Probability Theory 1 – Proposed solution of moed bet exam 2020b

1. **(a)** It readily follows from the definitions of X and Y that the support of both random variables is $\{0,1,2\}$. For every $0 \le i \le 2$ and $0 \le j \le 2$, it holds that $\mathbb{P}(X = i, Y = j) = 0$ if $i + j > 2$ and

$$\mathbb{P}(X = i, Y = j) = \mathbb{P}(X = i, Y = j, Z = 2 - i - j) = \frac{\binom{2}{i}\binom{2}{j}\binom{2}{2-i-j}}{\binom{6}{2}}$$

otherwise.

The values of $\mathbb{P}(X = i, Y = j)$ can be seen in the table below.

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	$\frac{\binom{2}{0}\binom{2}{0}\binom{2}{2}}{\binom{6}{2}} = \frac{1}{15}$	$\frac{\binom{2}{0}\binom{2}{1}\binom{2}{1}}{\binom{6}{2}} = \frac{4}{15}$	$\frac{\binom{2}{0}\binom{2}{2}\binom{2}{0}}{\binom{6}{2}} = \frac{1}{15}$
$X = 1$	$\frac{\binom{2}{1}\binom{2}{0}\binom{2}{1}}{\binom{6}{2}} = \frac{4}{15}$	$\frac{\binom{2}{1}\binom{2}{1}\binom{2}{0}}{\binom{6}{2}} = \frac{4}{15}$	0
$X = 2$	$\frac{\binom{2}{2}\binom{2}{0}\binom{2}{0}}{\binom{6}{2}} = \frac{1}{15}$	0	0

(b) Note that $X+Y+Z$ is the total number of balls that were drawn. Hence, $X+Y+Z = 2$. It follows by a claim that was proved in the lectures that $Var(X+Y+Z) = Var(2) = 0$.

(c) We have

$$\mathbb{P}(Z = 1 | X \ge Y) = \frac{\mathbb{P}(Z = 1, X \ge Y)}{\mathbb{P}(X \ge Y)}.$$

It is evident that

$$\mathbb{P}(Z = 1, X \ge Y) = \mathbb{P}(Z = 1, X = 1, Y = 0) = \frac{\binom{2}{1}\binom{2}{1}\binom{2}{0}}{\binom{6}{2}} = \frac{4}{15}.$$

Moreover, looking at the table we calculated in Part (a) of this exercise we see that

$$\begin{aligned}\mathbb{P}(X \ge Y) &= \mathbb{P}(X = 0, Y = 0) + \mathbb{P}(X = 1, Y = 0) + \mathbb{P}(X = 2, Y = 0) + \mathbb{P}(X = 1, Y = 1) \\ &= \frac{1}{15} + \frac{4}{15} + \frac{1}{15} + \frac{4}{15} = \frac{10}{15}.\end{aligned}$$

We conclude that

$$\mathbb{P}(Z = 1 | X \ge Y) = \frac{4/15}{10/15} = 2/5.$$

(d) It is intuitively clear that X and Y are dependent since if one is "large", then the other must be "small". Formally, looking at the table we calculated in Part (a) of this exercise we see that

$$\mathbb{P}(X = 2, Y = 2) = 0 \neq (1/15 + 0 + 0)(1/15 + 0 + 0) = \mathbb{P}(X = 2)\mathbb{P}(Y = 2).$$

Since X and Y are independent if and only if $\mathbb{P}(X = a, Y = b) = \mathbb{P}(X = a)\mathbb{P}(Y = b)$ for every a and b, we conclude that they are in fact dependent.

2. **(a)** Since $Y \sim \text{Poi}(1)$, it follows by a claim proved in the lectures that

$$\mathbb{E}\left(\frac{1}{Y+1}\right) = \sum_{i=0}^{\infty} \frac{1}{i+1}\mathbb{P}(Y=i) = \sum_{i=0}^{\infty} \frac{1}{i+1} e^{-1}\frac{1^i}{i!}$$
$$= e^{-1}\sum_{i=0}^{\infty} \frac{1}{(i+1)!} = e^{-1}\left[\sum_{i=0}^{\infty}\frac{1}{i!} - 1\right] = e^{-1}(e-1) = 1-e^{-1},$$

where the penultimate equality holds by the Taylor series of e.

(b) Since X and Y are independent, so are X and $\frac{1}{Y+1}$. Indeed, for every a and b it holds that

$$\mathbb{P}\left(X=a, \frac{1}{Y+1}=b\right) = \mathbb{P}\left(X=a, Y=1/b-1\right) = \mathbb{P}(X=a)\mathbb{P}\left(Y=1/b-1\right)$$
$$= \mathbb{P}(X=a)\mathbb{P}\left(\frac{1}{Y+1}=b\right).$$

In particular, X and $\frac{1}{Y+1}$ are uncorrelated and thus $\mathbb{E}\left(\frac{X}{Y+1}\right) = \mathbb{E}(X)\cdot\mathbb{E}\left(\frac{1}{Y+1}\right)$. We showed in Part (a) of this exercise that $\mathbb{E}\left(\frac{1}{Y+1}\right) = 1-e^{-1}$. Moreover, $X \sim \text{Bin}\left(n, \frac{1}{n+1}\right)$ implies that $\mathbb{E}(X) = n\cdot\frac{1}{n+1}$. We conclude that $\mathbb{E}\left(\frac{X}{Y+1}\right) = \frac{n}{n+1}(1-e^{-1})$.

(c) This statement is false. There are several ways to see this, but the simplest one is probably via the following simple calculation.

$$\lim_{n\to\infty}\mathbb{P}(X>1.1) \geq \lim_{n\to\infty}\mathbb{P}(X=2) = \lim_{n\to\infty}\binom{n}{2}\left(\frac{1}{n+1}\right)^2\left(1-\frac{1}{n+1}\right)^{n-2}$$
$$= \lim_{n\to\infty}\frac{n(n-1)}{2(n+1)^2}\cdot\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1}\cdot\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{-3}$$
$$= 1/2\cdot e^{-1}\cdot 1 = e^{-1}/2 > 0.$$

(d) The solution of this part is essentially the same as the solution of part (a), but is technically more involved.

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\begin{aligned}
\mathbb{E}\left(\frac{1}{X+1}\right) &= \sum_{i=0}^{n} \frac{1}{i+1}\mathbb{P}(X=i) = \sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i}\left(\frac{1}{n+1}\right)^{i}\left(1-\frac{1}{n+1}\right)^{n-i} \\
&= \sum_{i=0}^{n} \frac{1}{i+1}\cdot\frac{n!}{i!(n-i)!}\left(\frac{1}{n+1}\right)^{i}\left(1-\frac{1}{n+1}\right)^{n-i} \\
&= \sum_{i=0}^{n} \frac{(n+1)!}{(i+1)!(n-i)!}\left(\frac{1}{n+1}\right)^{i+1}\left(1-\frac{1}{n+1}\right)^{n-i} \\
&= \sum_{i=0}^{n} \binom{n+1}{i+1}\left(\frac{1}{n+1}\right)^{i+1}\left(1-\frac{1}{n+1}\right)^{(n+1)-(i+1)} \\
&= \sum_{k=1}^{n+1} \binom{n+1}{k}\left(\frac{1}{n+1}\right)^{k}\left(1-\frac{1}{n+1}\right)^{(n+1)-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k}\left(\frac{1}{n+1}\right)^{k}\left(1-\frac{1}{n+1}\right)^{(n+1)-k} - \binom{n+1}{0}\left(\frac{1}{n+1}\right)^{0}\left(1-\frac{1}{n+1}\right)^{n+1} \\
&= 1-\left(1-\frac{1}{n+1}\right)^{n+1},
\end{aligned}
$$

where the sixth equality holds by the substitution $k = i + 1$.

3. **(a)** Since the die is fair, the probability that the outcome of each die roll will be 6 is 1/6. Since, moreover, all die rolls are independent and we continue rolling the die until the first time the outcome is 6, it follows that $X \sim \text{Geom}(1/6)$.

   (b) For every positive integer i, let Y_i denote the outcome of the ith die roll. Since all die rolls are mutually independent, it follows that

$$Var(Y|X=10) = \sum_{i=1}^{10} Var(Y_i|X=10).$$

   Note that $(Y_{10}|X=10) = 6$ and thus $Var(Y_{10}|X=10) = 0$. Moreover, $(Y_i|X=10) \sim U(1,5)$ for every $1 \le i \le 9$ (since the die is fair and its outcome cannot be 6). Therefore $Var(Y_i|X=10) = \frac{(5-1+1)^2-1}{12} = 2$ for every $1 \le i \le 9$. We conclude that

$$Var(Y|X=10) = 9 \cdot 2 + 0 = 18.$$

   (c) Similarly to Part (b) of this exercise, for every positive integer k, it holds that $(Y_k|X=k) = 6$ and thus $\mathbb{E}(Y_k|X=k) = 6$. Moreover, for every $1 \le i \le k-1$, it holds that $(Y_i|X=k) \sim U(1,5)$. In particular $\mathbb{E}(Y_i|X=k) = \frac{1+5}{2} = 3$ for every such k and i. It then follows by the linearity of expectation that

$$\mathbb{E}(Y|X=k) = \sum_{i=1}^{k} \mathbb{E}(Y_i|X=k) = \sum_{i=1}^{k-1} 3 + 6 = 3(k+1)$$

holds for every positive integer k.

Finally, it follows by the law of total expectation that

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \sum_{k=1}^{\infty} \mathbb{E}(Y|X=k)\mathbb{P}(X=k) = \sum_{k=1}^{\infty} 3(k+1) \cdot 1/6 \cdot (5/6)^{k-1}$$
$$= \frac{1}{2} \cdot \frac{6}{5} \left[\sum_{k=0}^{\infty} (k+1) \cdot (5/6)^k - 1 \right] = \frac{3}{5} \left[\frac{1}{(1-5/6)^2} - 1 \right] = 21,$$

where the penultimate equality holds by the formula $\sum_{i=0}^{\infty}(i+1)x^i = \frac{1}{(1-x)^2}$ that was given in the exam.

(d) Since Y is a non-negative random variable, applying Markov's inequality we obtain

$$\mathbb{P}(Y \geq 100) \leq \frac{\mathbb{E}(Y)}{100} = \frac{21}{100} < \frac{1}{4}.$$