# Practical 2

Exercise 1   10 couples are seated uniformly at random around a round table. Calculate the probability that no couple is seated one next to the other?

Solution

For every $1 \leq i \leq 10$, let $A_i$ be the event that the members of the $i$-th couple are seated next to each other. Since the probability space is uniform, it follows that

$$\mathbb{P}\left(\bigcap_{i=1}^{10} A_i^c\right) = \frac{\left|\bigcap_{i=1}^{10} A_i^c\right|}{|\Omega|}.$$

Since the table is round, we can choose one person who will be seated in some specific seat, and then arrange the rest of the people uniformly at random. This is equivalent to arranging 19 people in a row, where in any arrangement in $\bigcap_{i=1}^{10} A_i^c$, one person (the one whose partner was already seated) cannot sit at either end of the row. Hence,

$$\left|\bigcap_{i \in \mathcal{I}} A_i\right| = 2^{|\mathcal{I}|} \cdot (19 - |\mathcal{I}|)!$$

For every $\mathcal{I} \subseteq \{1, 2, \ldots, 10\}$. It follows by inclusion-exclusion that

$$\begin{aligned}
\left|\bigcap_{i=1}^{10} A_i^c\right| &= 19! - \sum_{\emptyset \neq \mathcal{I} \subseteq \{1,2,\ldots,10\}} (-1)^{|\mathcal{I}|-1} \left|\bigcap_{i \in \mathcal{I}} A_i\right| \\
&= 19! + \sum_{\emptyset \neq \mathcal{I} \subseteq \{1,2,\ldots,10\}} (-1)^{|\mathcal{I}|} \cdot 2^{|\mathcal{I}|} \cdot (19 - |\mathcal{I}|)! \\
&= \sum_{k=0}^{10} \binom{10}{k} (-2)^k (19 - k)!.
\end{aligned}$$

We conclude that

$$\mathbb{P}\left(\bigcap_{i=1}^{10} A_i^c\right) = \frac{\sum_{k=0}^{10} \binom{10}{k} (-2)^k (19-k)!}{19!}.$$

Exercise 2   A bin contains $N$ balls, $B$ of which are black. $n$ balls are drawn uniformly at random from the bin without replacement. For every non-negative integer $k$ calculate the probability that exactly $k$ black balls were drawn.

Solution

For every non-negative integer $k$, let $A_k$ be the event that exactly $k$ black balls were drawn. We first observe that it cannot be the case that more black balls were drawn than there are in the bin, i.e., if $k > B$, then $\mathbb{P}(A_k) = 0$. Similarly, it cannot be the case that more non-black balls were drawn than there are in the bin, i.e., if $n - k > N - B$, then $\mathbb{P}(A_k) = 0$. Now, for every $n + B - N \le k \le B$, it holds that $|A_k| = \binom{B}{k} \cdot \binom{N-B}{n-k}$, since for we first choose $k$ black balls, and then $n - k$ non-black balls from the bin. Since the probability space is uniform, it follows that

$$\mathbb{P}(A_k) = \frac{|A_k|}{\binom{N}{n}} = \frac{\binom{B}{k} \cdot \binom{N-B}{n-k}}{\binom{N}{n}}.$$

Exercise 3 80 indistinguishable balls are placed uniformly at random into 5 different bins. Calculate the probability that no bin contains more than 24 balls.

Solution

For $1 \le i \le 5$, let $A_i$ be the event that bin $i$ contains at least 25 balls. We first observe that $|\Omega| = \binom{80+5-1}{5-1} = \binom{84}{4}$. Moreover, since there are 80 balls, at most 3 bins can contain at least 25 balls each. Since the probability space is uniform, for every $1 \le i < j < k \le 5$, it holds that

$$\begin{aligned}
\mathbb{P}(A_i) &= \frac{|A_i|}{|\Omega|} = \frac{\binom{55+5-1}{5-1}}{\binom{84}{4}} = \frac{\binom{59}{4}}{\binom{84}{4}}, \\
\mathbb{P}(A_i \cap A_j) &= \frac{|A_i \cap A_j|}{|\Omega|} = \frac{\binom{30+5-1}{5-1}}{\binom{84}{4}} = \frac{\binom{34}{4}}{\binom{84}{4}}, \\
\mathbb{P}(A_i \cap A_j \cap A_k) &= \frac{|A_i \cap A_j \cap A_k|}{|\Omega|} = \frac{\binom{5+5-1}{5-1}}{\binom{84}{4}} = \frac{\binom{9}{4}}{\binom{84}{4}}.
\end{aligned}$$

It follows by inclusion-exclusion that

$$\mathbb{P}\left(\bigcap_{i=1}^{5} A_i^c\right) = 1 - \sum_{\emptyset \neq \mathcal{I} \subseteq \{1,2,3,4,5\}} (-1)^{|\mathcal{I}|-1} \cdot \mathbb{P}\left(\bigcap_{i \in \mathcal{I}} A_i\right) = 1 - 5 \cdot \frac{\binom{59}{4}}{\binom{84}{4}} + \binom{5}{2} \cdot \frac{\binom{34}{4}}{\binom{84}{4}} - \binom{5}{3} \cdot \frac{\binom{9}{4}}{\binom{84}{4}}.$$

Exercise 4 An integer $i \in \{1, 2, \ldots, 1000\}$ is sampled uniformly at random. Calculate the probability that it is divisible by 2 or 3, but not by 5.

Solution

Clearly, $|\Omega| = 1000$. For every positive integer $i$, let $A_i$ be the event that the randomly chosen integer is divisible by $i$. Note that

$$\mathbb{P}(A_5^c \cap (A_2 \cup A_3)) = \mathbb{P}((A_2 \cap A_5^c) \cup (A_3 \cap A_5^c)) = \mathbb{P}(A_2 \cap A_5^c) + \mathbb{P}(A_3 \cap A_5^c) - \mathbb{P}(A_2 \cap A_3 \cap A_5^c),$$

where the last equality holds by inclusion-exclusion. Since the probability space is uniform, it follows that

$$\mathbb{P}\left(A_2 \cap A_5^c\right) = \frac{|A_2 \cap A_5^c|}{|\Omega|} = \frac{|A_2| - |A_2 \cap A_5|}{1000} = \frac{\lfloor 1000/2 \rfloor - \lfloor 1000/10 \rfloor}{1000} = \frac{400}{1000},$$
$$\mathbb{P}\left(A_3 \cap A_5^c\right) = \frac{|A_3 \cap A_5^c|}{|\Omega|} = \frac{|A_3| - |A_3 \cap A_5|}{1000} = \frac{\lfloor 1000/3 \rfloor - \lfloor 1000/15 \rfloor}{1000} = \frac{267}{1000},$$
$$\mathbb{P}\left(A_2 \cap A_3 \cap A_5^c\right) = \frac{|A_2 \cap A_3 \cap A_5^c|}{|\Omega|} = \frac{|A_2 \cap A_3| - |A_2 \cap A_3 \cap A_5|}{1000} = \frac{\lfloor 1000/6 \rfloor - \lfloor 1000/30 \rfloor}{1000} = \frac{133}{1000}.$$

We conclude that

$$\mathbb{P}\left(A_5^c \cap (A_2 \cup A_3)\right) = \frac{400}{1000} + \frac{267}{1000} - \frac{133}{1000} = \frac{267}{500}.$$