PROBLEM SET 10 - MATH 362 - FALL 2022
SOLUTIONS

Exercise 1

$$SSTO = \sum_{i=1}^{m}\sum_{j=1}^{n_i}(X_{ij}-\overline{X}_{..})^2$$

a) WTS $SSTO = \sum_{i=1}^{m}\sum_{j=1}^{n_i} X_{ij}^2 - n\overline{X}_{..}^2$

$$\begin{aligned}
SSTO &= \sum_i\sum_j (X_{ij}-\overline{X}_{..})^2 \\
&= \sum_i\sum_j (X_{ij}^2 - 2X_{ij}\overline{X}_{..} + \overline{X}_{..}^2) \\
&= \sum_i\sum_j X_{ij}^2 - 2\overline{X}_{..}\sum_i\sum_j X_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n_i}\overline{X}_{..}^2 \\
&= \sum\sum X_{ij}^2 - 2\overline{X}_{..}(n\overline{X}_{..}) + \overline{X}_{..}^2\sum_{i=1}^{m} n_i \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n_i} X_{ij}^2 - 2n\overline{X}_{..}^2 + n\overline{X}_{..}^2 \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n_i} X_{ij}^2 - n\overline{X}_{..}^2 \qquad //
\end{aligned}$$

$\overline{X}_{..} = \frac{1}{n}\sum_{i=1}^{m}\sum_{j=1}^{n_i} X_{ij}$

$n\overline{X}_{..} = \sum\sum X_{ij}$

b) WTS $SST \equiv \sum_{i=1}^{m}\sum_{j=1}^{n_i}(\overline{X}_{i.}-\overline{X}_{..})^2 = \sum_{i=1}^{m} n_i\overline{X}_{i.}^2 - n\overline{X}_{..}^2$

$$\begin{aligned}
SST &= \sum_{i=1}^{m}\sum_{j=1}^{n_i}(\overline{X}_{i.}-\overline{X}_{..})^2 \\
&= \sum_{i=1}^{m} n_i(\overline{X}_{i.}^2 - 2\overline{X}_{i.}\overline{X}_{..} + \overline{X}_{..}^2) \\
&= \sum_i n_i\overline{X}_{i.}^2 - 2\overline{X}_{..}\sum_i \underbrace{n_i\overline{X}_{i.}}_{①} + n\overline{X}_{..}^2 \\
&= \sum_i n_i\overline{X}_{i.}^2 - 2\overline{X}_{..}\sum_i^{m}\sum_{j=1}^{n_i} X_{ij} + n\overline{X}_{..}^2 \\
&= \sum_i n_i\overline{X}_{i.}^2 - 2\overline{X}_{..}\, n\overline{X}_{..} + n\overline{X}_{..}^2 \\
&= \sum_{i=1}^{m} n_i\overline{X}_{i.}^2 - n\overline{X}_{..}^2 \qquad //
\end{aligned}$$

$\overline{X}_{i.} = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$

① $\underline{n_i\overline{X}_{i.}} = \sum_{j=1}^{n_i} X_{ij}$

c) WTS $SSTO = SST + SSE$

$$SSTO = \sum_{i=1}^{m}\sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{..})^2$$

$$SST = \sum_{i=1}^{m}\sum_{j=1}^{n_i} (\bar{X}_{i.} - \bar{X}_{..})^2$$

$$SSE = \sum_{i=1}^{m}\sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.})^2$$

$$SSTO = \sum_{i=1}^{m}\sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{..})^2 = \sum_i\sum_j\Big[(X_{ij} \underbrace{- \bar{X}_{i.}) + (\bar{X}_{i.}}_{0} - \bar{X}_{..})\Big]^2$$

$$= \sum_i\sum_j\Big[(X_{ij} - \bar{X}_{i.})^2 + 2(X_{ij} - \bar{X}_{i.})(\bar{X}_{i.} - \bar{X}_{..}) + (\bar{X}_{i.} - \bar{X}_{..})^2\Big]$$

$$= \underbrace{\sum_i\sum_j (X_{ij} - \bar{X}_{i.})^2}_{①} + \underbrace{2\sum_i\sum_j(X_{ij} - \bar{X}_{i.})(\bar{X}_{i.} - \bar{X}_{..})}_{②} + \underbrace{\sum_i\sum_j(\bar{X}_{i.} - \bar{X}_{..})^2}_{③}$$

① is SSE, ③ is SST, so just need to show ② reduces to 0

② $2\sum_i\sum_j (X_{ij} - \bar{X}_{i.})(\bar{X}_{i.} - \bar{X}_{..}) = 2\sum_i (\bar{X}_{i.} - \bar{X}_{..})\underbrace{\sum_j (X_{ij} - \bar{X}_{i.})}_{③}$

③ $\sum_j (X_{ij} - \bar{X}_{i.}) = \sum_j X_{ij} - \sum_j \bar{X}_{i.}$

$= \sum_j X_{ij} - n_i \bar{X}_{i.}$

$= \sum_j X_{ij} - n_i \sum_j \frac{X_{ij}}{n_i} = 0$

$\Rightarrow$ ② $= 0$

$\Rightarrow SSTO = SSE + SST$ //

Exercise 2

$$MSE = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_{i\cdot})^2}{n-m} \qquad \text{WTS}: E(MSE)=\sigma^2$$

$$E(MSE) = \frac{1}{n-m} E\left[\sum_i\sum_j (X_{ij}-\bar{X}_{i\cdot})^2\right]$$

$$= \frac{1}{n-m} E\left[\sum_i\sum_j X_{ij}^2 - 2\underbrace{\sum_i\sum_j X_{ij}\bar{X}_{i\cdot}}_{①} + \sum_i\sum_{j=1}^{n_i}\bar{X}_{i\cdot}^2\right]$$

$$= \frac{1}{n-m} E\left[\sum_i\sum_j X_{ij}^2 - 2\sum_i n_i\bar{X}_{i\cdot}^2 + \sum_i n_i\bar{X}_{i\cdot}^2\right]$$

$$= \frac{1}{n-m} E\left[\sum_i\sum_j X_{ij}^2 - \sum_i n_i\bar{X}_{i\cdot}^2\right]$$

$$= \frac{1}{n-m}\left[\sum_i\sum_j \underbrace{E(X_{ij}^2)}_{②} - \sum_i n_i \underbrace{E(\bar{X}_{i\cdot}^2)}_{③}\right]$$

$$= \frac{1}{n-m}\left[\sum_i\sum_j (\sigma^2+\mu_i^2) - \sum_i n_i\left(\frac{\sigma^2}{n_i}+\mu_i^2\right)\right]$$

$$= \frac{1}{n-m}\left[\sum_i n_i(\sigma^2+\mu_i^2) - \sum_i \sigma^2 - \sum_i n_i\mu_i^2\right]$$

$$= \frac{1}{n-m}\left[\sigma^2\sum_{i=1}^{m} n_i + \cancel{\sum_i n_i\mu_i^2} - \sigma^2\sum_{i=1}^{m} 1 - \cancel{\sum_i n_i\mu_i^2}\right]$$

$$= \frac{1}{n-m}\left[n\sigma^2 - m\sigma^2\right]$$

$$= \frac{\sigma^2(n-m)}{n-m} = \sigma^2 \qquad //$$

① $\sum_i\sum_j X_{ij}\bar{X}_{i\cdot}$

$$= \sum_{i=1}^{m}\bar{X}_{i\cdot}\underbrace{\sum_{j=1}^{n_i}X_{ij}}$$

$$= \sum_{i=1}^{m}\bar{X}_{i\cdot}\,\underbrace{n_i\bar{X}_{i\cdot}}$$

$$= \sum_{i=1}^{m} n_i\bar{X}_{i\cdot}^2$$

② $E(X_{ij}^2) = V(X_{ij}) + (E(X_{ij}))^2$

$= \sigma^2 + \mu_i^2$

③ $E(\bar{X}_{i\cdot}^2) = V(\bar{X}_{i\cdot}) + E(\bar{X}_{i\cdot})^2$

$= \frac{\sigma^2}{n_i} + \mu_i^2$

Exercise 3

$$MST = \frac{\sum_{i=1}^{m}(n_i \bar{X}_{i.}^2) - n\bar{X}_{..}^2}{m-1}$$

a) WTS: $E(MST) = \sigma^2 + \frac{1}{m-1}\left[\sum n_i\mu_i^2 - n\bar{\mu}^2\right]$, where $\bar{\mu} = \frac{1}{n}\sum_{i=1}^{m} n_i\mu_i$

$$E(MST) = \frac{1}{m-1} E\left[\sum_i (n_i\bar{X}_{i.}^2) - n\bar{X}_{..}^2\right]$$

$$= \frac{1}{m-1}\left[\sum_i n_i \underbrace{E(\bar{X}_{i.}^2)}_{①} - n\underbrace{E(\bar{X}_{..}^2)}_{②}\right]$$

$$= \frac{1}{m-1}\left[\sum_i n_i\left(\frac{\sigma^2}{n_i} + \mu_i^2\right) - n\left(\frac{\sigma^2}{n} + \bar{\mu}^2\right)\right]$$

$$= \frac{1}{m-1}\left[\sum_i \sigma^2 + \sum_i n_i\mu_i^2 - \sigma^2 - n\bar{\mu}^2\right]$$

$$= \frac{1}{m-1}\left[m\sigma^2 - \sigma^2 + \sum n_i\mu_i^2 - n\bar{\mu}^2\right]$$

$$= \sigma^2 + \frac{\sum n_i\mu_i^2 - n\bar{\mu}^2}{m-1} \quad //$$

① $E(\bar{X}_{i.}^2) = \frac{\sigma^2}{n_i} + \mu_i^2$ from Exercise 2

② $E(\bar{X}_{..}^2) = V(\bar{X}_{..}) + (E(\bar{X}_{..}))^2$

$$V(\bar{X}_{..}) = V\left(\sum\sum \frac{X_{ij}}{n}\right)$$

$$= \frac{1}{n^2} V\left(\sum\sum X_{ij}\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{m}\sum_{j=1}^{n_i} V(X_{ij})$$

$$= \frac{1}{n^2}\sum_i n_i\sigma^2 = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$E(\bar{X}_{..}) = \frac{\sum\sum E(X_{ij})}{n} = \sum_i\sum_j \frac{\mu_i}{n}$$

$$= \sum_i \frac{n_i\mu_i}{n} = \bar{\mu}$$

$$\therefore ② = \frac{\sigma^2}{n} + \bar{\mu}^2$$

b) When $\mu_i = \bar{\mu}$ for all $i = 1, \ldots m$, then

$$\sum_i n_i\mu_i^2 = \sum_i n_i\bar{\mu}^2 = \bar{\mu}^2\sum n_i = n\bar{\mu}^2$$

$$\therefore \frac{\sum n_i\mu_i^2 - n\bar{\mu}^2}{m-1} = 0$$

$$\Longrightarrow E(MST) = \sigma^2$$

c) When $H_0: \mu_1 = \mu_2 = \cdots = \mu_m$ is true, $E(MST) = \sigma^2$, so $E\left(\frac{MST}{MSE}\right) \approx \frac{\sigma^2}{\sigma^2} = 1$, but when H_0 is not true, $E(MST) > \sigma^2$, so $F = \frac{MST}{MSE}$ will grow large and we can reject H_0 for large values of F.