Lecture 3

1 Inclusion-Exclusion

Theorem 1.1. *Let $A_1, A_2, \ldots, A_n$ be events in some probability space $(\Omega, \mathbb{P})$. Then*

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|-1} \cdot \mathbb{P}\left(\bigcap_{i \in I} A_i\right). \tag{1}$$

Proof. Recall that, for every event A, it holds that

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega).$$

We will apply this identity to both sides of Equation (1). For the left-hand side we have

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\omega \in \bigcup_{i=1}^{n} A_i} \mathbb{P}(\omega), \tag{2}$$

and for right-hand side we have

$$\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|-1} \cdot \mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|-1} \cdot \sum_{\omega \in \bigcap_{i \in I} A_i} \mathbb{P}(\omega). \tag{3}$$

We will show that the right-hand side of (2) is equal to the right-hand side of (3). We will do so by showing that for every $\omega \in \Omega$, $\mathbb{P}(\omega)$ appears the same number of times on the right-hand side of (2) and on the right-hand side of (3) (after summation). Fix some $\omega \in \Omega$. If $\omega \notin \bigcup_{i=1}^{n} A_i$, then $\omega \notin \bigcap_{i \in I} A_i$ for any non-empty $I \subseteq \{1, \ldots, n\}$. Hence, $\mathbb{P}(\omega)$ does not appear in (2) or (3) – in particular it appears the same number of times in (2) and (3). Therefore we can assume that $\omega \in \bigcup_{i=1}^{n} A_i$. Let $k = |\{1 \leq i \leq n : \omega \in A_i\}|$ be the number of events A_i that contain ω. Note that $\mathbb{P}(\omega)$ appears exactly once on the right-hand side of (2). On the right-hand side of (3), we have the following somewhat more complicated situation: Fix some $\emptyset \neq I \subseteq \{1, \ldots, n\}$. Then, by definition, $\omega \in \bigcap_{i \in I} A_i$ if and only if $\omega \in A_i$ for every $i \in I$. Hence, for every $1 \leq j \leq k$, there are exactly $\binom{k}{j}$ sets $\emptyset \neq I \subseteq \{1, \ldots, n\}$ such that $|I| = j$ and $\omega \in \bigcap_{i \in I} A_i$; for other values of j there are no such sets. We conclude that the number of times $\mathbb{P}(\omega)$ will appear on the right-hand side of (3) is exactly

$$\begin{aligned}
\sum_{j=1}^{k} \sum_{\substack{I\subseteq\{1,\ldots,n\},\\ |I|=j,\\ \omega\in\bigcap_{i\in I}A_i}} (-1)^{|I|-1} &= \sum_{j=1}^{k} \binom{k}{j}(-1)^{j-1} \\
&= (-1)\left(\left(\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\right)-1\right) \\
&= (-1)\left((1-1)^{k}-1\right) \\
&= (-1)(0-1) \\
&= 1,
\end{aligned}$$

where the third equality is by the binomial formula. $\square$

In some cases, it will be more convenient to use the following version of the Inclusion-Exclusion formula.

Theorem 1.2. *Let $A_1, A_2, \ldots, A_n$ be events in some probability space $(\Omega, \mathbb{P})$. Then*

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i^c\right) = \sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\cdot\mathbb{P}\left(\bigcap_{i\in I} A_i\right).$$

Proof. Recall that $\bigcap_{i=1}^{n} A_i^c = \left(\bigcup_{i=1}^{n} A_i\right)^c$ by De-Morgan's law. Hence

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{n} A_i^c\right) &= 1-\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \mathbb{P}(\Omega) - \sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} (-1)^{|I|-1}\cdot\mathbb{P}\left(\bigcap_{i\in I} A_i\right) \\
&= \sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\cdot\mathbb{P}\left(\bigcap_{i\in I} A_i\right),
\end{aligned}$$

where the second equality holds by Theorem 1.1 and the last equality holds since $\bigcap_{i\in\emptyset} A_i = \Omega$. $\square$

Example 1:

1. Given a positive integer N, an integer is drawn uniformly at random from the set $\{1,\ldots,N\}$. What is the probability that this random integer is divisible by 7, 10, or 15?

   Solution: Let

   $$\begin{aligned}
   &\Omega = \{1,\ldots,N\}, \\
   &\forall i\in\Omega,\ \mathbb{P}(i) = \frac{1}{N}.
   \end{aligned}$$

For every positive integer k, let A_k be the event that the random integer is divisible by k. Using this notation, our aim is to calculate

$$\mathbb{P}\left(A_7 \cup A_{10} \cup A_{15}\right).$$

For every positive integer k we have

$$\mathbb{P}\left(A_k\right) = \frac{|A_k|}{N} = \frac{1}{N} \cdot \left\lfloor \frac{N}{k} \right\rfloor,$$

where the first equality is due to the fact that the probability space is uniform. Moreover, observe that for positive integers $k \neq \ell$ we have

$$A_k \cap A_\ell = A_{\mathrm{lcm}(k,\ell)}.$$

Hence, by Inclusion-Exclusion (Theorem 1.1) we have

$$\begin{aligned}
\mathbb{P}\left(A_7 \cup A_{10} \cup A_{15}\right) &= \mathbb{P}\left(A_7\right) + \mathbb{P}\left(A_{10}\right) + \mathbb{P}\left(A_{15}\right) \\
&\quad - \mathbb{P}\left(A_7 \cap A_{10}\right) - \mathbb{P}\left(A_7 \cap A_{15}\right) - \mathbb{P}\left(A_{10} \cap A_{15}\right) \\
&\quad + \mathbb{P}\left(A_7 \cap A_{10} \cap A_{15}\right) \\
&= \mathbb{P}\left(A_7\right) + \mathbb{P}\left(A_{10}\right) + \mathbb{P}\left(A_{15}\right) \\
&\quad - \mathbb{P}\left(A_{70}\right) - \mathbb{P}\left(A_{105}\right) - \mathbb{P}\left(A_{30}\right) \\
&\quad + \mathbb{P}\left(A_{210}\right) \\
&= \frac{1}{N}\left(\left\lfloor \frac{N}{7} \right\rfloor + \left\lfloor \frac{N}{10} \right\rfloor + \left\lfloor \frac{N}{15} \right\rfloor - \left\lfloor \frac{N}{70} \right\rfloor - \left\lfloor \frac{N}{105} \right\rfloor - \left\lfloor \frac{N}{30} \right\rfloor + \left\lfloor \frac{N}{210} \right\rfloor\right).
\end{aligned}$$

For example, *for every* N which is divisible by 210, this probability equals

$$\frac{1}{7} + \frac{1}{10} + \frac{1}{15} - \frac{1}{70} - \frac{1}{105} - \frac{1}{30} + \frac{1}{210} = \frac{9}{35}.$$

2. A random permutation $\pi \in S_n$ is drawn uniformly at random. What is the probability that it has exactly k fixed points?

   Solution: The probability space is

$$\begin{aligned}
&\Omega = S_n \\
&\forall \pi \in \Omega,\ \mathbb{P}\left(\pi\right) = \frac{1}{n!}.
\end{aligned}$$

For every $0 \leq k \leq n$, let A_k be the event that the random permutation has exactly k fixed points. We begin by considering the case $k = 0$, that is, we wish to determine the probability of drawing a permutation that has no fixed points at all. For every

$i \in \{1, 2, \ldots, n\}$ let B_i be the event that i is a fixed point of the random permutation π. Then $A_0 = \bigcap_{i=1}^n B_i^c$ and thus by Theorem 1.2 we have

$$\mathbb{P}(A_0) = \mathbb{P}\left(\bigcap_{i=1}^n B_i^c\right) = \sum_{\mathcal{I} \subseteq \{1,\ldots,n\}} (-1)^{|\mathcal{I}|} \cdot \mathbb{P}\left(\bigcap_{i \in \mathcal{I}} B_i\right).$$

Observe that for every $\mathcal{I} \subseteq \{1, 2, \ldots, n\}$ it holds that

$$\mathbb{P}\left(\bigcap_{i \in \mathcal{I}} B_i\right) = \frac{\left|\bigcap_{i \in \mathcal{I}} B_i\right|}{n!} = \frac{(n - |\mathcal{I}|)!}{n!},$$

since we only need to permute the elements that are not in $\mathcal{I}$. Therefore

$$\begin{aligned}
\mathbb{P}(A_0) &= \sum_{\mathcal{I} \subseteq \{1,\ldots,n\}} (-1)^{|\mathcal{I}|} \cdot \mathbb{P}\left(\bigcap_{i \in \mathcal{I}} B_i\right) \\
&= \sum_{\mathcal{I} \subseteq \{1,\ldots,n\}} (-1)^{|\mathcal{I}|} \cdot \frac{(n - |\mathcal{I}|)!}{n!} \\
&= \sum_{i=0}^{n} \sum_{\substack{\mathcal{I} \subseteq \{1,\ldots,n\} \\ |\mathcal{I}| = i}} (-1)^i \cdot \frac{(n-i)!}{n!} \\
&= \sum_{i=0}^{n} \binom{n}{i} \cdot (-1)^i \cdot \frac{(n-i)!}{n!} \\
&= \sum_{i=0}^{n} \frac{(-1)^i}{i!}.
\end{aligned}$$

Note that we can approximate the above sum for large n as

$$\lim_{n \to \infty} \mathbb{P}(A_0) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} = \frac{1}{e}.$$

Moreover, observe that

$$|A_0| = n! \cdot \mathbb{P}(A_0) = n! \cdot \sum_{i=0}^{n} \frac{(-1)^i}{i!}. \tag{4}$$

Next we consider the case $k \neq 0$. Let J be the family of all subsets of $\{1, \ldots, n\}$ of size k. In particular, $|J| = \binom{n}{k}$. For every $T \in J$, let C_T denote the event: "i is a fixed point of the random permutation π if and only if $i \in T$". Then

$$A_k = \bigcup_{T \in J} C_T,$$

and the events $\{C_T : T \in J\}$ are pairwise disjoint. Therefore

$$\mathbb{P}(A_k) = \sum_{T \in J} \mathbb{P}(C_T) = \sum_{T \in J} \frac{|C_T|}{n!}.$$

It remains to calculate $|C_T|$. This quantity equals the number of permutations over the $n-|T|$ elements of $\{1,\ldots,n\}\setminus T$, *without any fixed points*, which we have already computed in Equation (4). Therefore

$$|C_T| = (n-|T|)! \cdot \sum_{i=0}^{n-|T|} \frac{(-1)^i}{i!}.$$

Since every such T is of size k, and there are $\binom{n}{k}$ such choices for T, we get that

$$\begin{aligned}
\mathbb{P}\left(A_k\right) &= \sum_{T\in J} \frac{(n-|T|)! \cdot \sum_{i=0}^{n-|T|} \frac{(-1)^i}{i!}}{n!} \\
&= \binom{n}{k} \frac{(n-k)! \cdot \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}}{n!} \\
&= \frac{1}{k!} \cdot \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}.
\end{aligned}$$

We conclude that, for every $0 \le k \le n$, it holds that

$$\mathbb{P}\left(A_k\right) = \frac{1}{k!} \cdot \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}.$$

This also implies that

$$\lim_{n\to\infty} \mathbb{P}\left(A_k\right) = e^{-1} \cdot \frac{1}{k!}.$$

This distribution, of the number of fixed points in a random permutation, is called the *Poisson distribution*, and we will discuss it later in the course.