Probability Theory 1 – Proposed solution of moed bet exam 2021

1. **(a)** It is evident that the support of the distribution of X is $\{0,1,2,3\}$ and the support of the distribution of Y is $\{0,1,2\}$. Moreover, $X+Y \leq 3$, implying that

$$\mathbb{P}(X=3, Y=2) = \mathbb{P}(X=3, Y=1) = \mathbb{P}(X=2, Y=2) = 0.$$

For any other ordered pair $(i,j) \in \{0,1,2,3\} \times \{0,1,2\}$ we have

$$\mathbb{P}(X=i, Y=j) = \frac{\binom{5}{i}\binom{2}{j}\binom{3}{3-i-j}}{\binom{10}{3}}.$$

The results are summarised in the table below.

	$X=0$	$X=1$	$X=2$	$X=3$
$Y=0$	$\frac{\binom{5}{0}\binom{2}{0}\binom{3}{3}}{\binom{10}{3}} = \frac{1}{120}$	$\frac{\binom{5}{1}\binom{2}{0}\binom{3}{2}}{\binom{10}{3}} = \frac{15}{120}$	$\frac{\binom{5}{2}\binom{2}{0}\binom{3}{1}}{\binom{10}{3}} = \frac{30}{120}$	$\frac{\binom{5}{3}\binom{2}{0}\binom{3}{0}}{\binom{10}{3}} = \frac{10}{120}$
$Y=1$	$\frac{\binom{5}{0}\binom{2}{1}\binom{3}{2}}{\binom{10}{3}} = \frac{6}{120}$	$\frac{\binom{5}{1}\binom{2}{1}\binom{3}{1}}{\binom{10}{3}} = \frac{30}{120}$	$\frac{\binom{5}{2}\binom{2}{1}\binom{3}{0}}{\binom{10}{3}} = \frac{20}{120}$	0
$Y=2$	$\frac{\binom{5}{0}\binom{2}{2}\binom{3}{1}}{\binom{10}{3}} = \frac{3}{120}$	$\frac{\binom{5}{1}\binom{2}{2}\binom{3}{0}}{\binom{10}{3}} = \frac{5}{120}$	0	0

(b) Using the table from (a) we conclude that

$$\begin{aligned}\mathbb{P}(X=0) &= \mathbb{P}(X=0, Y=0) + \mathbb{P}(X=0, Y=1) + \mathbb{P}(X=0, Y=2)\\ &= \frac{1}{120} + \frac{6}{120} + \frac{3}{120} = \frac{10}{120};\end{aligned}$$

$$\begin{aligned}\mathbb{P}(X=1) &= \mathbb{P}(X=1, Y=0) + \mathbb{P}(X=1, Y=1) + \mathbb{P}(X=1, Y=2)\\ &= \frac{15}{120} + \frac{30}{120} + \frac{5}{120} = \frac{50}{120};\end{aligned}$$

$$\begin{aligned}\mathbb{P}(X=2) &= \mathbb{P}(X=2, Y=0) + \mathbb{P}(X=2, Y=1) + \mathbb{P}(X=2, Y=2)\\ &= \frac{30}{120} + \frac{20}{120} + 0 = \frac{50}{120};\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(X=3) &= \mathbb{P}(X=3, Y=0) + \mathbb{P}(X=3, Y=1) + \mathbb{P}(X=3, Y=2)\\ &= \frac{10}{120} + 0 + 0 = \frac{10}{120}.\end{aligned}$$

Similarly

$$\begin{aligned}\mathbb{P}(Y=0) &= \mathbb{P}(X=0, Y=0) + \mathbb{P}(X=1, Y=0) + \mathbb{P}(X=2, Y=0) + \mathbb{P}(X=3, Y=2)\\ &= \frac{1}{120} + \frac{15}{120} + \frac{30}{120} + \frac{10}{120} = \frac{56}{120};\end{aligned}$$

$$\begin{aligned}\mathbb{P}(Y=1) &= \mathbb{P}(X=0,Y=1)+\mathbb{P}(X=1,Y=1)+\mathbb{P}(X=2,Y=1)+\mathbb{P}(X=3,Y=1)\\ &= \frac{6}{120}+\frac{30}{120}+\frac{20}{120}+0=\frac{56}{120};\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(Y=2) &= \mathbb{P}(X=0,Y=2)+\mathbb{P}(X=1,Y=2)+\mathbb{P}(X=2,Y=2)+\mathbb{P}(X=3,Y=2)\\ &= \frac{3}{120}+\frac{5}{120}+0+0=\frac{8}{120}.\end{aligned}$$

(c) One can calculate $\mathbb{E}(Z)$ by first calculating the distribution of Z. However, it is a bit shorter to use Part (b) of this question and to observe that $Z = 3 - X - Y$. By Part (b)

$$\mathbb{E}(X) = 0\cdot\frac{10}{120}+1\cdot\frac{50}{120}+2\cdot\frac{50}{120}+3\cdot\frac{10}{120}=3/2$$

and

$$\mathbb{E}(Y) = 0\cdot\frac{56}{120}+1\cdot\frac{56}{120}+2\cdot\frac{8}{120}=3/5.$$

Using the linearity of expectation we conclude that

$$\mathbb{E}(Z)=\mathbb{E}(3-X-Y)=3-\mathbb{E}(X)-\mathbb{E}(Y)=3-3/2-3/5=9/10.$$

2. **(a)** It is evident that the support of the distribution of Y is $\{0,1,2\}$. Since $X \sim \mathit{Geom}(1/2)$, it follows that

$$\begin{aligned}\mathbb{P}(Y=0) &= \sum_{k=1}^{\infty}\mathbb{P}(X=3k)=\sum_{k=1}^{\infty}1/2\cdot(1-1/2)^{3k-1}=\sum_{k=1}^{\infty}(1/2)^{3k}=\sum_{k=1}^{\infty}(1/8)^k\\ &= \frac{1/8}{1-1/8}=1/7.\end{aligned}$$

Similarly

$$\begin{aligned}\mathbb{P}(Y=1) &= \sum_{k=0}^{\infty}\mathbb{P}(X=3k+1)=\sum_{k=0}^{\infty}1/2\cdot(1-1/2)^{3k}=1/2\cdot\sum_{k=0}^{\infty}(1/2)^{3k}=1/2\cdot\sum_{k=0}^{\infty}(1/8)^k\\ &= \frac{1}{2}\cdot\frac{1}{1-1/8}=4/7\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(Y=2) &= \sum_{k=0}^{\infty}\mathbb{P}(X=3k+2)=\sum_{k=0}^{\infty}1/2\cdot(1-1/2)^{3k+1}=1/4\cdot\sum_{k=0}^{\infty}(1/2)^{3k}=1/4\cdot\sum_{k=0}^{\infty}(1/8)^k\\ &= \frac{1}{4}\cdot\frac{1}{1-1/8}=2/7.\end{aligned}$$

(b) We first calculate the probability of the complementary event. Since $X \geq 1$ and $Y \leq 2$, it is evident that

$$\mathbb{P}(Y \geq X) = \mathbb{P}(Y = 1, X = 1) + \mathbb{P}(Y = 2, X = 1) + \mathbb{P}(Y = 2, X = 2).$$

Since $Y = X \mod 3$, it is evident that $\mathbb{P}(Y = 2, X = 1) = 0$. Moreover

$$\mathbb{P}(Y = 1, X = 1) = \mathbb{P}(X = 1) = 1/2$$

and

$$\mathbb{P}(Y = 2, X = 2) = \mathbb{P}(X = 2) = 1/2 \cdot (1 - 1/2) = 1/4.$$

Therefore $\mathbb{P}(Y \geq X) = 1/2 + 0 + 1/4 = 3/4$, which in turn implies that

$$\mathbb{P}(Y < X) = 1 - \mathbb{P}(Y \geq X) = 1/4.$$

(c)

$$\mathbb{P}(X \geq 10|Y = 0) = \frac{\mathbb{P}(X \geq 10, Y = 0)}{\mathbb{P}(Y = 0)} = 7\sum_{k=4}^{\infty} \mathbb{P}(X = 3k) = 7\sum_{k=4}^{\infty} 1/2 \cdot (1 - 1/2)^{3k-1}$$
$$= 7\sum_{k=4}^{\infty}(1/2)^{3k} = 7\sum_{k=4}^{\infty}(1/8)^k = 7\frac{(1/8)^4}{1 - 1/8} = 1/8^3,$$

where the second equality holds by Part (a).

3. **(a)** This statement is false. Let $X \sim Ber(1/2)$ and let $Y = 2X - 1$. Then $\mathbb{E}(X) = 1/2$ and by the linearity of expectation $\mathbb{E}(Y) = \mathbb{E}(2X - 1) = 2\mathbb{E}(X) - 1 = 2 \cdot 1/2 - 1 = 0$. In particular, $\mathbb{E}(X) \geq \mathbb{E}(Y)$. On the other hand, $X^2 = X$ and so $\mathbb{E}\left(X^2\right) = \mathbb{E}(X) = 1/2$, but by the linearity of expectation

$$\mathbb{E}\left(Y^2\right) = \mathbb{E}\left[(2X - 1)^2\right] = \mathbb{E}\left(4X^2 - 4X + 1\right) = 4\mathbb{E}\left(X^2\right) - 4\mathbb{E}(X) + 1 = 1.$$

Hence, $\mathbb{E}\left(X^2\right) < \mathbb{E}\left(Y^2\right)$.

(b) This statement is false. Since $Var(X - Y) > 0$ holds for any $X \neq Y$ (as the variance of any random variable is non-negative, and equals 0 if and only if the random variable is constant), any X and Y such that $Var(X) < Var(Y)$ (and so, in particular, $X \neq Y$) will form a counter example. Take $X \sim Ber(1/2)$ and $Y = 2X$. Then $Var(X) = 1/2(1 - 1/2) = 1/4$ and $Var(Y) = Var(2X) = 2^2 Var(X) = 1$.

(c) This statement is false. Let $X \sim U(\{1, 2, 12\})$ and let $Y \sim U(\{3, 5\})$. Then

$$\mathbb{E}(X) = (1 + 2 + 12)/3 = 5 > 4 = (3 + 5)/2 = \mathbb{E}(Y).$$

However

$$\mathbb{P}(X \geq Y) = \mathbb{P}(X = 12) = 1/3 < 1/2.$$

4. **(a)** For every $1 \leq i \leq n-1$, let X_i be the indicator random variable for the event that $|Y_{i+1} - Y_i| \geq 2$ and $Y_{i+1} \cdot Y_i$ is odd. It is evident that $X_i = 1$ if and only if $Y_i, Y_{i+1} \in \{1, 3, 5\}$ and $Y_{i+1} \neq Y_i$. Hence, there are precisely $3 \cdot 2 = 6$ possibilities for the pair (Y_i, Y_{i+1}) for which $X_i = 1$. It follows that $\mathbb{E}(X_i) = \mathbb{P}(X_i = 1) = 6/6^2 = 1/6$ for every $1 \leq i \leq n-1$. Since $X = \sum_{i=1}^{n-1} X_i$, it follows by the linearity of expectation that $\mathbb{E}(X) = \sum_{i=1}^{n-1} \mathbb{E}(X_i) = (n-1)/6$.

(b) For every $1 \leq i \leq n-1$ it holds that $Var(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = \mathbb{E}(X_i) - (\mathbb{E}(X_i))^2 = 1/6 - 1/36 = 5/36$, where the second equality holds since X_i is an indicator random variable and thus $X_i^2 = X_i$.

Next, fix some $1 \leq i < j \leq n-1$. If $j > i+1$, then $\{i, i+1\} \cap \{j, j+1\} = \emptyset$, implying that X_i and X_j are independent. In particular, $Cov(X_i, X_j) = 0$. Suppose then that $j = i+1$. We then have

$$\begin{aligned} Cov(X_i, X_j) &= \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = \mathbb{P}(X_i = 1, X_{i+1} = 1) - 1/36 \\ &= \frac{3 \cdot 2 \cdot 2}{6^3} - \frac{1}{36} = 2/36 - 1/36 = 1/36, \end{aligned}$$

where the third equality holds since $X_{i+1} = X_i = 1$ if and only if $Y_i, Y_{i+1}, Y_{i+2} \in \{1, 3, 5\}$, $Y_{i+1} \neq Y_i$, and $Y_{i+2} \neq Y_{i+1}$.

We conclude that

$$\begin{aligned} Var(X) &= \sum_{i=1}^{n-1} Var(X_i) + 2 \sum_{1 \leq i < j \leq n-1} Cov(X_i, X_j) = \sum_{i=1}^{n-1} Var(X_i) + 2 \sum_{i=1}^{n-2} Cov(X_i, X_{i+1}) \\ &= 5(n-1)/36 + 2(n-2)/36 = (7n-9)/36. \end{aligned}$$

(c) Note that

$$\begin{aligned} \mathbb{P}(X \geq n/3) &\leq \mathbb{P}(X \geq \mathbb{E}(X) + n/6) \leq \mathbb{P}(|X - \mathbb{E}(X)| \geq n/6) \\ &\leq \frac{Var(X)}{(n/6)^2} \leq 7/n, \end{aligned}$$

where in the first inequality we used the value of $\mathbb{E}(X)$ as calculated in Part (a), the third inequality is Chebyshev's inequality, and in the last inequality we used the value of $Var(X)$ as calculated in Part (b). We conclude that

$$\lim_{n \to \infty} \mathbb{P}(X \geq n/3) = 0$$

as claimed.