Chapter 3 Part 2

STAT 5700: Probability

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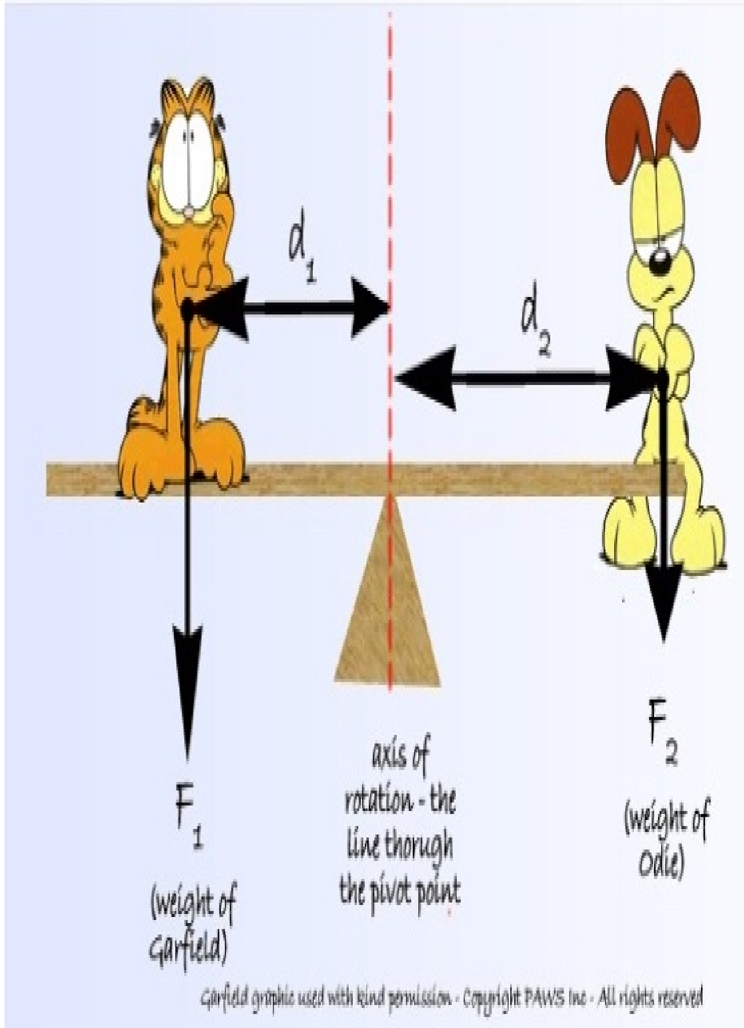
Special Expectations: Moments

Definition: The rth **moment** of a random variable X is the expected value of X^r and is denoted by $E(X^r)$, for each integer r. That is,

$$E(X^r) = \sum_{x \in \mathbb{S}} x^r p(x)$$

The term "moment" comes from physics: if the quantities $p(x)$ are point masses acting perpendicularly to the $x-$axis at distances y from the origin, $E(X^1)$ would be the $x-$coordinate of the center of gravity, and $E(X^2)$ would be the moment of inertia.



First Moment = Mean

Note that the first moment where $r = 1$, we have

$$\begin{aligned}
E(Y^1) &= \sum_{y \in \mathbb{S}} y^1 p(y) \\
&= E(Y) = \sum_{y \in \mathbb{S}} xp(y) \\
&= \mu
\end{aligned}$$

Therefore, we usually refer to the first moment as μ, the mean of Y.

Example:
Suppose Y is a random variable with support $\{1, 2, 3\}$ and *probabilitydistribution* is given by $p(1) = 0.5$, $p(2) = 0.2$, $p(3) = 0.3$. Find the mean and show that the negative distances from the mean balance the positive.

Special Expectations: Central Moments

Definition The rth **central moment** of a random variable Y is the expected value of $(Y - \mu)^r$ and is denoted by $E[(Y - \mu)^r]$, for each integer r. That is,

$$E[(Y - \mu)^r] = \sum_{y \in \mathbb{S}} (y - \mu)^r p(y)$$

Recall that $\mu = E(Y)$ is the mean of Y, so the central moments are sometimes refered to as **moments about the mean**.

Exercise:
What is $E(Y - \mu)$?

Moment-generating functions

Definition 2.3-1 Let Y be a discrete random variable with probability distribution $p(y)$ and support $\mathbb{S}$. If there is a positive number h such that

$$E(e^{tY}) = \sum_{y \in \mathbb{S}} e^{tx} p(y)$$

exists and is finite for $-h < t < h$, then the function defined by $M_Y(t) = E(e^{tY})$ is called the **moment-generating function** of Y. This function is often abbreviated as mgf.

$M_Y(t) = E(e^{tY})$ is called the moment-generating function, because by taking derivatives of $M_Y(t)$ at $t = 0$ can generate expressions for all the moments of a random variable Y!

Theorem

$$\frac{d^r}{dt^r} M_Y(t)|_{t=0} = E(Y^r)$$

That is, the rth moment of Y is equal to the rth derivative of $M_Y(t)$ evaluated at $t = 0$.

Example:
Let Y be a uniformly distributed random variable. Recall that the probability distribution of the uniform distribution is given by

$$p(y) = \frac{1}{m}, \qquad y = 1, 2, ..., m$$

Find an expression for the moment-generating function of the distribution. Then use the mgf to find the mean of Y.

Example:
If the moment-generating function of Y is $M_Y(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}$, find the mean, variance, and probability distribution of Y.

Moments of the Binomial Distribution

Exercise:

1. Find the mgf of the Binomial distribution.
2. Use the mgf to find the mean and the variance of the binomial distribution

Problem 7

Let Y be a random variable with probability distribution $p(y) = \frac{y}{6}, \quad y = 1, 2, 3$

a) Find an expression for the moment generating function of Y. That is, write $E(e^{tY})$ as a sum.
b) Use the mgf to show that $E(Y) = 7/3$
c) Use the mgf to show that $E(Y^2) = 6$
d) Find $V(Y)$