Practical 7

Exercise 1 A fair die is rolled indefinitely, all die rolls being mutually independent. Let X be the number of die rolls until the first time the outcome is 6, and let Y be the number of die rolls until the first time the outcome is 2 or 4.

1. Calculate the distributions of X and Y.
2. Calculate the distribution of $\min\{X, Y\}$.
3. Calculate the joint distribution of X and Y.
4. Calculate the distribution of $Y \mid X = 2$.

Solution

1. It is evident that $X \sim Geom(1/6)$, that is, $\mathbb{P}(X = k) = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}$ for every positive integer k. Similarly, $Y \sim Geom(1/3)$.
2. $\min\{X, Y\}$ counts the number of die rolls until the first time the outcome is 2, 4 or 6. Therefore, by the same reasoning as in the previous part of this exercise, we conclude that $X \sim Geom(1/2)$.
3. Let $x, y \in \mathbb{N}$. We wish to calculate $\mathbb{P}(X = x, Y = y)$. First, observe that $\mathbb{P}(X = x, Y = y) = 0$ whenever $x = y$, as no die roll can have its outcome in the set $\{6\} \cap \{2, 4\}$. We distinguish between the following two cases.

   1. $x > y$: In this case, the outcomes of the first $y - 1$ die rolls are in the set $\{1, 3, 5\}$, the outcome of the yth die roll is in the set $\{2, 4\}$, the outcomes of the next $x - y - 1$ die rolls are in the set $\{1, 2, 3, 4, 5\}$, and the outcome of the xth die roll is 6. We conclude that

   $$\mathbb{P}(X = x, Y = y) = \left(\frac{1}{2}\right)^{y-1} \cdot \frac{1}{3} \cdot \left(\frac{5}{6}\right)^{x-y-1} \cdot \frac{1}{6} = \left(\frac{5}{6}\right)^{x} \cdot \left(\frac{3}{5}\right)^{y} \cdot \frac{2}{15}.$$

   2. $x < y$: An analogous reasoning to the one used to analyze the previous case yields

   $$\mathbb{P}(X = x, Y = y) = \left(\frac{1}{2}\right)^{x-1} \cdot \frac{1}{6} \cdot \left(\frac{2}{3}\right)^{y-x-1} \cdot \frac{1}{3} = \left(\frac{3}{4}\right)^{x} \cdot \left(\frac{2}{3}\right)^{y} \cdot \frac{1}{6}.$$

4. Let $y \in \mathbb{N}$. We wish to calculate $\mathbb{P}(Y = y \mid X = 2)$. First, observe that $\mathbb{P}(Y = 2 \mid X = 2) = 0$. We distinguish between the following two cases.

   1. $y = 1$*:* Observe that $\mathbb{P}(Y = 1, X = 2) = \frac{2}{6} \cdot \frac{1}{6}$, as the outcome of the first die roll must be either 2 or 4, and the outcome of the second die roll must be 6. As we previously saw, $X \sim \text{Geom}(1/6)$ and thus

$$\mathbb{P}(Y = 1 \mid X = 2) = \frac{\mathbb{P}(Y = 1, X = 2)}{\mathbb{P}(X = 2)} = \frac{\frac{2}{6} \cdot \frac{1}{6}}{\frac{5}{6} \cdot \frac{1}{6}} = \frac{2}{5}.$$

   2. $y \geq 3$*:* In this case, the event $\{Y = y, X = 2\}$ occurs if and only if the outcome of the first die roll is in the set $\{1, 3, 5\}$, the outcome of the second die roll is 6, the outcomes of the next $(y-1) - 2 = y - 3$ die rolls are in the set $\{1, 3, 5, 6\}$, and the outcome of the yth die roll is in the set $\{2, 4\}$. We conclude that

$$\mathbb{P}(Y = y \mid X = 2) = \frac{\mathbb{P}(Y = y, X = 2)}{\mathbb{P}(X = 2)} = \frac{\frac{3}{6} \cdot \frac{1}{6} \cdot \left(\frac{4}{6}\right)^{y-3} \cdot \frac{2}{6}}{\frac{5}{6} \cdot \frac{1}{6}} = \frac{\left(\frac{2}{3}\right)^{y-3}}{5}.$$

Exercise 2 n kids are playing the following game. In his/her turn each kid, independently of the rest, tries to jump over a puddle of water until he/she succeeds, or until they had 3 tries (whichever happens first). On each single try, every kid successfully jumps over the puddle with probability $p \in (0, 1)$. Calculate the probability that

1. Exactly one kid successfully jumped over the puddle on his/her first try.
2. Exactly three kids successfully jumped over the puddle with at most 2 tries each.

Solution

1. Let X be the number of kids that successfully jumped over the puddle on their first try. Then $X \sim \text{Bin}(n, p)$. Therefore

$$\mathbb{P}(X = 1) = \binom{n}{1} \cdot p \cdot (1-p)^{n-1} = np(1-p)^{n-1}.$$

2. Let Y be the number of kids that successfully jumped over the puddle with at most 2 tries each. Observe that a kid will successfully jump over the puddle with at most 2 tries with probability $p + (1-p)p = 2p - p^2$, since the kid can either succeed on the first try, or fail on the first try and succeed on the second. Therefore $Y \sim \text{Bin}(n, 2p - p^2)$ implying that

$$\mathbb{P}(Y = 3) = \binom{n}{3} \cdot (2p - p^2)^3 \cdot \left(1 - 2p + p^2\right)^{n-3} = \binom{n}{3} \cdot (2p - p^2)^3 \cdot (1-p)^{2(n-3)}.$$

Exercise 3 A cafe has 10 chocolate cakes and 6 cheese cakes. 5 diners go into the cafe, each ordering one cake which they choose uniformly at random from the available cakes. Calculate the probability that at least 5 cheese cakes remain.

Solution

Let X be the number of cheese cakes that were ordered by the diners. Then, our aim is to calculate $\mathbb{P}(6 - X \geq 5) = \mathbb{P}(X \leq 1)$. Since $X \sim \text{Hyp}(16, 6, 5)$, for every $0 \leq k \leq 5$, it holds that

$$\mathbb{P}(X = k) = \frac{\binom{6}{k} \cdot \binom{10}{5-k}}{\binom{16}{5}}.$$

We conclude that

$$\mathbb{P}(X \leq 1) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) = \frac{\binom{6}{0} \cdot \binom{10}{5}}{\binom{16}{5}} + \frac{\binom{6}{1} \cdot \binom{10}{4}}{\binom{16}{5}} = \frac{9}{26}.$$

Exercise 4 The following data was collected from a certain population.

- The fraction of 'A' blooded type people is 0.4.
- The fraction of 'B' blooded type people is 0.1.
- The fraction of 'O' blooded type people is 0.45.
- The fraction of 'AB' blooded type people is 0.05.

1. Two people are chosen uniformly at random and independently from the population. Calculate the probability that the two chosen people have different blood types.
2. Consider the following experiment. Two people are chosen uniformly at random and independently from the population. If they have different blood types, the experiment is over. Otherwise, the two people are returned to the population and we sample two more people. This is repeated until the first time the two chosen people have different blood types. Determine the smallest integer k for which the probability that we will need to sample more than k pairs of people is at most 0.001.

Solution

1. Let E be the event that the two chosen people have different blood types. Then

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c) = 1 - (0.4^2 + 0.1^2 + 0.45^2 + 0.05^2) = 0.625.$$

2. We will present two different solutions to this exercise.

   In our first solution, we let X be the number of pairs we sampled until we obtain a pair of people with different blood types. It follows by Part 1 of this exercise that $X \sim \text{Geom}(0.625)$. Our goal is to determine the smallest integer k such that

$$\mathbb{P}(X > k) \leq 0.001.$$

It holds that

$$\mathbb{P}(X > k) = \sum_{j=k+1}^{\infty} \mathbb{P}(X = j) = \sum_{j=k+1}^{\infty} 0.375^{j-1} \cdot 0.625 = 0.625 \cdot \frac{0.375^k}{1 - 0.375} = 0.375^k.$$

Therefore, we are looking for the smallest integer k such that

$$0.375^k \leq 0.001,$$

which implies that

$$k \geq \frac{\log(0.001)}{\log(0.375)} > 7.$$

Since k is an integer, we conclude that $k = 8$.

Next, we present our second solution. Let $Y \sim \mathrm{Bin}(k, 0.625)$. Our goal is to determine the smallest integer k such that

$$\mathbb{P}(Y = 0) \leq 0.001.$$

Indeed, we need to sample more than k pairs of people if and only if the two people consisting the ith pair have the same blood type for every $1 \leq i \leq k$, that is, if and only if $Y = 0$. It holds that

$$\mathbb{P}(Y = 0) = \binom{k}{0} \cdot 0.625^0 \cdot 0.375^k = 0.375^k.$$

The remainder of this solution is the same as in the first solution.