Assignment 1

If you wish to submit your solutions to any of these questions, please hand them to your TA during the examples class (TIRGUL) on 11/08/17. This deadline is strict!

Exercise 1 Two fair dices are rolled. We then output the sum of the results mod 6. Prove that the corresponding probability space is uniform.

Exercise 2 A computer samples a sequence of n bits uniformly (i.e.,the probability space is uniform over all sequences of length n).

1. What is the probability that there are exactly k 1's in the sequence?
2. Let E be the event that the number of 1's in the sequence is even, and let O be the event that the number of 1's in the sequence is odd. Prove that $\mathbb{P}(E) = \mathbb{P}(O)$.
3. Prove that $\mathbb{P}(E) = \frac{1}{2}$.

Exercise 3 A fair coin is to be tossed until a head appears twice.

1. What is the sample space for this experiment?
2. What is the probability that it will be tossed exactly k times for some $k \in \mathbb{N}$?
3. Prove that the experiment forms a probability space.

Exercise 4 Let $A_1, A_2, \ldots, A_n$ be events in some arbitrary probability space. Prove the following:

1. $$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}(A_i^c).$$
2. $$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - n + 1.$$

Exercise 5 Four men went to a party and hung their coats in a closet. When they left, each of them randomly and uniformly picked a coat. What is the probability that no one got the coat they came with to the party?

Exercise 6 We are given n different bins and $k \geq n$ balls. Each ball is thrown into a uniformly chosen bin. What is the probability that no bin is empty if

1. The balls are different?
2. The balls are identical?

Assignment 1 – Solutions

Exercise 1 Two fair dices are rolled. We then output the sum of the results mod 6. Prove that the corresponding probability space is uniform.

Proof. Let $(\Omega, \mathbb{P})$ be the corresponding probability space, where $\Omega = \{1, 2, 3, 4, 5, 6\}$. We will show that for every $k \in \Omega$ it holds that $\mathbb{P}(k) = \frac{1}{6}$. Let $(\Omega', \mathbb{P}')$ be the probability space for the pair of dices, i.e.,$\Omega' = \{(a, b) : 1 \leq a, b \leq 6\}$. By the fact that the dices are fair we get that:

$$\mathbb{P}(1) = \mathbb{P}'(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = \frac{6}{36} = \frac{1}{6}.$$
$$\mathbb{P}(2) = \mathbb{P}'(\{(1,1),(2,6),(3,5),(4,4),(5,3),(6,2)\}) = \frac{6}{36} = \frac{1}{6}.$$
$$\mathbb{P}(3) = \mathbb{P}'(\{(1,2),(2,1),(3,6),(4,5),(5,4),(6,3)\}) = \frac{6}{36} = \frac{1}{6}.$$
$$\mathbb{P}(4) = \mathbb{P}'(\{(1,3),(2,2),(3,1),(4,6),(5,5),(6,4)\}) = \frac{6}{36} = \frac{1}{6}.$$
$$\mathbb{P}(5) = \mathbb{P}'(\{(1,4),(2,3),(3,2),(4,1),(5,6),(6,5)\}) = \frac{6}{36} = \frac{1}{6}.$$
$$\mathbb{P}(6) = \mathbb{P}'(\{(1,5),(2,4),(3,3),(4,2),(5,1),(6,6)\}) = \frac{6}{36} = \frac{1}{6}.$$

□

Exercise 2 A computer samples a sequence of n bits uniformly (i.e.,the probability space is uniform over all sequences of length n).

1. What is the probability that there are exactly k 1's in the sequence?
2. Let E be the event that the number of 1's in the sequence is even, and let O be the event that the number of 1's in the sequence is odd. Prove that $\mathbb{P}(E) = \mathbb{P}(O)$.
3. Prove that $\mathbb{P}(E) = \frac{1}{2}$.

Proof. Let $(\Omega, \mathbb{P})$ be the corresponding probability space. Then

$$|\Omega| = 2^n.$$

1. Let A_k be the event that there exactly k 1's. Then

$$|A_k| = \binom{n}{k}.$$

Since the probability space is uniform, we get that

$$\mathbb{P}(A_k) = \frac{|A_k|}{|\Omega|} = \binom{n}{k} \cdot \frac{1}{2^n}.$$

2. The probability space is uniform, therefore it is enough to show that $|E| = |O|$. We observe that

$$|E| = \sum_{\substack{0 \le k \le n \\ k \text{ is even}}} \binom{n}{k},$$

and that

$$|O| = \sum_{\substack{0 \le k \le n \\ k \text{ is odd}}} \binom{n}{k}.$$

Hence

$$|E| - |O| = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k = (1-1)^n = 0,$$

where the second equality is by Newton's binomial formula.

Another approach of proving $|E| = |O|$ is by giving a bijection between E and O. Flipping the first bit gives such a bijection, as flipping it again forms the inverse of that function.

3. Since $E \cap O = \emptyset$, it follows that

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(E \cup O) = \mathbb{P}(E) + \mathbb{P}(O) = 2\mathbb{P}(E),$$

where the last equality follows from the previous part of the exercise. We conclude that $\mathbb{P}(E) = \frac{1}{2}$.

$\square$

Exercise 3 A fair coin is to be tossed until a head appears twice.

1. What is the sample space for this experiment?

2. What is the probability that it will be tossed exactly k times for some $k \in \mathbb{N}$?

3. Prove that the experiment forms a probability space.

Proof.

1. One possible way to describe the sample space

$$\Omega = \left\{ (b_1, b_2, \ldots, b_n, 1) : n \in \mathbb{N} \wedge \forall i \; b_i \in \{0,1\} \wedge \sum_{i=1}^{n} b_i = 1 \right\},$$

where we use 1 for heads and 0 for tails.

2. Let A_k be the event that the coin was tossed k times. Since the last coin is always heads we get that:

$$\mathbb{P}(A_k) = \sum_{(b_1 \ldots b_{k-1}, 1) \in \Omega} \mathbb{P}((b_1 \ldots b_{k-1}, 1))$$

For a fixed sequence $(b_1 \ldots b_{k-1}, 1) \in \Omega$ it holds that

$$\mathbb{P}((b_1 \ldots b_{k-1}, 1)) = \frac{1}{2^k},$$

due to the fact that the coin is fair. As the sum of the b_i's is 1, it follows that there are $k-1$ possible sequences, hence

$$\sum_{(b_1 \ldots b_{k-1}, 1) \in \Omega} \frac{1}{2^k} = \frac{k-1}{2^k}.$$

3. We need to prove that

$$\sum_{n=1}^{\infty} \Big(\sum_{(b_1 \ldots b_n, 1) \in \Omega} \mathbb{P}((b_1 \ldots b_n, 1)) \Big) = 1.$$

By the previous exercise, for every $n \in \mathbb{N}$ it holds that

$$\sum_{(b_1 \ldots b_n, 1) \in \Omega} \mathbb{P}((b_1 \ldots b_n, 1)) = \frac{n}{2^{n+1}}.$$

By the formula of an infinite geometric sum we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

for all $|x| < 1$. Taking the derivative in both sides leads to

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2},$$

which is equivalent to

$$\sum_{n=1}^{\infty} n x^{n+1} = \frac{x^2}{(1-x)^2}.$$

By setting $x = \frac{1}{2}$ we get

$$\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = \frac{\frac{1}{2^2}}{(1-\frac{1}{2})^2} = 1.$$

$\square$

Exercise 4 Let $A_1, A_2, \ldots, A_n$ be events in some arbitrary probability space. Prove the following:

1. $$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}\left(A_i^c\right).$$

2. $$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} \mathbb{P}\left(A_i\right) - n + 1.$$

Proof.

1. Recall that one of the De-Morgan laws states that
$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.$$
Taking the probability of both sides we get that
$$\mathbb{P}\left(\left(\bigcap_{i=1}^{n} A_i\right)^c\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i^c\right).$$
Since $\mathbb{P}\left(A^c\right) = 1 - \mathbb{P}\left(A\right)$ for all events A, it holds that
$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = 1 - \mathbb{P}\left(\left(\bigcap_{i=1}^{n} A_i\right)^c\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{n} A_i^c\right).$$
By the union bound it follows that
$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{n} A_i^c\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}\left(A_i^c\right).$$

2. By the previous part it holds that
$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}\left(A_i^c\right).$$
As for every event A, it holds that $\mathbb{P}\left(A^c\right) = 1 - \mathbb{P}\left(A\right)$, it follows that
$$1 - \sum_{i=1}^{n} \mathbb{P}\left(A_i^c\right) = 1 - \sum_{i=1}^{n} \left(1 - \mathbb{P}\left(A_i\right)\right) = \sum_{i=1}^{n} \mathbb{P}\left(A_i\right) - n + 1.$$
The claim follows.

□

Exercise 5 Four men went to a party and hung their coats in a closet. When they left, each of them randomly and uniformly picked a coat. What is the probability that no one got the coat they came with to the party?

Solution

For $i \in \{1,2,3,4\}$ let A_i be the event that person i took his coat. Then

$$\mathbb{P}\left(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c\right) = 1 - \mathbb{P}\left(A_1 \cup A_2 \cup A_3 \cup A_4\right).$$

By inclusion-exclusion we get that

$$\mathbb{P}\left(A_1 \cup A_2 \cup A_3 \cup A_4\right) = \sum_{i=1}^{4} \mathbb{P}\left(A_i\right) - \sum_{1 \le i < j \le 4} \mathbb{P}\left(A_i \cap A_j\right) + \sum_{1 \le i < j < k \le 4} \mathbb{P}\left(A_i \cap A_j \cap A_k\right) + \mathbb{P}\left(A_1 \cap A_2 \cap A_3 \cap A_4\right).$$

Person i will take his coat in $3!$ cases out of all $4!$ possibilities. Therefore $\mathbb{P}\left(A_i\right) = \frac{6}{24} = \frac{1}{4}$ for all i. Similarly $\mathbb{P}\left(A_i \cap A_j\right) = \frac{2}{24} = \frac{1}{12}$, $\mathbb{P}\left(A_i \cap A_j \cap A_k\right) = \frac{1}{24}$, and $\mathbb{P}\left(A_1 \cap A_2 \cap A_3 \cap A_4\right) = \frac{1}{24}$. Therefore

$$\mathbb{P}\left(A_1 \cup A_2 \cup A_3 \cup A_4\right) = 4 \cdot \frac{1}{4} - \binom{4}{2} \cdot \frac{1}{12} + \binom{4}{3} \cdot \frac{1}{24} - \frac{1}{24} = \frac{5}{8},$$

hence

$$\mathbb{P}\left(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c\right) = \frac{3}{8}.$$

Exercise 6 We are given n different bins and $k \ge n$ balls. Each ball is thrown into a uniformly chosen bin. What is the probability that no bin is empty if

1. The balls are different?
2. The balls are identical?

Solution

1. Let $(\Omega_1, \mathbb{P}_1)$ be the probability space. Since the balls are different, it holds that $|\Omega_1| = n^k$. Let A_1 be the event that no bin is empty. For $i \in \{1,2,3,\ldots,n\}$ let B_i be the event that bin i is empty. Then

$$A_1 = \bigcap_{i=1}^{n} B_i^c.$$

By inclusion-exclusion

$$\mathbb{P}_1(A_1) = 1 - \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|-1} \cdot \mathbb{P}_1\left(\bigcap_{i \in I} B_i\right).$$

Since the balls were uniformly distributed, it holds that for all $\emptyset \neq I \subseteq \{1, \ldots, n\}$

$$\mathbb{P}_1 \left(\bigcap_{i \in I} B_i \right) = \frac{(n - |I|)^k}{n^k}.$$

Therefore

$$\begin{aligned}
\mathbb{P}_1 (A_1) &= 1 - \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|-1} \left(\frac{n - |I|}{n} \right)^k \\
&= 1 - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \left(\frac{n-i}{n} \right)^k \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(1 - \frac{i}{n} \right)^k.
\end{aligned}$$

2. Let $(\Omega_2, \mathbb{P}_2)$ be the probability space. Since the balls are identical, it holds that the size of the sample space is equal to the number of solutions in the naturals for

$$x_1 + x_2 + \ldots + x_n = k,$$

where we consider 0 as a natural number. The equation has $\binom{k+n-1}{k}$ solutions. Let A_2 be the event that no bin is empty. Therefore $|A_2|$ is equal to the number of solutions in the naturals for

$$x_1 + x_2 + \ldots + x_n = k,$$

where $x_i \geq 1$ for all $i \in \{1, \ldots, n\}$. By setting $y_i = x_i - 1 \geq 0$ for all i, we get that the equation is equivalent to

$$y_1 + y_2 + \ldots + y_n = k - n,$$

for which there are $\binom{k-1}{k-n}$ solutions. Therefore

$$\mathbb{P}_2 (A_2) = \frac{\binom{k-1}{k-n}}{\binom{k+n-1}{k}}.$$

Assignment 2

If you wish to submit your solutions to any of these questions, please hand them to your TA during the examples class (TIRGUL) on 18/08/17. This deadline is strict!

Exercise 1 Three identical chests, each with two drawers are examined. One has a gold coin in each of its two drawers, another has a silver coin in each drawer and the third has a silver coin in one drawer and a gold coin in the other. A chest is chosen uniformly at random, and in that chest a drawer is chosen uniformly at random. Given that the chosen coin is gold, what is the probability that the second drawer of the chosen chest contains a gold coin as well?

Exercise 2 The weather at a given planet can be either cloudy or clear, with a constant probability. In 60% of the cloudy days, the next day was clear and in 30% of the clear days, the next day was cloudy. What percent of the days are cloudy?

Exercise 3 Let S be a set with n elements. Out of the 2^n subsets of S two sets A and B are randomly chosen with replacement.

1. What is the probability that $|A| = k$?
2. Use your solution to Part 1 of this question to find the probability that A is a subset of B.

Exercise 4 Bowl I contains 2 black balls and a single white ball. Bowl II contains a single black ball and 3 white balls. A bowl is chosen uniformly at random and then a ball is chosen uniformly at random from that bowl.

1. What is the probability that the chosen ball is white?
2. What is the probability that Bowl I was chosen, given that the chosen ball is white?
3. The chosen ball is placed back into its bowl and a new ball is chosen uniformly at random from the same bowl. What is the probability that the second chosen ball is white, given that the first one was white?
4. Same as Part 3 except that the first ball that was chosen is not placed back into any of the bowls.

Exercise 5 Prove that an event is independent of all other events if and only if its probability is either 0 or 1.

Exercise 6 Prove the following statements:

1. Event A is independent of itself if and only if $\mathbb{P}(A) \in \{0, 1\}$.
2. If the events A and B are mutually exclusive and independent then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.
3. If the events A and B are independent it follows that A and B^c are independent.

Exercise 7 For each of the four equations listed below, give an example of a probability space and three events A, B, C in this space which will uphold the other three equations but not this one.

1. $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$
2. $\mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C)$
3. $\mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C)$
4. $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$

Assignment 2
Solutions

Exercise 1 Three identical chests, each with two drawers are examined. One has a gold coin in each of the two drawers, the other has a silver coin in each drawer and the third one has a silver coin in one drawer and a gold coin in the other. A chest is chosen at random (with equal chance for each chest), and in that chest a drawer is chosen at random. If a golden coin was found, what is the probability that the second drawer contains a gold coin as well?

Solution
Let C_2 be the event that chosen chest contains 2 gold coins, let C_1 be the event that chosen chest contains 1 gold coin, and let C_0 be the event that chosen chest contains no gold coins. Let G_1 be the event that the chosen drawer contains a gold coin, and let G_2 be the event that the other drawer contains a gold coin. Then

$$\mathbb{P}(G_2|G_1) = \frac{\mathbb{P}(G_1 \cap G_2)}{\mathbb{P}(G_1)} = \frac{\mathbb{P}(C_2)}{\mathbb{P}(G_1)} = \frac{1/3}{\sum_{i=0}^{2} \mathbb{P}(G_1|C_i) \cdot \mathbb{P}(C_i)} = \frac{1/3}{0 \cdot 1/3 + 1/2 \cdot 1/3 + 1 \cdot 1/3} = \frac{2}{3},$$

where the third equality is by the Law of total probabilities.

Exercise 2 The weather at a given planet can be either cloudy or clear, with a constant probability. In 60% of the cloudy days, the next day was clear and in 30% of the clear days, the next day was cloudy. What percent of the days are cloudy?

Solution
For a given day i, let A_i be the event that the i-th day was cloudy. Then $\mathbb{P}(A_i) = \mathbb{P}(A_j)$ for all i and j. Let $p := \mathbb{P}(A_i)$. Then by the Law of total probabilities it holds that

$$p = \mathbb{P}(A_{i+1}) = \mathbb{P}(A_{i+1} \mid A_i) \cdot \mathbb{P}(A_i) + \mathbb{P}(A_{i+1} \mid A_i^c) \cdot \mathbb{P}(A_i^c) = (1 - 0.6) \cdot p + 0.3 \cdot (1 - p) = 0.3 + 0.1p,$$

hence $p = 1/3$.

Exercise 3 Let S be a set with n elements. Out of the 2^n subsets of S two sets A and B are randomly chosen with replacement.

1. What is the probability that A contains k elements?
2. Use the previous part to find the probability that A is a subset of B.

Solution

1. For $0 \leq k \leq n$, let A_k be the event that $|A| = k$. Since the sample space is uniform it follows that
$$\mathbb{P}(A_k) = \frac{|A_k|}{|\Omega|} = \frac{\binom{n}{k}}{2^n}.$$

2. For $0 \leq k \leq n$, let B_k be the event that $|B| = k$. If $|B| = k$ then $A \subseteq B$ if and only if A is one of the 2^k possible subsets of B. Therefore for all $0 \leq k \leq n$ it holds that
$$\mathbb{P}(A \subseteq B) = \frac{2^k}{2^n}.$$
Then by the Law of total probabilities it holds that
$$\begin{aligned}\mathbb{P}(A \subseteq B) &= \sum_{k=0}^{n} \mathbb{P}(A \subseteq B \mid B_k) \cdot \mathbb{P}(B_k) \\ &= \sum_{k=0}^{n} \frac{2^k}{2^n} \cdot \frac{\binom{n}{k}}{2^n} \\ &= \frac{1}{4^n} \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k \\ &= \frac{3^n}{4^n},\end{aligned}$$
where the last equality is due to Newton's binomial formula.

Exercise 4 Bowl 'a' contains 2 black balls and a single white ball. Bowl 'b' contains a single black ball and 3 white balls. A bowl is randomly selected and then a ball is randomly selected from that bowl.

1. What is the probability that the ball is white?
2. What is the probability that bawl 'a' was chosen, given that the ball is white?
3. The ball is placed back and a new ball is randomly selected. What is the probability that the second ball is white, given the first one was white?
4. Same as part 3 only the first ball is not returned to the bowl.

Solution

Let W_1 be the event that the first ball is white, let W_2 be the event that the second ball is white, let U_1 be the event that the bowl 'a' was chosen, and let U_2 be the event that the bowl 'b' was chosen.

1. By the Law of total probabilities it holds that
$$\mathbb{P}(W_1) = \mathbb{P}(W_1 \mid U_1) \cdot \mathbb{P}(U_1) + \mathbb{P}(W_1 \mid U_2) \cdot \mathbb{P}(U_2) = \frac{1}{3} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{13}{24}.$$

2. Using Bayes' rule we get that

$$\mathbb{P}(U_1 \mid W_1) = \mathbb{P}(W_1 \mid U_1) \cdot \frac{\mathbb{P}(U_1)}{\mathbb{P}(W_1)} = \frac{1}{3} \cdot \frac{1/2}{13/24} = \frac{4}{13}.$$

3. By the Law of total probabilities, over the conditional probability space $(\Omega, \mathbb{P}(* \mid W_1))$ it follows that

$$\mathbb{P}(W_2 \mid W_1) = \mathbb{P}(W_2 \mid W_1 \cap U_1) \cdot \mathbb{P}(U_1 \mid W_1) + \mathbb{P}(W_2 \mid W_1 \cap U_2) \cdot \mathbb{P}(U_2 \mid W_1) = \frac{1}{3} \cdot \frac{4}{13} + \frac{3}{4} \cdot \left(1 - \frac{4}{13}\right) = \frac{97}{156}.$$

4. By the Law of total probabilities, over the conditional probability space $(\Omega, \mathbb{P}(* \mid W_1))$ it follows that

$$\mathbb{P}(W_2 \mid W_1) = \mathbb{P}(W_2 \mid W_1 \cap U_1) \cdot \mathbb{P}(U_1 \mid W_1) + \mathbb{P}(W_2 \mid W_1 \cap U_2) \cdot \mathbb{P}(U_2 \mid W_1) = 0 \cdot \frac{4}{13} + \frac{2}{3} \cdot \left(1 - \frac{4}{13}\right) = \frac{6}{13}.$$

Exercise 5 Prove that an event is independent of all other events if and only if itâĂŹs probability is either 0 or 1.

Solution

Let A be some event. For the first direction we assume that A is independent of all other events. Then

$$0 = \mathbb{P}(A \cap A^c) = \mathbb{P}(A) \cdot \mathbb{P}(A^c) = \mathbb{P}(A) \cdot (1 - \mathbb{P}(A)).$$

It follows that $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

We now prove the second direction, that is we assume that $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$, and prove that A is independent of all other events. Let B be some event. First assume that $\mathbb{P}(A) = 0$. Since $0 \leq \mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 0$, it follows that $\mathbb{P}(A \cap B) = 0$. Hence

$$\mathbb{P}(A \cap B) = 0 = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Assume now that $\mathbb{P}(A) = 1$ and thus $\mathbb{P}(A^c) = 0$. Then $\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B) = \mathbb{P}(A \cap B)$. Hence

$$\mathbb{P}(A \cap B) = 1 \cdot \mathbb{P}(B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Exercise 6 Prove the following properties:

1. Event A is independent of itself if and only if $\mathbb{P}(A) \in \{0, 1\}$.

2. If the events A and B are mutually exclusive and independent then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

3. If the events A and B are independent then A and B^c are independent.

Solution

1. It holds that
$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A) \cdot \mathbb{P}(A) \iff \mathbb{P}(A) = \mathbb{P}(A)^2 \iff \mathbb{P}(A) \in \{0,1\}.$$

2. It holds that
$$0 = \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

3. Observe that the events $A \cap B$ and $A \cap B^c$ are disjoint, and their union equals A. Therefore
$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A) \cdot \mathbb{P}(B) + \mathbb{P}(A \cap B^c),$$
hence
$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A) \cdot \mathbb{P}(B) = \mathbb{P}(A) \cdot (1 - \mathbb{P}(B)) = \mathbb{P}(A) \cdot \mathbb{P}(B^c).$$

Exercise 7 For the following four events, show that the correctness of any three of them, doe's not necessarily imply the fourth. That is, for each of the four equations, give an example of a probability space and three events A, B, and C in this space, which will uphold the other three equations but not this one.

1. $$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

2. $$\mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C).$$

3. $$\mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C).$$

4. $$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C).$$

Solution

We first show that the first three equations doe's not necessarily imply the fourth. Consider two independent tosses of a fair coin. Let A be the event that on the first toss the outcome was heads, let B be the event that on the second toss the outcome was heads, and let C be the event that both tosses had the same outcome. Then
$$\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{2},$$
and
$$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap C) = \mathbb{P}(A \cap B) = \frac{1}{4}.$$
Therefore the first three equations hold, yet the fourth doe's not holds.

We next show that the last three equations doe's not necessarily imply the first one. By symmetry, this will imply the other requirements. Consider the outcome of a roll of a fair dice. Let $A = \{1, 2, 3\}$, let $B = \{3, 4, 5\}$ and let $C = \{2, 3, 4, 6\}$. Then

$$\begin{aligned}
&\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}, \\
&\mathbb{P}(C) = \frac{2}{3}, \\
&\mathbb{P}(A \cap B) = \frac{1}{6}, \\
&\mathbb{P}(A \cap C) = \mathbb{P}(B \cap C) = \frac{1}{3}, \\
&\mathbb{P}(A \cap B \cap C) = \frac{1}{6}.
\end{aligned}$$

And indeed it holds that

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \frac{1}{6} \neq \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A) \cdot \mathbb{P}(B), \\
\mathbb{P}(A \cap C) &= \frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3} = \mathbb{P}(A) \cdot \mathbb{P}(C), \\
\mathbb{P}(B \cap C) &= \frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3} = \mathbb{P}(B) \cdot \mathbb{P}(C), \\
\mathbb{P}(A \cap B \cap C) &= \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C).
\end{aligned}$$

Assignment 3

If you wish to submit your solutions to any of these questions, please hand them to your TA during the examples class (TIRGUL) on 25/08/17. This deadline is strict!

Exercise 1 A bin holds n balls, labeled with the numbers $1, 2, \ldots, n$. Exactly m balls are being sampled uniformly at random from the bin. Let M be the maximum number that was drawn.

1. Compute the distribution of M, when the samples are being made without replacement.
2. Compute the distribution of M, when the samples are being made with replacement and independently.

Exercise 2 A machine M is capable of sampling from $\{0, 1\}$ such that $\mathbb{P}(M = 1) = p$ and $\mathbb{P}(M = 0) = 1 - p$, for some **unknown** $p \in (0, 1)$. For $n \in \mathbb{N}$ let $(L_n, R_n) \leftarrow M^2$ (i.e., we sample pairs of bits), be sampled independently of one another, and independently of all other samples. Define the algorithm A as follows: A will sample (L_n, R_n) until the first time $L_n \neq R_n$, and output the left element. Prove that A will output 1 with probability 1/2.

Exercise 3 Let X and Y be two random variables over the same distribution space. Prove that for every m it holds that

$$|\mathbb{P}(X = m) - \mathbb{P}(Y = m)| \leq \mathbb{P}(X \neq Y).$$

Exercise 4 In a library there are N books in total. N_1 of the books are in English and N_2 of books are in Hebrew (N could be larger than $N_1 + N_2$). Alice walked in the library and randomly picked n books. Let X_1 be the number of books in English Alice chose, and let X_2 be the number of books in Hebrew Alice chose.

1. Compute the distribution of $X_1 + X_2$.
2. After Alice returned all her books, Bob came to the library, and he chose the books in the following way: For every book in the library he flipped a coin that has probability $p \in (0, 1)$ to land on heads, and he took the book if and only if the coin laded on heads. Let Y_1 be the number of books in English Bob chose, and let Y_2 be the number of books in Hebrew Bob chose. Prove that the distribution $Y_1 + Y_2$, conditioned on the event that Bob took exactly n books, is equal to the distribution of $X_1 + X_2$.

Exercise 5 Let X be a random variable over some arbitrary probability space $(\Omega, \mathbb{P})$. For a subset $A \subseteq \mathbb{R}$ we define the event $\{X \in A\} = \{\omega \in \Omega : X(\omega) \in A\}$.

1. Prove that for every $A \subseteq \mathbb{R}$ it holds that

$$\{X \in A\}^c = \{X \in A^c\}.$$

2. Prove that for every infinite sequence of sets $A_1, A_2, \ldots \subseteq \mathbb{R}$ it holds that

   (a)
$$\bigcup_{i \in \mathbb{N}} \{X \in A_i\} = \left\{X \in \bigcup_{i \in \mathbb{N}} A_i\right\}.$$

   (b)
$$\bigcap_{i \in \mathbb{N}} \{X \in A_i\} = \left\{X \in \bigcap_{i \in \mathbb{N}} A_i\right\}.$$

Exercise 6 A computer sends a word of length of 8 bits to another computer using some channel. Since the channel is malfunctioning, each bit has a chance of 0.1 of being flipped (i.e., a 0 is sent but the other computer receives a 1 or a 1 is sent but the other computer receives a 0), independently of the other bits. In order to overcome this problem, the computer sends each bit 3 times, and the receiving computer deciphers by taking their majority.

1. Compute the probability that the word will get deciphered correctly.

2. What is distribution of the number of words that will get deciphered correctly, in a file of 1000 words, each of length 8?

Assignment 3
Solutions

Exercise 1 A bin holds n balls, labeled with the numbers $1, 2, \dots, n$. Exactly m balls are being sampled uniformly at random from the bin. Let M be the maximum number that was drawn.

1. Compute the distribution of M, when the samples are being made without replacement.
2. Compute the distribution of M, when the samples are being made with replacement and independently.

Solution

Let $k \leq n$. We compute $\mathbb{P}(M = k)$.

1. In this case, the possible values for M are $m, m+1, \dots, n$. It follows that k is the maximal value drawn if and only if k is drawn and the remaining $m-1$ balls have values in $\{1, 2, \dots, k-1\}$, meaning that there are $\binom{k-1}{m-1}$ possible samples for which k is the maximum value. Since the samples are being made without replacement, every subset of the balls is equally likely to appear, i.e., the sample space is uniform. We conclude that

$$\mathbb{P}(M = k) = \frac{|\{M = k\}|}{\binom{n}{m}} = \frac{\binom{k-1}{m-1}}{\binom{n}{m}}.$$

   We remind you that $\{M = k\}$ is the event $\{\omega \in \Omega : M(\omega) = k\}$.

2. In this case, the possible values for M are $1, 2, \dots, n$. Let X_i be the value of the i-th ball. Then for all $0 \leq k \leq n$ and for all $1 \leq i \leq m$, it holds that

$$\mathbb{P}(X_i \leq k) = \frac{k}{n}.$$

   As the samples are independent of one another, it follows that

$$\mathbb{P}(M \leq k) = \prod_{i=1}^{m} \mathbb{P}(X_i \leq k) = \left(\frac{k}{n}\right)^m.$$

   Therefore

$$\mathbb{P}(M = k) = \mathbb{P}(M \leq k) - \mathbb{P}(M \leq k-1) = \left(\frac{k}{n}\right)^m - \left(\frac{k-1}{n}\right)^m.$$

Exercise 2 A machine M is capable of sampling from $\{0,1\}$ such that $\mathbb{P}(M=1)=p$ and $\mathbb{P}(M=0)=1-p$, for some **unknown** $p\in(0,1)$. For $n\in\mathbb{N}$ let $(L_n,R_n)\leftarrow M^2$ (i.e., we sample pairs of bits), be sampled independently of one another, and independently of all other samples. Define the algorithm A as follows: A will sample (L_n,R_n) until the first time $L_n\neq R_n$, and output the left element. Prove that A will output 1 with probability $1/2$.

Solution

In the following we abuse notation and let A be both the algorithm, and its output.

For every n it holds that

$$\mathbb{P}(L_n=R_n)=\mathbb{P}(L_n=0,R_n=0)+\mathbb{P}(L_n=1,R_n=1)=p^2+(1-p)^2,$$

hence

$$\mathbb{P}(A=1)=\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n-1}\mathbb{P}(L_i=R_i)\cdot\mathbb{P}(L_n=1,R_n=0)\right)=\sum_{n=1}^{\infty}\left(p^2+(1-p)^2\right)^{n-1}\cdot p(1-p).$$

Observe that

$$\mathbb{P}(A=0)=\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n-1}\mathbb{P}(L_i=R_i)\cdot\mathbb{P}(L_n=0,R_n=1)\right)=\sum_{n=1}^{\infty}\left(p^2+(1-p)^2\right)^{n-1}\cdot p(1-p),$$

holds as well, hence $\mathbb{P}(A=1)=\mathbb{P}(A=0)$. Since these events form a partition of the sample space, we conclude that the probabilities are equal to $1/2$.

Exercise 3 Let X and Y be two random variables over the same distribution space. Prove that for every m it holds that

$$|\mathbb{P}(X=m)-\mathbb{P}(Y=m)|\leq\mathbb{P}(X\neq Y).$$

Solution

Fix some m. Then

$$\begin{aligned}|\mathbb{P}(X=m)-\mathbb{P}(Y=m)|&=|\mathbb{P}(X=m,Y=m)+\mathbb{P}(X=m,Y\neq m)-\mathbb{P}(X=m,Y=m)-\mathbb{P}(X\neq m,Y=m)|\\&\leq\mathbb{P}(X=m,Y\neq m)+\mathbb{P}(X\neq m,Y=m)\\&=\sum_{k\neq m}\left(\mathbb{P}(X=m\wedge Y=k)+\mathbb{P}(Y=m\wedge X=k)\right)\\&\leq\sum_{k}\mathbb{P}(X=k\wedge Y\neq k)\\&=\mathbb{P}(X\neq Y),\end{aligned}$$

where the first inequality is by the triangle inequality and the last inequality is due to the fact that we added more positive terms

Exercise 4 In a library there are N books in total. N_1 of the books are in English and N_2 of books are in Hebrew (N could be larger than $N_1 + N_2$). Alice walked in the library and randomly picked n books. Let X_1 be the number of books in English Alice chose, and let X_2 be the number of books in Hebrew Alice chose.

1. Compute the distribution of $X_1 + X_2$.
2. After Alice returned all her books, Bob came to the library, and he chose the books in the following way: For every book in the library he flipped a coin that has probability $p \in (0,1)$ to land on heads, and he took the book if and only if the coin laded on heads. Let Y_1 be the number of books in English Bob chose, and let Y_2 be the number of books in Hebrew Bob chose. Prove that the distribution $Y_1 + Y_2$, conditioned on the event that Bob took exactly n books, is equal to the distribution of $X_1 + X_2$.

Solution

1. Let $X = X_1 + X_2$. Then $\{X = k\}$ is the event that exactly k of the books that Alice chose are either in English or in Hebrew. There are $\binom{N_1+N_2}{k}\binom{N-N_1-N_2}{n-k}$ such choices. There are $\binom{N}{n}$ total ways to choose n books, hence
$$\mathbb{P}(X = k) = \frac{\binom{N_1+N_2}{k}\binom{N-N_1-N_2}{n-k}}{\binom{N}{n}}.$$

2. Let Z be the number of books that Bob took, which are not in English or Hebrew and let $Y = Y_1 + Y_2$. By Bayes' rule
$$\mathbb{P}(Y = k \mid Y + Z = n) = \frac{\mathbb{P}(Y + Z = n \mid Y = k)\,\mathbb{P}(Y = k)}{\mathbb{P}(Y + Z = n)}.$$
Note that $Y_1 \sim \mathrm{Bin}(N_1, p)$, since Bob will take k books in English if and only if in exactly k of the tosses, the outcome is heads, regardless of the outcome in the other $N - N_1$ tosses. Similarly $Y_2 \sim \mathrm{Bin}(N_2, p)$. As it was shown in class, $Y_1 = W_1 + W_2 + \ldots + W_{N_1}$ and $Y_2 = W_1' + W_2' + \ldots + W_{N_2}'$, where the W_i and W_i' are distributed according to $\mathrm{Ber}(p)$. Therefore $Y = Y_1 + Y_2 = W_1 + W_2 + \ldots + W_{N_1} + W_1' + W_2' + \ldots + W_{N_2}' \sim \mathrm{Bin}(N_1 + N_2, p)$. Similarly $Y + Z \sim \mathrm{Bin}(N, p)$ and $Z \sim \mathrm{Bin}(N - N_1 - N_2, p)$. This implies that
$$\mathbb{P}(Y + Z = n) = \binom{N}{n} p^n (1-p)^{N-n},$$
$$\mathbb{P}(Y + Z = n \mid Y = k) = \mathbb{P}(Z = n - k) = \binom{N - N_1 - N_2}{n - k} p^{n-k} (1-p)^{N-N_1-N_2-n+k},$$
$$\mathbb{P}(Y = k) = \binom{N_1 + N_2}{k} p^k (1-p)^{N_1+N_2-k}.$$

We conclude that

$$\begin{aligned}\mathbb{P}\left(Y=k \mid Y+Z=n\right) &= \frac{\binom{N-N_1-N_2}{n-k}p^{n-k}(1-p)^{N-N_1-N_2-n+k}\cdot\binom{N_1+N_2}{k}p^k(1-p)^{N_1+N_2-k}}{\binom{N}{n}p^n(1-p)^{N-n}} \\ &= \frac{\binom{N_1+N_2}{k}\binom{N-N_1-N_2}{n-k}}{\binom{N}{n}} \\ &= \mathbb{P}\left(X=k\right).\end{aligned}$$

Exercise 5 Let X be a random variable over some arbitrary probability space $(\Omega,\mathbb{P})$. For a subset $A\subseteq\mathbb{R}$ we define the event $\{X\in A\}=\{\omega\in\Omega : X(\omega)\in A\}$.

1. Prove that for every $A\subseteq\mathbb{R}$ it holds that
$$\{X\in A\}^c=\{X\in A^c\}.$$

2. Prove that for every infinite sequence of sets $A_1,A_2,\ldots\subseteq\mathbb{R}$ it holds that

   (a)
   $$\bigcup_{i\in\mathbb{N}}\{X\in A_i\}=\left\{X\in\bigcup_{i\in\mathbb{N}}A_i\right\}.$$

   (b)
   $$\bigcap_{i\in\mathbb{N}}\{X\in A_i\}=\left\{X\in\bigcap_{i\in\mathbb{N}}A_i\right\}.$$

Solution

1. It hold that
$$\{X\in A\}^c=\{\omega\in\Omega : X(\omega)\in A\}^c=\Omega\backslash\{\omega\in\Omega : X(\omega)\in A\}=\{\omega\in\Omega : X(\omega)\notin A\}=\{X\in A^c\}.$$

2. We will only show (a) as the proof for (b) will follow similar lines, or by using one of De-Morgan's laws. It holds that
$$\begin{aligned}\bigcup_{i\in\mathbb{N}}\{X\in A_i\} &= \bigcup_{i\in\mathbb{N}}\{\omega\in\Omega : X(\omega)\in A_i\} \\ &= \bigcup_{i\in\mathbb{N}}\left\{\omega\in\Omega : X(\omega)\in\bigcup_{j\in\mathbb{N}}A_j\right\} \\ &= \left\{\omega\in\Omega : X(\omega)\in\bigcup_{j\in\mathbb{N}}A_j\right\} \\ &= \left\{X\in\bigcup_{i\in\mathbb{N}}A_i\right\}.\end{aligned}$$

Exercise 6 A computer sends a word of length of 8 bits to another computer using some channel. Since the channel is malfunctioning, each bit has a chance of 0.1 of being flipped (i.e., a 0 is sent but the other computer receives a 1 or a 1 is sent but the other computer receives a 0), independently of the other bits. In order to overcome this problem, the computer sends each bit 3 times, and the receiving computer deciphers by taking their majority.

1. Compute the probability that the word will get deciphered correctly.
2. What is distribution of the number of words that will get deciphered correctly, in a file of 1000 words, each of length 8?

Solution

1. Let p be the probability that a single bit is sent correctly. Then

$$\begin{aligned} p &= \mathbb{P}\left(\text{at least 2 bits are correct}\right) \\ &= \mathbb{P}\left(\text{exactly 2 bits are correct}\right) + \mathbb{P}\left(\text{exactly 3 bits are correct}\right) \\ &= \binom{3}{2} \cdot 0.9^2 \cdot 0.1^1 + \binom{3}{3} \cdot 0.9^3 \cdot 0.1^0 \\ &= 0.972. \end{aligned}$$

   Since the word is of length 8 it follows that

$$\mathbb{P}\left(\text{the word was deciphered correctly}\right) = p^8 = 0.972^8.$$

2. Let X be the number of words sent incorrectly when 1000 words are sent. Using the previous part of the exercise, a single word is sent correctly with probability 0.972^8. Since each bit is independently sent, the correctness of different word is independent, hence

$$X \sim \text{Bin}\left(1000, 0.972^8\right).$$

Assignment 4

If you wish to submit your solutions to any of these questions, please hand them to your TA during the examples class (TIRGUL) on 01/09/17. This deadline is strict!

Exercise 1 Let $X \sim \text{Geom}\left(\lambda n^{-1}\right)$, for some $\lambda \geq 0$.

1. Compute $\mathbb{P}\left(X > k\right)$, for every $k \in \mathbb{N}$.
2. Prove that
$$\mathbb{P}\left(n^{-1}X > t\right) = \left(1 - \frac{\lambda}{n}\right)^{\lfloor tn \rfloor},$$
for all $t \geq 0$.
3. Conclude that
$$\lim_{n\to\infty} \mathbb{P}\left(n^{-1}X > t\right) = e^{-\lambda t},$$
for all $t \geq 0$.

Exercise 2 The number of cars crossing a particular bridge is a random variable with Poisson distribution, with an average of $\lambda = 0.3$ cars per minute, i.e., if X counts the number of cars crossing the bridge in any given minute, then $X \sim \text{Poi}\left(0.3\right)$. Calculate the probability that within 5 minutes:

1. No cars have crossed the bridge.
2. More than one car has crossed the bridge.
3. The number of cars that crossed the bridge is between 1 and 3.
4. Exactly 3 cars have crossed the bridge.

Exercise 3 Let $X \sim \text{Geom}\left(p\right)$ and $Y \sim \text{U}\left(0, 1, \ldots, n\right)$ be random variables. Find the probability distribution of $Z = X + Y$.

Exercise 4 A fair coin with 0 on one side and 1 on the other side is tossed. If the result is 0, a fair dice is rolled, otherwise the coin is tossed again. Let X, Y be the numbers that show up on the first and second experiment, respectively. What is the joint distribution of X and Y?

Exercise 5 In a given year the percent of defected oranges was 30%. Each shipment of oranges is checked as follows: oranges are drawn one by one uniformly at random with replacement, until 2 defected oranges are found or until 4 oranges are checked, whichever happens first. Let X, Y be the total number of oranges checked, and the number of defected oranges that were found, respectively. What is the joint distribution of X and Y?

Assignment 4
Solutions

Exercise 1 Let $X \sim \mathrm{Geom}\left(\lambda n^{-1}\right)$, for some $\lambda \geq 0$.

1. Compute $\mathbb{P}\left(X > k\right)$, for every $k \in \mathbb{N}$.
2. Prove that
$$\mathbb{P}\left(n^{-1}X > t\right) = \left(1 - \frac{\lambda}{n}\right)^{\lfloor tn \rfloor},$$
for all $t \geq 0$.
3. Conclude that
$$\lim_{n\to\infty} \mathbb{P}\left(n^{-1}X > t\right) = e^{-\lambda t},$$
for all $t \geq 0$.

Solution

1. Let $p = \lambda n^{-1}$. Since $\{X = i\}$ and $\{X = j\}$ are disjoint for all $i \neq j$, it holds that
$$\begin{aligned}\mathbb{P}\left(X > k\right) = \sum_{i=k+1}^{\infty} \mathbb{P}\left(X = i\right) &= \sum_{i=k+1}^{\infty} p \cdot (1-p)^{i-1} = p \cdot \frac{(1-p)^k}{1-(1-p)} \\ &= (1-p)^k = \left(1 - \frac{\lambda}{n}\right)^k.\end{aligned}$$
2. It holds that
$$\begin{aligned}\mathbb{P}\left(n^{-1}X > t\right) &= \mathbb{P}\left(X > tn\right) \\ &= \mathbb{P}\left(X > \lfloor tn \rfloor\right) \\ &= \left(1 - \frac{\lambda}{n}\right)^{\lfloor tn \rfloor},\end{aligned}$$
where the last equality is by the previous part of the exercise.

3. By the previous part, for all t it holds that

$$\mathbb{P}\left(n^{-1}X > t\right) = \left(1 - \frac{\lambda}{n}\right)^{\lfloor tn \rfloor} = \left(1 - \frac{\lambda}{n}\right)^{n(\lfloor tn \rfloor / n)}.$$

Therefore

$$\log\left(\mathbb{P}\left(n^{-1}X > t\right)\right) = \frac{\lfloor tn \rfloor}{n} \cdot \log\left(\left(1 - \frac{\lambda}{n}\right)^{n}\right).$$

Since $\lim_{n\to\infty} \lfloor tn \rfloor / n = t$ and $\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$, it follows that

$$\lim_{n\to\infty} \log\left(\mathbb{P}\left(n^{-1}X > t\right)\right) = -\lambda t.$$

Which is equivalent to

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-1}X > t\right) = e^{-\lambda t}.$$

Exercise 2 The number of cars crossing a particular bridge is a random variable with Poisson distribution, with an average of $\lambda = 0.3$ cars per minute, i.e., if X counts the number of cars crossing the bridge in any given minute, then $X \sim \text{Poi}(0.3)$. Calculate the probability that within 5 minutes:

1. No cars have crossed the bridge.

2. More than one car has crossed the bridge.

3. The number of cars that crossed the bridge is between 1 and 3.

4. Exactly 3 cars have crossed the bridge.

Solution

For $1 \le i \le 5$ let X_i be the number of cars that passed in the i-th minute. Then $X_i \sim \text{Poi}(0.3)$ for all such i. As was shown in the practical sessions, $Y := \sum_{i=1}^{5} X_i$ satisfies $Y \sim \text{Poi}(0.3 \cdot 5)$, i.e., $Y \sim \text{Poi}(1.5)$. Therefore, for all $k \ge 0$ it holds that

$$\mathbb{P}(Y = k) = e^{-1.5} \cdot \frac{1.5^k}{k!}.$$

1. $\mathbb{P}(Y = 0) = e^{-1.5}$.

2. $\mathbb{P}(Y > 1) = 1 - (\mathbb{P}(Y = 0) + \mathbb{P}(Y = 1)) = 1 - e^{-1.5}(1 + 1.5) = 2.5 \cdot e^{-1.5}$.

3. $\mathbb{P}(1 \le Y \le 3) = \mathbb{P}(Y = 1) + \mathbb{P}(Y = 2) + \mathbb{P}(Y = 3) = e^{-1.5}\left(1.5 + \frac{1.5^2}{2} + \frac{1.5^3}{3!}\right) = \frac{51}{16} \cdot e^{-1.5}$.

4. $\mathbb{P}(Y = 3) = e^{-1.5} \cdot \frac{1.5^3}{3!} = \frac{9}{16} \cdot e^{-1.5}$.

Exercise 3 Let $X \sim \mathrm{Geom}(p)$ and $Y \sim \mathrm{U}(0, 1, \ldots, n)$ be random variables. Find the probability distribution of $Z = X + Y$.

Solution
Let $k \in \mathbb{N} \setminus \{0\}$. For $k \leq n$ it holds that

$$
\begin{aligned}
\mathbb{P}(Z = k) &= \mathbb{P}(X + Y = k) \\
&= \sum_{j=1}^{k} \mathbb{P}(X + Y = k \mid X = j) \cdot \mathbb{P}(X = j) \\
&= \sum_{j=1}^{k} \mathbb{P}(Y = k - j) \cdot \mathbb{P}(X = j) \\
&= \sum_{j=1}^{k} \frac{1}{n+1} \cdot (1-p)^{j-1} \cdot p \\
&= \frac{p}{n+1} \cdot \sum_{j=1}^{k} (1-p)^{j-1} \\
&= \frac{p}{n+1} \cdot \frac{1-(1-p)^k}{1-(1-p)} \\
&= \frac{1-(1-p)^k}{n+1},
\end{aligned}
$$

where the second equality is by the Law of total probability. For $k > n$ it holds that

$$
\begin{aligned}
\mathbb{P}(Z = k) &= \mathbb{P}(X + Y = k) \\
&= \sum_{j=0}^{n} \mathbb{P}(X + Y = k \mid Y = j) \cdot \mathbb{P}(Y = j) \\
&= \sum_{j=0}^{n} \mathbb{P}(X = k - j) \cdot \mathbb{P}(Y = j) \\
&= \sum_{j=0}^{n} (1-p)^{k-j-1} \cdot p \cdot \frac{1}{n+1} \\
&= \frac{p(1-p)^{k-n-1}}{n+1} \cdot \sum_{j=0}^{n} (1-p)^j \\
&= \frac{p(1-p)^{k-n-1}}{n+1} \cdot \frac{1-(1-p)^{n+1}}{1-(1-p)} \\
&= \frac{(1-p)^{k-n-1} - (1-p)^k}{n+1},
\end{aligned}
$$

where the second equality is by the Law of total probability.

Exercise 4 A fair coin with 0 on one side and 1 on the other side is tossed. If the result is 0, a fair dice is rolled, otherwise the coin is tossed again. Let X and Y be the numbers that show up on the first and second experiment, respectively. What is the joint distribution of X and Y?

Solution

If $X = 1$ in the second toss we will toss the coin again, hence the support of Y conditioned on $X = 1$ is $\{0, 1\}$, implying that

$$\mathbb{P}(X = 1, Y = 0) = \mathbb{P}(X = 1, Y = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

If $X = 0$ in the second toss we will toss a fair dice, hence for $1 \leq k \leq 6$

$$\mathbb{P}(X = 0, Y = k) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}.$$

Exercise 5 In a given year the percent of defected oranges was 30%. Each shipment of oranges is checked as follows: oranges are drawn one by one uniformly at random with replacement, until 2 defected oranges are found or until 4 oranges are checked, whichever happens first. Let X and Y be the total number of oranges checked, and the number of defected oranges that were found, respectively. What is the joint distribution of X and Y?

Solution

The support of X is $\{2, 3, 4\}$ while the support of Y is $\{0, 1, 2\}$. Since the experiment is over once two defected oranges are found, it follows that $\mathbb{P}(Y < 2 \mid X < 4) = 0$.

- If $X = 2$ then so is Y, since both oranges must be defected in this case, hence $\mathbb{P}(X = 2, Y = 2) = 0.3^2 = 0.09$.
- If $X = 3$ then Y must be 2, since 2 of the 3 oranges must be defected in this case. Moreover, in the first two oranges, one must be fine, that is, it is not defected, as otherwise the experiment would have been ended sooner. Therefore $\mathbb{P}(X = 3, Y = 2) = (2 \cdot 0.7 \cdot 0.3) \cdot 0.3 = 0.126$.
- If $X = 4$ then one of the following events must happen:
  - $Y = 0$, meaning that all 4 oranges were fine, which happens with probability 0.7^4.
  - $Y = 1$, meaning that one of the 4 oranges was defected, which happens with probability $4 \cdot 0.7^3 \cdot 0.3 = 0.4116$.
  - $Y = 2$, meaning that two of the 4 oranges were defected, but the second defected orange was sampled in the forth sample. This event happens with probability $(3 \cdot 0.7^2 \cdot 0.3) \cdot 0.3 = 0.1323$.

We conclude that

$Y \backslash X$	2	3	4
0	0	0	0.7^4
1	0	0	0.4116
2	0.09	0.126	0.1323

Assignment 5

If you wish to submit your solutions to any of these questions, please hand them to your TA during the examples class (TIRGUL) on 08/09/17. This deadline is strict!

Exercise 1 Let $X \sim \text{Bin}(n, p)$ be a random variable. Find the expected value of X using two methods – by direct calculation according to the definition of expectation and by depicting X as a sum of n independent Bernoulli random variables.

Exercise 2 Let π be a permutation of $\{1, 2, \ldots, n\}$, chosen uniformly at random. Let $X = |\{i : \pi(i) = i\}|$ be the number of fixed points of π. Calculate $\mathbb{E}(X)$.

Exercise 3 A bar is positioned on a vertex v of the complete graph K_n.

1. A drunk comes out of the bar and wants to reach his home located at vertex $u \neq v$. In each step, if he is not already at u, he chooses another vertex (i.e., not the vertex he is currently at) uniformly at random, independently of his previous choices, and walks towards it. Let X denote the number of steps the drunk made until he reached u. Calculate $\mathbb{E}(X)$.

2. A friend of the drunk, which is a little less drunk, wants to go with his friend to his house at u. Being more sober, he recalls which vertices he visited before. Hence, in each step, if he is not already at u, he chooses a vertex he did not previously visit uniformly at random, and walks towards it. Let Y denote the number of steps the friend did until he reached u. Calculate $\mathbb{E}(Y)$.

Exercise 4 An eager student has n days to study for his exam in Probability for Computer Science 1. Since he does not remember the material from Discrete Mathematics and Calculus, he wants to refresh his memory. Hence, every day until the exam, with equal probability and independently of his previous choices, he does one of the following:

(a) Solve 1 exercise in Calculus and 2 in Probability Theory.

(b) Solve 2 exercises in Discrete Mathematics and 1 in Probability Theory.

(c) Solve 3 exercises in Probability Theory.

(d) Solve 1 exercise in each of the three subjects.

Let X_C be the number of exercises the student solved in Calculus, let X_D be the number of exercises the student solved in Discrete Mathematics, and let X_P be the number of exercises the student solved in Probability Theory. Calculate the expectation of each of the random variables X_C, X_D and X_P.

Exercise 5 Let $X \sim \mathrm{Geom}\,(p)$ for some $p \in (0,1)$, and let $t \in \mathbb{R}$. Calculate $\mathbb{E}\left(e^{tX}\right)$ for every t.

Exercise 6 Define the following procedure:

procedure 1. (BubbleSortOnePass(A, n))
for $i = 1$ to $n - 1$ do
If $A[i] > A[i+1]$ then
Swap $A[i]$ and $A[i+1]$.

Suppose that A starts out as a uniform random permutation of distinct elements. What is the expected number of swaps performed by the above procedure?

Assignment 5
Solutions

Exercise 1 Let $X \sim \text{Bin}(n,p)$ be a random variable. Find the expected value of X using two methods – by direct calculation according to the definition of expectation and by depicting X as a sum of n independent Bernoulli random variables.

Solution

We start with the direct computation. It holds that for all $1 \le k \le n$

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}.$$

Therefore

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=0}^{n} k \cdot \mathbb{P}(X = k) \\
&= \sum_{k=1}^{n} k \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k \cdot \frac{n}{k} \cdot \binom{n-1}{k-1} \cdot p^k \cdot (1-p)^{n-k} \\
&= np \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} \cdot p^{k-1} \cdot (1-p)^{(n-1)-(k-1)} \\
&= np \cdot \sum_{m=0}^{n-1} \binom{n-1}{m} \cdot p^m \cdot (1-p)^{n-1-m} \\
&= np,
\end{aligned}$$

where the last equality is by Newton's binomial formula.

We next compute $\mathbb{E}(X)$ using the second method. For $1 \le i \le n$ let $X_i \sim \text{Ber}(p)$ be independent Bernoulli random variables. Then $X = \sum_{i=1}^{n} X_i$. Therefore by linearity of expectation

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}(X_i) = n \cdot \mathbb{P}(X_1 = 1) = np.$$

Exercise 2 Let π be a permutation of $\{1, 2, \ldots, n\}$, chosen uniformly at random. Let $X = |\{i : \pi(i) = i\}|$ be the number of fixed points of π. Calculate $\mathbb{E}(X)$.

Solution

For every $1 \le i \le n$ let

$$X_i = \begin{cases} 1 & \text{if } \pi(i) = i \\ 0 & \text{otherwise} \end{cases}$$

be the indicator for the event "i is a fixed point of π". Then for all $1 \le i \le n$

$$\mathbb{E}(X_i) = \mathbb{P}(X_i = 1) = \frac{1}{n},$$

which, by the linearity of expectation, implies that

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i) = 1.$$

Exercise 3 A bar is positioned on a vertex v of the complete graph K_n.

1. A drunk comes out of the bar and wants to reach his home located at vertex $u \neq v$. In each step, if he is not already at u, he chooses another vertex (i.e., not the vertex he is currently at) uniformly at random, independently of his previous choices, and walks towards it. Let X denote the number of steps the drunk made until he reached u. Calculate $\mathbb{E}(X)$.

2. A friend of the drunk, which is a little less drunk, wants to go with his friend to his house at u. Being more sober, he recalls which vertices he visited before. Hence, in each step, if he is not already at u, he chooses a vertex he did not previously visit uniformly at random, and walks towards it. Let Y denote the number of steps the friend did until he reached u. Calculate $\mathbb{E}(Y)$.

Solution

1. Observe that $X \sim \text{Geom}\left(\frac{1}{n-1}\right)$, since every step the drunk man takes, with probability $\frac{1}{n-1}$ he reaches u. Therefore $\mathbb{E}(X) = n - 1$.

2. For every $1 \le k \le n-1$ it holds that

$$\mathbb{P}(Y = k) = \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \cdots \frac{n-k}{n-k+1} \cdot \frac{1}{n-k} = \frac{1}{n-1},$$

while for other values of k the probability is 0. Therefore $Y \sim \text{U}(1, 2, \ldots, n-1)$, which implies that $\mathbb{E}(Y) = n/2$.

Exercise 4 An eager student has n days to study for his exam in Probability for Computer Science 1. Since he does not remember the material from Discrete Mathematics and Calculus, he wants to refresh his memory. Hence, every day until the exam, with equal probability and independently of his previous choices, he does one of the following:

(a) Solve 1 exercise in Calculus and 2 in Probability Theory.

(b) Solve 2 exercises in Discrete Mathematics and 1 in Probability Theory.

(c) Solve 3 exercises in Probability Theory.

(d) Solve 1 exercise in each of the three subjects.

Let X_C be the number of exercises the student solved in Calculus, let X_D be the number of exercises the student solved in Discrete Mathematics, and let X_P be the number of exercises the student solved in Probability Theory. Calculate the expectation of each of the random variables X_C, X_D and X_P.

Solution

We start with computing $\mathbb{E}(X_C)$. For $1 \leq i \leq n$ let

$$X_C^i = \begin{cases} 1 & \text{the student solved an exercise in Calculus on day } i \\ 0 & \text{otherwise} \end{cases}$$

Then $X_C = \sum_{i=1}^n X_C^i$. Moreover, for all $1 \leq i \leq n$ it holds that

$$\mathbb{E}\left(X_C^i\right) = \mathbb{P}\left(X_C^i = 1\right) = \frac{1}{2}.$$

Therefore, by linearity of expectation

$$\mathbb{E}(X_C) = \sum_{i=1}^n \mathbb{E}\left(X_C^i\right) = \frac{n}{2}.$$

We next compute $\mathbb{E}(X_D)$. For $1 \leq i \leq n$ let

$$X_D^i = \begin{cases} 2 & \text{the student solved 2 exercises in Discrete math on day } i \\ 1 & \text{the student solved an exercise in Discrete math on day } i \\ 0 & \text{otherwise} \end{cases}$$

Then $X_D = \sum_{i=1}^n X_D^i$. Moreover, for all $1 \leq i \leq n$ it holds that

$$\mathbb{E}\left(X_D^i\right) = \mathbb{P}\left(X_D^i = 1\right) + 2 \cdot \mathbb{P}\left(X_D^i = 2\right) = \frac{1}{4} + 2 \cdot \frac{1}{4} = \frac{3}{4}.$$

Therefore, by linearity of expectation

$$\mathbb{E}(X_D) = \sum_{i=1}^n \mathbb{E}\left(X_D^i\right) = \frac{3n}{4}.$$

We next compute $\mathbb{E}(X_P)$. For $1 \leq i \leq n$ let

$$X_P^i = \begin{cases} 3 & \text{the student solved 3 exercises in Probability on day } i \\ 2 & \text{the student solved 2 exercises in Probability on day } i \\ 1 & \text{the student solved an exercise in Probability on day } i \\ 0 & \text{otherwise} \end{cases}$$

Then $X_P = \sum_{i=1}^{n} X_P^i$. Moreover, for all $1 \le i \le n$ it holds that

$$\mathbb{E}\left(X_P^i\right) = \mathbb{P}\left(X_P^i = 1\right) + 2 \cdot \mathbb{P}\left(X_P^i = 2\right) + 3 \cdot \mathbb{P}\left(X_P^i = 3\right) = \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = \frac{7}{4}.$$

Therefore, by linearity of expectation

$$\mathbb{E}\left(X_P\right) = \sum_{i=1}^{n} \mathbb{E}\left(X_P^i\right) = \frac{7n}{4}.$$

Exercise 5 Let $X \sim \mathrm{Geom}\left(p\right)$ for some $p \in (0,1)$, and let $t \in \mathbb{R}$. Calculate $\mathbb{E}\left(e^{tX}\right)$ for every t.

Solution
We have

$$\begin{aligned}
\mathbb{E}\left(e^{tX}\right) &= \sum_{k=1}^{\infty} e^{tk} \cdot \mathbb{P}\left(X = k\right) \\
&= \sum_{k=1}^{\infty} e^{tk}(1-p)^{k-1}p \\
&= \frac{p}{1-p} \cdot \sum_{k=1}^{\infty} \left(e^t(1-p)\right)^k.
\end{aligned}$$

This is a sum of an infinite geometric sequence, hence the series converges if and only if $e^t(1-p) < 1$ if and only if $t < -\ln(1-p)$. For those values of t it holds that

$$\mathbb{E}\left(e^{tX}\right) = \frac{p}{1-p} \cdot \frac{e^t(1-p)}{1-e^t(1-p)} = \frac{e^t p}{1-e^t+e^t p}.$$

For $t \ge -\ln(1-p)$ the expectation in infinite. We conclude that

$$\mathbb{E}\left(e^{tX}\right) = \begin{cases} \frac{e^t p}{1-e^t+e^t p} & \text{if } t < -\ln(1-p) \\ \infty & \text{otherwise} \end{cases}$$

Exercise 6 Define the following procedure:

procedure 1. (BubbleSortOnePass(A, n))
for $i = 1$ to $n-1$ do
If $A[i] > A[i+1]$ then
Swap $A[i]$ and $A[i+1]$.

Suppose that A starts out as a uniform random permutation of distinct elements. What is the expected number of swaps performed by the above procedure?

Solution

We count how many values are not swapped from $A[i]$ to $A[i-1]$. We can then subtract from n to get the number that are. Let A_i represent the original contents of $A[i]$, before doing any swaps. Let

$$X_i = \begin{cases} 1 & A_i \text{ is not swapped into } A[i-1] \\ 0 & \text{otherwise} \end{cases}$$

be the indicator variable for the event that A_i is not swapped into $A[i-1]$. This occurs if, when testing $A[i-1]$ against $A[i]$, $A[i]$ is larger. Since we know that at this point $A[i-1]$ is the largest value among $A_1, \ldots, A_{i-1}$, $X_i = 1$ if and only if A_i is larger than all of $A_1, \ldots, A_{i-1}$, or equivalently if A_i is the largest value in $A_1, \ldots, A_i$. By symmetry we have $\mathbb{E}\left(X_i\right) = \mathbb{P}\left(X_i = 1\right) = \frac{1}{i}$. Summing over all i gives the result $\sum_{i=1}^{n} \frac{1}{i}$, hence the expected number of swaps is $n - \sum_{i=1}^{n} \frac{1}{i}$.

Assignment 6

If you wish to submit your solutions to any of these questions, please hand them to your TA during the examples class (TIRGUL) on 15/09/17. This deadline is strict!

Exercise 1 Let S be a set with n elements. A set $A \subseteq S$ is selected uniformly at random among all 2^n subsets of S. Let $X = |A|$.

1. Calculate the probability distribution of X.
2. Calculate the expected value of X using two methods – by direct calculation according to the definition of expectation and by depicting X as a Binomial random variable.

Exercise 2 Let $X \sim \mathrm{Bin}\left(n, p\right)$, for some $n \in \mathbb{N}$ and $p \in [0, 1]$. Find $\mathrm{Var}\left(X\right)$ using two methods: by direct calculation according to the definition of variance, and by depicting X as a sum of n mutually independent Bernoulli random variables.

Exercise 3 A fair coin is being tossed $n + 2$ times. Let X be the number of times 3 consecutive heads appeared, for example, in the sequence $HHHHHTTHHH$, $X = 4$. Compute $\mathbb{E}\left(X\right)$ and $\mathrm{Var}\left(X\right)$.

Exercise 4 A fair coin is being tossed n times. If the result is heads, you win 1 token. If the result is tails, you lose 5 tokens. R is the profit. What is the expected value and variance of R?

Exercise 5 Let X be a random variable which can get the values $\{a, b\}$. It is known that $\mathbb{E}\left(X\right) = \mu$. Prove that $\mathrm{Var}\left(X\right) = \left(\mu - a\right)\left(b - \mu\right)$.

Exercise 6 Let A, B, C be events with the probabilities $0.5,\ 0.25,\ 0.2$, respectively. Let N be the number of events that occur out of the three. Calculate $\mathbb{E}\left(N\right)$ and $\mathrm{Var}\left(N\right)$ for:

1. A, B, C are independent.
2. A, B, C are pairwise disjoint.
3. $C \in B \in A$

Assignment 6
Solutions

Exercise 1 Let S be a set with n elements. A set $A \subseteq S$ is selected uniformly at random among all 2^n subsets of S. Let $X = |A|$.

1. Calculate the probability distribution of X.
2. Calculate the expected value of X using two methods âĂŞ by direct calculation according to the definition of expectation and by depicting X as a Binomial random variable.

Solution

1. For all $0 \leq k \leq n$ the event $\{X = k\}$ is of size $\binom{n}{k}$. Since the sample space is uniform it follows that
$$\mathbb{P}\left(X = k\right) = \frac{\binom{n}{k}}{2^n}.$$
2. We start with the direct computation. It holds that for all $1 \leq k \leq n$
$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}.$$
Therefore
$$\begin{aligned}
\mathbb{E}\left(X\right) &= \sum_{k=0}^{n} k \cdot \mathbb{P}\left(X = k\right) \\
&= \sum_{k=1}^{n} k \cdot \binom{n}{k} \cdot \frac{1}{2^n} \\
&= \sum_{k=1}^{n} k \cdot \frac{n}{k} \cdot \binom{n-1}{k-1} \cdot \frac{1}{2^n} \\
&= \frac{n}{2} \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} \cdot \frac{1}{2^{n-1}} \\
&= \frac{n}{2} \cdot \sum_{m=0}^{n-1} \binom{n-1}{m} \cdot \frac{1}{2^{n-1}} \\
&= \frac{n}{2},
\end{aligned}$$

where the last equality is by Newton's binomial formula.

We next compute $\mathbb{E}(X)$ using the second method. Observe that

$$\mathbb{P}(X=k)=\frac{\binom{n}{k}}{2^n}=\mathbb{P}(X=k)=\binom{n}{k}\cdot\frac{1}{2^k}\cdot\frac{1}{2^{n-k}},$$

for all $0\leq k\leq n$, hence $X\sim\mathrm{Bin}(n,1/2)$, which implies that $\mathbb{E}(X)=n/2$.

Exercise 2 Let $X\sim\mathrm{Bin}(n,p)$, for some $n\in\mathbb{N}$ and $p\in[0,1]$. Find $\mathrm{Var}(X)$ using two methods: by direct calculation according to the definition of variance, and by depicting X as a sum of n mutually independent Bernoulli random variables.

Solution

We start with a direct calculation. It holds that

$$\mathrm{Var}(X)=\mathbb{E}\left((X-\mathbb{E}(X))^2\right)=\mathbb{E}\left(X^2\right)-(\mathbb{E}(X))^2,$$

where $\mathbb{E}(X)=np$ as was computed in the previous assignment. It is left to compute $\mathbb{E}(X^2)$. We do so by computing $\mathbb{E}(X^2-X)$.

$$\begin{aligned}
\mathbb{E}\left(X^2-X\right)&=\sum_{k=0}^{n}(k^2-k)\cdot\mathbb{P}(X=k)\\
&=\sum_{k=2}^{n}k(k-1)\cdot\binom{n}{k}p^k(1-p)^{n-k}\\
&=\sum_{k=2}^{n}k(k-1)\cdot\frac{n}{k}\binom{n-1}{k-1}p^k(1-p)^{n-k}\\
&=n\cdot\sum_{k=2}^{n}(k-1)\cdot\binom{n-1}{k-1}p^k(1-p)^{n-k}\\
&=n\cdot\sum_{k=2}^{n}(k-1)\cdot\frac{n-1}{k-1}\cdot\binom{n-2}{k-2}p^k(1-p)^{n-k}\\
&=n(n-1)\cdot\sum_{k=2}^{n}\binom{n-2}{k-2}p^k(1-p)^{n-k}\\
&=n(n-1)p^2\cdot\sum_{m=0}^{n-2}\binom{n-2}{m}p^m(1-p)^{(n-2)-m}\\
&=n(n-1)p^2,
\end{aligned}$$

where the third and fifth equalities follow from the identity $\binom{n}{k}=\frac{n}{k}\cdot\binom{n-1}{k-1}$, the penultimate equality follows from the substitution $m=k-2$, and the last equality follows from the binomial formula. Therefore, from linearity of expectation $\mathbb{E}(X^2)=\mathbb{E}(X^2-X+X)=\mathbb{E}(X^2-X)+\mathbb{E}(X)=n(n-1)p^2+np=(np)^2+np(1-p)$. We conclude that

$$\mathrm{Var}(X)=\mathbb{E}\left(X^2\right)-(\mathbb{E}(X))^2=np(1-p).$$

We now compute the variance by depicting X as a sum of n mutually independent Bernoulli random variables. For $i \in \{1, 2, \dots, n\}$ let $X_i \sim \text{Ber}(p)$, where the random variables are mutually independent. Then $X = \sum_{i=1}^{n} X_i$, and from independence it follows that

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i).$$

Since $X_i \sim \text{Ber}(p)$ it follows that $\text{Var}(X_i) = p(1-p)$, hence $\text{Var}(X) = np(1-p)$.

Exercise 3 A fair coin is being tossed $n+2$ times. Let X be the number of times 3 consecutive heads appeared, for example, in the sequence $HHHHHTTHHH$, $X = 4$. Compute $\mathbb{E}(X)$ and $\text{Var}(X)$.

Solution

For every $1 \le i \le n+2$, let C_i be the outcome of the i-th toss, that is, $C_i = H$ if the i-th toss is heads, and $C_i = T$ otherwise. For every $1 \le i \le n$, let

$$I_i = \begin{cases} 1 & \text{if } C_i C_{i+1} C_{i+2} = HHH \\ 0 & \text{otherwise} \end{cases}$$

be the indicator variable for 3 consecutive heads starting at position i. Note that $X = \sum_{i=1}^{n} I_i$, and that $\mathbb{E}(I_i) = \mathbb{P}(I_i = 1) = 1/8$ for all $1 \le i \le n$. Hence by linearity of expectation it holds that $\mathbb{E}(X) = n/8$.

We now compute the variance. It holds that

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} \text{Var}(I_i) + 2\sum_{i<j} \text{Cov}(I_i, I_j).$$

Since $I_i \sim \text{Ber}(1/8)$, it follows that $\text{Var}(I_i) = 7/64$. Observe that for all $1 \le i \le n$ and for all $j > i+2$ the random variables I_i and I_j are independent, since the sets of coin flips $C_i C_{i+1} C_{i+2}$ and $C_j C_{j+1} C_{j+2}$ are disjoint. Hence $\text{Cov}(I_i, I_j) = 0$ for such choices of i and j. For all $1 \le i \le n-1$ it holds that

$$\text{Cov}(I_i, I_{i+1}) = \mathbb{E}(I_i \cdot I_{i+1}) - \mathbb{E}(I_i)\,\mathbb{E}(I_{i+1}) = \mathbb{P}(I_i = 1, I_{i+1} = 1) - \frac{1}{64} = \frac{1}{16} - \frac{1}{64} = \frac{3}{64},$$

where the third equality follows from the fact that $I_i = I_{i+1} = 1$ if and only if $C_i C_{i+1} C_{i+2} C_{i+3} = HHHH$. Similarly, for all $1 \le i \le n-2$ it holds that

$$\text{Cov}(I_i, I_{i+2}) = \mathbb{E}(I_i \cdot I_{i+2}) - \mathbb{E}(I_i)\,\mathbb{E}(I_{i+2}) = \frac{1}{32} - \frac{1}{64} = \frac{1}{64}.$$

We conclude that

$$\text{Var}(X) = \sum_{i=1}^{n} \text{Var}(I_i) + 2\sum_{i<j} \text{Cov}(I_i, I_j) = n \cdot \frac{7}{64} + 2\left((n-1) \cdot \frac{3}{64} + (n-2) \cdot \frac{1}{64}\right) = \frac{15n - 10}{64}.$$

Exercise 4 A fair coin is being tossed n times. If the result is heads, you win 1 token. If the result is tails, you lose 5 tokens. Let R be the profit. What is the expected value and variance of R?

Solution

Let X be the number of times the coin landed on heads. Since the number of times the coin landed on tails is exactly $n - X$, it follows that $R = X - 5(n - X) = 6X - 5n$. It also holds that $X \sim \mathrm{Bin}\,(n, 1/2)$, hence $\mathbb{E}\,(X) = n/2$ and $\mathrm{Var}\,(X) = n/4$. By linearity of expectation it holds that

$$\mathbb{E}\,(R) = \mathbb{E}\,(6X - 5n) = 6\mathbb{E}\,(X) - 5n = -2n,$$

and, since $5n$ is a constant,

$$\mathrm{Var}\,(R) = \mathrm{Var}\,(6X - 5n) = \mathrm{Var}\,(6X) = 36\,\mathrm{Var}\,(X) = 9n.$$

Exercise 5 Let X be a random variable which can get the values $\{a, b\}$. It is known that $\mathbb{E}\,(X) = \mu$. Prove that $\mathrm{Var}\,(X) = (\mu - a)(b - \mu)$.

Solution

Let $p := \mathbb{P}\,(X = a)$. Then

$$\mu = \mathbb{E}\,(X) = a \cdot p + b \cdot (1 - p) \Rightarrow p = \frac{\mu - b}{a - b}.$$

Therefore

$$\mathbb{E}\left(X^2\right) = a^2 \cdot p + b^2 \cdot (1 - p) = b^2 + p(a - b)(a + b) = b^2 + (\mu - b)(a + b) = a\mu + b\mu - ab,$$

which implies that

$$\mathrm{Var}\,(X) = \mathbb{E}\left(X^2\right) - (\mathbb{E}\,(X))^2 = a\mu + b\mu - ab - \mu^2 = (\mu - a)(b - \mu).$$

Exercise 6 Let A, B, and C be events with the probabilities 0.5, 0.25, 0.2 respectively. Let N be the number of events that occur out of the three. Calculate $\mathbb{E}\,(N)$ and $\mathrm{Var}\,(N)$ for:

1. A, B, and C are independent.
2. A, B, and C are pairwise disjoint.
3. $C \subseteq B \subseteq A$.

Solution

We first start with some computation, common to all parts of the exercise:

Let $\mathcal{I} = \{A, B, C\}$ and for $I \in \mathcal{I}$ let $X_I = \mathbf{1}_I$ be the indicator for the event I, that is

$$X_I = \begin{cases} 1 & \text{if } I \text{ occurred} \\ 0 & \text{otherwise} \end{cases}$$

Then $N = X_A + X_B + X_C$. Therefore

$$\mathbb{E}(N) = \mathbb{E}(X_A + X_B + X_C) = \mathbb{E}(X_A) + \mathbb{E}(X_B) + \mathbb{E}(X_C) = \frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20},$$

and

$$\mathrm{Var}(N) = \mathrm{Var}(X_A + X_B + X_C) = \mathrm{Var}(X_A) + \mathrm{Var}(X_B) + \mathrm{Var}(X_C) + \sum_{I_1 \neq I_2} \mathrm{Cov}(X_{I_1}, X_{I_2}).$$

Since $X_I \sim \mathrm{Ber}(\mathbb{P}(I))$ for all events $I \in \mathcal{I}$, it follows that

$$\mathrm{Var}(X_I) = \mathbb{P}(I)(1 - \mathbb{P}(I)),$$

hence

$$\mathrm{Var}(X_A) + \mathrm{Var}(X_B) + \mathrm{Var}(X_C) = \frac{239}{400}.$$

We are now ready to solve the three parts of the exercise:

1. When the 3 events are independent $\mathrm{Cov}(X_{I_1}, X_{I_2}) = 0$ for all $I_1 \neq I_2$ hence in this case $\mathrm{Var}(N) = \mathrm{Var}(X_A) + \mathrm{Var}(X_B) + \mathrm{Var}(X_C) = \frac{239}{400}$.

2. In the case that the three events are pairwise disjoint, it holds that for all $I_1 \neq I_2$

$$\begin{aligned} \mathrm{Cov}(X_{I_1}, X_{I_2}) &= \mathbb{E}(X_{I_1} \cdot X_{I_2}) - \mathbb{E}(X_{I_1}) \cdot \mathbb{E}(X_{I_2}) \\ &= \mathbb{P}(X_{I_1} = 1, X_{I_2} = 1) - \mathbb{P}(X_{I_1} = 1) \cdot \mathbb{P}(X_{I_2} = 1) \\ &= \mathbb{P}(I_1 \cap I_2) - \mathbb{P}(I_1) \cdot \mathbb{P}(I_2) \\ &= \mathbb{P}(\emptyset) - \mathbb{P}(I_1) \cdot \mathbb{P}(I_2) \\ &= -\mathbb{P}(I_1) \cdot \mathbb{P}(I_2). \end{aligned}$$

   Therefore

$$\mathrm{Var}(N) = \frac{239}{400} - \frac{11}{40} = \frac{129}{400}.$$

3. If $C \subseteq B \subseteq A$ then for all $I_1 \subseteq I_2 \in \mathcal{I}$ it holds that

$$\begin{aligned} \mathrm{Cov}(X_{I_1}, X_{I_2}) &= \mathbb{E}(X_{I_1} \cdot X_{I_2}) - \mathbb{E}(X_{I_1}) \cdot \mathbb{E}(X_{I_2}) \\ &= \mathbb{P}(X_{I_1} = 1, X_{I_2} = 1) - \mathbb{P}(X_{I_1} = 1) \cdot \mathbb{P}(X_{I_2} = 1) \\ &= \mathbb{P}(I_1 \cap I_2) - \mathbb{P}(I_1) \cdot \mathbb{P}(I_2) \\ &= \mathbb{P}(I_1) - \mathbb{P}(I_1) \cdot \mathbb{P}(I_2) \\ &= \mathbb{P}(I_1) \cdot (1 - \mathbb{P}(I_2)). \end{aligned}$$

   Therefore

$$\mathrm{Var}(N) = \frac{239}{400} + \frac{3}{8} = \frac{389}{400}.$$

Assignment 7

Note: This assignment is not for submission, it is for your enjoyment only!

Exercise 1 Let $X \sim \text{Bin}(n, p)$, for some $n \in \mathbb{N}$ and $p \in [0, 1]$. Prove that for every t satisfying $0 \leq t < n$, it holds that:

$$\mathbb{P}(X > t) \geq \frac{np - t}{n - t}.$$

Exercise 2 A computer samples independently, uniformly, and with replacement 100 natural numbers from the set $\{1, 2, \ldots, 100\}$. Let $\bar{X}$ denote their average. Prove, using Chebyshev's inequality, that

$$\mathbb{P}\left(45.5 < \bar{X} < 55.5\right) > 2/3.$$

Exercise 3 Show that it is enough to toss a coin $\frac{1}{4(1-q)\alpha^2}$ times to determine its probability for heads within α of its true value, with probability at least q. That is, give a way to determine a value p', using only n tosses, such that its distance from the probability that the coin would land on heads, is less than α with probability at least q.

Exercise 4 Alice wants to send a message (i.e., a binary string) to Bob. However, the communication channel she uses is noisy and so, every bit she sends is flipped (i.e., changed from 0 to 1 or from 1 to 0) with some probability $0 < p < 1/2$, independently of all other bits. in order to overcome this difficulty, Alice and Bob agree that she will send him each bit k times, where k is some odd integer known to both of them. Once Bob receives the message, he divides it into blocks, each of size k, and decode it into the bit that appears more frequently in that block, that is, he takes the majority of the bits in each block. Prove that the probability that Bob fails to fully recover the message tends to zero as k tends to infinity.

Exercise 5 A biased coin is being tossed indefinitely, each toss is independent of the other. Let $0 < p < 1$ be the probability it lands on heads, and let $q = 1 - p$ be the probability it lands on tails. Define the sequence of random variables $\{X_n\}_{n=0}^{\infty}$ as follows: X_0 is the constant random variable 0, and for $n \geq 1$ let

$$X_n = \begin{cases} X_{n-1} + 1 & \text{in the } n\text{-th toss, the coin landed on heads} \\ X_{n-1} - 1 & \text{in the } n\text{-th toss, the coin landed on tails} \end{cases}$$

Next, define $Y_n = (q/p)^{X_n}$ for all $n \geq 0$. Compute $\mathbb{E}(Y_n)$.

Exercise 6 The number of people that enter a store is a non-negative random variable N satisfying $\mathbb{E}(N) = \mu$ and $\mathrm{Var}(N) = \sigma^2$. Each person either buys a shirt with probability 1/2 or a pair of pants with probability 1/2, independent of the other people in the store, and independently of N. Let X be the number of shirts that were bought and let Y be the number of pairs of pants that were bought.

1. Compute $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.
2. Compute $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.
3. Compute $\rho(X, Y)$.

Assignment 7
Solutions

Exercise 1 Let $X \sim \mathrm{Bin}(n,p)$, for some $n \in \mathbb{N}$ and $p \in [0,1]$. Prove that for every t satisfying $0 \le t < n$, it holds that:

$$\mathbb{P}(X > t) \ge \frac{np-t}{n-t}.$$

Solution

Let t be such that $0 \le t < n$. We make the following observation:

$$\mathbb{P}(X > t) = \mathbb{P}(-X < -t) = \mathbb{P}(n - X < n - t) = 1 - \mathbb{P}(n - X \ge n - t).$$

Since $X \sim \mathrm{Bin}(n,p)$, it follows that $n-X$ is a non-negative random variable, and that $\mathbb{E}(X) = np$. Therefore we can apply Markov's inequality on the random variable $n - X$, hence

$$\mathbb{P}(n - X \ge n - t) \le \frac{\mathbb{E}(n-X)}{n-t} = \frac{n-np}{n-t}.$$

Applying that to the above equation to get

$$\mathbb{P}(X > t) = 1 - \mathbb{P}(n - X \ge n - t) \ge 1 - \frac{n-np}{n-t} = \frac{np-t}{n-t}.$$

Exercise 2 A computer samples independently, uniformly, and with replacement 100 natural numbers from the set $\{1, 2, \dots, 100\}$. Let $\bar{X}$ denote their average. Prove, using Chebyshev's inequality, that

$$\mathbb{P}\left(45.5 < \bar{X} < 55.5\right) > 2/3.$$

Solution

For every $1 \le i \le 100$ let X_i be uniform on $[1, 100]$. Then $\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i$. First observe that for every $1 \le i \le 100$ it holds that $\mathbb{E}(X_i) = 50.5$ and $\mathrm{Var}(X_i) = \frac{100^2-1}{12} = \frac{3333}{4}$. Then by linearity of expectation we get that $\mathbb{E}\left(\bar{X}\right) = 50.5$, and since the random variables are mutually independent it holds that

$$\mathrm{Var}\left(\bar{X}\right) = \mathrm{Var}\left(\frac{1}{100} \cdot \sum_{i=1}^{100} X_i\right) = \frac{1}{100^2} \cdot \sum_{i=1}^{100} \mathrm{Var}(X_i) = \frac{3333}{400}.$$

Therefore

$$\mathbb{P}\left(45.5 < \bar{X} < 55.5\right) = 1 - \mathbb{P}\left(\left|\bar{X} - 50.5\right| \geq 5\right) \geq 1 - \frac{\mathrm{Var}\left(\bar{X}\right)}{25} = \frac{6667}{10000} > \frac{2}{3},$$

where the first inequality is by Chebyshev's inequality.

Exercise 3 Show that it is enough to toss a coin $\frac{1}{4(1-q)\alpha^2}$ times to determine its probability for heads within α of its true value, with probability at least q. That is, give a way to determine a value p', using only n tosses, such that its distance from the probability that the coin would land on heads, is less than α with probability at least q.

Solution

Let n be the number of tosses required to achieve the above. Let p be the probability to get heads, let $X_i \sim Ber(p)$ for $1 \leq i \leq n$ be mutually independent, and let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. We show that $\bar{X}$ satisfies the requirements for every $n \geq \frac{1}{4(1-q)\alpha^2}$, that is we show that

$$\mathbb{P}\left(\left|\bar{X} - p\right| < \alpha\right) \geq q,$$

which is equivalent to

$$\mathbb{P}\left(\left|\bar{X} - p\right| \geq \alpha\right) \leq 1 - q.$$

Since $X_i \sim Ber(p)$ for all $1 \leq i \leq n$, and they are all mutually independent, it follows that $\sum_{i=1}^{n} X_i \sim Bin(n,p)$, therefore $\mathbb{E}\left(\bar{X}\right) = p$ and $\mathrm{Var}\left(\bar{X}\right) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$. By Chebyshev's inequality we get that

$$\begin{aligned}\mathbb{P}\left(\left|\bar{X} - p\right| \geq \alpha\right) &\leq \frac{\mathrm{Var}\left(\bar{X}\right)}{\alpha^2} \\ &= \frac{p(1-p)}{n\alpha^2}\end{aligned}$$

Letting $\frac{p(1-p)}{n\alpha^2} \leq 1 - q$, we get that $n \geq \frac{p(1-p)}{(1-q)\alpha^2}$. Since the numerator is maximized at $p = 1/2$, it follows that $n \geq \frac{1}{4(1-q)\alpha^2}$.

Exercise 4 Alice wants to send a message (i.e., a binary string) to Bob. However, the communication channel she uses is noisy and so, every bit she sends is flipped (i.e., changed from 0 to 1 or from 1 to 0) with some probability $0 < p < 1/2$, independently of all other bits. in order to overcome this difficulty, Alice and Bob agree that she will send him each bit k times, where k is some odd integer known to both of them. Once Bob receives the message, he divides it into blocks, each of size k, and decode it into the bit that appears more frequently in that block, that is, he takes the majority of the bits in each block. Prove that the probability that Bob fails to fully recover the message tends to zero as k tends to infinity.

Solution

An error occurs when at least $\lceil k/2 \rceil$ bits have been altered due to the noise incurred by the channel. The number of positions of the block of length k where errors occur is a random variable that follows the binomial distribution with parameters k and p. In other words, it is the sum of k Bernoulli variables each having probability of success p (âĂIJsuccessâĂİ here means âĂIJan error has occurredâĂİ). Denote the number of errors by $E_{k,p}$, then

$$\mathbb{P}\left(E_{k,p} \geq k/2\right) = \mathbb{P}\left(E_{k,p} - kp \geq k/2 - kp\right) \leq \mathbb{P}\left(|E_{k,p} - kp| \geq k(1/2 - p)\right).$$

Since $E_{k,p} \sim \text{Bin}\left(k, p\right)$, it follows that $\mathbb{E}\left(E_{k,p}\right) = kp$ and $\text{Var}\left(E_{k,p}\right) = kp(1-p)$. Since $p < 1/2$, we can apply Chebyshev's inequality and obtain

$$\mathbb{P}\left(|E_{k,p} - kp| \geq k(1/2 - p)\right) \leq \frac{\text{Var}\left(E_{k,p}\right)}{k^2(1/2-p)^2} = \frac{kp(1-p)}{k^2(1-p)^2} \leq \frac{1}{k} \cdot \frac{p}{(1/2-p)^2},$$

which tends to 0 as k tends to infinity.

Exercise 5 A biased coin is being tossed indefinitely, each toss is independent of the other. Let $0 < p < 1$ be the probability it lands on heads, and let $q = 1 - p$ be the probability it lands on tails. Define the sequence of random variables $\{X_n\}_{n=0}^{\infty}$ as follows: X_0 is the constant random variable 0, and for $n \geq 1$ let

$$X_n = \begin{cases} X_{n-1} + 1 & \text{in the } n\text{-th toss, the coin landed on heads} \\ X_{n-1} - 1 & \text{in the } n\text{-th toss, the coin landed on tails} \end{cases}$$

Next, define $Y_n = (q/p)^{X_n}$ for all $n \geq 0$. Compute $\mathbb{E}\left(Y_n\right)$.

Solution

Observe that for all $n \geq 0$ it holds that

$$\mathbb{E}\left(Y_{n+1} \mid X_n\right) = \mathbb{E}\left((q/p)^{X_{n+1}} \mid X_n\right) = p(q/p)^{X_n+1} + q(q/p)^{X_n-1} = (q/p)^{X_n} \cdot (p \cdot (q/p) + q \cdot (p/q)) = (q/p)^{X_n}.$$

Therefore, by the law of total expectation it follows that

$$\mathbb{E}\left(Y_{n+1}\right) = \mathbb{E}\left(\mathbb{E}\left(Y_{n+1} \mid X_n\right)\right) = \mathbb{E}\left((q/p)^{X_n}\right) = \mathbb{E}\left(Y_n\right),$$

which, by an inductive argument, implies that $\mathbb{E}\left(Y_{n+1}\right) = \mathbb{E}\left(Y_0\right) = \mathbb{E}\left((q/p)^{X_0}\right) = 1$.

Exercise 6 The number of people that enter a store is a non-negative random variable N satisfying $\mathbb{E}\left(N\right) = \mu$ and $\text{Var}\left(N\right) = \sigma^2$. Each person either buys a shirt with probability 1/2 or a pair of pants with probability 1/2, independent of the other people in the store, and independently of N. Let X be the number of shirts that were bought and let Y be the number of pairs of pants that were bought.

1. Compute $\mathbb{E}\left(X\right)$ and $\mathbb{E}\left(Y\right)$.
2. Compute $\text{Var}\left(X\right)$ and $\text{Var}\left(Y\right)$.
3. Compute $\rho\left(X, Y\right)$.

Solution

First observe that $X + Y = N$ and that for all $n \geq 0$ it holds that $(X \mid N = n), (Y \mid N = n) \sim \text{Bin}\left(n, 1/2\right)$.

1. By the law of total expectation

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}(\mathbb{E}(X \mid N)) \\
&= \sum_{n=0}^{\infty} \mathbb{E}(X \mid N = n) \cdot \mathbb{P}(N = n) \\
&= \sum_{n=0}^{\infty} n/2 \cdot \mathbb{P}(N = n) \\
&= \frac{1}{2} \cdot \mathbb{E}(N) \\
&= \frac{\mu}{2}.
\end{aligned}$$

   Since X and Y are identically distributed it follows that $\mathbb{E}(Y) = \mu/2$ as well.

2. By the law of total variance and similar computation to the previous part of the exercise, it holds that

$$\begin{aligned}
\operatorname{Var}(X) &= \mathbb{E}(\operatorname{Var}(X \mid N)) + \operatorname{Var}(\mathbb{E}(X \mid N)) \\
&= \mathbb{E}(N \cdot 1/2 \cdot (1 - 1/2)) + \operatorname{Var}(N \cdot 1/2) \\
&= \frac{1}{4}\mathbb{E}(N) + \frac{1}{4}\operatorname{Var}(N) \\
&= (\mu + \sigma^2)/4
\end{aligned}$$

   Since X and Y are identically distributed it follows that $\operatorname{Var}(Y) = (\mu + \sigma^2)/4$ as well.

3. Since

$$\rho(X, Y) = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X) \cdot \operatorname{Var}(Y)}},$$

   then $\rho(X, Y)$ is undefined when $\operatorname{Var}(X)$ or $\operatorname{Var}(Y) = 0$. This happens when $\mu = -\sigma^2$, and as both are non-negative, $\rho(X, Y)$ is if and only if $\mu = \sigma = 0$.

   Assume that $\mu \neq -\sigma^2$ (and in particular one of them is positive). In this case we can compute $\rho(X, Y)$. We start with computing $\operatorname{Cov}(X, Y)$. It holds that

$$\begin{aligned}
\operatorname{Cov}(X, Y) &= \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y) \\
&= \mathbb{E}(\mathbb{E}(XY \mid N)) - \mu^2/4 \\
&= \mathbb{E}(\mathbb{E}(X(N - X) \mid N)) - \mu^2/4 \\
&= \mathbb{E}(\mathbb{E}(XN \mid N)) - \mathbb{E}\left(\mathbb{E}\left(X^2 \mid N\right)\right) - \mu^2/4.
\end{aligned}$$

Observe that $\mathbb{E}\left(\mathbb{E}\left(X^2 \mid N\right)\right) = \mathbb{E}\left(X^2\right) = \operatorname{Var}\left(X\right) + \left(\mathbb{E}\left(X\right)\right)^2 = (\mu + \mu^2 + \sigma^2)/4$ and that

$$\begin{aligned}
\mathbb{E}\left(\mathbb{E}\left(XN \mid N\right)\right) &= \sum_{n=0}^{\infty} n\mathbb{E}\left(X \mid N = n\right) \cdot \mathbb{P}\left(N = n\right) \\
&= \sum_{n=0}^{\infty} n^2/2 \cdot \mathbb{P}\left(N = n\right) \\
&= \frac{1}{2} \cdot \mathbb{E}\left(N^2\right) \\
&= \frac{1}{2} \cdot \left(\operatorname{Var}\left(N\right) + \left(\mathbb{E}\left(N\right)\right)^2\right) \\
&= \frac{1}{2}(\sigma^2 + \mu^2).
\end{aligned}$$

Therefore

$$\operatorname{Cov}\left(X, Y\right) = \frac{1}{2}(\sigma^2 + \mu^2) - (\mu + \mu^2 + \sigma^2)/4 - \mu^2/4 = (\sigma^2 - \mu)/4,$$

which implies that

$$\rho\left(X, Y\right) = \frac{\operatorname{Cov}\left(X, Y\right)}{\sqrt{\operatorname{Var}\left(X\right) \cdot \operatorname{Var}\left(Y\right)}} = \frac{\operatorname{Cov}\left(X, Y\right)}{\operatorname{Var}\left(X\right)} = \frac{\sigma^2 - \mu}{\sigma^2 + \mu}.$$