

HW 11 SOLUTIONS

Practice Problems

5.103

$$E(3Y_1 + 4Y_2 - 6Y_3) = 3E(Y_1) + 4E(Y_2) - 6E(Y_3) = 3(2) + 4(-1) - 6(-4) = -22.$$

$$\begin{aligned} V(3Y_1 + 4Y_2 - 6Y_3) &= 9V(Y_1) + 16V(Y_2) + 36V(Y_3) \\ &\quad + 24Cov(Y_1, Y_2) - 36Cov(Y_1, Y_3) - 48Cov(Y_2, Y_3) \\ &= 9(4) + 16(6) + 36(8) + 24(1) - 36(-1) - 48(0) \\ &= 480. \end{aligned}$$

5.119

a. Using the multinomial distribution with $p_1 = p_2 = p_3 = \frac{1}{3}$,

$$P(Y_1 = 3, Y_2 = 1, Y_3 = 2) = \frac{6!}{3!1!2!} \left(\frac{1}{3}\right)^6 = 0.0823.$$

b.

$$E(Y_1) = \frac{n}{3}, \quad V(Y_1) = n \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) = \frac{2n}{9}.$$

c.

$$Cov(Y_2, Y_3) = -n \left(\frac{1}{3}\right) \left(\frac{1}{3}\right) = -\frac{n}{9}.$$

d.

$$E(Y_2 - Y_3) = \frac{n}{3} - \frac{n}{3} = 0,$$

$$V(Y_2 - Y_3) = V(Y_2) + V(Y_3) - 2Cov(Y_2, Y_3) = \frac{2n}{3}.$$

5.123

Let Y_1 be the number of family home fires, Y_2 the number of apartment fires, and Y_3 the number of fires in other types. Then (Y_1, Y_2, Y_3) has a multinomial distribution with

$$n = 4, \quad p_1 = 0.73, \quad p_2 = 0.20, \quad p_3 = 0.07.$$

Thus,

$$P(Y_1 = 2, Y_2 = 1, Y_3 = 1) = \frac{4!}{2! 1! 1!} (0.73)^2 (0.20) (0.07) = 6(0.73)^2 (0.20) (0.07) = 0.08953.$$

5.125

Let Y_1 be the number of planes with no wing cracks, Y_2 the number of planes with detectable wing cracks, and Y_3 the number of planes with critical wing cracks. Then (Y_1, Y_2, Y_3) has a multinomial distribution with

$$n = 5, \quad p_1 = 0.70, \quad p_2 = 0.25, \quad p_3 = 0.05.$$

a.

$$P(Y_1 = 2, Y_2 = 2, Y_3 = 1) = \frac{5!}{2! 2! 1!} (0.7)^2 (0.25)^2 (0.05) = 30(0.7)^2 (0.25)^2 (0.05) = 0.046.$$

b. The marginal distribution of Y_3 is binomial with parameters $n = 5$ and $p_3 = 0.05$. Thus,

$$P(Y_3 \geq 1) = 1 - P(Y_3 = 0) = 1 - (0.95)^5 = 0.2262.$$

5.133

From Ex. 5.27, $f(y_1 | y_2) = \frac{1}{y_2}$, $0 \leq y_1 \leq y_2$ and $f_2(y_2) = 6y_2(1 - y_2)$, $0 \leq y_2 \leq 1$.

a. To find $E(Y_1 | Y_2 = y_2)$, note that the conditional distribution of Y_1 given Y_2 is uniform on the interval $(0, y_2)$. So,

$$E(Y_1 | Y_2 = y_2) = \frac{y_2}{2}.$$

b. To find $E(E(Y_1 | Y_2))$, note that the marginal distribution is beta with $\alpha = 2$ and $\beta = 2$. So, from part a,

$$E(E(Y_1 | Y_2)) = E\left(\frac{Y_2}{2}\right) = \frac{1}{4}.$$

This is the same answer as in Ex. 5.77.

5.139**a.**

$$E(T \mid N = n) = E\left(\sum_{i=1}^n Y_i\right) = \sum_{i=1}^n E(Y_i) = n\alpha\beta.$$

b.

$$E(T) = E[E(T \mid N)] = E(N\alpha\beta) = \lambda\alpha\beta.$$

Note that this is $E(N)E(Y)$.

5.141

$$E(Y_2) = E(E(Y_2 \mid Y_1)) = E\left(\frac{Y_1}{2}\right) = \frac{\lambda}{2}.$$

$$V(Y_2) = E[V(Y_2 \mid Y_1)] + V[E(Y_2 \mid Y_1)] = E\left[\frac{Y_1^2}{12}\right] + V\left(\frac{Y_1}{2}\right) = \frac{2\lambda^2}{12} + \frac{\lambda^2}{2} = \frac{2\lambda^2}{3}.$$

Submitted Problems**5.102**

The quantity $3Y_1 + 5Y_2$ represents the dollar amount spent per week. Thus:

$$E(3Y_1 + 5Y_2) = 3E(Y_1) + 5E(Y_2) = 3(40) + 5(65) = 445.$$

$$V(3Y_1 + 5Y_2) = 9V(Y_1) + 25V(Y_2) = 9(4) + 25(8) = 236.$$

5.110**a.**

$$V(1 + 2Y_1) = 4V(Y_1), \quad V(3 + 4Y_2) = 16V(Y_2), \quad \text{and} \quad \text{Cov}(1 + 2Y_1, 3 + 4Y_2) = 8\text{Cov}(Y_1, Y_2).$$

So,

$$\rho = \frac{8\text{Cov}(Y_1, Y_2)}{\sqrt{4V(Y_1)}\sqrt{16V(Y_2)}} = 0.2.$$

b.

$$V(1 + 2Y_1) = 4V(Y_1), \quad V(3 - 4Y_2) = 16V(Y_2), \quad \text{and } \text{Cov}(1 + 2Y_1, 3 - 4Y_2) = -8\text{Cov}(Y_1, Y_2).$$

So,

$$\rho = \frac{-8\text{Cov}(Y_1, Y_2)}{\sqrt{4V(Y_1)}\sqrt{16V(Y_2)}} = -0.2.$$

c.

$$V(1 - 2Y_1) = 4V(Y_1), \quad V(3 - 4Y_2) = 16V(Y_2), \quad \text{and } \text{Cov}(1 - 2Y_1, 3 - 4Y_2) = 8\text{Cov}(Y_1, Y_2).$$

So,

$$\rho = \frac{8\text{Cov}(Y_1, Y_2)}{\sqrt{4V(Y_1)}\sqrt{16V(Y_2)}} = 0.2.$$

5.120

$$E(C) = E(Y_1) + 3E(Y_2) = np_1 + 3np_2.$$

$$V(C) = V(Y_1) + 9V(Y_2) + 6\text{Cov}(Y_1, Y_2) = np_1q_1 + 9np_2q_2 - 6np_1p_2.$$

5.126

Using formulas for means, variances, and covariances for the multinomial:

$$\begin{aligned} E(Y_1) &= 10(0.1) = 1, & V(Y_1) &= 10(0.1)(0.9) = 0.9, \\ E(Y_2) &= 10(0.05) = 0.5, & V(Y_2) &= 10(0.05)(0.95) = 0.475, \\ \text{Cov}(Y_1, Y_2) &= -10(0.1)(0.05) = -0.05. \end{aligned}$$

So:

$$\begin{aligned} E(Y_1 + 3Y_2) &= 1 + 3(0.5) = 2.5, \\ V(Y_1 + 3Y_2) &= 0.9 + 9(0.475) + 6(-0.05) = 4.875. \end{aligned}$$

5.138

If Y = number of bacteria per cubic centimeter:

a.

$$E(Y) = E(E(Y | \lambda)) = E(\lambda) = \alpha\beta.$$

b.

$$V(Y) = E[V(Y | \lambda)] + V[E(Y | \lambda)] = \alpha\beta + \alpha\beta^2 = \alpha\beta(1 + \beta).$$

Thus:

$$\sigma = \sqrt{\alpha\beta(1 + \beta)}.$$

5.142

a.

$$E(Y) = E(E(Y | p)) = E(np) = nE(p) = \frac{n\alpha}{\alpha + \beta}.$$

b.

$$V(Y) = E[V(Y | p)] + V[E(Y | p)] = E[np(1 - p)] + V(np) = nE(p - p^2) + n^2V(p).$$

Now:

$$nE(p - p^2) = \frac{n\alpha}{\alpha + \beta} - \frac{n\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)},$$

$$n^2V(p) = \frac{n^2\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

So:

$$V(Y) = \frac{n\alpha}{\alpha + \beta} - \frac{n\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)} + \frac{n^2\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} = \frac{n\alpha\beta(\alpha + \beta + n)}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

Additional Problem

a. Total weight for 12 eggs

$$E(\text{Total}) = 12 \times 90 = 1080 \text{ grams},$$

$$\sigma_{\text{Total}} = \sqrt{12 \times (10)^2} = \sqrt{1200} \approx 34.64 \text{ grams}.$$

b. Average weight for 12 eggs

$$E(\text{Average}) = 90 \text{ grams},$$

$$\sigma_{\text{Average}} = \sqrt{\frac{(10)^2}{12}} = \sqrt{\frac{100}{12}} \approx 2.89 \text{ grams}.$$