Assignment 4
Solutions

Exercise 1 Let $A_1, \ldots, A_n$ be events in some probability space $(\Omega, \mathbb{P})$. For every $1 \le i \le n$, let 1_{A_i} denote the indicator of A_i (i.e., $1_{A_i} = 1$ if A_i occurs and $1_{A_i} = 0$ otherwise). Prove that the events $A_1, \ldots, A_n$ are mutually independent if and only if the random variables $1_{A_1}, \ldots, 1_{A_n}$ are mutually independent.

Solution

Fix $x_1, \ldots, x_n$. If $x_i \notin \{0, 1\}$ for some $1 \le i \le n$, then

$$\mathbb{P}\left(1_{A_1} = x_1, \ldots, 1_{A_n} = x_n\right) = 0 = \mathbb{P}\left(1_{A_1} = x_1\right) \cdot \ldots \cdot \mathbb{P}\left(1_{A_n} = x_n\right).$$

Suppose then that $x_i \in \{0, 1\}$ for every $1 \le i \le n$. For every $1 \le i \le n$ let $B_i = A_i$ if $x_i = 1$ and $B_i = A_i^c$ if $x_i = 0$. Observe that $\{1_{A_i} = x_i\} = B_i$ for every $1 \le i \le n$. Therefore

$$\mathbb{P}\left(1_{A_1} = x_1, \ldots, 1_{A_n} = x_n\right) = \mathbb{P}\left(\bigcap_{i=1}^{n} B_i\right),$$

and

$$\mathbb{P}\left(1_{A_1} = x_1\right) \cdot \ldots \cdot \mathbb{P}\left(1_{A_n} = x_n\right) = \prod_{i=1}^{n} \mathbb{P}\left(B_i\right).$$

By considering all possible values of $(x_1, \ldots, x_n) \in \{0, 1\}^n$, we conclude that the events $A_1, \ldots, A_n$ are independent if and only if the random variables $1_{A_1}, \ldots, 1_{A_n}$ are independent.

Exercise 2 Prove that the Hypergeometric distribution is in fact a probability distribution.

Solution

Let $X \sim \text{Hyp}\left(N, D, n\right)$. Then, for every integer $0 \le k \le n$ it holds that

$$\mathbb{P}\left(X = k\right) = \frac{\binom{D}{k} \cdot \binom{N-D}{n-k}}{\binom{N}{n}}$$

(as usual, we are using the convention that $\binom{a}{b} = 0$ if $a < b$).

We need to prove that

$$\sum_{k} \frac{\binom{D}{k} \cdot \binom{N-D}{n-k}}{\binom{N}{n}} = 1,$$

or, equivalently, that

$$\sum_{k=0}^{n} \binom{D}{k} \cdot \binom{N-D}{n-k} = \binom{N}{n}. \tag{1}$$

We provide a combinatorial proof of (1). Suppose that we want to choose n people out of D men and $N-D$ women. We claim that both sides of (1) equal the number of ways of making such a choice. This is obvious for the right-hand side. For every $0 \leq k \leq n$, there are $\binom{D}{k} \cdot \binom{N-D}{n-k}$ ways to choose exactly k men and $n-k$ women. Summing over all possible values of k and observing that no choice of n people can have exactly i men and exactly j men if $i \neq j$, proves our claim for the left-hand side of (1) as well.

Exercise 3 The number of cars crossing a particular bridge is a random variable with Poisson distribution with an average of $\lambda = 0.3$ cars per minute, i.e., if X counts the number of cars crossing the bridge in any given minute, then $X \sim \text{Poi}(0.3)$. Calculate the probability that within 5 minutes:

1. No cars have crossed the bridge.
2. More than one car has crossed the bridge.
3. The number of cars that have crossed the bridge is between 1 and 3.
4. Exactly 3 cars have crossed the bridge.

Solution

For every $1 \leq i \leq 5$ let X_i be the number of cars that have crossed the bridge during the ith minute. Then $X_i \sim \text{Poi}(0.3)$ for all such i. As was shown in the practical sessions, $Y := \sum_{i=1}^{5} X_i$ satisfies $Y \sim \text{Poi}(0.3 \cdot 5)$, i.e., $Y \sim \text{Poi}(1.5)$. Therefore, for every $k \geq 0$ it holds that

$$\mathbb{P}(Y = k) = e^{-1.5} \cdot \frac{1.5^k}{k!}.$$

1. $\mathbb{P}(Y = 0) = e^{-1.5}$.
2. $\mathbb{P}(Y > 1) = 1 - (\mathbb{P}(Y = 0) + \mathbb{P}(Y = 1)) = 1 - e^{-1.5}(1 + 1.5) = 1 - 2.5 \cdot e^{-1.5}$.
3. $\mathbb{P}(1 \leq Y \leq 3) = \mathbb{P}(Y = 1) + \mathbb{P}(Y = 2) + \mathbb{P}(Y = 3) = e^{-1.5}\left(1.5 + \frac{1.5^2}{2} + \frac{1.5^3}{3!}\right) = \frac{51}{16} \cdot e^{-1.5}$.
4. $\mathbb{P}(Y = 3) = e^{-1.5} \cdot \frac{1.5^3}{3!} = \frac{9}{16} \cdot e^{-1.5}$.

Exercise 4 Let $X \sim \text{Geom}(p)$ and $Y \sim \text{U}(\{0, 1, \ldots, n\})$ be independent random variables. Find the probability distribution of $Z := X + Y$.

Solution

Clearly, the support of Z is $\mathbb{N}$. Fix some $k \in \mathbb{N}$. Assume first that $k \leq n$. Then

$$\begin{aligned}
\mathbb{P}(Z = k) &= \mathbb{P}(X + Y = k) \\
&= \sum_{j=1}^{k} \mathbb{P}(X + Y = k \mid X = j) \cdot \mathbb{P}(X = j) \\
&= \sum_{j=1}^{k} \mathbb{P}(Y = k - j) \cdot \mathbb{P}(X = j) \\
&= \sum_{j=1}^{k} \frac{1}{n+1} \cdot (1-p)^{j-1} \cdot p \\
&= \frac{p}{n+1} \cdot \sum_{j=1}^{k} (1-p)^{j-1} \\
&= \frac{p}{n+1} \cdot \frac{1-(1-p)^k}{1-(1-p)} \\
&= \frac{1-(1-p)^k}{n+1},
\end{aligned}$$

where the second equality holds by the Law of total probability, and the fourth equality holds since $k \leq n$.

Assume now that $k > n$. Then

$$\begin{aligned}
\mathbb{P}(Z = k) &= \mathbb{P}(X + Y = k) \\
&= \sum_{j=0}^{n} \mathbb{P}(X + Y = k \mid Y = j) \cdot \mathbb{P}(Y = j) \\
&= \sum_{j=0}^{n} \mathbb{P}(X = k - j) \cdot \mathbb{P}(Y = j) \\
&= \sum_{j=0}^{n} (1-p)^{k-j-1} \cdot p \cdot \frac{1}{n+1} \\
&= \frac{p(1-p)^{k-n-1}}{n+1} \cdot \sum_{j=0}^{n} (1-p)^j \\
&= \frac{p(1-p)^{k-n-1}}{n+1} \cdot \frac{1-(1-p)^{n+1}}{1-(1-p)} \\
&= \frac{(1-p)^{k-n-1} - (1-p)^k}{n+1},
\end{aligned}$$

where the second equality holds by the Law of total probability, and the fourth equality holds since $k > n$.

Exercise 5 A fair coin with 0 on one side and 1 on the other side is tossed. If the outcome of the coin toss is 0, a fair die is rolled, otherwise the coin is tossed again. Let X and Y be the outcomes of the first and second experiments, respectively. Calculate the joint distribution of X and Y.

Solution

If $X = 1$, in the second round we will toss the coin again. Hence the support of Y, conditioned on the event $\{X = 1\}$, is $\{0, 1\}$. Therefore

$$\mathbb{P}(X = 1, Y = 0) = \mathbb{P}(X = 1, Y = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

If $X = 0$, in the second round we will roll a fair die. Hence, for $1 \leq k \leq 6$, we have

$$\mathbb{P}(X = 0, Y = k) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}.$$

Exercise 6 A coin with probability p to come up heads, is tossed indefinitely, all coin tosses being mutually independent. For every positive integer r, let X_r be the number of tosses until the rth time the outcome of the coin toss is heads.

1. Calculate the distribution of X_r for every positive integer r.
2. Calculate the joint distribution of X_r and X_s for every $1 \leq r < s$.

Solution

1. For every positive integer r it holds that $X \sim NB(r, p)$, that is,

$$\mathbb{P}(X_r = k) = \binom{k-1}{r-1} \cdot (1-p)^{k-r} \cdot p^r,$$

   for every integer $k \geq r$.

2. Let $s > k \geq r$, and $\ell \geq s$ be integers, and let $t = \ell - k$. First, observe that

$$\mathbb{P}(X_s = t + k \mid X_r = k) = \binom{t-1}{s-r-1} \cdot (1-p)^{t-s+r} \cdot p^{s-r}$$

   Indeed, conditioned on the event $\{X_r = k\}$, there are t additional coin tosses, in which we expect to see the rest of the $s - r$ heads. Since the outcome of the sth coin toss must be heads, it remains to choose the location of $s - r - 1$ additional coin tosses whose outcome is heads. Hence, by the previous part of the exercise we obtain

$$\begin{aligned}
\mathbb{P}(X_r = k, X_s = \ell) &= \mathbb{P}(X_r = k) \cdot \mathbb{P}(X_s = \ell \mid X_r = k) \\
&= \binom{k-1}{r-1} \cdot (1-p)^{k-r} \cdot p^r \cdot \binom{t-1}{s-r-1} \cdot (1-p)^{t-s+r} \cdot p^{s-r} \\
&= \binom{k-1}{r-1} \cdot \binom{\ell-k-1}{s-r-1} \cdot (1-p)^{\ell-s} \cdot p^s.
\end{aligned}$$

   For all other values of k and ℓ, it holds that $\mathbb{P}(X_r = k, X_s = \ell) = 0$.