$$\boxed{\begin{array}{l}\textbf{Def 0}: \textit{Let } V, W \textit{ be vector spaces } \left(\dim_{\mathbb{R}} V, \dim_{\mathbb{R}} W < \infty\right). \\ \textit{A map } T: V \to W \textit{ is called a } \textbf{linear transformation}, \textit{ if }: \\ 1)\ v_1, v_2 \in V \Rightarrow T(v_1+v_2) = T(v_1)+T(v_2), \\ 2)\ v \in V, \alpha \in \mathbb{R} \Rightarrow T(\alpha v) = \alpha T(v)\end{array}}$$

$T(0) = T(\alpha 0) = \alpha T(0),\ \alpha \in F_2,\ char\ F_2 = 2,\ \underbrace{1+1}_{2-times} = 0$

$0 = 1 \cdot T(0) - \alpha T(0) = (1-\alpha)T(0)$

$1-\alpha \neq 0 \Rightarrow T(0) = 0.$

$$\boxed{\begin{array}{l}T(v_1+v_2) = T(v_1)+T(v_2) \\ T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0})+T(\mathbf{0}) \Rightarrow T(\mathbf{0}) = \mathbf{0}.\end{array}}$$

1.1) $(T+S)(u+v) = (T+S)(u)+(T+S)(v)$

1.2) $(T+S)(\alpha u) = \alpha \cdot (T+S)(u)$? ------ (*Exercise*)

2.2) $(\alpha T)(u+v) = (\alpha T)(u)+(\alpha T)(v)$? ------ (*Exercise*)

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0) $\dim_{\mathbb{R}} \mathbb{R} = 1$

1) $\dim_{\mathbb{R}} \mathbb{C}^1 = 2,$

2) $\dim_{\mathbb{C}} \mathbb{C} = 1.$

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$\dim_{\mathbb{C}} \mathbb{C}$: $\{1, \cancel{\sqrt{-1}}\} \Rightarrow a \cdot 1 + b \cdot \sqrt{-1} \in \mathbb{C},\ \ a, b \in \mathbb{C}.$

$\mathbb{C}^n, F: \{e_1, ..., e_n, \sqrt{-1}e_1, ..., \sqrt{-1}e_n\} \Rightarrow \boxed{\dim_{\mathbb{R}} \mathbb{C}^n = 2n,\ \ \dim_{\mathbb{C}} \mathbb{C}^n = n}$

$\dim_{\mathbb{R}} \mathbb{C}^1$: $\{1, \sqrt{-1}\}$ *is a basis of* $\mathbb{C}$ *over* $\mathbb{R}$

$\dim_{\mathbb{C}} \mathbb{C}$: $\{1\}$ *is a basis of* $\mathbb{C}$ *over* $\mathbb{C}$

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Example: $\dim_{\mathbb{R}} \mathbb{C}^2 = \dim_{\mathbb{R}} (\mathbb{C} \times \mathbb{C}) = 4$

Solution: write $\alpha \begin{bmatrix}1\\0\end{bmatrix} + \beta \begin{bmatrix}0\\1\end{bmatrix} + \gamma \begin{bmatrix}\sqrt{-1}\\0\end{bmatrix} + \delta \begin{bmatrix}0\\\sqrt{-1}\end{bmatrix},\ \alpha, \beta, \gamma, \delta \in F.$

1) $\alpha, \beta, \gamma, \delta \in \mathbb{R} \Rightarrow B = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}\sqrt{-1}\\0\end{bmatrix}, \begin{bmatrix}0\\\sqrt{-1}\end{bmatrix} \right)$ linear independent and spanning $\mathbb{C}$

$\Rightarrow$ B is a basis and $\dim_{\mathbb{R}} \mathbb{C}^2 = 4.$

2) $\alpha, \beta, \gamma, \delta \in \mathbb{C} \Rightarrow \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}\sqrt{-1}\\0\end{bmatrix}$ linear dependent, and $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\\sqrt{-1}\end{bmatrix}$ also linear dependent

$\Rightarrow$ only $B = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right)$ linear independent. But $\mathbb{C} = \operatorname{span}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right) = \operatorname{span}\left(1, \sqrt{-1}\right)$

$\Rightarrow$ B is a basis and $\dim_{\mathbb{C}} \mathbb{C}^2 = 2.$