Probability Theory 1 – Proposed solution of moed aleph exam

1. **(a)** It readily follows from the definitions of X_1, X_2, Y and Z that the support of Y is $\{0,1,2\}$ and that the support of Z is $\{1,2,3\}$. For every $0 \leq i \leq 2$ and $1 \leq j \leq 3$, the table below shows the value of $P(Y=i, Z=j)$. We explain two of these calculations in greater detail:

$$\begin{aligned} P(Y=0, Z=2) &= P(X_1=0, X_2=2) = P(X_1=0)\cdot P(X_2=2) \\ &= \binom{2}{0}(1/2)^2\cdot 1/3 = 1/12, \end{aligned}$$

where the first equality holds since $P(X_2=0)=0$ and the second equality holds since X_1 and X_2 are independent.

$$\begin{aligned} P(Y=1, Z=2) &= P(X_1=1, X_2=2) + P(X_1=2, X_2=1) \\ &= P(X_1=1)\cdot P(X_2=2) + P(X_1=2)\cdot P(X_2=1) \\ &= \binom{2}{1}(1/2)^2\cdot 1/3 + \binom{2}{2}(1/2)^2\cdot 1/3 = 1/6 + 1/12 = 1/4. \end{aligned}$$

(b) One can calculate the distributions of Y and Z directly or deduce them from the joint distribution calculated in part (a). The marginal distributions appear in the table below as well. We calculate $P(Z=2)$ in greater detail in both ways:

$$\begin{aligned} P(Z=2) &= P(X_1=0, X_2=2) + P(X_1=1, X_2=2) + P(X_1=2, X_2=2) + P(X_1=2, X_2=1) \\ &= \binom{2}{0}(1/2)^2\cdot 1/3 + \binom{2}{1}(1/2)^2\cdot 1/3 + \binom{2}{2}(1/2)^2\cdot 1/3 + \binom{2}{2}(1/2)^2\cdot 1/3 \\ &= 1/12 + 1/6 + 1/12 + 1/12 = 5/12. \end{aligned}$$

$$P(Z=2) = \sum_{i=0}^{2} P(Y=i, Z=2) = 1/12 + 1/4 + 1/12 = 5/12.$$

(c) We see, for example, that

$$P(Y=2, Z=1) = 0 \neq \frac{1}{6}\cdot\frac{1}{4} = P(Y=2)\cdot P(Z=1).$$

Hence, Y and Z are not independent.

	Y = 0	Y = 1	Y = 2	Z
Z = 1	1/12	1/6	0	1/4
Z = 2	1/12	1/4	1/12	5/12
Z = 3	1/12	1/6	1/12	1/3
Y	1/4	7/12	1/6	

2. **(a)**

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{t=0}^{\infty} t \cdot P(X=t) = \sum_{t=1}^{\infty} t \cdot P(X=t) = \sum_{k=1}^{\infty}\sum_{t=k}^{\infty} P(X=t) \\
&= \sum_{k=0}^{\infty}\sum_{t=k+1}^{\infty} P(X=t) = \sum_{k=0}^{\infty} P(X>k),
\end{aligned}$$

where the third equality holds since, for every positive integer t, the term $P(X=t)$ appears on the left hand side with coefficient t and on the right hand side with coefficient 1 for every $1 \leq k \leq t$.

(b) First, note that the requested probability is 1 for $k \in \{1,2\}$. For every $k \geq 3$ let R_k denote the event "none of the first k die rolls resulted in a red color", let B_k denote the event "none of the first k die rolls resulted in a blue color", and let Y_k denote the event "none of the first k die rolls resulted in a Yelow color". The requested probability is $P(R_k \cup B_k \cup Y_k)$. Using inclusion-exclusion we get

$$\begin{aligned}
P(R_k \cup B_k \cup Y_k) &= P(R_k) + P(B_k) + P(Y_k) - P(R_k \cap B_k) - P(R_k \cap Y_k) - P(B_k \cap Y_k) \\
&+ P(R_k \cap B_k \cap Y_k) = 3P(R_k) - 3P(R_k \cap B_k) = 3\left[(2/3)^k - (1/3)^k\right].
\end{aligned}$$

(c) Since the only values X takes are non-negative integers, we can use part (a) to conclude that

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=0}^{\infty} P(X>k) = P(X>0) + P(X>1) + P(X>2) + \sum_{k=3}^{\infty} P(X>k) \\
&= 3 + \sum_{k=3}^{\infty} 3\left[(2/3)^k - (1/3)^k\right] = 3 + 3\sum_{k=3}^{\infty}(2/3)^k - 3\sum_{k=3}^{\infty}(1/3)^k \\
&= 3 + 3 \cdot \left(\frac{2}{3}\right)^3 \cdot \frac{1}{1-2/3} - 3 \cdot \left(\frac{1}{3}\right)^3 \cdot \frac{1}{1-1/3} = 3 + \frac{8}{3} - \frac{1}{6} = \frac{11}{2},
\end{aligned}$$

where the third equality holds by part (b) and since, by definition, $P(X>k) = 1$ for $k \in \{0,1,2\}$.

3. **(a)** By going through all possible pairs of ball types (i.e., two white balls, two black balls, two red balls, one white and one red, one white and one black, one red and one black) we see that the support of X is $\{-2,-1,0,1,2,4\}$. A direct calculation shows that

$$P(X=-2) = P(\text{drawing two white balls}) = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}.$$

$$P(X=-1) = P(\text{drawing one white ball and one red ball}) = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}.$$

$$P(X=0) = P(\text{drawing two red balls}) = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}.$$

$$P(X=1) = P(\text{drawing one black ball and one white ball}) = \frac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \frac{32}{91}.$$

$$P(X=2) = P(\text{drawing one black ball and one red ball}) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}.$$

$$P(X=4) = P(\text{drawing two black balls}) = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}.$$

Remark: It is easy and recommended to verify that this is indeed a distribution, i.e., that

$$P(X=-2) + P(X=-1) + P(X=0) + P(X=1) + P(X=2) + P(X=4) = 1.$$

(b) Let B_2 denote the event: "we drew 2 black balls". By definition, we have

$$P(B_2|X \geq 0) = \frac{P(B_2, X \geq 0)}{P(X \geq 0)} = \frac{P(B_2)}{P(X \geq 0)}.$$

By the calculations made in part (a) we have

$$P(B_2) = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}.$$

and

$$\begin{aligned} P(X \geq 0) &= P(X=0) + P(X=1) + P(X=2) + P(X=4) \\ &= \frac{1}{91} + \frac{32}{91} + \frac{8}{91} + \frac{6}{91} = \frac{47}{91} \end{aligned}$$

Therefore, $P(B_2|X \geq 0) = 6/47$.

(c) By part (a) (it is just as easy to use part (b) instead) we see that the probability of losing money in any single round is

$$P(X < 0) = P(X=-2) + P(X=-1) = \frac{28}{91} + \frac{16}{91} = \frac{44}{91}.$$

Let Y be the random variable counting the number of rounds played by the gambler, then $Y \sim Geom(44/91)$. Therefore $\mathbb{E}(Y) = 91/44$.

4. **(a)** For every $1 \leq i \leq n-1$, we define random variables X_i and Y_i as follows. $X_i = 1$ if the result of the ith dice roll is different than the result of the $(i+1)$st dice roll and $X_i = 0$ otherwise. Similarly, $Y_i = 1$ if the results of the ith and $(i+1)$st dice rolls are of the same parity and $Y_i = 0$ otherwise. We then have $X = \sum_{i=1}^{n-1} X_i$ and $Y = \sum_{i=1}^{n-1} Y_i$. Hence

$$Cov(X, Y) = Cov\left(\sum_{i=1}^{n-1} X_i, \sum_{i=1}^{n-1} Y_i\right) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} Cov(X_i, Y_j).$$

For every $1 \leq i \neq j \leq n-1$, the random variables X_i and Y_j are independent. This is clear when $|i-j| > 1$ as then the two corresponding variables are determined by disjoint pairs of dice rolls. When $|i-j| = 1$, X_i and Y_j are independent since $P(X_i = 1) = 5/6$ holds regardless of the result of the ith dice roll, and, similarly, $P(Y_j = 1) = 1/2$ holds regardless of the result of the jth dice roll. In particular $Cov(X_i, Y_j) = 0$ for every $1 \leq i \neq j \leq n-1$. On the other hand, for every $1 \leq i, j \leq n-1$ we have

$$\begin{aligned} Cov(X_i, Y_i) &= \mathbb{E}(X_i Y_i) - \mathbb{E}(Y_i)\mathbb{E}(Y_i) = P(X_i = 1, Y_i = 1) - P(X_i = 1)P(Y_i = 1) \\ &= \frac{2}{6} - \frac{5}{6} \cdot \frac{3}{6} = \frac{12 - 15}{36} = -\frac{3}{36} = -\frac{1}{12}, \end{aligned}$$

where the third equality holds since, for every $1 \leq j \leq 6$, the probability that a die roll results in a number $i \in \{1, \ldots, 6\} \setminus \{j\}$ which has the same parity as j is $2/6$.
We conclude that

$$Cov(X, Y) = \sum_{i=1}^{n-1} Cov(X_i, Y_i) = -\frac{n-1}{12}.$$

(b) If X and Y were independent, then we would have, in particular, $Cov(X, Y) = 0$. It thus follows by part (a) that X and Y are not independent.

(c) Recall that

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$

(since X and Y receive only finitely many values, it is evident that both $Var(X)$ and $Var(Y)$ are finite). Since both $\sqrt{Var(X)}$ and $\sqrt{Var(Y)}$ are positive by definition and by the assumption that they are non-zero, and $Cov(X, Y)$ is negative by part (a), it follows that $\rho(X, Y) < 0$, that is, X and Y are negatively correlated.