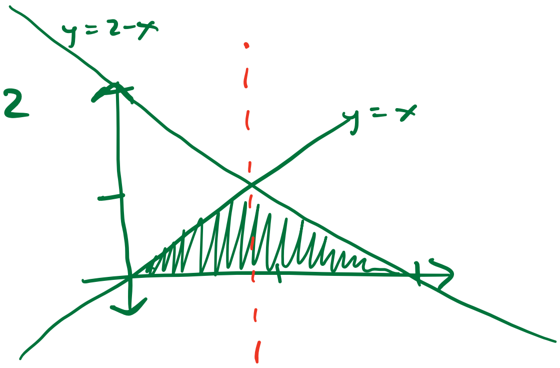
Practice Final Exam

① $f(x) = \frac{x}{12} \qquad -1 < x < 5$

$$\int_{-1}^{5} \frac{x}{12}\,dx = \frac{x^2}{24}\Big|_{-1}^{5} = \frac{1}{24}\left(25 - (-1)^2\right) = \frac{24}{24} = 1 \checkmark \quad \text{valid pdf}$$

② $f(x,y) = 1 \qquad 0 < y < x < 2, \qquad x + y < 2$

$y < 2 - x$



$$f_x(x) = \begin{cases} \int_0^x 1\,dy & 0 < x < 1 \\ \int_0^{2-x} 1\,dy & 1 < x < 2 \end{cases}$$

$$\boxed{f_x(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}}$$

③ $f(x,y) = x + y \qquad 0 < x < 1, \qquad 0 < y < 1$

$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$$E(X) = \int_0^1\int_0^1 x(x+y)\,dx\,dy = \int_0^1\int_0^1 (x^2 + xy)\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^3}{3}\Big|_0^1 + \frac{x^2}{2}y\Big|_0^1 \right) dy = \int_0^1 \frac{1}{3} + \frac{1}{2}y \; dy$$

$$= \frac{1}{3}y\Big|_0^1 + \frac{1}{4}y^2\Big|_0^1 = \frac{1}{3} + \frac{1}{4} = \boxed{\frac{7}{12}}$$

$E(Y)$ = same b/c X, Y interchangeable

$$E(XY) = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int\int (x^2y + xy^2)\,dx\,dy$$

$$\int \left[\frac{x^3}{3}y + \frac{x^2}{2}y^2 \Big|_0^1 \, dy = \int_0^1 \left(\frac{1}{3}y + \frac{1}{2}y^2 \right) dy$$

$$= \left[\frac{1}{6}y^2 + \frac{1}{6}y^3 \Big|_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$\mathrm{Cov}(X,Y) = \frac{2}{6} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = \frac{48}{144} - \frac{49}{144} = \boxed{\frac{-1}{144}}$$

④ X = # of bets required to reach 4^{th} win

$X \sim \text{negbinom}(r=4, p=18/38)$ $\quad p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$

a) $P(X=9) = \binom{8}{3}\left(\frac{18}{38}\right)^4\left(\frac{20}{38}\right)^5 = \boxed{0.1139}$

b) Winnings $= W = 5(4) - 5(x-4)$

$= 20 - 5x + 20 = 40 - 5x$

$E(W) = E(40-5X) = 40 - 5E(X) = 40 - 5\left(\frac{r}{p}\right)$

$= 40 - 5\left(\frac{4}{18/38}\right) = \boxed{-\$2.22}$

c) $V(W) = V(40-5X) = 25V(X) = 25\left(\frac{r(1-p)}{p^2}\right)$

$= 25\left(\frac{4(20/38)}{(18/38)^2}\right) = \boxed{234.57}$

⑤ W_1 = # w/ one band $\quad p_1 = .12$

W_2 = # w/ multiple $\quad p_2 = .06$

W_3 = # untagged $\quad p_3 = .82$

multinomial w/ $n=15$

$E(W_i) = np_i$ $\quad \text{Cov}(W_i, W_j) = -np_ip_j$

$V(W_i) = np_iq_i$

$E(T) = E(5W_1 + 12W_2) = 5E(W_1) + 12E(W_2)$

$= 5(15(.12)) + 12(15(.06))$ by multin. dist.

$= 19.8$

$V(T) = V(5W_1 + 12W_2) = 25V(W_1) + 144V(W_2) + 2(5)(12)\text{Cov}(W_1, W_2)$

$= 25(15(.12)(.88)) + 144(15(.06)(.94))$

$+ 2(5)(12)[-15(.12)(.06)]$

$= 39.6 + 121.824 - 12.96$

$= \boxed{148.464}$

⑥ $X \sim \text{Poisson}(\lambda = 2)$ $\quad f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

$$P(X \geq 4 \mid X > 2) = \frac{P(X \geq 4 \cap X > 2)}{P(X > 2)} = \frac{P(X \geq 4)}{P(X > 2)}$$

$P(X \geq 4) = 1 - P(X \leq 3) = 1 - [\quad] = 0.1429$

$P(X=0) = \dfrac{2^0 e^{-2}}{0!} = .1353$

$P(X=1) = \dfrac{2^1 e^{-2}}{1!} = .2707$

$P(X=2) = \dfrac{2^2 e^{-2}}{2!} = .2707$

$P(X=3) = \dfrac{2^3 e^{-2}}{3!} = .1804$

$P(X > 2) = 1 - P(X \leq 2)$

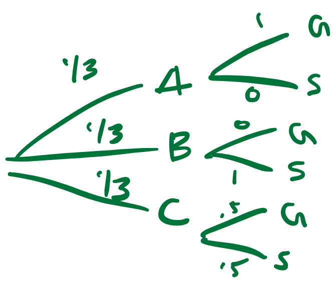
$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= .3233$

So $P(X \geq 4 \mid X > 2) = \dfrac{.1429}{.3233}$

$= \boxed{0.442}$

⑦



$$P(A \mid G) = \frac{P(A \cap G)}{P(G)}$$

$$= \frac{1/3}{1/3 + 1/3(0) + 1/3(1/2)} = \frac{1/3}{3/6} = \frac{6}{9} = \boxed{\frac{2}{3}}$$

⑧

2nd

1st	1	2	3	4	6
1	.	2	3	4	6
2	2	.	6	8	12
3	3	6	.	12	18
4	4	8	12	.	24
6	6	12	18	24	.

x	$P(X=x)$
2	2/20
3	2/20
4	2/20
6	4/20
8	2/20
12	4/20
18	2/20
24	2/20
	20/20 ✓

(9) $N = 28$ $r = 16$ $N-r = 12$ k = sample size = n

$y = 5$ stats majors among k students in sample

Hypergeometric.

$$P(Y=5) = \frac{\binom{16}{5}\binom{12}{k-5}}{\binom{28}{k}}$$

k can be anything from 5 to 28 - plug in to see what maximizes this probability. increases up to $k=9$, then decreases, so $\boxed{k=9}$

(10) $P(A) = 0.8$ $P(B) = 0.6$ $P(B|A) = 0.7$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B|A)P(A) = (0.7)(0.8) = .56$

$= .8 + .6 - .56 = \boxed{0.84}$

(11) prob of miss = 0.1

a) $Y \sim \text{binomial}(n=10, p=0.1)$ Y = # of misses

$P(Y \geq 2) = 1 - [P(Y=0) + P(Y=1)]$

$P(Y=0) = \binom{10}{0} .1^0 .9^{10} = .3487$

$P(Y=1) = \binom{10}{1} .1^1 .9^9 = .3874$

$= \boxed{0.2639}$

b) $P(Y = \text{even}) = 1 - P(\text{odd})$

odd: $Y = 2k+1$ $P(Y=2k+1) = \sum_{k=0}^{\infty} \underbrace{(.9)^{2k}}_{2k \text{ makes}} \underbrace{(.1)}_{1 \text{ miss}}$ $= 2k+1$ attempts

$= .1 \underbrace{\sum_{k=0}^{\infty} (.9^2)^k}_{\text{geometric series}} = .1\left(\frac{1}{1-.9^2}\right)$

$= .5263$

$P(Y = \text{even}) = 1 - .5263 = \boxed{.4737}$

(12) $f(x) = \begin{cases} \frac{x}{3} & 0 < x < 2 \quad (1) \\ \frac{1}{9} & 2 < x < 5 \quad (2) \end{cases}$

For (1): $\int_0^x \frac{t}{3}\,dt = \frac{t^2}{6}\Big|_0^x = \frac{x^2}{6} \qquad 0 < x < 2$

at $x=2$: $\frac{2^2}{6} = \frac{2}{3}$

For (2): $\frac{2}{3} + \int_2^x \frac{1}{9}\,dt = \frac{2}{3} + \frac{1}{9}t\Big|_2^x = \frac{2}{3} + \frac{1}{9}(x-2)$

$= \frac{4+x}{9} \qquad 2 < x < 5$

$$\boxed{F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{6} & 0 < x < 2 \\ \frac{4+x}{9} & 2 < x < 5 \\ 1 & x > 5 \end{cases}}$$

note both equal $\frac{2}{3}$ when $x=2$

b) median is $F(x) = 0.5$ note $F(2) = \frac{4}{6} > 0.5$ so median < 2

set $0.5 = \frac{x^2}{6}$ and solve for x

$3 = x^2$

$\sqrt{3} = x = \boxed{1.732}$