Practical 6

Exercise 1 A bin contains 9 white balls and 1 black ball. The balls are drawn uniformly at random from the bin one by one. Let X be the number of balls that were drawn until the first time (inclusive) the black ball was drawn, and let Y be the number of balls that were drawn until the third time the black ball was drawn. For every positive integer k, calculate

1. $\mathbb{P}(X = k)$ when the samples are being made with replacement.
2. $\mathbb{P}(X = k)$ when the samples are being made without replacement.
3. $\mathbb{P}(Y = k)$ when the samples are being made with replacement.

Solution

1. The event $\{X = k\}$ occurs if and only if the first $k-1$ balls we draw are white and the kth ball we draw is black. Since the samples are made with replacement, each time we draw a ball the probability that it is white is $9/10$ and the probability that it is black is $1/10$. We conclude that
$$\mathbb{P}(X = k) = \left(\frac{9}{10}\right)^{k-1} \cdot \frac{1}{10} = \frac{9^{k-1}}{10^k}.$$

2. Since there are only 10 balls and the samples are made without replacement, $\mathbb{P}(X = k) = 0$ for every $k > 10$. On the other hand, for every integer $1 \leq k \leq 10$, the probability that the first $k-1$ balls are white is $\frac{11-k}{10}$. Indeed, this probability is trivially 1 if $k = 1$ and is
$$\frac{9}{10} \cdot \frac{8}{9} \cdots \frac{10-k+1}{10-k+2} = \frac{11-k}{10},$$
otherwise. Given that the first $k-1$ balls to be drawn were white, the probability that the next balls will be black is $\frac{1}{10-k+1}$. We conclude that
$$\mathbb{P}(X = k) = \frac{11-k}{10} \cdot \frac{1}{10-k+1} = \frac{1}{10}.$$

3. The event $\{Y = k\}$ occurs if and only if out of the first $k-1$ balls to be drawn, the black ball is drawn exactly twice and, moreover, the kth ball to be drawn is black. The probability that out of the first $k-1$ balls to be drawn the black ball is drawn exactly twice is
$$\binom{k-1}{2} \cdot \left(\frac{9}{10}\right)^{k-1-2} \cdot \left(\frac{1}{10}\right)^2.$$

We conclude that

$$\mathbb{P}(Y=k) = \binom{k-1}{2} \cdot \left(\frac{9}{10}\right)^{k-1-2} \cdot \left(\frac{1}{10}\right)^2 \cdot \frac{1}{10} = \binom{k-1}{2} \cdot \frac{9^{k-3}}{10^k}.$$

Exercise 2 Ben tosses 5 fair coins, all coin tosses being mutually independent. He then tosses each coin which came up heads once more, all coin tosses being mutually independent. What is the probability that, in the second round, the outcome of exactly 3 of the coin tosses will be heads?

Solution

We will present two different solutions of this exercise. Our first solution uses the Law of total probability. Let X_1 be the number of coins that came up heads in the first round of coin tosses and let X_2 be the number of coins that came up heads in the second round. Observe that, for every $3 \leq i \leq 5$, it holds that

$$\mathbb{P}(X_1 = i) = \binom{5}{i} \cdot \left(\frac{1}{2}\right)^5,$$

since out of the 5 coins Ben tosses, we need to choose which ones will come up heads. Similarly, for every $0 \leq i \leq 5$, it holds that

$$\mathbb{P}(X_2 = 3 \mid X_1 = i) = \binom{i}{3} \cdot \left(\frac{1}{2}\right)^i.$$

In particular, $\mathbb{P}(X_2 = 3 \mid X_1 = i) = 0$ whenever $i < 3$. Therefore, it follows by the Law of total probability that

$$\begin{aligned}
\mathbb{P}(X_2 = 3) &= \sum_{i=3}^{5} \mathbb{P}(X_2 = 3 \mid X_1 = i) \cdot \mathbb{P}(X_1 = i) \\
&= \binom{3}{3}\left(\frac{1}{2}\right)^3 \cdot \binom{5}{3}\left(\frac{1}{2}\right)^5 + \binom{4}{3}\left(\frac{1}{2}\right)^4 \cdot \binom{5}{4}\left(\frac{1}{2}\right)^5 + \binom{5}{3}\left(\frac{1}{2}\right)^5 \cdot \binom{5}{5}\left(\frac{1}{2}\right)^5 \\
&= \frac{45}{512}.
\end{aligned}$$

Next, we present our second solution; it involves less calculations. Let X_2 be defined as in the first solution. In order for a coin to come up heads when tossed in the second round, it has to come up heads in both the first and second rounds. Therefore, the probability of any single coin to come up heads when tossed in the second round is $1/2 \cdot 1/2 = 1/4$. Since the event $\{X_2 = 3\}$ occurs if and only if exactly 3 of the 5 coins come up heads when tossed in the second round, we conclude that

$$\mathbb{P}(X_2 = 3) = \binom{5}{3} \cdot \left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right)^2 = \frac{45}{512}.$$

Exercise 3 A drunk makes a random walk on $\mathbb{Z}$, starting at 0. That is, each step the drunk takes will be one step to the right with probability $p \in [0,1]$, and one step to the left with probability $1-p$, where the steps he takes are independent of one another. Let N be some positive integer.

Let X be the position of the drunk man after making precisely N steps, and let Z be the number of steps to the right made by the drunk during the first N steps. Calculate the distributions of X and of Z.

Solution

For every $k \in \{0, 1, \ldots, N\}$, it holds that

$$\mathbb{P}(Z = k) = \binom{N}{k} \cdot p^k \cdot (1-p)^{N-k}.$$

Observe that the position of the drunk is the number of right steps he took minus the number of left steps he took, i.e., $X = Z - (N - Z) = 2Z - N$. In particular, this implies that X and N have the same parity. Therefore, for every $k \in \{-N, -N+1, \ldots, N\}$ whose parity differs from the parity of N, it holds that $\mathbb{P}(X = k) = 0$. For all other integers $k \in \{-N, -N+1, \ldots, N\}$, it holds that

$$\mathbb{P}(X = k) = \mathbb{P}(2Z - N = k) = \mathbb{P}\left(Z = \frac{N+k}{2}\right) = \binom{N}{\frac{N+k}{2}} \cdot p^{\frac{N+k}{2}} \cdot (1-p)^{\frac{N-K}{2}}.$$