# Chapter 5 Part 1 Group Work

SOLUTIONS

## Problem 1

Suppose that you randomly select a student from a large population of 4th grade students and record the student's sex and number of siblings. Let M = 1 if the selected student is male and M = 0 if the selected student is female. Let S record the student's number of siblings. The proportion of the population of students falling into each category of M and S is recorded in the table below.

| # of siblings: | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Female | .10 | .18 | .12 | .07 | .04 |
| Male | .12 | .18 | .14 | .03 | .02 |

a. Verify that the above table is a valid joint pmf.
b. Find the probability that the randomly selected student will be male with two siblings
c. Find $P(M = 0, S \leq 2)$
d. Find the probability that the randomly selected student will be an only child
e. Find $F(1, 1)$

### Solution

#### Part a

0.10 + 0.12 + 0.18 + 0.18 + 0.12 + 0.14 + 0.07 + 0.03 + 0.04 + 0.02 = 1

#### Part b

$P(M = 1, S = 2) = 0.14$

#### Part c

$P(M = 0, S \leq 2) = 0.12 + 0.18 + 0.14 = 0.44$

#### Part d

$P(S = 0) = 0.10 + 0.12 = 0.22$

#### Part e

$F(1, 1) = P(M \leq 1, S \leq 1) = P(S \leq 1) = 0.10 + 0.12 + 0.18 + 0.18 = 0.58$

# Problem 2

Of nine executives in a business firm, four are married, three have never married, and two are divorced. Three of the executives are to be selected for promotion. Let $Y_1$ denote the number of married executives and $Y_2$ denote the number of never married executives among the three selected for promotion. Assuming that the three are randomly selected from the nine available, find the joint pmf of $Y_1$ and $Y_2$.

### Solution

Note that using material from Chapter 3, the joint probability function is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}, \quad 0 \le y_1, 0 \le y_2, y_1 + y_2 \le 3$$

Given table format, this is

| y2 y1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 3/84 | 6/84 | 1/84 |
| 1 | 4/84 | 24/84 | 12/84 | 0 |
| 2 | 12/84 | 18/84 | 0 | 0 |
| 3 | 4/84 | 0 | 0 | 0 |

# Problem 3

Let $Y_1$ and $Y_2$ have the joint pdf

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0 \\ 0, & elsewhere \end{cases}$$

a. What is $P(Y_1 < 1, Y_2 > 5)$? b. What is $P(Y_1 + Y_2 < 3)$?

### Part a

$$\begin{aligned} P(Y_1 < 1, Y_2 > 5) &= \int_0^1 \int_5^\infty e^{-(y_1+y_2)}\, dy_2\, dy_1 \\ &= \int_0^1 \left(-e^{-(y_1+y_2)}|_{y_2=5}^{\infty}\right) dy_1 \\ &= \int_0^1 e^{-(y_1+5)} dy_1 \\ &= -e^{-(1+5)} + e^{-5} \\ &= .00426. \end{aligned}$$

### Part b

$$\begin{aligned} P(Y_1 + Y_2 < 3) &= P(Y_1 < 3 - Y_2) \\ &= \int_0^3 \left(\int_0^{3-y_1} e^{-(y_1+y_2)}\, dy_2\right) dy_1 \\ &= 1 - 4e^{-3} \\ &= .8009. \end{aligned}$$

## Problem 4

Return to the scenario in Problem 1

Return to the scenario in Problem 1.

a. Find the marginal distribution of $M$
b. Find the marginal distribution of $S$
c. Find $P(S \leq 2|M = 1)$
d. Find $P(M = 1|S \leq 2)$

### Solutions

#### Part a

| | P(M = m) |
|---|---|
| Female (M = 0) | 0.51 |
| Male (M = 1) | 0.49 |

#### Part b

| # of siblings (S): | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| P(S = s) | .22 | .36 | .26 | .10 | .06 |

#### Part c

$$P(S \leq 2 \mid M = 1) = \frac{P(S \leq 2, M = 1)}{P(M = 1)}$$

- $P(M = 1) = 0.49$
- $P(S \leq 2, M = 1) = P(S = 0, M = 1) + P(S = 1, M = 1) + P(S = 2, M = 1) = 0.12 + 0.18 + 0.14 = 0.44$

$$P(S \leq 2 \mid M = 1) = \frac{0.44}{0.49} \approx 0.898$$

#### Part d

$$P(M = 1 \mid S \leq 2) = \frac{P(S \leq 2, M = 1)}{P(S \leq 2)}$$

- $P(S \leq 2) = P(S = 0) + P(S = 1) + P(S = 2) = 0.22 + 0.36 + 0.26 = 0.84$
- $P(S \leq 2, M = 1) = 0.44$ (from above)

$$P(M = 1 \mid S \leq 2) = \frac{0.44}{0.84} \approx 0.524$$

## Problem 5

Let $f(x,y) = \frac{4}{3}(1-xy)$, $\quad 0 \le x \le 1, \quad 0 \le y \le 1$.

a. Find the marginal pdfs of $X$ and $Y$.
b. Find $P(X \le Y/2)$

$$f(x,y) = \frac{4}{3}(1-xy), \quad 0 \le x \le 1,\ 0 \le y \le 1$$

$$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 \frac{4}{3}(1-xy)\,dy = \frac{4}{3}\left[y - \frac{1}{2}xy^2\right]_0^1 = \frac{4}{3}\left(1-\frac{1}{2}x\right) = \frac{4}{3} - \frac{2}{3}x,\ 0<x<1$$

$$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 \frac{4}{3}(1-xy)\,dx = \frac{4}{3}\left[x - \frac{1}{2}yx^2\right]_0^1 = \frac{4}{3}\left(1-\frac{1}{2}y\right) = \frac{4}{3} - \frac{2}{3}y,\ 0<y<1$$

$$P(X \le Y/2) = \int_0^1 \int_0^{y/2} f(x,y)\,dx\,dy = \int_0^1 \int_0^{y/2} \frac{4}{3}(1-xy)\,dx\,dy = \frac{7}{24}$$

## Problem 6

Let the joint pmf of $X$ and $Y$ be

$$f(x,y) = \frac{xy^2}{30}, \quad x = 1,2,3, \quad y = 1,2$$

a. What are the marginal distributions of $X$ and $Y$?
b. Are X and Y independent?
c. What is $P(X>1)$?
d. What is $E(X)$?

### Part a

$$f_X(x) = \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x\cdot 1^2}{30} + \frac{x \cdot 2^2}{30} = \frac{x}{30} + \frac{4x}{30} = \frac{5x}{30} = \frac{x}{6}, x = 1,2,3$$

$$f_Y(y) = \sum_{x=1}^{3} \frac{xy^2}{30} = \frac{1\cdot y^2}{30} + \frac{2\cdot y^2}{30} + \frac{3\cdot y^2}{30} = \frac{6y^2}{30} = \frac{y^2}{5},\ y = 1,2$$

### Part b

Since $f_X(x) f_Y(y) = \frac{x}{6}\frac{y^2}{5} = \frac{xy^2}{30} = f(x,y)$ for all $x, y$, $X$ **and** $Y$ **are independent**.

### Part c

$$P(X>1) = f_X(2) + f_X(3) = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$$

### Part d

$$E(X) = \sum_{x=1}^{3} x f_X(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{3}{6} = \frac{14}{6} = \frac{7}{3}$$

## Problem 7

Let $X$ and $Y$ be two continuous random variables with joint pdf $f(x, y) = \frac{3}{16} x y^2, \quad 0 \leq x \leq 2, \ \ 0 \leq y \leq 2$. Are the two random variables independent?

### Solution

Both are bounded between constants, and we can define $g(x) = \frac{3}{16} x$ and $h(y) = y^2$, so by Theorem 5.5 they are independent.

## Problem 8

Let $X$ and $Y$ have the joint pdf $f(x, y) = x + y, 0 \leq x \leq 1, 0 \leq y \leq 1$. Find the marginal pdfs $f_X(x)$ and $f_Y(y)$ and show that $X$ and $Y$ are dependent.

## Solution

$$f_x(x) = \int_0^1 (x + y) dy = x + \frac{1}{2}, \ \ 0 \leq x \leq 1$$

$$f_y(y) = \int_0^1 (x + y) dx = y + \frac{1}{2}, \ \ 0 \leq y \leq 1$$

$$f_x(x) f_y(y) = (x + \frac{1}{2})(y + \frac{1}{2}) \neq x + y$$