Chapter 2 Group Work Solutions

Problem 1

How many four-letter code words are possible using the letters in IOWA if:

a. The letters may not be repeated?
b. The letters may be repeated?

SOLUTIONS

(a) $4 \cdot 3 \cdot 2 \cdot 1 = 24$

(b) $4 \cdot 4 \cdot 4 \cdot 4 = 256$

Problem 2

1. Three students (S) and six faculty members (F) are on a panel discussing a new college policy.

   a. In how many different ways can the nine participants be lined up at a table in front of the auditorium?
   b. How many lineups are possible if you only care about where the students are placed in relation to faculty? In other words, how many ways are there to choose 3 seats to place the students in (or alternatively, choose 6 seats to place the faculty in)?
   c. For each of the nine participants, you are to decide whether the participant did a good job or a poor job stating their opinion of the new policy; that is, give each of the nine participants a grade of G or P. How many different "scorecards" are possible?

Solution Part a:

Think in terms of "slots." We have 9 people we are placing in 9 slots - we have 9 people to choose from for the first slot, 8 people to choose from for the second slot, etc.

$$9! = 9 \cdot 8 \cdot 7 \cdots 2 \cdot 1 = 362,880$$

Solution Part b:

Another way to think of this is "how many ways are there to place 3 Ss in 9 slots?" (the rest get filled in with Fs). Or stated more explicitly, "how many ways are there to choose 3 seats (out of 9 seats) for the students to sit in?"

$$\binom{9}{3} = \frac{9!}{3!6!} = \frac{9 \cdot 8 \cdot 7}{6} = 84$$

Solution Part c:

For each person there are 2 possibilities

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^9 = 512$$

Problem 3

The NBA Finals continues until either the Western Conference team or the Eastern Conference team wins four games. How many different orders are possible (e.g. WEEWWW means the Western Conferences team wins in six games) if the series lasts. . . .

a. four games?
b. five games?
c. six games?
d. seven games?

Solution Part a:

2 orders (WWWW or EEEE)

Solution Part b:

Say that the West wins the series; this means the fifth game must be won by the West, and the East must win one of the first four games. There are $\binom{4}{1} = \frac{4!}{3!1!} = 4$ ways for the East to win 1 of 4 games. Therefore there are 4 ways for the West to win in 5 games. Similarly, there are 4 ways for the East to win in 5 games, by the same logic. Therefore there are $2 \cdot \binom{4}{1} = 8$ ways for the series to end in 5 games.

An equivalent way of approaching it is thinking about needing to place 4 Ws in 5 slots, but since there has to be a W in the 5th slot, this reduces to placing 3 Ws in 4 slots (i.e. the West has to win exactly 3 of the first 4 games). And $\binom{4}{3} = \frac{4!}{1!3!} = 4$.

Solution Part c:

Say that the West wins the series; this means that the 6th game must be won by the West, and the East must win 2 of the first 5 games. By the same logic as above, there are $2 \cdot \binom{5}{2} = 2 * \frac{5!}{3!2!} = 20$ ways for the series to end in 6 games.

Solution Part d:

By the same logic, $2 \cdot \binom{6}{3} = 2 \cdot \frac{6!}{3!3!} = 40$ ways for the series to end in 7 games.

Problem 4

Suppose that 78% of the students at a particular college have a TikTok account and 43% have a Facebook account. Using only this information,

a. what is the largest possible value for the percentage who have both a TikTok account and a Facebook account? Describe the (unrealistic) situation in which this occurs.
b. what is the smallest possible value for the percentage who have both a TikTok account and a Facebook account? Describe the (unrealistic) situation in which this occurs.

	Facebook	No Facebook	Total
TikTok			0.78
No TikTok			0.22
Total	0.43	0.57	1.00

Now assume you know that 36% of students have both a TikTok and a Facebook account.

c. What percentage of students have at least one of these accounts?
d. What percentage of students have neither of these accounts?
e. What percentage of students have one of these accounts but not both?

Solution Part a:

Let A = the event that a student as a TikTok and B = the event that a student has a Facebook. The intersection $P(A \cap B)$ cannot be larger than $P(A)$ or $P(B)$, so the largest it can be is 0.43, which happens in the unlikely scenario that every person who has a TikTok also has a Facebook.

Solution Part b:

We want to find a lower bound on $P(A \cap B)$. First note that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1,$$

by definition of union and the fact that all probabilities must be less than or equal to 1. By rearranging, we then have:

$$\begin{aligned} P(A) + P(B) - 1 &\leq P(A \cap B) \\ 0.78 + 0.43 - 1 &\leq P(A \cap B) \\ 0.21 &\leq P(A \cap B) \end{aligned}$$

The percentage of students who have both a TikTok and a Facebook has to be at least 21%, which happens in the unlikely scenario that everyone has either a TikTok or a Facebook (i.e. no students have neither).

Solution Part c:

We can fill in the rest of the table

	Facebook	No Facebook	Total
TikTok	0.36	0.42	0.78
No TikTok	0.07	0.15	0.22
Total	0.43	0.57	1.00

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.78 + 0.43 - 0.36 \\ &= 0.85 \end{aligned}$$

Solution part d:

$$P[(A \cup B)'] = 1 - P(A \cup B) = 1 - 0.85 = 0.15$$

Solution Part e:

$$\begin{gathered} P(A \cap B') + P(B \cap A') \\ P(\text{TikTok but not Facebook}) + P(\text{Facebook but not TikTok}) \\ 0.42 + 0.07 \\ = 0.49 \end{gathered}$$

Problem 5

Let A_1 and A_2 be the events that a person is left- eye dominant or right-eye dominant, respectively. When a person folds their hands, let B_1 and B_2 be the events that the left thumb and right thumb, respectively, are on top. A survey in one statistics class yielded the following table:

	B_1	B_2	Totals
A_1	5	7	12
A_2	14	9	23
Totals	19	16	35

If a student is selected randomly, find the following probabilities:

Solution:

(a) $P(A_1 \cap B_1) = 5/35 = 1/7$,

(b) $P(A_1 \cup B_1) = (12 + 19 - 5)/35 = 26/35$,

(c) $P(A_1|B_1) = 5/19$,

(d) $P(B_2|A_2) = 9/23$.

(e) If the students had their hands folded and you hoped to select a right-eye-dominant student, would you select a "right thumb on top" or a "left thumb on top" student? Why? $P(A_2|B_1) = 14/19 > P(A_2|B_2) = 9/16$ so we should choose a left-thumb person.

Problem 6

Let A and B be independent events. Prove that A' and B are also independent.

Solution

We want to show that $P(A' \cap B) = P(A')P(B)$ to prove they are independent.

The proof is provided below. To see the first line, draw a Venn Diagram. The 2nd line relies on the fact that A and B are independent, and the 3rd line applies the rule of complements.

$$\begin{aligned} P(A' \cap B) &= P(B) - P(B \cap A) \\ &= P(B) - P(B)P(A) \\ &= P(B)[1 - P(A)] \\ &= P(B)P(A') \end{aligned}$$

Problem 7

Suppose that every day you play a lottery game in which a three-digit number is randomly selected. Your probability of winning for each day is 1/1000.

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a) Is it reasonable to assume that whether you win or lose is independent from day to day?
b) Determine the probability that you win at least once in a 7-day week.  Report your answer with five
c) Determine the probability that you win at least once in a 365-day year.
d) Suppose that your friend says that because there are only 1000 three-digit numbers, you're guarantee
e) Express the probability of winning at least once as a function of the number of days that you play.
f) For how many days would you have to play in order to have at least a 90% chance of winning at least
g) Suppose that the lottery game costs $1 to play and pays $500 when you win.  If you were to play for
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Solution

Part a

Yes, the three-digit number is selected randomly each day. So whether or not you win on one day doesn't have any effect on the probability of winning on any other day.

Part b

P(win at least once) = 1 - P(lose every day) = $1 - (999/1000)^7 = 0.00698$

Part c

P(win at least once) = 1 - P(lose every day) = $1 - (999/1000)^{365} = 0.306$

Part d

No, it's certainly possible you could lose on all 1000 days. In fact, that prospect is not terribly unlikely:

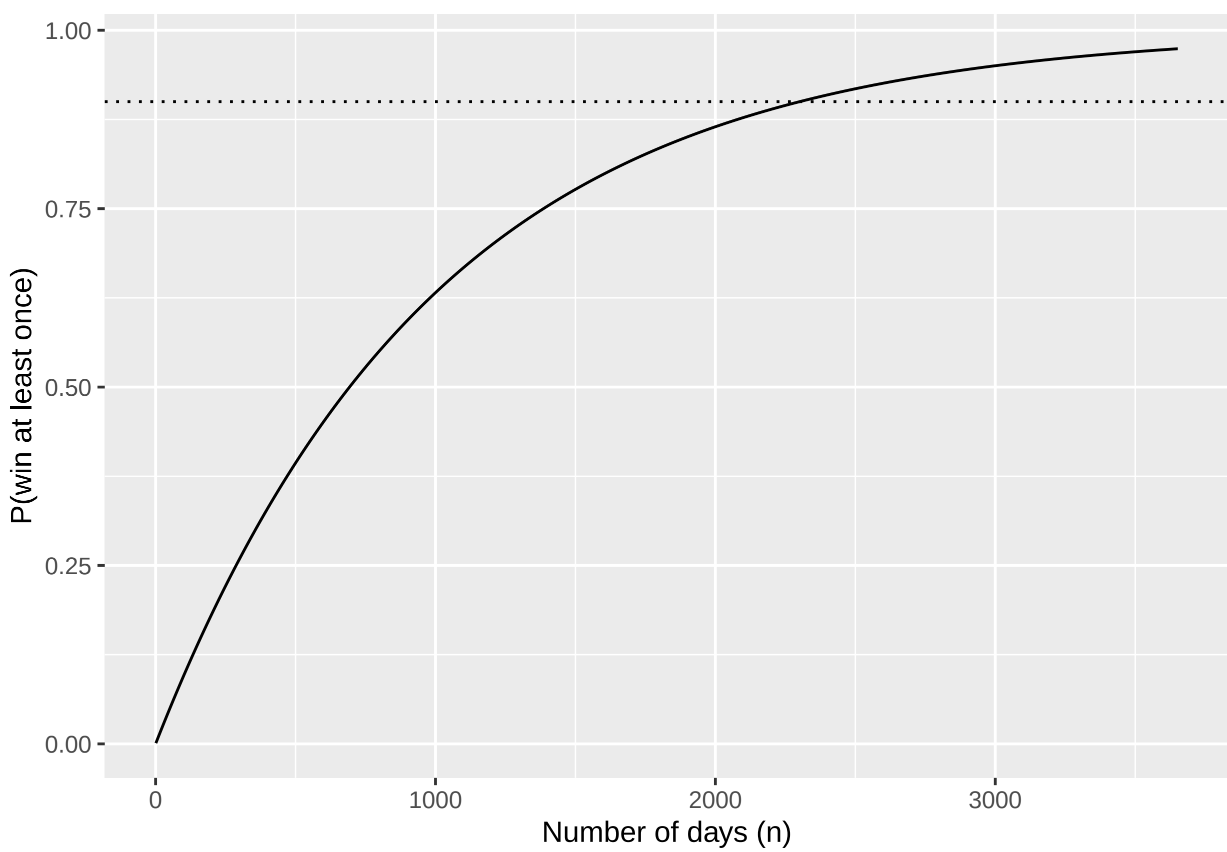
$1 - (999/1000)^{1000} = 0.632$. It's more than 1/2, but closer to 1/2 than to 1!

Part e

$$P(\text{win at least once in n days}) = 1 - (999/1000)^n$$

```r
library(tidyverse)
data <- data.frame(x = 1:3652) |>
  mutate(y = 1 - (999/1000)^x)

ggplot(data, aes(x = x, y = y)) +
  geom_line() +
  geom_hline(yintercept = 0.9, linetype = 21) +
  labs(x = "Number of days (n)",
       y = "P(win at least once)")
```



Part f

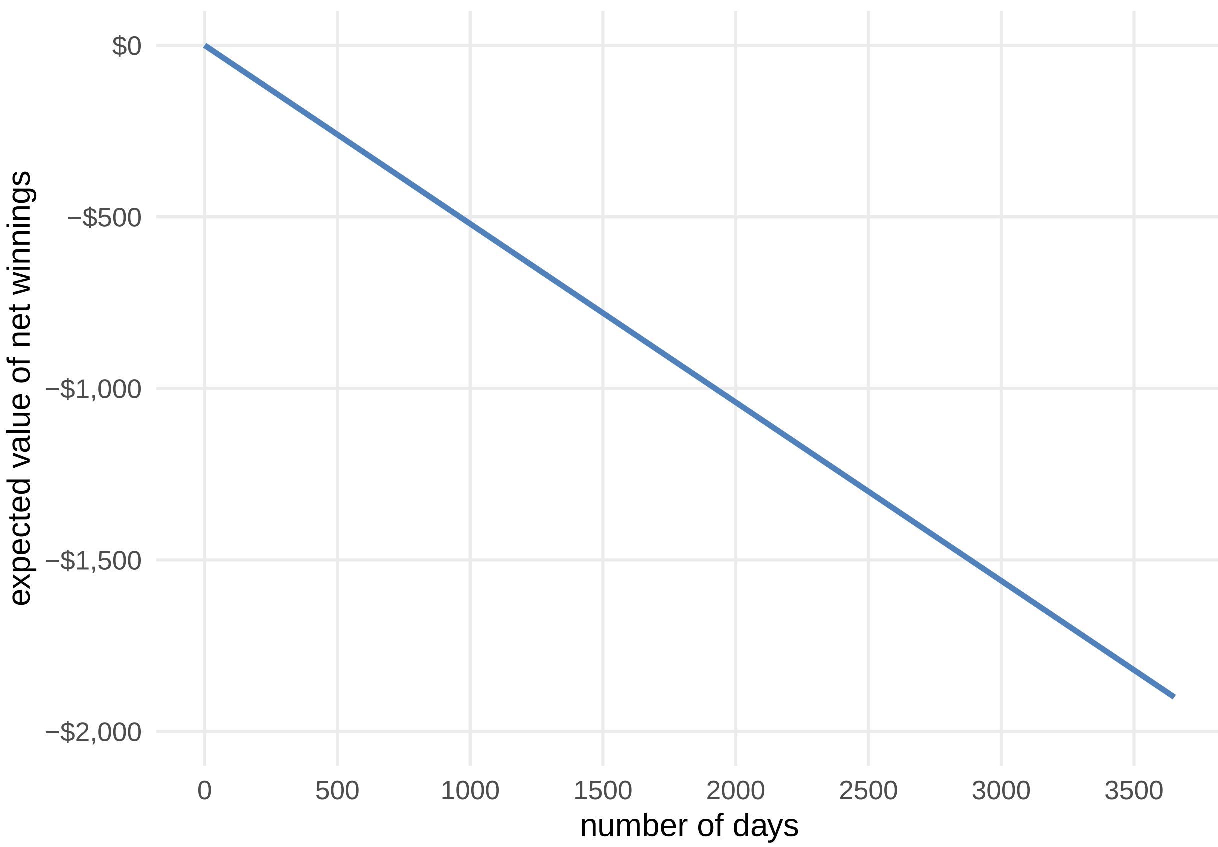
$$
\begin{aligned}
1 - \left(\frac{999}{1000}\right)^n &\geq 0.9 \\
0.1 &\geq \left(\frac{999}{1000}\right)^n \\
ln(0.1) &\geq n * ln\left(\frac{999}{1000}\right) \\
\frac{ln(0.1)}{ln\left(\frac{999}{1000}\right)} &\geq n \\
2301.434 \text{ days} &\geq n
\end{aligned}
$$

Part g

For any given day, expected winnings is

$$-1(0.999) + 500(0.001) = -0.499$$

So, after n days, the expected winnings is -0.499*n



Problem 8

Two processes of a company produce rolls of materials: The rolls of Process I are 3% defective and the rolls of Process II are 1% defective. Process I produces 60% of the company's output, Process II 40%. A roll is selected at random from the total output. Given that this roll is defective, what is the conditional probability that it is from Process I?

Solution

Let

- A_1: Roll is from Process I
- A_2: Roll is from Process II
- D: Roll is defective

The known probabilities are:

- $P(A_1) = 0.6$, $P(A_2) = 0.4$
- $P(D \mid A_1) = 0.03$, $P(D \mid A_2) = 0.01$

We are asked to find $P(A_1 \mid D)$. Using Bayes Theorem,

$$P(A_1 \mid D) = \frac{P(D|A_1)P(A_1)}{P(D)} = \frac{P(D|A_1)P(A_1)}{P(D|A_1)P(A_1) + P(D|A_2)P(A_2)} = \frac{0.03(0.6)}{0.03(0.6) + 0.01(0.4)} = 0.978$$