Practical 7

Exercise 1 There are X cars that enter a junction. Assume that $X \sim \mathrm{Poi}(\lambda)$, for some positive λ. Each car turns right at the junction with probability p and turns left with probability $1-p$, for some $p \in [0,1]$. Let X_R be the number of cars that turned right, and let X_L be the number of cars that turned left.

1. Compute the distributions of X_R and X_L.
2. Compute the joint distribution of X_R and X_L.

Solution

1. Let $k \in \mathbb{N}$ (we consider 0 to be a natural number). By the Law of total probabilities it holds that

$$\mathbb{P}(X_R = k) = \sum_{n=0}^{\infty} \mathbb{P}(X_R = k \mid X = n) \cdot \mathbb{P}(X = n) = \sum_{n=0}^{\infty} \mathbb{P}(X_R = k \mid X = n) \cdot \frac{e^{-\lambda}\lambda^n}{n!}.$$

Observe that when $n \geq k$ it holds that $X_R \mid X = n \sim \mathrm{Bin}(n,p)$, since out of the n cars that passed through the junction, we count how many turned right. Therefore for all $n \in \mathbb{N}$, $n \geq k$ it holds that

$$\mathbb{P}(X_R = k \mid X = n) = \binom{n}{k} p^k (1-p)^{n-k}.$$

For $n < k$ it holds that $\mathbb{P}(X_R = k \mid X = n) = 0$.

Combine the above and compute:

$$
\begin{aligned}
\mathbb{P}\left(X_R = k\right) &= \sum_{n=0}^{\infty} \mathbb{P}\left(X_R = k \mid X = n\right) \cdot \frac{e^{-\lambda}\lambda^n}{n!} \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{e^{-\lambda}\lambda^n}{n!} \\
&= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \cdot \frac{e^{-\lambda}\lambda^n}{n!} \\
&= \frac{e^{-\lambda}p^k}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}\lambda^n}{(n-k)!} \\
&= \frac{e^{-\lambda}p^k}{k!} \sum_{m=0}^{\infty} \frac{(1-p)^m\lambda^{m+k}}{m!} \\
&= \frac{e^{-\lambda}(\lambda p)^k}{k!} \sum_{m=0}^{\infty} \frac{(\lambda(1-p))^m}{m!} \\
&= \frac{e^{-\lambda}(\lambda p)^k}{k!} \cdot e^{\lambda(1-p)} \\
&= \frac{e^{-\lambda p}(\lambda p)^k}{k!},
\end{aligned}
$$

where the penultimate equation follows from the taylor expansion for the exponent function. Hence $X_R \sim \text{Poi}\left(\lambda p\right)$, and similarly (or by substituting p for $1-p$) we get that $X_L \sim \text{Poi}\left(\lambda(1-p)\right)$.

2. Let $r, l \in \mathbb{N}$. Compute

$$
\begin{aligned}
\mathbb{P}\left(X_R = r, X_L = l\right) &= \mathbb{P}\left(X_R = r, X_L = l \mid X = r+l\right) \cdot \mathbb{P}\left(X = r+l\right) \\
&= \binom{r+l}{r} p^r (1-p)^l \cdot \frac{e^{-\lambda}\lambda^{r+l}}{(r+l)!} \\
&= \frac{(r+l)!}{r! \cdot l!} p^r (1-p)^l \cdot \frac{e^{-\lambda}\lambda^{r+l}}{(r+l)!} \\
&= e^{-\lambda p} \cdot \frac{(\lambda p)^r}{r!} \cdot e^{-\lambda(1-p)} \frac{(\lambda(1-p))^l}{l!} \\
&= \mathbb{P}\left(X_R = r\right) \cdot \mathbb{P}\left(X_l = l\right).
\end{aligned}
$$

Exercise 2 Let $X_1 \sim \text{Poi}\left(\lambda_1\right)$ and let $X_2 \sim \text{Poi}\left(\lambda_2\right)$. Prove that $X_1 + X_2 \sim \text{Poi}\left(\lambda_1 + \lambda_2\right)$.

Proof. Since X_1 and X_2 take values in $\mathbb{N}$, it follows that $\mathbb{P}(X_1 + X_2 \notin \mathbb{N}) = 0$. Let $k \in \mathbb{N}$. Then

$$\begin{aligned}
\mathbb{P}(X_1 + X_2 = k) &= \sum_{n=0}^{k} \mathbb{P}(X_1 + X_2 = k \mid X_2 = n) \cdot \mathbb{P}(X_2 = n) \\
&= \sum_{n=0}^{k} \mathbb{P}(X_1 = k - n) \cdot \mathbb{P}(X_2 = n) \\
&= \sum_{n=0}^{k} e^{-\lambda_1} \frac{\lambda_1^{k-n}}{(k-n)!} \cdot e^{-\lambda_2} \frac{\lambda_2^n}{n!} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{1}{k!} \sum_{n=0}^{k} \frac{k!}{n! \cdot (k-n)!} \cdot \lambda_1^{k-n} \cdot \lambda_2^n \\
&= e^{-(\lambda_1+\lambda_2)} \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} \cdot \lambda_1^{k-n} \cdot \lambda_2^n \\
&= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^k}{k!},
\end{aligned}$$

where the first equality is due to the Law of total probabilities, and the last equality is by the binomial formula. □

Exercise 3 A pair of fair dices are tossed independently. Let X be the result in the first dice, and let Y be the maximum result from between the two dices. Compute the joint distribution of X and Y and their marginal distributions.

Solution

Let $x, y \in \{1, 2, 3, 4, 5, 6\}$. Then

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(Y = y \mid X = x) \cdot \mathbb{P}(X = x) = \frac{1}{6}\mathbb{P}(Y = y \mid X = x).$$

If $x = y$, then the outcome of the second dice must be less then or equal to x. Hence in this case $\mathbb{P}(Y = y \mid X = x) = x/6$. If $y > x$ then $\mathbb{P}(Y = y \mid X = x) = 1/6$, since in this case the outcome of the second dice must be equal to a known value y. The case $y < x$ happens with probability 0 since x has been sampled. Therefore

$$\mathbb{P}(X = x, Y = y) = \begin{cases} \frac{x}{36} & \text{if } 1 \le x = y \le 6 \\ \frac{1}{36} & \text{if } 1 \le x < y \le 6 \\ 0 & \text{otherwise} \end{cases}$$

We now compute their marginal distributions. $X \sim U[1,6]$ and for Y we compute:

$$\begin{aligned}
\mathbb{P}(Y=y) &= \sum_{x=1}^{6} \mathbb{P}(X=x, Y=y) \\
&= \sum_{x=1}^{y} \mathbb{P}(X=x, Y=y) \\
&= \sum_{x=1}^{y-1} \mathbb{P}(X=x, Y=y) + \mathbb{P}(X=Y=y) \\
&= \frac{1}{36}(y-1) + \frac{y}{36} \\
&= \frac{2y-1}{36}.
\end{aligned}$$

Exercise 4 A fair dice is being tossed indefinitely, each toss is independent of the other tosses. Let X be the number of tosses until the first time 6 appears, and let Y be the number of tosses until the first time 2 or 4 appears.

1. Compute the distributions of X and Y.
2. Compute the distribution of $\min\{X, Y\}$.
3. Compute the joint distribution of X and Y.
4. Compute the distribution of $Y \mid X = 2$.

Solution

1. $X \sim \text{Geom}(1/6)$ since the probability for 6 is $1/6$, and $Y \sim \text{Geom}(1/3)$ since the probability for 2 or 4 is $2/6 = 1/3$.
2. $\min\{X, Y\}$ counts the number of tosses until 2, or 4, or 6 appears. Therefore $\min\{X, Y\} \sim \text{Geom}(1/2)$.
3. Let $x, y \in \mathbb{N} \setminus \{0\}$. We compute $\mathbb{P}(X=x, Y=y)$. First observe that the case where $x = y$, happens with probability 0, since 2 or 4, cannot appears in the same toss as 6 appears in. We separate into two cases:

   1. $x > y$: In this case, 1,3, and 5 are the only possible results in the first $y-1$ tosses, 2 and 4 are the only possible results in the y-th toss, and 1,2,3,4, and 5 are the only possible results in the next $x-y-1$ tosses. After that 6 must appear the x-th toss. Hence

   $$\mathbb{P}(X=x, Y=y) = \left(\frac{1}{2}\right)^{y-1} \cdot \frac{1}{3} \cdot \left(\frac{5}{6}\right)^{x-y-1} \cdot \frac{1}{6}.$$

   2. $x < y$: A similar reasoning to the previous case yields that

   $$\mathbb{P}(X=x, Y=y) = \left(\frac{1}{2}\right)^{x-1} \cdot \frac{1}{6} \cdot \left(\frac{2}{3}\right)^{y-x-1} \cdot \frac{1}{3}.$$

4. Let $y \in \mathbb{N} \setminus \{0\}$. We compute $\mathbb{P}(Y = y \mid X = 2)$. First note that for $y = 2$ the probability is 0. We separate into two cases:

   1. $y = 1$*:* In this case it holds that $\mathbb{P}(Y = 1, X = 2) = \frac{2}{6} \cdot \frac{1}{6}$ since the first outcome must be either 2 or 4, and the second outcome must be 6. Since $X \sim \mathrm{Geom}(1/6)$ it follows that

$$\mathbb{P}(Y = 1 \mid X = 2) = \frac{\mathbb{P}(Y = 1, X = 2)}{\mathbb{P}(X = 2)} = \frac{\frac{2}{6} \cdot \frac{1}{6}}{\frac{5}{6} \cdot \frac{1}{6}} = \frac{2}{5}.$$

   1. $y \geq 3$*:* In this case it holds that $\{Y = y, X = 2\}$ happens if and only if the first toss is either 1,3 or 5, the second toss is 6, the next $y - 1 - 2 = y - 3$ tosses are 1,3,5, or 6, and the y-th toss is 2 or 4. Therefore

$$\mathbb{P}(Y = y \mid X = 2) = \frac{\mathbb{P}(Y = y, X = 2)}{\mathbb{P}(X = 2)} = \frac{\frac{3}{6} \cdot \frac{1}{6} \cdot \left(\frac{4}{6}\right)^{y-3} \cdot \frac{2}{6}}{\frac{5}{6} \cdot \frac{2}{6}} = \frac{\left(\frac{2}{3}\right)^{y-3}}{5}.$$

Exercise 5 The number of eggs a chicken lays is distributed according to the uniform distribution over $1, 2, 3, 4$. From every egg, there is a chance of $\frac{1}{3}$ that a chick will come out, independently of the rest of the eggs and the number of eggs the chicken lay. Let X be the number of eggs and let Y be the number of chicks. Compute the joint distribution of X and Y.

Solution

It holds that

$$\mathrm{Supp}(X, Y) = \left\{(x, y) \in \mathbb{N}^2 : 1 \leq y \leq x \leq 4\right\}.$$

Let $(x, y) \in \mathrm{Supp}(X, Y)$. Observe that $Y \mid X = x \sim \mathrm{Bin}(x, 1/3)$. Therefore

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y \mid X = x) = \frac{1}{4} \cdot \binom{x}{y} \left(\frac{1}{3}\right)^y \left(\frac{2}{3}\right)^{x-y}.$$