HW 10 SOLUTIONS

Practice Problems

5.45

No. Counterexample:

$$P(Y_1 = 2, Y_2 = 2) = 0 \neq P(Y_1 = 2)P(Y_2 = 2) = \left(\frac{1}{9}\right)\left(\frac{1}{9}\right).$$

5.49

Note that

$$f_1(y_1) = \int_0^{y_1} 3y_1\, dy_2 = 3y_1^2, \qquad 0 \leq y_1 \leq 1,$$

and

$$f_2(y_2) = \int_{y_2}^{1} 3y_1\, dy_1 = \frac{3}{2}\left(1 - y_2^2\right), \qquad 0 \leq y_2 \leq 1.$$

Thus,

$$f(y_1, y_2) \neq f_1(y_1)f_2(y_2),$$

so that Y_1 and Y_2 are dependent.

5.53

The ranges of y_1 and y_2 depend on each other so Y_1 and Y_2 cannot be independent.

5.63

$$P(Y_1 > Y_2,\, Y_1 < 2Y_2) = \int_0^\infty \int_{y_1/2}^{y_1} e^{-(y_1+y_2)}\, dy_2\, dy_1 = \frac{1}{6},$$

and

$$P(Y_1 < 2Y_2) = \int_0^\infty \int_{y_1/2}^{\infty} e^{-(y_1+y_2)}\, dy_2\, dy_1 = \frac{2}{3}.$$

So,

$$P(Y_1 > Y_2 \mid Y_1 < 2Y_2) = \frac{1}{4}.$$

5.77

Assume uniform distributions for the call times over the 1-hour period.

(a)

$$P(Y_1 \le 1/2,\, Y_2 \le 1/2) = P(Y_1 \le 1/2)P(Y_2 \le 1/2) = (1/2)(1/2) = 1/4.$$

(b)

Note that 5 minutes $= 1/12$ hour. To find $P(|Y_1 - Y_2| \le 1/12)$, we break the region into three parts:

$$\begin{aligned} P(|Y_1 - Y_2| \le 1/12) = & \int_0^{1/12} \int_0^{y_1+1/12} 1\, dy_2\, dy_1 \\ & + \int_{1/12}^{11/12} \int_{y_1-1/12}^{y_1+1/12} 1\, dy_2\, dy_1 \\ & + \int_{11/12}^{1} \int_{y_1-1/12}^{1} 1\, dy_2\, dy_1 \\ = & \frac{23}{144}. \end{aligned}$$

5.81

Since Y_1 and Y_2 are independent,

$$E\left(\frac{Y_2}{Y_1}\right) = E(Y_2)\, E\left(\frac{1}{Y_1}\right).$$

Thus, using the marginal densities found in Exercise 5.61,

$$
\begin{aligned}
E\left(\frac{Y_2}{Y_1}\right) &= E(Y_2)\, E\left(\frac{1}{Y_1}\right) \\
&= \frac{1}{2}\int_0^{\infty} y_2 e^{-y_2/2}\, dy_2 \left[\frac{1}{4}\int_0^{\infty} e^{-y_1/2}\, dy_1\right] \\
&= 2\left(\frac{1}{2}\right) \\
&= 1.
\end{aligned}
$$

5.87

(a)

$$E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = \nu_1 + \nu_2.$$

(b)

By independence,

$$\mathrm{Var}(Y_1 + Y_2) = \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) = 2\nu_1 + 2\nu_2.$$

5.89

$$
\begin{aligned}
\mathrm{Cov}(Y_1, Y_2) &= E(Y_1Y_2) - E(Y_1)E(Y_2) \\
&= \sum_{y_1}\sum_{y_2} y_1 y_2\, p(y_1, y_2) - \left[2\left(\frac{1}{3}\right)\right]^2 \\
&= \frac{2}{9} - \frac{4}{9} \\
&= -\frac{2}{9}.
\end{aligned}
$$

As the value of Y_1 increases, the value of Y_2 tends to decrease.

5.93

a. From Ex. 5.55 and 5.79, $E(Y_1Y_2) = 0$ and $E(Y_1) = 0$. So,

$$\mathrm{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 0 - 0 \cdot E(Y_2) = 0.$$

b. Y_1 and Y_2 are dependent.

c. Since $\text{Cov}(Y_1, Y_2) = 0$, $\rho = 0$.

d. If $\text{Cov}(Y_1, Y_2) = 0$, Y_1 and Y_2 are not necessarily independent.

5.95

Note that the marginal distributions for Y_1 and Y_2 are

y_1	-1	0	1
$p_1(y_1)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

y_2	0	1
$p_2(y_2)$	$\frac{2}{3}$	$\frac{1}{3}$

So, Y_1 and Y_2 are not independent since

$$p(-1, 0) \neq p_1(-1)p_2(0).$$

However, $E(Y_1) = 0$ and

$$\begin{aligned} E(Y_1Y_2) &= (-1)(0)\left(\tfrac{1}{3}\right) + (0)(1)\left(\tfrac{1}{3}\right) + (1)(0)\left(\tfrac{1}{3}\right) \\ &= 0, \end{aligned}$$

so

$$\text{Cov}(Y_1, Y_2) = 0.$$

Submitted Problems

5.48

Dependent. For example,

$$P(Y_1 = 0, Y_2 = 0) \neq P(Y_1 = 0)\, P(Y_2 = 0).$$

5.52

Note that $f(y_1, y_2)$ can be factored and the ranges of y_1 and y_2 do not depend on each other so by Theorem 5.5 Y_1 and Y_2 are independent.

5.60

From Exercise 5.36,

$$f_1(y_1) = y_1 + 1/2, \quad 0 \le y_1 \le 1,$$

and

$$f_2(y_2) = y_2 + 1/2, \quad 0 \le y_2 \le 1.$$

But,

$$f(y_1, y_2) \neq f_1(y_1) f_2(y_2),$$

so Y_1 and Y_2 are dependent.

5.64

$P(Y_1 > Y_2,\ Y_1 < 2Y_2) = \int_{1/2}^{1} \int_{0}^{y_1} 1\, dy_2\, dy_1 = \frac{1}{4}$

$P(Y_1 < 2Y_2) = 1 - P(Y_1 \ge 2Y_2) = 1 - \int_0^1 \int_0^{y_1/2} 1\, dy_2\, dy_1 = \frac{3}{4}$

So, $P(Y_1 > Y_2 \mid Y_1 < 2Y_2) = \frac{1}{3}$

5.76

From Ex. 5.52, we found that Y_1 and Y_2 are independent. So,

a.

$$E(Y_1) = \int_0^1 2y_1^2\, dy_1 = \frac{2}{3}.$$

b.

$$E(Y_1^2) = \int_0^1 2y_1^3\, dy_1 = \frac{2}{4}, \text{ so } V(Y_1) = \frac{2}{4} - \frac{4}{9} = \frac{1}{18}.$$

c.

$$E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 0.$$

5.80

From Ex. 5.36, $f_1(y_1) = y_1 + \frac{1}{2},\ 0 \le y_1 \le 1$, and $f_2(y_2) = y_2 + \frac{1}{2},\ 0 \le y_2 \le 1$. Thus,

$$E(Y_1) = \frac{7}{12} \text{ and } E(Y_2) = \frac{7}{12}.$$

So,

$$E(30Y_1 + 25Y_2) = 30\left(\frac{7}{12}\right) + 25\left(\frac{7}{12}\right) = 32.08.$$

5.92

From Ex. 5.77, $E(Y_1) = \frac{1}{4}$ and $E(Y_2) = \frac{1}{2}$.

$$E(Y_1Y_2) = \int_0^1 \int_0^{y_2} 6y_1y_2(1-y_2)\,dy_1\,dy_2 = \frac{3}{20}.$$

So,

$\mathrm{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = \dfrac{3}{20} - \dfrac{1}{8} = \dfrac{1}{40}$ as expected since Y_1 and Y_2 are dependent.

5.94

a.

$$\begin{aligned}
\mathrm{Cov}(U_1, U_2) &= E[(Y_1+Y_2)(Y_1-Y_2)] - E(Y_1+Y_2)E(Y_1-Y_2) \\
&= E(Y_1^2) - E(Y_2^2) - [E(Y_1)]^2 - [E(Y_2)]^2 \\
&= (\sigma_1^2 + \mu_1^2) - (\sigma_2^2 + \mu_2^2) - (\mu_1^2 - \mu_2^2) = \sigma_1^2 - \sigma_2^2.
\end{aligned}$$

b. Since $V(U_1) = V(U_2) = \sigma_1^2 + \sigma_2^2$ (Y and Y are uncorrelated),

$$\rho = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

c. If $\sigma_1^2 = \sigma_2^2$, U_1 and U_2 are uncorrelated.

Additional Problems

Problem 1

Let X and Y be independent random variables with $\mu_X = 1$, $\sigma_X = 10$, $\mu_Y = 2$, and $\sigma_Y = 4$. Compute the mean and standard deviation for:

a. $X + Y$

$$\mu = \mu_X + \mu_Y = 1 + 2 = 3, \quad \sigma = \sqrt{\sigma_X^2 + \sigma_Y^2} = \sqrt{10^2 + 4^2} = \sqrt{116} \approx 10.77.$$

b. $X - Y$

$$\mu = \mu_X - \mu_Y = 1 - 2 = -1, \quad \sigma = \sqrt{\sigma_X^2 + \sigma_Y^2} = \sqrt{10^2 + 4^2} = \sqrt{116} \approx 10.77.$$

c. $X + 4Y$

$\mu = \mu_X + 4\mu_Y = 1 + 4(2) = 9, \quad \sigma = \sqrt{\sigma_X^2 + (4^2)\sigma_Y^2} = \sqrt{10^2 + 16(4^2)} = \sqrt{100 + 256} = \sqrt{356} \approx 18.87.$

d. $2X - 5Y$

$\mu = 2\mu_X - 5\mu_Y = 2(1) - 5(2) = -8, \quad \sigma = \sqrt{(2^2)\sigma_X^2 + (5^2)\sigma_Y^2} = \sqrt{4(10^2) + 25(4^2)} = \sqrt{400 + 400} = \sqrt{800} \approx 28.28$

Problem 2

We are given:

$$V(X) = 5,\ V(Y) = 4,\ \text{and } \mathrm{Cov}(X, Y) = -2.$$

Recall:

$$\mathrm{Cov}(aX + bY, cX + dY) = ac\,V(X) + bd\,V(Y) + (ad + bc)\,\mathrm{Cov}(X, Y).$$

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a. Cov$(X + Y, X - Y)$ Here, $a = 1, b = 1, c = 1, d = -1$:

$$\mathrm{Cov}(X + Y, X - Y) = (1)(1)(5) + (1)(-1)(4) + (1)(-1) + (1)(1)$$

$$= 5 - 4 + (-1 + 1)(-2) = 5 - 4 + 0 = 1.$$

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b. Cov$(X - Y, -2X + 5Y)$ Here, $a = 1, b = -1, c = -2, d = 5$:

$$\mathrm{Cov}(X - Y, -2X + 5Y) = (1)(-2)(5) + (-1)(5)(4) + (1)(5) + (-1)(-2)$$

$$= -10 - 20 + (5 + 2)(-2) = -10 - 20 - 14 = -44.$$

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c. Corr$(X - Y, -2X + 5Y)$ First compute variances:

For $X - Y$:

$$V(X - Y) = V(X) + V(Y) - 2\mathrm{Cov}(X, Y) = 5 + 4 - 2(-2) = 9 + 4 = 13.$$

For $-2X+5Y$:

$$V(-2X+5Y) = (-2)^2V(X) + (5)^2V(Y) + 2(-2)(5)\text{Cov}(X,Y)$$

$$= 4(5) + 25(4) + 2(-10)(-2) = 20 + 100 + 40 = 160.$$

So:

$$\text{Corr}(X-Y, -2X+5Y) = \frac{\text{Cov}(X-Y, -2X+5Y)}{\sqrt{V(X-Y)}\sqrt{V(-2X+5Y)}}$$

$$= \frac{-44}{\sqrt{13}\cdot\sqrt{160}} = \frac{-44}{\sqrt{2080}} \approx \frac{-44}{45.6} \approx -0.965.$$

Problem 3

We are given that $\rho_{XY} \neq 0$, and $W = bX$ where $b > 0$. Show that $\rho_{WY} = \rho_{XY}$.

Recall the correlation formula:

$$\rho_{WY} = \frac{\text{Cov}(W,Y)}{\sqrt{V(W)}\sqrt{V(Y)}}.$$

Since $W = bX$:

$$\text{Cov}(W,Y) = \text{Cov}(bX,Y) = b\,\text{Cov}(X,Y).$$

Also:

$$V(W) = V(bX) = b^2V(X).$$

So:

$$\rho_{WY} = \frac{b\,\text{Cov}(X,Y)}{\sqrt{b^2V(X)}\sqrt{V(Y)}} = \frac{b\,\text{Cov}(X,Y)}{b\sqrt{V(X)}\sqrt{V(Y)}} = \frac{\text{Cov}(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \rho_{XY}.$$

Thus:

$$\rho_{WY} = \rho_{XY}.$$