Probability Theory 1 – Proposed solution of moed aleph exam 2019

1. **(a)** It is evident that the support of the distribution of Z_1 is $\{0, 1, 2\}$ and that the support of the distribution of Z_2 is $\{-1, 0, 1\}$. For every $0 \le i \le 2$ and $-1 \le j \le 1$, the table below shows the value of $P(Z_1 = i, Z_2 = j)$. The actual calculations can be found below the table.

	$Z_1 = 0$	$Z_1 = 1$	$Z_1 = 2$
$Z_2 = -1$	0	1/9	0
$Z_2 = 0$	2/9	0	2/9
$Z_2 = 1$	0	4/9	0

If $Z_2 = -1$, then it must hold that $X = 0$ and $Y = 1$ and thus $Z_1 = 1$. Therefore, $P(Z_1 = 0, Z_2 = -1) = P(Z_1 = 2, Z_2 = -1) = 0$. Moreover

$$P(Z_1 = 1, Z_2 = -1) = P(X = 0, Y = 1) = P(X = 0)P(Y = 1) = 1/3 \cdot 1/3 = 1/9,$$

where the second equality holds by our assumption that X and Y are independent. Similarly, if $Z_2 = 1$, then it must hold that $X = 1$ and $Y = 0$ and thus $Z_1 = 1$. Therefore, $P(Z_1 = 0, Z_2 = 1) = P(Z_1 = 2, Z_2 = 1) = 0$. Moreover

$$P(Z_1 = 1, Z_2 = 1) = P(X = 1, Y = 0) = P(X = 1)P(Y = 0) = 2/3 \cdot 2/3 = 4/9.$$

Finally, if $Z_2 = 0$, then it must hold that $X = Y$ and thus $Z_1 = 0$ or $Z_1 = 2$. Therefore, $P(Z_1 = 1, Z_2 = 0) = 0$. Moreover

$$P(Z_1 = 0, Z_2 = 0) = P(X = 0, Y = 0) = P(X = 0)P(Y = 0) = 1/3 \cdot 2/3 = 2/9$$

and

$$P(Z_1 = 2, Z_2 = 0) = P(X = 1, Y = 1) = P(X = 1)P(Y = 1) = 2/3 \cdot 1/3 = 2/9.$$

(b) Using the table from (a) we conclude that

$$\begin{aligned} P(Z_2 = -1) &= P(Z_2 = -1, Z_1 = 0) + P(Z_2 = -1, Z_1 = 1) + P(Z_2 = -1, Z_1 = 2) \\ &= 0 + 1/9 + 0 = 1/9, \end{aligned}$$

$$\begin{aligned} P(Z_2 = 0) &= P(Z_2 = 0, Z_1 = 0) + P(Z_2 = 0, Z_1 = 1) + P(Z_2 = 0, Z_1 = 2) \\ &= 2/9 + 0 + 2/9 = 4/9, \end{aligned}$$

and

$$\begin{aligned} P(Z_2 = 1) &= P(Z_2 = 1, Z_1 = 0) + P(Z_2 = 1, Z_1 = 1) + P(Z_2 = 1, Z_1 = 2) \\ &= 0 + 4/9 + 0 = 4/9. \end{aligned}$$

(c) It holds that

$$P(Z_1 > Z_2) = P(X + Y > X - Y) = P(Y > 0) = P(Y = 1) = 1/3.$$

2. **(a)** This statement is false. Consider the probability space (Ω, P), where $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $P(i) = 1/6$ for every $1 \leq i \leq 6$. Let $A = \{1, 2\}$, $B = \{2, 4, 6\}$, and $C = \{5, 6\}$. Then

$$P(A \cap B) = P(\{2\}) = 1/6 = 1/3 \cdot 1/2 = P(\{1, 2\}) \cdot P(\{2, 4, 6\}) = P(A) \cdot P(B)$$

implying that A and B are independent, and

$$P(B \cap C) = P(\{6\}) = 1/6 = 1/2 \cdot 1/3 = P(\{2, 4, 6\}) \cdot P(\{5, 6\}) = P(B) \cdot P(C)$$

implying that B and C are independent. On the other hand

$$P(A \cap C) = P(\emptyset) = 0 \neq 1/3 \cdot 1/3 = P(\{1, 2\}) \cdot P(\{5, 6\}) = P(A) \cdot P(C)$$

implying that A and C are dependent.

(b) This statement is true. It follows by the law of total probability that

$$P(A) = P(A|C)P(C) + P(A|C^c)P(C^c) > 2/3 \cdot (P(C) + P(C^c)) = 2/3.$$

(c) This statement is false. Consider the probability space (Ω, P), where $\Omega = \{1, 2, 3, 4\}$ and $P(i) = 1/4$ for every $1 \leq i \leq 4$. Let $A = \{1\}$, $B = \{3\}$, and $C = \{1, 2\}$. Note that

$$0 \leq P(A \cap B \cap C) \leq P(B \cap C) = P(\emptyset) = 0.$$

It follows that

$$P(A \cap B|C) = \frac{P(A \cap B \cap C)}{P(C)} = 0$$

and

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = 0.$$

We conclude that

$$P(A \cap B|C) = P(A|C) \cdot P(B|C),$$

that is, A and B are independent given C. Similarly

$$0 \leq P(A \cap B \cap C^c) \leq P(A \cap C^c) = P(\emptyset) = 0.$$

It follows that

$$P(A \cap B|C^c) = \frac{P(A \cap B \cap C^c)}{P(C^c)} = 0$$

and

$$P(A|C^c) = \frac{P(A \cap C^c)}{P(C^c)} = 0.$$

We conclude that

$$P(A \cap B|C^c) = P(A|C^c) \cdot P(B|C^c),$$

that is, A and B are independent given C^c. On the other hand

$$P(A \cap B) = P(\emptyset) = 0 \neq 1/4 \cdot 1/4 = P(A) \cdot P(B)$$

and thus A and B are dependent.

3. **(a)** For every $1 \leq i \leq n$, since the three coin tosses made in the ith round are independent, the probability that the outcome of all 3 is 0, is $(1/2)^3 = 1/8$. Similarly, by symmetry, the probability that the outcome of at least 2 of the 3 tosses made in the ith round is 1, is $1/2$. Since the random experiments made in different rounds are independent, it follows that $X \sim Bin(n, 1/8)$ and $Y \sim Bin(n, 1/2)$.

   (b) There are various ways to solve this part, but the simplest one is to ignore the rounds and observe that we make $3n$ independent tosses of fair coins. Hence $Z \sim Bin(3n, 1/2)$ and thus $\mathbb{E}(Z) = 3n/2$.

   (c) Note first that

$$\mathbb{E}(Z - X) = \mathbb{E}(Z) - \mathbb{E}(X) = 3n/2 - n/8 = 11n/8,$$

   where the first equality holds by the linearity of expectation and the second equality holds by parts (a) and (b) of this question.

   Now, observe that $Z - X$ is a non-negative random variable. Indeed, each of the n rounds contributes either 0 or 1 to X and at least 0 to Z. Moreover, every round which contributes 1 to X, contributes 3 to Z. Therefore, we can apply Markov's inequality to deduce that

$$P(Z - X \geq 2n) \leq \frac{\mathbb{E}(Z - X)}{2n} = \frac{11n/8}{2n} < 3/4.$$

4. **(a)** For every $1 \leq i \leq n$, let X_i be the indicator random variable for the event that the outcome of the ith die roll is in $\{1, 2, 3\}$, and let Y_i be the indicator random variable for the event that the outcome of the ith die roll is in $\{3, 4\}$. Then $X = \sum_{i=1}^n X_i$ and $Y = \sum_{i=1}^n Y_i$. Since the die rolls are independent, it follows that $X \sim Bin(n, 1/2)$ and $Y \sim Bin(n, 1/3)$. In particular, $Var(X) = n \cdot 1/2 \cdot (1 - 1/2) = n/4$ and $Var(Y) = n \cdot 1/3 \cdot (1 - 1/3) = 2n/9$.

   (b) It holds that

$$\begin{aligned} Var(X - Y) &= Var(X) + Var(Y) - 2Cov(X, Y) \\ &= Var(X) + Var(Y) - 2\sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, Y_j). \end{aligned}$$

   Since $Var(X)$ and $Var(Y)$ were calculated in part (a) of this question, it remains to calculate $Cov(X_i, Y_j)$ for every $1 \leq i, j \leq n$. Fix some $1 \leq i \neq j \leq n$. Since the die rolls are independent, it follows that X_i and Y_j are independent and thus, in particular, $Cov(X_i, Y_j) = 0$. For every $1 \leq i \leq n$, it holds that

$$\begin{aligned} Cov(X_i, Y_i) &= \mathbb{E}(X_i Y_i) - \mathbb{E}(X_i)\mathbb{E}(Y_i) = P(X_i = 1, Y_i = 1) - P(X_i = 1)P(Y_i = 1) \\ &= 1/6 - 1/2 \cdot 1/3 = 0. \end{aligned}$$

It follows that $Cov(X, Y) = 0$ and thus

$$Var(X - Y) = Var(X) + Var(Y) = n/4 + 2n/9 = 17n/36.$$

(c) Fix some positive integer n. It follows by part (a) of this question and by the linearity of expectation that

$$\mathbb{E}(X - Y) = \mathbb{E}(X) - \mathbb{E}(Y) = n/2 - n/3 = n/6.$$

It then follows by Chebyshev's inequality that

$$\begin{aligned} P(X \leq Y) &= P(X - Y - n/6 \leq -n/6) \leq P(|(X - Y) - \mathbb{E}(X - Y)| \geq n/6) \\ &\leq \frac{Var(X - Y)}{n^2/36} = \frac{17n/36}{n^2/36} = \frac{17}{n}, \end{aligned}$$

where the first inequality holds since $X - Y - n/6 \leq -n/6 \implies |(X-Y) - \mathbb{E}(X-Y)| \geq n/6$, and the second equality holds by part (b) of this question. We conclude that

$$1 \geq \lim_{n \to \infty} P(X > Y) = 1 - \lim_{n \to \infty} P(X \leq Y) \geq 1 - \lim_{n \to \infty} \frac{17}{n} = 1$$

implying that $\lim_{n \to \infty} P(X > Y) = 1$ as claimed.