Assignment 3
Solutions

Exercise 1 A bin contains n balls, labeled with the numbers $1, 2, \ldots, n$. Exactly m balls are drawn uniformly at random from the bin. Let M be the maximum number of a ball that was drawn.

1. Calculate the distribution of M, when the samples are being made without replacement.
2. Calculate the distribution of M, when the samples are being made independently with replacement.

Solution

We will calculate $\mathbb{P}(M = k)$ for every $1 \leq k \leq n$.

1. In this case, it is evident that $\mathbb{P}(M = k) = 0$ for every $1 \leq k \leq m - 1$. Moreover, for every $m \leq k \leq n$, it holds that $M = k$ if and only if the ball labeled k is drawn and the remaining $m - 1$ balls that are drawn have their labels in $\{1, 2, \ldots, k-1\}$. Hence $|\{M = k\}| = \binom{k-1}{m-1}$. Since the probability space is uniform, we conclude that
$$\mathbb{P}(M = k) = \frac{|\{M = k\}|}{\binom{n}{m}} = \frac{\binom{k-1}{m-1}}{\binom{n}{m}}.$$

2. For every $1 \leq i \leq m$, let X_i be the label of the ith ball that was drawn. Then
$$\mathbb{P}(X_i \leq k) = \frac{k}{n}$$
holds for every $0 \leq k \leq n$ and every $1 \leq i \leq m$. Since the samples are independent of one another, it follows that
$$\mathbb{P}(M \leq k) = \mathbb{P}(X_i \leq k \text{ for every } 1 \leq i \leq m) = \prod_{i=1}^{m} \mathbb{P}(X_i \leq k) = \left(\frac{k}{n}\right)^m.$$
We conclude that
$$\mathbb{P}(M = k) = \mathbb{P}(M \leq k) - \mathbb{P}(M \leq k-1) = \left(\frac{k}{n}\right)^m - \left(\frac{k-1}{n}\right)^m$$
holds for every $1 \leq k \leq n$.

Exercise 2 A machine M is capable of sampling from $\{0,1\}$ such that $\mathbb{P}(M=1)=p$ and $\mathbb{P}(M=0)=1-p$ for some **unknown** $p\in(0,1)$. For every positive integer n, let $(L_n,R_n)\leftarrow M^2$ (i.e., we sample pairs of bits), be sampled independently of one another, and independently of all other samples. Define the algorithm A as follows: A will sample (L_n,R_n) until the first time $L_n\neq R_n$, and will then output the left element. Prove that A will output 1 with probability $1/2$.

Solution

We abuse notation and let A denote both the algorithm and its output. For every positive integer n it holds that

$$\mathbb{P}(L_n=R_n)=\mathbb{P}(L_n=0,R_n=0)+\mathbb{P}(L_n=1,R_n=1)=(1-p)^2+p^2.$$

Hence

$$\mathbb{P}(A=1)=\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n-1}\mathbb{P}(L_i=R_i)\cdot\mathbb{P}(L_n=1,R_n=0)\right)=\sum_{n=1}^{\infty}\left(p^2+(1-p)^2\right)^{n-1}\cdot p(1-p).$$

Similarly

$$\mathbb{P}(A=0)=\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n-1}\mathbb{P}(L_i=R_i)\cdot\mathbb{P}(L_n=0,R_n=1)\right)=\sum_{n=1}^{\infty}\left(p^2+(1-p)^2\right)^{n-1}\cdot p(1-p).$$

In particular $\mathbb{P}(A=1)=\mathbb{P}(A=0)$. Since, clearly, $\{A=1\}$ and $\{A=0\}$ are disjoint events and $\mathbb{P}(A=1\vee A=0)=1$, we conclude that $\mathbb{P}(A=1)=1/2$.

Exercise 3 Let $(\Omega,\mathbb{P})$ be a probability space and let $X,Y:\Omega\to\mathbb{R}$ be random variables. Prove that for every $m\in\mathbb{R}$ it holds that

$$|\mathbb{P}(X=m)-\mathbb{P}(Y=m)|\leq\mathbb{P}(X\neq Y).$$

Solution

Let $m\in\mathbb{R}$ be arbitrary. Then

$$\begin{aligned}
|\mathbb{P}(X=m)-\mathbb{P}(Y=m)| &= |\mathbb{P}(X=m,Y=m)+\mathbb{P}(X=m,Y\neq m)-\mathbb{P}(X=m,Y=m)-\mathbb{P}(X\neq m,Y=m)|\\
&\leq \mathbb{P}(X=m,Y\neq m)+\mathbb{P}(X\neq m,Y=m)\\
&= \sum_{k\in\mathbb{R}\setminus\{m\}}\left(\mathbb{P}(X=m\wedge Y=k)+\mathbb{P}(Y=m\wedge X=k)\right)\\
&\leq \sum_{k\in\mathbb{R}}\mathbb{P}(X=k\wedge Y\neq k)\\
&= \mathbb{P}(X\neq Y),
\end{aligned}$$

where the first inequality holds by the triangle inequality and the last inequality holds since we added more non-negative terms.

Exercise 4 A library has a total of N books. N_1 of the books are in English and N_2 of the books are in Hebrew (N could be larger than $N_1 + N_2$). Alice chooses n different books from the library uniformly at random. Let X_1 be the number of books in English that Alice chose and let X_2 be the number of books in Hebrew that Alice chose.

1. Calculate the distribution of $X_1 + X_2$.

2. After Alice returned all the books she borrowed, Bob came to the library and chose books to borrow in the following way: For every book in the library, he tossed a coin whose outcome is heads with some probability $p \in (0,1)$, all coin tosses being mutually independent. He borrowed each book if and only if the outcome of the corresponding coin toss was heads. Let Y_1 be the number of books in English that Bob chose and let Y_2 be the number of books in Hebrew that Bob chose. Prove that the distribution of Y_1+Y_2, conditioned on the event that Bob took exactly n books, is equal to the distribution of $X_1 + X_2$.

Solution

1. Let $X = X_1 + X_2$. Then $\{X = k\}$ is the event that exactly k of the books that Alice chose are either in English or in Hebrew. There are $\binom{N_1+N_2}{k}\binom{N-N_1-N_2}{n-k}$ such choices and $\binom{N}{n}$ ways to choose n books from the library. Since the probability space is uniform, we conclude that
$$\mathbb{P}(X = k) = \frac{\binom{N_1+N_2}{k}\binom{N-N_1-N_2}{n-k}}{\binom{N}{n}}.$$

2. Let Z be the number of books that Bob borrowed which are not in English or Hebrew, and let $Y = Y_1 + Y_2$. It follows by Bayes' rule that
$$\mathbb{P}(Y = k \mid Y + Z = n) = \frac{\mathbb{P}(Y + Z = n \mid Y = k)\,\mathbb{P}(Y = k)}{\mathbb{P}(Y + Z = n)}.$$
Note that $Y_1 \sim \mathrm{Bin}(N_1, p)$, since exactly k of the books Bob borrowed will be in English if and only if in exactly k of the N_1 coin tosses corresponding to the English books in the library, the outcome is heads, regardless of the outcome in the remaining $N - N_1$ coin tosses. Similarly $Y_2 \sim \mathrm{Bin}(N_2, p)$. As was proved in Lecture 7, $Y_1 = W_1 + W_2 + \ldots + W_{N_1}$ and $Y_2 = W'_1 + W'_2 + \ldots + W'_{N_2}$, where $W_i \sim \mathrm{Ber}(p)$ for every $1 \le i \le N_1$ and $W'_i \sim \mathrm{Ber}(p)$ for every $1 \le i \le N_2$. Moreover, $W_1, \ldots, W_{N_1}, W'_1, \ldots, W'_{N_2}$ are mutually independent. Therefore $Y = Y_1 + Y_2 = W_1 + W_2 + \ldots + W_{N_1} + W'_1 + W'_2 + \ldots + W'_{N_2} \sim \mathrm{Bin}(N_1 + N_2, p)$. Similarly $Y + Z \sim \mathrm{Bin}(N, p)$ and $Z \sim \mathrm{Bin}(N - N_1 - N_2, p)$. Therefore
$$\mathbb{P}(Y + Z = n) = \tbinom{N}{n} p^n (1-p)^{N-n},$$
$$\begin{aligned}\mathbb{P}(Y + Z = n \mid Y = k) &= \frac{\mathbb{P}(Y + Z = n \wedge Y = k)}{\mathbb{P}(Y = k)} = \frac{\mathbb{P}(Z = n - k \wedge Y = k)}{\mathbb{P}(Y = k)} \\ &= \frac{\mathbb{P}(Z = n - k) \cdot \mathbb{P}(Y = k)}{\mathbb{P}(Y = k)} = \mathbb{P}(Z = n - k) \\ &= \binom{N - N_1 - N_2}{n - k} p^{n-k} (1-p)^{N - N_1 - N_2 - n + k},\end{aligned}$$

$$\mathbb{P}(Y=k) = \binom{N_1+N_2}{k} p^k (1-p)^{N_1+N_2-k}.$$

We conclude that

$$\begin{aligned}
\mathbb{P}(Y=k \mid Y+Z=n) &= \frac{\binom{N-N_1-N_2}{n-k} p^{n-k}(1-p)^{N-N_1-N_2-n+k} \cdot \binom{N_1+N_2}{k} p^k (1-p)^{N_1+N_2-k}}{\binom{N}{n} p^n (1-p)^{N-n}} \\
&= \frac{\binom{N_1+N_2}{k}\binom{N-N_1-N_2}{n-k}}{\binom{N}{n}} \\
&= \mathbb{P}(X=k).
\end{aligned}$$

Exercise 5 Let $X \sim \mathrm{Geom}\left(\lambda n^{-1}\right)$, for some real number $\lambda \geq 0$.

1. Calculate $\mathbb{P}(X > k)$ for every non-negative integer k.
2. Prove that
$$\mathbb{P}\left(n^{-1}X > t\right) = \left(1 - \frac{\lambda}{n}\right)^{\lfloor tn \rfloor},$$
for all $t \geq 0$.
3. Conclude that
$$\lim_{n\to\infty} \mathbb{P}\left(n^{-1}X > t\right) = e^{-\lambda t},$$
for all $t \geq 0$.

Solution

1. Let $p = \lambda n^{-1}$. Since the events $\{X = i\}$ and $\{X = j\}$ are disjoint for all $i \neq j$, it holds that
$$\mathbb{P}(X > k) = \sum_{i=k+1}^{\infty} \mathbb{P}(X = i) = \sum_{i=k+1}^{\infty} p \cdot (1-p)^{i-1} = p \cdot \frac{(1-p)^k}{1-(1-p)} = (1-p)^k = \left(1 - \frac{\lambda}{n}\right)^k.$$
2. For every $t \geq 0$ we have
$$\begin{aligned}
\mathbb{P}\left(n^{-1}X > t\right) &= \mathbb{P}(X > tn) \\
&= \mathbb{P}(X > \lfloor tn \rfloor) \\
&= \left(1 - \frac{\lambda}{n}\right)^{\lfloor tn \rfloor},
\end{aligned}$$
where the second equality holds since the support of X consists of non-negative integers and the last equality holds by the previous part of this exercise.
3. By the previous part of this exercise
$$\mathbb{P}\left(n^{-1}X > t\right) = \left(1 - \frac{\lambda}{n}\right)^{\lfloor tn \rfloor} = \left(1 - \frac{\lambda}{n}\right)^{n(\lfloor tn \rfloor / n)}$$

holds for every $t \geq 0$. Therefore

$$\log\left(\mathbb{P}\left(n^{-1}X > t\right)\right) = \frac{\lfloor tn \rfloor}{n} \cdot \log\left(\left(1 - \frac{\lambda}{n}\right)^n\right).$$

Since $\lim_{n\to\infty} \lfloor tn \rfloor / n = t$, it follows that

$$\lim_{n\to\infty} \log\left(\mathbb{P}\left(n^{-1}X > t\right)\right) = -\lambda t,$$

which is equivalent to

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-1}X > t\right) = e^{-\lambda t}.$$