Practical 4

Exercise 1 A library holds 10 different books on probability theory, of which 5 include solutions to all exercises and 5 do not. In an unexplained burst of anger, the librarian chose one of these books uniformly at random and destroyed it. Alon comes to the library, chooses one of the remaining 9 books on probability theory uniformly at random, borrows it and then returns it one week later. Two months later, Ben comes to the library, chooses one of the remaining 9 books on probability theory uniformly at random, borrows it and then returns it one week later. Let A be the event that Alon borrows a book with solutions and let B be the event that Ben borrows a book with solutions. Are the events A and B independent?

Solution

The intuition of most people would suggest that A and B are independent. However, we will show that this intuition is false. Let C be the event that the book which was destroyed by the librarian included solutions. Then, it follows by the Law of total probability that

$$\mathbb{P}\left(A\right) = \mathbb{P}\left(A \mid C\right) \cdot \mathbb{P}\left(C\right) + \mathbb{P}\left(A \mid C^c\right) \cdot \mathbb{P}\left(C^c\right) = \frac{4}{9} \cdot \frac{1}{2} + \frac{5}{9} \cdot \frac{1}{2} = \frac{1}{2}.$$

Similarly $\mathbb{P}\left(B\right) = \frac{1}{2}$.

Consider now the conditional probability space $(\Omega, \mathbb{P}(\cdot \mid C))$. We index the remaining 9 probability theory books with the integers $1, 2, \ldots, 9$, such that the books which include solutions are indexed $1, 2, 3, 4$. It is then evident that $|\Omega| = 81$ and $|A \cap B| = 16$. Since $(\Omega, \mathbb{P}(\cdot \mid C))$ is a uniform probability space, it follows that $\mathbb{P}\left(A \cap B \mid C\right) = 16/81$. Similarly, $\mathbb{P}\left(A \cap B \mid C^c\right) = 25/81$. Hence, it follows by the Law of total probability that

$$\mathbb{P}\left(A \cap B\right) = \mathbb{P}\left(A \cap B \mid C\right) \cdot \mathbb{P}\left(C\right) + \mathbb{P}\left(A \cap B \mid C^c\right) \cdot \mathbb{P}\left(C^c\right) = \frac{16}{81} \cdot \frac{1}{2} + \frac{25}{81} \cdot \frac{1}{2} > \frac{1}{4} = \mathbb{P}\left(A\right) \cdot \mathbb{P}\left(B\right).$$

We conclude that the events A and B are not independent.

Exercise 2 Alon has a red coin and Ben has a blue coin. The outcome of a toss of the red coin is heads with probability q, and the outcome of a toss of the blue coin is heads with probability p. Alon and Ben play the following game: in every round, both toss the coin they currently hold. If the outcomes of both coin tosses are heads, then Alon and Ben switch coins, otherwise, they keep the coins they have. For every positive integer n, let A_n be the event that at the end of the nth round Alon holds the red coin.

1. Calculate $\mathbb{P}\left(A_2\right)$.

2. Determine all values of p and q for which the events A_2 and A_3 are independent.

Solution

1. It follows by the Law of total probability that

$$\begin{aligned}\mathbb{P}(A_2) &= \mathbb{P}(A_2 \mid A_1) \cdot \mathbb{P}(A_1) + \mathbb{P}(A_2 \mid A_1^c) \cdot \mathbb{P}(A_1^c) \\ &= (1-pq) \cdot (1-pq) + pq \cdot pq \\ &= 2p^2q^2 - 2pq + 1.\end{aligned}$$

   Note that, formally, this only holds if $p \neq 0$ and $q \neq 0$ (as, otherwise $\mathbb{P}(A_1^c) = 0$ so one cannot condition on A_1^c). However, if $p = 0$ or $q = 0$, then trivially $\mathbb{P}(A_2) = 1$.

   A different approach would be to say that either Alon and Ben switched twice, which happens with probability $pq \cdot pq$, or they never switched, which happens with probability $(1-pq) \cdot (1-pq)$. Since the events are disjoint $\mathbb{P}(A_2)$ equals the sum of the probabilities of these events.

2. Let $\alpha = pq$ be the probability that Alon and Ben switch coins in any given round. If $\alpha = 0$, they never switch and so $\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = 1$. Hence, if $p = 0$ or $q = 0$, then A_2 and A_3 are independent. Similarly, if $\alpha = 1$, they switch coins in every round and so $\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_3) = 0$. Hence, if $p = 1$ and $q = 1$, then A_2 and A_3 are independent. It remains to check for which values of $0 < \alpha < 1$ it holds that $\mathbb{P}(A_3 \mid A_2) = \mathbb{P}(A_3)$. Note that $\mathbb{P}(A_3 \mid A_2) = 1 - \alpha$ and $\mathbb{P}(A_3 \mid A_2^c) = \alpha$ hold by definition, and $\mathbb{P}(A_2) = 2\alpha^2 - 2\alpha + 1$ and $\mathbb{P}(A_2^c) = 2\alpha - 2\alpha^2$ hold by the previous part of this exercise. Hence, it follows by the Law of total probability that

$$\begin{aligned}\mathbb{P}(A_3) &= \mathbb{P}(A_3 \mid A_2) \cdot \mathbb{P}(A_2) + \mathbb{P}(A_3 \mid A_2^c) \cdot \mathbb{P}(A_2^c) \\ &= (1-\alpha) \cdot (2\alpha^2 - 2\alpha + 1) + \alpha \cdot (2\alpha - 2\alpha^2) \\ &= (1-\alpha) \cdot (4\alpha^2 - 2\alpha + 1).\end{aligned}$$

   Note that, similarly to the previous part of this exercise, there is an alternative way to calculate $\mathbb{P}(A_3)$. Indeed, either Alon and Ben never switched, which happens with probability $(1-\alpha)^3$, or they switched exactly twice, which happens with probability $3\alpha^2 \cdot (1-\alpha)$, where the 3 comes from choosing the round in which they did not switch.

   It thus remains to check for which values of α it holds that

$$1 - \alpha = (1-\alpha) \cdot (4\alpha^2 - 2\alpha + 1).$$

   Since $\alpha < 1$ by assumption, this is equivalent to

$$1 = 4\alpha^2 - 2\alpha + 1,$$

   which is equivalent to

$$0 = 2\alpha(2\alpha - 1).$$

   Since $\alpha > 0$ by assumption, it follows that $\alpha = 1/2$. We conclude that A_2 and A_3 are independent if and only if p and q satisfy $pq \in \{0, 1/2, 1\}$.

Exercise 3 The integers 1 through n are ordered in a row uniformly at random. For every $1 \le i \le n$, let A_i be the event that the ith number in the random ordering is larger than all the numbers that were placed before it (i.e., in places 1 through $i-1$).

1. Calculate $\mathbb{P}(A_i)$ for every $1 \le i \le n$.
2. Determine which of the following statements are true for A_2, A_3, and A_7.
   (a) They are mutually independent.
   (b) They are dependent, but pair-wise independent.
   (c) A_2 and A_3 are independent, but A_3 and A_7 are dependent.

Solution

1. Fix some $i \in \{1, 2, \ldots, n\}$. Intuitively, for every set of i pairwise-distinct numbers, there is a unique maximum, which has equal probability to be placed in any given position. Hence, $\mathbb{P}(A_i) = 1/i$.

   Formally, for every $\mathcal{I} \subseteq \{1, 2, \ldots, n\}$ of size i, let $B_{\mathcal{I}}$ be the event that the elements of $\mathcal{I}$ are placed (in some order) in the first i positions. Then, it follows by the Law of total probability that
   $$\mathbb{P}(A_i) = \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,n\} \\ |I| = i}} \mathbb{P}(A_i \mid B_{\mathcal{I}}) \cdot \mathbb{P}(B_{\mathcal{I}}) = \frac{(i-1)!}{i!} \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,n\} \\ |\mathcal{I}| = i}} \mathbb{P}(B_{\mathcal{I}}) = \frac{1}{i}.$$

2. We will show that the events A_2, A_3, and A_7 are mutually independent; this will imply that only (a) is correct. $A_2 \cap A_3$ is the event that the first 3 numbers are arranged in increasing order. Therefore, conditioning on any specific choice of 3 numbers, this event is of size 1. It thus follows by the fact that the probability space is uniform that
   $$\mathbb{P}(A_2 \cap A_3) = \frac{1}{3!} = \frac{1}{2} \cdot \frac{1}{3} = \mathbb{P}(A_2) \cdot \mathbb{P}(A_3).$$
   $A_2 \cap A_7$ is the event that the second number is larger than the first, and the seventh number is larger than all 6 numbers that are placed in positions 1 through 6. Therefore, conditioning on any specific choice of 7 numbers, this event is of size $1 \cdot \binom{6}{2} \cdot 4!$, since we fix the position of the largest of these numbers, we then choose the first two numbers with specific order, and then permute the remaining 4 numbers. It thus follows by the fact that the probability space is uniform that
   $$\mathbb{P}(A_2 \cap A_7) = \frac{1 \cdot \binom{6}{2} \cdot 4!}{7!} = \frac{1}{14} = \frac{1}{2} \cdot \frac{1}{7} = \mathbb{P}(A_2) \cdot \mathbb{P}(A_7).$$
   Similar calculations (make them!) show that $\mathbb{P}(A_3 \cap A_7) = \mathbb{P}(A_3) \cdot \mathbb{P}(A_7)$ and that $\mathbb{P}(A_2 \cap A_3 \cap A_7) = \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \cdot \mathbb{P}(A_7)$.

Exercise 4 On a table there are two bins. In the first bin there are 3 white balls and 5 red balls, and in the second bin there 6 white balls and 3 red balls. Alon tosses a fair coin once. If the outcome of this coin toss is heads, then he draws 2 balls from the first bin uniformly at random with replacement, and if the outcome of this coin toss is tails, then he draws 2 balls from the second bin uniformly at random with replacement. Let A be the event that the first ball that was drawn is white, let B be the event that the second ball that was drawn is white, and let C be the event that the outcome of the coin toss is heads.

1. Are the events A and B independent?
2. Are the events A and B independent if we condition on C?

Solution

1. It follows by the Law of total probability that

$$\mathbb{P}(A) = \mathbb{P}(A \mid C) \cdot \mathbb{P}(C) + \mathbb{P}(A \mid C^c) \cdot \mathbb{P}(C^c) = \frac{3}{8} \cdot \frac{1}{2} + \frac{6}{9} \cdot \frac{1}{2} = \frac{75}{144}.$$

Since the 2 balls are drawn from the bins with replacement, it follows that $\mathbb{P}(B) = \mathbb{P}(A)$. Moreover, it follows by the Law of total probability that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap B \mid C) \cdot \mathbb{P}(C) + \mathbb{P}(A \cap B \mid C^c) \cdot \mathbb{P}(C^c) = \frac{9}{64} \cdot \frac{1}{2} + \frac{36}{81} \cdot \frac{1}{2} \neq \left(\frac{75}{144}\right)^2.$$

We conclude that the events A and B are dependent.

2. It holds that

$$\mathbb{P}(A \mid C) = \mathbb{P}(B \mid C) = \frac{3}{8},$$

and that

$$\mathbb{P}(A \cap B \mid C) = \frac{9}{64} = \mathbb{P}(A \mid C) \cdot \mathbb{P}(B \mid C).$$

Therefore the events are independent conditioned on C.

Exercise 5 A fair die is rolled again and again, all die rolls being mutually independent. The highest result until the nth roll is called a “temporary maximum”.

1. What is the probability that 3 will be a “temporary maximum” at least once?
2. What is the probability that every number from 1 to 6 will be a “temporary maximum” at least once?

Solution

For every $1 \leq i \leq 6$, let M_i be the event that i is a “temporary maximum” at some point.

1. Note that M_3 occurs if and only if there exists some positive integer n such that the outcome of each of the $n-1$ first die rolls is less than 3 and the outcome of the nth die roll is 3. Since the die rolls are mutually independent, it follows that

$$\mathbb{P}(M_3) = \sum_{n=1}^{\infty} \left(\frac{2}{6}\right)^{n-1} \frac{1}{6} = \frac{1}{6} \cdot \frac{1}{1-\frac{2}{6}} = \frac{1}{4}.$$

An alternative approach to solving this exercise would be to observe that the first outcome which is not 1 or 2 is in $\{3,4,5,6\}$, and exactly one of these results (namely, 3) yields M_3. Formally, let T be the event that there is a die roll whose outcome is not 1 or 2. Note that

$$\mathbb{P}(T^c) = \lim_{n\to\infty} \left(\frac{2}{6}\right)^{n} = 0$$

and thus $\mathbb{P}(T) = 1$. Hence

$$\mathbb{P}(M_3) = \frac{\mathbb{P}(M_3 \cap T)}{\mathbb{P}(T)} = \mathbb{P}(M_3 \mid T) = \frac{|\{3\}|}{|\{3,4,5,6\}|} = \frac{1}{4}.$$

2. Observe that $\bigcap_{i=1}^{6} M_i$ occurs if and only if 1 appears before all the other numbers, 2 appears before the remaining numbers and so on. Putting it differently, the first outcome is 1, the first time that 1 was not the outcome of the die roll, 2 was the outcome, the first time that 1 and 2 were not the outcome of the die roll, 3 was the outcome, and so on. Since the die rolls are mutually independent and since, similarly to the previous part of this exercise, the probability that some number in $\{1,2,\ldots,6\}$ is never the outcome is 0, we conclude that

$$\mathbb{P}\left(\bigcap_{i=1}^{6} M_i\right) = \frac{1}{6} \cdot \frac{1}{5} \cdot \ldots \cdot \frac{1}{1} = \frac{1}{6!}.$$