	10,000	4,000	10	100	—	1,000	1,000	—
5	4	2	30,000	12,000	16	60	—	2,000	1,000	—
6	5	2	10,000	—	10	25	10	—	—	—
7	5	2	30,000	—	20	120	10	—	—	—
8	6	2	30,000	12,000	1	120	11	83	56	56
9	6	2	10,000	4,000	5	96	9	64	28	80

The output of a single computer run is listed on the following pages, which include both the input data and the calculated results for all of the examples. The input data are printed in the usual decimal form, but the results are printed as decimal numbers times powers of ten because their magnitudes are unknown beforehand. Of course, those structures for which $NLS > 1$ have more than one set of calculated results.


```
                    ANALYSIS OF FRAMED STRUCTURES--FR1

STRUCTURE NO.  1 CONTINUOUS BEAM

STRUCTURE DATA
 M   N  NJ  NR  NRJ      E         G
 3   3   4   5   3  10000.0

MEMBER INFORMATION
MEMBER JJ JK   AX        IX        IY        IZ   AA     L
   1    1  2                               1000.00    100.00
   2    2  3                               2000.00    100.00
   3    3  4                               2000.00    200.00

JOINT RESTRAINTS
JOINT RL1 RL2 RL3 RL4 RL5 RL6
  1    1   1
  3    1   0
  4    1   1

LOADING NO.  1
NLJ  NLM
  2    3

ACTIONS APPLIED AT JOINTS
JOINT        A1          A2          A3          A4          A5          A6
   2      -10.000    1000.000
   3      -10.000       0.000

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
MEMBER       AML1        AML2        AML3        AML4        AML5        AML6
   1       10.000     250.000      10.000    -250.000
   2       10.000     250.000      10.000    -250.000
   3       10.000     333.333      10.000    -333.333

JOINT DISPLACEMENTS
JOINT        D1          D2          D3          D4          D5          D6
   1     0.00000@+00 0.00000@+00
   2    -1.31614@-01 1.21032@-03
   3     0.00000@+00 8.43254@-04
   4     0.00000@+00 0.00000@+00

MEMBER END-ACTIONS
MEMBER       AM1         AM2         AM3         AM4         AM5         AM6
   1     3.30556@+01 1.28175@+03-1.30556@+01 1.02381@+03
   2     3.05556@+00-2.38096@+01 1.69444@+01-6.70635@+02
   3     1.25298@+01 6.70635@+02 7.47024@+00-1.64682@+02

SUPPORT REACTIONS
JOINT        AR1         AR2         AR3         AR4         AR5         AR6
   1     3.30556@+01 1.28175@+03
   3     3.94742@+01 0.00000@+00
   4     7.47024@+00-1.64682@+02
```

STRUCTURE NO. 2 PLANE TRUSS

STRUCTURE DATA

M	N	NJ	NR	NRJ	E	G
6	4	4	4	2	10000.0	

COORDINATES OF JOINTS

JOINT	X	Y	Z
1	0.00	80.00	
2	60.00	80.00	
3	0.00	0.00	
4	60.00	0.00	

MEMBER INFORMATION

MEMBER	JJ	JK	AX	IX	IY	IZ	AA	L
1	1	2	6.00					60.00
2	3	4	6.00					60.00
3	3	1	8.00					80.00
4	4	2	8.00					80.00
5	1	4	10.00					100.00
6	3	2	10.00					100.00

JOINT RESTRAINTS

JOINT	RL1	RL2	RL3	RL4	RL5	RL6
3	1	1				
4	1	1				

LOADING NO. 1

NLJ	NLM
1	0

ACTIONS APPLIED AT JOINTS

JOINT	A1	A2	A3	A4	A5	A6
2	20.000	10.000				

JOINT DISPLACEMENTS

JOINT	D1	D2	D3	D4	D5	D6
1	3.50427E-02	1.02564E-02				
2	4.27350E-02	-6.41026E-03				
3	0.00000E+00	0.00000E+00				
4	0.00000E+00	0.00000E+00				

MEMBER END-ACTIONS

MEMBER	AM1	AM2	AM3	AM4	AM5	AM6
1	-7.69231E+00	0.00000E+00	7.69231E+00	0.00000E+00		
2	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00		
3	-1.02564E+01	0.00000E+00	1.02564E+01	0.00000E+00		
4	6.41026E+00	0.00000E+00	-6.41026E+00	0.00000E+00		
5	1.28205E+01	0.00000E+00	-1.28205E+01	0.00000E+00		
6	-2.05128E+01	0.00000E+00	2.05128E+01	0.00000E+00		

SUPPORT REACTIONS

JOINT	AR1	AR2	AR3	AR4	AR5	AR6
3	-1.23077E+01	-2.66667E+01				
4	-7.69231E+00	1.66667E+01				

LOADING NO. 2

NLJ	NLM
0	4

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS

MEMBER	AML1	AML2	AML3	AML4	AML5	AML6
1	0.000	-20.000	0.000	20.000		
2	0.000	10.000	0.000	10.000		
3	0.000	10.000	0.000	10.000		
4	5.000	5.000	5.000	5.000		

JOINT DISPLACEMENTS

JOINT	D1	D2	D3	D4	D5	D6
1	6.49634E-02	3.12088E-02				
2	6.33700E-02	-3.37912E-02				
3	0.00000E+00	0.00000E+00				
4	0.00000E+00	0.00000E+00				

MEMBER END-ACTIONS

MEMBER	AM1	AM2	AM3	AM4	AM5	AM6
1	1.59341E+00	-2.00000E+01	-1.59341E+00	2.00000E+01		
2	0.00000E+00	1.00000E+01	0.00000E+00	1.00000E+01		
3	-3.12088E+01	1.00000E+01	3.12088E+01	1.00000E+01		
4	3.87912E+01	5.00000E+00	-2.87912E+01	5.00000E+00		
5	1.40110E+01	0.00000E+00	-1.40110E+01	0.00000E+00		
6	-1.09890E+01	0.00000E+00	1.09890E+01	0.00000E+00		

SUPPORT REACTIONS

JOINT	AR1	AR2	AR3	AR4	AR5	AR6
3	-1.65934E+01	-3.00000E+01				
4	-1.34066E+01	6.00000E+01				

```
STRUCTURE NO.  3 PLANE FRAME

STRUCTURE DATA
  M   N  NJ  NR  NRJ        E          G
  2   3   3   6   2  10000.0

COORDINATES OF JOINTS
JOINT     X        Y             Z
  1  100.00    75.00
  2    0.00    75.00
  3  200.00     0.00

MEMBER INFORMATION
MEMBER  JJ  JK     AX          IX          IY          IZ       AA       L
   1     2   1   10.00                               1000.00          100.00
   2     1   3   10.00                               1000.00          125.00

JOINT RESTRAINTS
JOINT RL1 RL2 RL3 RL4 RL5 RL6
  2    1   1   1
  3    1   1   1

LOADING NO.  1
NLJ  NLM
  1    2

ACTIONS APPLIED AT JOINTS
JOINT           A1            A2            A3            A4            A5            A6
  1          0.000       -10.000     -1000.000

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
MEMBER        AML1          AML2          AML3          AML4          AML5          AML6
   1         0.000        12.000       200.000         0.000        12.000      -200.000
   2        -6.000         8.000       250.000        -6.000         8.000      -250.000

JOINT DISPLACEMENTS
JOINT           D1            D2            D3            D4            D5            D6
  1      -2.02608@-02-9.93600@-02-1.79756@-03
  2       0.00000@+00 0.00000@+00 0.00000@+00
  3       0.00000@+00 0.00000@+00 0.00000@+00

MEMBER END-ACTIONS
MEMBER          AM1           AM2           AM3           AM4           AM5           AM6
   1      2.02608@+01 1.31378@+01 4.36648@+02-2.02608@+01 1.08622@+01-3.22865@+02
   2      2.87259@+01-4.53328@+00-6.77135@+02-4.07259@+01 2.05333@+01-8.89525@+02

SUPPORT REACTIONS
JOINT          AR1           AR2           AR3           AR4           AR5           AR6
  2       2.02608@+01 1.31378@+01 4.36648@+02
  3      -2.02608@+01 4.08622@+01-8.89525@+02
```



```
STRUCTURE NO.  4 GRID

STRUCTURE DATA
  M   N  NJ  NR  NRJ        E          G
  2   3   3   6   2  10000.0     4000.0

COORDINATES OF JOINTS
JOINT     X        Y             Z
  1  100.00    75.00
  2    0.00    75.00
  3  200.00     0.00

MEMBER INFORMATION
MEMBER  JJ  JK     AX          IX          IY          IZ       AA       L
   1     2   1             1000.00     1000.00                        100.00
   2     1   3             1000.00     1000.00                        125.00

JOINT RESTRAINTS
JOINT RL1 RL2 RL3 RL4 RL5 RL6
  2    1   1   1
  3    1   1   1

LOADING NO.  1
NLJ  NLM
  1    2

ACTIONS APPLIED AT JOINTS
JOINT           A1            A2            A3            A4            A5            A6
  1          0.000      1000.000       -10.000

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
MEMBER        AML1          AML2          AML3          AML4          AML5          AML6
   1         0.000      -200.000        12.000         0.000       200.000        12.000
   2         0.000      -312.500        10.000         0.000       312.500        10.000

JOINT DISPLACEMENTS
JOINT           D1            D2            D3            D4            D5            D6
  1      -7.59854@-03 5.09489@-03-3.55075@-01
  2       0.00000@+00 0.00000@+00 0.00000@+00
  3       0.00000@+00 0.00000@+00 0.00000@+00

MEMBER END-ACTIONS
MEMBER          AM1           AM2           AM3           AM4           AM5           AM6
   1      3.03942@+02-1.31147@+03 2.40397@+01-3.03942@+02 1.07504@+02-3.97015@-02
   2     -2.92344@+02 8.96362@+02-9.96030@+00 2.92344@+02 1.59868@+03 2.99603@+01

SUPPORT REACTIONS
JOINT          AR1           AR2           AR3           AR4           AR5           AR6
  2       3.03942@+02-1.31147@+03 2.40397@+01
  3       1.19308@+03 1.10353@+03 2.99603@+01
```

```
STRUCTURE NO.  5 GRID

STRUCTURE DATA
  M   N  NJ  NR  NRJ      E          G
  5   6   6  12    4  30000.0    12000.0

COORDINATES OF JOINTS
JOINT    X         Y          Z
  1  180.00      0.00
  2  180.00    120.00
  3    0.00      0.00
  4  300.00      0.00
  5   60.00    120.00
  6  180.00    240.00

MEMBER INFORMATION
MEMBER JJ JK    AX        IX        IY       IZ    AA      L
   1    3  1             2000.00  1000.00                180.00
   2    1  4             2000.00  1000.00                120.00
   3    1  2             2000.00  1000.00                120.00
   4    5  2             2000.00  1000.00                120.00
   5    2  6             2000.00  1000.00                120.00

JOINT RESTRAINTS
JOINT RL1 RL2 RL3 RL4 RL5 RL6
  3    1   1   1
  4    1   1   1
  5    1   1   1
  6    1   1   1

LOADING NO.  1
NLJ  NLM
  1    0

ACTIONS APPLIED AT JOINTS
JOINT       A1          A2           A3            A4            A5            A6
  1       0.000      960.000      16.000

JOINT DISPLACEMENTS
JOINT        D1          D2          D3            D4            D5            D6
  1     -3.86878E-04 7.70580E-04 6.99044E-02
  2     -3.09257E-04-3.51589E-05 1.62671E-02
  3      0.00000E+00 0.00000E+00 0.00000E+00
  4      0.00000E+00 0.00000E+00 0.00000E+00
  5      0.00000E+00 0.00000E+00 0.00000E+00
  6      0.00000E+00 0.00000E+00 0.00000E+00

MEMBER END-ACTIONS
MEMBER       AM1         AM2         AM3         AM4         AM5         AM6
   1      5.15838E+01 6.45218E+02-8.59608E+00-5.15838E+01 9.02078E+02 8.59608E+00
   2     -7.73757E+01-1.03225E+02 4.93117E+00 7.73757E+01-4.88515E+02-4.93117E+00
   3      1.61148E+02-1.28959E+02 2.47275E+00-1.61148E+02-1.67770E+02-2.47275E+00
   4      6.18514E+01 1.85759E+02-2.94949E+00-6.18514E+01 1.68180E+02 2.94949E+00
   5     -7.03178E+00 1.05919E+02-4.76740E-01 7.03178E+00-4.87099E+01 4.76740E-01
```



```
SUPPORT REACTIONS
JOINT        AR1         AR2         AR3           AR4           AR5           AR6
  3      5.15838E+01 6.45218E+02-8.59608E+00
  4      7.73757E+01-4.88515E+02-4.93117E+00
  5      6.18514E+01 1.85759E+02-2.94949E+00
  6      4.87099E+01 7.03178E+00 4.76740E-01

LOADING NO.  2
NLJ  NLM
  0    2

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
MEMBER       AML1        AML2         AML3          AML4          AML5          AML6
   1        0.000    -1080.000      24.000         0.000      1080.000        24.000
   4        0.000     -960.000      48.000         0.000       960.000        48.000

JOINT DISPLACEMENTS
JOINT        D1          D2          D3            D4            D5            D6
  1     -2.96631E-04-1.05812E-03-1.40075E-01
  2      8.63299E-04 3.87985E-04-1.37184E-01
  3      0.00000E+00 0.00000E+00 0.00000E+00
  4      0.00000E+00 0.00000E+00 0.00000E+00
  5      0.00000E+00 0.00000E+00 0.00000E+00
  6      0.00000E+00 0.00000E+00 0.00000E+00

MEMBER END-ACTIONS
MEMBER       AM1         AM2         AM3         AM4         AM5         AM6
   1      3.95508E+01-2.21090E+03 3.85251E+01-3.95508E+01-4.03607E+02 9.47495E+00
   2     -5.93262E+01 6.92828E+02-1.59559E+01 5.93262E+01 1.22189E+03 1.59559E+01
   3     -2.89220E+02-9.88770E+01 6.48100E+00 2.89220E+02-6.78842E+02-6.48100E+00
   4     -1.72660E+02-2.48081E+03 7.17302E+01 1.72660E+02-3.66817E+02 2.42698E+01
   5      7.75969E+01 8.51502E+02-1.77888E+01-7.75969E+01 1.28315E+03 1.77888E+01

SUPPORT REACTIONS
JOINT        AR1         AR2         AR3           AR4           AR5           AR6
  3      3.95508E+01-2.21090E+03 3.85251E+01
  4      5.93262E+01 1.22189E+03 1.59559E+01
  5     -1.72660E+02-2.48081E+03 7.17302E+01
  6     -1.28315E+03-7.75969E+01 1.77888E+01
```

```
STRUCTURE NO.  6 SPACE TRUSS

STRUCTURE DATA
  M    N  NJ  NR  NRJ       E            G
  6    3   4   9    4  10000.0

COORDINATES OF JOINTS
JOINT     X         Y         Z
  1     0.00      0.00      0.00
  2    75.00      0.00      0.00
  3     0.00    100.00      0.00
  4     0.00      0.00    100.00

MEMBER INFORMATION
MEMBER  JJ  JK    AX        IX        IY        IZ    AA       L
  1      1   2  10.00                                       75.00
  2      1   3  10.00                                      100.00
  3      1   4  10.00                                      100.00
  4      2   3  10.00                                      125.00
  5      2   4  10.00                                      125.00
  6      3   4  10.00                                      141.42

JOINT RESTRAINTS
JOINT RL1 RL2 RL3 RL4 RL5 RL6
  1    1   1   1
  2    0   1   1
  3    1   1   1
  4    0   0   1

LOADING NO.  1
NLJ  NLM
  1    0

ACTIONS APPLIED AT JOINTS
JOINT        A1           A2           A3           A4           A5           A6
  4        30.000       40.000        0.000

JOINT DISPLACEMENTS
JOINT        D1           D2           D3           D4           D5           D6
  1      0.00000@+00 0.00000@+00 0.00000@+00
  2      1.85033@-02 0.00000@+00 0.00000@+00
  3      0.00000@+00 0.00000@+00 0.00000@+00
  4      1.22670@-01 1.13137@-01 0.00000@+00

MEMBER END-ACTIONS
MEMBER       AM1          AM2          AM3          AM4          AM5          AM6
  1     -2.46711@+01 0.00000@+00 0.00000@+00 2.46711@+01 0.00000@+00 0.00000@+00
  2      0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00
  3      0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00
  4     -8.88158@+00 0.00000@+00 0.00000@+00 8.88158@+00 0.00000@+00 0.00000@+00
  5      5.00000@+01 0.00000@+00 0.00000@+00-5.00000@+01 0.00000@+00 0.00000@+00
  6      5.65685@+01 0.00000@+00 0.00000@+00-5.65685@+01 0.00000@+00 0.00000@+00
```



```
SUPPORT REACTIONS
JOINT        AR1          AR2          AR3          AR4          AR5          AR6
  1     -2.46711@+01 0.00000@+00 0.00000@+00
  2      0.00000@+00-7.10526@+00 4.00000@+01
  3     -5.32895@+00-3.28947@+01 4.00000@+01
  4      0.00000@+00 0.00000@+00-8.00000@+01

LOADING NO.  2
NLJ  NLM
  0    2

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
MEMBER       AML1         AML2         AML3         AML4         AML5         AML6
  2        -20.000       20.000      -10.000      -20.000       20.000      -10.000
  5          5.000       10.000        5.000        5.000       10.000        5.000

JOINT DISPLACEMENTS
JOINT        D1           D2           D3           D4           D5           D6
  1      0.00000@+00 0.00000@+00 0.00000@+00
  2      8.63487@-03 0.00000@+00 0.00000@+00
  3      0.00000@+00 0.00000@+00 0.00000@+00
  4      3.29404@-02-2.82843@-02 0.00000@+00

MEMBER END-ACTIONS
MEMBER       AM1          AM2          AM3          AM4          AM5          AM6
  1     -1.15132@+01 0.00000@+00 0.00000@+00 1.15132@+01 0.00000@+00 0.00000@+00
  2     -2.00000@+01 2.00000@+01-1.00000@+01-2.00000@+01 2.00000@+01-1.00000@+01
  3      0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00
  4     -4.14474@+00 0.00000@+00 0.00000@+00 4.14474@+00 0.00000@+00 0.00000@+00
  5      1.66667@+01 1.00000@+01 5.00000@+00-6.66667@+00 1.00000@+01 5.00000@+00
  6     -1.41421@+01 0.00000@+00 0.00000@+00 1.41421@+01 0.00000@+00 0.00000@+00

SUPPORT REACTIONS
JOINT        AR1          AR2          AR3          AR4          AR5          AR6
  1     -3.15132@+01-2.00000@+01-1.00000@+01
  2      0.00000@+00 6.68421@+00 1.03333@+01
  3     -2.24868@+01-6.68421@+00-2.00000@+01
  4      0.00000@+00 0.00000@+00 1.66667@+00
```

```
STRUCTURE NO.  7 SPACE TRUSS

STRUCTURE DATA
  M   N  NJ  NR  NRJ      E          G
  7   6   5   9   3  30000.0

COORDINATES OF JOINTS
JOINT     X         Y         Z
  1  120.00      0.00      0.00
  2    0.00    240.00    240.00
  3  240.00    240.00    240.00
  4  240.00    240.00      0.00
  5    0.00    240.00      0.00

MEMBER INFORMATION
MEMBER JJ JK    AX        IX        IY        IZ    AA      L
   1    1  2  10.00                                      360.00
   2    1  3  10.00                                      360.00
   3    2  3  10.00                                      240.00
   4    2  5  10.00                                      240.00
   5    3  4  10.00                                      240.00
   6    2  4  10.00                                      339.41
   7    3  5  10.00                                      339.41

JOINT RESTRAINTS
JOINT RL1 RL2 RL3 RL4 RL5 RL6
  1    1   1   1
  4    1   1   1
  5    1   1   1

LOADING NO.  1
NLJ  NLM
 2    0

ACTIONS APPLIED AT JOINTS
JOINT         A1           A2           A3           A4           A5           A6
  2         0.000      -20.000        0.000
  3         0.000        0.000      -20.000

JOINT DISPLACEMENTS
JOINT         D1           D2           D3           D4           D5           D6
  1      0.00000@+00 0.00000@+00 0.00000@+00
  2     -1.08268@-03-6.60793@-02 1.15379@-02
  3      2.45526@-03 1.12344@-02-1.24621@-02
  4      0.00000@+00 0.00000@+00 0.00000@+00
  5      0.00000@+00 0.00000@+00 0.00000@+00

MEMBER END-ACTIONS
MEMBER        AM1          AM2          AM3          AM4          AM5          AM6
  1      3.00000@+01 0.00000@+00 0.00000@+00-3.00000@+01 0.00000@+00 0.00000@+00
  2     -2.91038@-11 0.00000@+00 0.00000@+00 2.91038@-11 0.00000@+00 0.00000@+00
  3     -4.42242@+00 0.00000@+00 0.00000@+00 4.42242@+00 0.00000@+00 0.00000@+00
  4     -1.44224@+01 0.00000@+00 0.00000@+00 1.44224@+01 0.00000@+00 0.00000@+00
  5      1.55776@+01 0.00000@+00 0.00000@+00-1.55776@+01 0.00000@+00 0.00000@+00
  6     -7.88789@+00 0.00000@+00 0.00000@+00 7.88789@+00 0.00000@+00 0.00000@+00
  7      6.25425@+00 0.00000@+00 0.00000@+00-6.25425@+00 0.00000@+00 0.00000@+00
```



```
SUPPORT REACTIONS
JOINT         AR1          AR2          AR3          AR4          AR5          AR6
  1     -1.00000@+01 2.00000@+01 2.00000@+01
  4      5.57758@+00 0.00000@+00 1.00000@+01
  5      4.42242@+00 0.00000@+00-1.00000@+01

LOADING NO.  2
NLJ  NLM
 0    2

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
MEMBER        AML1         AML2         AML3         AML4         AML5         AML6
  4       -10.000        0.000      -10.000      -10.000        0.000      -10.000
  5         0.000       10.000        0.000        0.000       10.000        0.000

JOINT DISPLACEMENTS
JOINT         D1           D2           D3           D4           D5           D6
  1      0.00000@+00 0.00000@+00 0.00000@+00
  2      1.58550@-02 9.69649@-03-1.76897@-03
  3      1.40861@-02-3.62741@-02 2.23103@-03
  4      0.00000@+00 0.00000@+00 0.00000@+00
  5      0.00000@+00 0.00000@+00 0.00000@+00

MEMBER END-ACTIONS
MEMBER        AM1          AM2          AM3          AM4          AM5          AM6
  1      5.09317@-11 0.00000@+00 0.00000@+00-5.09317@-11 0.00000@+00 0.00000@+00
  2      1.50000@+01 0.00000@+00 0.00000@+00-1.50000@+01 0.00000@+00 0.00000@+00
  3      2.21121@+00 0.00000@+00 0.00000@+00-2.21121@+00 0.00000@+00 0.00000@+00
  4     -7.78879@+00 0.00000@+00-1.00000@+01-1.22112@+01 0.00000@+00-1.00000@+01
  5     -2.78879@+00 1.00000@+01 0.00000@+00 2.78879@+00 1.00000@+01 0.00000@+00
  6      1.10150@+01 0.00000@+00 0.00000@+00-1.10150@+01 0.00000@+00 0.00000@+00
  7     -1.01982@+01 0.00000@+00 0.00000@+00 1.01982@+01 0.00000@+00 0.00000@+00

SUPPORT REACTIONS
JOINT         AR1          AR2          AR3          AR4          AR5          AR6
  1      5.00000@+00 1.00000@+01 1.00000@+01
  4     -7.78879@+00 1.00000@+01 5.00000@+00
  5     -1.72112@+01 0.00000@+00 5.00000@+00
```

```
STRUCTURE NO.  8 SPACE FRAME

STRUCTURE DATA
  M   N  NJ  NR  NRJ      E          G
  3  12   4  12    2  30000.0   12000.0

COORDINATES OF JOINTS
JOINT    X       Y       Z
  1    0.00  120.00    0.00
  2  240.00  120.00    0.00
  3    0.00    0.00    0.00
  4  360.00    0.00  120.00

MEMBER INFORMATION
MEMBER JJ JK    AX      IX      IY      IZ   AA    L
   1    1  2  11.00   83.00   56.00   56.00  0  240.00
   2    3  1  11.00   83.00   56.00   56.00  0  120.00
   3    2  4  11.00   83.00   56.00   56.00  0  207.85

JOINT RESTRAINTS
JOINT RL1 RL2 RL3 RL4 RL5 RL6
  3    1   1   1   1   1   1
  4    1   1   1   1   1   1

LOADING NO.  1
NLJ NLM
 2   0

ACTIONS APPLIED AT JOINTS
JOINT        A1          A2          A3          A4          A5          A6
  1        2.000       0.000       0.000       0.000       0.000       0.000
  2        0.000      -1.000       0.000       0.000       0.000    -120.000

JOINT DISPLACEMENTS
JOINT        D1          D2          D3          D4          D5          D6
  1     2.22671@-01 1.57170@-04-1.71823@-01-2.55327@-03 2.16542@-03-2.13387@-03
  2     2.22020@-01-4.81189@-01-7.01606@-01-8.02487@-03 1.00766@-03-4.34716@-03
  3     0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00
  4     0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00

MEMBER END-ACTIONS
MEMBER       AM1         AM2         AM3         AM4         AM5         AM6
             AM7         AM8         AM9         AM10        AM11        AM12
  1     8.95878@-01-4.32217@-01 2.17311@-01 2.27071@+01-1.79730@+01-3.63731@+01
       -8.95878@-01 4.32217@-01-2.17311@-01-2.27071@+01-3.41817@+01-6.73590@+01
  2    -4.32217@-01 1.10412@+00 2.17311@-01-1.79730@+01-4.87845@+01 9.61216@+01
        4.32217@-01-1.10412@+00-2.17311@-01 1.79730@+01 2.27071@+01 3.63731@+01
  3     1.46959@+00-7.14943@-01-4.79819@-01-3.70171@+01 1.56888@+01-5.32791@+01
       -1.46959@+00 7.14943@-01 4.79819@-01 3.70171@+01 8.40397@+01-9.53189@+01

SUPPORT REACTIONS
JOINT        AR1         AR2         AR3         AR4         AR5         AR6
  3    -1.10412@+00-4.32217@-01 2.17311@-01 4.87845@+01-1.79730@+01 9.61216@+01
  4    -8.95878@-01 1.43222@+00-2.17311@-01 1.23082@+02 4.72463@+01-1.17197@+01
```



```
LOADING NO.  2
NLJ NLM
 0   1

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
MEMBER       AML1        AML2        AML3        AML4        AML5        AML6
             AML7        AML8        AML9        AML10       AML11       AML12
  1        0.000       0.000      -2.000       0.000     120.000       0.000
           0.000       0.000      -2.000       0.000    -120.000       0.000

JOINT DISPLACEMENTS
JOINT        D1          D2          D3          D4          D5          D6
  1    -3.75484@-01 8.63819@-05 7.98144@-01 1.00894@-02-7.62797@-03 4.80737@-03
  2    -3.76222@-01 9.37343@-01 1.31546@+00 1.16091@-02 4.74040@-03 1.64567@-03
  3     0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00
  4     0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00

MEMBER END-ACTIONS
MEMBER       AM1         AM2         AM3         AM4         AM5         AM6
             AM7         AM8         AM9         AM10        AM11        AM12
  1     1.01549@+00-2.37550@-01-2.24909@+00-6.30654@+00 6.33122@+01-6.37409@+00
       -1.01549@+00 2.37550@-01-1.75091@+00 6.30654@+00-3.53064@+00-5.06379@+01
  2    -2.37550@-01-1.01549@+00-2.24909@+00 6.33122@+01 2.76197@+02-1.28232@+02
        2.37550@-01 1.01549@+00 2.24909@+00-6.33122@+01-6.30654@+00 6.37409@+00
  3     1.73433@+00 9.35417@-01 5.20024@-01 2.35563@+01 2.09810@+01 4.02658@+01
       -1.73433@+00-9.35417@-01-5.20024@-01-2.35563@+01-1.29066@+02 1.54157@+02

SUPPORT REACTIONS
JOINT        AR1         AR2         AR3         AR4         AR5         AR6
  3     1.01549@+00-2.37550@-01-2.24909@+00-2.76197@+02 6.33122@+01-1.28232@+02
  4    -1.01549@+00 2.37550@-01-1.75091@+00-1.75297@+02-9.17817@+01 4.27143@+01
```

```
STRUCTURE NO.  9 SPACE FRAME

STRUCTURE DATA
  M    N  NJ  NR  NRJ       E           G
  3    6   4  18    3  10000.0      4000.0

COORDINATES OF JOINTS
JOINT     X         Y         Z
  1  128.00     96.00     72.00
  2  128.00    192.00      0.00
  3    0.00      0.00      0.00
  4  256.00      0.00      0.00

MEMBER INFORMATION
MEMBER JJ JK    AX        IX        IY        IZ   AA      L
  1     1  2   9.00     64.00     28.00     80.00   0  120.00
  2     3  1   9.00     64.00     28.00     80.00   1  175.45
  3     1  4   9.00     64.00     28.00     80.00   1  175.45

JOINT RESTRAINTS
JOINT RL1 RL2 RL3 RL4 RL5 RL6
  2    1   1   1   1   1   1
  3    1   1   1   1   1   1
  4    1   1   1   1   1   1

LOADING NO.  1
NLJ  NLM
 1    0

ACTIONS APPLIED AT JOINTS
JOINT        A1          A2          A3          A4          A5          A6
  1       0.000       0.000      -5.000    -120.000       0.000       0.000

JOINT DISPLACEMENTS
JOINT        D1          D2          D3          D4          D5          D6
  1     1.85739E-15-7.72853E-04-1.07548E-02-2.77936E-03 5.46897E-15 2.98654E-15
  2     0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00
  3     0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00
  4     0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00

MEMBER END-ACTIONS
MEMBER      AM1         AM2         AM3         AM4         AM5         AM6
            AM7         AM8         AM9         AM10        AM11        AM12
  1     4.37593E+00-9.76828E-01-6.57960E-13 5.51094E-12 5.27091E-11-7.71388E+01
       -4.37593E+00 9.76828E-01 6.57960E-13-5.51094E-12 2.62462E-11-4.00806E+01
  2     2.48078E+00 2.42930E-01 5.01800E-02 2.95849E+00-2.94600E+00 1.37078E+01
       -2.48078E+00-2.42930E-01-5.01800E-02-2.95849E+00-5.85826E+00 2.89151E+01
  3     2.48078E+00-2.42930E-01-5.01800E-02-2.95849E+00 5.85826E+00-2.89151E+01
       -2.48078E+00 2.42930E-01 5.01800E-02 2.95849E+00 2.94600E+00-1.37078E+01

SUPPORT REACTIONS
JOINT       AR1         AR2         AR3         AR4         AR5         AR6
  2     6.57960E-13-2.91464E+00 3.40702E+00-4.00806E+01 1.13390E-11 2.43035E-11
  3     1.85946E+00 1.45732E+00 7.96491E-01-7.03349E+00 1.18596E+01 3.90058E+00
  4    -1.85946E+00 1.45732E+00 7.96491E-01-7.03349E+00-1.18596E+01-3.90058E+00
```



```
LOADING NO.  2
NLJ  NLM
 0    2

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
MEMBER      AML1        AML2        AML3        AML4        AML5        AML6
            AML7        AML8        AML9        AML10       AML11       AML12
  1        -1.500       2.000       0.000       0.000       0.000      60.000
           -1.500       2.000       0.000       0.000       0.000     -60.000
  3        -1.368      -0.616       2.000       0.000     -87.727      27.000
           -1.368       0.616       2.000       0.000      87.727     -27.000

JOINT DISPLACEMENTS
JOINT        D1          D2          D3          D4          D5          D6
  1    -2.17639E-04-3.28905E-03-5.98127E-03-2.42285E-03 1.87041E-04-4.49532E-03
  2     0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00
  3     0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00
  4     0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00

MEMBER END-ACTIONS
MEMBER      AM1         AM2         AM3         AM4         AM5         AM6
            AM7         AM8         AM9         AM10        AM11        AM12
  1    -7.81858E-01 1.15484E+00 4.06048E-01 6.07323E+00-3.24923E+01-6.86209E+00
       -2.21814E+00 2.84516E+00-4.06048E-01-6.07323E+00-1.62334E+01-9.45575E+01
  2     2.26362E+00 2.41873E-01-1.81269E-01 5.12126E+00 1.05559E+01 1.39081E+01
       -2.26362E+00-2.41873E-01 1.81269E-01-5.12126E+00 2.12483E+01 2.85294E+01
  3     7.32728E-01 4.21044E-01 1.72864E+00-3.67330E-02-5.60367E+01 3.95106E+00
       -3.46873E+00 8.10956E-01 2.27136E+00 3.67330E-02 1.03648E+02-3.81568E+01

SUPPORT REACTIONS
JOINT       AR1         AR2         AR3         AR4         AR5         AR6
  2    -4.06048E-01-6.74159E-02 3.60702E+00-9.45575E+01-1.45986E+01-9.34281E+00
  3     1.83956E+00 1.15308E+00 7.08340E-01-1.14331E+00 1.65277E+01-7.52459E+00
  4    -1.43398E+00 3.91468E+00 6.83915E-01-5.68939E+01-5.02556E+00-9.45335E+01
```

Chapter 4

ANALYSIS BY SOLUTION OF SIMULTANEOUS EQUATIONS

4.1 Introduction. Program $FR2$, described in this chapter, is oriented toward the analysis of large structures containing numerous members and joints. Such structures represent a special challenge to the programmer because they require large blocks of storage, and the calculations may consume excessive amounts of computer time. This program is especially designed to conserve both core storage and computer time. Consequently, it can accommodate a much larger structure than can Program $FR1$ in a given amount of time and storage. An essential feature of Program $FR2$ is the fact that all operations for a large structure are performed completely in core without resorting to the use of auxiliary storage units, such as magnetic disks or tapes. A program which is oriented toward the use of auxiliary storage appears in Chapter 5 and is designated as Program $FR3$.

Program $FR2$ conserves storage by generating only the upper band of the symmetric stiffness matrix $\mathbf{S}$. This part of the matrix is stored as a rectangular array in the pattern indicated previously in Fig. 2-3. The program saves calculation time by obtaining joint displacements through the process of solving the equations of joint equilibrium simultaneously instead of inverting the stiffness matrix. Procedures *DECOMPOSEBAND* and *SOLVEBAND* (see Arts. 2.7 and 2.8) serve to accomplish this objective. Procedure *DECOMPOSEBAND* need only be called once for a given structure, but procedure *SOLVEBAND* must be called for each system of applied loads. Furthermore, the matrix $\mathbf{S}_{RD}$ is not generated in this program, but reactions are computed by summing the end-actions of members which frame into the supports.

The stiffness matrix $\mathbf{S}$ for a given structure has a certain upper band width *UBW*, which is determined by the system for numbering joints. The value of this parameter may be calculated from the formula*:

$$UBW = [n_{dj}(k - j + 1) - r]_{\max} \qquad (k > j) \qquad (4\text{-}1)$$

As in previous discussions, the identifiers j and k in Eq. (4-1) denote the joint numbers at the two ends of a member, and n_{dj} is the number of displacements per joint. The symbol r represents the number of support restraints which have joint displacement indexes in the range of indexes spanned by the term $n_{dj}(k - j + 1)$. The member for which the right-hand side of Eq. (4-1) is a maximum determines the upper band width *UBW*.

*This formula may be used by the analyst to determine the amount of storage required for a given structure.

Since the magnitude of *UBW* dictates the amount of storage required for the upper band portion of the matrix $\mathbf{S}$, the joints of a structure (to be analyzed by Program $FR2$) should be numbered in a sequence which minimizes that parameter. If the term r is equal to zero, minimization of the difference $k - j$ serves to minimize *UBW* also.

4.2 Conversion of Program *FR*1 to *FR*2. Once Program $FR1$ has been coded, it may be easily converted to Program $FR2$ by changing certain parts, as described in this article. The form of the input data is the same for both programs, but the following identifiers are used in Program $FR2$ in addition to those listed previously for $FR1$ (see Art. 3.2):

Identifier	*Definition*	*Type*
UBW	Upper band width of stiffness matrix	*I*
ROW	Row index of stiffness matrix	*I*
COL	Column index of stiffness matrix	*I*
AMD[ ]	Actions at ends of members due to displacements of joints	*D*

The first significant change in the program occurs in Sec. 2a (see the flow chart in Art. 3.5), where the stiffness matrix is generated. Parts (10) through (14) of that section, which deal with the re-indexing of joint displacements, may be omitted; and parts (26) through (32), which involve transferring elements from $\mathbf{S}_{MD}$ to $\mathbf{S}$, are to be replaced by new statements. These statements include the calculation of row and column indexes of elements in the upper band of the matrix $\mathbf{S}$ (stored as a rectangular array). The row index *ROW* for a particular degree of freedom is computed by the expression:

$$ROW = (\text{Old index}) - (\text{Cumulative number of support restraints}) \qquad (4\text{-}2)$$

The column index *COL* is determined by:

$$COL = (\text{Old index}) - (\text{Cumulative number of support restraints}) - (ROW - 1) \qquad (4\text{-}3)$$

Equation (4-2) is the same as Eq. (1-71) in Chapter 1, but Eq. (4-3) reflects the fact that elements of the upper band are shifted $(ROW - 1)$ places to the left when the matrix is stored as a rectangular array (see Figs. 2-2 and 2-3).

The second significant change in converting program $FR1$ to $FR2$ appears in Sec. 2b. The title of this section must now be shortened to:

2*b*. Decomposition of stiffness matrix

In this section the upper band of the stiffness matrix is to be decomposed using the procedure *DECOMPOSEBAND* from Art. 2.7. Later in the program (Sec. 5a) the procedure *SOLVEBAND* from Art. 2.8 is used to compute the unknown joint displacements.

Another important modification of the program arises in Sec. 5b, where member end-actions are calculated. Equation (1-69) was used in Program *FR*1 to compute member end-actions in one step, but this calculation is broken into two parts in Program *FR*2. For this purpose, a new vector $\{A_{MD}\}_i$ is defined to be the second term on the right-hand side of Eq. (1-69).

$$\{A_{MD}\}_i = [S_M]_i[R_T]_i\{D_J\}_i \qquad (4\text{-}4)$$

This vector is computed separately for each member i in the structure. Then the vector of final member end-actions is obtained as the sum:

$$\{A_M\}_i = \{A_{ML}\}_i + \{A_{MD}\}_i \qquad (4\text{-}5)$$

Subsequently, the restraint list is checked to determine whether the member frames into a support. If this is the case, the contributions of the member end-actions $\{A_{MD}\}_i$ to the reactions at the support are extracted from the following expression:

$$[R'_T]\{A_{MD}\}_i \qquad (4\text{-}6)$$

When such contributions to support reactions have been accumulated for all members in the structure, the results are equivalent to the product $\mathbf{S}_{RD}\mathbf{D}$, which is the second term on the right-hand side of Eq. (1-6). (The first term, $\mathbf{A}_{RL}$, is obtained later in Sec. 5c.) Of course, if the structure is a beam, the rotation transformation matrices in Eq. (4-4) and expression (4-6) are not needed.

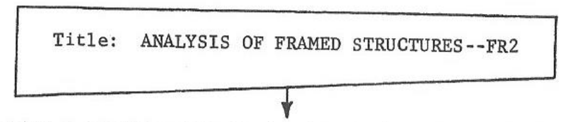
Finally, the calculation of support reactions in Sec. 5c of Program *FR*2 is accomplished by adding the negatives of the loads applied at the supports to the values of the reactions accumulated previously in Sec. 5b.

The modifications mentioned above are given in the flow chart for Program *FR*2 in the next article, and further explanatory remarks appear in Art. 4.4.

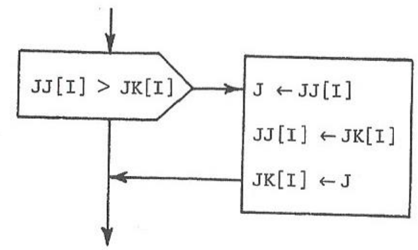
4.3 Flow Chart for Program *FR*2.

1. Input and Print Control and Structure Data

a. Control data, structure parameters, and elastic moduli



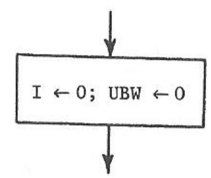
Section 1 of this program is almost the same as in Program FR1, except that in Sec. 1c the joint indexes JJ[I] and JK[I] must be placed in ascending order. For this purpose, the following statements must be inserted after part 1c(4) and after part 1c(10):



2. Structure Stiffness Matrix

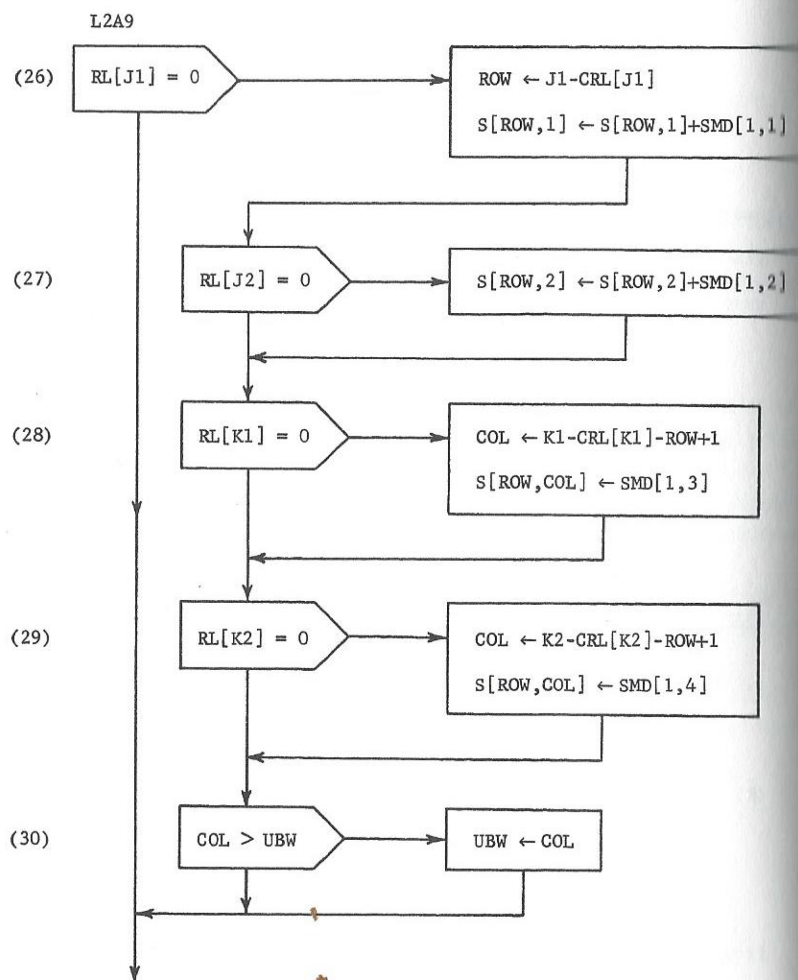
a. Generation of stiffness matrix

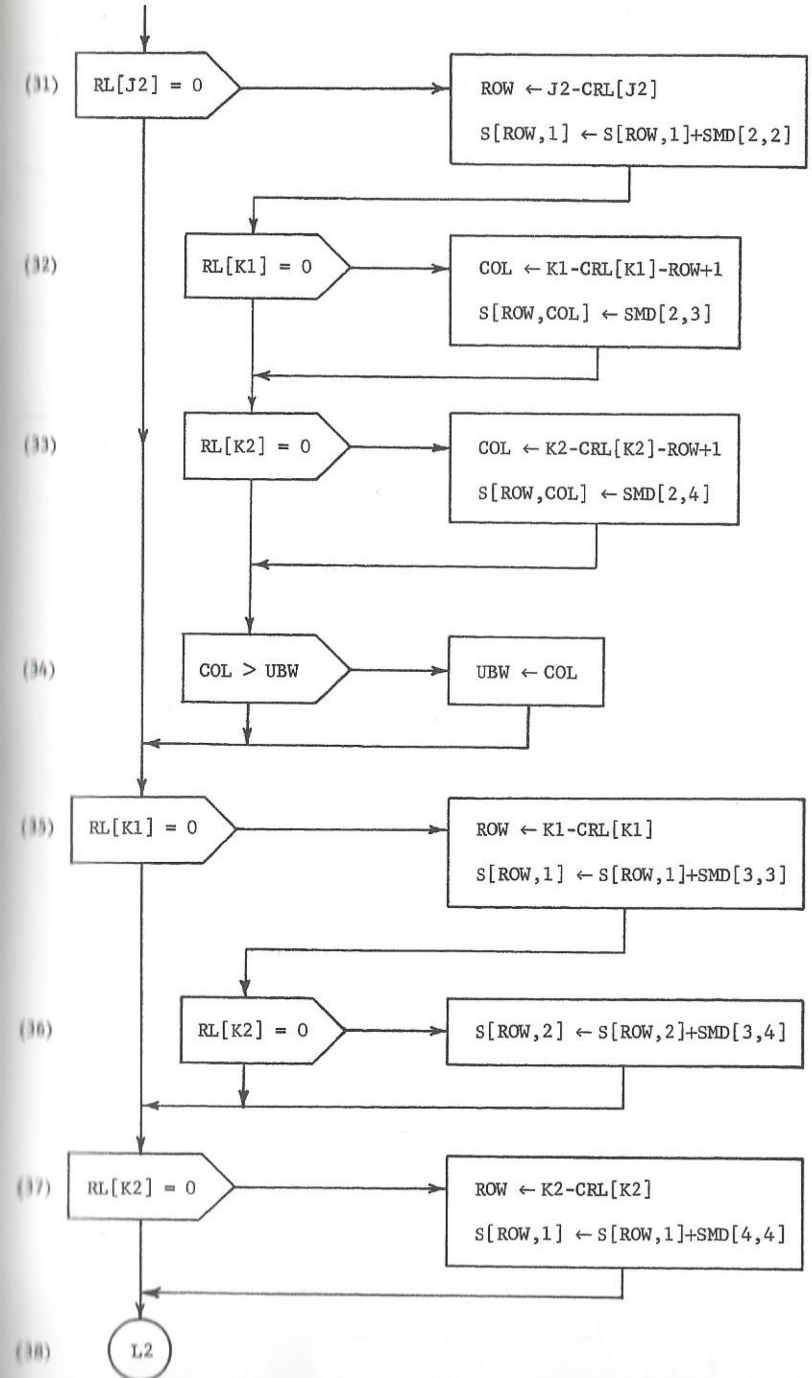
(1)

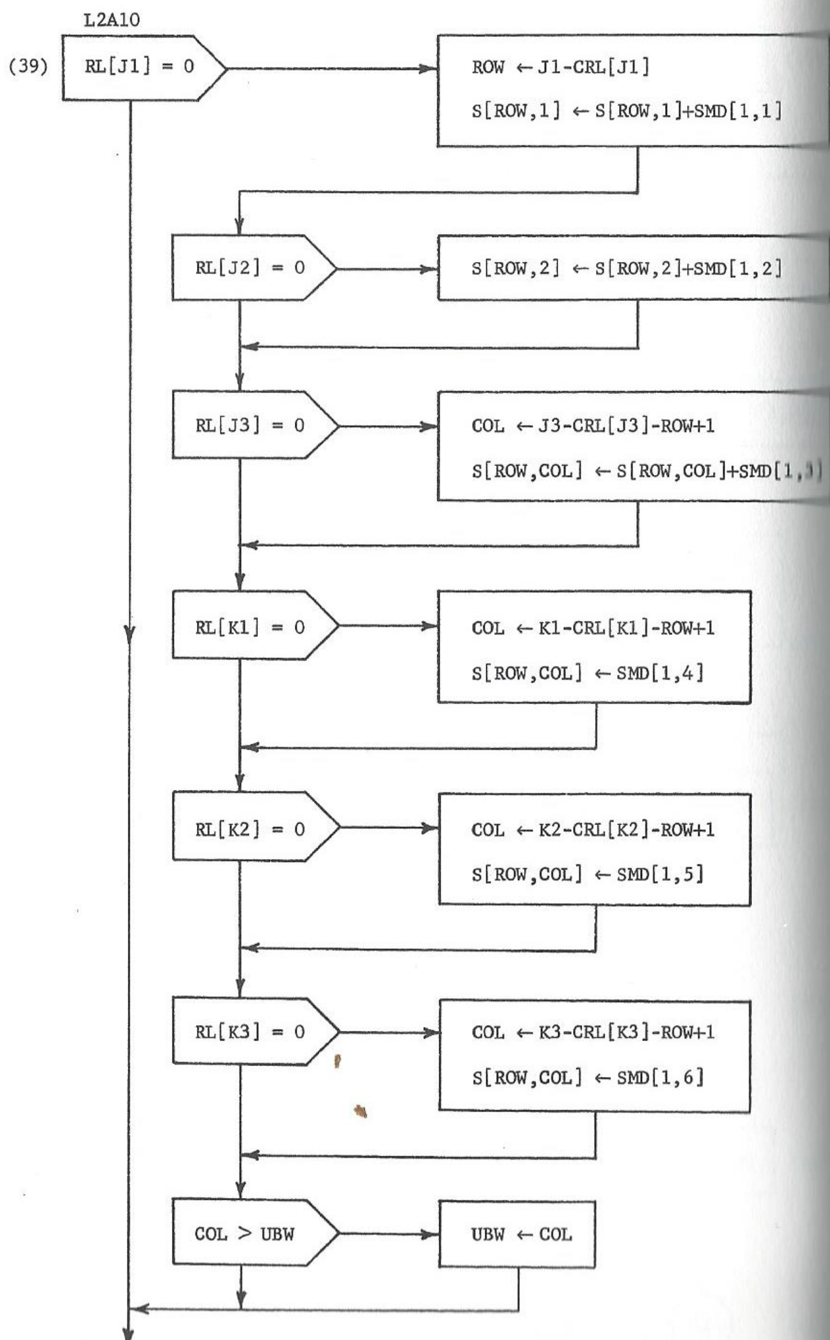


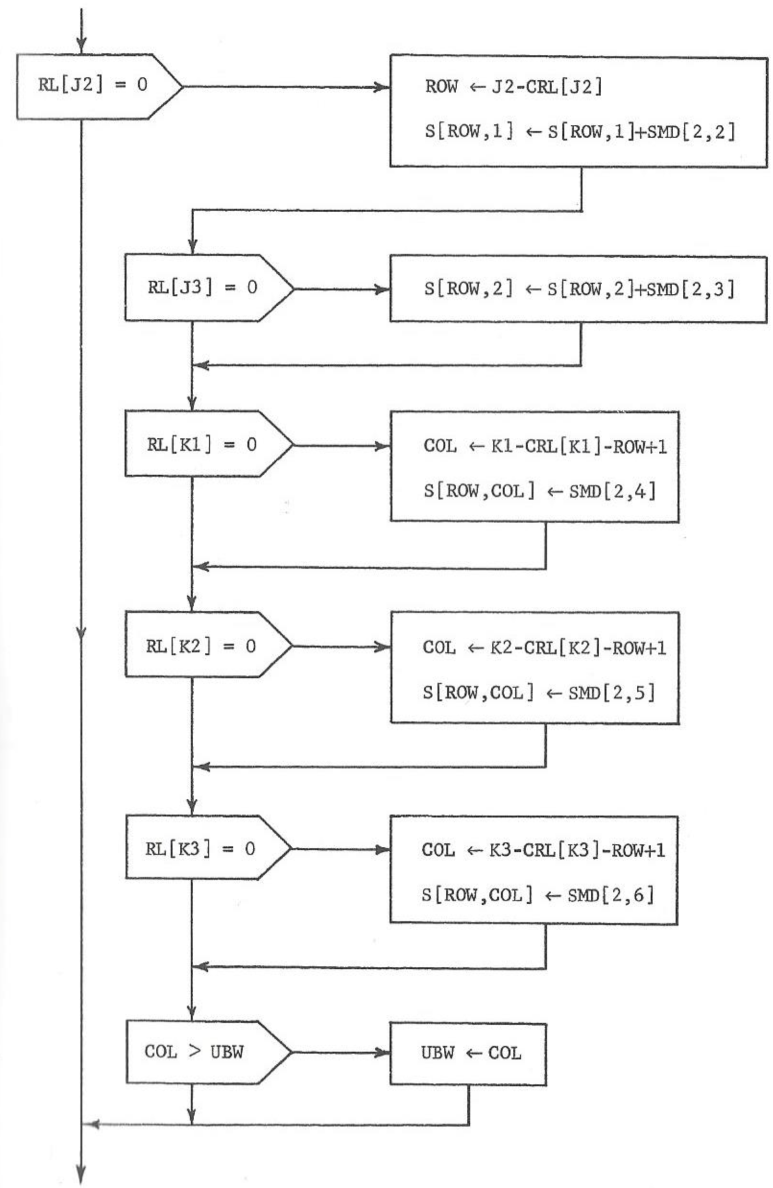
Parts (2) through (9) of this section are the same as in Program FR1; parts (10) through (14) are to be omitted, and the statement label L2A may be removed from statement (15). In addition, parts (16) through (25) are the same as in Program FR1.

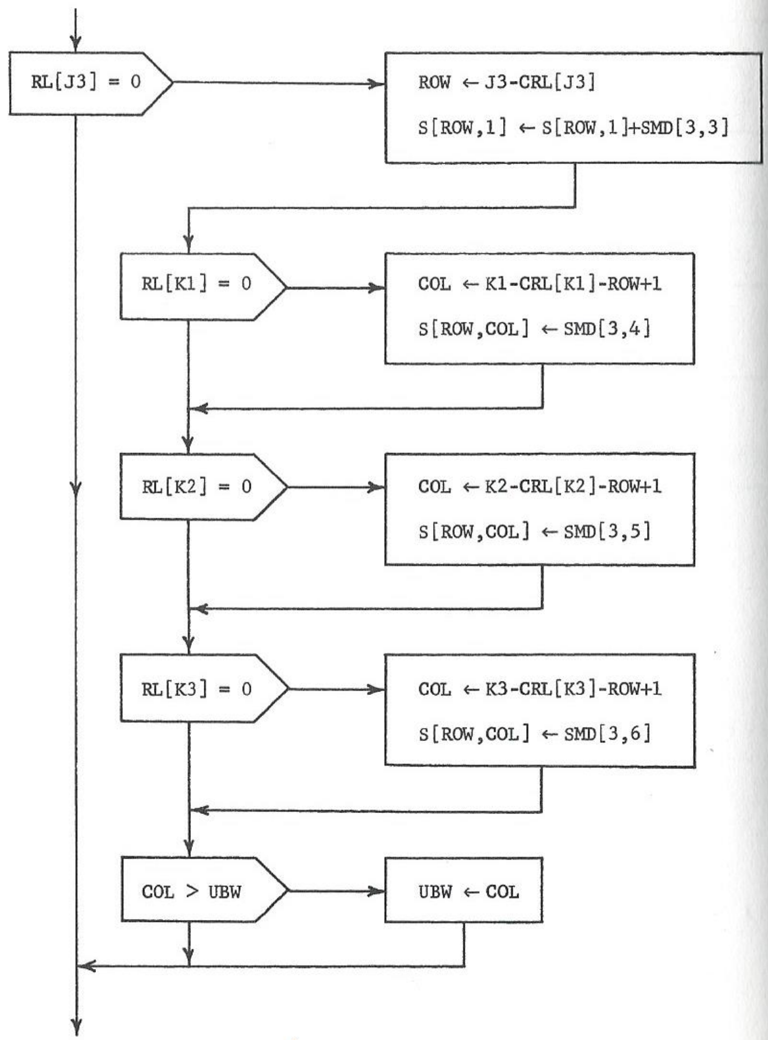
The remainder of Sec. 2a follows:

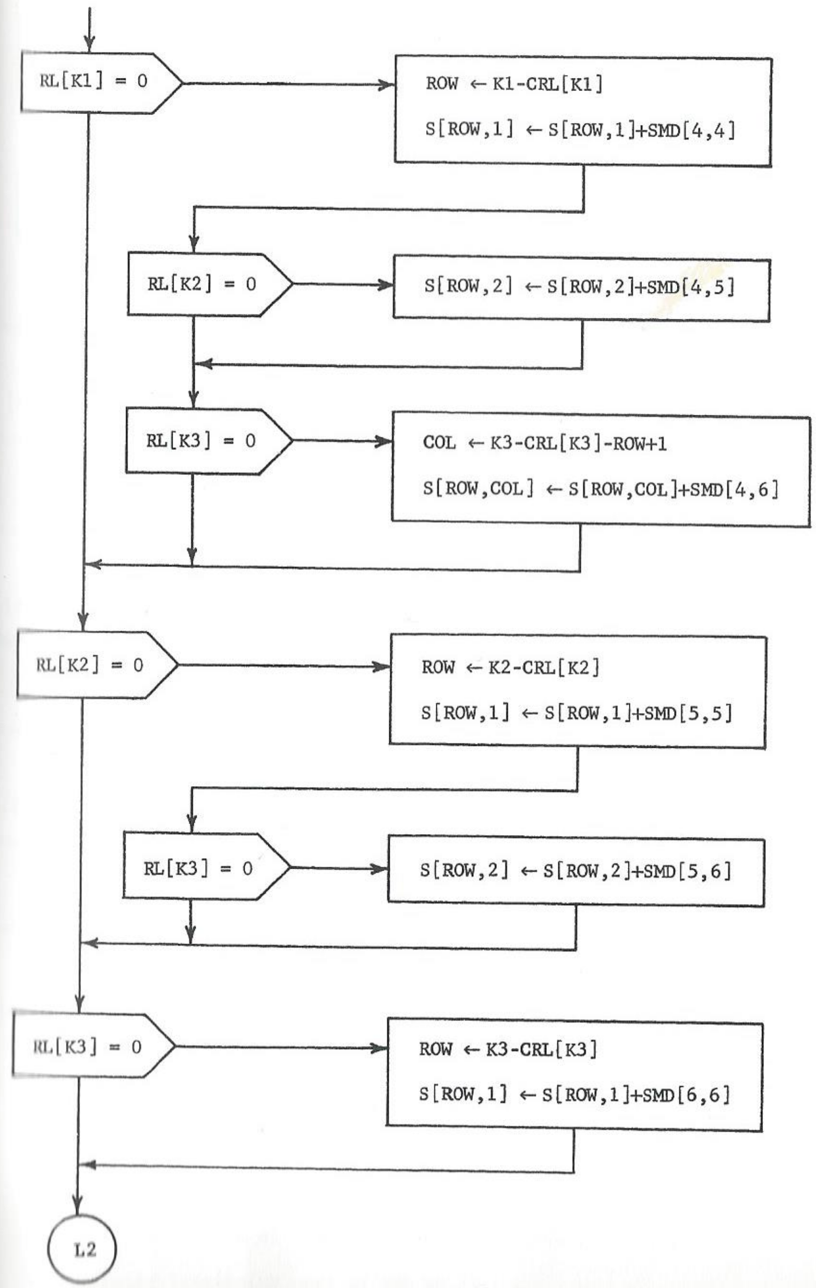


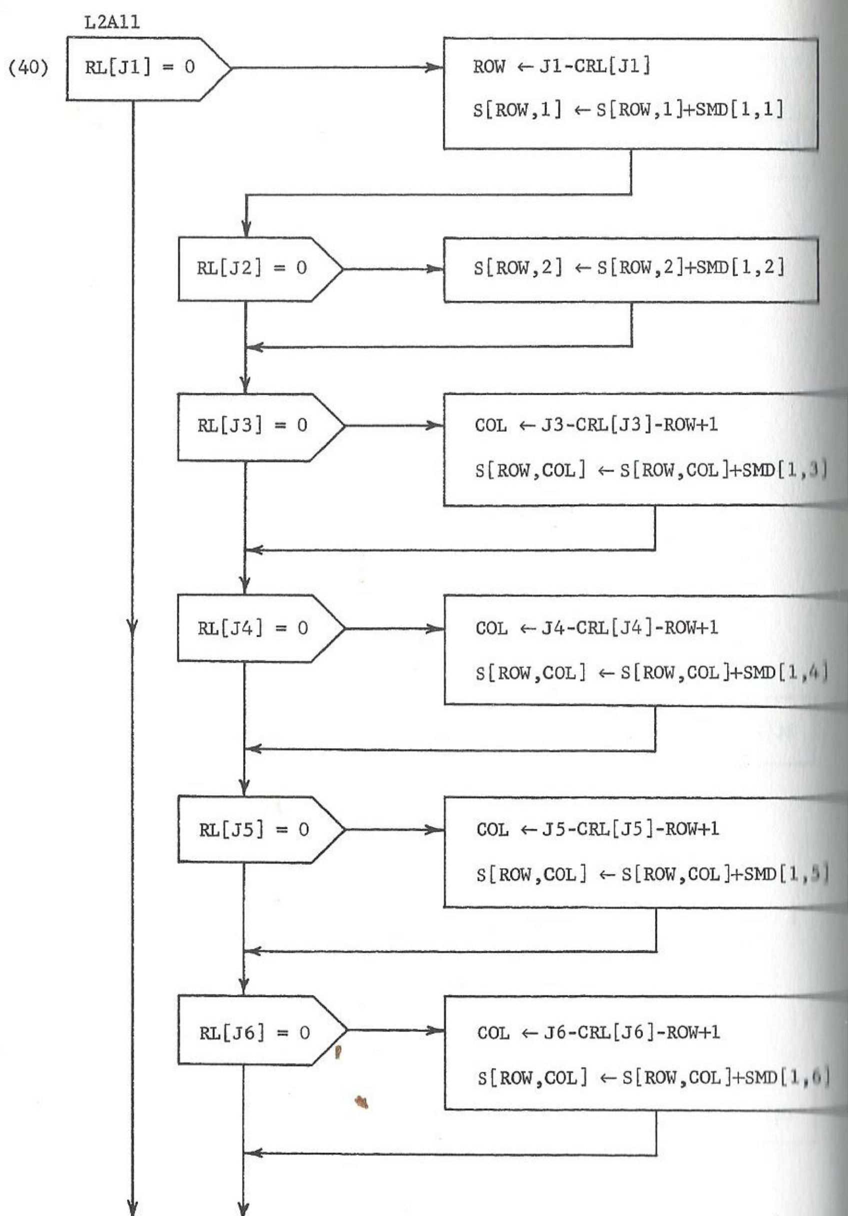


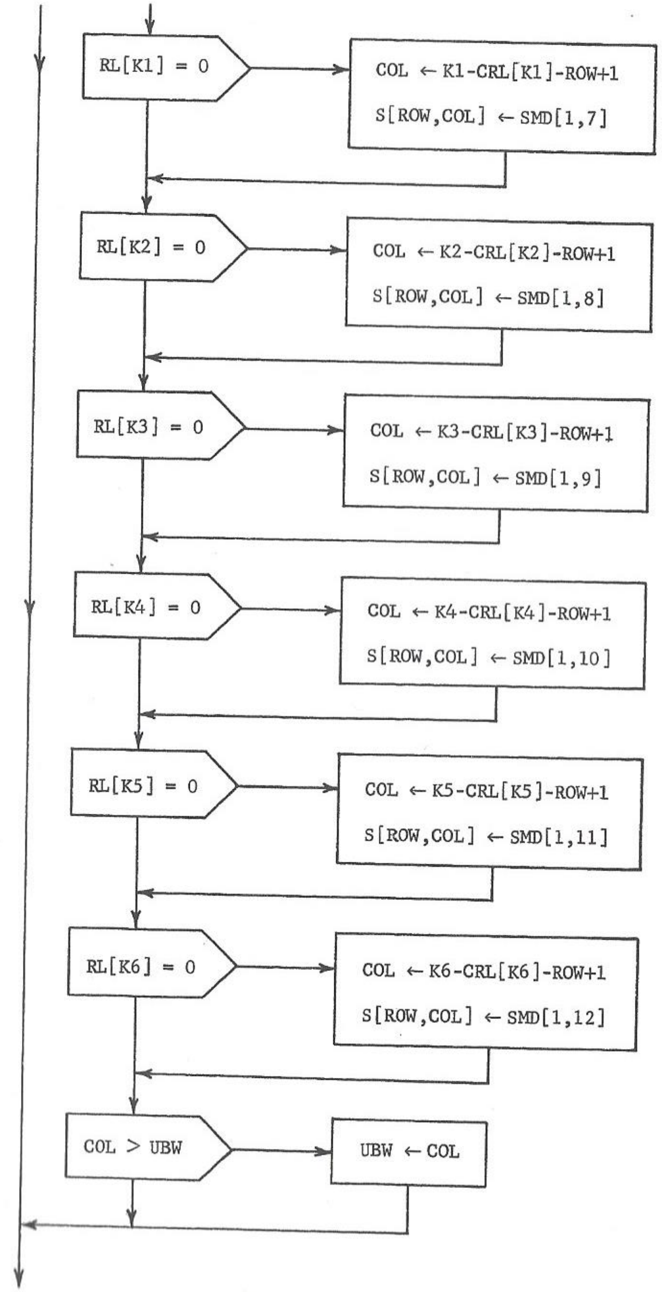

L2A10
(39)
RL[J1] = 0
ROW ← J1-CRL[J1]
S[ROW,1] ← S[ROW,1]+SMD[1,1]
RL[J2] = 0
S[ROW,2] ← S[ROW,2]+SMD[1,2]
RL[J3] = 0
COL ← J3-CRL[J3]-ROW+1
S[ROW,COL] ← S[ROW,COL]+SMD[1,3]
RL[K1] = 0
COL ← K1-CRL[K1]-ROW+1
S[ROW,COL] ← SMD[1,4]
RL[K2] = 0
COL ← K2-CRL[K2]-ROW+1
S[ROW,COL] ← SMD[1,5]
RL[K3] = 0
COL ← K3-CRL[K3]-ROW+1
S[ROW,COL] ← SMD[1,6]
COL > UBW
UBW ← COL


RL[J2] = 0
ROW ← J2-CRL[J2]
S[ROW,1] ← S[ROW,1]+SMD[2,2]
RL[J3] = 0
S[ROW,2] ← S[ROW,2]+SMD[2,3]
RL[K1] = 0
COL ← K1-CRL[K1]-ROW+1
S[ROW,COL] ← SMD[2,4]
RL[K2] = 0
COL ← K2-CRL[K2]-ROW+1
S[ROW,COL] ← SMD[2,5]
RL[K3] = 0
COL ← K3-CRL[K3]-ROW+1
S[ROW,COL] ← SMD[2,6]
COL > UBW
UBW ← COL

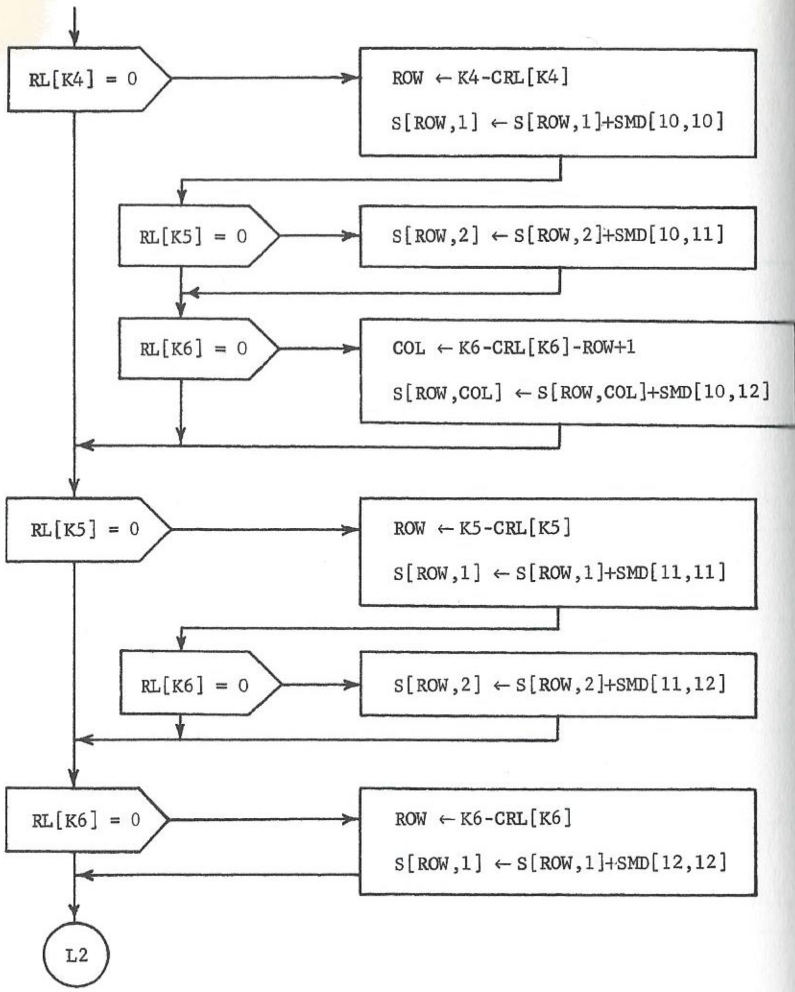

RL[J3] = 0
ROW ← J3-CRL[J3]
S[ROW,1] ← S[ROW,1]+SMD[3,3]
RL[K1] = 0
COL ← K1-CRL[K1]-ROW+1
S[ROW,COL] ← SMD[3,4]
RL[K2] = 0
COL ← K2-CRL[K2]-ROW+1
S[ROW,COL] ← SMD[3,5]
RL[K3] = 0
COL ← K3-CRL[K3]-ROW+1
S[ROW,COL] ← SMD[3,6]
COL > UBW
UBW ← COL


RL[K1] = 0
ROW ← K1-CRL[K1]
S[ROW,1] ← S[ROW,1]+SMD[4,4]
RL[K2] = 0
S[ROW,2] ← S[ROW,2]+SMD[4,5]
RL[K3] = 0
COL ← K3-CRL[K3]-ROW+1
S[ROW,COL] ← S[ROW,COL]+SMD[4,6]
RL[K2] = 0
ROW ← K2-CRL[K2]
S[ROW,1] ← S[ROW,1]+SMD[5,5]
RL[K3] = 0
S[ROW,2] ← S[ROW,2]+SMD[5,6]
RL[K3] = 0
ROW ← K3-CRL[K3]
S[ROW,1] ← S[ROW,1]+SMD[6,6]
L2

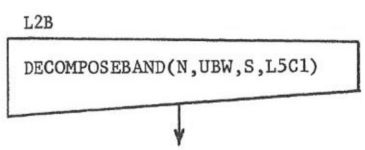




(Total of 12 sets of statements for transferring 12 rows of the upper triangular part of SMD to S).



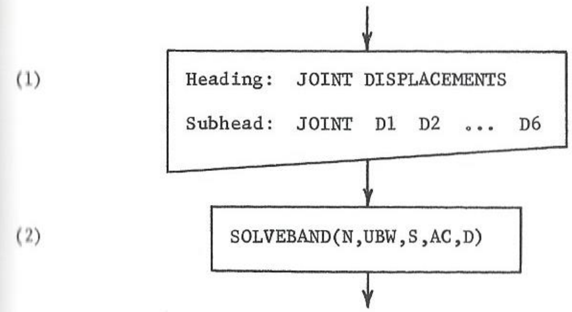
b. Decomposition of stiffness matrix



Sections 3 and 4 of this program are the same as in Program FR1, except that the vector AR must be cleared in addition to those specified previously in Sec. 3a(3).

Calculation and Output of Results

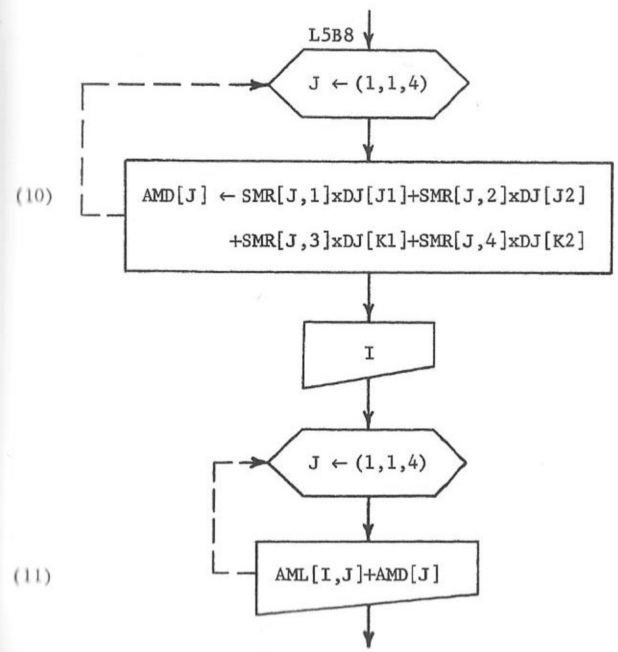
a. Joint displacements

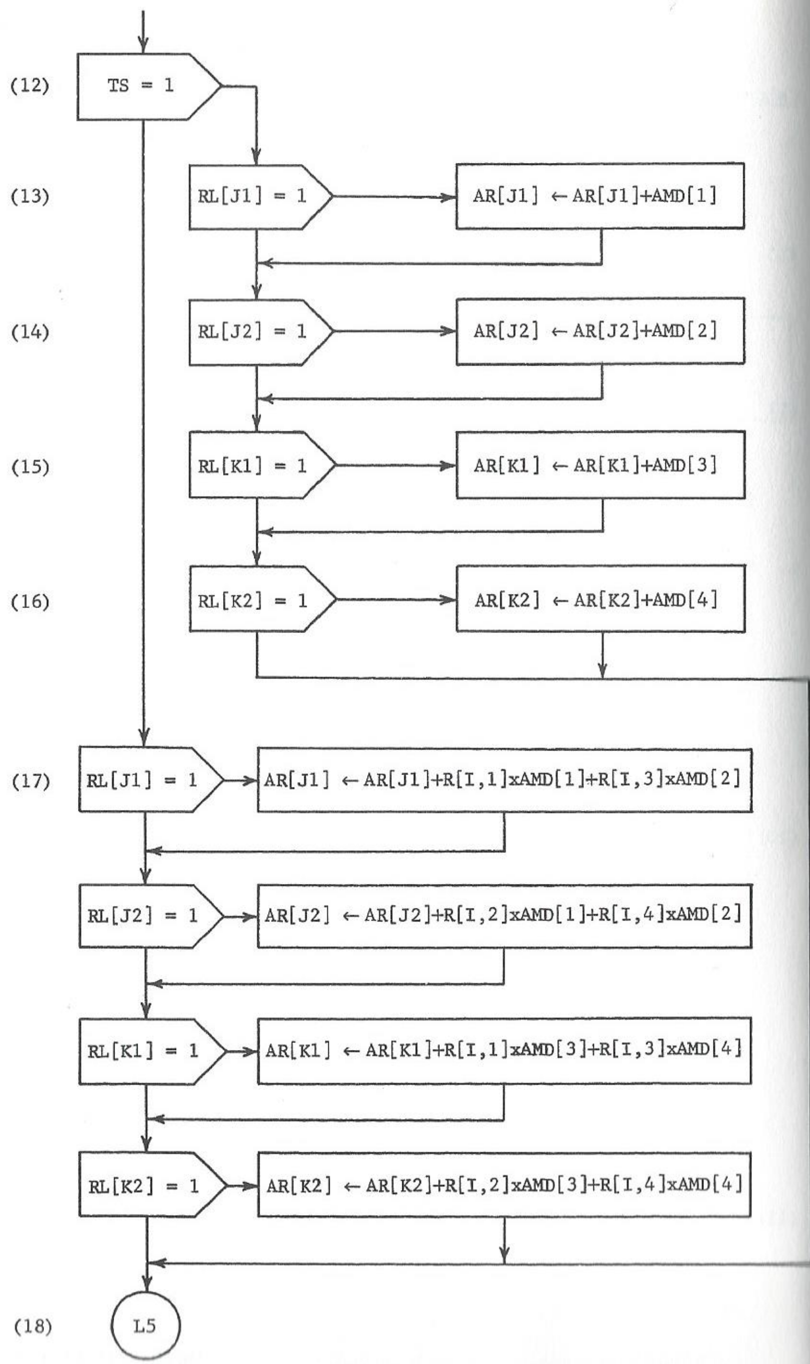


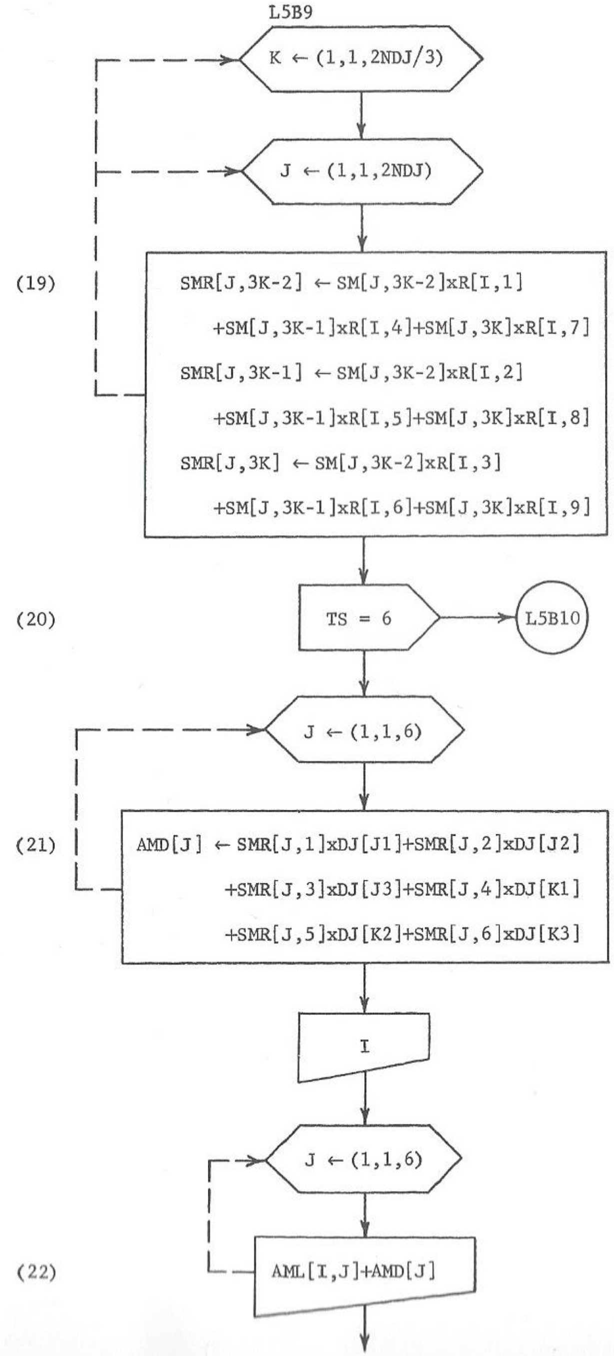
The remainder of this section is the same as in Program FR1.

b. Member end-actions

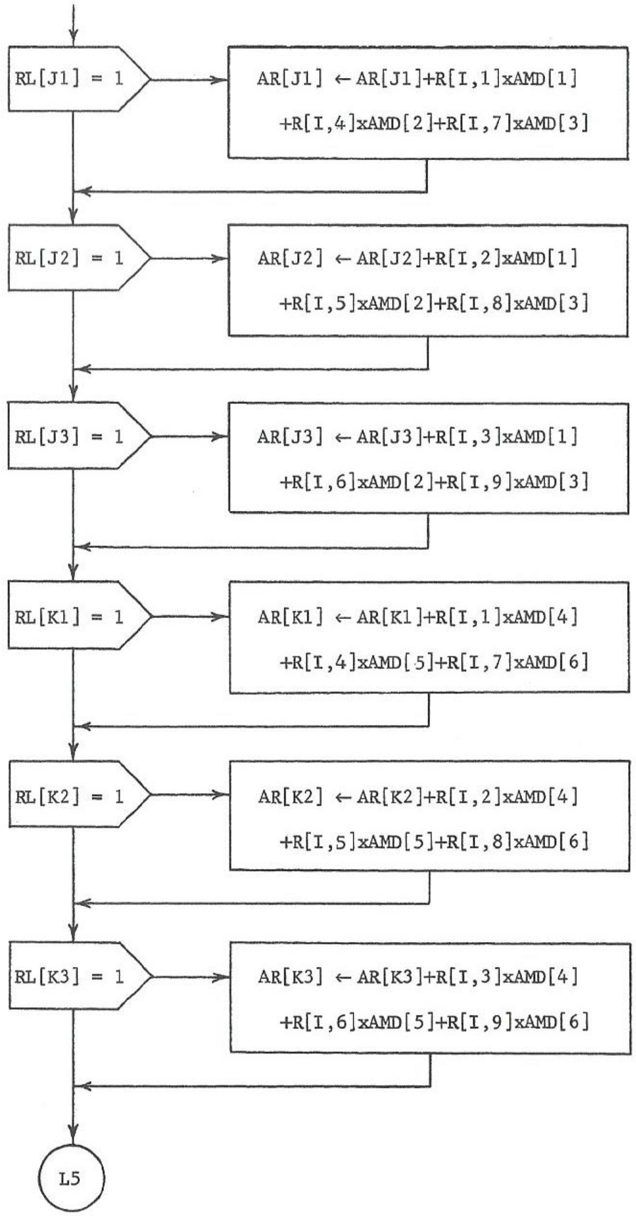
Parts (1) through (9) of this section are the same as in Program FR1.



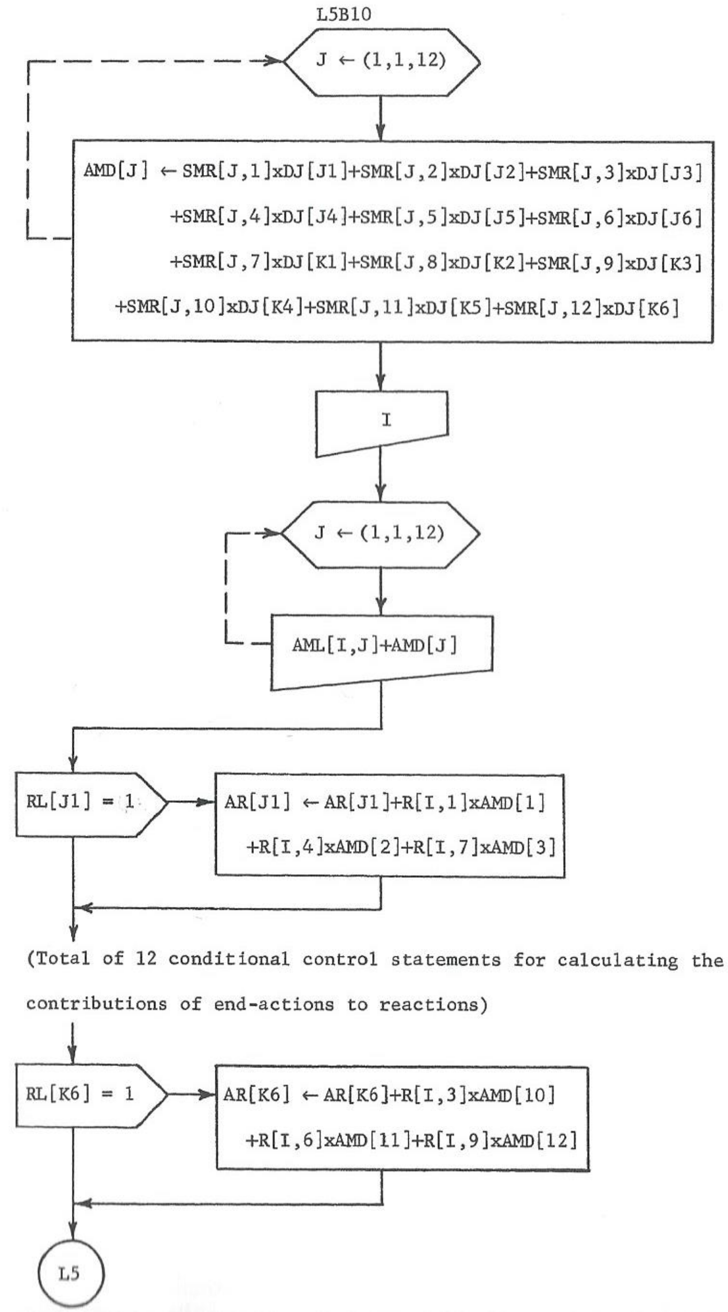

(12)
TS = 1
(13)
RL[J1] = 1
AR[J1] ← AR[J1]+AMD[1]
(14)
RL[J2] = 1
AR[J2] ← AR[J2]+AMD[2]
(15)
RL[K1] = 1
AR[K1] ← AR[K1]+AMD[3]
(16)
RL[K2] = 1
AR[K2] ← AR[K2]+AMD[4]
(17)
RL[J1] = 1
AR[J1] ← AR[J1]+R[I,1]xAMD[1]+R[I,3]xAMD[2]
RL[J2] = 1
AR[J2] ← AR[J2]+R[I,2]xAMD[1]+R[I,4]xAMD[2]
RL[K1] = 1
AR[K1] ← AR[K1]+R[I,1]xAMD[3]+R[I,3]xAMD[4]
RL[K2] = 1
AR[K2] ← AR[K2]+R[I,2]xAMD[3]+R[I,4]xAMD[4]
(18)
L5


L5B9
K ← (1,1,2NDJ/3)
J ← (1,1,2NDJ)
(19)
SMR[J,3K-2] ← SM[J,3K-2]xR[I,1]
+SM[J,3K-1]xR[I,4]+SM[J,3K]xR[I,7]
SMR[J,3K-1] ← SM[J,3K-2]xR[I,2]
+SM[J,3K-1]xR[I,5]+SM[J,3K]xR[I,8]
SMR[J,3K] ← SM[J,3K-2]xR[I,3]
+SM[J,3K-1]xR[I,6]+SM[J,3K]xR[I,9]
(20)
TS = 6
L5B10
J ← (1,1,6)
(21)
AMD[J] ← SMR[J,1]xDJ[J1]+SMR[J,2]xDJ[J2]
+SMR[J,3]xDJ[J3]+SMR[J,4]xDJ[K1]
+SMR[J,5]xDJ[K2]+SMR[J,6]xDJ[K3]
I
J ← (1,1,6)
(22)
AML[I,J]+AMD[J]

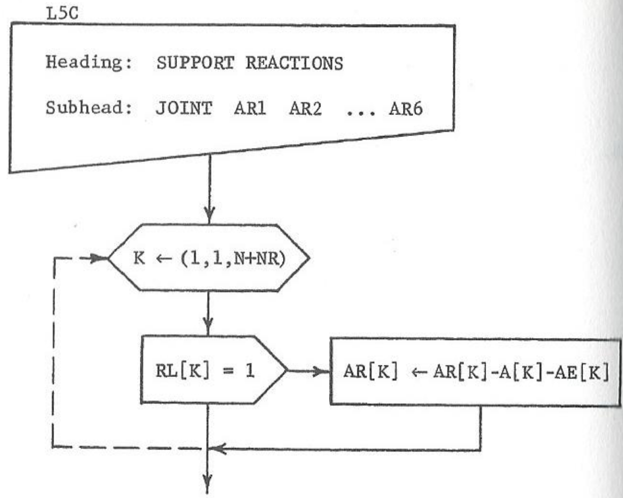
(23)


RL[J1] = 1
AR[J1] ← AR[J1]+R[I,1]xAMD[1]
+R[I,4]xAMD[2]+R[I,7]xAMD[3]
RL[J2] = 1
AR[J2] ← AR[J2]+R[I,2]xAMD[1]
+R[I,5]xAMD[2]+R[I,8]xAMD[3]
RL[J3] = 1
AR[J3] ← AR[J3]+R[I,3]xAMD[1]
+R[I,6]xAMD[2]+R[I,9]xAMD[3]
RL[K1] = 1
AR[K1] ← AR[K1]+R[I,1]xAMD[4]
+R[I,4]xAMD[5]+R[I,7]xAMD[6]
RL[K2] = 1
AR[K2] ← AR[K2]+R[I,2]xAMD[4]
+R[I,5]xAMD[5]+R[I,8]xAMD[6]
RL[K3] = 1
AR[K3] ← AR[K3]+R[I,3]xAMD[4]
+R[I,6]xAMD[5]+R[I,9]xAMD[6]
L5

(24)


L5B10
J ← (1,1,12)
AMD[J] ← SMR[J,1]xDJ[J1]+SMR[J,2]xDJ[J2]+SMR[J,3]xDJ[J3]
+SMR[J,4]xDJ[J4]+SMR[J,5]xDJ[J5]+SMR[J,6]xDJ[J6]
+SMR[J,7]xDJ[K1]+SMR[J,8]xDJ[K2]+SMR[J,9]xDJ[K3]
+SMR[J,10]xDJ[K4]+SMR[J,11]xDJ[K5]+SMR[J,12]xDJ[K6]
I
J ← (1,1,12)
AML[I,J]+AMD[J]
RL[J1] = 1
AR[J1] ← AR[J1]+R[I,1]xAMD[1]
+R[I,4]xAMD[2]+R[I,7]xAMD[3]
(Total of 12 conditional control statements for calculating the contributions of end-actions to reactions)
RL[K6] = 1
AR[K6] ← AR[K6]+R[I,3]xAMD[10]
+R[I,6]xAMD[11]+R[I,9]xAMD[12]
L5

c. Support reactions



The remainder of this section is the same as in parts 5c(5) through 5c(7) of Program FR1.

4.4 Explanation of Flow Chart for Program *FR*2.

Item No. *Remarks*

1. In this program the joint indexes $JJ[I]$ and $JK[I]$ must be placed in ascending order so that the upper triangular part of the member stiffness matrix SMD will contribute to the upper triangular part of the joint stiffness matrix S.

2. a. (1) The member index I and the upper band width identifier UBW are initialized to zero in preparation for generating the upper band of the structure stiffness matrix from member stiffness matrices.

(26) If the index $J1$ corresponds to a degree of freedom, the first row of a 4×4 member stiffness matrix SMD is to be transferred to the joint stiffness matrix S for the structure. In this program the upper band portion of the matrix S is stored as a rectangular matrix consisting of contributions from the upper triangular portions of member stiffness matrices. The row index ROW for the matrix S is calculated by subtracting from the index $J1$ the corresponding element of CRL (see Eq. 4-2). Then the element $SMD[1,1]$ is transferred to the first position in that row. The element $S[ROW,1]$ is cumulative because, in general, it receives contributions from more than one member.

(27) If the index $J2$ also corresponds to a degree of freedom, the element $SMD[1,2]$ is transferred to the second position in the same row of the matrix S. The element $S[ROW,2]$ is also cumulative.

(28) If $K1$ corresponds to a degree of freedom, the column index COL for the matrix S is calculated using Eq. (4-3). Then the element $SMD[1,3]$ is transferred to the appropriate location $S[ROW,COL]$.

(29) If $K2$ corresponds to a degree of freedom, the column index is computed, and the element $SMD[1,4]$ is transferred to the appropriate location in the matrix S. Note that elements of S involving the coupling of $J1$ with $K1$ and $K2$ receive only single contributions from SMD.

(30) After the last element of the first row of SMD has been processed, the index COL is compared with the current value of the upper band width UBW. If the value of COL exceeds that of UBW, the latter is replaced by the former. Thus, after all members have been processed, the maximum column index for the matrix S will have been determined and stored as the upper band width.

(31) If $RL[J2]$ is equal to zero, elements of the second row of the upper triangular part of SMD are to be transferred to the joint stiffness matrix S. The row index ROW for the matrix S is calculated by subtracting from the index $J2$ the corresponding element of CRL. Then the element $SMD[2,2]$ is transferred to the first position in that row.

(32) If $RL[K1]$ is equal to zero, the column index COL is computed; and $SMD[2,3]$ is transferred to $S[ROW,COL]$.

(33) If $RL[K2]$ is equal to zero, COL is computed; and $SMD[2,4]$ is transferred to S.

(34) The current value of UBW is replaced by COL if necessary, as described in (30) above.

(35) If $RL[K1]$ is equal to zero, elements of the third row of the upper triangular part of SMD are to be transferred to the matrix S. The transfer of $SMD[3,3]$ is similar to that for $SMD[1,1]$ in (26) above.

(36) The transfer of $SMD[3,4]$ to S is similar to that for $SMD[1,2]$ in (27) above.

(37) If $RL[K2]$ is equal to zero, element $SMD[4,4]$ is transferred to S in a manner similar to that for $SMD[1,1]$ in (26) above.

(38) Control is transferred back to the statement labeled $L2$ (see part 2a(2) of Program $FR1$) for the purpose of processing the next member in the structure.

(39) If the index $J1$ corresponds to a degree of freedom, the first row of the upper triangular part of a 6×6 member stiffness matrix is to be transferred to the joint stiffness matrix S for the structure. This transfer process is carried out for all six rows of SMD, and control is then returned to $L2$.

(40) Similar to (39) but for a 12×12 member stiffness matrix (space frame).

b. The upper band portion of the joint stiffness matrix (stored as a rectangular array of order $N \times UBW$) is decomposed using the procedure *DECOMPOSEBAND* from Art. 2.7. If the procedure fails, control is transferred to the last statement in the program (labeled $L5C1$).

5. a. (1) A heading and a subheading for joint displacements are printed in a form sufficiently general to accommodate all types of framed structures.

(2) Joint displacements D are calculated from the decomposed stiffness matrix S in conjunction with the first N elements in the combined load vector AC, using the procedure *SOLVEBAND* (see Art. 2.8).

b. (10) Member end-actions AMD, due only to the displacements of the ends of a beam or plane truss member, are computed using Eq. (4-4).

(11) The member number I is printed, and the four end-actions for a beam or plane truss member are computed and printed on the same line. Each end-action consists of the sum of the action AML (due to loads on the restrained member) and the action AMD (due to displacements of the ends), according to Eq. (4-5).

(12) If TS is equal to 1, the contributions of member end-actions AMD for a beam are to be transferred to the reaction vector AR.
(13) If the index $J1$ corresponds to a restraint, the element $AR[J1]$ is incremented by $AMD[1]$. The calculation of AR is cumulative because more than one member may contribute to a given reaction.
(14) Similar to (13) but for $J2$.
(15) Similar to (13) but for $K1$.
(16) Similar to (13) but for $K2$.
(17) If TS is not equal to 1 in (12) above, the structure is a plane truss; and the contributions of member end-actions to support reactions must be calculated using the rotation matrix R. These calculations are determined by expression (4-6), and the process of transferring the contributions of AMD to AR follows the pattern established in (13) through (16).
(18) Control is transferred back to the statement labeled $L5$ (see part 5b(4) of Program $FR1$) for the purpose of processing the next member in the structure.
(19) The matrix SMR is computed for structures having three or six possible displacements per joint.

(20) If TS is equal to 6, control is transferred forward to the statement labeled $L5B10$.
(21) Otherwise, TS is equal to 3, 4, or 5, and the member end-actions AMD due only to the displacements of the ends of a member are computed as in (10) above.
(22) The member number I is printed, and the six member end-actions for a plane frame, a grid, or a space truss are computed and printed on the same line, as in (11) above.
(23) The process of calculating and transferring the contributions of AMD to AR is similar to that for plane trusses in (17). After the necessary transfers are completed, control is returned to $L5$.
(24) The calculation of AMD and the subsequent transfer of contributions to AR for a space frame follows the pattern established for other types of structures in (21) through (23). However, the member end-actions are printed in two lines (6 actions per line), as in Program $FR1$. After the necessary transfers are completed, control is returned to $L5$.

c. (1) A heading and a subheading for support reactions are printed in a form sufficiently general to accommodate all types of framed structures.
(2) Using the restraint list RL as a guide to indicate where restraints exist, the negatives of joint loads A and equivalent joint loads AE, corresponding to support restraints, are added to the reaction vector AR. Note that the combined load vector cannot be utilized for this purpose because it has previously been arranged in the form illustrated by Eq. (1-64).

4.5 Examples. The reader is invited to run the data for the examples in Art. 3.7 in order to assure himself that Program $FR2$ yields the same results as Program $FR1$ for those problems. It would be superfluous to reproduce these results in this article. The reader is also encouraged to check the program with other problems of his own choice, for which he knows the complete solutions. One should appreciate the fact that the computer output for a large structure with many loadings could exceed the length of this book. For this reason, only small or moderate-sized problems are presentable, and the two examples chosen for this article are designed to serve dual purposes in this chapter and the next.

Example 1: Figure 4-1a shows a plane truss which is to be analyzed for the applied loads, using Program $FR2$. An orderly system for numbering members and joints is indicated for the restrained structure in Fig. 4-1b. The joints are numbered in vertical sweeps across the structure in order to minimize the band width of the stiffness matrix.

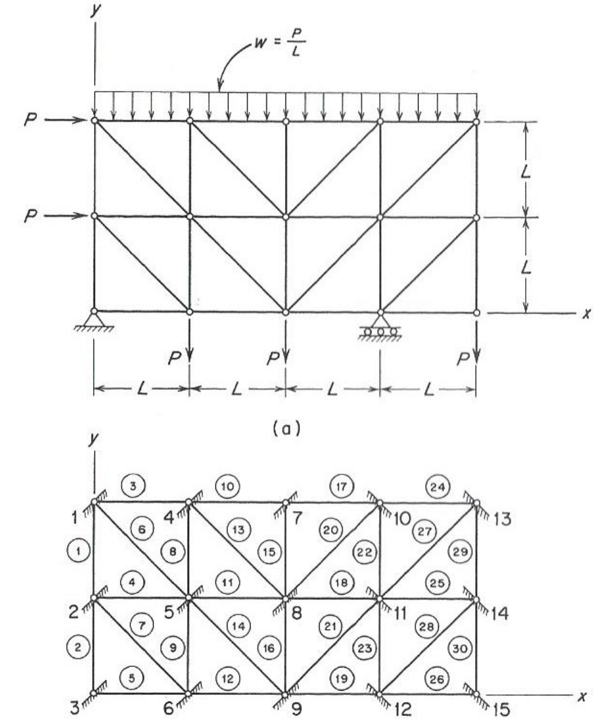


FIG. 4-1. Plane truss example

Assume that all members of the truss have the same cross-sectional area and that the numerical values in the problem are as follows:

$$E = 10{,}000 \text{ ksi} \qquad L = 100 \text{ in.} \qquad P = 100 \text{ kips} \qquad AX = 10 \text{ in.}^2$$

The structure is to be analyzed for three systems of loads. The first system consists of the loads applied at the joints, the second involves only the loads applied to the top chord members, and the third system is a combination of the first two. Input data for this example are listed in Table 4-1, which follows the format given previously in Chapter 3 (see Table 3-2). The three different loading systems in the table are identified by the numbers 1, 2, and 3 adjacent to the heading *Load Data*.

Example 2: A braced plane frame for a four-story building appears in Fig. 4-2a, and a numbering system for members and joints is indicated for the restrained structure in Fig. 4-2b. Joints are numbered story-by-story in order to minimize the band width of the stiffness matrix.

TABLE 4-1

Data Cards for Plane Truss Example

Type of Data		Numerical Data on Card					Card No.
Control Data		1	2	3			1
Structure Data	(a)	30	15	3	2	10000.0	2
	(b)	1	0	200.0			3
		2	0	100.0			4
		3	0	0			5
		4	100.0	200.0			6
		5	100.0	100.0			7
		6	100.0	0			8
		7	200.0	200.0			9
		8	200.0	100.0			10
		9	200.0	0			11
		10	300.0	200.0			12
		11	300.0	100.0			13
		12	300.0	0			14
		13	400.0	200.0			15
		14	400.0	100.0			16
		15	400.0	0			17
	(c)	1	1	2	10.0		18
		2	2	3	10.0		19
		3	1	4	10.0		20
		4	2	5	10.0		21
		5	3	6	10.0		22
		6	1	5	10.0		23
		7	2	6	10.0		24
		8	4	5	10.0		25
		9	5	6	10.0		26
		10	4	7	10.0		27
		11	5	8	10.0		28
		12	6	9	10.0		29
		13	4	8	10.0		30
		14	5	9	10.0		31
		15	7	8	10.0		32
		16	8	9	10.0		33
		17	7	10	10.0		34
		18	8	11	10.0		35
		19	9	12	10.0		36
		20	8	10	10.0		37

TABLE 4-1

(cont.)

Type of Data			Numerical Data on Card					Card No.
Structure Data (cont.)		(c)	21	9	11	10.0		38
			22	10	11	10.0		39
			23	11	12	10.0		40
			24	10	13	10.0		41
			25	11	14	10.0		42
			26	12	15	10.0		43
			27	11	13	10.0		44
			28	12	14	10.0		45
			29	13	14	10.0		46
			30	14	15	10.0		47
		(d)	3	1	1			48
			12	0	1			49
Load Data	1	(a)	5	0				50
		(b)	1	100.0	0			51
			2	100.0	0			52
			6	0	−100.0			53
			9	0	−100.0			54
			15	0	−100.0			55
	2	(a)	0	4				56
		(c)	3	0	50.0	0	50.0	57
			10	0	50.0	0	50.0	58
			17	0	50.0	0	50.0	59
			24	0	50.0	0	50.0	60
	3	(a)	5	4				61
		(b)	1	100.0				62
			2	100.0				63
			6	0	−100.0			64
			9	0	−100.0			65
			15	0	−100.0			66
		(c)	3	0	50.0	0	50.0	67
			10	0	50.0	0	50.0	68
			17	0	50.0	0	50.0	69
			24	0	50.0	0	50.0	70

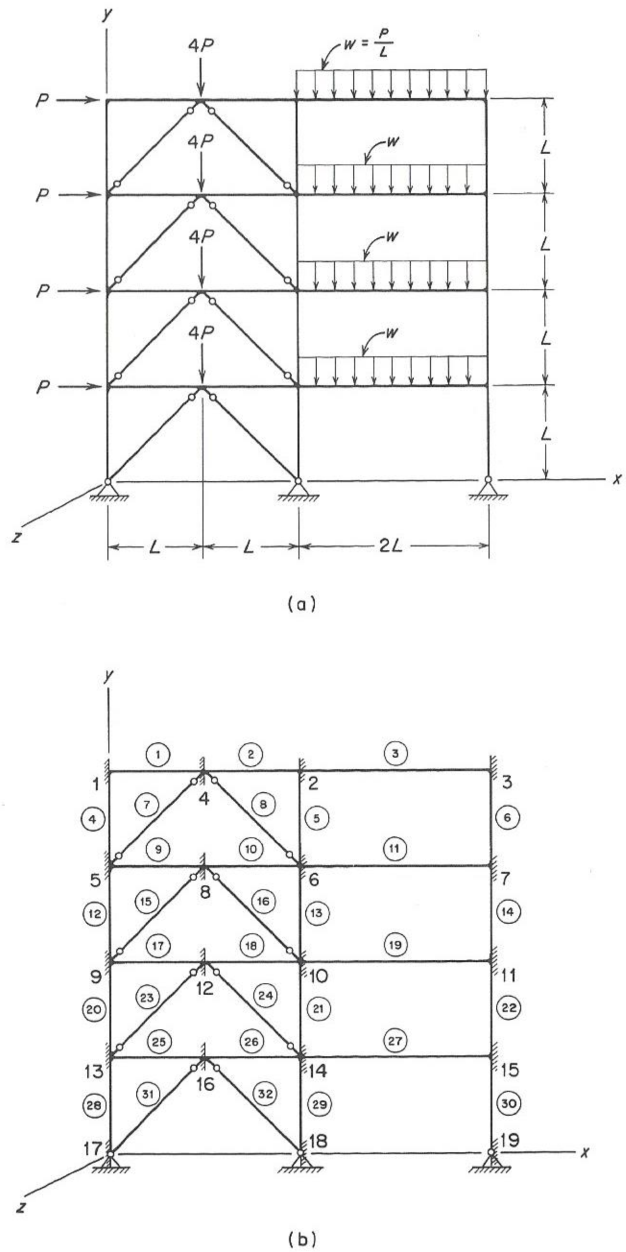


FIG. 4-2. Plane frame example

TABLE 4-2

Data Cards for Plane Frame Example

Type of Data		Numerical Data on Card					Card No.
Control Data		2	3	3			1
Structure Data	(a)	32	19	6	3	30000.0	2
	(b)	1	0	576.0			3
		2	288.0	576.0			4
		3	576.0	576.0			5
		4	144.0	576.0			6
		5	0	432.0			7
		6	288.0	432.0			8
		7	576.0	432.0			9
		8	144.0	432.0			10
		9	0	288.0			11
		10	288.0	288.0			12
		11	576.0	288.0			13
		12	144.0	288.0			14
		13	0	144.0			15
		14	288.0	144.0			16
		15	576.0	144.0			17
		16	144.0	144.0			18
		17	0	0			19
		18	288.0	0			20
		19	576.0	0			21
	(c)	1	1	4	10.0	500.0	22
		2	4	2	10.0	500.0	23
		3	2	3	10.0	500.0	24
		4	1	5	15.0	250.0	25
		5	2	6	15.0	250.0	26
		6	3	7	15.0	250.0	27
		7	5	4	5.0	0	28
		8	4	6	5.0	0	29
		9	5	8	10.0	500.0	30
		10	8	6	10.0	500.0	31
		11	6	7	10.0	500.0	32
		12	5	9	15.0	250.0	33
		13	6	10	15.0	250.0	34
		14	7	11	15.0	250.0	35
		15	9	8	5.0	0	36
		16	8	10	5.0	0	37

TABLE 4-2

(cont.)

Type of Data			Numerical Data on Card							Card No.
Structure Data (cont.)		(c)	17	9	12	10.0	500.0			38
			18	12	10	10.0	500.0			39
			19	10	11	10.0	500.0			40
			20	9	13	15.0	250.0			41
			21	10	14	15.0	250.0			42
			22	11	15	15.0	250.0			43
			23	13	12	5.0	0			44
			24	12	14	5.0	0			45
			25	13	16	10.0	500.0			46
			26	16	14	10.0	500.0			47
			27	14	15	10.0	500.0			48
			28	13	17	15.0	250.0			49
			29	14	18	15.0	250.0			50
			30	15	19	15.0	250.0			51
			31	17	16	5.0	0			52
			32	16	18	5.0	0			53
		(d)	17	1	1	0				54
			18	1	1	0				55
			19	1	1	0				56
Load Data	1	(a)	4	4						57
		(b)	4	0	−80.0	0				58
			8	0	−80.0	0				59
			12	0	−80.0	0				60
			16	0	−80.0	0				61
		(c)	3	0	20.0	960.0	0	20.0	−960.0	62
			11	0	20.0	960.0	0	20.0	−960.0	63
			19	0	20.0	960.0	0	20.0	−960.0	64
			27	0	20.0	960.0	0	20.0	−960.0	65
	2	(a)	4	0						66
		(b)	1	20.0	0	0				67
			5	20.0	0	0				68
			9	20.0	0	0				69
			13	20.0	0	0				70

TABLE 4-2

(cont.)

Type of Data			Numerical Data on Card							Card No.
Load Data (cont.)	3	(a)	8	4						71
		(b)	4	0	−80.0	0				72
			8	0	−80.0	0				73
			12	0	−80.0	0				74
			16	0	−80.0	0				75
			1	20.0	0	0				76
			5	20.0	0	0				77
			9	20.0	0	0				78
			13	20.0	0	0				79
		(c)	3	0	20.0	960.0	0	20.0	−960.0	80
			11	0	20.0	960.0	0	20.0	−960.0	81
			19	0	20.0	960.0	0	20.0	−960.0	82
			27	0	20.0	960.0	0	20.0	−960.0	83

Parameters in the problem have the following numerical values:

$$E = 30{,}000 \text{ ksi} \qquad L = 144 \text{ in.} \qquad P = 20 \text{ kips}$$

	AX(in.^{2})	IZ(in.^{4})
Beams:	10	500
Columns:	15	250
Braces:	5	0

Note that the moment of inertia IZ for the diagonal braces is set equal to zero because they are assumed to be pinned at their ends and do not transmit bending couples.

This frame is to be analyzed for the vertical and horizontal loads separately, but the two loading systems are also to be combined in a third analysis. Table 4-2 contains the required input data, which are listed in the form specified earlier in Table 3-3.

The computer output from Program $FR2$ for Examples 1 and 2 is reproduced on succeeding pages.

```
                                  ANALYSIS OF FRAMED STRUCTURES--FR2

STRUCTURE NO.  1 PLANE TRUSS

STRUCTURE DATA
  M    N   NJ   NR   NRJ       E            G
 30   27   15    3    2   10000.0

COORDINATES OF JOINTS
JOINT     X        Y          Z
   1     0.00   200.00
   2     0.00   100.00
   3     0.00     0.00
   4   100.00   200.00
   5   100.00   100.00
   6   100.00     0.00
   7   200.00   200.00
   8   200.00   100.00
   9   200.00     0.00
  10   300.00   200.00
  11   300.00   100.00
  12   300.00     0.00
  13   400.00   200.00
  14   400.00   100.00
  15   400.00     0.00

MEMBER INFORMATION
MEMBER  JJ  JK    AX          IX         IY         IZ      AA      L
    1    1   2   10.00                                            100.00
    2    2   3   10.00                                            100.00
    3    1   4   10.00                                            100.00
    4    2   5   10.00                                            100.00
    5    3   6   10.00                                            100.00
    6    1   5   10.00                                            141.42
    7    2   6   10.00                                            141.42
    8    4   5   10.00                                            100.00
    9    5   6   10.00                                            100.00
   10    4   7   10.00                                            100.00
   11    5   8   10.00                                            100.00
   12    6   9   10.00                                            100.00
   13    4   8   10.00                                            141.42
   14    5   9   10.00                                            141.42
   15    7   8   10.00                                            100.00
   16    8   9   10.00                                            100.00
   17    7  10   10.00                                            100.00
   18    8  11   10.00                                            100.00
   19    9  12   10.00                                            100.00
   20    8  10   10.00                                            141.42
   21    9  11   10.00                                            141.42
   22   10  11   10.00                                            100.00
   23   11  12   10.00                                            100.00
   24   10  13   10.00                                            100.00
   25   11  14   10.00                                            100.00
   26   12  15   10.00                                            100.00
```



```
     11  13   10.00                                              141.42
     12  14   10.00                                              141.42
     13  14   10.00                                              100.00
     14  15   10.00                                              100.00

 RESTRAINTS
 RL1  RL2  RL3  RL4  RL5  RL6
   1    1
   0    1

ING NO.  1
 NLM
   0

NS APPLIED AT JOINTS
           A1            A2            A3            A4            A5            A
       100.000         0.000
       100.000         0.000
         0.000      -100.000
         0.000      -100.000
         0.000      -100.000

 DISPLACEMENTS
           D1            D2            D3            D4            D5            D
     1.08801@+00  3.45361@-02
     7.87143@-01  3.33333@-02
     0.00000@+00  0.00000@+00
     9.89215@-01 -2.84477@-01
     7.19274@-01 -3.30800@-01
     2.00000@-01 -4.62931@-01
     9.36742@-01 -4.05457@-01
     7.37211@-01 -4.05457@-01
     3.67869@-01 -4.36105@-01
     8.84268@-01 -3.34804@-01
     6.31853@-01 -2.57832@-01
     2.92368@-01  0.00000@+00
     9.08767@-01 -6.04038@-01
     7.07354@-01 -6.28537@-01
     2.92368@-01 -7.28537@-01

R END-ACTIONS
R          AM1           AM2           AM3           AM4           AM5           AM
    -1.20274@+00  0.00000@+00  1.20274@+00  0.00000@+00
    -3.33333@+01  0.00000@+00  3.33333@+01  0.00000@+00
     9.87973@+01  0.00000@+00 -9.87973@+01  0.00000@+00
     6.78694@+01  0.00000@+00 -6.78694@+01  0.00000@+00
    -2.00000@+02  0.00000@+00  2.00000@+02  0.00000@+00
     1.70093@+00  0.00000@+00 -1.70093@+00  0.00000@+00
     4.54395@+01  0.00000@+00 -4.54395@+01  0.00000@+00
    -4.63238@+01  0.00000@+00  4.63238@+01  0.00000@+00
    -1.32131@+02  0.00000@+00  1.32131@+02  0.00000@+00
     5.24734@+01  0.00000@+00 -5.24734@+01  0.00000@+00
    -1.79374@+01  0.00000@+00  1.79374@+01  0.00000@+00
    -1.67869@+02  0.00000@+00  1.67869@+02  0.00000@+00
     6.55118@+01  0.00000@+00 -6.55118@+01  0.00000@+00
```

```
 14      1.23050@+02 0.00000@+00-1.23050@+02 0.00000@+00
 15     -3.72529@-09 0.00000@+00 3.72529@-09 0.00000@+00
 16     -3.06482@+01 0.00000@+00 3.06482@+01 0.00000@+00
 17      5.24734@+01 0.00000@+00-5.24734@+01 0.00000@+00
 18      1.05358@+02 0.00000@+00-1.05358@+02 0.00000@+00
 19      7.55014@+01 0.00000@+00-7.55014@+01 0.00000@+00
 20     -1.08855@+02 0.00000@+00 1.08855@+02 0.00000@+00
 21     -2.21128@+02 0.00000@+00 2.21128@+02 0.00000@+00
 22      7.69720@+01 0.00000@+00-7.69720@+01 0.00000@+00
 23      2.57832@+02 0.00000@+00-2.57832@+02 0.00000@+00
 24     -2.44986@+01 0.00000@+00 2.44986@+01 0.00000@+00
 25     -7.55014@+01 0.00000@+00 7.55014@+01 0.00000@+00
 26      0.00000@+00 0.00000@+00 0.00000@+00 0.00000@+00
 27      3.46462@+01 0.00000@+00-3.46462@+01 0.00000@+00
 28      1.06775@+02 0.00000@+00-1.06775@+02 0.00000@+00
 29     -2.44986@+01 0.00000@+00 2.44986@+01 0.00000@+00
 30     -1.00000@+02 0.00000@+00 1.00000@+02 0.00000@+00

SUPPORT REACTIONS
JOINT       AR1          AR2            AR3            AR4            AR5
  3     -2.00000@+02-3.33333@+01
 12      0.00000@+00 3.33333@+02

LOADING NO.  2
NLJ  NLM
  0    4

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
MEMBER        AML1          AML2           AML3           AML4           AML5
  3          0.000        50.000          0.000         50.000
 10          0.000        50.000          0.000         50.000
 17          0.000        50.000          0.000         50.000
 24          0.000        50.000          0.000         50.000

JOINT DISPLACEMENTS
JOINT         D1            D2             D3             D4             D5
  1      1.58100@-01-2.24797@-01
  2      1.24207@-01-1.33333@-01
  3      0.00000@+00 0.00000@+00
  4      1.16636@-01-5.08279@-01
  5      8.23371@-02-4.17836@-01
  6     -2.18224@-12-3.75966@-01
  7      8.47297@-02-6.08843@-01
  8      8.90403@-02-5.08843@-01
  9      4.18698@-02-4.38194@-01
 10      5.28233@-02-3.62577@-01
 11      4.72782@-02-2.23669@-01
 12     -1.12830@-03 0.00000@+00
 13      5.98252@-02-2.56020@-01
 14      9.02763@-02-2.13022@-01
 15     -1.12830@-03-2.13022@-01
```



```
[illegible] END-ACTIONS
           AM1          AM2            AM3            AM4            AM5            AM
     9.14635@+01 0.00000@+00-9.14635@+01 0.00000@+00
     1.33333@+02 0.00000@+00-1.33333@+02 0.00000@+00
     4.14635@+01 5.00000@+01-4.14635@+01 5.00000@+01
     4.18698@+01 0.00000@+00-4.18698@+01 0.00000@+00
     2.18224@-09 0.00000@+00-2.18224@-09 0.00000@+00
    -5.86382@+01 0.00000@+00 5.86382@+01 0.00000@+00
    -5.92129@+01 0.00000@+00 5.92129@+01 0.00000@+00
     9.04429@+01 0.00000@+00-9.04429@+01 0.00000@+00
     4.18698@+01 0.00000@+00-4.18698@+01 0.00000@+00
     3.19064@+01 5.00000@+01-3.19064@+01 5.00000@+01
    -6.70323@+00 0.00000@+00 6.70323@+00 0.00000@+00
    -4.18698@+01 0.00000@+00 4.18698@+01 0.00000@+00
     1.35158@+01 0.00000@+00-1.35158@+01 0.00000@+00
     1.00545@+01 0.00000@+00-1.00545@+01 0.00000@+00
     1.00000@+02 0.00000@+00-1.00000@+02 0.00000@+00
     7.06488@+01 0.00000@+00-7.06488@+01 0.00000@+00
     3.19064@+01 5.00000@+01-3.19064@+01 5.00000@+01
     4.17621@+01 0.00000@+00-4.17621@+01 0.00000@+00
     4.29981@+01 0.00000@+00-4.29981@+01 0.00000@+00
    -5.50246@+01 0.00000@+00 5.50246@+01 0.00000@+00
    -1.09967@+02 0.00000@+00 1.09967@+02 0.00000@+00
     1.38908@+02 0.00000@+00-1.38908@+02 0.00000@+00
     2.23669@+02 0.00000@+00-2.23669@+02 0.00000@+00
    -7.00187@+00 5.00000@+01 7.00187@+00 5.00000@+01
    -4.29981@+01 0.00000@+00 4.29981@+01 0.00000@+00
     3.92902@-10 0.00000@+00-3.92902@-10 0.00000@+00
     9.90214@+00 0.00000@+00-9.90214@+00 0.00000@+00
     6.08085@+01 0.00000@+00-6.08085@+01 0.00000@+00
     4.29981@+01 0.00000@+00-4.29981@+01 0.00000@+00
     1.86265@-09 0.00000@+00-1.86265@-09 0.00000@+00

RT REACTIONS
           AR1          AR2            AR3            AR4            AR5            AR
     2.18224@-09 1.33333@+02
     0.00000@+00 2.66667@+02

NO.  3
 NLM
  4

APPLIED AT JOINTS
           A1             A2            A3             A4             A5             A
        100.000         0.000
        100.000         0.000
          0.000      -100.000
          0.000      -100.000
          0.000      -100.000

AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
          AML1          AML2           AML3           AML4           AML5           AM
         0.000        50.000          0.000         50.000
         0.000        50.000          0.000         50.000
         0.000        50.000          0.000         50.000
         0.000        50.000          0.000         50.000
```

JOINT DISPLACEMENTS

JOINT	D1	D2	D3	D4	D5
1	1.24611@+00	-1.90261@-01			
2	9.11350@-01	-1.00000@-01			
3	0.00000@+00	0.00000@+00			
4	1.10585@+00	-7.92755@-01			
5	8.01611@-01	-7.48636@-01			
6	2.00000@-01	-8.38897@-01			
7	1.02147@+00	-1.01430@+00			
8	8.26252@-01	-9.14300@-01			
9	4.09739@-01	-8.74299@-01			
10	9.37092@-01	-6.97381@-01			
11	6.79131@-01	-4.81500@-01			
12	2.91240@-01	0.00000@+00			
13	9.68592@-01	-8.60058@-01			
14	7.97631@-01	-8.41558@-01			
15	2.91240@-01	-9.41558@-01			

MEMBER END-ACTIONS

MEMBER	AM1	AM2	AM3	AM4	AM5
1	9.02608@+01	0.00000@+00	-9.02608@+01	0.00000@+00	
2	1.00000@+02	0.00000@+00	-1.00000@+02	0.00000@+00	
3	1.40261@+02	5.00000@+01	-1.40261@+02	5.00000@+01	
4	1.09739@+02	0.00000@+00	-1.09739@+02	0.00000@+00	
5	-2.00000@+02	0.00000@+00	2.00000@+02	0.00000@+00	
6	-5.69373@+01	0.00000@+00	5.69373@+01	0.00000@+00	
7	-1.37734@+01	0.00000@+00	1.37734@+01	0.00000@+00	
8	4.41191@+01	0.00000@+00	-4.41191@+01	0.00000@+00	
9	-9.02608@+01	0.00000@+00	9.02608@+01	0.00000@+00	
10	8.43798@+01	5.00000@+01	-8.43798@+01	5.00000@+01	
11	-2.46406@+01	0.00000@+00	2.46406@+01	0.00000@+00	
12	-2.09739@+02	0.00000@+00	2.09739@+02	0.00000@+00	
13	7.90275@+01	0.00000@+00	-7.90275@+01	0.00000@+00	
14	1.33104@+02	0.00000@+00	-1.33104@+02	0.00000@+00	
15	1.00000@+02	0.00000@+00	-1.00000@+02	0.00000@+00	
16	4.00006@+01	0.00000@+00	-4.00006@+01	0.00000@+00	
17	8.43798@+01	5.00000@+01	-8.43798@+01	5.00000@+01	
18	1.47121@+02	0.00000@+00	-1.47121@+02	0.00000@+00	
19	1.18500@+02	0.00000@+00	-1.18500@+02	0.00000@+00	
20	-1.63879@+02	0.00000@+00	1.63879@+02	0.00000@+00	
21	-3.31095@+02	0.00000@+00	3.31095@+02	0.00000@+00	
22	2.15880@+02	0.00000@+00	-2.15880@+02	0.00000@+00	
23	4.81500@+02	0.00000@+00	-4.81500@+02	0.00000@+00	
24	-3.15004@+01	5.00000@+01	3.15004@+01	5.00000@+01	
25	-1.18500@+02	0.00000@+00	1.18500@+02	0.00000@+00	
26	-1.86265@-09	0.00000@+00	1.86265@-09	0.00000@+00	
27	4.45483@+01	0.00000@+00	-4.45483@+01	0.00000@+00	
28	1.67584@+02	0.00000@+00	-1.67584@+02	0.00000@+00	
29	1.84996@+01	0.00000@+00	-1.84996@+01	0.00000@+00	
30	-1.00000@+02	0.00000@+00	1.00000@+02	0.00000@+00	

SUPPORT REACTIONS

JOINT	AR1	AR2	AR3	AR4	AR5
3	-2.00000@+02	1.00000@+02			
12	0.00000@+00	6.00000@+02			

STRUCTURE NO. 2 PLANE FRAME

STRUCTURE DATA

M	N	NJ	NR	NRJ	E	G
32	51	19	6	3	30000.0	

COORDINATES OF JOINTS

JOINT	X	Y	Z
1	0.00	576.00	
2	288.00	576.00	
3	576.00	576.00	
4	144.00	576.00	
5	0.00	432.00	
6	288.00	432.00	
7	576.00	432.00	
8	144.00	432.00	
9	0.00	288.00	
10	288.00	288.00	
11	576.00	288.00	
12	144.00	288.00	
13	0.00	144.00	
14	288.00	144.00	
15	576.00	144.00	
16	144.00	144.00	
17	0.00	0.00	
18	288.00	0.00	
19	576.00	0.00	

MEMBER INFORMATION

MEMBER	JJ	JK	AX	IX	IY	IZ	AA	L
1	1	4	10.00			500.00		144.00
2	2	4	10.00			500.00		144.00
3	2	3	10.00			500.00		288.00
4	1	5	15.00			250.00		144.00
5	2	6	15.00			250.00		144.00
6	3	7	15.00			250.00		144.00
7	4	5	5.00			0.00		203.65
8	4	6	5.00			0.00		203.65
9	5	8	10.00			500.00		144.00
10	6	8	10.00			500.00		144.00
11	6	7	10.00			500.00		288.00
12	5	9	15.00			250.00		144.00
13	6	10	15.00			250.00		144.00
14	7	11	15.00			250.00		144.00
15	8	9	5.00			0.00		203.65
16	8	10	5.00			0.00		203.65
17	9	12	10.00			500.00		144.00
18	10	12	10.00			500.00		144.00
19	10	11	10.00			500.00		288.00
20	9	13	15.00			250.00		144.00
21	10	14	15.00			250.00		144.00
22	11	15	15.00			250.00		144.00
23	12	13	5.00			0.00		203.65
24	12	14	5.00			0.00		203.65
25	13	16	10.00			500.00		144.00

```
  26    14 16   10.00                          500.00      144.00
  27    14 15   10.00                          500.00      288.00
  28    13 17   15.00                          250.00      144.00
  29    14 18   15.00                          250.00      144.00
  30    15 19   15.00                          250.00      144.00
  31    16 17    5.00                            0.00      203.65
  32    16 18    5.00                            0.00      203.65

JOINT RESTRAINTS
JOINT RL1 RL2 RL3 RL4 RL5 RL6
 17    1   1   0
 18    1   1   0
 19    1   1   0

LOADING NO.  1
NLJ  NLM
  4    4

ACTIONS APPLIED AT JOINTS
JOINT          A1             A2              A3              A4              A5
   4         0.000        -80.000          0.000
   8         0.000        -80.000          0.000
  12         0.000        -80.000          0.000
  16         0.000        -80.000          0.000

ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
MEMBER        AML1           AML2            AML3            AML4            AML5
   3         0.000         20.000         960.000           0.000          20.000
  11         0.000         20.000         960.000           0.000          20.000
  19         0.000         20.000         960.000           0.000          20.000
  27         0.000         20.000         960.000           0.000          20.000

JOINT DISPLACEMENTS
JOINT           D1              D2              D3              D4              D5
   1       5.93321@-02-8.14321@-02-1.06721@-03
   2       5.60279@-02-1.50649@-01-9.79104@-04
   3       4.85656@-02-6.26383@-02 2.51205@-03
   4       5.82521@-02-2.19593@-01 1.51075@-04
   5       1.41162@-02-8.00394@-02-5.97598@-04
   6       4.74519@-02-1.41449@-01-5.86587@-04
   7       4.94553@-02-5.65584@-02 1.08220@-03
   8       3.12225@-02-2.00475@-01-2.37932@-05
   9      -6.84120@-03-6.59414@-02-5.01161@-04
  10       2.67754@-02-1.13413@-01-6.71380@-04
  11       2.61180@-02-4.40458@-02 1.21394@-03
  12       1.03753@-02-1.65496@-01 4.58898@-05
  13      -1.84271@-02-3.91864@-02-3.10996@-04
  14       1.66848@-02-6.62976@-02-9.69590@-04
  15       2.09311@-02-2.51544@-02 1.64984@-03
  16      -7.40270@-04-9.74181@-02 1.78942@-04
  17       0.00000@+00 0.00000@+00 3.47447@-04
  18       0.00000@+00 0.00000@+00 3.10995@-04
  19       0.00000@+00 0.00000@+00-1.04295@-03
```



```
END-ACTIONS
     AM1            AM2            AM3            AM4            AM5            AM6
 2.25002@+00 4.35229@+00 1.86460@+02-2.25002@+00-4.35229@+00 4.40269
 4.63393@+00-7.74995@+00-6.75723@+02-4.63393@+00 7.74995@+00-4.40269
 7.77321@+00 2.10002@+01 9.22195@+02-7.77321@+00 1.89998@+01-6.34143
 4.35229@+00-2.25002@+00-1.86460@+02-4.35229@+00 2.25002@+00-1.37542
 2.87501@+01-3.13928@+00-2.46472@+02-2.87501@+01 3.13928@+00-2.05585
 1.89998@+01 7.77321@+00 6.34143@+02-1.89998@+01-7.77321@+00 4.85200
 4.96967@+01 0.00000@+00 0.00000@+00-4.96967@+01 0.00000@+00 0.00000
 4.63253@+01 0.00000@+00 0.00000@+00-4.63253@+01 0.00000@+00 0.00000
-3.56381@+01 4.56301@+00 2.68766@+02 3.56381@+01-4.56301@+00 3.88308
-3.38113@+01-6.20740@+00-5.05557@+02 3.38113@+01 6.20740@+00-3.88308
-2.08686@+00 1.98981@+01 8.58412@+02 2.08686@+00 2.01019@+01-8.87756
 4.40561@+01-1.75279@+00-1.31223@+02-4.40561@+01 1.75279@+00-1.21178
 8.76126@+01-2.10676@+00-1.47270@+02-8.76126@+01 2.10676@+00-1.56103
 3.91017@+01 5.68636@+00 4.02556@+02-3.91017@+01-5.68636@+00 4.16279
 5.02445@+01 0.00000@+00 0.00000@+00-5.02445@+01 0.00000@+00 0.00000
 4.76610@+01 0.00000@+00 0.00000@+00-4.76610@+01 0.00000@+00 0.00000
-3.58677@+01 4.02532@+00 2.32839@+02 3.58677@+01-4.02532@+00 3.46808
-3.41668@+01-5.85449@+00-4.96239@+02 3.41668@+01 5.85449@+00-3.46808
 6.84790@-01 2.00660@+01 8.71314@+02-6.84790@-01 1.99340@+01-8.52299
 8.36097@+01-1.41329@+00-1.11661@+02-8.36097@+01 1.41329@+00-9.18521
 1.47234@+02-3.25700@+00-2.18972@+02-1.47234@+02 3.25700@+00-2.50035
 5.90357@+01 6.37115@+00 4.36019@+02-5.90357@+01-6.37115@+00 4.81426
 5.07852@+01 0.00000@+00 0.00000@+00-5.07852@+01 0.00000@+00 0.00000
 4.83798@+01 0.00000@+00 0.00000@+00-4.83798@+01 0.00000@+00 0.00000
-3.68475@+01 2.93716@+00 1.60440@+02 3.68475@+01-2.93716@+00 2.62510
-3.63022@+01-5.30763@+00-5.01788@+02 3.63022@+01 5.30763@+00-2.62510
-4.42323@+00 2.04281@+01 8.85218@+02 4.42323@+00 1.95719@+01-7.61925
 1.22457@+02-4.76304@-01-6.85878@+01-1.22457@+02 4.76304@-01-1.86265
 2.07180@+02-9.26349@-01-1.33394@+02-2.07180@+02 9.26349@-01-1.86265
 7.86076@+01 1.94791@+00 2.80499@+02-7.86076@+01-1.94791@+00 2.79397
 5.11242@+01 0.00000@+00 0.00000@+00-5.11242@+01 0.00000@+00 0.00000
 5.03530@+01 0.00000@+00 0.00000@+00-5.03530@+01 0.00000@+00 0.00000

REACTIONS
     AR1            AR2            AR3            AR4            AR5            AR6
 3.66265@+01 1.58608@+02 0.00000@+00
-3.46786@+01 2.42785@+02 0.00000@+00
-1.94791@+00 7.86076@+01 0.00000@+00

NO.  2
NLM
 0

APPLIED AT JOINTS
     A1             A2             A3             A4             A5             A6
   20.000         0.000          0.000
   20.000         0.000          0.000
   20.000         0.000          0.000
   20.000         0.000          0.000

DISPLACEMENTS
     D1             D2             D3             D4             D5             D6
 3.82320@-01 3.09585@-02-2.53421@-04
 3.72181@-01-2.75405@-02-1.90572@-04
```

```
  3        3.71402@-01-2.46794@-03-1.06606@-04
  4        3.72904@-01 5.26769@-03-1.93684@-04
  5        3.23705@-01 3.08331@-02-3.83700@-04
  6        3.14294@-01-2.78033@-02-2.93274@-04
  7        3.14138@-01-2.30430@-03-3.14833@-04
  8        3.10065@-01 4.81820@-03-1.36155@-04
  9        2.37714@-01 2.77022@-02-5.01241@-04
 10        2.28271@-01-2.53618@-02-3.81287@-04
 11        2.28006@-01-1.86802@-03-4.32928@-04
 12        2.19534@-01 3.79165@-03-5.57430@-05
 13        1.29901@-01 1.84293@-02-5.28754@-04
 14        1.21342@-01-1.71322@-02-3.39261@-04
 15        1.22139@-01-1.09238@-03-4.61558@-04
 16        1.06750@-01-8.68317@-04 3.17874@-05
 17        0.00000@+00 0.00000@+00-1.08875@-03
 18        0.00000@+00 0.00000@+00-1.09435@-03
 19        0.00000@+00 0.00000@+00-1.04151@-03

MEMBER END-ACTIONS
MEMBER        AM1              AM2              AM3              AM4              AM5              AM6
  1        1.96159@+01-3.91876@-01-3.44377@+01-1.96159@+01 3.91876@-01-2.199
  2        1.50617@+00 3.09953@-01 2.26407@+01-1.50617@+00-3.09953@-01 2.199
  3        8.11404@-01-5.11386@-01-7.80128@+01-8.11404@-01 5.11386@-01-6.926
  4       -3.91876@-01 3.84060@-01 3.44377@+01 3.91876@-01-3.84060@-01 2.086
  5       -8.21339@-01 6.94765@-01 5.53721@+01 8.21339@-01-6.94765@-01 4.467
  6        5.11386@-01 8.11404@-01 6.92663@+01-5.11386@-01-8.11404@-01 4.757
  7       -1.23093@+01 0.00000@+00 0.00000@+00 1.23093@+01 0.00000@+00 0.000
  8        1.33018@+01 0.00000@+00 0.00000@+00-1.33018@+01 0.00000@+00 0.000
  9        2.84166@+01-6.88093@-01-7.53286@+01-2.84166@+01 6.88093@-01-2.375
 10       -8.80972@+00 1.02641@-01-8.97641@+00 8.80972@+00-1.02641@-01 2.375
 11        1.61946@-01-8.51979@-01-1.21562@+02-1.61946@-01 8.51979@-01-1.238
 12       -9.78394@+00 6.71385@-01 5.44617@+01 9.78394@+00-6.71385@-01 4.221
 13        7.62984@+00 1.12889@+00 8.58644@+01-7.62984@+00-1.12889@+00 7.669
 14        1.36336@+00 9.73350@-01 7.62320@+01-1.36336@+00-9.73350@-01 6.393
 15       -2.57639@+01 0.00000@+00 0.00000@+00 2.57639@+01 0.00000@+00 0.000
 16        2.68822@+01 0.00000@+00 0.00000@+00-2.68822@+01 0.00000@+00 0.000
 17        3.78748@+01-9.76098@-01-1.16685@+02-3.78748@+01 9.76098@-01-2.387
 18       -1.82018@+01-1.39414@-01-4.39487@+01 1.82018@+01 1.39414@-01 2.387
 19        2.76395@-01-1.06051@+00-1.50024@+02-2.76395@-01 1.06051@+00-1.554
 20       -2.89779@+01 1.01437@+00 7.44673@+01 2.89779@+01-1.01437@+00 7.160
 21        2.57173@+01 1.65923@+00 1.17276@+02-2.57173@+01-1.65923@+00 1.216
 22        2.42387@+00 1.24974@+00 9.14727@+01-2.42387@+00-1.24974@+00 8.849
 23       -3.90606@+01 0.00000@+00 0.00000@+00 3.90606@+01 0.00000@+00 0.000
 24        4.02438@+01 0.00000@+00 0.00000@+00-4.02438@+01 0.00000@+00 0.000
 25        4.82293@+01-9.93681@-01-1.29935@+02-4.82293@+01 9.93681@-01-1.315
 26       -3.03999@+01-3.54105@-01-6.41465@+01 3.03999@+01 3.54105@-01 1.315
 27       -8.30221@-01-9.89808@-01-1.36163@+02 8.30221@-01 9.89808@-01-1.489
 28       -5.75915@+01 4.05093@-01 5.83333@+01 5.75915@+01-4.05093@-01-9.313
 29        5.35383@+01 5.46218@-01 7.86554@+01-5.35383@+01-5.46218@-01 9.313
 30        3.41368@+00 4.19524@-01 6.04114@+01-3.41368@+00-4.19524@-01 9.313
 31       -5.51470@+01 0.00000@+00 0.00000@+00 5.51470@+01 0.00000@+00 0.000
 32        5.60515@+01 0.00000@+00 0.00000@+00-5.60515@+01 0.00000@+00 0.000
```



```
REACTIONS
     AR1              AR2              AR3              AR4              AR5              AR6
  -3.93999@+01-9.65863@+01 0.00000@+00
  -4.01806@+01 9.31726@+01 0.00000@+00
  -4.19524@-01 3.41368@+00 0.00000@+00

NO.  3
NLM
 4

APPLIED AT JOINTS
      A1               A2               A3               A4               A5               A6
     0.000          -80.000            0.000
     0.000          -80.000            0.000
     0.000          -80.000            0.000
     0.000          -80.000            0.000
    20.000            0.000            0.000
    20.000            0.000            0.000
    20.000            0.000            0.000
    20.000            0.000            0.000

AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
     AML1             AML2             AML3             AML4             AML5             AML6
     0.000           20.000          960.000            0.000           20.000         -960.000
     0.000           20.000          960.000            0.000           20.000         -960.000
     0.000           20.000          960.000            0.000           20.000         -960.000
     0.000           20.000          960.000            0.000           20.000         -960.000

DISPLACEMENTS
      D1               D2               D3               D4               D5               D6
  4.41652@-01-5.04737@-02-1.32063@-03
  4.28209@-01-1.78189@-01-1.16968@-03
  4.19968@-01-6.51063@-02 2.40545@-03
  4.31156@-01-2.14325@-01-4.26084@-05
  3.37821@-01-4.92063@-02-9.81298@-04
  3.61746@-01-1.69252@-01-8.79861@-04
  3.63594@-01-5.88627@-02 7.67368@-04
  3.41288@-01-1.95656@-01-1.59948@-04
  2.30873@-01-3.82392@-02-1.00240@-03
  2.55047@-01-1.38774@-01-1.05267@-03
  2.54124@-01-4.59139@-02 7.81014@-04
  2.29910@-01-1.61705@-01-9.85316@-06
  1.11473@-01-2.07571@-02-8.39750@-04
  1.38027@-01-8.34298@-02-1.30885@-03
  1.43071@-01-2.62468@-02 1.18828@-03
  1.06010@-01-9.82864@-02 2.10730@-04
  0.00000@+00 0.00000@+00-7.41307@-04
  0.00000@+00 0.00000@+00-7.83359@-04
  0.00000@+00 0.00000@+00-2.08446@-03

END-ACTIONS
     AM1              AM2              AM3              AM4              AM5              AM6
  2.18660@+01 3.96041@+00 1.52022@+02-2.18660@+01-3.96041@+00 4.18277@+
  6.14010@+00-7.43999@+00-6.53082@+02-6.14010@+00 7.43999@+00-4.18277@+
  8.58462@+00 2.04888@+01 8.44182@+02-8.58462@+00 1.95112@+01-7.03409@+
  3.96041@+00-1.86596@+00-1.52022@+02-3.96041@+00 1.86596@+00-1.16675@+
  2.79288@+01-2.44452@+00-1.91100@+02-2.79288@+01 2.44452@+00-1.60911@+
```

```
 6     1.95112@+01 8.58462@+00 7.03409@+02-1.95112@+01-8.58462@+00 5.327
 7     3.73874@+01 0.00000@+00 0.00000@+00-3.73874@+01 0.00000@+00 0.000
 8     5.96271@+01 0.00000@+00 0.00000@+00-5.96271@+01 0.00000@+00 0.000
 9    -7.22143@+00 3.87492@+00 1.93437@+02 7.22143@+00-3.87492@+00 3.645
10    -4.26210@+01-6.10476@+00-5.14533@+02 4.26210@+01 6.10476@+00-3.645
11    -1.92491@+00 1.90461@+01 7.36850@+02 1.92491@+00 2.09539@+01-1.011
12     3.42722@+01-1.08140@+00-7.67617@+01-3.42722@+01 1.08140@+00-7.896
13     9.52424@+01-9.77862@-01-6.14057@+01-9.52424@+01 9.77862@-01-7.940
14     4.04651@+01 6.65971@+00 4.78788@+02-4.04651@+01-6.65971@+00 4.802
15     2.44806@+01 0.00000@+00 0.00000@+00-2.44806@+01 0.00000@+00 0.000
16     7.45431@+01 0.00000@+00 0.00000@+00-7.45431@+01 0.00000@+00 0.000
17     2.00714@+00 3.04922@+00 1.16154@+02-2.00714@+00-3.04922@+00 3.229
18    -5.23687@+01-5.99390@+00-5.40188@+02 5.23687@+01 5.99390@+00-3.229
19     9.61185@-01 1.90055@+01 7.21290@+02-9.61185@-01 2.09945@+01-1.007
20     5.46318@+01-3.98920@-01-3.71937@+01-5.46318@+01 3.98920@-01-2.025
21     1.72952@+02-1.59776@+00-1.01696@+02-1.72952@+02 1.59776@+00-1.283
22     6.14596@+01 7.62089@+00 5.27492@+02-6.14596@+01-7.62089@+00 5.699
23     1.17246@+01 0.00000@+00 0.00000@+00-1.17246@+01 0.00000@+00 0.000
24     8.86236@+01 0.00000@+00 0.00000@+00-8.86236@+01 0.00000@+00 0.000
25     1.13818@+01 1.94347@+00 3.05052@+01-1.13818@+01-1.94347@+00 2.493
26    -6.67022@+01-5.66173@+00-5.65934@+02 6.67022@+01 5.66173@+00-2.493
27    -5.25345@+00 1.94383@+01 7.49055@+02 5.25345@+00 2.05617@+01-9.108
28     6.48658@+01-7.12115@-02-1.02545@+01-6.48658@+01 7.12115@-02 9.313
29     2.60718@+02-3.80130@-01-5.47388@+01-2.60718@+02 3.80130@-01-4.656
30     8.20213@+01 2.36744@+00 3.40911@+02-8.20213@+01-2.36744@+00-9.313
31    -4.02281@+00 0.00000@+00 0.00000@+00 4.02281@+00 0.00000@+00 0.000
32     1.06405@+02 0.00000@+00 0.00000@+00-1.06405@+02 0.00000@+00 0.000

SUPPORT REACTIONS
JOINT       AR1         AR2         AR3          AR4          AR5          AR6
  17   -2.77334@+00 6.20213@+01 0.00000@+00
  18   -7.48592@+01 3.35957@+02 0.00000@+00
  19   -2.36744@+00 8.20213@+01 0.00000@+00
```

Chapter 5

ANALYSIS BY DIVISION OF STRUCTURE

5.1 Introduction. A structure of arbitrarily large size may be analyzed by dividing it into parts, which will be referred to as *substructures.* Such a physical idealization has a mathematical counterpart in the form of partitioned matrices. Methods of analysis that involve partitioning and condensation of the matrices associated with a divided structure are discussed in this chapter (see Art. 5.2). From among the various techniques available, the method of series elimination is chosen and explained in detail in Art. 5.3. Program *FR*3 utilizes this approach, and the remainder of the chapter is devoted to documenting that program.

Program *FR*3 will handle structures for which the required storage exceeds the available core storage. It is designed to use auxiliary storage units for the purpose of saving large blocks of information pertaining to individual substructures. When auxiliary storage is utilized by a program, the benefits to be derived from a larger capacity are partially offset by an increase in computer time. This loss of efficiency is due to the fact that access time is typically much greater for auxiliary storage than for core storage. Therefore, Program *FR*3 requires much more time to perform a given analysis than would be required by Program *FR*2. For this reason the program in this chapter should be applied only to the analysis of structures that are too large to be accommodated within the core of the computer.

5.2 Methods of Matrix Condensation. When a structure contains too many members and joints to be analyzed directly, it may be divided into substructures of smaller size that can be handled more conveniently. The structural analyst unconsciously does this when he considers a framed structure to be composed of a number of individual members. A substructure is analogous to a single member, but it is usually taken to be a larger unit consisting of a group (or subassemblage) of members. The interactions between such groups of members at connection points play a role which is similar to the interactions of individual members framing into the joints. Equilibrium equations for the connection points are solved for unknown displacements that are common to two or more substructures framing into those points. Methods for formulating condensed stiffness and load matrices for such equilibrium equations are briefly described in this article.

Assume that a given structure is divided into two substructures by introducing *temporary connection restraints* at nodes that join the two parts. Figure 5-1 shows three of the many possible choices for dividing the plane truss example of Art. 4.5 into two portions. (Numbers enclosed in