Probability Theory 1 – Proposed solution of moed bet exam 2019

1. **(a)** It is evident that the support of the distribution of Y is $\{1,2,3\}$ and that the support of the distribution of Z is $\{1,2,3\}$. For every $1 \leq i,j \leq 3$, the table below shows the value of $P(Y=i, Z=j)$. The actual calculations can be found below the table.

	$Z=1$	$Z=2$	$Z=3$
$Y=1$	0	0	6/36
$Y=2$	2/36	2/36	1/36
$Y=3$	10/36	10/36	5/36

If $Y=1$, then the result of the first die roll is 6. This event occurs with probability 1/6 and immediately implies that $Z=3$. Therefore, $P(Y=1, Z=1) = P(Y=1, Z=2) = 0$ and $P(Y=1, Z=3) = 1/6$.
If $Y=2$, then the result of the first die roll is not 6 and the result of the second die roll is 6. Since the die rolls are independent, this event occurs with probability $(1-1/6)\cdot 1/6 = 5/36$. Moreover $P(Z=1|Y=2)$ is the probability that the result of the first die roll is 1 or 2, given that it is not 6. Therefore $P(Z=1|Y=2) = 2/5$, implying that

$$P(Z=1, Y=2) = P(Y=2)P(Z=1|Y=2) = 5/36 \cdot 2/5 = 2/36.$$

Similarly, $P(Z=2, Y=2) = 2/36$ and $P(Z=3, Y=2) = 1/36$.
Finally, $P(Y=3) = 1 - P(Y=1) - P(Y=2) = 1 - 1/6 - 5/36 = 25/36$ (an alternative way to see this is to observe that $Y=3$ if and only if the results of the first two die rolls are not 6, which happens with probability $(1-1/6)^2 = 25/36$). Moreover, for every $1 \leq i \leq 3$, it holds that $P(Z=i|Y=3) = P(Z=i|Y=2)$ (as Z depends only on the first die roll). We conclude that $P(Z=1, Y=3) = 25/36 \cdot 2/5 = 10/36$, $P(Z=2, Y=3) = 10/36$, and $P(Z=3, Y=3) = 5/36$.

(b) As shown in intermediate calculations in part (a) of this question, $P(Y=1) = 1/6$, $P(Y=2) = 5/36$, and $P(Y=3) = 25/36$. Moreover, using the table from part (a) of this question we see that

$$P(Z=1) = P(Y=1, Z=1) + P(Y=2, Z=1) + P(Y=3, Z=1) = 0 + 2/36 + 10/36 = 1/3,$$

$$P(Z=2) = 0 + 2/36 + 10/36 = 1/3,$$

and

$$P(Z=3) = 6/36 + 1/36 + 5/36 = 1/3.$$

It then follows by the linearity of expectation that

$$\mathbb{E}(Y+Z) = \mathbb{E}(Y) + \mathbb{E}(Z) = (1\cdot 1/6 + 2\cdot 5/36 + 3\cdot 25/36) + (1\cdot 1/3 + 2\cdot 1/3 + 3\cdot 1/3) = 163/36.$$

(c) Using the table from part (a) of this question we see that

$$P(Z > Y) = P(Z=3, Y=1) + P(Z=3, Y=2) + P(Z=2, Y=1) = 6/36 + 1/36 + 0 = 7/36.$$

2. **(a)** This statement is true. Indeed

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega)P(\omega) \leq \sum_{\omega \in \Omega} Y(\omega)P(\omega) = \mathbb{E}(Y),$$

where the inequality holds by our assumption that $X(\omega) \leq Y(\omega)$ for every $\omega \in \Omega$.

(b) This statement is false. Consider the probability space (Ω, P), where $\Omega = \{1, 2\}$ and $P(1) = P(2) = 1/2$. Let $X : \Omega \to \mathbb{R}$ be the random variable satisfying $X(1) = 1$ and $X(2) = 2$, and let $Y : \Omega \to \mathbb{R}$ be the random variable satisfying $Y(1) = Y(2) = 10$. Clearly, $X(i) \leq Y(i)$ for every $i \in \{1, 2\}$. However, we will show that $Var(X) > Var(Y)$. Indeed, it was proved in the lectures that $Var(Z) \geq 0$ holds for every random variable Z and, moreover, $Var(Z) = 0$ if and only if $P(Z = \mathbb{E}(Z)) = 1$. Observe that $\mathbb{E}(X) = 1 \cdot 1/2 + 2 \cdot 1/2 = 3/2$ and thus $P(X = \mathbb{E}(X)) = 0 < 1$. It follows that $Var(X) > 0$. On the other hand, $\mathbb{E}(Y) = 10 \cdot 1/2 + 10 \cdot 1/2 = 10$ and thus $P(Y = \mathbb{E}(Y)) = 1$. It follows that $Var(Y) = 0$.

(c) This statement is false. Consider the probability space (Ω, P), where $\Omega = \{1, 2\}$ and $P(1) = P(2) = 1/2$. Let $X : \Omega \to \mathbb{R}$ be the random variable satisfying $X(1) = X(2) = -2$, and let $Y : \Omega \to \mathbb{R}$ be the random variable satisfying $Y(1) = Y(2) = -1$. Clearly, $X(i) \leq Y(i)$ for every $i \in \{1, 2\}$. However

$$\mathbb{E}(|X|) = 2 \cdot 1/2 + 2 \cdot 1/2 = 2 > 1 = 1 \cdot 1/2 + 1 \cdot 1/2 = \mathbb{E}(|Y|).$$

3. **(a)** Given that $X = 3$, we toss a fair coin 3 times, all coin tosses being mutually independent. Hence, $(Y|X = 3) \sim Bin(3, 1/2)$. In particular, $P(Y = 1|X = 3) = \binom{3}{1}(1/2)^3 = 3/8$.

(b) Similarly to part (a), for every $n \in \{1, \ldots, 100\}$ it holds that $(Y|X = n) \sim Bin(n, 1/2)$; in particular, $\mathbb{E}(Y|X = n) = n/2$. It thus follows by the law of total expectation that

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \sum_{n=1}^{100} \mathbb{E}(Y|X = n) \cdot P(X = n) = \frac{1}{100} \cdot \frac{1}{2} \cdot \sum_{n=1}^{100} n = \frac{101}{4}.$$

(c) By definition, Y is a non-negative random variable. Hence, we can apply Markov's inequality to obtain

$$P(Y \geq 60) \leq \frac{\mathbb{E}(Y)}{60} = \frac{101}{240} < \frac{1}{2}.$$

4. **(a)** For every $1 \leq i \leq n - 1$, there are 36 possible values for the ordered pair (X_i, X_{i+1}), of which exactly 5 sum to 8 (namely, $(2, 6)$, $(3, 5)$, $(4, 4)$, $(5, 3)$, $(6, 2)$). Since the die is fair and the die rolls are independent, all those possibilities have the same probability. Hence

$$\mathbb{E}(Y_i) = P(Y_i = 1) = P(X_i + X_{i+1} = 8) = 5/36.$$

It then follows by the linearity of expectation that

$$\mathbb{E}(Y) = \sum_{i=1}^{n-1} \mathbb{E}(Y_i) = 5(n - 1)/36.$$

(b) Recall that

$$Var(Y) = Var\left(\sum_{i=1}^{n-1} Y_i\right) = \sum_{i=1}^{n-1} Var(Y_i) + 2\sum_{1\le i<j\le n-1} Cov(Y_i, Y_j).$$

Note first, that for every $1 \le i \le n-1$, it holds that

$$Var(Y_i) = \mathbb{E}\left(Y_i^2\right) - (\mathbb{E}(Y_i))^2 = \mathbb{E}(Y_i) - (\mathbb{E}(Y_i))^2 = \frac{5}{36} - \frac{5^2}{6^4} = \frac{155}{6^4}.$$

Now, if $j > i+1$, then Y_i and Y_j are independent as they rely of disjoint pairs of independent die rolls. In particular, $Cov(Y_i, Y_j) = 0$ holds in this case. On the other hand, for every $1 \le i \le n-2$, it holds that

$$Cov(Y_i, Y_{i+1}) = \mathbb{E}(Y_i \cdot Y_{i+1}) - \mathbb{E}(Y_i)\cdot\mathbb{E}(Y_{i+1}) = P(Y_i = 1, Y_{i+1} = 1) - \frac{5^2}{6^4} = \frac{5}{6^3} - \frac{5^2}{6^4} = \frac{5}{6^4},$$

where the third equality holds since if $X_i + X_{i+1} = 8$ and $X_{i+1} + X_{i+2} = 8$, then $(X_i, X_{i+1}, X_{i+2}) \in \{(6,2,6), (5,3,5), (4,4,4), (3,5,3), (2,6,2)\}$. We conclude that

$$Var(Y) = (n-1)\frac{155}{6^4} + 2(n-2)\frac{5}{6^4} = \frac{165n}{6^4} - \frac{175}{6^4}.$$

(c) Fix any integer $n \ge 2$. Then

$$P(Y \ge n/6) \le P(|Y - \mathbb{E}(Y)| \ge n/36) \le \frac{Var(Y)}{(n/36)^2} \le \frac{165n/6^4}{n^2/6^4} = \frac{165}{n},$$

where the first inequality holds by part (a) of this question, the second inequality holds by Chebyshev's inequality, and the third inequality holds by part (b) of this question. We conclude that

$$1 \ge \lim_{n\to\infty} P(Y < n/6) = 1 - \lim_{n\to\infty} P(Y \ge n/6) \ge 1 - \lim_{n\to\infty} 165/n = 1$$

and thus $\lim_{n\to\infty} P(Y < n/6) = 1$ as required.