Probability Theory 1 – Proposed solution of model exam

1. **(a)** It readily follows from the definitions of X_1, X_2, Y and Z that the support of Y is $\{-2, 1, 0, 1, 2\}$ and that the support of Z is $\{0, 1, 2\}$. For every $-2 \leq i \leq 2$ and $0 \leq j \leq 2$, the table below shows the value of $P(Y = i, Z = j)$. We explain two of these calculations in greater detail:

$$\begin{aligned} P(Y = 0, Z = 2) &= P(\{(0,0), (1,-1), (-1,1)\} \cap \{(1,-1), (-1,1), (1,1), (-1,-1)\}) \\ &= P(\{(1,-1), (-1,1)\}) = 2 \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}. \end{aligned}$$

$$P(Y = 1, Z = 0) = P(\{(1,0), (0,1)\} \cap \{(0,0)\}) = P(\emptyset) = 0.$$

(b) One can calculate the distributions of Y and Z directly or deduce them from the joint distribution calculated in part (a). The marginal distributions appear in the table below as well. We calculate $P(Z = 2)$ in greater detail in both ways:

$$P(Z = 2) = P(\{(1,-1), (-1,1), (1,1), (-1,-1)\}) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}.$$

$$P(Z = 2) = \sum_{i=-2}^{2} P(Y = i, Z = 2) = 1/9 + 0 + 2/9 + 0 + 1/9 = 4/9.$$

(c) We see, for example, that

$$P(Y = 0, Z = 2) = \frac{2}{9} \neq \frac{3}{9} \cdot \frac{4}{9} = P(Y = 0) \cdot P(Z = 2).$$

Hence, Y and Z are not independent.

	Y = -2	Y = -1	Y = 0	Y = 1	Y = 2	Z
Z = 0	0	0	1/9	0	0	1/9
Z = 1	0	2/9	0	2/9	0	4/9
Z = 2	1/9	0	2/9	0	1/9	4/9
Y	1/9	2/9	3/9	2/9	1/9	

2. **(a)** Since $X \sim Geom(p)$ we have

$$\begin{aligned} P(X > k) &= \sum_{t=k+1}^{\infty} P(X = t) = \sum_{t=k+1}^{\infty} p(1-p)^{t-1} = p(1-p)^k \sum_{t=0}^{\infty} (1-p)^t \\ &= p(1-p)^k \frac{1}{1-(1-p)} = (1-p)^k. \end{aligned}$$

(b) Fix integers $n \geq 0$ and $k > 0$. We have

$$\begin{aligned} P(X = n+k | X > n) &= \frac{P(X = n+k, X > n)}{P(X > n)} = \frac{P(X = n+k)}{P(X > n)} \\ &= \frac{p(1-p)^{n+k-1}}{(1-p)^n} = p(1-p)^{k-1} = P(X = k). \end{aligned}$$

where the second equality holds since, for $k > 0$, if $X = n+k$, then $X > n$, and the third equality holds by part (a) of this question.

(c) Since $X \sim Geom(p)$ it follows that $\mathbb{E}(X) = 1/p$ and $Var(X) = (1-p)/p^2$. Hence

$$\begin{aligned} \mathbb{E}(Y) &= \mathbb{E}\left(\frac{pX - 1}{\sqrt{1-p}}\right) = \frac{1}{\sqrt{1-p}} \cdot (p\mathbb{E}(X) - 1) \\ &= \frac{1}{\sqrt{1-p}} \cdot (p \cdot 1/p - 1) = 0. \end{aligned}$$

Similarly

$$Var(Y) = Var\left(\frac{pX - 1}{\sqrt{1-p}}\right) = \frac{1}{1-p} \cdot p^2 Var(X) = 1.$$

3. **(a)** By definition $N \sim Geom(1/6)$ and thus $\mathbb{E}(N) = 6$.

(b) X and N are not independent. For example, $P(X = 1, N = 1) = 0$ (as if $N = 1$, then there was only one die roll and it resulted in a 6 entailing $X = 0$). On the other hand, $P(N = 1) = 1/6$ and $P(X = 1) > 0$ (as, for example, the probability that the first die roll results in a 1 and the second in a 6 is $1/36$). Therefore $P(X = 1, N = 1) \neq P(X = 1) \cdot P(N = 1)$.

(c) Observe that for any positive integer n we have $X|N = n \sim Bin(n-1, 1/5)$ and thus, in particular, $\mathbb{E}(X|N = n) = (n-1)/5$. Indeed, the fact that $N = n$ entails that the result of the nth die roll is 6 and the results of the ith roll is in $\{1, \ldots, 5\}$ for every $1 \leq i \leq n-1$. By the law of total expectation we have

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}(\mathbb{E}(X|N)) = \sum_{n=1}^{\infty} \mathbb{E}(X|N = n) P(N = n) = \sum_{n=1}^{\infty} \frac{n-1}{5} \cdot \frac{1}{6}\left(\frac{5}{6}\right)^{n-1} \\ &= \frac{1}{30} \sum_{m=0}^{\infty} m \cdot \left(\frac{5}{6}\right)^m = \frac{1}{30} \sum_{m=1}^{\infty} (m - 1 + 1) \cdot \left(\frac{5}{6}\right)^m \\ &= \frac{1}{30} \cdot \frac{5}{6} \sum_{m=1}^{\infty} (m-1) \cdot \left(\frac{5}{6}\right)^{m-1} + \frac{1}{30} \sum_{m=1}^{\infty} \left(\frac{5}{6}\right)^m \\ &= \frac{5}{6} \cdot \frac{1}{30} \sum_{m=2}^{\infty} (m-1) \cdot \left(\frac{5}{6}\right)^{m-1} + \frac{1}{30} \cdot \frac{5/6}{1 - 5/6} \\ &= \frac{5}{6} \cdot \frac{1}{30} \sum_{k=1}^{\infty} k \cdot \left(\frac{5}{6}\right)^k + \frac{1}{6} = \frac{5}{6} \cdot \mathbb{E}(X) + \frac{1}{6}. \end{aligned}$$

Therefore $\mathbb{E}(X) = \frac{5}{6} \cdot \mathbb{E}(X) + \frac{1}{6}$ entailing $\mathbb{E}(X) = 1$.

4. **(a)** By the properties of ρ that were proved in the lecture we have

$$\rho(Y, 10 - Y) = \rho(Y, -Y) = -\rho(Y, Y) = -1.$$

(b) For every $1 \le i \le 420$, let X_i be the value of the ith die roll. We observe that $X_i \sim U(1, 2, \ldots, 6)$ and thus $\mathbb{E}(X_i) = \frac{1+6}{2} = \frac{7}{2}$ and $Var(X_i) = \frac{(6-1+1)^2-1}{12} = \frac{35}{12}$. Note that $X = \sum_{i=1}^{420} X_i$ and so, by linearity of expectation we have

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{420} X_i\right) = \sum_{i=1}^{420} \mathbb{E}(X_i) = 420 \cdot \frac{7}{2} = 1470.$$

Moreover, since the X_i's are independent we have

$$Var(X) = Var\left(\sum_{i=1}^{420} X_i\right) = \sum_{i=1}^{420} Var(X_i) = 420 \cdot \frac{35}{12} = 1225.$$

We can now use Chebyshev's inequality to conclude that

$$\begin{aligned} P(1400 < X < 1540) &= 1 - P(|X - 1470| \ge 70) = 1 - P(|X - \mathbb{E}(X)| \ge 70) \\ &\ge 1 - \frac{Var(X)}{70^2} = 1 - \frac{1225}{4900} = \frac{3}{4} \end{aligned}$$

as claimed.