Lecture 1

Intuitive notion	Mathematical notion
Experiment	Probability space
Possible outcomes	Sample space
Single outcome	Member of the sample space
Set of outcomes	Event
Random quantity	Random variable
Average	Expectation
Dispersion index	Standard deviation (Variance)

Table 1: correspondence between intuitive notions and mathematical notions

1 Probability Space

Definition 1.1 (Probability space)**.** *A* probability space *is a pair* $(\Omega, \mathbb{P})$*, where* Ω *is a set (in this course we will assume it is finite or countably infinite) called the* sample space*, and* $\mathbb{P} : \Omega \to [0, 1]$ *is a function, called the* probability function*, which satisfies*

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$$

Example 1:

1. *Rolling a fair die:*

$$\Omega = \{1, 2, 3, 4, 5, 6\},$$
$$\forall i \in \Omega \ \ \mathbb{P}(i) = \frac{1}{6}.$$

2. *Tossing an unfair coin:* the probability of its result being heads is $\frac{1}{3}$.

$$\Omega = \{h, t\},$$
$$\mathbb{P}(h) = \frac{1}{3},\ \mathbb{P}(t) = \frac{2}{3}.$$

3. *Tossing two unfair coins:* probability of $\frac{1}{3}$ for heads in each.

$$\Omega = \{(h,h), (h,t), (t,h), (t,t)\},$$
$$\mathbb{P}((h,h)) = \frac{1}{9},\ \mathbb{P}((h,t)) = \frac{2}{9},\ \mathbb{P}((t,h)) = \frac{2}{9},\ \mathbb{P}((t,t)) = \frac{4}{9}.$$

   Observe that $\frac{1}{9} + \frac{2}{9} + \frac{2}{9} + \frac{4}{9} = 1$.

4. *Tossing an unfair coin until the first time it comes up heads:* probability of $\frac{1}{3}$ for heads per toss.

$$\Omega = \{1, 2, 3, \ldots\},$$
$$\mathbb{P}(k) = \left(\frac{2}{3}\right)^{k-1} \cdot \frac{1}{3}.$$

   Observe that

$$\sum_{k\in\Omega} \mathbb{P}(k) = \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1} \cdot \frac{1}{3} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1.$$

5. *Sampling a point uniformly at random from a segment of real numbers:* we will consider the segment $[0, 1]$.

$$\Omega = [0, 1],$$
$$\forall \omega \in \Omega\ \ \mathbb{P}(\omega) = 0.$$

   This is problematic with our current definition requiring $\sum_{\omega\in\Omega} \mathbb{P}(\omega) = 1$. We will not deal with this problem in this course. Note that the same problem occurs if we wish to sample a rational number uniformly, even though the set of all rational numbers is countable.

1.1 Events

Definition 1.2 (Event). *An* event *is a subset of the sample space Ω. We will extend the definition of $\mathbb{P}$ for events in the following way: For an event $A \subseteq \Omega$ define*

$$\mathbb{P}(A) = \sum_{\omega\in A} \mathbb{P}(\omega).$$

Example 2:

1. *Tossing a fair die:* Let A be the event that the outcome of the die is even. Using the modeling of the probability space from the previous example, we get that

$$\begin{aligned} A &= \{2,4,6\}, \\ \mathbb{P}(A) &= \mathbb{P}(2)+\mathbb{P}(4)+\mathbb{P}(6) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{1}{2}. \end{aligned}$$

2. *Tossing two unfair coins:* probability of $\frac{1}{3}$ for heads in each. Let A be the event that the result of at least one of the coin tosses is heads. Using the modeling of the probability space from the previous example, we get that

$$\begin{aligned} &\Omega = \{(h,h),(h,t),(t,h),(t,t)\}, \\ &\mathbb{P}((h,h)) = \frac{1}{9},\ \mathbb{P}((h,t)) = \frac{2}{9},\ \mathbb{P}((t,h)) = \frac{2}{9},\ \mathbb{P}((t,t)) = \frac{4}{9}, \\ \\ &A = \{(h,h),(h,t),(t,h)\}, \\ &\mathbb{P}(A) = \mathbb{P}((h,h))+\mathbb{P}((h,t))+\mathbb{P}((t,h)) = \frac{1}{9}+\frac{2}{9}+\frac{2}{9} = \frac{5}{9}. \end{aligned}$$

Now, let B be the event that the result of the first coin toss is heads. Then

$$\begin{aligned} B &= \{(h,h),(h,t)\}, \\ \mathbb{P}(B) &= \mathbb{P}((h,h))+\mathbb{P}((h,t)) = \frac{1}{9}+\frac{2}{9} = \frac{1}{3}, \end{aligned}$$

which matches our intuition – this event is similar to tossing one coin which comes up heads. In the following lectures we will explain why this intuition is correct.

1.2 Basic Properties Of Probability Spaces

Lemma 1.3. *Let $(\Omega,\mathbb{P})$ be a probability space. Then*

$$\begin{aligned} \mathbb{P}(\emptyset) &= 0, \\ \mathbb{P}(\Omega) &= 1. \end{aligned}$$

Proof. Follows immediately from the definition of an event, and from the fact that

$$\sum_{\omega\in\Omega} \mathbb{P}(\omega) = 1.$$

□

Definition 1.4. *For an event A define its complementary event A^c by $A^c = \Omega \setminus A$.*

Lemma 1.5. *Let $(\Omega, \mathbb{P})$ be a probability space. Then for all events A*

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

Proof.

$$\mathbb{P}(A^c) = \sum_{\omega \in A^c} \mathbb{P}(\omega) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) - \sum_{\omega \in A} \mathbb{P}(\omega) = 1 - \mathbb{P}(A).$$

□

Lemma 1.6 (Monotonicity)**.** *Let $(\Omega, \mathbb{P})$ be a probability space. If $A \subseteq B$ are events, then*

$$\mathbb{P}(A) \leq \mathbb{P}(B).$$

Proof.

$$\mathbb{P}(B) = \sum_{\omega \in B} \mathbb{P}(\omega) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in B \setminus A} \mathbb{P}(\omega) \geq \sum_{\omega \in A} \mathbb{P}(\omega) = \mathbb{P}(A),$$

where the inequality is due to the fact that $\mathbb{P}$ is a non-negative function, i.e., $\mathbb{P}(\omega) \geq 0$ for every $\omega \in \Omega$. □

Lemma 1.7. *Let $(\Omega, \mathbb{P})$ be a probability space, and let A and B be disjoint events. Then*

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

More generally, if $A_1, A_2, \ldots$ is an infinite sequence of pairwise disjoint events (i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$), then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Proof. We start with proving the statement for two events. It holds that

$$\mathbb{P}(A \cup B) = \sum_{\omega \in A \cup B} \mathbb{P}(\omega) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in B} \mathbb{P}(\omega) = \mathbb{P}(A) + \mathbb{P}(B),$$

where the second equality is due to the fact that A and B are disjoint. The proof of the general statement follows similar lines:

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{\omega \in \bigcup_{n=1}^{\infty} A_n} \mathbb{P}(\omega) = \sum_{n=1}^{\infty} \sum_{\omega \in A_n} \mathbb{P}(\omega) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

□

Lemma 1.8. *Let $(\Omega, \mathbb{P})$ be a probability space, and let A and B be events. Then*

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

first proof of Lemma 1.8. We will prove the equivalent statement $\mathbb{P}(A)+\mathbb{P}(B) = \mathbb{P}(A \cup B)+\mathbb{P}(A \cap B)$. Indeed

$$\begin{aligned}
\mathbb{P}(A) + \mathbb{P}(B) &= \mathbb{P}((A \setminus B) \cup (A \cap B)) + \mathbb{P}((B \setminus A) \cup (A \cap B)) \\
&= \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B) \\
&= \mathbb{P}((A \setminus B) \cup (A \cap B) \cup (B \setminus A)) + \mathbb{P}(A \cap B) \\
&= \mathbb{P}(A \cup B) + \mathbb{P}(A \cap B),
\end{aligned}$$

where the second and third equalities hold by Lemma 1.7. □

second proof of Lemma 1.8.

$$\mathbb{P}(A \cup B) = \sum_{\omega \in A \cup B} \mathbb{P}(\omega) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in B} \mathbb{P}(\omega) - \sum_{\omega \in A \cap B} \mathbb{P}(\omega) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

□

Lemma 1.9 (Union Bound)**.** *Let $(\Omega, \mathbb{P})$ be a probability space, and let A and B be events. Then*

$$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B).$$

More generally, if $A_1, A_2, \ldots$ is an infinite sequence of events, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Proof. We will prove the general case. It holds that

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{\omega \in \bigcup_{n=1}^{\infty} A_n} \mathbb{P}(\omega) \leq \sum_{n=1}^{\infty} \sum_{\omega \in A_n} \mathbb{P}(\omega) = \sum_{n=1}^{\infty} \mathbb{P}(A_n),$$

where the inequality is due to the fact that every term $\mathbb{P}(\omega)$ in the sum on the left-hand side appears at least once in the sum on the right-hand side, and $\mathbb{P}$ is a non-negative function. □