Chapter 5 Part 2 Group Work

SOLUTIONS

Problem 1

Show that for any constants a and b, $Cov(a+X, b+Y) = Cov(X,Y)$. That is, shifting by a constant does not change the covariance. *Note: this fact will be useful on HW 5.110*

SOLUTION

By definition,

$$Cov(a+X,\ b+Y) = E[(a+X)(b+Y)] - E[a+X]E[b+Y].$$

Compute:

$$E[(a+X)(b+Y)] = ab + aE[Y] + bE[X] + E[XY],$$

and

$$E[a+X]E[b+Y] = ab + aE[Y] + bE[X] + E[X]E[Y].$$

Thus,

$$Cov(a+X,\ b+Y) = E[XY] - E[X]E[Y] = Cov(X,Y).$$

Problem 2

To estimate a proportion of units that meet a given criteria, we often use the estimator $\hat{p} = \frac{Y}{n}$, where $Y \sim binomial(n,p)$. Find the expected value and variance of $\hat{p}$, assuming n is fixed.

SOLUTION

Let

$$\hat{p} = \frac{Y}{n}, \quad Y \sim \text{Binomial}(n,p),$$

with n fixed.

Since $E[Y] = np$,

$$E[\hat{p}] = E\left[\frac{Y}{n}\right] = \frac{1}{n}E[Y] = \frac{np}{n} = p.$$

Since $V(Y) = np(1-p)$,

$$V(\hat{p}) = V\left(\frac{Y}{n}\right) = \frac{1}{n^2}V(Y) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}.$$

Problem 3

A learning experiment requires a rat to run a maze (a network of pathways) until it locates one of three possible exits. Exit 1 presents a reward of food, but exits 2 and 3 do not. (If the rat eventually selects exit 1 almost every time, learning may have taken place). Let Y_i denote the number of times exit i is chosen in successive runnings. For the following, assume that the rate chooses an exit at random on each run.

a. Find the probability that $n = 6$ runs result in $Y_1 = 3$, $Y_2 = 1$ and $Y_3 = 2$.
b. For general n, find $E(Y_1)$ and $V(Y_1)$.
c. For general n, find $Cov(Y_2, Y_3)$
d. To check for the fat's preference between exits 2 and 3, we may look at $Y_2 - Y_3$. Find $E(Y_2 - Y_3)$ and $V(Y_2 - Y_3)$ for general n.

SOLUTION

a. Using the multinomial distribution with $p_1 = p_2 = p_3 = \frac{1}{3}$,

$$P(Y_1 = 3,\, Y_2 = 1,\, Y_3 = 2) = \frac{6!}{3!\,1!\,2!}\left(\frac{1}{3}\right)^6 = 0.0823.$$

b.

$$E(Y_1) = \frac{n}{3}, \qquad V(Y_1) = n\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2n}{9}.$$

c.

$$Cov(Y_2, Y_3) = -n\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{n}{9}.$$

d.

$$E(Y_2 - Y_3) = \frac{n}{3} - \frac{n}{3} = 0,$$

$$V(Y_2 - Y_3) = V(Y_2) + V(Y_3) - 2Cov(Y_2, Y_3) = \frac{2n}{3}.$$

Problem 4

The number of defects per yard in a certain fabric, Y, is known to have a Poisson distribution with parameter λ. The parameter λ is assumed to be a random variable with a pdf given by

$$f(\lambda) = e^{-\lambda}, \quad \lambda \geq 0$$

a. Find the expected number of defects per yard by first finding the conditional expectation of Y for given λ.
b. Find the variance of Y.

SOLUTION

Let

$$Y \mid \lambda \sim \text{Poisson}(\lambda), \qquad f(\lambda) = e^{-\lambda},\ \lambda \geq 0.$$

a. Expected value

The conditional expectation is $E(Y \mid \lambda) = \lambda$. by properties of Poisson distribution. Thus,

$$E(Y) = E[E(Y \mid \lambda)] = E(\lambda).$$

Since $\lambda \sim \text{Exponential}(1)$,

$$E(\lambda) = \int_0^\infty \lambda e^{-\lambda}\, d\lambda = 1.$$

Therefore,

$$E(Y) = 1.$$

b. Variance

Using the law of total variance,

$$V(Y) = E[V(Y \mid \lambda)] + V(E(Y \mid \lambda))\,.$$

For a Poisson random variable,

$$V(Y \mid \lambda) = \lambda, \qquad E(Y \mid \lambda) = \lambda.$$

Hence,

$$V(Y) = E(\lambda) + V(\lambda).$$

For $\lambda \sim \text{Exponential}(1)$,

$$E(\lambda) = 1, \qquad V(\lambda) = 1.$$

Therefore,

$$V(Y) = 1 + 1 = 2.$$