Probability Theory 1 – Proposed solution of moed aleph exam 2018

1. **(a)** It is evident that the support of X is $\{0,1\}$ and that the support of Y is $\{0,1,2,3\}$. For every $0 \le i \le 1$ and $0 \le j \le 3$, the table below shows the value of $P(X=i, Y=j)$. We explain three of these calculations in greater detail.

   Since there are only two black balls and we draw three balls, at least one of these balls must be red or white. Hence, $P(X=0, Y=0) = 0$.

$$P(X=0,Y=2) = \frac{\binom{1}{0}\binom{2}{1}\binom{3}{2}}{\binom{6}{3}} = \frac{1\cdot 2\cdot 3}{20} = \frac{3}{10}.$$

$$P(X=1,Y=0) = \frac{\binom{1}{1}\binom{2}{2}\binom{3}{0}}{\binom{6}{3}} = \frac{1\cdot 1\cdot 1}{20} = \frac{1}{20}.$$

	Y = 0	Y = 1	Y = 2	Y = 3
X = 0	0	3/20	3/10	1/20
X = 1	1/20	3/10	3/20	0

   (b) Using the table from (a) we conclude that

$$\begin{aligned} P(X=0) &= P(X=0,Y=0)+P(X=0,Y=1)+P(X=0,Y=2)+P(X=0,Y=3) \\ &= 0+3/20+3/10+1/20 = 1/2 \end{aligned}$$

   and

$$\begin{aligned} P(X=1) &= P(X=1,Y=0)+P(X=1,Y=1)+P(X=1,Y=2)+P(X=1,Y=3) \\ &= 1/20+3/10+3/20+0 = 1/2. \end{aligned}$$

   (c) Using the table from (a) we conclude that

$$P(X>Y) = P(X=1,Y=0) = 1/20.$$

2. **(a)** By the definition of the process, the values that X can have (with positive probability) are positive integers. For every positive integer i we have $X=i$ if and only if the result of the first $i-1$ coin flips is 0, then, for some positive integer j, the result of the next j coin flips is 1, and finally the result of the $(i+j)$th coin flip is 0. Since the coin is fair and all coin flips are mutually independent, summing over all possible j's we obtain

$$P(X=i) = \sum_{j=1}^{\infty} P(0^{i-1}1^j0) = \sum_{j=1}^{\infty}(1/2)^{i+j} = (1/2)^i\sum_{j=1}^{\infty}(1/2)^j = (1/2)^i,$$

   where $0^{i-1}1^j0$ denotes the binary word consisting of $i-1$ zeroes, followed by j ones and ending with an additional zero.

(b) Similarly to (a), the values that Y can have (with positive probability) are positive integers. For every positive integer j we have $Y = j$ if and only if, for some positive integer i, the result of the first $i-1$ coin flips is 0, the result of the next j coin flips is 1, and finally the result of the $(i+j)$th coin flip is 0. Since the coin is fair and all coin flips are mutually independent, summing over all possible i's we obtain

$$P(Y = j) = \sum_{i=1}^{\infty} P(0^{i-1}1^j0) = \sum_{i=1}^{\infty} (1/2)^{i+j} = (1/2)^j \sum_{i=1}^{\infty} (1/2)^i = (1/2)^j.$$

(c) It follows from the result of Parts (a) and (b) that $X \sim \mathit{Geom}(1/2)$ and $Y \sim \mathit{Geom}(1/2)$. By linearity of expectation we thus have $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 2+2 = 4$. Clearly $X+Y$ is a non-negative random variable. Hence, we can apply Markov's inequality to conclude that

$$P(X+Y \geq 12) \leq \frac{\mathbb{E}(X+Y)}{12} = \frac{4}{12} = \frac{1}{3}.$$

3. **(a)** It is evident from the definition of X that it only takes values of the form $6i+j$ where i is a non-negative integer and $j \in \{1, \ldots, 5\}$. That is, $P(X = k) = 0$ for every $k \in \mathbb{N} \setminus \{6i + j : i \in \mathbb{N} \cup \{0\}, j \in \{1, \ldots, 5\}\}$. Now, fix some $i \in \mathbb{N} \cup \{0\}$ and $j \in \{1, \ldots, 5\}$. Then $X = 6i + j$ if and only if the result of the first i die rolls is 6 and the result of the $(i+1)$st die roll is j. Since all coin flips are mutually independent, we have $P(X = 6i+j) = (1/6)^i \cdot 1/6 = (1/6)^{i+1}$.

(b) It follows by Part (a) of this question that

$$\sum_{k=1}^{\infty} P(X = k) = \sum_{i=0}^{\infty} \sum_{j=1}^{5} (1/6)^{i+1} = 5 \sum_{i=1}^{\infty} (1/6)^i = \frac{5}{6} \cdot \frac{1}{1-1/6} = 1.$$

(c) We have

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=1}^{\infty} k \cdot P(X = k) = \sum_{i=0}^{\infty} \sum_{j=1}^{5} (6i+j)(1/6)^{i+1} = \sum_{i=0}^{\infty} \sum_{j=1}^{5} i \cdot (1/6)^i + \sum_{i=0}^{\infty} \sum_{j=1}^{5} j \cdot (1/6)^{i+1} \\
&= 5 \sum_{i=1}^{\infty} i \cdot (1/6)^i + \sum_{i=0}^{\infty} (1/6)^{i+1} \sum_{j=1}^{5} j = \sum_{i=1}^{\infty} i \cdot (5/6) \cdot (1/6)^{i-1} + 15 \sum_{i=1}^{\infty} (1/6)^i \\
&= 6/5 + 15 \cdot \frac{1/6}{1-1/6} = 21/5,
\end{aligned}$$

where the first equality holds by a result which was proved in the lecture and in the fifth equality we used the fact that $\sum_{i=1}^{\infty} i \cdot (5/6) \cdot (1/6)^{i-1}$ is the expectation of a random variable $Y \sim \mathit{Geom}(5/6)$ and thus equals $6/5$.

4. **(a)** If $n = 3$, then we rolled the die three times and $X = X_1 + X_2$. Since $X_i \in \{-1, 0, 1\}$ for $i \in \{1, 2\}$, it follows that $P(X = 1) = P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 0)$. Note that $P(X_1 = 0, X_2 = 1)$ is the probability that the results of the three die rolls are a, a, b, where $1 \leq a < b \leq 6$. Hence, $P(X_1 = 0, X_2 = 1) = \frac{\binom{6}{2}}{6^3} = 5/72$. Similarly $P(X_1 = 1, X_2 = 0) = 5/72$ and thus $P(X = 1) = 5/36$.

(b) Observe that, for every $1 \leq i \leq n-1$, we have $P(X_i = 1) = P(X_i = -1)$. Indeed, switching the results of the ith and $(i+1)$th die rolls is a bijection from $\{X_i = 1\}$ to $\{X_i = -1\}$. Hence, for every $1 \leq i \leq n-1$, we have

$$\mathbb{E}(X_i) = -1 \cdot P(X_i = -1) + 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = 0.$$

By linearity of expectation we conclude that

$$\mathbb{E}(X) = \sum_{i=1}^{n-1} \mathbb{E}(X_i) = 0.$$

(c) We will use the formula $Var(X) = \sum_{i=1}^{n-1} Var(X_i) + 2\sum_{1 \leq i < j \leq n-1} Cov(X_i, X_j)$. First, for every $1 \leq i \leq n-1$ we have

$$\begin{aligned} Var(X_i) &= \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = P(X_i^2 = 1) = 1 - P(X_i^2 = 0) = 1 - P(X_i = 0) \\ &= 1 - \frac{6}{6^2} = 5/6. \end{aligned}$$

Next, fix some $1 \leq i < j \leq n-1$. If $j \geq i+2$, then X_i and X_j are determined by disjoint pairs of die rolls. Hence, X_i and X_j are independent and, in particular, $Cov(X_i, X_j) = 0$. On the other hand, if $j = i+1$, then

$$\begin{aligned} Cov(X_i, X_j) &= \mathbb{E}(X_i X_{i+1}) - \mathbb{E}(X_i)\mathbb{E}(X_{i+1}) = P(X_i X_{i+1} = 1) - P(X_i X_{i+1} = -1) \\ &= P(X_i = 1, X_{i+1} = 1) + P(X_i = -1, X_{i+1} = -1) \\ &\quad - P(X_i = 1, X_{i+1} = -1) - P(X_i = -1, X_{i+1} = 1). \end{aligned} \tag{1}$$

Observe that $P(X_i = 1, X_{i+1} = 1)$ is the probability that the results of the ith, $(i+1)$th and $(i+2)$th die rolls are a, b, c, where $1 \leq a < b < c \leq 6$. Hence, $P(X_i = 1, X_{i+1} = 1) = \frac{\binom{6}{3}}{6^3} = 5/54$. Similarly, $P(X_i = -1, X_{i+1} = -1) = 5/54$. Now, $P(X_i = 1, X_{i+1} = -1)$ is the probability that the results of the ith, $(i+1)$th and $(i+2)$th die rolls are a, b, c, where $1 \leq a, c < b \leq 6$. We distinguish between two cases, namely, $a = c$ and $a \neq c$. In the former case there are $\binom{6}{2} = 15$ choices for a, b and c. In the latter case there are $2\binom{6}{3} = 40$ choices for a, b and c. Hence, $P(X_i = 1, X_{i+1} = -1) = \frac{15+40}{6^3} = 55/216$. Similarly, $P(X_i = -1, X_{i+1} = 1) = 55/216$. Combining this with (1) yields $Cov(X_i, X_{i+1}) = 2 \cdot 5/54 - 2 \cdot 55/216 = -35/108$. Putting everything together, we conclude that

$$Var(X) = 5(n-1)/6 - 35(n-2)/54 = (10n - 25)/54.$$