1. $T: \mathbb{R}^3 \to \mathbb{R}^4$ linear transformation; $B = \left(b_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}, b_2 = \begin{pmatrix}1\\1\\3\end{pmatrix}, b_3 = \begin{pmatrix}4\\2\\7\end{pmatrix} \right)$ basis of $\mathbb{R}^3$.

$C = \left(c_1 = \begin{pmatrix}1\\0\\1\\1\end{pmatrix}, c_2 = \begin{pmatrix}0\\1\\1\\-1\end{pmatrix}, c_3 = \begin{pmatrix}0\\0\\-1\\1\end{pmatrix}, c_4 = \begin{pmatrix}0\\0\\0\\-1\end{pmatrix} \right)$ basis of $\mathbb{R}^4$. $[T]^B_C = \begin{pmatrix}-1 & 1 & 5\\ 1 & 0 & -1\\ 3 & 1 & 2\\ 2 & -1 & -3\end{pmatrix}$, $X = \begin{pmatrix}x\\y\\z\end{pmatrix}$, $TX = ?$

Answer. To find: $[TX]_{E_4} = [T]^{E_3}_{E_4}[X]_{E_3}$, $[X]_{E_3} = \begin{pmatrix}x\\y\\z\end{pmatrix}$.

$$[T]^{E_3}_{E_4} = [I]^C_{E_4}[T]^B_C[I]^{E_3}_B \; ; \; [I]^{E_3}_B = \left([I]^B_{E_3}\right)^{-1} = \begin{pmatrix}1 & 1 & 4\\ 0 & 1 & 2\\ 1 & 3 & 7\end{pmatrix}^{-1} = \begin{pmatrix}-1 & -5 & 2\\ -2 & -3 & 2\\ 1 & 2 & -1\end{pmatrix}$$

$$\Rightarrow [T]^{E_3}_{E_4} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 1 & -1 & 0\\ 1 & -1 & 1 & -1\end{pmatrix}\begin{pmatrix}-1 & 1 & 5\\ 1 & 0 & -1\\ 3 & 1 & 2\\ 2 & -1 & -3\end{pmatrix}\begin{pmatrix}-1 & -5 & 2\\ -2 & -3 & 2\\ 1 & 2 & -1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 1 & -1 & 0\\ 1 & -1 & 1 & -1\end{pmatrix}\begin{pmatrix}4 & 12 & -5\\ -2 & -7 & 3\\ -3 & -14 & 6\\ -3 & -13 & 5\end{pmatrix} = \begin{pmatrix}4 & 12 & -5\\ -2 & -7 & 3\\ 5 & 19 & -8\\ 6 & 18 & -7\end{pmatrix}$$

$$\Rightarrow TX = \begin{pmatrix}4x + 12y - 5z\\ -2x - 7y + 3z\\ 5x + 19y - 8z\\ 6x + 18y - 7z\end{pmatrix}$$

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2. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x-y\\7x-3y\end{pmatrix} = \begin{pmatrix}1 & -1\\ 7 & -3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow [T]^E_E = \begin{pmatrix}1 & -1\\ 7 & -3\end{pmatrix}$, $B = \left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\right)$

$$[T]^B_B = [I]^E_B[T]^E_E[I]^B_E = \left([I]^B_E\right)^{-1}\begin{pmatrix}1 & -1\\ 7 & -3\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix} = \begin{pmatrix}-3 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}-1 & -1\\ 1 & 5\end{pmatrix} = \begin{pmatrix}5 & 13\\ -3 & -7\end{pmatrix}.$$

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3. V vector space$/\mathbb{R}$, $B = (u, v, w)$ basis for V.

a. $C = (u+v+2w,\ u+v+3w,\ u+2v+2w)$ is also a basis for V since these 3 vectors are linearly independent: $a(u+v+2w) + b(u+v+3w) + c(u+2v+2w) = 0 \Rightarrow a+b+c = 0,\ a+b+2c = 0,\ 2a+3b+2c = 0$
$\Rightarrow a = b = c = 0$. They make a basis since $\dim V = 3$.

b. $[I]^C_C = I$.

c. $[I]^B_C = \left([I]^C_B\right)^{-1} = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 2\\ 2 & 3 & 2\end{pmatrix}^{-1} = \begin{pmatrix}4 & -1 & -1\\ -2 & 0 & 1\\ -1 & 1 & 0\end{pmatrix}$

d. $[3u - v + 8w]_C = 3[u]_C - [v]_C + 8[w]_C = \left([u]_C\ [v]_C\ [w]_C\right)\begin{pmatrix}3\\-1\\8\end{pmatrix} = [I]^B_C\begin{pmatrix}3\\-1\\8\end{pmatrix} = \begin{pmatrix}4 & -1 & -1\\ -2 & 0 & 1\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix}3\\-1\\8\end{pmatrix} = \begin{pmatrix}5\\2\\-4\end{pmatrix}$.

Check: $5(u+v+2w) + 2(u+v+3w) - 4(u+2v+2w) = 3u - v + 8w$.

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4. V, W vector spaces over a field F; $T, S: V \to W$ transformations.
Define $T+S: V \to W$ by $(T+S)(v) = T(v) + S(v)$ for all $v \in V$. Assume: T, S are linear.
Show: $T+S$ is also linear.

T-ו S לינאריות

Proof. $(T+S)(av_1 + bv_2) \underset{\text{הגדרה}}{=} T(av_1+bv_2) + S(av_1+bv_2) = aT(v_1) + bT(v_2) + aS(v_1) + bS(v_2)$

$= aT(v_1) + aS(v_1) + bT(v_2) + bS(v_2) = a(T+S)(v_1) + b(T+S)(v_2) \quad \forall a, b \in F,\ \forall v_1, v_2 \in V.$

קומוטטיביות ב-W

5. $T: V \to V$ linear transformation (operator) on a vector space V over F.

Given: $\forall v \in V \ \exists u \in \operatorname{Im} T, w \in \operatorname{Ker} T$, s.t. $v = u + w$. Show: $Tv = 0 \Leftrightarrow T(T(v)) = 0$.

Answer: ($\Rightarrow$) If $Tv = 0$ then $T(Tv) = T(0) = 0$.

($\Leftarrow$) Recall the Rank-Nullity theorem ("משפט המימדים"): $\dim \operatorname{Im} T + \dim \operatorname{Ker} T = \dim V$.

Put $r = \dim \operatorname{Im} T$, $k = \dim \operatorname{Ker} T$, $n = \dim V$: $r + k = n$.

Our assumption implies that if $v_1, \ldots, v_r$ is a basis of $\operatorname{Im} T$, and $w_1, \ldots, w_k$ is a basis of $\operatorname{Ker} T$, then $v_1, \ldots, v_r, w_1, \ldots, w_k$ spans V.

But $r + k = n = \dim V$ by the thm, hence $v_1, \ldots, v_r, w_1, \ldots, w_k$ must be a basis of V.

Hence $\operatorname{Im} T \cap \operatorname{Ker} T = \{0\}$, so $V = \operatorname{Im} T \oplus \operatorname{Ker} T$ is a direct sum.

If $T(Tv) = 0$ then $Tv \in \operatorname{Ker} T \cap \operatorname{Im} T = \{0\}$, [illegible].