Assignment 5
Solutions

Exercise 1 Let $X \sim \mathrm{Bin}(n,p)$, for some $n \in \mathbb{N}$ and $p \in [0,1]$, be a random variable. Find the expected value of X using two methods: by direct calculation (i.e., using the identity $\mathbb{E}(X) = \sum_x x \cdot \mathbb{P}(X = x)$) and by depicting X as a sum of n independent Bernoulli random variables.

Solution

We start with the direct calculation. Observe that

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$

holds for every $1 \leq k \leq n$. Therefore

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=0}^{n} k \cdot \mathbb{P}(X = k) \\
&= \sum_{k=1}^{n} k \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k \cdot \frac{n}{k} \cdot \binom{n-1}{k-1} \cdot p^k \cdot (1-p)^{n-k} \\
&= np \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} \cdot p^{k-1} \cdot (1-p)^{(n-1)-(k-1)} \\
&= np \cdot \sum_{m=0}^{n-1} \binom{n-1}{m} \cdot p^m \cdot (1-p)^{n-1-m} \\
&= np,
\end{aligned}$$

where the penultimate equality holds by the substitution $m = k - 1$ and the last equality follows by the binomial formula.

Next, we calculate $\mathbb{E}(X)$ using the second method. Let $X_1, \ldots, X_n$ be mutually independent random variables, where $X_i \sim \mathrm{Ber}(p)$ for every $1 \leq i \leq n$. Then $\mathbb{E}(X_i) = p$ for every $1 \leq i \leq n$, and $X = \sum_{i=1}^{n} X_i$. It thus follows by the linearity of expectation that

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}(X_i) = np.$$

Exercise 2 Let S be a set with n elements. A set $A \subseteq S$ is selected uniformly at random among all 2^n subsets of S. Let $X = |A|$.

1. Calculate the probability distribution of X.
2. Calculate the expected value of X using two methods: by direct calculation and by depicting X as a Binomial random variable.

Solution

1. Observe that $|\{X = k\}| = \binom{n}{k}$ for every $0 \leq k \leq n$. Since the probability space is uniform, it follows that
$$\mathbb{P}(X = k) = \frac{\binom{n}{k}}{2^n}.$$

2. We start with the direct calculation. As in the solution of the previous exercise,
$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$
holds for every $1 \leq k \leq n$. Therefore
$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=0}^{n} k \cdot \mathbb{P}(X = k) \\
&= \sum_{k=1}^{n} k \cdot \binom{n}{k} \cdot \frac{1}{2^n} \\
&= \sum_{k=1}^{n} k \cdot \frac{n}{k} \cdot \binom{n-1}{k-1} \cdot \frac{1}{2^n} \\
&= \frac{n}{2} \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} \cdot \frac{1}{2^{n-1}} \\
&= \frac{n}{2} \cdot \sum_{m=0}^{n-1} \binom{n-1}{m} \cdot \frac{1}{2^{n-1}} \\
&= \frac{n}{2},
\end{aligned}$$
where the penultimate equality holds by the substitution $m = k - 1$ and the last equality follows by the binomial formula.

   Next, we calculate $\mathbb{E}(X)$ using the second method. Observe that
$$\mathbb{P}(X = k) = \frac{\binom{n}{k}}{2^n} = \binom{n}{k} \cdot \frac{1}{2^k} \cdot \frac{1}{2^{n-k}}$$
holds for every integer $0 \leq k \leq n$. That is $X \sim \mathrm{Bin}(n, 1/2)$ and thus $\mathbb{E}(X) = n/2$ by the previous exercise.

Exercise 3 Let $X \sim \text{Bin}(n,p)$, for some $n \in \mathbb{N}$ and $p \in [0,1]$, be a random variable. Find $\text{Var}(X)$ using two methods: by direct calculation according to the definition of variance, and by depicting X as a sum of n mutually independent Bernoulli random variables.

Solution

We start with the direct calculation. Recall that

$$\text{Var}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right) = \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2$$

and that $\mathbb{E}(X) = np$ (as was shown in the first exercise). In order to calculate $\mathbb{E}(X^2)$, we will first calculate $\mathbb{E}(X^2 - X)$.

$$\begin{aligned}
\mathbb{E}\left(X^2 - X\right) &= \sum_{k=0}^{n} (k^2 - k) \cdot \mathbb{P}(X = k) \\
&= \sum_{k=2}^{n} k(k-1) \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=2}^{n} k(k-1) \cdot \frac{n}{k} \binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= n \cdot \sum_{k=2}^{n} (k-1) \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= n \cdot \sum_{k=2}^{n} (k-1) \cdot \frac{n-1}{k-1} \cdot \binom{n-2}{k-2} p^k (1-p)^{n-k} \\
&= n(n-1) \cdot \sum_{k=2}^{n} \binom{n-2}{k-2} p^k (1-p)^{n-k} \\
&= n(n-1)p^2 \cdot \sum_{m=0}^{n-2} \binom{n-2}{m} p^m (1-p)^{(n-2)-m} \\
&= n(n-1)p^2,
\end{aligned}$$

where the third and fifth equalities follow by the identity $\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$, the penultimate equality holds by the substitution $m = k-2$, and the last equality follows by the binomial formula. Therefore, it follows by the linearity of expectation that

$$\mathbb{E}\left(X^2\right) = \mathbb{E}\left(X^2 - X + X\right) = \mathbb{E}\left(X^2 - X\right) + \mathbb{E}(X) = n(n-1)p^2 + np = (np)^2 + np(1-p).$$

. We conclude that

$$\text{Var}(X) = \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2 = np(1-p).$$

Next, we calculate $\text{Var}(X)$ by depicting X as a sum of n mutually independent Bernoulli random variables. Let $X_1, \ldots, X_n$ be mutually independent random variables, where $X_i \sim \text{Ber}(p)$ for every $1 \le i \le n$. Then $X = \sum_{i=1}^{n} X_i$ and $\text{Var}(X_i) = p(1-p)$ for every $1 \le i \le n$. Moreover, since $X_1, \ldots, X_n$ are mutually independent, it follows that

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) = np(1-p).$$

Exercise 4 A fair coin is being tossed $n+2$ times, all coin tosses being mutually independent. Let X be the number of times 3 consecutive heads appeared, for example, in the sequence $HHHHHTTHHH$, $X = 4$. Calculate $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

Solution

For every $1 \le i \le n+2$, let C_i be the outcome of the ith coin toss, that is, $C_i = H$ if the outcome of the ith toss is heads, and $C_i = T$ otherwise. For every $1 \le i \le n$, let

$$I_i = \begin{cases} 1 & \text{if } C_i C_{i+1} C_{i+2} = HHH \\ 0 & \text{otherwise} \end{cases}$$

be the indicator random variable for the event: "there are 3 consecutive heads starting at position i". Note that $X = \sum_{i=1}^{n} I_i$ and that $\mathbb{E}(I_i) = \mathbb{P}(I_i = 1) = 1/8$ for every $1 \le i \le n$. It follows by the linearity of expectation that $\mathbb{E}(X) = n/8$.

Next, we will calculate the variance of X. It holds that

$$\mathrm{Var}(X) = \mathrm{Var}\left(\sum_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} \mathrm{Var}(I_i) + 2 \cdot \sum_{1 \le i < j \le n} \mathrm{Cov}(I_i, I_j).$$

Since $I_i \sim \mathrm{Ber}(1/8)$, it follows that $\mathrm{Var}(I_i) = 1/8 \cdot (1 - 1/8) = 7/64$. Observe that the random variables I_i and I_j are independent for every $1 \le i \le n$ and $j > i+2$ as the sets of coin flips $\{i, i+1, i+2\}$ and $\{j, j+1, j+2\}$ are disjoint. In particular, $\mathrm{Cov}(I_i, I_j) = 0$ for all such choices of i and j. On the other hand, for every $1 \le i \le n-1$, it holds that

$$\mathrm{Cov}(I_i, I_{i+1}) = \mathbb{E}(I_i \cdot I_{i+1}) - \mathbb{E}(I_i)\,\mathbb{E}(I_{i+1}) = \mathbb{P}(I_i = 1, I_{i+1} = 1) - \frac{1}{64} = \frac{1}{16} - \frac{1}{64} = \frac{3}{64},$$

where the third equality holds since $I_i = I_{i+1} = 1$ if and only if $C_i C_{i+1} C_{i+2} C_{i+3} = HHHH$. Similarly, for every $1 \le i \le n-2$, it holds that

$$\mathrm{Cov}(I_i, I_{i+2}) = \mathbb{E}(I_i \cdot I_{i+2}) - \mathbb{E}(I_i)\,\mathbb{E}(I_{i+2}) = \frac{1}{32} - \frac{1}{64} = \frac{1}{64}.$$

We conclude that

$$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(I_i) + 2 \cdot \sum_{1 \le i < j \le n} \mathrm{Cov}(I_i, I_j) = n \cdot \frac{7}{64} + 2\left((n-1) \cdot \frac{3}{64} + (n-2) \cdot \frac{1}{64}\right) = \frac{15n - 10}{64}.$$

Exercise 5 A computer samples uniformly at random, independently, and with replacement 100 natural numbers from the set $\{1, 2, \ldots, 100\}$. Let $\bar{X}$ denote their average. Prove, using Chebyshev's inequality, that

$$\mathbb{P}\left(45.5 < \bar{X} < 55.5\right) > 2/3.$$

Solution

For every $1 \leq i \leq 100$, let $X_i \sim U(1, 100)$. Then $\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i$ and thus $\mathbb{E}(X_i) = 50.5$ and $\text{Var}(X_i) = \frac{100^2-1}{12} = \frac{3333}{4}$ hold for every $1 \leq i \leq 100$. It thus follows by the linearity of expectation that $\mathbb{E}\left(\bar{X}\right) = 50.5$. Moreover, since the random variables $X_1, \ldots, X_n$ are mutually independent, it follows that

$$\text{Var}\left(\bar{X}\right) = \text{Var}\left(\frac{1}{100} \cdot \sum_{i=1}^{100} X_i\right) = \frac{1}{100^2} \cdot \sum_{i=1}^{100} \text{Var}(X_i) = \frac{3333}{400}.$$

We conclude that

$$\mathbb{P}\left(45.5 < \bar{X} < 55.5\right) = 1 - \mathbb{P}\left(\left|\bar{X} - 50.5\right| \geq 5\right) = 1 - \mathbb{P}\left(\left|\bar{X} - \mathbb{E}\left(\bar{X}\right)\right| \geq 5\right)$$
$$\geq 1 - \frac{\text{Var}\left(\bar{X}\right)}{5^2} = \frac{6667}{10000} > \frac{2}{3},$$

where the first inequality holds by Chebyshev's inequality.

Exercise 6 Let $X \sim \text{Bin}(n, p)$, for some $n \in \mathbb{N}$ and $p \in [0, 1]$, be a random variable. Prove that for every t satisfying $0 \leq t < n$, it holds that

$$\mathbb{P}(X > t) \geq \frac{np - t}{n - t}.$$

Solution

Fix some $0 \leq t < n$. Note that

$$\mathbb{P}(X > t) = \mathbb{P}(-X < -t) = \mathbb{P}(n - X < n - t) = 1 - \mathbb{P}(n - X \geq n - t). \tag{1}$$

Since $X \sim \text{Bin}(n, p)$, it follows that $n - X$ is a non-negative random variable, and that $\mathbb{E}(X) = np$. Therefore, we can apply Markov's inequality with the random variable $n - X$ to obtain

$$\mathbb{P}(n - X \geq n - t) \leq \frac{\mathbb{E}(n - X)}{n - t} = \frac{n - np}{n - t}. \tag{2}$$

Combining (1) and (2) we obtain

$$\mathbb{P}(X > t) = 1 - \mathbb{P}(n - X \geq n - t) \geq 1 - \frac{n - np}{n - t} = \frac{np - t}{n - t}.$$

Exercise 7 Let $X \sim \text{Geom}(p)$, for some $p \in (0, 1)$, be a random variable. Calculate $\mathbb{E}\left(e^{tX}\right)$ for every $t \in \mathbb{R}$.

Solution

Note that

$$\begin{aligned}
\mathbb{E}\left(e^{tX}\right) &= \sum_{k=1}^{\infty} e^{tk} \cdot \mathbb{P}\left(X = k\right) \\
&= \sum_{k=1}^{\infty} e^{tk}(1-p)^{k-1}p \\
&= \frac{p}{1-p} \cdot \sum_{k=1}^{\infty} \left(e^{t}(1-p)\right)^{k}.
\end{aligned}$$

This is a sum of an infinite geometric series. It converges if and only if $e^t(1-p) < 1$ which is equivalent to $t < -\ln(1-p)$. For these values of t, it holds that

$$\mathbb{E}\left(e^{tX}\right) = \frac{p}{1-p} \cdot \frac{e^t(1-p)}{1-e^t(1-p)} = \frac{e^t p}{1-e^t+e^t p}.$$

For all $t \geq -\ln(1-p)$, the expectation in infinite. We conclude that

$$\mathbb{E}\left(e^{tX}\right) = \begin{cases} \frac{e^t p}{1-e^t+e^t p} & \text{if } t < -\ln(1-p) \\ \infty & \text{otherwise} \end{cases}$$