Chapter 5 Part 2

STAT 5700: Probability

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ADD 5.8

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5.11 Conditional Expectation

Conditional distributions can be used to computed probabilities and expected values just as with any distribution:

$$P(a < Y < b|X = x) = \sum_{y=a}^{b} h(y|x)$$

$$E[u(Y)|X = x] = \sum_{y} u(y)h(y|x)$$

For example, the **conditional mean** and **conditional variance** of Y given $X = x$ is defined as

$$\mu_{Y|x} = E(Y|x) = \sum_{y} yh(y|x)$$

$$\sigma^2_{Y|x} = V(Y|x) = \sum_{y} (y - \mu_{Y|x})^2 h(y|x)$$

The "shortcut" formula also applies: $\sigma^2_{Y|x} = E[Y^2|x] - [E(Y|x)]^2$

[ADAPT FOR MEAN/VARIANCE ONLY] ::: {.activitybox data-latex=" "} **Example:**

Let

$$f(x, y) = \frac{x + y}{21}, \qquad x = 1, 2, 3, \qquad y = 1, 2.$$

In 4.1 we found marginal pmf's:

$$f_x(x) = \frac{2x + 3}{21}, \qquad x = 1, 2, 3, \qquad \text{and} \quad f_y(y) = \frac{y + 2}{7}, \qquad y = 1, 2.$$

1. Produce a table for the joint/marginal pmfs
2. Find the conditional pmfs of X given $Y = y$ and add a table for this
3. Find the conditional mean and variance of X when $Y = 2$.
4. Find the conditional pmfs of Y given $X = x$ and add a table for this
5. Find the conditional mean and variance of Y when $X = 2$.

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[ADAPT FOR JUST MEAN/VARIANCE]

Example:
From Hogg, Tanis, and Zimmerman: Exercise 4.3-9
Let X and Y have a uniform distribution on the set of points with integer coordinates in $S = \{(x, y) : 0 \le x \le 7, \quad x \le y \le x + 2\}$. That is, $f(x, y) = 1/24, \quad (x, y) \in S$, and both x and y are integers. Find:

1. $f_X(x)$ and $f_Y(y)$
2. $h(y|x)$
3. $E(Y|x)$
4. $\sigma^2_{Y|x}$

UPDATE Chapter 5 Group Work

Problem 1

Suppose that you randomly select a student from a large population of 4th grade students and record the student's sex and number of siblings. Let M = 1 if the selected student is male and M = 0 if the selected student is female. Let S record the student's number of siblings. The proportion of the population of students falling into each category of M and S is recorded in the table below.

# of siblings:	0	1	2	3	4
Female	.10	.18	.12	.07	.04
Male	.12	.18	.14	.03	.02

a. What is the support of M? Of S?
b. Find the probability that the randomly selected student will be male with two siblings
c. Find $P(M = 0, S \leq 2)$
d. Find the probability that the randomly selected student will be an only child
e. Find the marginal distribution of M
f. Find the marginal distribution of S
g. Are M and S independent?

Problem 2

Let the joint pmf of X and Y be

$$f(x, y) = \frac{xy^2}{30}, \quad x = 1, 2, 3, \quad y = 1, 2$$

a. What are the marginal distributions of X and Y?
b. Are X and Y independent?
c. What is $P(X > 1)$?
d. What is $E(X)$?

Problem 3

Let X and Y be two continuous random variables with joint pdf $f(x, y) = \frac{3}{16}xy^2, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 2$. Are the two random variables independent?

Problem 4

[HTZ 4.4-2 - homework problem!] Let X and Y have the joint pdf $f(x, y) = x + y, 0 \leq x \leq 1, 0 \leq y \leq 1$

a. Find the marginal pdfs $f_X(x)$ and $f_Y(y)$ and show that X and Y are dependent.
b. Compute:
   i. μ_X
   ii. μ_Y
   iii. σ_X^2
   iv. σ_Y^2

Problem 5

Show that $Cov(X, Y) = E(XY) - \mu_X \mu_Y$.

> *Hint: start with the definition* $Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$

Problem 6

Show that $E(XY) = \mu_X \mu_Y + \rho_{XY} \sigma_X \sigma_Y$

Problem 7

Show that if two random variables X and Y are independent, then their covariance is 0.

> *Hint: start with the shortcut formula* $Cov(X, Y) = E(XY) - \mu_X \mu_Y$ *and re-write* $E(XY)$ *as a double sum.*

Problem 8

Return to the scenario in Problem 1.

a. Find $E(M)$ and $V(M)$
b. Find $E(S)$ and $V(S)$
c. Find $Cov(M, S)$
d. Suppose that instead of just one student, you sample 12 students. Let $S_1, \ldots S_{12}$ represent the number of siblings reported by each student (you can assume that these random variables are independent). Let $\overline{S} = \sum_{i=1}^{12} \frac{S_i}{12}$ be the average number of siblings in the sample. Find $E(\overline{S})$ and $V(\overline{S})$