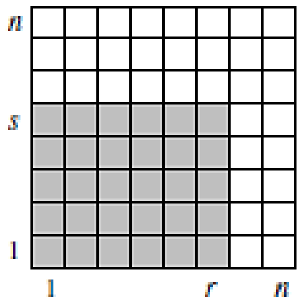
Theorem 4.5 (Erdős–Szekeres 1935). *Let $A = (a_1, \ldots, a_n)$ be a sequence of n different real numbers. If $n \geq sr + 1$ then either A has an increasing subsequence of $s + 1$ terms or a decreasing subsequence of $r + 1$ terms (or both).*

Proof (due to Seidenberg 1959). Associate to each term a_i of A a pair of "scores" (x_i, y_i) where x_i is the number of terms in the longest *increasing* subsequence *ending* at a_i, and y_i is the number of terms in the longest *decreasing* subsequence *starting* at a_i. Observe that no two terms have the same score, i.e., that $(x_i, y_i) \neq (x_j, y_j)$ whenever $i \neq j$. Indeed, if we have $\cdots a_i \cdots a_j \cdots$, then either $a_i < a_j$ and the longest increasing subsequence ending at a_i can be extended by adding on a_j (so that $x_i < x_j$), or $a_i > a_j$ and the longest decreasing subsequence starting at a_j can be preceded by a_i (so that $y_i > y_j$).

Now make a grid of n^2 pigeonholes:



Place each term a_i in the pigeonhole with coordinates (x_i, y_i). Each term of A can be placed in some pigeonhole, since $1 \leq x_i, y_i \leq n$ for all $i = 1, \ldots, n$. Moreover, no pigeonhole can have more than one term because $(x_i, y_i) \neq (x_j, y_j)$ whenever $i \neq j$. Since $|A| = n \geq sr + 1$, we have more items than the

pigeonholes shaded in the above picture. So some term a_i will lie outside this shaded region. But this means that either $x_i \geq s + 1$ or $y_i \geq r + 1$ (or both), exactly what we need. $\square$