coordinate axes Ox and Oy (for $C = 0$). **1175.** A family of planes $x + y + 3z = C$. **1176.** A family of spheres $x^2+y^2+z^2=C$. **1177.** A family of two-sheet hyperboloids $x^2-y^2-z^2=C$ (for $C>0$); a family of one-sheet hyperboloids $x^2-y^2-z^2=C$ (for $C<0$); a cone $x^2-y^2-z^2=0$ (for $C = 0$). **1183.** $\frac{\partial u}{\partial x} = 2x - 3y - 4$, $\frac{\partial u}{\partial y} = 4y - 3x + 2$. **1184.** $\frac{\partial r}{\partial \rho} = 2\rho\sin^4\theta\,\frac{\partial r}{\partial \theta} = 4\rho^2\sin^3\theta\cos\theta$. **1185.** $\frac{\partial u}{\partial x} = \frac{2x}{y^2} - \frac{1}{y}, \frac{\partial u}{\partial y} = \frac{x}{y^2} - \frac{2x^2}{y^3}$.

1186. $\frac{\partial z}{\partial x} = e^{xy(x^2+y^2)}(3x^2y + y^3), \frac{\partial z}{\partial y} = e^{xy(x^2+y^2)}(x^3 + 3y^2)$.

1187. $\frac{\partial u}{\partial x} = \frac{y}{\sqrt{x}}, \frac{\partial u}{\partial y} = 2\sqrt{x} + 6y\sqrt[3]{z^2}, \frac{\partial u}{\partial z} = \frac{2y^2}{\sqrt[3]{z}}$.

1188. $\frac{\partial u}{\partial x} = \frac{1}{y}e^{x/y}; \frac{\partial u}{\partial y} = -\frac{x}{y^2}e^{x/y} + \frac{z}{y^2}e^{-z/y}, \frac{\partial u}{\partial z} = -\frac{1}{y}e^{-z/y}$.

1189. $\frac{\partial z}{\partial x} = -\frac{2xy}{(1+x^2)^2+y^2}, \frac{\partial z}{\partial y} = \frac{1+x^2}{(1+x^2)^2+y^2}$.

1190. $\frac{\partial z}{\partial x} = 6x^2(x^3+y^2)e^{(x^3+y^2)^2}, \frac{\partial z}{\partial y} = 4y(x^3+y^2)e^{(x^3+y^2)^2}$.

1191. $\frac{\partial u}{\partial x} = (y - z)(2x - y - z), \frac{\partial u}{\partial y} = (x - z)(x - 2y + z), \frac{\partial u}{\partial z} = (x - y)(-y + 2z - x)$.

1192. $\frac{\partial u}{\partial x} = (6x - y)e^{3x^2+2y^2-xy}, \frac{\partial u}{\partial y} = (4y - x)e^{3x^2+2y^2-xy}$.

1193. $\frac{\partial u}{\partial y} = xze^{xyz}\sin\frac{y}{x} + \frac{1}{x}e^{xyz}\cos\frac{y}{x}$. **1195.** ρ. **1200.** $dz = \frac{2(xdx + ydy)}{x^2+y^2}$.

1201. $dz = \frac{2(xdy - ydx)}{x^2 + \sin(2y/x)}$. **1202.** $2(xdx+ydy)\cos(x^2+y^2)$. **1203.** $dz = xy\left(\frac{y}{x}dx + \ln x dy\right)$.

1204. $du = \frac{1}{\sqrt{x^2+y^2}}\left(dx + \frac{ydy}{x + \sqrt{x^2+y^2}}\right)$.

1205. $dz = e^x[(x\cos y - \sin y)dy + (\sin y + \cos y + x\sin y)dx]$. **1206.** $dz = e^{x+y}\{[(x + 1)\cos y + y(\sin x + \cos x)]dx + [x(\cos y - \sin y) + (y + 1)\sin x]dy\}$. **1207.** $dz = \frac{2dx}{x^2-4} + \frac{2\cos ydy}{\sin^2 y + 4}$. **1208.** $du = e^{xyz}(yzdx + xzdy + xydz)$. **1209.** 1.08. **1210.** $-$0.03. **1211.** 1.013. **1212.** 3.037. **1213.** 1.05. **1218.** $6(x+y)$. **1219.** $-\sin(x+y)$. **1210.** $-4\cos(2x+2y)/\sin^2(2x + 2y)$. **1221.** 0. **1222.** $x(x+2y)/(x+y)^2$. **1223.** $y(2-y^2)\cos xy - xy^2\sin xy$. **1224.** $\sin y\cos(x + \cos y)$. **1225.** $\frac{x^2-y^2}{(x^2+y^2)^2}[(dy)^2 - (dx)^2] - \frac{4xydxdy}{(x^2+y^2)^2}$. **1226.** $-\cos(x + y)(dx + dy)^2$. **1227.** $ae^y\{e^y\sin(ax + e^y) - \cos(ax + e^y)\}$. **1228.** 4. **1230.** $2[(dx)^2 - dxdy + (dy^2)]$. **1235.**

$4/\sin 2x$. **1236.** $2x(3x + 2)/(x^2 + 3x + 1)^2$. **1237.** $\frac{\partial z}{\partial x} = 2x\cos x, \frac{dz}{dx} = x(2\cos x - x\sin x)$.

1238. 0. **1239.** $\frac{\partial z}{\partial \xi} = 4\xi, \frac{\partial z}{\partial \eta} = 4\eta$. **1240.** $\frac{\partial u}{\partial \xi} = \frac{2}{\xi}, \frac{\partial u}{\partial \eta} = \frac{2(\eta^4 - 1)}{\eta(\eta^4 + 1)}$.

1246. $\left(\frac{\partial z}{\partial l}\right)_M = \frac{7}{5}$. **1247.** $\left(\frac{\partial u}{\partial l}\right)_M = \frac{1}{6}$.

1248. $\left(\frac{\partial u}{\partial l}\right)_M = \frac{7}{9}$. **1249.** $|\mathrm{grad}\, u|_M = 1/r_0^2$; $\cos\alpha = -x_0/r_0$, $\cos\beta = -y_0/r_0$, $\cos\gamma = -z_0/r_0$, where $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$.
1250. $|\mathrm{grad}\, u|_M = 3$, $\cos\alpha = 1/3$, $\cos\beta = \cos\gamma = 2/3$. **1251.** 1/3. **1256.** $-x/y$. **1257.** y/x. **1258.** $y' = -y/x, y'' = 2y/x^2$ **1259.** $(y\sqrt{2xy} - x^2)/(2y^2 - x\sqrt{2xy})$. **1260.** y/x. **1261.** $(a^2 + b^2)/(2b^2 - a^2)$. **1262.** $y/(2x)$. **1263.** $(x + y)/(x - y)$. **1264.** $1/(2^y\ln 2)$.

1265. $y' = -1, y'' = 0$, **1266.** $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = \frac{1}{x + y + z - 1}$. **1267.** $\frac{\partial z}{\partial x} = -\frac{x^2 - yz}{z^2 - xy}, \frac{\partial z}{\partial y} = -\frac{y^2 - xz}{z^2 - xy}$. **1268.** $\frac{dx + (z/y)dy}{1 + \ln(z/y)}$. **1269.** $-\frac{x\cos y + \sin x}{\sin x}$. **1270.** $-\frac{(y+z)dx + (x+z)dy}{x + y}$.
1271. $-y-z-e^{y-x}$. **1272.** 1. **1275.** $2x+2y-z=1$, $(x-1)/2=(y-1)/2=(z - 3)/(-1)$. **1276.** $2x + 2y - 3z + 1 = 0$, $(x - 2)/2 = (y - 2)/2 = (z - 3)/(-3)$. **1277.** $z - 2x + 2 = 0$, $(x-1)/2=y/0=z/(-1)$. **1278.** $x - y - 2z + 1 = 0$, $(x - \pi/4)/1 = (y - \pi/4)/(-1) = (z-1/2)/(-2)$. **1279.** $x+4y+6z\pm21=0$. **1281.** $(4/3; 4/3; 1/3)$ and $(-4/3; -4/3; -1/3)$. **1285.** $z_{max} = 1/64$. **1286.** $z_{min} = -125$. **1287.** $z_{max} = 4$. **1288.** $z_{min} = 0$. **1289.** $z_{max} = a\sqrt{3}/9$ at $x=y=2a/3$. **1293.** $z_{min}=144/25$ at the point $(36/25; 48/25)$. **1294.** $z_{least} = -16/3$, $z_{gr} = 16$. **1295.** $z_{least}=5$, $z_{gr}=11$. **1296.** $z_{least} = -1/2$, $z_{gr} = 1/2$. **1297.** $z_{least} = 1$, $z_{gr}=4$. **1298.** $z_{least} = -2(\sqrt{2} + 1) \cong -4.8$, $z_{gr} = 2(\sqrt{2} - 1) \cong 0.8$. **1299.** $z_{least} = 0$, $z_{gr} = 3\sqrt{3}/2$. **1300.** $z_{least} = -3$ for $x = y = 3\pi/2$, $z_{gr} = 1 + \sqrt{3}/2$ for $x = y = 5\pi/6$. **1301.** $z_{least} = -1/8$, $z_{gr} = 1$. **1302.** Equilateral. **1303.** Equilateral. **1304.** A square; $P_{least} = 4\sqrt{S}$. **1305.** A cube; $V_{max} = (S/6)\sqrt{S/6}$.

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1315. $(2/5)x^2\sqrt{x} + C$. **1316.** $(5/4)\sqrt[5]{x^4} + C$. **1317.** $2\arcsin x - x + C$. **1318.** $\arctan x + x - x^3/3 + C$. **1319.** $e^{3x}\cdot 3^x/(3 + \ln 3) + C$. **1320.** $\tan x - x + C$. **1321.** $\cosh x + \cos x + C$. **1322.** $x^2/2 - 2x + \ln|x| + C$. **1323.** $4\tan x - 9\cot x - x + C$. **1324.** $(1/2)\sin(x^2) + C$. **1325.** $\ln|\ln x| + C$. **1326.** $3(ax^2+b)^{4/3}/(8a)+C$. **1327.** $(2/3)\sin x\sqrt{\sin x}+C$. **1328.** $-(1/b)\cos(a + bx) + C$. **1329.** $\sin(\sin x) + C$. **1346.** $e^{\sqrt{2x-1}} + C$. **1347.** $-(1/32)(1 - 2x^4)^4 + C$. **1348.** $(1/3)\cos(2 - 3x) + C$. **1349.** $(1/10)\sinh(5x^2+3)+C$. **1350.** $2\arctan\sqrt{x} + C$. **1351.** $(1/5)(x^2 + 1)^{5/2} + C$. **1352.** $(1/2)\ln|x^2 - 1| + C$. **1353.** $(1/2)\ln|x^2 + \sqrt{x^4 - 1}| + C$. **1354.** $(-1/4)\arctan(0.5\cos^2 2x) + C$. **1355.** $(1/\sqrt{7})\ln|(\sqrt{x} - \sqrt{7})/(\sqrt{x} + \sqrt{7})| + C$. **1356.** $2\arcsin(e^{x/2}/4)+C$. **1357.** $\ln|x+\sqrt{2+x^2}| + \arcsin(x/\sqrt{2})+C$. **1358.** $(-2/9)\sqrt{2 - 3x^3} + C$. **1359.** $-5\sqrt{3 - x^2} + 3\arcsin(x/\sqrt{3}) + C$. **1360.** $(1/4)\arctan(x - 3)/4 + C$. **1361.** $2\sqrt{3x+5} + \sqrt{5}\ln|(\sqrt{3x+5} - \sqrt{5})/(\sqrt{3x+5} + \sqrt{5})| + C$. **1362.** $1/(2\sqrt{10})\arctan(x^2\sqrt{2/5}) + C$. **1370.** $(x^2/4)(2\ln x - 1) + C$. **1371.** $x\arcsin x + \sqrt{1 - x^2} + C$. **1372.** $(x^2/3)\arctan x - (1/6)x^2 +$

$+ (1/6)\ln(x^2+1) + C$. **1373.** $xe^x + C$. **1374.** $-x^2\cos x + 2x\sin x + 2\cos x + C$. **1375.** $(1/2)e^{x^2}(x^4 - 2x^2 + 2) + C$. **1376.** $(x+1)^2\sin x + 2(x+1)\cos x + C$ **1377.** $(e^{2x}/5)(\sin x + 2\cos x) + C$. **1378.** $(x/2)(\sin\ln x - \cos\ln x) + C$. **1379.** $-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$. **1387.** $-1/[3(x-1)^3] + C$. **1388.** $-1/[4(2x+3)^2] + C$. **1389.** $(1/3)\arctan(x-3)/3 + C$. **1390.** $1/(3\sqrt{2})\arctan(x^3+1)/\sqrt{2} + C$. **1391.** $(1/2)\ln(x^2 - 4x + 7) + C$. **1392.** $(5/2)\ln(x^2 + 10x + 29) - 11\arctan(x+5)/2 + C$. **1393.** $(1/10)\ln(5x^2 + 2x + 1) + (2/5)\arctan(5x+1)/2 + C$. **1394.** $x/[8(x^2+2)^2] + 3x/[32(x^2+1)] + (3\sqrt{2}/64)\arctan(x/\sqrt{2}) + C$. **1395.** $(x-7)/[8(x^2 + 2x + 5)] + (1/16)\arctan(x+1)/2 + C$. **1405.** $-(2/3)\ln|x| + (5/3)\ln|x-3| + C$. **1406.** $-(1/2)\ln(x^2 + x + 1) + 3\ln|x+2| + (1/\sqrt{3})\arctan(2x+1)/\sqrt{3} + C$. **1407.** $1/[2(x-1)^2] + 2\ln|x-1| + 3\ln|x-2| + C$. **1408.** $(1/12)\ln|x-2| - (1/24)\ln(x^2 + 2x + 4) - 1/(4\sqrt{3})\arctan(x+1)\sqrt{3} + C$. **1409.** $(31/108)\ln|x-3|x + (29/108)\ln|x+3| + (2/9)\ln(x^2+9) - (1/54)\arctan(x/3) + C$. **1410.** $(1/4)\ln|x/(x-2)| - (x-1)/[2x(x-2)] + C$. **1411.** $(1/16)\,[\ln(x^2+1)/(x^2+9)] + (1/8)\arctan(x/2) - (1/24)\arctan(x/3) + C$. **1412.** $(1/4)\ln[(x^2+4)/(x^2+2x+5)] + (1/8)\arctan(x/2) + (7/32)\arctan(x+1)/2 + C$. **1413.** $(1/2)[\arctan(x-1) + \arctan(x+1)] + C$. **1414.** $x + (9/2)\ln|x-3| - (1/2)\ln|x-1| + C$. **1415.** $x^2/2 + 7x + (75/2)\ln|x-5| - (1/2)\ln|x-1| + C$. **1416.** $x + (1/2)\ln|(x-2)/(x+2)| - \arctan(x/2) + C$. **1417.** $3x + \ln|x| + 2\arctan x + C$. **1427.** $-\sqrt{1-2x} - 2\sqrt[4]{1-2x} - 2\ln \times \times |\sqrt[4]{1-2x} - 1| + C$. **1428.** $(6/5)\sqrt[4]{x^5} - 2\sqrt{x} + 6\sqrt[6]{x} - 6\arctan\sqrt[6]{x} + C$. **1429.** $\ln|x - 1/2 + \sqrt{x^2 - x - 1}| + C$. **1430.** $\arcsin(x+1)/3 + C$. **1431.** $-5\sqrt{-x^2 + 4x + 5} + 13\arcsin(x-2)/3 + C$. **1432.** $3\sqrt{x^2 + x + 2} + (1/2)\ln|x + 1/2 + \sqrt{x^2 + x + 2}| + C$. **1433.** $\sqrt{x/(x+2)} + C$. **1434.** $-\arcsin[(x+1)/(x\sqrt{3})] + C$. **1435.** $\ln|x + \sqrt{x^2+1}| + \sqrt{2}\ln\left|\dfrac{1 - x + \sqrt{2(x^2+1}}{2(x+1)}\right| + C$. **1436.** $-(x/2 + 5\times\sqrt{-x^2 + 4x} + 13\arcsin(x-2)/2 + C$. **1440.** $3/\sqrt[3]{x+1} + \ln[x/(\sqrt[3]{x} + 1)^3] + C$. **1441.** $4\sqrt{\sqrt{x}+1}[(1/5)(\sqrt{x}+1)^2 - (2/3)(\sqrt{x}+1) + 1] + C$. **1442.** $(1/6)\ln[(t^2 + t + 1)/(t^2 - 2t + 1)] - (1/\sqrt{3})\arctan(2t+1)/\sqrt{3} + C$, where $t = \sqrt[3]{1 + x^3}/x$. **1443.** $(1/3)\ln[(\sqrt{1+x^3} - 1)^2/|x^3|] + C$. **1444.** $(1/10)(5x^{4/3} + 3)^{3/2} + C$. **1445.** $-(2 - x^3)^{2/3}/(4x^2) + C$. **1463.** $(1/5)\ln|5\tan(x/2) + 3| + C$. **1464.** $-2/[\tan(x/2) - 1] + C$. **1465.** $-(1/17)x + (1/4)\ln|\sin x| - (1/68)\ln|\sin x + 4\cos x| + C$. **1466.** $\ln|\sin x| - \sin x + C$. **1467.** $-(2/5)\ln(1 - \cos x) + (1/5)\ln(\cos^2 x + 2\cos x + 2) - (6/5)\arctan(1 + \cos x) + C$. **1468.** $(1/3)\cos^3 x - \cos x + C$. **1469.** $\ln|\sin x| - \sin^2 x + (1/4)\sin^4 x + C$. **1470.** $(1/8)x - (1/8)\sin x + C$. **1471.** $(3/8)x + (1/4)\sin 2x + (1/32)\sin 4x + C$. **1472.** $(2/3)\tan^3(x/2) - 2\tan(x/2) + x + C$. **1473.** $-(1/6)\cot^2 3x - (1/3)\ln|\sin 3x| + C$. **1474.** $\tan x + (2/3)\tan^3 x + (1/5)\tan^5 x + C$. **1475.** $-(1/3)\cot^3 x + C$. **1476.** $(1/2)\tan x\sec x + (1/2)\ln|\tan(x/2 + \pi/4)| + C$. **1477.** $-(1/2)\cot x\operatorname{cosec} x - (1/2)\ln|\tan(x/2)| + C$. **1478.** $(1/4)\sin 2x - (1/8)\sin 4x + C$. **1479.** $(3/5)\sin(5x/6) + 3\sin(x/6) + C$. **1483.** $x/\sqrt{1 - x^2} + C$. **1484.** $x/(a^2\sqrt{a^2 + x^2}) + C$. **1485.** $(1/2)(\arccos(1/x) + \sqrt{x^2 - 1}/x^2) + C$. **1486.** $-(1/6)\cos 3x + (1/20)\cos 5x + (1/4) \times \cos x + C$. **1487.** $-2(2x^2 + 6x + 13)e^{-x/2} + C$. **1488.** $-(2\ln x + 1)/(4x^2) + C$. **1489.** $x\arctan x - (3/2)(\arctan x)^2 - (1/2)\ln(1 + x^2) + C$. **1490.** $(x^2 - 1)\sin 2x + x\cos 2x + C$. **1491.** $(1/4)x^2(2\ln^2 x - 2\ln x + 1) + C$. **1492.** $x + \ln|e^x - 3| + C$. **1493.** $(x+1)\arctan\sqrt{x} - \sqrt{x} + C$. **1494.** $(2/\ln 2)(\sqrt{2^x - 1} - \arctan\sqrt{2^x - 1}) + C$. **1495.** $(1/4)\ln|(t+1)/(t-1)| - (1/2) \times \arctan t + C$, where $t^4 = 1 + x^{-4}$. **1496.** $\sqrt{2}[(1/2)(x-1)\sqrt{3 + 2x - x^2} + 2\arcsin(x-1)/2] + C$. **1497.** $\sin e^x - e^x\cos e^x + C$. **1498.** $(1/\sqrt{5})\ln|\tan x + \sqrt{\tan^2 x + 2/5}| + C$. **1499.** $(1/2) \times \cos^2 x(1 - 2\ln\cos x) + C$. **1500.** $(1/2)\sin(x^2 + 4x + 1) + C$. **1501.** $-0.5(x/\sin^2 x + \cot x) + C$.

1502. $2(x-2)\sqrt{1 + e^x} - 2\ln[(\sqrt{1 + e^x} - 1)/(\sqrt{1 + e^x} + 1)] + C$. **1503.** $x\ln(x^2 + x) + \ln|x + 1| - x + C$. **1504.** $-1/x - \arctan x + C$. **1505.** $(x/2)(\cos\ln x + \sin\ln x) + C$. **1506.** $12[\sqrt[12]{x} + \ln[(\sqrt[12]{x} - 1)^2/\sqrt[12]{x}]] + C$. **1507.** $[e^{\alpha x}/(\alpha^2 + \beta^2)] \times (\alpha\sin\beta x - \beta\cos\beta x) + C$. **1508.** $[e^{\alpha x}/\alpha^2 + \beta^2)](\beta\sin\beta x + \alpha\cos\beta x) + C$. **1509.** $[1/(ab)]\arctan[(b/a)\tan x] + C$. **1510.** $\tan x - \cot x + C$. **1511.** $0.5(\arctan x + x/(1 + x^2)) + C$.

Chapter 10

1521. $1/2$. **1522.** $e - 1$. **1523.** $0 < I \leqslant 4/27$. **1524.** $\pi/2 \leqslant I \leqslant e\pi/2$. **1525.** $0 < I < 1$. **1526.** $464\sqrt{2}/15$. **1527.** $\pi/8$. **1528.** $e - \sqrt{e}$. **1529.** $e^e - e$. **1530.** $(e^{\pi/2} - 1)/2$. **1531.** $(\ln 3 - 1)/2$. **1532.** $\ln 1.5$. **1533.** 0. **1534.** $2/5$. **1535.** $\pi/2$. **1536.** $\ln(4/3)$. **1537.** $(e^{\pi/2} - 1)/2$. **1538.** 0. **1539.** $\pi/2 - 1$. **1546.** $\pi^2/8$. **1547.** $\pi/4$. **1548.** $256/15$. **1549.** π. **1550.** $+\infty$. **1551.** $1/4$. **1552.** $\pi/6$. **1559.** Diverges. **1560.** Converges. **1561.** Diverges. **1562.** Converges. **1563.** Diverges. **1564.** Converges. **1565.** Diverges. **1569.** 4.5 (sq. units). **1570.** 18 (sq. units). **1571.** 2/15 (sq. units). **1572.** $(41/2)\arcsin(9/41) + 20\ln 0.8$ (sq. units). **1573.** $\sqrt{2} - 1$ (sq. units). **1574.** 8 (sq. units). **1575.** $(9\pi/4) - \sqrt{2} + 4\sqrt{2}\ln 2 - (9/2)\arcsin(1/3)$ (sq. units). **1576.** 169π (sq. units). **1577.** $(3/8)\pi a^2$. **1578.** $8\sqrt{3}/3$ (sq. units). **1579.** $(3/2)\pi a^2$. **1580.** $(3\pi - 8)/32$ (sq. units). **1581.** $\pi a^2/12$. **1582.** $\pi/3 + \sqrt{3}/2$ (sq. units). **1586.** $(1/2)\ln 3$. **1587.** $(20/9)\sqrt{5/3}$. **1588.** $0.5[\sqrt{2} + \ln(1 + \sqrt{2})]$. **1589.** $(1/2)\ln 3$. **1590.** $\sinh 1 \approx 1.17$. **1591.** 12. **1592.** $\sqrt{2}(\pi - 1)$. **1593.** 5π. **1594.** 72. **1595.** $[(\pi^2 + 4)\sqrt{\pi^2 + 4} - 8]/3$. **1596.** πa. **1597.** $a(2\pi + 3\sqrt{3})/8$. **1598.** 8. **1601.** $16\pi(5\pi + 8)/5$ (cu. units). **1602.** 0.3π (cu. units). **1603.** $\pi(e^2 + 1)/4$ (cu. units). **1604.** $4\pi/35$ (cu. units). **1605.** 72 (cu. units). **1606.** $2a^2h/3$. **1607.** $\pi r^2 h/2$. **1609.** $\pi(e^2 - e^{-2} + 4)$ (sq. units). **1610.** $61\pi/1728$ (sq. units). **1611.** $2\pi b\,[b + (a^2/c^2)\arcsin(c/a)]$, where $c^2 = a^2 - b^2$. **1612.** $64\pi/3$ (sq. units). **1617.** $M_x = a^2(e^2 - e^{-2} + 4)/8$; $I_x = a^3(e - e^{-1})(e^2 + e^{-2} + 10)/24$. **1618.** $M_a = ah^2/6$; $I_a = ah^3/12$. **1619.** $I_x = 1628/105$. **1620.** $I_x = ab^3/12$; $I_y = a^3b/12$. **1621.** $I_0 = \pi d^4/32$. **1626.** $\bar{x} = 0, \bar{y} = 2r/\pi$ (for a semicircle); $\bar{x} = 0, \bar{y} = 4r/(3\pi)$ (for half a disc). **1627.** $\bar{x} = (\pi - 2)/2, \bar{y} = \pi/8$. **1628.** $\bar{x} = 0, \bar{y} = 8/5$. **1629.** $\bar{x} = \bar{y} = 2a/5$. **1630.** $\bar{x} = 1, \bar{y} = 2/5$. **1631.** $\bar{x} = 1, \bar{y} = \pi/8$. **1632.** $\pi r^3(3\pi - 4)/3$. **1634.** 480π (cu. units). **1643.** $\pi\rho g r^2 h^2/4$. **1644.** $\pi\rho g a d^3/8$. **1645.** $51\,450\pi$ J. **1646.** 547.8π J. **1647.** 50.7 J. **1648.** $\rho g a h^2/3$. **1649.** 17.64π kPa; 70.56π kPa; 158.76π kPa; 282.24π kPa. **1650.** $\rho g\pi d^3/8$. **1651.** 150 kg. **1652.** 1400 m. **1653.** $x = e^{10}$. **1654.** 36 m. **1663.** (1) $y' = \sinh x/\sqrt{\cosh^2 x + 1}$; (2) $y' = \sinh^2(x/15)\,\cosh^3(x/15)$; (3) $y' = 1/\cosh x$; (4) $y' = 1/\cosh^6 x$; (5) $y' = 1/\cosh x$; (6) $y' = x/\sinh(x^2/2)$. **1664.** $M[\ln(1 + \sqrt{2}); \sqrt{2}]$; **1665.** $y_{\min} = 0$ at $x = 0$. **1666.** (1) $x^2\sinh x - 2x\cosh x + 2\sinh x + C$; (2) $(1/32)\sinh 4x - (1/4)\sinh 2x + 6x + C$; (3) $2\arctan\sqrt{\cosh x - 1} + C$; (4) $(1/2)(\cosh x\sinh x - \sin x\cos x) + C$; (5) $\tanh^2(x/2) + C$; (6) $(3/5)\cosh^5(x/3) - \cosh^3(x/3) + C$. **1667.** (1) $\pi/6$; (2) $\ln 2 - 0.6$; (3) $2(2\ln 3 - 1)/3$. **1668.** $\sinh(x - a) = \sinh x\cosh a - \cosh x\sinh a$; $\cosh(x - a) = \cosh x\cosh a - \sinh x\sinh a$. **1669.** $\tanh(x + a) = (\tanh x + \tanh a)/(1 + \tanh x\tanh a)$; $\tanh(x - a) = (\tanh x - \tanh a)/(1 - \tanh x\tanh a)$; $\tanh 2x = 2\tanh x/(1 + \tanh^2 x)$. **1670.** $\sinh(x/2) = \pm\sqrt{(\cosh x - 1)/2}$; $\cosh(x/2) = \sqrt{(\cosh x + 1)/2}$; $\tanh(x/2) = \pm\sqrt{(\cosh x - 1)/(\cosh x + 1)}$. **1671.** $2\sinh(x \pm y)/2 \cdot \cosh(x \mp y)/2$; $2\cosh(x + y)/2 \cdot \cosh(x - y)/2$; $2\sinh(x + y)/2 \cdot \sin(x - y)/2$; $\sinh(x \pm$

We have lowered the degree of x by unity. To find $\int xe^x dx$, we shall once again apply integration by parts. Putting $u = x$, $dv = e^x dx$, we obtain $du = dx$, $u = e^x$ and

$$\int x^2 e^x dx = x^2 e^x - 2(xe^x - \int e^x dx)$$
$$= x^2 e^x - 2xe^x + 2e^x + C = e^x(x^2 - 2x + 2) + C.$$

1367. Find the integral $I = \int e^x \sin x dx$.

Solution. Assume $u = e^x$, $dv = \sin x dx$; then $du = e^x dx$, $v = -\cos x$. Consequently,

$$I = -e^x \cos x + \int e^x \cos x dx.$$

It seems we have not attained our aim by integrating by parts since we have not simplified the integral. Let us, however, apply integration by parts once again. Assuming $u = e^x$, $dv = \cos x dx$, whence $du = e^x dx$, $v = \sin x$, we get

$$I = -e^x \cos x + (e^x \sin x - I), \text{ i.e. } I = -e^x \cos x + e^x \sin x - I.$$

Applying twice integration by parts, we again obtain the original integral on the right-hand side of the equation. Thus, we arrive at an equation with an unknown integral I, from which we find

$$2I = -e^x \cos x + e^x \sin x, \text{ i.e. } I = \frac{e^x}{2}(\sin x - \cos x) + C.$$

In the final result we have added an arbitrary constant to the primitive we have obtained.

1368. Find the integral $\int \sqrt{a^2 - x^2} dx$, if $a > 0$.

Solution. We put $u = \sqrt{a^2 - x^2}$, $dv = dx$, whence $du = -\frac{xdx}{\sqrt{a^2 - x^2}}$, $v = x$. Consequently,

$$\int \sqrt{a^2 - x^2} dx = x\sqrt{a^2 - x^2} - \int \frac{-x^2 dx}{\sqrt{a^2 - x^2}} = x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}} dx,$$

or

$$\int \sqrt{a^2 - x^2} dx = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2} dx + a^2 \arcsin \frac{x}{a}.$$

Hence it follows that

$$2\int \sqrt{a^2 - x^2} dx = x\sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a},$$

i.e.

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C.$$

1369. Derive the recurrence relation for the integral $\int \frac{dx}{(x^2 + a^2)^n}$.

Solution. The integral can be transformed as follows:

$$I_n = \int \frac{dx}{(x^2 + a^2)^n} = \frac{1}{a^2} \int \frac{a^2 + x^2 - x^2}{(x^2 + a^2)^n} dx = \frac{1}{a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}}$$

$$-\frac{1}{a^2} \int \frac{x \cdot xdx}{(x^2 + a^2)^n} = \frac{1}{a^2} \cdot I_{n-1} - \frac{1}{a^2} \int x \cdot \frac{xdx}{(x^2 + a^2)^n}.$$

Let us put $u = x$, $dv = \frac{xdx}{(x^2 + a^2)^n}$; then $du = dx$, and

$$v = \frac{1}{2} \int (x^2 + a^2)^{-n} \cdot d(x^2 + a^2) = -\frac{1}{2(n-1)} \cdot \frac{1}{(x^2 + a^2)^{n-1}},$$

whence

$$I_n = \frac{1}{a^2} I_{n-1} + \frac{1}{a^2} \left[\frac{x}{2(n-1)(x^2 + a^2)^{n-1}} - \frac{1}{2(n-1)} \cdot \int \frac{dx}{(x^2 + a^2)^{n-1}} \right],$$

or

$$I_n = \frac{1}{a^2} \cdot I_{n-1} + \frac{x}{2a^2(n-1)(x^2 + a^2)^{n-1}} - \frac{1}{2a^2(n-1)} I_{n-1},$$

that is

$$I_n = \frac{x}{2a^2(n-1)(x^2 + a^2)^{n-1}} + \frac{1}{a^2} \frac{2n-3}{2n-2} \cdot I_{n-1}.$$

Putting $n = 2$, we obtain an expression for the integral I_2 in terms of the elementary functions. Assuming now $n = 3$, we find the integral I_3 (the integral I_2 having been found). In this way we can find I_n for any integral positive n.

Find the following integrals:

1370. $\int x \ln x dx$.	**1371.** $\int \arcsin x dx$.
1372. $\int x^2 \arctan x dx$.	**1373.** $\int (x + 1)e^x dx$.
1374. $\int x^2 \sin x dx$.	**1375.** $\int x^5 e^{x^2} dx$. *Hint.* Put $x^2 = t$.
1376. $\int (x^2 + 2x + 3) \cos x dx$.	**1377.** $\int e^{2x} \cos x dx$.
1378. $\int \sin \ln x dx$.	**1379.** $\int \sin \sqrt{x} dx$. *Hint.* Put $\sqrt{x} = t$.

9.2. Integration of Rational Fractions

9.2.1. Integration of partial fractions. A *rational fraction* is a fraction of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials. A rational fraction is said to be *proper* if the polynomial $P(x)$ is of lower degree than the polynomial $Q(x)$; otherwise the fraction is said to be *improper*.

The term ***partial*** (***elementary***) fractions is used for proper fractions of the following types:

1345. Find the integral $\int \frac{\sin\sqrt{x} + \cos\sqrt{x}}{\sqrt{x} \cdot \sin 2\sqrt{x}} dx$.

Solution. Putting $\sqrt{x} = t$, $x = t^2$, $dx = 2tdt$, we get

$$\int \frac{\sin\sqrt{x} + \cos\sqrt{x}}{\sqrt{x} \cdot \sin 2\sqrt{x}} dx = \int \frac{(\sin t + \cos t) \cdot 2t}{t \sin 2t} dt =$$
$$= \int \frac{\sin t + \cos t}{\sin t \cdot \cos t} dt = \int \left(\frac{1}{\cos t} + \frac{1}{\sin t}\right) dt$$
$$= \ln \left|\tan\left(\frac{t}{2} + \frac{\pi}{4}\right)\right| + \ln\left|\tan\frac{t}{2}\right| + C$$

(see formulas XXII and XXIII).

Returning to the old variable, we get

$$\int \frac{(\sin\sqrt{x} + \cos\sqrt{x})}{\sqrt{x} \cdot \sin 2\sqrt{x}} dx = \ln\left|\tan\left(\frac{\sqrt{x}}{2} + \frac{\pi}{4}\right)\right| + \ln\left|\tan\frac{\sqrt{x}}{2}\right| + C.$$

Find the following integrals:

1346. $\int \frac{e^{\sqrt{2x-1}}}{\sqrt{2x-1}} dx$. **1347.** $\int x^3(1 - 2x^4)^3 dx$.

1348. $\int \sin(2 - 3x) dx$. **1349.** $\int x \cosh(5x^2 + 3) dx$.

1350. $\int \frac{dx}{(x+1)\sqrt{x}}$. **1351.** $\int x(x^2 + 1)^{3/2} dx$.

1352. $\int \frac{x dx}{x^2 - 1}$. **1353.** $\int \frac{x dx}{\sqrt{x^4 - 1}}$.

1354. $\int \frac{\sin 4x dx}{\cos^4 2x + 4}$. **1355.** $\int \frac{dx}{(x - 7)\sqrt{x}}$.

1356. $\int \frac{e^{x/2} dx}{\sqrt{16 - e^x}}$. **1357.** $\int \frac{\sqrt{2 - x^2} + \sqrt{2 + x^2}}{\sqrt{4 - x^4}} dx$.

Hint. Represent the integral as a sum of integrals.

1358. $\int \frac{x^2 dx}{\sqrt{2 - 3x^3}}$. **1359.** $\int \frac{5x + 3}{\sqrt{3 - x^2}} dx$.

1360. $\int \frac{dx}{x^2 - 6x + 25}$. **1361.** $\int \frac{\sqrt{3x + 5}}{x} dx$.

1362. $\int \frac{x dx}{2x^4 + 5}$.

9.1.3. Integration by parts. *Integration by parts* is the calculation of an integral by the formula

$$\int u dv = uv - \int v du,$$

where $u = \varphi(x)$, $u = \psi(x)$ are continuously differentiable functions of x. With the aid of this formula, calculation of the integral $\int u dv$ reduces to finding another integral, $\int v du$; its application is expedient in the cases when the latter integral is either simpler than the former or is identical to it.

Then a function which becomes simpler under differentiation is taken as u and a part of the element of integration whose integral is known or can be found is taken as dv.

Thus, for instance, for integrals of the form $\int P(x)e^{ax}dx$, $\int P(x)\sin ax dx$, $\int P(x)\cos ax dx$, where $P(x)$ is a polynomial, it is advisable to take $P(x)$ as u and the expressions $e^{ax}dx$, $\sin ax dx$, $\cos ax dx$, respectively, as dv; for integrals of the form $\int P(x)\ln x dx$, $\int P(x) \arcsin x dx$, $\int P(x) \arccos x dx$, the functions $\ln x$, $\arcsin x$, $\arccos x$, respectively, should be taken as u and the expression $P(x)dx$ as dv.

1363. Find the integral $\int \ln x dx$.

Solution. We put $u = \ln x$, $dv = dx$; then $v = x$, $du = \frac{dx}{x}$. Employing the formula for integration by parts, we obtain

$$\int \ln x dx = x \ln x - \int x \frac{dx}{x} = x \ln x - \int dx = x \ln x - x + C = x(\ln x - 1) + C.$$

1364. Find the integral $\int \arctan x dx$.

Solution. Assume $u = \arctan x$, $dv = dx$; then $du = \frac{dx}{1 + x^2}$, $v = x$. We get by the formula for integration by parts

$$\int \arctan x dx = x \arctan x - \int \frac{x dx}{1 + x^2} = x \arctan x - \frac{1}{2}\ln(1 + x^2) + C.$$

1365. Find the integral $\int x \sin x dx$.

Solution. We put $u = x$, $dv = \sin x dx$; then $du = dx$, $v = -\cos x$. Hence, we have

$$\int x \sin x dx = -x\cos x + \int \cos x dx = -x \cos x + \sin x + C.$$

Should we choose some other expressions for u and dv, say, $u = \sin x$, $dv = x dx$, we would have got $du = \cos x dx$, $v = (1/2)x^2$, whence

$$\int x \sin x dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x dx = \frac{1}{2}x^2 \sin x - \frac{1}{2}\int x^2 \cos x dx,$$

and we would have arrived at an integral more complex than the original one, since the power of the factor before the trigonometric function has become higher by unity.

1366. Find the integral $\int x^2 e^x dx$.

Solution. We put $u = x^2$, $dv = e^x dx$; then $du = 2x dx$, $v = e^x$. We apply the formula for integration by parts:

$$\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx.$$