

Chapter 5 Part 2
STAT 5700: Probability

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Contents

5.6 & 5.7 Review	2
5.8 Expected value and variance of linear functions of r.v's	3
Sample statistics	6
5.9 The Multinomial Probability Distribution	7
5.11 Conditional Expectation	9
Chapter 5 Group Work	12

5.6 & 5.7 Review

Recall the Theorems from 5.6.

Example:

Let Y_1, Y_2, Y_3 be three continuous random variables with joint pdf $f(y_1, y_2, y_3)$. Find an expression for $E(2Y_1 + 3Y_2^2 - Y_3)$

Example: Let Y_1, Y_2, Y_3 be three independent random variables with binomial distributions $b(4, 1/2)$, $b(6, 1/3)$, and $b(12, 1/6)$ respectively. Find $E(Y_1 Y_2 Y_3)$

Example: Let X, Y be two independent random variables. Show that $Cov(X, Y) = 0$.

Recall: *independence $\implies Cov(X, Y) = 0$, but the converse is NOT true. It is possible to have $Cov(X, Y) = 0$ when X and Y are dependent.*

5.8 Expected value and variance of linear functions of r.v's

Theorem 5.12 Let Y_1, Y_2, \dots, Y_n and X_1, X_2, \dots, X_m be random variables with $E(Y_i) = \mu_i$ and $E(X_j) = \xi_j$. Define

$$U_1 = \sum_{i=1}^n a_i Y_i \quad \text{and} \quad U_2 = \sum_{j=1}^m b_j X_j$$

for constants a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_m . Then the following hold:

- (a) $E(U_1) = \sum_{i=1}^n a_i \mu_i$
- (b) $V(U_1) = \sum_{i=1}^n a_i^2 V(Y_i) + 2 \sum \sum_{1 \leq i < j \leq n} a_i a_j \text{Cov}(Y_i, Y_j)$, where the double sum is over all pairs (i, j) with $i < j$
- (c) $\text{Cov}(U_1, U_2) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \text{Cov}(Y_i, X_j)$

Note, when $n = 2$, (b) is simply $V(a_1 Y_1 + a_2 Y_2) = a_1^2 V(Y_1) + a_2^2 V(Y_2) + 2a_1 a_2 \text{Cov}(Y_1, Y_2)$

Example: Let Y_1 and Y_2 be random variables with means μ_1, μ_2 , variances σ_1^2, σ_2^2 , and covariance σ_{12} respectively. Suppose $U = 2Y_1 + 3Y_2$. Find $E(U)$ and $V(U)$.

Example (cont'd): Return to the above example where Y_1, Y_2, Y_3 are three independent random variables with binomial distributions $b(4, 1/2), b(6, 1/3)$, and $b(12, 1/6)$ respectively. Find the mean and variance of $W = Y_1 + Y_2 + Y_3$.

Example:

Let the independent random variables X_1 and X_2 have respective means $\mu_1 = -4$ and $\mu_2 = 3$ and variances $\sigma_1^2 = 4$ and $\sigma_2^2 = 9$. Find the mean and variance of $Y = 3X_1 - 2X_2$.

If X_3 is a third random variable that has the same mean and variance as X_2 but is correlated with X_1 by the amount $\rho = 0.3$, what is the mean and variance of $W = 3X_1 - 2X_3$?

Example In the 5.2 gasoline example, we were interested in X and Y that had the joint and marginal pdfs below:

$$f(x, y) = 3x, \quad 0 \leq y \leq x \leq 1$$

$$f_x(x) = 3x^2, \quad 0 \leq x \leq 1$$

$$f_y(y) = 3/2(1 - y^2), \quad 0 \leq y \leq 1$$

Supposed we are interested in $W = X - Y$, which in context represents the proportional amount of gasoline remaining at the end of the week. Find the mean and variance of W .

Sample statistics

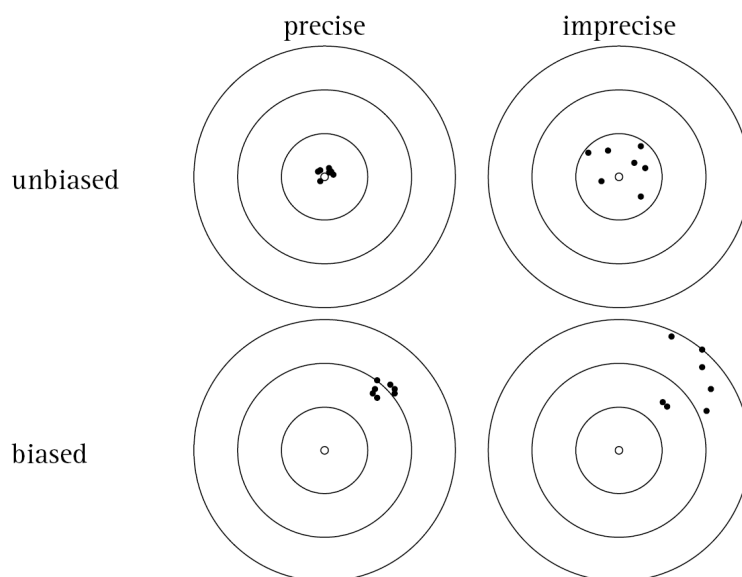
Any function of the sample observations X_1, X_2, \dots, X_n is called a **statistic**. The **sample mean** of a random sample X_1, X_2, \dots, X_n is given by

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

and the **sample variance** is given by

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

\bar{X} and S^2 are two examples of **statistics**. We will see that \bar{X} is an **unbiased estimator** for the population mean μ and S^2 is an **unbiased estimator** for the population variance σ^2 .



When X_1, X_2, \dots, X_n are independent random variables with mean μ and variance σ^2 , find the mean and variance of \bar{X} .

5.9 The Multinomial Probability Distribution

Recall a binomial random variable results from an experiment with n trials, with two possible outcomes per trial (success/failure). Sometimes, we're interested in situations where there are more than 2 outcomes per trial. For example, there are at least 4 possible blood types a person can have.

A **multinomial experiment** is a generalization of the binomial experiment.

A **multinomial experiment** possesses the following properties:

1. The experiment consists of n identical trials
2. The outcome of each trial falls into one of k classes or cells
3. The probability that the outcomes of a single trial falls into cell i is p_i , $i = 1, 2, \dots, k$ and remains the same from trial to trial. Notice $p_1 + p_2 + \dots + p_k = 1$
4. The trials are independent
5. The random variables of interest are Y_1, Y_2, \dots, Y_k where Y_i equals the number of trials for which the outcome falls into cell i . Notice that $Y_1 + Y_2 + \dots + Y_k = n$

The joint pmf for Y_1, Y_2, \dots, Y_k is given by

$$p(y_1, y_2, \dots, y_k) = \frac{n!}{y_1! y_2! \dots y_k!} p_1^{y_1} p_2^{y_2} \dots p_k^{y_k},$$

where $\sum_{i=1}^k p_i = 1$ and $\sum_{i=1}^k y_i = n$

According to recent census figures, the proportions of adults (persons over 18 years of age) in the United States associated with five age categories are as given in the following table.

Age	Proportion
18-24	0.18
25 - 34	0.23
35 - 44	0.16
45 - 64	0.27
65+	0.16

If these figures are accurate and five adults are randomly sampled, find the probability that the sample contains one person between ages of 18 and 24, two between the ages of 25 and 34, and two between the ages of 45 and 64.

Theorem 5.13 If Y_1, Y_2, \dots, Y_k have a multinomial distribution with parameters n and p_1, p_2, \dots, p_k , then

1. $E(Y_i) = np_i$
2. $V(Y_i) = np_i q_i$
3. $Cov(Y_s, Y_t) = -np_s p_t$ if $s \neq t$

5.11 Conditional Expectation

Conditional distributions can be used to compute probabilities and expected values just as with any distribution.

Definition 5.13 If X and Y are any two random variables, the conditional expectation of $g(X)$ given that $Y = y$, is defined to be

$$E(g(X)|Y = y) = \int_{-\infty}^{\infty} g(x)f(x|y)dx$$

if X and Y are jointly continuous, and

$$E(g(X)|Y = y) = \sum_x g(x)p(x|y)$$

if X and Y are jointly discrete.

Refer back to our example in 5.3 with $f(x, y) = 1/2, 0 \leq x \leq y \leq 2$, where we found $f(x|y) = 1/y, 0 < x \leq y$. Find the conditional expectation of the amount of sales, X , given that $Y = 1.5$.

Theorem 5.14 Let X and Y denote random variables. Then

$$E(X) = E[E(X|Y)],$$

where on the right-hand side the inner expectation is with respect to the conditional distribution of $X|Y$ and the outside expectation is with respect to the distribution of Y .

PROOF

The conditional variance of X given Y is defined by

$$V(X|Y = y) = E(X^2|Y = y) - [E(X|Y = y)]^2$$

Theorem 5.15 Let X and Y denote random variables. Then

$$V(X) = E[V(X|Y)] + V[E(X|Y)]$$

PROOF:

A quality control plan for an assembly line involves sampling $n = 10$ finished items per day and counting Y , the number of defectives. If p denotes the probability of observing a defective, then Y has a binomial distribution, assuming that a large number of items are produced by the line. But p varies from day to day and is assumed to have a uniform distribution on the interval from 0 to $1/4$. Find the expected value and variance of Y .

Chapter 5 Group Work

Problem 1

Show that for any constants a and b , $Cov(a + X, b + Y) = Cov(X, Y)$. That is, shifting by a constant does not change the covariance. *Note: this fact will be useful on HW 5.110*

Problem 2

To estimate a proportion of units that meet a given criteria, we often use the estimator $\hat{p} = \frac{Y}{n}$, where $Y \sim binomial(n, p)$. Find the expected value and variance of \hat{p} , assuming n is fixed.

Problem 3

A learning experiment requires a rat to run a maze (a network of pathways) until it locates one of three possible exits. Exit 1 presents a reward of food, but exits 2 and 3 do not. (If the rat eventually selects exit 1 almost every time, learning may have taken place). Let Y_i denote the number of times exit i is chosen in successive runnings. For the following, assume that the rat chooses an exit at random on each run.

- Find the probability that $n = 6$ runs result in $Y_1 = 3$, $Y_2 = 1$ and $Y_3 = 2$.
- For general n , find $E(Y_1)$ and $V(Y_1)$.
- For general n , find $Cov(Y_2, Y_3)$.
- To check for the rat's preference between exits 2 and 3, we may look at $Y_2 - Y_3$. Find $E(Y_2 - Y_3)$ and $V(Y_2 - Y_3)$ for general n .

Problem 4

The number of defects per yard in a certain fabric, Y , is known to have a Poisson distribution with parameter λ . The parameter λ is assumed to be a random variable with a pdf given by

$$f(\lambda) = e^{-\lambda}, \quad \lambda \geq 0$$

- Find the expected number of defects per yard by first finding the conditional expectation of Y for given λ .
- Find the variance of Y .