Probability Theory 1 – Proposed solution of moed bet summer exam

1. **(a)** It readily follows from the definitions of X and Y that the support of both random variables is $\{0,1,2\}$. For every $0 \le i \le 2$ and $0 \le j \le 2$, the table below shows the value of $P(X=i, Y=j)$. We explain two of these calculations in greater detail:

   If $X=1$ and $Y=1$, then either the result of the second coin toss was 1 and the results of the first and third coin tosses were 0, or the result of the second coin toss was 0 and the results of the first and third coin tosses were 1. Since all coin tosses are independent, the first event happens with probability $1/3 \cdot (2/3)^2$ and the second with probability $2/3 \cdot (1/3)^2$. These two events are clearly disjoint and thus

   $$Pr(X=1, Y=1) = 1/3 \cdot (2/3)^2 + 2/3 \cdot (1/3)^2 = 2/9 \cdot (2/3 + 1/3) = 2/9.$$

   If $X=2$, then, in particular, the result of the second coin toss must be 1 and so $Y \ge 1$. Hence $Pr(X=2, Y=0) = 0$.

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$(2/3)^3$	$(2/3)^2 \cdot 1/3$	0
$X=1$	$(2/3)^2 \cdot 1/3$	$1/3 \cdot (2/3)^2 + 2/3 \cdot (1/3)^2$	$2/3 \cdot (1/3)^2$
$X=2$	0	$2/3 \cdot (1/3)^2$	$(1/3)^3$

   (b) Since there are zero entries in the table above, it is evident that X and Y are not independent. Formally, by the table below we have $Pr(X=0, Y=2) = 0$. On the other hand $Pr(X=0) = (2/3)^2 > 0$ and $Pr(Y=2) = (1/3)^2 > 0$. Therefore $Pr(X=0, Y=2) \neq Pr(X=0)Pr(Y=2)$ and so X and Y are not independent.

   (c) For every $(\omega_1, \omega_2, \omega_3, \omega_4) \in \{0,1\}^4$, let $Pr(\omega_1, \omega_2, \omega_3, \omega_4)$ denote the probability that the result of the ith coin toss was ω_i for every $1 \le i \le 4$. Observe that the result of the ith coin toss is counted twice in $X+Y+Z$ for $i \in \{2,3\}$ and once for $i \in \{1,4\}$. Then

   $$Pr(X+Y+Z=4) = Pr(0,1,1,0) + Pr(1,0,1,1) + Pr(1,1,0,1) = (1/3)^2(2/3)^2 + 2(1/3)^3 2/3 = 8/81.$$

   (d) Clearly $Pr(Z=1) = 2 \cdot 1/3 \cdot 2/3$. Moreover, using the notation of Part (c) of this question we have

   $$Pr(X=1, Y=1, Z=1) = Pr(0,1,0,1) + Pr(1,0,1,0) = 2(1/3)^2(2/3)^2.$$

   Hence

   $$Pr(X=1, Y=1 | Z=1) = \frac{Pr(X=1, Y=1, Z=1)}{Pr(Z=1)} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}.$$

2. **(a)** Since we sample with replacement, the probability of drawing a yellow ball in any single round is $2/(4+4+2) = 1/5$. Since, moreover, the sampling in any given round is independent of all previous rounds, we conclude that $X \sim Geom(1/5)$.

(b) Let us first calculate $Pr(Y = 2|X = 5)$. If $X = 5$, then we drew exactly 5 balls, the fifth one was yellow, and the first four were not yellow. That is, out of 4 red balls and 4 blue balls we drew a blue ball twice and a red ball twice. Since there are 4 blue balls, the probability of drawing a blue ball in each of those four attempts is $1/2$. Since these attempts are independent, we conclude that $Pr(Y = 2|X = 5) = \binom{4}{2}(1/2)^4 = 3/8$. Finally

$$Pr(X = 5, Y = 2) = Pr(Y = 2|X = 5)Pr(X = 5) = \frac{3}{8} \cdot \frac{1}{5}\left(\frac{4}{5}\right)^4 = \frac{32}{5^5}.$$

(c) Since we drew n balls in total and one of them (the last one) was yellow, it is evident that $Pr(Y = k|X = n) = 0$ for every $k \notin \{0, 1, \ldots, n-1\}$. Fix some $0 \leq k \leq n-1$. The analysis we did in Part (b) of this question (which was just a warmup for part (c) – there is nothing special about $n = 5$ and $k = 2$) shows that $Pr(Y = k|X = n) = \binom{n-1}{k}(1/2)^{n-1}$, that is, $(Y|X = n) \sim Bin(n-1, 1/2)$.

(d) Since $(Y|X = n) \sim Bin(n-1, 1/2)$ by Part (c), it follows that $\mathbb{E}(Y|X = n) = (n-1)/2$. Hence, by the law of total expectation

$$\begin{aligned}\mathbb{E}(Y) &= \mathbb{E}(\mathbb{E}(Y|X)) = \sum_{i=1}^{\infty} \mathbb{E}(Y|X = i)Pr(X = i) = \sum_{i=1}^{\infty} \frac{i-1}{2} \cdot \frac{1}{5}\left(\frac{4}{5}\right)^{i-1} \\ &= \frac{1}{10}\sum_{t=1}^{\infty} t \cdot (4/5)^t = \frac{1}{10} \cdot \frac{4/5}{(1/5)^2} = 2,\end{aligned}$$

where the penultimate equality holds by the formula which was stated in the question.

3. **(a)** $X \leq 3$ in exactly five cases, namely, when $A = \emptyset$, $A = \{1\}$, $A = \{2\}$, $A = \{3\}$, or $A = \{1, 2\}$. Since the coin is fair and the coin flips are mutually independent, the probability of each such event is $(1/2)^{20}$. Hence $Pr(X \leq 3) = 5(1/2)^{20}$.

(b) For every $1 \leq i \leq 20$, let $X_i = 1$ if $i \in A$ and $X_i = 0$ otherwise. Then $\mathbb{E}(X_i) = Pr(X_i = 1) = 1/2$ for every $1 \leq i \leq 20$ and $X = \sum_{i=1}^{20} iX_i$. By linearity of expectation we then have

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{20} iX_i\right) = \sum_{i=1}^{20} \mathbb{E}(iX_i) = \sum_{i=1}^{20} i\mathbb{E}(X_i) = 1/2\sum_{i=1}^{20} i = \frac{1}{2} \cdot \frac{20 \cdot 21}{2} = 105.$$

(c) For every $1 \leq i \leq 20$ we have

$$Var(iX_i) = i^2 Var(X_i) = i^2\left(\mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2\right) = i^2(1/2 - 1/4) = i^2/4.$$

Moreover, since X_i and X_j are independent for every $1 \leq i < j \leq 20$, it follows that

$$Cov(iX_i, jX_j) = ijCov(X_i, X_j) = 0.$$

We conclude that

$$Var(X) = \sum_{i=1}^{20} i^2 Var(X_i) = \frac{1}{4}\sum_{i=1}^{20} i^2 = \frac{1}{4} \cdot \frac{20 \cdot 21 \cdot 41}{6} = \frac{35 \cdot 41}{2}$$

where the penultimate equality holds by the formula which was stated in the question.

(d) Using the results from parts (b) and (c) of this question, Chebyshev's inequality implies that

$$Pr(|X - 105| \geq 41) = Pr(|X - \mathbb{E}(X)| \geq 41) \leq \frac{Var(X)}{41^2} = \frac{35}{82}.$$