HW 09 SOLUTIONS

Practice Problems

5.3

Note that using material from Chapter 3, the joint probability function is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}, \quad 0 \le y_1, 0 \le y_2, y_1 + y_2 \le 3$$

Given table format, this is

y2 y1	0	1	2	3
0	0	3/84	6/84	1/84
1	4/84	24/84	12/84	0
2	12/84	18/84	0	0
3	4/84	0	0	0

5.5

(a)

$$P(Y_1 \le 1/2, Y_2 \le 1/3) = \int_0^{1/2} \int_0^{1/3} 3y_1 \, dy_1 \, dy_2 = 0.1065.$$

(b)

$$P(Y_2 \le Y_1/2) = \int_0^1 \int_0^{y_1/2} 3y_1 \, dy_1 \, dy_2 = 0.5.$$

1

5.9

(a)

Since the density must integrate to 1:

$$1 = \int_0^1 \int_0^{y_2} k(1 - y_2)\, dy_1\, dy_2 = k/6,$$

so $k = 6$

(b)

Because $Y_1 \leq Y_2$, we split into two regions (drawing a picture is useful):

$$P(Y_1 \leq \tfrac{3}{4}, Y_2 \geq \tfrac{1}{2}) = \int_{1/2}^1 \int_{1/2}^1 6(1 - y_2)dy_1 dy_2 + \int_{1/2}^{3/4} \int_{y_1}^1 6(1 - y_2)dy_2 dy_1 = \frac{24}{64} + \frac{7}{64} = \frac{31}{64}.$$

5.11

The area of the triangular region is 1, so with a uniform distribution this is the value of the density function. Again, using geometry (drawing a picture is again useful):

(a)

$$P(Y_1 \leq \tfrac{3}{4}, Y_2 \leq \tfrac{3}{4}) = 1 - P(Y_1 > \tfrac{3}{4}) - P(Y_2 > \tfrac{3}{4}) = 1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{29}{32}.$$

(b)

$$P(Y_1 - Y_2 \geq 0) = P(Y_1 \geq Y_2)$$

The region specified in this probability statement represents 1/4 of the total region of support, so $P(Y_1 \geq Y_2) = \tfrac{1}{4}$.

5.17

This can be found using integration (polar coordinates are helpful). But, note that this is a bivariate uniform distribution over the unit circle (radius 1), and the probability of interest represents 50% of the support. Thus, the probability is 0.5.

5.25

(a)

The marginals are: $f_1(y_1) = e^{-y_1}, \quad y_1 > 0$, and $f_2(y_2) = e^{-y_2}, \quad y_2 > 0$. These are exponential densities with $\beta = 1$.

(b)

$$P(1 < Y_1 < 2.5) = P(1 < Y_2 < 2.5) = e^{-1} - e^{-2.5} = 0.2858$$

(c)

Support: $y_2 > 0$.

(d)

Conditional density:

$$f(y_1 \mid y_2) = f_1(y_1) = e^{-y_1}, \quad y_1 > 0.$$

(e)

Similarly:

$$f(y_2 \mid y_1) = f_2(y_2) = e^{-y_2}, \quad y_2 > 0.$$

(f)

The answers are the same.

(g)

The probabilities are the same.

5.27

(a)

$$f_1(y_1) = \int_0^{y_1} 6(1 - y_2) \, dy_2, = 3(1 - y_1)^2, 0 \le y_1 \le 1$$

$$f_2(y_2) = \int_0^{y_2} 6(1 - y_2) \, dy_1, = 6y_2(1 - y_2), 0 \le y_2 \le 1$$

(b)

$$P(Y_2 \le \tfrac{1}{2} \mid Y_2 \le \tfrac{3}{4}) = \frac{\int_0^{1/2} \int_0^{y_2} 6(1 - y_2) dy_1 dy_2}{\int_0^{3/4} 3(1 - y_1)^2 dy_1} = \frac{32}{63}.$$

(c)

$f(y_1|y_2) = 1/y_2, 0 \le y_1 \le y_2 \le 1.$

(d)

$f(y_2|y_1) = 2(1 - y_2)/(1 - y_1)^2, 0 \le y_1 \le y_2 \le 1.$

(e)

From part (d): $f(y_2|1/2) = 8(1 - y_2), = 1/2 \le y_2 \le 1.$ Thus $P(Y_2 \ge 3/4|Y_1 = 1/2) = 1/4$

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5.33

Refer to Ex 5.15: ### (a)

$$f_1(y_1) = \int_0^{y_1} e^{-(y_1)} \, dy_2 = y_1 e^{-y_1}, y_1 \geq 0.$$

$$f_2(y_2) = \int_{y_2}^{\infty} e^{-(y_1)} \, dy_1 = e^{-y_2}, y_2 \geq 0.$$

(b)

$$f(y_1 \mid y_2) = e^{-(y_1 - y_2)}, \quad y_1 \geq y_2.$$

(c)

$$f(y_2 \mid y_1) = 1/y_1, \quad 0 \leq y_2 \leq y_1.$$

(d)

The density functions are different.

(e)

The marginal and conditional probabilities can differ.