Chapter 3 Group Work

SOLUTIONS

Contents

Simulation Activity Solutions

Part 1 - psuedocode

Answers may vary.

Part A

```
toss1 <- rbernoulli(1, 0.7)
toss2 <- rbernoulli(1, 0.7)
toss3 <- rbernoulli(1, 0.7)
```

Part B

```
X <- toss1 + toss2 + toss3 # number of heads
Y <- toss2 # logical for whether toss2 is heads
tails <- number of tosses - number of heads
Z <- number of heads - number of tails

#alternative way of calculating Z
Z <- X - (3 - X)
```

Part C

```
#just change 1st argument (# of sims)
toss1 <- rbernoulli(10000, 0.7)
toss2 <- rbernoulli(10000, 0.7)
toss3 <- rbernoulli(10000, 0.7)

#X, Y, Z are then calculated in same way as above
X <- toss1 + toss2 + toss3 # number of heads
Y <- toss2 # logical for whether toss2 is heads
Z <- X - (3 - X)
```

Part D

For X, count the number of times the result is 0, and divide by 10,000. This proportion will be a good estimate of $P(X = 0) = p(0)$. The same is done for X = 1,2,3. This will define the probability distribution. Plotting the 10,000 values of X would give you a visual of the distribution.

For Y, follow the same logic to count the number of times $Y = 0$ and $Y = 1$ and divide each by 10,000. These two proportions specify the pmf.

For Z, follow the same logic to count the number of times Z = -3, -1, 1, 3 and divide by 10,000. These 4 proportions specify the pmf.

Part 2 - pmfs

Distribution of X

Support of X is

$$\mathbb{S} = \{0, 1, 2, 3\}$$

- 1 way to have 0 heads: ttt, with probability $(0.25)^3$
- 3 ways to have 1 head: htt, tht, tth, each with probability $(0.7)(0.3)^2$
- 3 ways to have 2 heads: hht, hth, thh, each with probability $(0.7)^2(0.3)$
- 1 way to have 3 heads: hhh, with probability $(0.75)^3$

So the pmf of X is given by:

$$f(x) = \begin{cases} 0.027, & x = 0 \\ 0.189, & x = 1 \\ 0.441, & x = 1 \\ 0.343, & x = 3 \end{cases}$$

Distribution of Y

Support of Y is

$$\mathbb{S} = \{0, 1\}$$

Y is simply the result of the second coin toss, so the pmf of Y is defined simply as:

$$f(y) = \begin{cases} 0.3, & y = 0 \\ 0.7, & y = 1 \end{cases}$$

Distribution of Z

Note that since Z is the number of heads minus the number of tails, it can be written as $Z = X - (3 - X) = 2X - 3$

Therefore, the support and pmf of Z can be derived easily from the support and pmf of X.

Since X can be 0,1,2,3, the support of Z is

$$\mathbb{S} = \{-3, -1, 1, 3\}$$

Z will take on each of these values with the same probabilities as the pmf of X.

X	Z = 2X - 3	f(x) = f(z)
0	-3	0.027
1	-1	0.189
2	1	0.441
3	3	0.343

Back of Packet Group Work Solutions

Problem 1

For each of the following, determine the constant c so that $f(x)$ satisfies the conditions of being a pmf for a random variable X. That is, find c such that $\sum_{x \in S} f(x) = 1$.

a. $f(x) = x/c, \quad x = 1, 2, 3, 4$
b. $f(x) = cx, \quad x = 1, 2, 3, \ldots, 10$
c. $f(x) = c(1/4)^x, \quad x = 1, 2, 3, \ldots$

Hint: use the infinite series identity from calculus that tells us

$$\sum_{n=1}^{\infty} a_1 (r)^{n-1} = \frac{a_1}{1-r}$$

Solution part a

$$\begin{aligned} 1 &= \sum_{x=1}^{4} f(x) \\ &= \sum_{x=1}^{4} \frac{x}{c} \\ &= \frac{1}{c} + \frac{2}{c} + \frac{3}{c} + \frac{4}{c} \\ &= \frac{10}{c} & \implies c = 10 \end{aligned}$$

Solution part b

$$\begin{aligned} 1 &= \sum_{x=1}^{10} f(x) \\ &= \sum_{x=1}^{10} cx \\ &= c \sum_{x=1}^{10} x \\ &= c(1+2+3+4+5+6+7+8+9+10) \\ &= c55 \\ &\implies c = \frac{1}{55} \end{aligned}$$

Solution part c

$$\begin{aligned}
1 &= \sum_{x=1}^{\infty} f(x) \\
&= \sum_{x=1}^{\infty} c\left(\frac{1}{4}\right)^x \\
&= \sum_{x=1}^{\infty} c\frac{1}{4}\left(\frac{1}{4}\right)^{x-1}
\end{aligned}$$

Then $a_1 = \frac{c}{4}$ and $r = \frac{1}{4}$, and we can use the fact that the geometric series $\sum_{n=1}^{\infty} a_1(r)^{n-1} = \frac{a_1}{1-r}$. So we have

$$\begin{aligned}
1 &= \frac{c/4}{1-1/4} \\
&= \frac{c/4}{3/4} \\
&= \frac{c}{3} \\
&\implies c = 3
\end{aligned}$$

Problem 2

Recall the non-uniform example where we rolled a fair four-sided die twice and let X be the maximum of the two outcomes. We determined that the pmf of X was given by

$$f(x) = \frac{2x-1}{16}, \quad x = 1, 2, 3, 4$$

. Define the cdf of X. That is define $F(x) = P(X \leq x)$ for each value of x in the support.

Solution

First recall:

- $f(1) = P(X = 1) = 1/16$
- $f(2) = P(X = 2) = 3/16$
- $f(3) = P(X = 3) = 5/16$
- $f(4) = P(X = 4) = 7/16$

To define the cdf, we define $F(x) = P(X \leq x)$ for each value $x = 1, 2, 3, 4$:

$$\begin{aligned}
F(1) &= P(X \leq 1) \\
&= P(X = 1) \\
&= 1/16
\end{aligned}$$

$$\begin{aligned}
F(2) &= P(X \leq 2) \\
&= P(X = 1) + P(X = 2) \\
&= 1/16 + 3/16 \\
&= 4/16
\end{aligned}$$

$F(3) = P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3) = 1/16 + 3/16 + 5/16 = 9/16$

$F(4) = P(X \leq 4) = P(X = 1)+P(X = 2)+P(X = 3)+P(X = 4) = 1/16+3/16+5/16+7/16 = 16/16 = 1$

Note that this can be written as a general expression $F(x) = \frac{x^2}{16}, \quad x = 1, 2, 3, 4$

Problem 3

$$\begin{aligned} E(X) &= 0(.027) + 1(.189) + 2(.441) + 3(.343) \\ &= 2.1 \end{aligned}$$

$$\begin{aligned} E(Y) &= 0(.3) + 1(.7) \\ &= 0.7 \end{aligned}$$

$$\begin{aligned} E(Z) &= -3(.027) - 1(.189) + 1(.441) + 3(.343) \\ &= 1.2 \end{aligned}$$

Problem 4

Let $E(X) = 4$. Find

a. $E(Y)$ when $Y = 2X + 3$
b. $E(Z)$ when $Z = 7 - 5X$
c. $E(32)$

Solution part a

$$\begin{aligned} E(Y) &= E(2X + 3) \\ &= E(2X) + E(3) \\ &= 2E(X) + 3 \\ &= 2(4) + 3 \\ &= 11 \end{aligned}$$

Alternatively, could show it from definition of Expected value:

$$\begin{aligned} E(Y) &= E(2X + 3) \\ &= \sum_{x \in S} (2x + 3)f(x) \\ &= \sum_{x \in S} (2xf(x) + 3f(x)) \\ &= \sum_{x \in S} 2xf(x) + \sum_{x \in S} 3f(x) \\ &= 2\sum_{x \in S} xf(x) + 3\sum_{x \in S} f(x) \\ &= 2E(X) + 3(1) \\ &= 2(4) + 3 \\ &= 11 \end{aligned}$$

Solution part b

$$\begin{aligned} E(Z) &= E(7 - 5X) \\ &= 7 - 5E(X) \\ &= 7 - 5(4) \\ &= -13 \end{aligned}$$

Solution part c

$E(32) = 32$ by part 1 of Theorem 2.2-1

Problem 5

Let $f(x) = \frac{x}{10}, \quad x = 1, 2, 3, 4$. Find:

a. $E(X)$
b. $E(X^2)$
c. $E(X(5 - X))$

Solution part a

$$\begin{aligned} E(X) &= \sum x f(x) \\ &= \sum x \frac{x}{10} \\ &= \frac{1}{10} \sum x^2 \\ &= \frac{1}{10}(1^2 + 2^2 + 3^2 + 4^2) \\ &= \frac{1}{10}(30) \\ &= 3 \end{aligned}$$

Solution part b

$$\begin{aligned} E(X^2) &= \sum x^2 f(x) \\ &= \sum x^2 \frac{x}{10} \\ &= \frac{1}{10} \sum x^3 \\ &= \frac{1}{10}(1^3 + 2^3 + 3^3 + 4^3) \\ &= \frac{1}{10}(100) \\ &= 10 \end{aligned}$$

Solution part c

$$\begin{aligned}
E[X(5-X)] &= E[5X - X^2] \\
&= E(5X) - E(X^2) \\
&= 5E(X) - E(X^2) \\
&= 5(3) - 10 \\
&= 5
\end{aligned}$$

Problem 6

Let $E(X) = 5$ and $V(X) = 36$. Find:

a) $V(3X+7)$
b) $V(2-X)$
c) $E(X^2)$
d) $E(5X+2X^2)$

Solution part a

$$V(3X+7) = 3^2 V(X) = 9 * 36 = 324$$

Solution part b

$$V(2-X) = V(-X+2) = (-1)^2 V(X) = V(X) = 36$$

Solution part c

Note

$$V(X) = E(X^2) - [E(X)]^2 \implies E(X^2) = V(X) + [E(X)]^2$$

Therefore,

$$E(X^2) = 36 + 5^2 = 61$$

Solution part d

$$\begin{aligned}
E(5X+2X^2) &= E(5X) + E(2X^2) \\
&= 5E(X) + 2E(X^2) \\
&= 5(5) + 2(61) \\
&= 147
\end{aligned}$$