Lecture 2

1 Uniform Probability Spaces

Definition 1.1 (Uniform Probability Space)**.** *A probability space* $(\Omega, \mathbb{P})$ *is called* uniform*, if* Ω *is finite and for all* $\omega \in \Omega$ *it holds that*

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}.$$

Claim 1.2. *In a uniform probability space, for every event* $A \subseteq \Omega$ *it holds that*

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}.$$

Proof. By definition:

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}.$$

$\square$

Example 1: 52 cards of a regular deck are divided between 4 players uniformly at random such that each player receives 13 cards. What is the probability that each player receives one ace?

The corresponding probability space $(\Omega, \mathbb{P})$ is uniform, that is, $\mathbb{P}(\omega) = \frac{1}{|\Omega|}$ for every $\omega \in \Omega$. The sample space Ω consists of all possible ways of dividing 52 cards between 4 players such that each player receives 13 cards. Hence

$$|\Omega| = \binom{52}{13, 13, 13, 13} = \frac{52!}{(13!)^4}.$$

Let $A \subseteq \Omega$ denote the event: each player receives one ace. Then

$$|A| = 4! \cdot \binom{48}{12, 12, 12, 12} = \frac{4! \cdot 48!}{(12!)^4},$$

where the first equality holds since we first decide which player receives which ace (in 4! ways) and then divide the remaining 48 cards equally between the players. Since $(\Omega, \mathbb{P})$ is uniform, we conclude that

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\frac{4! \cdot 48!}{(12!)^4}}{\frac{52!}{(13!)^4}} = \frac{4! \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{13^3}{17 \cdot 25 \cdot 49}.$$

Example 2: Chevelier de Mere (1607–1684) was a gentleman gambler in France who made it to the history books by turning to Blaise Pascal, an eminent mathematician of his time, for help in finding a mathematical answer to why he consistently lost money in a certain game of dice. Unlike other gamblers who might just chalk it up to bad luck, he pursued the cause of the problem with the help of Pascal. As a result of Pascal's efforts combined with those of Pierre de Fermat, the area of probability theory subsequently emerged as an academic field of study.

Chevelier de Mere's predicament involved two games of dice. In the first game, de Mere bet with even odds on getting at least one 6 on four rolls of a fair die. He reasoned correctly that the chance of getting a 6 in one roll of a die is $\frac{1}{6}$. He then incorrectly thought that in four rolls of a die, the chance of getting at least one 6 would be $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{2}{3}$. Though his reasoning was faulty, he made considerable money over the years in making this bet.

With the success of the first game, de Mere modified the game by betting with even odds on getting at least one "double six" on 24 rolls of a pair of fair dice. He reasoned correctly that the chance of getting a double six when rolling a pair of dice once is $\frac{1}{36}$. Similarly to his faulty reasoning concerning the first game, he then thought that in 24 rolls of a pair of dice, the chance of getting at least one double six would be $\frac{24}{36}=\frac{2}{3}$.

Based on empirical data (i.e., he lost a lot of money), he knew something was not quite right with his calculations. He challenged his renowned friend Blaise Pascal to help him find an explanation. In a series of letters between Pascal and Pierre de Fermat, the problem of de Mere was solved. Out of this joint effort, a foundation was laid for the theory of probability. Nowadays, anyone with basic knowledge of probability theory would be able to spot the faulty reasoning of de Mere. Let's see why the first game was profitable for de Mere whereas the second game was not.

The First Game: The corresponding sample space is

$$\Omega = \{(x_1, x_2, x_3, x_4) : 1 \leq x_1, x_2, x_3, x_4 \leq 6\}.$$

Let $A \subseteq \Omega$ be the required event, that is, A is the event of getting at least one 6 on four rolls of a fair die. Then

$$A^c = \{(x_1, x_2, x_3, x_4) : 1 \leq x_1, x_2, x_3, x_4 \leq 5\}.$$

Since the die is fair, we have

$$\mathbb{P}(A^c) = \frac{|A^c|}{|\Omega|} = \frac{5^4}{6^4}.$$

We conclude that $\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - 5^4/6^4$. Observe that this probability is strictly larger than $1/2$ (it is approximately 0.517747) which explains why de Mere made a lot of money over time playing this game.

The Second Game: The corresponding sample space is

$$\Omega = \{((x_1, y_1), \dots, (x_{24}, y_{24})) : 1 \leq x_i, y_i \leq 6 \text{ for every } 1 \leq i \leq 24\}.$$

Let $A \subseteq \Omega$ be the required event, that is, A is the event of getting at least one double six on 24 rolls of a pair of fair dice. Then

$$A^c = \{((x_1, y_1), \ldots, (x_{24}, y_{24})) : (x_i, y_i) \in \{1, \ldots, 6\}^2 \setminus \{(6,6)\} \text{ for every } 1 \leq i \leq 24\}.$$

Since the dice are fair, we have

$$\mathbb{P}(A^c) = \frac{|A^c|}{|\Omega|} = \frac{35^{24}}{36^{24}}.$$

We conclude that $\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - 35^{24}/36^{24}$. Observe that this probability is strictly smaller than $1/2$ (it is approximately 0.4914) which explains why de Mere lost a lot of money over time playing this game.

Example 3: Recall that a permutation over the set $\{1, 2, \ldots, n\}$ is a bijection

$$\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

Denote the set of all permutations over $\{1, \ldots, n\}$ by S_n. In Discrete math it was shown that

$$|S_n| = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!.$$

For a permutation $\pi \in S_n$ we let $C_1(\pi)$ be the cycle of π which contains 1, i.e., $C_1(\pi) = (1, a_2, a_3, \ldots, a_k)$ where

$$\begin{aligned} &\pi(1) = a_2 \\ &\pi(a_i) = a_{i+1} \text{ for every } 2 \leq i \leq k-1 \\ &\pi(a_k) = 1. \end{aligned}$$

Here the length of the cycle which contains 1 is k.

Let n be a positive integer. A random permutation is chosen uniformly at random from the set S_n (that is, every permutation has equal probability of being chosen). What is the probability that the cycle of the chosen permutation which contains 1 is of length k, for a given $1 \leq k \leq n$? Surprisingly, the answer does not depend on k.

Claim 1.3. *The probability that the cycle of a randomly chosen permutation which contains 1 is of length k equals $1/n$ for every $1 \leq k \leq n$, that is, the length of the cycle which contains 1 is uniformly distributed over $\{1, \ldots, n\}$.*

Proof. For every $1 \leq k \leq n$ let A_k be the event that the cycle of the permutation which contains 1 is of length k. Since π was chosen uniformly at random from the set S_n, we can take $\Omega = S_n$ and $\mathbb{P}(\pi) = \frac{1}{|S_n|} = \frac{1}{n!}$ for every $\pi \in S_n$, to be our probability space. It further holds that

$$\mathbb{P}(A_k) = \frac{|A_k|}{|S_n|} = \frac{|A_k|}{n!}.$$

Hence, in order to calculate $\mathbb{P}(A_k)$ it is enough to count the number of permutations whose cycle which contains 1 is of length k.

As a warm-up, consider the case $k = 1$. If $\pi \in S_n$ is such that $|C_1(\pi)| = 1$, then $\pi(1) = 1$ and the remaining elements can be permuted among themselves in any possible way. It follows that $|A_1| = (n-1)!$. We conclude that, in this case,

$$\mathbb{P}(A_1) = \frac{|A_1|}{n!} = \frac{(n-1)!}{n!} = \frac{1}{n},$$

As claimed.

Next, we consider the general case, that is, $2 \le k \le n$. For every vector $(a_2, \ldots, a_k)$ of $k-1$ distinct integers $2 \le a_2, \ldots, a_k \le n$, let $B_{(a_2,\ldots,a_k)}$ denote the event: $C_1(\pi) = (1, a_2, \ldots, a_k)$, that is, $B_{(a_2,\ldots,a_k)}$ is the set of all permutations $\pi \in S_n$ that satisfy $C_1(\pi) = (1, a_2, \ldots, a_k)$. Then

$$A_k = \bigcup_{(a_2,\ldots,a_k)\in\{2,\ldots,n\}^{k-1}} B_{(a_2,\ldots,a_k)}.$$

Moreover, these events are pairwise disjoint and thus

$$|A_k| = \sum_{(a_2,\ldots,a_k)\in\{2,\ldots,n\}^{k-1}} |B_{(a_2,\ldots,a_k)}|.$$

We next show that $B_{(a_2,\ldots,a_k)} = (n-k)!$ for every choice of $2 \le a_2, \ldots, a_k \le n$. Indeed, there is a natural bijection between the permutations of $B_{(a_2,\ldots,a_k)}$ and the permutations over $\{1, \ldots, n\} \setminus \{1, a_2, \ldots, a_k\}$ (figure out for yourself what it is!). Since there are $(n-k)!$ permutations over $\{1, \ldots, n\} \setminus \{1, a_2, \ldots, a_k\}$, it follows that $|B_{(a_2,\ldots,a_k)}| = (n-k)!$. Overall we get that

$$\begin{aligned}
\mathbb{P}(A_k) &= \frac{|A_k|}{n!} \\
&= \frac{\sum_{(a_2,\ldots,a_k)\in\{2,\ldots,n\}^{k-1}} |B_{(a_2,\ldots,a_k)}|}{n!} \\
&= \frac{(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)\cdot(n-k)!}{n!} \\
&= \frac{(n-1)!}{n!} \\
&= \frac{1}{n}.
\end{aligned}$$

$\square$