Assignment 1 – Solutions

Exercise 1 Two fair dice are rolled. We then output the sum of the results mod 6. Prove that the corresponding probability space is uniform.

Solution

Let $(\Omega, \mathbb{P})$ be the corresponding probability space, where $\Omega = \{1, 2, 3, 4, 5, 6\}$. We will show that for every $k \in \Omega$ it holds that $\mathbb{P}(k) = \frac{1}{6}$. Let $(\Omega', \mathbb{P}')$ be the probability space for the pair of dices, i.e.,$\Omega' = \{(a, b) : 1 \leq a, b \leq 6\}$. By the fact that the dices are fair we get that:

$$\mathbb{P}(1) = \mathbb{P}'(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = \frac{6}{36} = \frac{1}{6}.$$
$$\mathbb{P}(2) = \mathbb{P}'(\{(1,1),(2,6),(3,5),(4,4),(5,3),(6,2)\}) = \frac{6}{36} = \frac{1}{6}.$$
$$\mathbb{P}(3) = \mathbb{P}'(\{(1,2),(2,1),(3,6),(4,5),(5,4),(6,3)\}) = \frac{6}{36} = \frac{1}{6}.$$
$$\mathbb{P}(4) = \mathbb{P}'(\{(1,3),(2,2),(3,1),(4,6),(5,5),(6,4)\}) = \frac{6}{36} = \frac{1}{6}.$$
$$\mathbb{P}(5) = \mathbb{P}'(\{(1,4),(2,3),(3,2),(4,1),(5,6),(6,5)\}) = \frac{6}{36} = \frac{1}{6}.$$
$$\mathbb{P}(6) = \mathbb{P}'(\{(1,5),(2,4),(3,3),(4,2),(5,1),(6,6)\}) = \frac{6}{36} = \frac{1}{6}.$$

Exercise 2 A computer samples a binary string of length n uniformly at random (i.e., the probability space is uniform over all binary strings of length n).

1. For every $0 \leq k \leq n$, calculate the probability that there are exactly k 1's in the chosen string?
2. Let E be the event that the number of 1's in the chosen string is even, and let O be the event that the number of 1's in the chosen string is odd. Prove that $\mathbb{P}(E) = \mathbb{P}(O)$.
3. Prove that $\mathbb{P}(E) = \frac{1}{2}$.

Solution

Let $(\Omega, \mathbb{P})$ be the corresponding probability space. Then

$$|\Omega| = 2^n.$$

1. Let A_k be the event that there exactly k 1's. Then

$$|A_k| = \binom{n}{k}.$$

Since the probability space is uniform, we get that

$$\mathbb{P}(A_k) = \frac{|A_k|}{|\Omega|} = \binom{n}{k} \cdot \frac{1}{2^n}.$$

2. The probability space is uniform, therefore it is enough to show that $|E| = |O|$. We observe that

$$|E| = \sum_{\substack{0 \le k \le n \\ k \text{ is even}}} \binom{n}{k},$$

and that

$$|O| = \sum_{\substack{0 \le k \le n \\ k \text{ is odd}}} \binom{n}{k}.$$

Hence

$$|E| - |O| = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k = (1-1)^n = 0,$$

where the second equality is by Newton's binomial formula.

Another approach of proving $|E| = |O|$ is by giving a bijection between E and O. Flipping the first bit gives such a bijection, as flipping it again forms the inverse of that function.

3. Since $E \cap O = \emptyset$, it follows that

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(E \cup O) = \mathbb{P}(E) + \mathbb{P}(O) = 2\mathbb{P}(E),$$

where the last equality follows from the previous part of the exercise. We conclude that $\mathbb{P}(E) = \frac{1}{2}$.

Exercise 3 A fair coin is tossed repeatedly until the second time the outcome is heads.

1. Describe the sample space for this experiment.

2. For some $k \in \mathbb{N}$, calculate the probability that the coin will be tossed exactly k times.

3. Prove that the experiment forms a probability space.

Solution

1. One possible way to describe the sample space
$$\Omega = \left\{ (b_1, b_2, \dots, b_n, 1) : n \in \mathbb{N} \wedge \forall i\ b_i \in \{0,1\} \wedge \sum_{i=1}^{n} b_i = 1 \right\},$$
where we use 1 for heads and 0 for tails.

2. Let A_k be the event that the coin was tossed k times. Since the last coin is always heads we get that:
$$\mathbb{P}(A_k) = \sum_{(b_1 \dots b_{k-1}, 1) \in \Omega} \mathbb{P}((b_1 \dots b_{k-1}, 1))$$
For a fixed sequence $(b_1 \dots b_{k-1}, 1) \in \Omega$ it holds that
$$\mathbb{P}((b_1 \dots b_{k-1}, 1)) = \frac{1}{2^k},$$
due to the fact that the coin is fair. As the sum of the b_i's is 1, it follows that there are $k-1$ possible sequences, hence
$$\sum_{(b_1 \dots b_{k-1}, 1) \in \Omega} \frac{1}{2^k} = \frac{k-1}{2^k}.$$

3. We need to prove that
$$\sum_{n=1}^{\infty} \Big(\sum_{(b_1 \dots b_n, 1) \in \Omega} \mathbb{P}((b_1 \dots b_n, 1)) \Big) = 1.$$
By the previous exercise, for every $n \in \mathbb{N}$ it holds that
$$\sum_{(b_1 \dots b_n, 1) \in \Omega} \mathbb{P}((b_1 \dots b_n, 1)) = \frac{n}{2^{n+1}}.$$
By the formula of an infinite geometric sum we have
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$
for all $|x| < 1$. Taking the derivative in both sides leads to
$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2},$$
which is equivalent to
$$\sum_{n=1}^{\infty} n x^{n+1} = \frac{x^2}{(1-x)^2}.$$
By setting $x = \frac{1}{2}$ we get
$$\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = \frac{\frac{1}{2^2}}{(1-\frac{1}{2})^2} = 1.$$

Exercise 4 Let $A_1, A_2, \dots, A_n$ be events in an arbitrary probability space $(\Omega, \mathbb{P})$. Prove that

1.
$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}\left(A_i^c\right).$$

2.
$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} \mathbb{P}\left(A_i\right) - n + 1.$$

Solution

1. Recall that one of the De-Morgan laws states that
$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.$$
Taking the probability of both sides we get that
$$\mathbb{P}\left(\left(\bigcap_{i=1}^{n} A_i\right)^c\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i^c\right).$$
Since $\mathbb{P}\left(A^c\right) = 1 - \mathbb{P}\left(A\right)$ for all events A, it holds that
$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = 1 - \mathbb{P}\left(\left(\bigcap_{i=1}^{n} A_i\right)^c\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{n} A_i^c\right).$$
By the union bound it follows that
$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{n} A_i^c\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}\left(A_i^c\right).$$

2. By the previous part it holds that
$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}\left(A_i^c\right).$$
As for every event A, it holds that $\mathbb{P}\left(A^c\right) = 1 - \mathbb{P}\left(A\right)$, it follows that
$$1 - \sum_{i=1}^{n} \mathbb{P}\left(A_i^c\right) = 1 - \sum_{i=1}^{n} \left(1 - \mathbb{P}\left(A_i\right)\right) = \sum_{i=1}^{n} \mathbb{P}\left(A_i\right) - n + 1.$$
The claim follows.

Exercise 5 Four men go to a party and hang their coats in a closet. Upon leaving, each of the four men picks one of these four coats uniformly at random. Calculate the probability that none of them picks the coat they came with to the party.

Solution

For $i \in \{1,2,3,4\}$ let A_i be the event that person i took his coat. Then

$$\mathbb{P}\left(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c\right) = 1 - \mathbb{P}\left(A_1 \cup A_2 \cup A_3 \cup A_4\right).$$

By inclusion-exclusion we get that

$$\begin{aligned}\mathbb{P}\left(A_1 \cup A_2 \cup A_3 \cup A_4\right) = &\sum_{i=1}^{4} \mathbb{P}\left(A_i\right) - \sum_{1 \le i < j \le 4} \mathbb{P}\left(A_i \cap A_j\right) + \\ &\sum_{1 \le i < j < k \le 4} \mathbb{P}\left(A_i \cap A_j \cap A_k\right) + \mathbb{P}\left(A_1 \cap A_2 \cap A_3 \cap A_4\right).\end{aligned}$$

Person i will take his coat in $3!$ cases out of all $4!$ possibilities. Therefore $\mathbb{P}\left(A_i\right) = \frac{6}{24} = \frac{1}{4}$ for all i. Similarly $\mathbb{P}\left(A_i \cap A_j\right) = \frac{2}{24} = \frac{1}{12}$, $\mathbb{P}\left(A_i \cap A_j \cap A_k\right) = \frac{1}{24}$, and $\mathbb{P}\left(A_1 \cap A_2 \cap A_3 \cap A_4\right) = \frac{1}{24}$. Therefore

$$\mathbb{P}\left(A_1 \cup A_2 \cup A_3 \cup A_4\right) = 4 \cdot \frac{1}{4} - \binom{4}{2} \cdot \frac{1}{12} + \binom{4}{3} \cdot \frac{1}{24} - \frac{1}{24} = \frac{5}{8},$$

hence

$$\mathbb{P}\left(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c\right) = \frac{3}{8}.$$

Exercise 6 We are given n different bins and $k \ge n$ balls. Each ball is thrown into a bin chosen uniformly at random. Calculate the probability that no bin is empty if

1. The balls are different.
2. The balls are identical.

Solution

1. Let $(\Omega_1, \mathbb{P}_1)$ be the probability space. Since the balls are different, it holds that $|\Omega_1| = n^k$. Let A_1 be the event that no bin is empty. For $i \in \{1,2,3,\ldots,n\}$ let B_i be the event that bin i is empty. Then

$$A_1 = \bigcap_{i=1}^{n} B_i^c.$$

By inclusion-exclusion

$$\mathbb{P}_1(A_1) = 1 - \sum_{\emptyset \ne I \subseteq \{1,\ldots,n\}} (-1)^{|I|-1} \cdot \mathbb{P}_1\left(\bigcap_{i \in I} B_i\right).$$

Since the balls were uniformly distributed, it holds that for all $\emptyset \ne I \subseteq \{1,\ldots,n\}$

$$\mathbb{P}_1\left(\bigcap_{i \in I} B_i\right) = \frac{(n-|I|)^k}{n^k}.$$

Therefore

$$
\begin{aligned}
\mathbb{P}_1 (A_1) &= 1 - \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|-1} \left(\frac{n - |I|}{n} \right)^k \\
&= 1 - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \left(\frac{n-i}{n} \right)^k \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(1 - \frac{i}{n} \right)^k .
\end{aligned}
$$

2. Let $(\Omega_2, \mathbb{P}_2)$ be the probability space. Since the balls are identical, it holds that the size of the sample space is equal to the number of solutions in the naturals for

$$x_1 + x_2 + \ldots + x_n = k,$$

where we consider 0 as a natural number. The equation has $\binom{k+n-1}{k}$ solutions. Let A_2 be the event that no bin is empty. Therefore $|A_2|$ is equal to the number of solutions in the naturals for

$$x_1 + x_2 + \ldots + x_n = k,$$

where $x_i \geq 1$ for all $i \in \{1, \ldots, n\}$. By setting $y_i = x_i - 1 \geq 0$ for all i, we get that the equation is equivalent to

$$y_1 + y_2 + \ldots + y_n = k - n,$$

for which there are $\binom{k-1}{k-n}$ solutions. Therefore

$$\mathbb{P}_2 (A_2) = \frac{\binom{k-1}{k-n}}{\binom{k+n-1}{k}}.$$