Practical 10

Exercise 1 Let $X \sim \text{Poi}(\lambda)$, for some non-negative real number λ. Calculate $\text{Var}(X)$.

Solution
Recall that

$$\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

for every non-negative integer k. Moreover, it was shown in Practical session 8 that $\mathbb{E}(X) = \lambda$. We begin by calculating $\mathbb{E}(X^2 - X)$.

$$\begin{aligned}
\mathbb{E}\left(X^2 - X\right) &= \mathbb{E}(X(X-1)) \\
&= \sum_{k=0}^{\infty} k(k-1) \cdot \mathbb{P}(X = k) \\
&= \sum_{k=2}^{\infty} k(k-1) \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\
&= \sum_{k=2}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-2)!} \\
&= \lambda^2 \cdot \sum_{m=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^m}{m!} \\
&= \lambda^2,
\end{aligned}$$

where the penultimate equality holds by the substitution $m = k - 2$, and the last equality holds since the sum of probabilities of a Poisson random variable over its support is 1. It is now easy to calculate $\text{Var}(X)$:

$$\text{Var}(X) = \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2 = \mathbb{E}\left(X^2 - X\right) + \mathbb{E}(X) - (\mathbb{E}(X))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda,$$

where the second equality holds by the linearity of expectation.

Exercise 2 Let $X \sim \text{Hyp}(N, D, n)$, for some $N, D, n \in \mathbb{N}$ satisfying $D, n \leq N$. Calculate $\text{Var}(X)$.

Solution
Recall that

$$\mathbb{P}(X = k) = \frac{\binom{D}{k} \cdot \binom{N-D}{n-k}}{\binom{N}{n}}$$

for every integer k satisfying $\max\{0, n+D-N\} \le k \le D$. Moreover, it was shown in Practical session 8 that $\mathbb{E}(X) = n \cdot \frac{D}{N}$. Finally, recall that it was shown in Lecture 9 that

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$

holds for every integer $1 \le k \le n$. Clearly, this also implies that

$$\binom{n}{k} = \frac{n(n-1)}{k(k-1)} \cdot \binom{n-2}{k-2}$$

holds for every integer $2 \le k \le n$.

Let S denote the support of X. Then

$$\begin{aligned}
\mathbb{E}\left(X^2 - X\right) &= \mathbb{E}\left(X(X-1)\right) \\
&= \sum_{k \in S} k(k-1) \cdot \mathbb{P}(X = k) \\
&= \sum_{\substack{k \in S \\ k \ge 2}} k(k-1) \cdot \frac{\binom{D}{k} \cdot \binom{N-D}{n-k}}{\binom{N}{n}} \\
&= \sum_{\substack{k \in S \\ k \ge 2}} k(k-1) \cdot \frac{\frac{D(D-1)}{k(k-1)} \cdot \binom{D-2}{k-2} \cdot \binom{N-D}{n-k}}{\frac{N(N-1)}{n(n-1)} \cdot \binom{N-2}{n-2}} \\
&= n(n-1) \cdot \frac{D}{N} \cdot \frac{D-1}{N-1} \cdot \sum_{\substack{k \in S \\ k \ge 2}} \frac{\binom{D-2}{k-2} \cdot \binom{(N-2)-(D-2)}{(n-2)-(k-2)}}{\binom{N-2}{n-2}} \\
&= n(n-1) \cdot \frac{D}{N} \cdot \frac{D-1}{N-1},
\end{aligned}$$

where the last equality follows since the sum of probabilities of a Hypergeometric random variable with parameters $N-2$, $D-2$ and $n-2$ over its support is 1. It is now easy to calculate $\mathrm{Var}(X)$:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\left(X^2\right) - \left(\mathbb{E}(X)\right)^2 \\
&= \mathbb{E}\left(X^2 - X\right) + \mathbb{E}(X) - \left(\mathbb{E}(X)\right)^2 \\
&= n(n-1) \cdot \frac{D}{N} \cdot \frac{D-1}{N-1} + n \cdot \frac{D}{N} - n^2 \cdot \frac{D^2}{N^2} \\
&= \frac{D \cdot n \cdot (N-D) \cdot (N-n)}{N^2 \cdot (N-1)},
\end{aligned}$$

where the second equality holds by the linearity of expectation.

Exercise 3 Let $0 < p < 1$ be a rel number and let X a random variable satisfying

$$\mathbb{P}(X = 1) = p \text{ and } \mathbb{P}(X = -1) = 1 - p.$$

Find all real numbers c for which $\mathbb{E}\left(c^X\right) = 1$.

Solution
Observe that
$$\mathbb{E}\left(c^X\right) = c^1 \cdot \mathbb{P}(X = 1) + c^{-1} \cdot \mathbb{P}(X = -1) = cp + \frac{1-p}{c}.$$
Hence, our aim is to solve the equation
$$cp + \frac{1-p}{c} = 1$$
for c. This equation is equivalent to
$$c^2 p - c + 1 - p = 0,$$
whose solutions are easily seen to be $c = 1$ and $c = \frac{1-p}{p}$.

Exercise 4 Let $X \sim \mathrm{Poi}(\lambda)$, for some non-negative real number λ.

1. Prove that
$$\mathbb{E}(X^n) = \lambda \cdot \mathbb{E}\left((X+1)^{n-1}\right)$$
holds for every positive integer n.

2. Calculate $\mathbb{E}(X^3)$.

Solution

1. Fix some $n \in \mathbb{N}$. Then
$$\begin{aligned}
\mathbb{E}(X^n) &= \sum_{k=0}^{\infty} k^n \cdot \mathbb{P}(X = k) \\
&= \sum_{k=1}^{\infty} k^n \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\
&= \sum_{k=1}^{\infty} k^{n-1} \cdot e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!} \\
&= \sum_{m=0}^{\infty} (m+1)^{n-1} \cdot e^{-\lambda} \cdot \frac{\lambda^{m+1}}{m!} \\
&= \lambda \cdot \sum_{m=0}^{\infty} (m+1)^{n-1} \cdot e^{-\lambda} \cdot \frac{\lambda^m}{m!} \\
&= \lambda \cdot \sum_{m=0}^{\infty} (m+1)^{n-1} \cdot \mathbb{P}(X = m) \\
&= \lambda \cdot \mathbb{E}\left((X+1)^{n-1}\right),
\end{aligned}$$
where the fourth equality holds by the substitution $m = k - 1$.

2. It follows by the previous part of this exercise that

$$\begin{aligned}
\mathbb{E}\left(X^3\right) &= \lambda \cdot \mathbb{E}\left((X+1)^2\right) \\
&= \lambda \cdot \mathbb{E}\left(X^2 + 2X + 1\right) \\
&= \lambda \cdot \mathbb{E}\left(X^2\right) + 2\lambda \cdot \mathbb{E}(X) + \lambda \\
&= \lambda^2 \cdot \mathbb{E}(X+1) + 2\lambda \cdot \mathbb{E}(X) + \lambda \\
&= \lambda^2 \cdot \mathbb{E}(X) + \lambda^2 + 2\lambda \cdot \mathbb{E}(X) + \lambda \\
&= \lambda^3 + \lambda^2 + 2\lambda^2 + \lambda \\
&= \lambda \cdot \left(\lambda^2 + 3\lambda + 1\right),
\end{aligned}$$

where the third and fifth equalities hold by the linearity of expectation.

Exercise 5 A fair coin is tossed 5 times, all coin tosses being mutually independent. Let X be the number of coin tosses whose outcome was heads and let Y be the number of coin tosses whose outcome was tails. Calculate

1. $\mathbb{E}(X) \cdot \mathbb{E}(Y)$.
2. $\mathbb{E}(X \cdot Y)$.

Solution

1. Observe that $X, Y \sim \text{Bin}\left(5, \frac{1}{2}\right)$ and thus

$$\mathbb{E}(X) = \mathbb{E}(Y) = 5 \cdot \frac{1}{2}.$$

Therefore

$$\mathbb{E}(X) \cdot \mathbb{E}(Y) = \frac{25}{4}.$$

2. We will first calculate the joint distribution of X and Y, i.e., we will calculate $\mathbb{P}(X = x, Y = y)$ for all $x, y \in \{0, 1, \ldots, 5\}$. Observe that $X + Y = 5$. Hence, $\mathbb{P}(X = x, Y = y) = 0$ whenever $x + y \neq 5$. Fix some $x \in \{0, 1, \ldots, 5\}$ and let $y = 5 - x$. Then

$$\begin{aligned}
\mathbb{P}(X = x, Y = y) &= \mathbb{P}(X = x, Y = 5 - x) \\
&= \mathbb{P}(Y = 5 - x \mid X = x) \cdot \mathbb{P}(X = x) \\
&= 1 \cdot \binom{5}{x} \cdot 2^{-5}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}\left(X \cdot Y\right) &= \sum_{x=0}^{5} x(5-x) \cdot \mathbb{P}\left(X = x, Y = 5-x\right) \\
&= \sum_{x=1}^{4} x(5-x)\binom{5}{x} \cdot 2^{-5} \\
&= 2^{-5} \cdot (1 \cdot 4 \cdot 5 + 2 \cdot 3 \cdot 10 + 3 \cdot 2 \cdot 10 + 4 \cdot 1 \cdot 5) = 5.
\end{aligned}$$