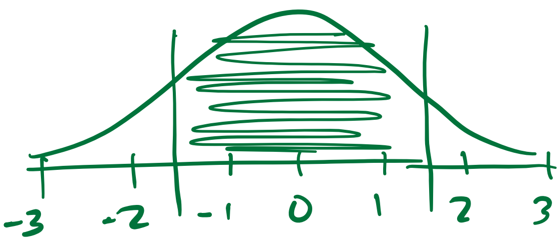
Practice Exam 2

① $X \sim N(1200, 200^2)$

a) $P(900 < X < 1500)$

$P\left(\frac{900-1200}{200} < \frac{X-\mu}{\sigma} < \frac{1500-1200}{200}\right)$ convert to z-scores

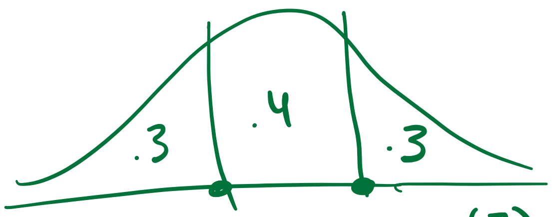
$P\left(-\frac{3}{2} < Z < \frac{3}{2}\right)$



pnorm(1.5) - pnorm(-1.5)

$0.933 - 0.067 = \boxed{0.866}$

b) middle 40%



qnorm(.3) qnorm(.7)

-.524 .524

$z = .524 = \frac{X-\mu}{\sigma} = \frac{X-1200}{200}$

solve for x

$200(.524) + 1200 = X$

1304.8

$1200 - 200(.524) = 1095.2$

$[1095.2, 1304.8]$

c) X = # of times sales are outside (\$900, \$1500) out of 5

$X \sim \text{binomial}(n=5,\ p = 1 - .866)$

from part a

$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X=0) - P(X=1)$

$= 1 - \binom{5}{0}(.134)^0(.866)^5 - \binom{5}{1}.134^1(.866)^4$

$= \boxed{0.1361}$

② $S = \{1, 2, 3, 4\}$

a) 12, 13, 14, 23, 24, 34 — 6 possible pairs (order doesn't matter)

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{4\cdot 3\cdot 2!}{2!\,2!} = 6$$

			X	
		1	2	3
Y	2	1/6	0	0
	3	1/6	1/6	0
	4	1/6	1/6	1/6

← joint dist. of X, Y

can also write as
$p(x,y) = 1/6$ for $1 \leq x < y \leq 4$

b)

X	P(X=x)
1	3/6
2	2/6
3	1/6

Y	P(Y=y)
2	1/6
3	2/6
4	3/6

c) $p(x \mid y=3) = \dfrac{p(x,3)}{P(Y=3)} = \dfrac{p(x,3)}{2/6}$

X	p(x | y=3)
1	1/6 / 2/6 = 1/2
2	1/6 / 2/6 = 1/2
3	0 / 2/6 = 0

③ $f(x) = \begin{cases} x^2 & -1 < x < 0 \\ cx & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

a) $1 = \int_{-1}^{0} x^2\,dx + \int_0^2 cx\,dx$

$= \frac{x^3}{3}\Big|_{-1}^{0} + \frac{c}{2}x^2\Big|_0^2$

$1 = \frac{1}{3} + 2c$

$\frac{2}{3} = 2c \implies \boxed{c = \frac{1}{3}}$

b) $F(x) = \begin{cases} \int_{-1}^{x} t^2\,dt = \frac{t^3}{3}\Big|_{-1}^{x} = \frac{x^3}{3} + \frac{1}{3} & -1 < x < 0 \\ F(0) + \int_0^x \frac{1}{3}t\,dt = \frac{1}{6}t^2\Big|_0^x = \frac{1}{6}x^2 & 0 < x < 2 \end{cases}$

add this b/c its a <u>cumulative</u> probability

$F(x) = \begin{cases} \frac{x^3+1}{3} & -1 < x < 0 \\ \frac{1}{3} + \frac{1}{6}x^2 & 0 < x < 2 \\ 1 & x > 2 \\ 0 & x < -1 \end{cases}$

note these should return same values at $x=0$

④ $S \sim U[2,4]$

a) $f(s) = \frac{1}{4-2} = \frac{1}{2} \qquad 2 < s < 4$

b) $E(A) = E(S^2)$ $\qquad$ use $V(S) = E(S^2) - (E(S))^2$

$E(S) = \frac{2+4}{2} = 3$

$V(S) = \frac{(\theta_2 - \theta_1)^2}{12} = \frac{(4-2)^2}{12} = \frac{4}{12} = \frac{1}{3}$ $\Big]$ properties of uniform dist.

$E(S^2) = V(S) + (E(S))^2$

$= \frac{1}{3} + 3^2 = \boxed{\frac{28}{3}}$

c) $V(S^2) = E(S^4) - (E(S^2))^2$

$E(S^4) = \int_2^4 s^4 f(s)\,ds = \int_2^4 s^4 \frac{1}{2}\,ds = \frac{s^5}{5}\frac{1}{2}\Big|_2^4$

$= \frac{4^5 - 2^5}{10} = 99.2$

$V(S^2) = 99.2 - \left(\frac{28}{3}\right)^2 = \boxed{12.09}$

5 $X \sim \text{Gamma}(\alpha, \beta)$

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \qquad x > 0$$

Note that $f(x)$ is a pdf so $\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx = 1$

$$E(X^3) = \int x^3 \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \int \frac{1}{\Gamma(\alpha)\beta^\alpha} \underline{x^{(\alpha+3)-1}}\ \underline{e^{-x/\beta}} dx \qquad (1)$$

this looks close to gamma pdf w/ parameters $\alpha+3$ and β. we just need to manipulate the constant out front.

Need: $\frac{1}{\Gamma(\alpha+3)\beta^{\alpha+3}} = c \frac{1}{\Gamma(\alpha)\beta^\alpha}$ solve for c

$$\frac{\Gamma(\alpha)\beta^\alpha}{\Gamma(\alpha+3)\beta^{\alpha+3}} = c$$

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha+3)\beta^3} = c$$

multiply (1) by $\frac{1}{c} \cdot c = 1$

$$\frac{\Gamma(\alpha+3)\beta^3}{\Gamma(\alpha)} \int \frac{\cancel{\Gamma(\alpha)}}{\Gamma(\alpha+3)\beta^3} \frac{1}{\cancel{\Gamma(\alpha)}\beta^\alpha} x^{\alpha+3-1} e^{-x/\beta} dx$$

$$= \frac{\Gamma(\alpha+3)\beta^3}{\Gamma(\alpha)} \underbrace{\int \frac{1}{\Gamma(\alpha+3)\beta^{\alpha+3}} x^{\alpha+3-1} e^{-x/\beta} dx}_{= 1 \text{ b/c gamma pdf w/ } \alpha+3,\ \beta}$$

$$E(X^3) = \frac{\Gamma(\alpha+3)\beta^3}{\Gamma(\alpha)} = \frac{(\alpha+3-1)!\,\beta^3}{(\alpha-1)!} = \frac{(\alpha+2)(\alpha+1)(\alpha)\cancel{(\alpha-1)!}}{\cancel{(\alpha-1)!}}\beta^3$$

$$= \boxed{(\alpha+2)(\alpha+1)\alpha\,\beta^3}$$

(6) $m(t) = (0.5 + 0.5e^{2t})^2$

$m'(t) = 2(0.5 + 0.5e^{2t})(e^{2t}) =$

$m'(0) = 2(0.5 + 0.5e^{0})(e^{0})$

$= 2(1)(1) = \boxed{2 = E(X)}$

$V(X) = m''(t=0) - (m'(t=0))^2$

simplify $m'(t) = e^{2t} + e^{4t}$

$m''(t) = 2e^{2t} + 4e^{4t}$

$m''(0) = 2e^{0} + 4e^{0} = 6 = E(X^2)$

$V(X) = 6 - 2^2 = \boxed{2}$

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a) $f(y) = 2(1-y) \qquad 0 \le y \le 1$

$E(y) = \int y\,2(1-y)\,dy = \int_0^1 2y - 2y^2\,dy$

$= y^2 \Big|_0^1 - \frac{2}{3}y^3 \Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$

$\frac{1}{3}(\$100{,}000) = \boxed{\$33{,}333.33}$ average claim

b) $F(y) = 0.9$ solve for y

$F(y) = \int_0^y 2(1-t)\,dt = 2\left(t - \frac{t^2}{2}\right)\Big|_0^y$

$= 2\left(y - \frac{y^2}{2}\right) = 2y - y^2$

solve $0.9 = 2y - y^2$

$y^2 - 2y + 0.9 = 0 \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\frac{2 \pm \sqrt{4 - 4(1)(0.9)}}{2}$

$1 \pm \frac{\sqrt{.4}}{2}$

$1 \pm .316$

$\boxed{.684}$ (the other solution 1.316 is not in support)

c) $P(Y < .75) = F(.75) = 2(.75) - .75^2$

$\$175{,}000$

$= \boxed{.9375}$

(8) can't be cdf b/c $h(150) \neq 1$ for $y > 150$

to check if pdf, see if it integrates to 1

$$1 \stackrel{?}{=} \int_{100}^{150} h(y)\,dy = \int_{100}^{150} (0.02y - 2)\,dy$$

$$= \frac{.02}{2} y^2 \Big|_{100}^{150} - 2y \Big|_{100}^{150}$$

$$= 125 - 2(150 - 100)$$

$$= 100 \neq 1$$

$\boxed{\text{(C) neither pdf or cdf}}$