Chapter 5 Part 1 Group Work

SOLUTIONS

Problem 1

Suppose that you randomly select a student from a large population of 4th grade students and record the student's sex and number of siblings. Let M = 1 if the selected student is male and M = 0 if the selected student is female. Let S record the student's number of siblings. The proportion of the population of students falling into each category of M and S is recorded in the table below.

# of siblings:	0	1	2	3	4
Female	.10	.18	.12	.07	.04
Male	.12	.18	.14	.03	.02

a. Verify that the above table is a valid joint pmf.
b. Find the probability that the randomly selected student will be male with two siblings
c. Find $P(M = 0, S \leq 2)$
d. Find the probability that the randomly selected student will be an only child
e. Find $F(1, 1)$

Solution

Part a

$0.10 + 0.12 + 0.18 + 0.18 + 0.12 + 0.14 + 0.07 + 0.03 + 0.04 + 0.02 = 1$

Part b

$P(M = 1, S = 2) = 0.14$

Part c

$P(M = 0, S \leq 2) = 0.12 + 0.18 + 0.14 = 0.44$

Part d

$P(S = 0) = 0.10 + 0.12 = 0.22$

Part e

$F(1, 1) = P(M \leq 1, S \leq 1) = P(S \leq 1) = 0.10 + 0.12 + 0.18 + 0.18 = 0.58$

Problem 2

Of nine executives in a business firm, four are married, three have never married, and two are divorced. Three of the executives are to be selected for promotion. Let Y_1 denote the number of married executives and Y_2 denote the number of never married executives among the three selected for promotion. Assuming that the three are randomly selected from the nine available, find the joint pmf of Y_1 and Y_2.

Solution

Note that using material from Chapter 3, the joint probability function is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}, \quad 0 \le y_1, 0 \le y_2, y_1 + y_2 \le 3$$

Given table format, this is

y2 y1	0	1	2	3
0	0	3/84	6/84	1/84
1	4/84	24/84	12/84	0
2	12/84	18/84	0	0
3	4/84	0	0	0

Problem 3

Let Y_1 and Y_2 have the joint pdf

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0 \\ 0, & elsewhere \end{cases}$$

a. What is $P(Y_1 < 1, Y_2 > 5)$? b. What is $P(Y_1 + Y_2 < 3)$?

Part a

$$\begin{aligned} P(Y_1 < 1, Y_2 > 5) &= \int_0^1 \int_5^\infty e^{-(y_1+y_2)}\, dy_2\, dy_1 \\ &= \int_0^1 \left(-e^{-(y_1+y_2)}|_{y_2=5}^{\infty}\right) dy_1 \\ &= \int_0^1 e^{-(y_1+5)} dy_1 \\ &= -e^{-(1+5)} + e^{-5} \\ &= .00426. \end{aligned}$$

Part b

$$\begin{aligned} P(Y_1 + Y_2 < 3) &= P(Y_1 < 3 - Y_2) \\ &= \int_0^3 \left(\int_0^{3-y_1} e^{-(y_1+y_2)}\, dy_2\right) dy_1 \\ &= 1 - 4e^{-3} \\ &= .8009. \end{aligned}$$

Problem 4

Return to the scenario in Problem 1

Return to the scenario in Problem 1.

a. Find the marginal distribution of M
b. Find the marginal distribution of S
c. Find $P(S \leq 2|M = 1)$
d. Find $P(M = 1|S \leq 2)$

Solutions

Part a

	P(M = m)
Female (M = 0)	0.51
Male (M = 1)	0.49

Part b

# of siblings (S):	0	1	2	3	4
P(S = s)	.22	.36	.26	.10	.06

Part c

$$P(S \leq 2 \mid M = 1) = \frac{P(S \leq 2, M = 1)}{P(M = 1)}$$

- $P(M = 1) = 0.49$
- $P(S \leq 2, M = 1) = P(S = 0, M = 1) + P(S = 1, M = 1) + P(S = 2, M = 1) = 0.12 + 0.18 + 0.14 = 0.44$

$$P(S \leq 2 \mid M = 1) = \frac{0.44}{0.49} \approx 0.898$$

Part d

$$P(M = 1 \mid S \leq 2) = \frac{P(S \leq 2, M = 1)}{P(S \leq 2)}$$

- $P(S \leq 2) = P(S = 0) + P(S = 1) + P(S = 2) = 0.22 + 0.36 + 0.26 = 0.84$
- $P(S \leq 2, M = 1) = 0.44$ (from above)

$$P(M = 1 \mid S \leq 2) = \frac{0.44}{0.84} \approx 0.524$$

Problem 5

Let $f(x,y) = \frac{4}{3}(1-xy), \quad 0 \le x \le 1, \quad 0 \le y \le 1.$

a. Find the marginal pdfs of X and Y.
b. Find $P(X \le Y/2)$

$$f(x,y) = \frac{4}{3}(1-xy), \quad 0 \le x \le 1,\ 0 \le y \le 1$$

$$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 \frac{4}{3}(1-xy)\,dy = \frac{4}{3}\left[y - \frac{1}{2}xy^2\right]_0^1 = \frac{4}{3}\left(1-\frac{1}{2}x\right) = \frac{4}{3} - \frac{2}{3}x,\ 0<x<1$$

$$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 \frac{4}{3}(1-xy)\,dx = \frac{4}{3}\left[x - \frac{1}{2}yx^2\right]_0^1 = \frac{4}{3}\left(1-\frac{1}{2}y\right) = \frac{4}{3} - \frac{2}{3}y,\ 0<y<1$$

$$P(X \le Y/2) = \int_0^1 \int_0^{y/2} f(x,y)\,dx\,dy = \int_0^1 \int_0^{y/2} \frac{4}{3}(1-xy)\,dx\,dy = \frac{7}{24}$$

Problem 6

Let the joint pmf of X and Y be

$$f(x,y) = \frac{xy^2}{30}, \quad x = 1,2,3, \quad y = 1,2$$

a. What are the marginal distributions of X and Y?
b. Are X and Y independent?
c. What is $P(X>1)$?
d. What is $E(X)$?

Part a

$$f_X(x) = \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x\cdot 1^2}{30} + \frac{x \cdot 2^2}{30} = \frac{x}{30} + \frac{4x}{30} = \frac{5x}{30} = \frac{x}{6}, x = 1,2,3$$

$$f_Y(y) = \sum_{x=1}^{3} \frac{xy^2}{30} = \frac{1\cdot y^2}{30} + \frac{2\cdot y^2}{30} + \frac{3\cdot y^2}{30} = \frac{6y^2}{30} = \frac{y^2}{5},\ y = 1,2$$

Part b

Since $f_X(x)f_Y(y) = \frac{x}{6}\frac{y^2}{5} = \frac{xy^2}{30} = f(x,y)$ for all x, y, X **and** Y **are independent**.

Part c

$$P(X>1) = f_X(2) + f_X(3) = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$$

Part d

$$E(X) = \sum_{x=1}^{3} x f_X(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{3}{6} = \frac{14}{6} = \frac{7}{3}$$

Problem 7

Let X and Y be two continuous random variables with joint pdf $f(x,y) = \frac{3}{16}xy^2, \quad 0 \leq x \leq 2, \;\; 0 \leq y \leq 2$. Are the two random variables independent?

Solution

Both are bounded between constants, and we can define $g(x) = \frac{3}{16}x$ and $h(y) = y^2$, so by Theorem 5.5 they are independent.

Problem 8

Let X and Y have the joint pdf $f(x,y) = x + y, 0 \leq x \leq 1, 0 \leq y \leq 1$. Find the marginal pdfs $f_X(x)$ and $f_Y(y)$ and show that X and Y are dependent.

Solution

$$f_x(x) = \int_0^1 (x+y)dy = x + \frac{1}{2}, \quad 0 \leq x \leq 1$$

$$f_y(y) = \int_0^1 (x+y)dx = y + \frac{1}{2}, \quad 0 \leq y \leq 1$$

$$f_x(x)f_y(y) = (x + \frac{1}{2})(y + \frac{1}{2}) \neq x + y$$

Problem 9

Show that $Cov(X,Y) = E(XY) - \mu_X \mu_Y$.

> *Hint: start with the definition* $Cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)]$

Solution

$$\begin{aligned} Cov(X,Y) &= E((X - \mu_X)(Y - \mu_Y)) \\ &= E(XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y) \\ &= E(XY) - \mu_Y E(X) - \mu_X E(Y) + \mu_X \mu_Y \\ &= E(XY) - \mu_Y \mu_X - \mu_X \mu_Y + \mu_X \mu_Y \\ &= E(XY) - \mu_X \mu_Y \end{aligned}$$

Problem 10

Show that $E(XY) = \mu_X \mu_Y + \rho_{XY}\sigma_X\sigma_Y$

Solution

By re-arranging the identity proven in Problem 1, we have

$$E(XY) = Cov(X,Y) + E(X)E(Y)$$

Since the correlation is defined as $\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X\sigma_y}$, we have $Cov(X,Y) = \rho_{XY}\sigma_X\sigma_y$. Plugging in gives

$$E(XY) = \rho_{XY}\sigma_X\sigma_y + E(X)E(Y)$$

Problem 11

Show that if two random variables X and Y are independent, then their covariance is 0.

> *Hint: start with the shortcut formula* $Cov(X,Y) = E(XY) - \mu_X\mu_Y$ and re-write $E(XY)$ as a double sum.

Solution

Let X, Y be independent. We will therefore use the fact that $f(x,y) = f_x(x)f_y(y)$

$$\begin{aligned}
Cov(X,Y) &= E((X-\mu_X)(Y-\mu_Y)) \\
&= \sum_x \sum_y xyf(x,y) - \mu_X\mu_Y \\
&= \sum_x \sum_y xyf_x(x)f_y(y) - \mu_X\mu_Y \\
&= \sum_x xf_x(x) \sum_y yf_y(y) - \mu_X\mu_Y \\
&= E(X)E(Y) - \mu_X\mu_Y \\
&= 0
\end{aligned}$$

Problem 12

Return to the scenario in Problem 1.

a. Are M and S independent?
b. Find $E(M)$ and $V(M)$
c. Find $E(S)$ and $V(S)$
d. Find $Cov(M,S)$
e. Suppose that instead of just one student, you sample 12 students. Let $S_1, \dots S_{12}$ represent the number of siblings reported by each student (you can assume that these random variables are independent). Let $\overline{S} = \sum_{i=1}^{12} \frac{S_i}{12}$ be the average number of siblings in the sample. Find $E(\overline{S})$ and $V(\overline{S})$

Part a

Independent if and only if $P(S = s)P(M = m) = P(S = s, M = m)$ for all s, m. For the coordinate (0,1), we have $P(S = 0)P(M = 0) = 0.22 * 0.51 = 0.1122 \neq P(S = 0, M = 0) = .10$, therefore S and M are NOT independent.

$$\sigma_y^2 = \frac{11}{144}$$

Part b

Using the marginal distribution found above, we have

$$\begin{aligned} E(M) &= 0(.51) + 1(.49) = 0.49 \\ E(M^2) &= 0^2(.51) + 1^2(.49) = 0.49 \\ V(M) &= 0.49 - (0.49)^2 = 0.2499 \end{aligned}$$

Part c

Using the marginal distribution found above, we have

$$E(S) = 0(.22) + 1(.36) + 2(.26) + 3(.10) + 4(.06) = 1.42$$

$$E(S^2) = 0^2(.22) + 1^2(.36) + 2^2(.26) + 3^2(.10) + 4^2(.06) = 3.26$$

$$V(S) = 3.26 - (1.42)^2 = 1.2436$$

Part d

$$\begin{aligned} Cov(M, S) &= \sum_m \sum_s msf(m, s) \\ &= (1)(1)(.18) + (1)(2)(.14) + (1)(3)(.03) + (1)(4)(.02) \\ &= 0.63 \end{aligned}$$

Note that the other 6 (m,s) coordinate pairs have at least one 0, so the term cancels out of double sum

Part e

$$
\begin{aligned}
E(\overline{S}) &= E(\sum_{i=1}^{12} \frac{S_i}{12}) \\
&= \sum_{i=1}^{12} E(\frac{S_i}{12}) \quad \text{b/c E() is a linear operator} \\
&= \sum_{i=1}^{12} \frac{1}{12} E(S_i) \\
&= \sum_{i=1}^{12} \frac{1}{12}(1.42) \\
&= \frac{1}{12}(1.42) \sum_{i=1}^{12} 1 \\
&= \frac{1}{12}(1.42)(12) \\
&= 1.42 = E(S_i)
\end{aligned}
$$