Assignment 3

If you wish to submit your solutions to any of these questions, please send them via email to your TA by 02/05/2021. This deadline is strict!

Exercise 1 A bin contains n balls, labeled with the numbers $1, 2, \ldots, n$. Exactly m balls are drawn uniformly at random from the bin. Let M be the maximum number of a ball that was drawn.

1. Calculate the distribution of M, when the samples are being made without replacement.

2. Calculate the distribution of M, when the samples are being made independently with replacement.

Exercise 2 A machine M is capable of sampling from $\{0, 1\}$ such that $\mathbb{P}(M = 1) = p$ and $\mathbb{P}(M = 0) = 1 - p$ for some **unknown** $p \in (0, 1)$. For every positive integer n, let $(L_n, R_n) \leftarrow M^2$ (i.e., we sample pairs of bits), be sampled independently of one another, and independently of all other samples. Define the algorithm A as follows: A will sample (L_n, R_n) until the first time $L_n \neq R_n$, and will then output the left element. Prove that A will output 1 with probability 1/2.

Exercise 3 Let $(\Omega, \mathbb{P})$ be a probability space and let $X, Y : \Omega \to \mathbb{R}$ be random variables. Prove that for every $m \in \mathbb{R}$ it holds that

$$|\mathbb{P}(X = m) - \mathbb{P}(Y = m)| \leq \mathbb{P}(X \neq Y).$$

Exercise 4 A library has a total of N books. N_1 of the books are in English and N_2 of the books are in Hebrew (N could be larger than $N_1 + N_2$). Alice chooses n different books from the library uniformly at random. Let X_1 be the number of books in English that Alice chose and let X_2 be the number of books in Hebrew that Alice chose.

1. Calculate the distribution of $X_1 + X_2$.

2. After Alice returned all the books she borrowed, Bob came to the library and chose books to borrow in the following way: For every book in the library, he tossed a coin whose outcome is heads with some probability $p \in (0, 1)$, all coin tosses being mutually independent. He borrowed each book if and only if the outcome of the corresponding coin toss was heads. Let Y_1 be the number of books in English that Bob chose and let Y_2 be the number of books in Hebrew that Bob chose. Prove that the distribution of $Y_1 + Y_2$, conditioned on the event that Bob took exactly n books, is equal to the distribution of $X_1 + X_2$.

Exercise 5 Let $X \sim \mathrm{Geom}\left(\lambda n^{-1}\right)$, for some real number $\lambda \geq 0$.

1. Calculate $\mathbb{P}\left(X > k\right)$ for every non-negative integer k.

2. Prove that
$$\mathbb{P}\left(n^{-1}X > t\right) = \left(1 - \frac{\lambda}{n}\right)^{\lfloor tn \rfloor},$$
for all $t \geq 0$.

3. Conclude that
$$\lim_{n\to\infty} \mathbb{P}\left(n^{-1}X > t\right) = e^{-\lambda t},$$
for all $t \geq 0$.