# Assignment 5

If you wish to submit your solutions to any of these questions, please send them via email to your TA by 10/06/2021. This deadline is strict!

**Exercise 1** Let $X \sim \text{Bin}(n, p)$, for some $n \in \mathbb{N}$ and $p \in [0, 1]$, be a random variable. Find the expected value of $X$ using two methods: by direct calculation (i.e., using the identity $\mathbb{E}(X) = \sum_x x \cdot \mathbb{P}(X = x)$) and by depicting $X$ as a sum of $n$ independent Bernoulli random variables.

**Exercise 2** Let $S$ be a set with $n$ elements. A set $A \subseteq S$ is selected uniformly at random among all $2^n$ subsets of $S$. Let $X = |A|$.

1. Calculate the probability distribution of $X$.
2. Calculate the expected value of $X$ using two methods: by direct calculation and by depicting $X$ as a Binomial random variable.

**Exercise 3** Let $X \sim \text{Bin}(n, p)$, for some $n \in \mathbb{N}$ and $p \in [0, 1]$, be a random variable. Find $\text{Var}(X)$ using two methods: by direct calculation according to the definition of variance, and by depicting $X$ as a sum of $n$ mutually independent Bernoulli random variables.

**Exercise 4** A fair coin is being tossed $n + 2$ times, all coin tosses being mutually independent. Let $X$ be the number of times 3 consecutive heads appeared, for example, in the sequence $HHHHHTTHHH$, $X = 4$. Calculate $\mathbb{E}(X)$ and $\text{Var}(X)$.

**Exercise 5** A computer samples uniformly at random, independently, and with replacement 100 natural numbers from the set $\{1, 2, \ldots, 100\}$. Let $\bar{X}$ denote their average. Prove, using Chebyshev's inequality, that

$$\mathbb{P}\left(45.5 < \bar{X} < 55.5\right) > 2/3.$$

**Exercise 6** Let $X \sim \text{Bin}(n, p)$, for some $n \in \mathbb{N}$ and $p \in [0, 1]$, be a random variable. Prove that for every $t$ satisfying $0 \leq t < n$, it holds that

$$\mathbb{P}(X > t) \geq \frac{np - t}{n - t}.$$

**Exercise 7** Let $X \sim \text{Geom}(p)$, for some $p \in (0, 1)$, be a random variable. Calculate $\mathbb{E}\left(e^{tX}\right)$ for every $t \in \mathbb{R}$.