Probability Theory 1 – Proposed solution of moed bet exam 2018

1. **(a)** It is evident that the support of X is $\{1, 2, 3\}$ and that the support of Y is $\{1, 2, 3, 4\}$. By definition, we have $P(X = i, Y = i) = 0$ for every $1 \leq i \leq 3$. On the other hand, for every $1 \leq i \leq 3$ and $j \in \{1, 2, 3, 4\} \setminus \{i\}$ we have

$$P(X = i, Y = j) = P(Y = j | X = i) \cdot P(X = i) = 1/3 \cdot 1/3 = 1/9.$$

These results are summarized in the table below.

	Y = 1	Y = 2	Y = 3	Y = 4
X = 1	0	1/9	1/9	1/9
X = 2	1/9	0	1/9	1/9
X = 3	1/9	1/9	0	1/9

(b) Using the table from (a) we conclude that

$$\begin{aligned} P(X = 1 | Y = 3) &= \frac{P(X = 1, Y = 3)}{P(Y = 3)} \\ &= \frac{P(X = 1, Y = 3)}{P(X = 1, Y = 3) + P(X = 2, Y = 3) + P(X = 3, Y = 3)} \\ &= \frac{1/9}{1/9 + 1/9 + 0} = \frac{1}{2}. \end{aligned}$$

(c) Using the table from (a) we see that

$$P(Y = 4) = P(X = 1, Y = 4) + P(X = 2, Y = 4) + P(X = 3, Y = 4) = 1/9 + 1/9 + 1/9 = 1/3.$$

Since Ariel's choice of clothes in a certain day is independent of his choices in all other days, it thus follows that $Z \sim Bin(90, 1/3)$. In particular, $\mathbb{E}(Z) = 90 \cdot 1/3 = 30$ and $Var(Z) = 90 \cdot 1/3 \cdot (1 - 1/3) = 20$. Therefore, it follows by Chebyshev's inequality that

$$P(|Z - 30| \geq 10) = P(|Z - \mathbb{E}(Z)| \geq 10) \leq \frac{Var(Z)}{10^2} = \frac{20}{100} = \frac{1}{5}$$

as claimed.

2. **(a)** By the inclusion-exclusion formula we have

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ + P(A \cap B \cap C). \qquad (1) \end{aligned}$$

We go on to calculate each of the terms appearing in (1). We have

$$P(A) = \frac{9 \cdot \binom{10}{2}}{\binom{90}{2}} = \frac{9}{89},$$

$$P(B) = \frac{\binom{45}{2}}{\binom{90}{2}} = \frac{22}{89}.$$

Moreover, let $S_1 = \{(i, i+1) : i \geq 10 \text{ and } i+1 \leq 99\}$ and let $S_2 = \{(i, i+2) : i \geq 10 \text{ and } i+2 \leq 99\}$. Then

$$P(C) = \frac{|S_1| + |S_2|}{\binom{90}{2}} = \frac{89+88}{45 \cdot 89} = \frac{59}{15 \cdot 89}.$$

Next

$$P(A \cap B) = \frac{9 \cdot \binom{5}{2}}{\binom{90}{2}} = \frac{2}{89},$$

$$P(A \cap C) = \frac{9 \cdot (9+8)}{\binom{90}{2}} = \frac{17}{5 \cdot 89}.$$

Moreover, let $S_3 = \{(i, i+2) : i \text{ is even}, i \geq 10 \text{ and } i+2 \leq 98\}$. Then

$$P(B \cap C) = \frac{|S_3|}{\binom{90}{2}} = \frac{44}{45 \cdot 89}.$$

Finally,

$$P(A \cap B \cap C) = \frac{9 \cdot 4}{\binom{90}{2}} = \frac{4}{5 \cdot 89}.$$

Using (1) we conclude that

$$\begin{aligned} P(A \cup B \cup C) &= \frac{9}{89} + \frac{22}{89} + \frac{59}{15 \cdot 89} - \frac{2}{89} - \frac{17}{5 \cdot 89} - \frac{44}{45 \cdot 89} + \frac{4}{5 \cdot 89} \\ &= \frac{45 \cdot 9 + 45 \cdot 22 + 3 \cdot 59 - 45 \cdot 2 - 9 \cdot 17 - 44 + 9 \cdot 4}{45 \cdot 89} = \frac{1321}{45 \cdot 89}. \end{aligned}$$

(b) Using the calculations made in (a) we obtain

$$P(A \cap B | C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{\frac{4}{5 \cdot 89}}{\frac{59}{15 \cdot 89}} = \frac{12}{59}.$$

(c) Using the calculations made in (a) we obtain

$$P(A | B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{\frac{4}{5 \cdot 89}}{\frac{44}{45 \cdot 89}} = \frac{9}{11}.$$

3. **(a)** For every $1 \leq i \leq n-2$, let Y_i be the indicator random variable for the event: "the result of the ith coin toss is 1, the result of the $(i+1)$th coin toss is 0, and the result of the $(i+2)$th coin toss is 1". For every $1 \leq i \leq n-2$ we then have

$$\mathbb{E}(Y_i) = P(Y_i = 1) = 1/2 \cdot 1/2 \cdot 1/2 = 1/8.$$

It thus follows by the linearity of expectation that

$$\mathbb{E}(Y) = \sum_{i=1}^{n-2} \mathbb{E}(Y_i) = (n-2)/8.$$

(b) For every $1 \leq i \leq n-1$, let X_i be the indicator random variable for the event: "the result of the ith coin toss is 1 and the result of the $(i+1)$th coin toss is 0". For every $1 \leq i \leq n-1$ we then have $\mathbb{E}(X_i) = P(X_i = 1) = 1/2 \cdot 1/2 = 1/4$ and thus

$$Var(X_i) = \mathbb{E}\left(X_i^2\right) - (\mathbb{E}(X_i))^2 = P(X_i = 1) - P(X_i = 1)^2 = 1/4 - 1/16 = 3/16.$$

Now, fix some $1 \leq i < j \leq n-1$. If $j > i+1$, then X_i and X_j rely on disjoint pairs of coin tosses and are thus independent; in particular, $Cov(X_i, X_j) = 0$. On the other hand

$$Cov(X_i, X_{i+1}) = \mathbb{E}(X_i X_{i+1}) - \mathbb{E}(X_i)\mathbb{E}(X_{i+1}) = P(X_i = 1, X_{i+1} = 1) - 1/16 = -1/16.$$

We conclude that

$$Var(X) = \sum_{i=1}^{n-1} Var(X_i) + 2 \sum_{1 \leq i < j \leq n-1} Cov(X_i, X_j) = \frac{3(n-1)}{16} - \frac{2(n-2)}{16} = \frac{n+1}{16}.$$

(c) Fix some $1 \leq i \leq n-1$ and $1 \leq j \leq n-2$. Then

$$\begin{aligned} Cov(X_j, Y_j) &= \mathbb{E}(X_j Y_j) - \mathbb{E}(X_j)\mathbb{E}(Y_j) = P(X_j = 1, Y_j = 1) - 1/4 \cdot 1/8 \\ &= 1/8 - 1/32 = 3/32, \end{aligned}$$

$$\begin{aligned} Cov(X_j, Y_{j+1}) &= \mathbb{E}(X_j Y_{j+1}) - \mathbb{E}(X_j)\mathbb{E}(Y_{j+1}) = P(X_j = 1, Y_{j+1} = 1) - 1/4 \cdot 1/8 \\ &= 0 - 1/32 = -1/32, \end{aligned}$$

$$\begin{aligned} Cov(X_j, Y_{j-1}) &= \mathbb{E}(X_j Y_{j-1}) - \mathbb{E}(X_j)\mathbb{E}(Y_{j-1}) = P(X_j = 1, Y_{j-1} = 1) - 1/4 \cdot 1/8 \\ &= 0 - 1/32 = -1/32, \end{aligned}$$

$$\begin{aligned} Cov(X_j, Y_{j-2}) &= \mathbb{E}(X_j Y_{j-2}) - \mathbb{E}(X_j)\mathbb{E}(Y_{j-2}) = P(X_j = 1, Y_{j-2} = 1) - 1/4 \cdot 1/8 \\ &= 1/16 - 1/32 = 1/32. \end{aligned}$$

For all other values of $1 \leq i \leq n-1$ and $1 \leq j \leq n-2$, the random variables X_i and Y_j rely on disjoint sets of coin tosses and are thus independent; in particular, $Cov(X_i, Y_j) = 0$. Using the properties of covariance which were proved in the lectures, we conclude that

$$\begin{aligned} Cov(X, Y) &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-2} Cov(X_i, Y_j) = \frac{3(n-2)}{32} - \frac{n-3}{32} - \frac{n-2}{32} + \frac{n-3}{32} \\ &= \frac{n-2}{16}. \end{aligned}$$

4. **(a)** It is evident that the support of X is contained in $\{0, 1, \dots, n\}$. Moreover, since the die is fair and the die rolls are independent, for every $1 \le i \le n$ it holds that $P(X = i) = (5/6)^{i-1} \cdot 1/6$. Finally,

$$P(X = 0) = 1 - \sum_{i=1}^{n} P(X = i) = 1 - \sum_{i=1}^{n} (5/6)^{i-1} \cdot 1/6 = 1 - \frac{1}{6} \cdot \frac{1 - (5/6)^n}{1 - 5/6} = (5/6)^n.$$

(b) For every $1 \le i \le n - 1$, let Y_i be the indicator random variable for the event: "both the ith and the $(i + 1)$th die rolls resulted in a 6". Then, for every $1 \le i \le n - 1$ we have

$$\mathbb{E}(Y_i | X = 0) = P(Y_i = 1 | X = 0) = 1/25,$$

where the last equality holds since, given that $X = 0$, the results of the ith and $(i+1)$th die rolls were not a 3 and thus the probability that both were a 6 is $1/5 \cdot 1/5$. Since, by definition we have $Y = \sum_{i=1}^{n-1} Y_i$, it follows by the linearity of expectation that

$$\mathbb{E}(Y | X = 0) = \sum_{i=1}^{n-1} \mathbb{E}(Y_i | X = 0) = \frac{n - 1}{25}.$$

(c) The random variables X and Y are dependent. For example $P(X = 1, Y = n - 1) \neq P(X = 1) \cdot P(Y = n - 1)$. Indeed, $P(X = 1, Y = n - 1) = 0$ as $Y = n - 1$ if and only if the result of every die roll was 6 and then, in particular, $X = 0$. On the other hand $P(X = 1) = 1/6$ and $P(Y = n - 1) = (1/6)^n$.