Practical 3

Exercise 1 A pair of fair dice is rolled n times, all die rolls being mutually independent. What is the probability that each of the outcomes $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, and $(6,6)$ appears at least once?

Solution

For every $1 \leq i \leq 6$, let A_i be the event that (i,i) did not appear. We first observe that $|\Omega| = 36^n$. Since the probability space is uniform, for every $\emptyset \neq \mathcal{I} \subseteq \{1,2,3,4,5,6\}$ it holds that

$$\mathbb{P}\left(\bigcap_{i\in\mathcal{I}} A_i\right) = \frac{|\bigcap_{i\in\mathcal{I}} A_i|}{|\Omega|} = \frac{(36-|\mathcal{I}|)^n}{36^n} = \left(1 - \frac{|\mathcal{I}|}{36}\right)^n.$$

Therefore, it follows by inclusion-exclusion that

$$\mathbb{P}\left(\bigcap_{i=1}^{6} A_i^c\right) = \sum_{\mathcal{I}\subseteq\{1,2,3,4,5,6\}} (-1)^{|\mathcal{I}|} \cdot \mathbb{P}\left(\bigcap_{i\in\mathcal{I}} A_i\right) = \sum_{k=0}^{6} \binom{6}{k} \cdot (-1)^k \cdot \left(1 - \frac{k}{36}\right)^n.$$

Exercise 2 [The boy or girl paradox]

A father has two children. Each child is a boy with probability 1/2.

1. What is the probability that the father has a girl, if we know that he has a boy?
2. What is the probability that the father has a girl, if we know that he has a boy that was born on Tuesday?

Solution

Let G be the event that the father has a girl.

1. The sample space is $\{bb, bg, gb, gg\}$. Let B be the event that the father has a boy. Then

   $$\mathbb{P}(G|B) = \frac{\mathbb{P}(G\cap B)}{\mathbb{P}(B)} = \frac{|G\cap B|}{|B|} = \frac{|\{bg, gb\}|}{|bb, bg, gb|} = \frac{2}{3},$$

   where the second equality is due to the fact that the probability space is uniform.

   An alternative approach to solving this question is by using Bayes' rule. Let BB be the event that the father has two boys. Then

   $$\mathbb{P}(G|B) = 1 - \mathbb{P}(BB|B) = 1 - \frac{\mathbb{P}(B|BB)\cdot\mathbb{P}(BB)}{\mathbb{P}(B)} = 1 - \frac{1\cdot\frac{1}{4}}{\frac{3}{4}} = \frac{2}{3}.$$

2. The sample space is

$$\Omega=\left\{b_i^1 b_j^2 : 1\leq i,j\leq 7\right\}\cup\left\{b_i^1 g_j^2 : 1\leq i,j\leq 7\right\}\cup\left\{g_i^1 b_j^2 : 1\leq i,j\leq 7\right\}\cup\left\{g_i^1 g_j^2 : 1\leq i,j\leq 7\right\}.$$

   Let B_3 be the event that the father has a boy that was born on Tuesday. Then $|B_3| = 27$ and $|G\cap B_3| = 14$. Therefore

$$\mathbb{P}\left(G|B_3\right)=\frac{\mathbb{P}\left(G\cap B_3\right)}{\mathbb{P}\left(B_3\right)}=\frac{|G\cap B_3|}{|B_3|}=\frac{14}{27},$$

   where the second equality is due to the fact that the probability space is uniform.

   Again, we can also solve this question using Bayes' rule.

$$\mathbb{P}\left(G|B_3\right)=1-\mathbb{P}\left(BB|B_3\right)=1-\frac{\mathbb{P}\left(B_3|BB\right)\cdot\mathbb{P}\left(BB\right)}{\mathbb{P}\left(B_3\right)}=1-\frac{\mathbb{P}\left(B_3|BB\right)\cdot\mathbb{P}\left(BB\right)}{\mathbb{P}\left(B_3|BB\right)\cdot\mathbb{P}\left(BB\right)+\mathbb{P}\left(B_3|G\right)\cdot\mathbb{P}\left(G\right)}.$$

   Note that

$$\mathbb{P}\left(B_3|BB\right)=1-\mathbb{P}\left(B_3^c|BB\right)=1-\left(\frac{6}{7}\right)^2=\frac{13}{49},$$

   and that $\mathbb{P}\left(G\right)=1-\mathbb{P}\left(BB\right)=\frac{3}{4}$ and $\mathbb{P}\left(B_3|G\right)=\frac{2}{21}$.

   We conclude that

$$\mathbb{P}\left(G|B_3\right)=1-\frac{\frac{13}{49}\cdot\frac{1}{4}}{\frac{13}{49}\cdot\frac{1}{4}+\frac{2}{21}\cdot\frac{3}{4}}=\frac{14}{27}.$$

Exercise 3 A box contains three coins: two regular coins and one fake two-headed coin (i.e., the probability for heads is 1).

1. A coin is chosen uniformly at random and tossed once. What is the probability that the outcome of this coin toss is heads?

2. A coin is chosen uniformly at random. It is tossed once and the outcome is heads. What is the probability that the chosen coin is the fake one?

Solution

Let C_1 be the event that a regular coin is chosen, let C_2 be the event that the two-headed coin is chosen, let H be the event that the outcome of the coin toss is heads, and let T be the event that the outcome of the coin toss is tails. Then

$$\begin{aligned}\mathbb{P}\left(H\mid C_1\right)&=1/2,\\ \mathbb{P}\left(H\mid C_2\right)&=1.\end{aligned}$$

1. It follows by the Law of total probability that

$$\begin{aligned}\mathbb{P}\left(H\right)&=\mathbb{P}\left(H\mid C_1\right)\cdot\mathbb{P}\left(C_1\right)+\mathbb{P}\left(H\mid C_2\right)\cdot\mathbb{P}\left(C_2\right)\\&=\frac{1}{2}\cdot\frac{2}{3}+1\cdot\frac{1}{3}\\&=\frac{2}{3}.\end{aligned}$$

2. Using Bayes' rule we obtain

$$\begin{aligned}
\mathbb{P}(C_2 \mid H) &= \frac{\mathbb{P}(H \mid C_2) \cdot \mathbb{P}(C_2)}{\mathbb{P}(H)} \\
&= \frac{1 \cdot \frac{1}{3}}{\frac{2}{3}} \\
&= \frac{1}{2}.
\end{aligned}$$

Exercise 4 [Monty Hall]
Suppose you play in a game show, and you are given the choice between three closed doors. Behind one door, chosen uniformly at random, there is a car; behind the other two, nothing. You pick a door, without loss of generality, say door number 1. The host of the game show, who knows behind which door the car is, chooses another door, without loss of generality, say door number 3, and opens it to reveal that the car is not behind it (we assume that if the player picked the correct door, i.e. the one hiding the car, then the host chooses one of the other two doors uniformly at random). The host then offers the player the chance to switch his choice, that is, to choose door number 2. Prove that by switching, the probability of winning the car is $\frac{2}{3}$.

Proof. For $1 \leq i \leq 3$, let C_i be the event that the car is behind door number i, let A_i be the event that the player chose door number i, and let H_i be the event that the host opened door number i. We first note that

$$\begin{aligned}
\mathbb{P}(H_3 \mid C_1, A_1) &= 1/2 \\
\mathbb{P}(H_3 \mid C_2, A_1) &= 1 \\
\mathbb{P}(H_3 \mid C_3, A_1) &= 0.
\end{aligned}$$

Since we assume the player chose door number 1 and the host opened door number 3, the probability of winning by switching is $\mathbb{P}(C_2 \mid A_1, H_3)$. Applying Bayes' rule in the conditional probability space $(\Omega, \mathbb{P}(* \mid A_1))$ we obtain

$$\begin{aligned}
\mathbb{P}(C_2 \mid A_1, H_3) &= \frac{\mathbb{P}(C_2 \mid A_1) \cdot \mathbb{P}(H_3 \mid C_2, A_1)}{\mathbb{P}(H_3 \mid A_1)} \\
&= \frac{\mathbb{P}(C_2 \mid A_1) \cdot \mathbb{P}(H_3 \mid C_2, A_1)}{\mathbb{P}(C_2 \mid A_1) \cdot \mathbb{P}(H_3 \mid C_2, A_1) + \mathbb{P}(C_1 \mid A_1) \cdot \mathbb{P}(H_3 \mid C_1, A_1) + \mathbb{P}(C_3 \mid A_1) \cdot \mathbb{P}(H_3 \mid C_3, A_1)} \\
&= \frac{\frac{1}{3} \cdot 1}{\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0} \\
&= \frac{2}{3}.
\end{aligned}$$

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