Inference with the CLT

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The Central Limit Theorem

For a population with a well-defined mean μ and standard deviation σ, these three properties hold for the distribution of sample average $\bar{X}$, assuming certain conditions hold:

✅ The distribution of the sample statistic is nearly normal

✅ The distribution is centered at the (often unknown) population parameter

✅ The variability of the distribution is inversely proportional to the square root of the sample size

Why do we care?

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Why do we care?

Knowing the distribution of the sample statistic $\bar{X}$ can help us

- estimate a population parameter as point **estimate $\pm$ margin of error**
  - the margin of error is comprised of a measure of how confident we want to be and how variable the sample statistic is
- test for a population parameter by evaluating how likely it is to obtain to observed sample statistic when assuming that the null hypothesis is true
  - this probability will depend on how variable the sampling distribution is

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Then the CLT tells us that $\bar{X}$ approximately has the distribution $N\left(\mu, \sigma/\sqrt{n}\right)$.

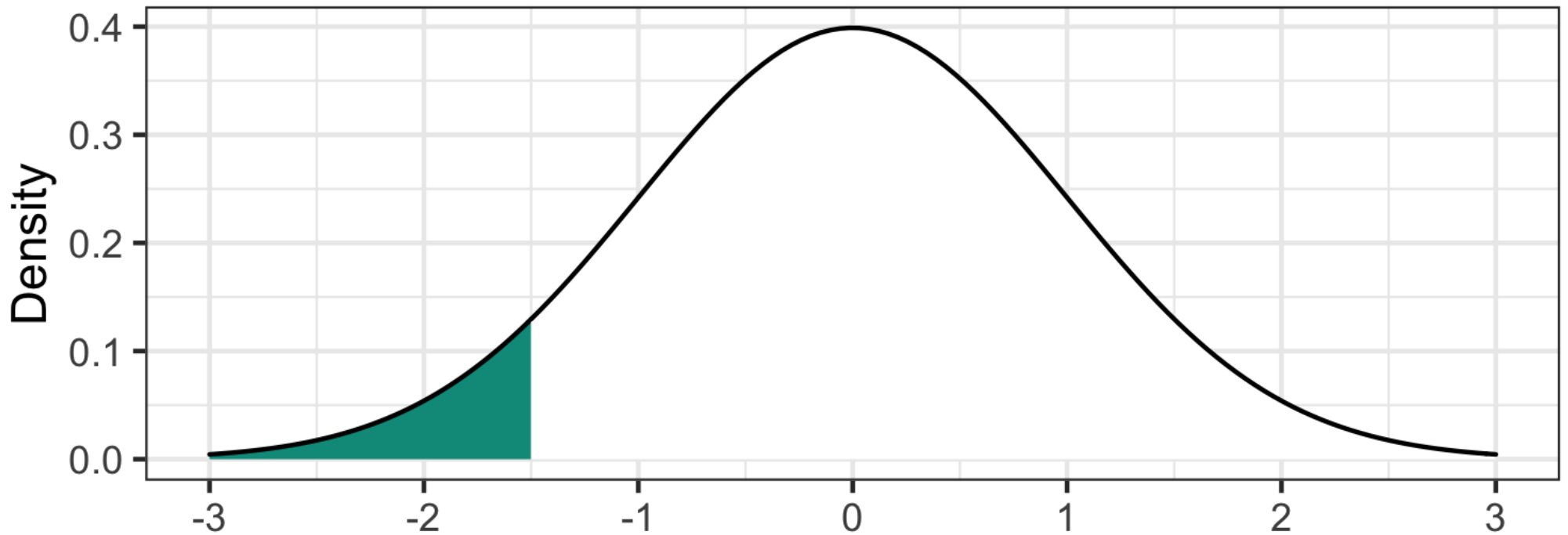
That is,

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Probabilities under N(0,1) curve

```
# P(Z < -1.5)
pnorm(-1.5)
```

```
## [1] 0.0668072
```



Probability between two values

If $Z \sim N(0, 1)$, what is $P(-1 < Z < 2)$?

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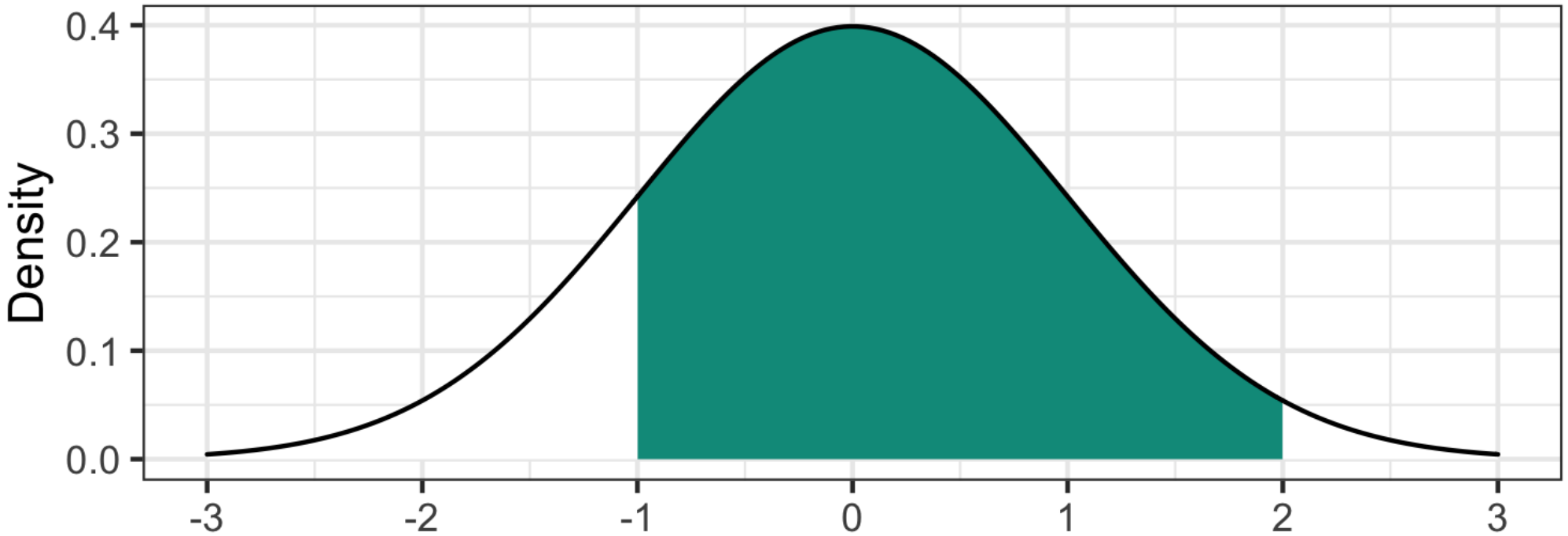

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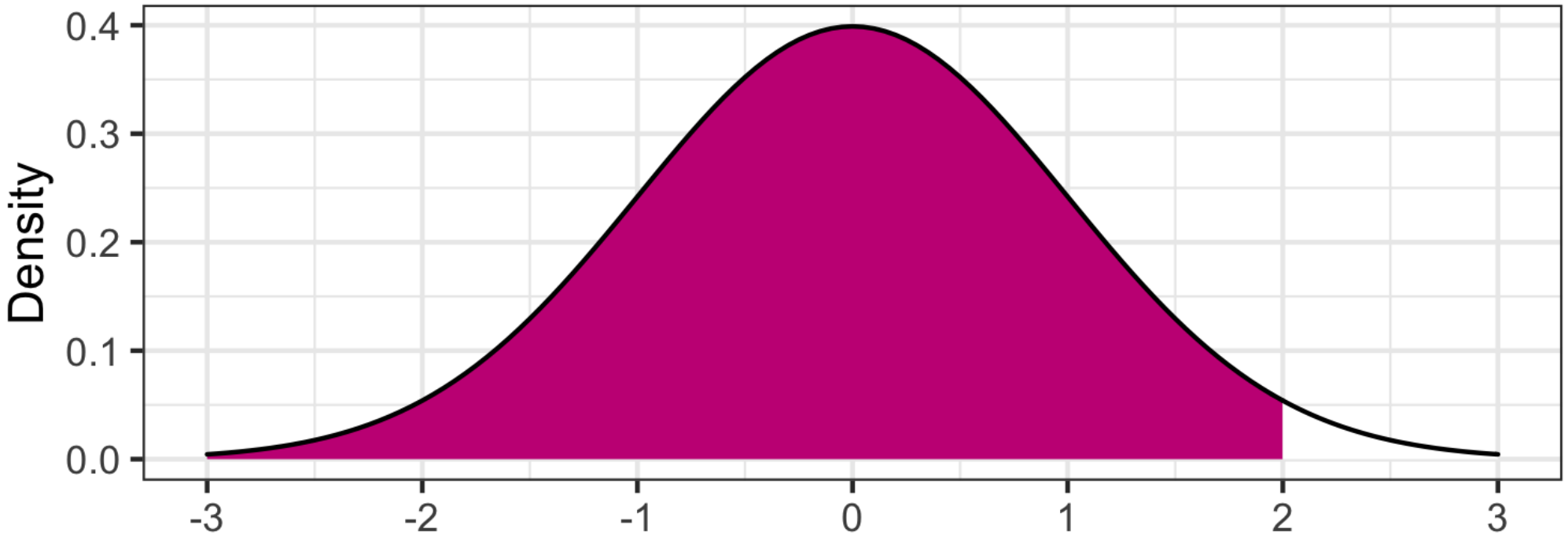

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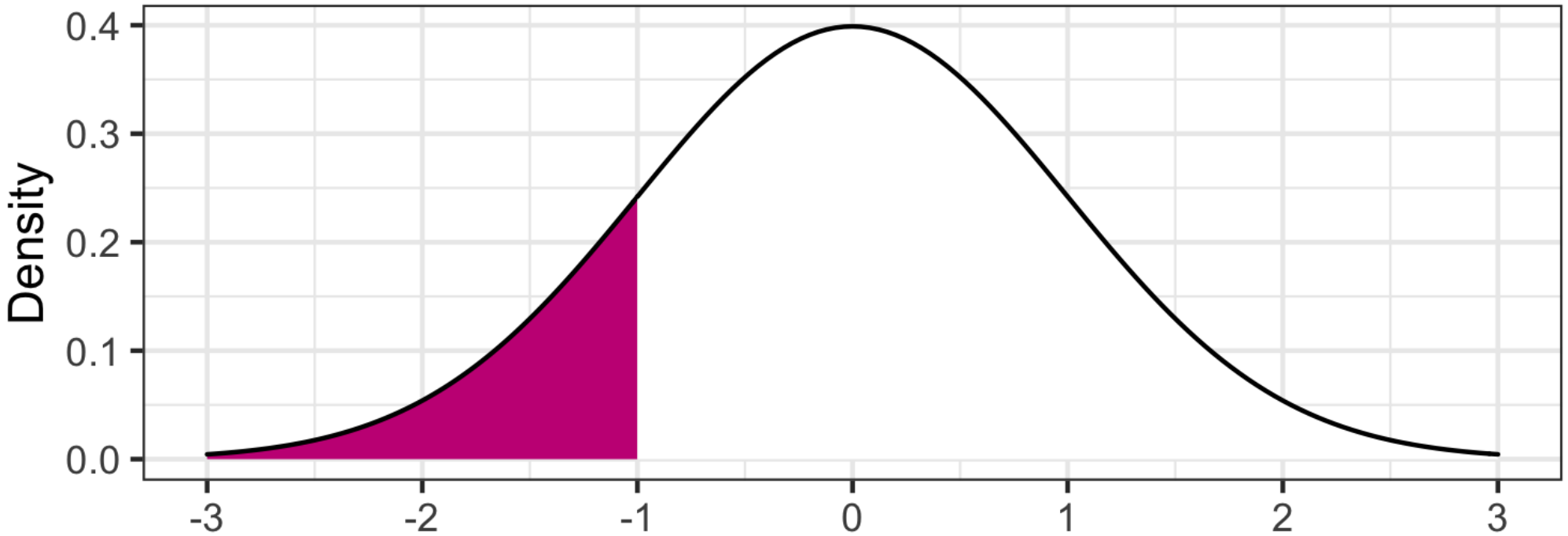

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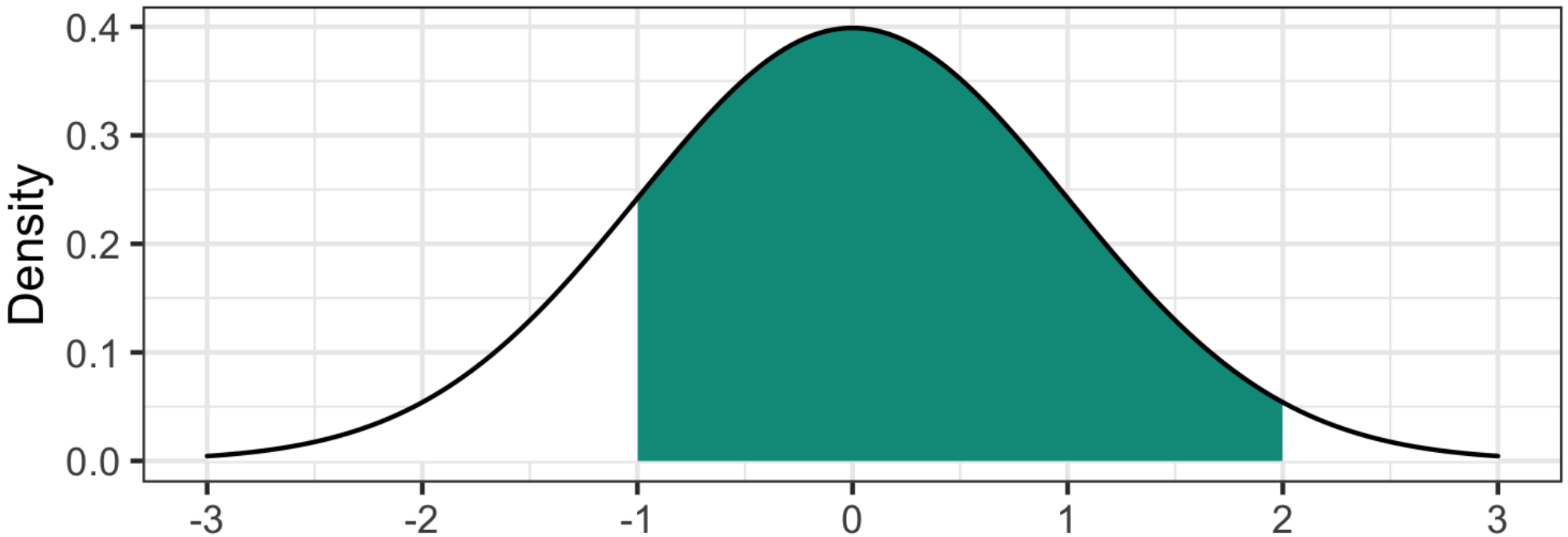


Probability between two values

If $Z \sim N(0, 1)$, what is $P(-1 < Z < 2)$?

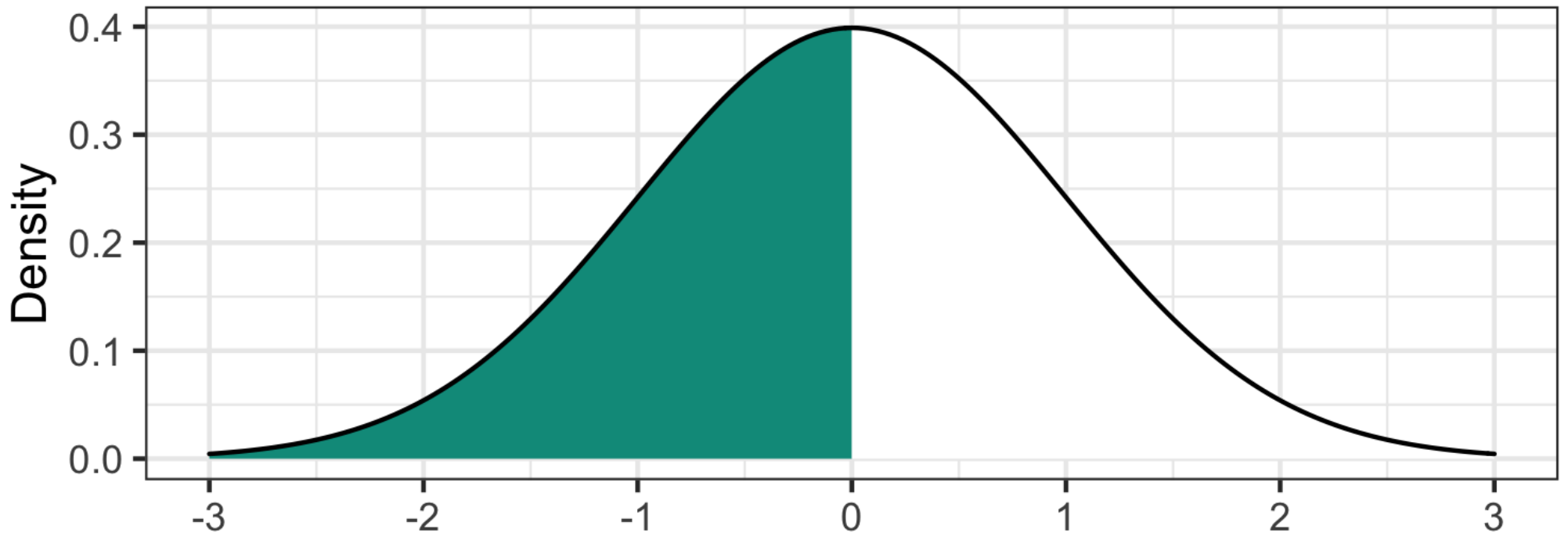
```
pnorm(2) - pnorm(-1)
```

```
## [1] 0.8185946
```

Finding cutoff values under N(0,1) curve

```
# find the median, Q2
qnorm(0.5)
```

```
## [1] 0
```



What if σ isn't known?

T distribution

- In practice, we never know the true value of σ, and so we estimate it from our data with s.

We can make the following test statistic for testing a single sample's population mean, which has a **t-distribution with n-1 degrees of freedom**:

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$

T distribution

The t-distribution is also unimodal and symmetric, and is centered at 0

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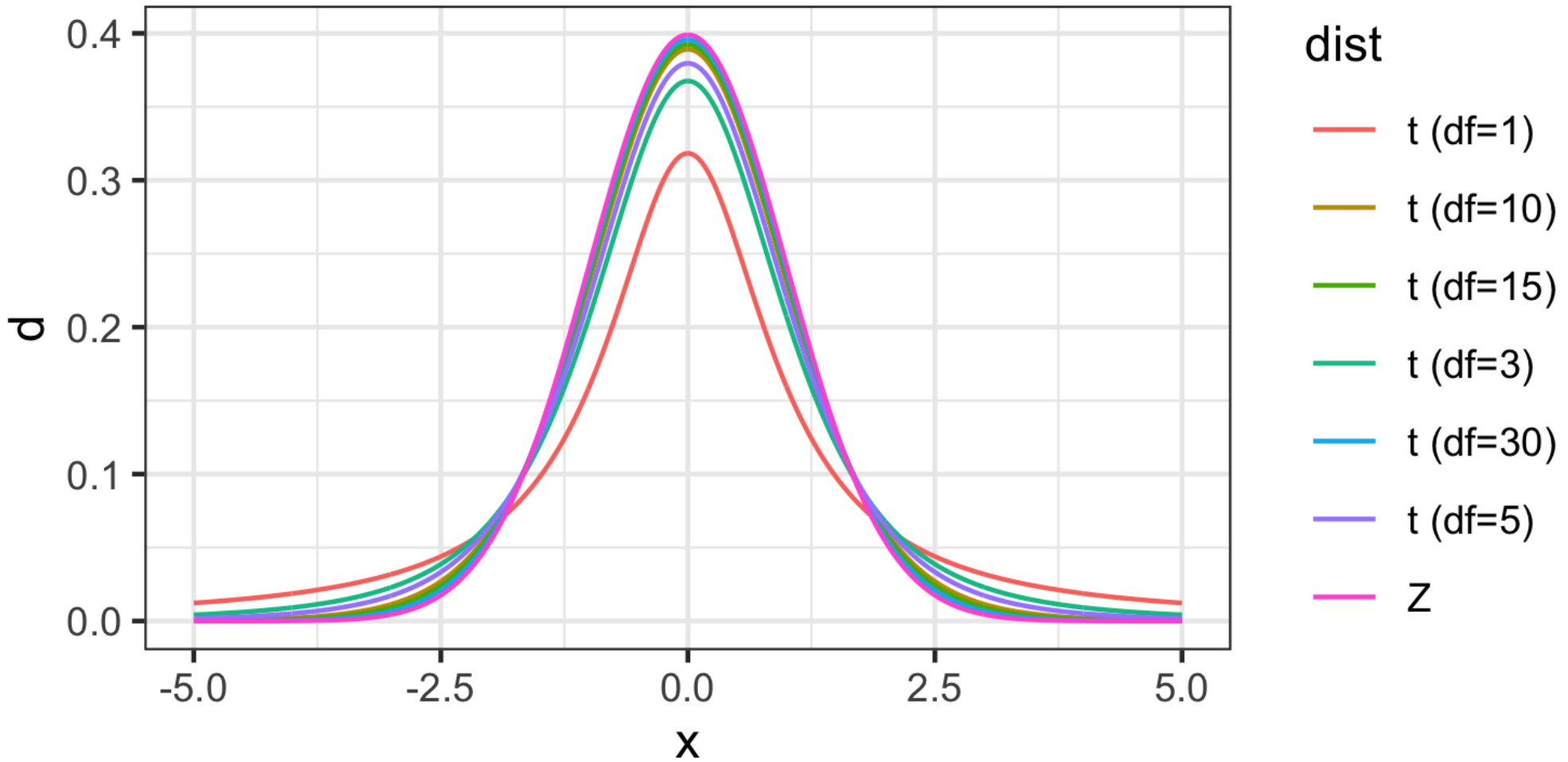
Thicker tails than the normal distribution

- This is to make up for additional variability introduced by using s instead of σ in calculation of the SE

T vs Z distributions



T distribution

Finding probabilities under the t curve:

```
#P(t < -1.96)
pt(-1.96, df = 9)
```

```
## [1] 0.0408222
```

```
#P(t > -1.96)
pt(-1.96, df = 9,
   lower.tail = FALSE)
```

```
## [1] 0.9591778
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```

```
## [1] 0.9591778
```

Finding cutoff values under the t curve:

```
# Find Q1
qt(0.25, df = 9)
```

```
## [1] -0.7027221
```

```
# Q3
qt(0.75, df = 9)
```

```
## [1] 0.7027221
```

Resident satisfaction in Durham

`durham_survey` contains resident responses to a survey given by the City of Durham in 2018. These are a randomly selected, representative sample of Durham residents.

Questions were rated 1 - 5, with 1 being "highly dissatisfied" and 5 being "highly satisfied."

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Questions were rated 1 - 5, with 1 being "highly dissatisfied" and 5 being "highly satisfied."

Is there evidence that, on average, Durham residents are generally satisfied (score greater than 3) with the quality of the public library system?

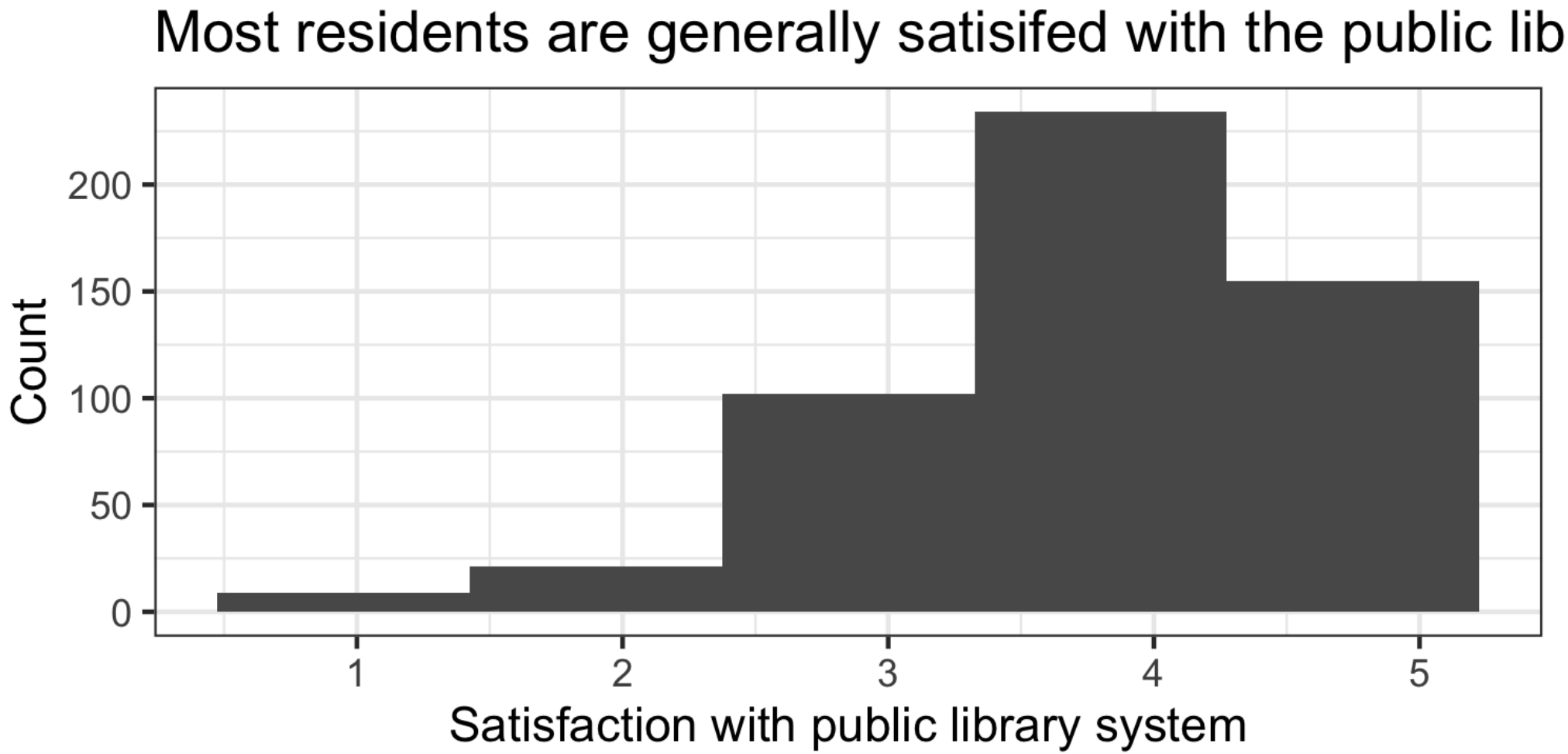
Exploratory Data Analysis

```
durham <- read_csv("data/durham_survey.csv") %>%
  filter(quality_library != 9)
```

```
durham %>%
  summarise(x_bar = mean(quality_library),
            med = median(quality_library),
            sd = sd(quality_library), n = n())
```

```
## # A tibble: 1 x 4
##   x_bar   med    sd     n
##   <dbl> <dbl> <dbl> <int>
## 1  3.97     4 0.900   521
```

Exploratory Data Analysis



Hypotheses

What are the hypotheses for evaluating if Durham residents, on average, are generally satisfied with the public library system?

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$$H_0 : \mu = 3$$

$$H_a : \mu > 3$$

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What conditions must be satisfied to conduct this hypothesis test using methods based on the CLT? Are these conditions satisfied?

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✅ The residents were randomly selected for the survey, and 521 is less than 10% of the Durham population (~ 270,000).

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Sample size / distribution?

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What conditions must be satisfied to conduct this hypothesis test using methods based on the CLT? Are these conditions satisfied?

Independence?

✅ The residents were randomly selected for the survey, and 521 is less than 10% of the Durham population (~ 270,000).

Sample size / distribution?

✅ 521 > 30, so the sample is large enough to apply the Central Limit Theorem.

Calculating the test statistic

Summary statistics from the sample:

```
## # A tibble: 1 x 3
##    xbar     s     n
##   <dbl> <dbl> <int>
## 1  3.97 0.900   521
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Calculating the test statistic

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## # A tibble: 1 x 3
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And the CLT says:

$$\bar{x} \sim N\left(mean = \mu, SE = \frac{\sigma}{\sqrt{n}}\right)$$

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How many standard errors away from the population mean is the observed sample mean?

Calculations

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```
(se <- durham_summary$s / sqrt(durham_summary$n)) # SE
```

```
## [1] 0.03944416
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(t <- (durham_summary$xbar - 3) / se) # Test statistic
```

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## [1] 24.57372
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```
(df <- durham_summary$n - 1) # Degrees of freedom
```

```
## [1] 520
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Calculations

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(t <- (durham_summary$xbar - 3) / se) # Test statistic
```

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## [1] 24.57372
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```
(df <- durham_summary$n - 1) # Degrees of freedom
```

```
## [1] 520
```

```
pt(t, df, lower.tail = FALSE) # P-value, P(T > t |H_0 true)
```

```
## [1] 2.247911e-89
```

Conclusion

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The data provide sufficient evidence at the $\alpha = 0.05$ level that Durham residents, on average, are satisfied with the quality of the public library system.

Would you expect a 95% confidence interval to include 3?

Confidence interval for a mean

General form of the confidence interval

$$point\ estimate \pm critical\ value \times SE$$

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Confidence interval for the mean

$$\bar{x} \pm t^*_{n-1} \times \frac{s}{\sqrt{n}}$$

Calculate 95% confidence interval

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```
# Critical value
t_star <- qt(0.975, df)
```

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```

```
# Point estimate
point_est <- durham_summary$xbar
```

Calculate 95% confidence interval

$$\bar{x} \pm t^*_{n-1} \times \frac{s}{\sqrt{n}}$$

```
# Critical value
t_star <- qt(0.975, df)
```

```
# Point estimate
point_est <- durham_summary$xbar
```

```
# Confidence interval
round(point_est + c(-1,1) * t_star * se, 2)
```

```
## [1] 3.89 4.05
```

Interpret 95% confidence interval

```
round(point_est + c(-1,1) * t_star * se, 2)
```

```
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```

Interpret this interval in context of the data.

Interpret 95% confidence interval

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round(point_est + c(-1,1) * t_star * se, 2)
```

```
## [1] 3.89 4.05
```

Interpret this interval in context of the data.

We are 95% confident that the true mean rating for Durham residents' satisfaction with the library system is between 3.89 and 4.05.

Inference with the CLT using `infer`

CLT-based hypothesis testing in `infer`

$$H_0 : \mu = 3 \text{ vs } H_a : \mu > 3$$

CLT-based hypothesis testing in `infer`

$$H_0 : \mu = 3 \text{ vs } H_a : \mu > 3$$

```
durham %>%
  t_test(response = quality_library,
         mu = 3,
         alternative = "greater",
         conf_int = FALSE)
```

```
## # A tibble: 1 x 4
##   statistic  t_df  p_value alternative
##       <dbl> <dbl>    <dbl> <chr>
## 1      24.6   520 2.25e-89 greater
```

CLT-based confidence intervals in `infer`

Calculate a 95% confidence interval for the mean satisfaction rating.

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```
durham %>%
  t_test(response = quality_library,
         alternative = "two-sided",
         conf_int = TRUE, conf_level = 0.95)
```

```
## # A tibble: 1 x 6
##   statistic  t_df p_value alternative lower_ci upper_ci
##       <dbl> <dbl>   <dbl> <chr>          <dbl>    <dbl>
## 1      101.   520       0 two.sided       3.89     4.05
```

Other built-in functionality in R

- There are more built in functions for doing some of these tests in R.
- However a learning goal is this course is not to go through an exhaustive list of all CLT based tests and how to implement them
- Instead the goal is to understand how these methods are / are not like the simulation based methods we learned about earlier

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- Instead the goal is to understand how these methods are / are not like the simulation based methods we learned about earlier

What is similar, and what is different, between CLT based test of means vs. simulation based test?