Probability Theory 1 – Proposed solution of moed bet exam

1. **(a)** It is evident that the support of both X and Y is $\{0, 1, 2\}$. For every $0 \leq i, j \leq 2$, the table below shows the value of $P(X = i, Y = j)$. We explain three of these calculations in greater detail.

   Since there are only two red balls and we draw three balls, at least one of these balls must be black or white. Hence, $P(X = 0, Y = 0) = 0$.

$$P(X = 0, Y = 2) = \frac{\binom{2}{0}\binom{2}{2}\binom{2}{1}}{\binom{6}{3}} = \frac{1 \cdot 1 \cdot 2}{20} = \frac{1}{10}.$$

$$P(X = 1, Y = 1) = \frac{\binom{2}{1}\binom{2}{1}\binom{2}{1}}{\binom{6}{3}} = \frac{2 \cdot 2 \cdot 2}{20} = \frac{4}{10}.$$

	Y = 0	Y = 1	Y = 2
X = 0	0	1/10	1/10
X = 1	1/10	4/10	1/10
X = 2	1/10	1/10	0

   (b) Using the table from (a) we conclude that

$$P(X > Y) = P(X = 1, Y = 0) + P(X = 2, Y = 0) + P(X = 2, Y = 1) = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10}.$$

   (c) Using the table from (a) we conclude that

$$\begin{aligned} \mathbb{E}(XY) &= \sum_{i=0}^{2}\sum_{j=0}^{2} ij \cdot P(X = i, Y = j) = \sum_{i=1}^{2}\sum_{j=1}^{2} ij \cdot P(X = i, Y = j) \\ &= 1 \cdot 1 \cdot 4/10 + 1 \cdot 2 \cdot 1/10 + 2 \cdot 1 \cdot 1/10 + 2 \cdot 2 \cdot 0 = 4/5. \end{aligned}$$

2. **(a)** Fix some $1 \leq k \leq n - 1$.

$$\begin{aligned} P(X = k | X + Y = n) &= \frac{P(X = k, X + Y = n)}{P(X + Y = n)} = \frac{P(X = k, Y = n - k)}{P(X + Y = n)} \\ &= \frac{P(X = k) \cdot P(Y = n - k)}{P(X + Y = n)} = \frac{p(1-p)^{k-1} \cdot p(1-p)^{n-k-1}}{P(X + Y = n)} \\ &= \frac{p^2(1-p)^{n-2}}{P(X + Y = n)}. \end{aligned}$$

   We see that $P(X = k|X+Y = n)$ does not depend on k, that is, $P(X = k|X+Y = n) = P(X = \ell|X + Y = n)$ for every $1 \leq k, \ell \leq n - 1$. Since, moreover, $P(X = k|X + Y = n) = 0$ whenever $k \notin \{1, \ldots, n - 1\}$, it follows that $\sum_{k=1}^{n-1} P(X = k|X + Y = n) = 1$. We conclude that $P(X = k|X + Y = n) = \frac{1}{n-1}$ for every $1 \leq k \leq n - 1$ as claimed.

(b) Assume first that X and Y are indepndent random variables, that is $P(X = x, Y = y) = P(X = x)P(Y = y)$ for every x and y. Fix some non-negative integers a and b. Then

$$\begin{aligned}
P(X \geq a, Y \geq b) &= \sum_{x=a}^{\infty}\sum_{y=b}^{\infty} P(X = x, Y = y) = \sum_{x=a}^{\infty}\sum_{y=b}^{\infty} P(X = x)P(Y = y) \\
&= \left(\sum_{x=a}^{\infty} P(X = x)\right)\left(\sum_{y=b}^{\infty} P(Y = y)\right) = P(X \geq a)P(Y \geq b),
\end{aligned}$$

where the second equality holds since X and Y are independent.

Assume now that $P(X \geq a, Y \geq b) = P(X \geq a)P(Y \geq b)$ holds for every non-negative integers a and b. Fix some non-negative integers x and y. Then

$$\begin{aligned}
P(X = x)P(Y = y) &= [P(X \geq x) - P(X \geq x+1)]\,[P(Y \geq y) - P(Y \geq y+1)] \\
&= P(X \geq x)P(Y \geq y) - P(X \geq x)P(Y \geq y+1) \\
&- P(X \geq x+1)P(Y \geq y) + P(X \geq x+1)P(Y \geq y+1) \\
&= P(X \geq x, Y \geq y) - P(X \geq x, Y \geq y+1) \\
&- P(X \geq x+1, Y \geq y) + P(X \geq x+1, Y \geq y+1) \\
&= \sum_{i=x}^{\infty}\sum_{j=y}^{\infty} P(X = i, Y = j) - \sum_{i=x}^{\infty}\sum_{j=y+1}^{\infty} P(X = i, Y = j) \\
&- \sum_{i=x+1}^{\infty}\sum_{j=y}^{\infty} P(X = i, Y = j) + \sum_{i=x+1}^{\infty}\sum_{j=y+1}^{\infty} P(X = i, Y = j) \\
&= P(X = x, Y = y).
\end{aligned}$$

Since x and y were arbitrary, it follows that X and Y are independent.

3. **(a)** It readily follows from the definition of X that, for every non-negative integer k, we have $(X|N = k) \sim Bin(k, 1/2)$. In particular, $\mathbb{E}(X|N = k) = k/2$. By the law of total expectation we then have

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}(\mathbb{E}(X|N)) = \sum_{k=0}^{\infty} \mathbb{E}(X|N = k)P(N = k) = \sum_{k=1}^{\infty} \frac{k}{2} \cdot \frac{\lambda^k e^{-\lambda}}{k!} \\
&= \sum_{k=1}^{\infty} \frac{\lambda}{2} \cdot \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} = \frac{\lambda}{2} \sum_{t=0}^{\infty} \frac{\lambda^t e^{-\lambda}}{t!} = \frac{\lambda}{2}.
\end{aligned}$$

(b) Since $X \leq N$ with probability 1, it is evident that X and N are not independent. We consider a concrete example. Clearly, $P(X = 2, N = 1) = 0$. On the other hand, $P(N = 1) = \lambda e^{-\lambda} > 0$ and

$$P(X = 2) \geq P(X = 2, N = 2) = P(X = 2|N = 2) \cdot P(N = 2) = (1/2)^2 \cdot \lambda^2 e^{-\lambda}/2 > 0.$$

Therefore $P(X = 2, N = 1) \neq P(X = 2) \cdot P(N = 1)$.

(c) Clearly X is a non-negative random variable. Moreover, it was shown in part (a) that $\mathbb{E}(X) = \lambda/2$. Hence, by Markov's inequality we have

$$P(X \geq \lambda) \leq \frac{\mathbb{E}(X)}{\lambda} = \frac{\lambda/2}{\lambda} = \frac{1}{2}.$$

4. **(a)** For every $1 \leq i \leq n$, let Z_i denote the result of the ith die roll. For every $1 \leq i \leq n-2$, let X_i be the indicator random variable for the event "$Z_i+Z_{i+1}+Z_{i+2} = 5$" (i.e., $X_i = 1$ if $Z_i + Z_{i+1} + Z_{i+2} = 5$ and $X_i = 0$ otherwise). Observe that for every $1 \leq i \leq n-2$ we have

$$\begin{aligned} \mathbb{E}(X_i) &= P(X_i = 1) = P(Z_i = 3, Z_{i+1} = 1, Z_{i+2} = 1) + P(Z_i = 1, Z_{i+1} = 3, Z_{i+2} = 1) \\ &+ P(Z_i = 1, Z_{i+1} = 1, Z_{i+2} = 3) + P(Z_i = 2, Z_{i+1} = 2, Z_{i+2} = 1) \\ &+ P(Z_i = 2, Z_{i+1} = 1, Z_{i+2} = 2) + P(Z_i = 1, Z_{i+1} = 2, Z_{i+2} = 2) \\ &= 6 \cdot \frac{1}{6^3} = \frac{1}{36}. \end{aligned}$$

Now, clearly, $X = \sum_{i=1}^{n-2} X_i$ and thus, by linearity of expectation, we have

$$\mathbb{E}(X) = \sum_{i=1}^{n-2} \mathbb{E}(X_i) = \frac{n-2}{36}.$$

(b) We will first calculate $Cov(X_i, X_j)$ for every $1 \leq i < j \leq n-2$. It is evident that X_i and X_j are independent whenever $j > i+2$, as they correspond to disjoint triples of dice rolls. In particular, $Cov(X_i, X_j) = 0$ in these cases. For every $1 \leq i \leq n-3$ we have

$$\begin{aligned} & Cov(X_i, X_{i+1}) = \mathbb{E}(X_i X_{i+1}) - \mathbb{E}(X_i)\mathbb{E}(X_{i+1}) = P(X_i = 1, X_{i+1} = 1) - \left(\frac{1}{36}\right)^2 \\ = \;& P(Z_i = 3, Z_{i+1} = 1, Z_{i+2} = 1, Z_{i+3} = 3) + P(Z_i = 1, Z_{i+1} = 3, Z_{i+2} = 1, Z_{i+3} = 1) \\ + \;& P(Z_i = 1, Z_{i+1} = 1, Z_{i+2} = 3, Z_{i+3} = 1) + P(Z_i = 2, Z_{i+1} = 2, Z_{i+2} = 1, Z_{i+3} = 2) \\ + \;& P(Z_i = 2, Z_{i+1} = 1, Z_{i+2} = 2, Z_{i+3} = 2) + P(Z_i = 1, Z_{i+1} = 2, Z_{i+2} = 2, Z_{i+3} = 1) \\ - \;& \frac{1}{6^4} = 6 \cdot \frac{1}{6^4} - \frac{1}{6^4} = \frac{5}{6^4}. \end{aligned}$$

A similar calculation shows that, for every $1 \leq i \leq n-4$, there are 14 different sequences of 5 dice rolls for which $X_i = 1$ and $X_{i+2} = 1$. Hence

$$\begin{aligned} & Cov(X_i, X_{i+2}) = \mathbb{E}(X_i X_{i+2}) - \mathbb{E}(X_i)\mathbb{E}(X_{i+2}) = P(X_i = 1, X_{i+2} = 1) - \left(\frac{1}{36}\right)^2 \\ = \;& 14 \cdot \frac{1}{6^5} - \frac{1}{6^4} = \frac{8}{6^5}. \end{aligned}$$

We can now calculate the variance of X.

$$
\begin{aligned}
Var(X) &= \sum_{i=1}^{n-2} Var(X_i) + 2 \sum_{1 \le i < j \le n-2} Cov(X_i, X_j) \\
&= \frac{n-2}{6^2} - \frac{n-2}{6^4} + 2 \sum_{i=1}^{n-3} Cov(X_i, X_{i+1}) + 2 \sum_{i=1}^{n-4} Cov(X_i, X_{i+2}) \\
&= \frac{n-2}{6^2} - \frac{n-2}{6^4} + \frac{10(n-3)}{6^4} + \frac{16(n-4)}{6^5} = \frac{286n - 664}{6^5}.
\end{aligned}
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