HW 09 SOLUTIONS

Practice Problems

5.3

Note that using material from Chapter 3, the joint probability function is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}, \quad 0 \le y_1, 0 \le y_2, y_1 + y_2 \le 3$$

Given table format, this is

y2 y1	0	1	2	3
0	0	3/84	6/84	1/84
1	4/84	24/84	12/84	0
2	12/84	18/84	0	0
3	4/84	0	0	0

5.5

(a)

$$P(Y_1 \le 1/2, Y_2 \le 1/3) = \int_0^{1/2} \int_0^{1/3} 3y_1 \, dy_1 \, dy_2 = 0.1065.$$

(b)

$$P(Y_2 \le Y_1/2) = \int_0^{1} \int_0^{y_1/2} 3y_1 \, dy_1 \, dy_2 = 0.5.$$

5.9

(a)

Since the density must integrate to 1: $1 = \int_0^1 \int_0^{y_2} k(1 - y_2)\, dy_1\, dy_2 = k/6$, so $k = 6$

(b)

Because $Y_1 \le Y_2$, we split into two regions (drawing a picture is useful):

$$P(Y_1 \le \tfrac{3}{4},\, Y_2 \ge \tfrac{1}{2}) = \int_{1/2}^{1} \int_{1/2}^{1} 6(1 - y_2) dy_1 dy_2 + \int_{1/2}^{3/4} \int_{y_1}^{1} 6(1 - y_2) dy_2 dy_1 = \frac{24}{64} + \frac{7}{64} = \frac{31}{64}.$$

5.11

The area of the triangular region is 1, so with a uniform distribution this is the value of the density function. Again, using geometry (drawing a picture is again useful):

(a)

$$P(Y_1 \le \tfrac{3}{4}, Y_2 \le \tfrac{3}{4}) = 1 - P(Y_1 > \tfrac{3}{4}) - P(Y_2 > \tfrac{3}{4}) = 1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{29}{32}.$$

(b)

$$P(Y_1 - Y_2 \ge 0) = P(Y_1 \ge Y_2)$$

The region specified in this probability statement represents 1/4 of the total region of support, so $P(Y_1 \ge Y_2) = \frac{1}{4}$.

5.17

This can be found using integration (polar coordinates are helpful). But, note that this is a bivariate uniform distribution over the unit circle (radius 1), and the probability of interest represents 50% of the support. Thus, the probability is 0.5.

5.25

(a)

The marginals are: $f_1(y_1) = e^{-y_1}, \quad y_1 > 0$, and $f_2(y_2) = e^{-y_2}, \quad y_2 > 0$. These are exponential densities with $\beta = 1$.

(b)

$$P(1 < Y_1 < 2.5) = P(1 < Y_2 < 2.5) = e^{-1} - e^{-2.5} = 0.2858$$

(c)

Support: $y_2 > 0$.

(d)

Conditional density: $f(y_1 \mid y_2) = f_1(y_1) = e^{-y_1}, \quad y_1 > 0$

(e)

$f(y_2 \mid y_1) = f_2(y_2) = e^{-y_2}, \quad y_2 > 0$.

(f)

The answers are the same.

(g)

The probabilities are the same.

5.27

(a)

$$f_1(y_1) = \int_{y_1}^{1} 6(1-y_2)\, dy_2, = 3(1-y_1)^2, 0 \le y_1 \le 1$$

$$f_2(y_2) = \int_{0}^{y_2} 6(1-y_2)\, dy_1, = 6y_2(1-y_2), 0 \le y_2 \le 1$$

(b)

$$P(Y_2 \le \tfrac{1}{2} \mid Y_2 \le \tfrac{3}{4}) = \frac{\int_0^{1/2}\int_0^{y_2} 6(1-y_2)dy_1dy_2}{\int_0^{3/4} 3(1-y_1)^2dy_1} = \frac{32}{63}.$$

(c)

$f(y_1|y_2) = 1/y_2, 0 \le y_1 \le y_2 \le 1.$

(d)

$f(y_2|y_1) = 2(1-y_2)/(1-y_1)^2, 0 \le y_1 \le y_2 \le 1.$

(e)

From part (d): $f(y_2|1/2) = 8(1-y_2), = 1/2 \le y_2 \le 1$. Thus $P(Y_2 \ge 3/4|Y_1 = 1/2) = 1/4$

5.33

Refer to Ex 5.15:

(a)

$$f_1(y_1) = \int_{0}^{y_1} e^{-(y_1)}\, dy_2 = y_1 e^{-y_1}, y_1 \ge 0.$$

$$f_2(y_2) = \int_{y_2}^{\infty} e^{-(y_1)}\, dy_1 = e^{-y_2}, y_2 \ge 0.$$

(b)

$$f_{(}y_1 \mid y_2) = e^{-(y_1 - y_2)}, \quad y_1 \geq y_2.$$

(c)

$$f_{(}y_2 \mid y_1) = 1/y_1, \quad 0 \leq y_2 \leq y_1.$$

(d)

The density functions are different.

(e)

The marginal and conditional probabilities can differ.

Submitted Problems

5.2

a. The sample space for the toss of three balanced coins w/ probabilities are below:

Outcome	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
(y_1, y_2)	$(3,1)$	$(3,1)$	$(2,1)$	$(1,1)$	$(2,2)$	$(1,2)$	$(1,3)$	$(0,-1)$
probability	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

	$y_1 = 0$	1	2	3
$y_2 = -1$	$\frac{1}{8}$	0	0	0
0	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$	0
3	0	$\frac{1}{8}$	0	0

b.

$$F(2,1) = p(0,-1) + p(1,1) + p(2,1) = \frac{1}{2}.$$

5.8

a. Since the density must integrate to 1, evaluate

$$\int_0^1 \int_0^1 k y_1 y_2 \, dy_1 \, dy_2 = k/4 = 1,$$

so $k = 4$.

b.

$$F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2) = 4 \int_0^{y_2} \int_0^{y_1} t_1 t_2 \, dt_1 \, dt_2 = y_1^2 y_2^2, \; 0 \le y_1 \le 1, \; 0 \le y_2 \le 1.$$

c.

$$P(Y_1 \le 1/2, \; Y_2 \le 3/4) = (1/2)^2 (3/4)^2 = 9/64.$$

5.14

a. Since $f(y_1, y_2) \ge 0$, simply show

$$\int_0^1 \int_{y_1}^{2-y_1} 6 y_1^2 y_2 \, dy_2 \, dy_1 = 1.$$

b.

$$P(Y_1 + Y_2 < 1) = P(Y_2 < 1 - Y_1) = \int_0^{0.5} \int_{y_1}^{1-y_1} 6 y_1^2 y_2 \, dy_2 \, dy_1 = \frac{1}{16}.$$

5.16

a.

$$P(Y_1 < 1/2, \; Y_2 > 1/4) = \int_{1/4}^{1} \int_0^{1/2} (y_1 + y_2) \, dy_1 \, dy_2 = \frac{21}{64} \approx 0.328125.$$

b.

$$P(Y_1 + Y_2 \le 1) = P(Y_1 \le 1 - Y_2) = \int_0^1 \int_0^{1-y_2} (y_1 + y_2) \, dy_1 \, dy_2 = \frac{1}{3}.$$

5.22

a. The marginal distributions for Y_1 and Y_2 are given in the margins of the table.

b.

$$P(Y_2 = 0 \mid Y_1 = 0) = \frac{.38}{.76} = .5 \qquad P(Y_2 = 1 \mid Y_1 = 0) = \frac{.14}{.76} = .18$$

$$P(Y_2 = 2 \mid Y_1 = 0) = \frac{.24}{.76} = .32$$

c.

$$\text{The desired probability is } P(Y_1 = 0 \mid Y_2 = 0) = \frac{.38}{.55} = .69.$$

5.32

a.

$$f_1(y_1) = \int_{y_1}^{2-y_1} 6y_1^2 y_2 \, dy_2 = 12y_1^2(1 - y_1), \ 0 \le y_1 \le 1.$$

b. This marginal density must be constructed in two parts:

$$f_2(y_2) = \begin{cases} \int_0^{y_2} 6y_1^2 y_2 \, dy_1 = 2y_2^4, & 0 \le y_2 \le 1, \\ \int_0^{2-y_2} 6y_1^2 y_2 \, dy_1 = 2y_2(2 - y_2)^3, & 1 \le y_2 \le 2. \end{cases}$$

c.

$$f(y_2 \mid y_1) = \tfrac{1}{2} y_2/(1 - y_1), \ y_1 \le y_2 \le 2 - y_1.$$

d. Using the density found in part c,

$$P(Y_2 < 1.1 \mid Y_1 = 0.6) = \tfrac{1}{2} \int_{0.6}^{1.1} \frac{y_2}{0.4} \, dy_2 \approx 0.53.$$

5.34

a. Given $Y_1 = y_1$, Y_2 has a uniform distribution on the interval $(0, y_1)$.

b. Since $f_1(y_1) = 1$, $0 \leq y_1 \leq 1$,

$$f(y_1, y_2) = f(y_2 \mid y_1) f_1(y_1) = \frac{1}{y_1},\ 0 \leq y_2 \leq y_1 \leq 1.$$

c.

$$f_2(y_2) = \int_{y_2}^{1} \frac{1}{y_1}\, dy_1 = -\ln(y_2),\ 0 \leq y_2 \leq 1.$$