# Assignment 2
# Solutions

Exercise 1 There are three identical chests, with two drawers each. One has a gold coin in each of its two drawers, another has a silver coin in each drawer and the third has a silver coin in one drawer and a gold coin in the other. A chest is chosen uniformly at random and then, in that chest, a drawer is chosen uniformly at random. Given that a gold coin was found, what is the probability that the second drawer of the chosen chest contains a gold coin as well?

Solution

Let $C_2$ be the event that the chosen chest contains 2 gold coins, let $C_1$ be the event that the chosen chest contains 1 gold coin, and let $C_0$ be the event that the chosen chest contains no gold coins. Let $G_1$ be the event that the chosen drawer contains a gold coin, and let $G_2$ be the event that the other drawer of the chosen chest contains a gold coin. Then

$$\mathbb{P}(G_2|G_1) = \frac{\mathbb{P}(G_1 \cap G_2)}{\mathbb{P}(G_1)} = \frac{\mathbb{P}(C_2)}{\mathbb{P}(G_1)} = \frac{1/3}{\sum_{i=0}^{2} \mathbb{P}(G_1|C_i) \cdot \mathbb{P}(C_i)} = \frac{1/3}{0 \cdot 1/3 + 1/2 \cdot 1/3 + 1 \cdot 1/3} = \frac{2}{3},$$

where the third equality holds by the Law of total probability.

Exercise 2 The weather at any given day on some planet can be either cloudy or clear, with a constant probability. In 60% of the cloudy days, the next day was clear and in 30% of the clear days, the next day was cloudy. What percent of the days are cloudy?

Solution

For any positive integer $i$, let $A_i$ be the event that the $i$th day was cloudy. Then $\mathbb{P}(A_i) = \mathbb{P}(A_j)$ for all $i$ and $j$. Let $p := \mathbb{P}(A_i)$. It thus follows by the Law of total probability that

$$\begin{aligned} p = \mathbb{P}(A_{i+1}) &= \mathbb{P}(A_{i+1} \mid A_i) \cdot \mathbb{P}(A_i) + \mathbb{P}(A_{i+1} \mid A_i^c) \cdot \mathbb{P}(A_i^c) \\ &= \left(1 - \mathbb{P}(A_{i+1}^c \mid A_i)\right) \cdot \mathbb{P}(A_i) + \mathbb{P}(A_{i+1} \mid A_i^c) \cdot \mathbb{P}(A_i^c) \\ &= (1 - 0.6) \cdot p + 0.3 \cdot (1 - p) = 0.3 + 0.1p. \end{aligned}$$

We conclude that $p = 1/3$.

Exercise 3 Let $S$ be a set of size $n$. Two sets $A$ and $B$ are chosen uniformly at random with replacement from the family of all subsets of $S$.

1. For any integer $0 \leq k \leq n$, calculate the probability that $|A| = k$.

2. Use the previous part of this exercise to calculate the probability that $A \subseteq B$.

Solution

1. For every $0 \le k \le n$, let $A_k$ be the event that $|A| = k$. Since the probability space is uniform, it follows that
$$\mathbb{P}(A_k) = \frac{|A_k|}{|\Omega|} = \frac{\binom{n}{k}}{2^n}.$$

2. For every $0 \le k \le n$, let $B_k$ be the event that $|B| = k$. If $|B| = k$, then $A \subseteq B$ if and only if $A$ is one of the $2^k$ possible subsets of $B$. Therefore, for every $0 \le k \le n$ it holds that
$$\mathbb{P}(A \subseteq B \mid B_k) = \frac{2^k}{2^n}.$$
It thus follows by the Law of total probability that
$$\begin{aligned}\mathbb{P}(A \subseteq B) &= \sum_{k=0}^{n} \mathbb{P}(A \subseteq B \mid B_k) \cdot \mathbb{P}(B_k) \\ &= \sum_{k=0}^{n} \frac{2^k}{2^n} \cdot \frac{\binom{n}{k}}{2^n} \\ &= \frac{1}{4^n} \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k \\ &= \frac{3^n}{4^n},\end{aligned}$$
where the last equality is due to Newton's binomial formula.

Exercise 4 Bowl 'a' contains 2 black balls and a single white ball. Bowl 'b' contains a single black ball and 3 white balls. A bowl is chosen uniformly at random and then a ball is chosen uniformly at random from that bowl.

1. What is the probability that the chosen ball is white?
2. What is the probability that Bowl 'a' was chosen, given that the chosen ball is white?
3. The chosen ball is placed back in its bowl and then a new ball is chosen uniformly at random. What is the probability that the second chosen ball is white, given that the first one was white?
4. Same as part 3 of this exercise, except that the first ball is not placed back in its bowl.

Solution

Let $W_1$ be the event that the first ball is white, let $W_2$ be the event that the second ball is white, let $U_1$ be the event that Bowl 'a' was chosen, and let $U_2$ be the event that Bowl 'b' was chosen.

1. It follows by the Law of total probability that

$$\mathbb{P}(W_1) = \mathbb{P}(W_1 \mid U_1) \cdot \mathbb{P}(U_1) + \mathbb{P}(W_1 \mid U_2) \cdot \mathbb{P}(U_2) = \frac{1}{3} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{13}{24}.$$

2. Using Bayes' rule we obtain

$$\mathbb{P}(U_1 \mid W_1) = \frac{\mathbb{P}(U_1) \cdot \mathbb{P}(W_1 \mid U_1)}{\mathbb{P}(W_1)} = \frac{1/2 \cdot 1/3}{13/24} = \frac{4}{13}.$$

3. Applying the Law of total probability in the conditional probability space $(\Omega, \mathbb{P}(\cdot \mid W_1))$, we obtain

$$\begin{aligned}\mathbb{P}(W_2 \mid W_1) &= \mathbb{P}(W_2 \mid W_1 \cap U_1) \cdot \mathbb{P}(U_1 \mid W_1) + \mathbb{P}(W_2 \mid W_1 \cap U_2) \cdot \mathbb{P}(U_2 \mid W_1) \\ &= \frac{1}{3} \cdot \frac{4}{13} + \frac{3}{4} \cdot \left(1 - \frac{4}{13}\right) = \frac{97}{156}.\end{aligned}$$

4. Applying the Law of total probability in the conditional probability space $(\Omega, \mathbb{P}(\cdot \mid W_1))$, we obtain

$$\begin{aligned}\mathbb{P}(W_2 \mid W_1) &= \mathbb{P}(W_2 \mid W_1 \cap U_1) \cdot \mathbb{P}(U_1 \mid W_1) + \mathbb{P}(W_2 \mid W_1 \cap U_2) \cdot \mathbb{P}(U_2 \mid W_1) \\ &= 0 \cdot \frac{4}{13} + \frac{2}{3} \cdot \left(1 - \frac{4}{13}\right) = \frac{6}{13}.\end{aligned}$$

**Exercise 5** Prove that an event is independent of all other events if and only if its probability is either 0 or 1.

**Solution**

Let $A$ be some event. For the first implication, we assume that $A$ is independent of all other events. Then

$$0 = \mathbb{P}(\emptyset) = \mathbb{P}(A \cap A^c) = \mathbb{P}(A) \cdot \mathbb{P}(A^c) = \mathbb{P}(A) \cdot (1 - \mathbb{P}(A)).$$

It follows that $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$ as claimed.

Next, we prove the opposite implication, that is, we assume that $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$, and prove that $A$ is independent of all other events. Let $B$ be some event. Assume first that $\mathbb{P}(A) = 0$. Since, by monotonicity, $0 \leq \mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 0$, it follows that $\mathbb{P}(A \cap B) = 0$. Therefore

$$\mathbb{P}(A \cap B) = 0 = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Assume now that $\mathbb{P}(A) = 1$. Then $0 \leq \mathbb{P}(A^c \cap B) \leq \mathbb{P}(A^c) = 0$ and thus

$$\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B) = \mathbb{P}(A \cap B).$$

Therefore

$$\mathbb{P}(A \cap B) = \mathbb{P}(B) = 1 \cdot \mathbb{P}(B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

**Exercise 6** Prove the following statements.

1. Event $A$ is independent of itself if and only if $\mathbb{P}(A) \in \{0, 1\}$.
2. If the events $A$ and $B$ are disjoint and independent, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.
3. If the events $A$ and $B$ are independent, then $A$ and $B^c$ are independent.

Solution

1. It holds that
$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A) \cdot \mathbb{P}(A) \iff \mathbb{P}(A) = \mathbb{P}(A)^2 \iff \mathbb{P}(A) \in \{0, 1\}.$$
2. It holds that
$$0 = \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$
implying that $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.
3. Observe that the events $A \cap B$ and $A \cap B^c$ are disjoint, and their union equals $A$. Therefore
$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A) \cdot \mathbb{P}(B) + \mathbb{P}(A \cap B^c),$$
and thus
$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A) \cdot \mathbb{P}(B) = \mathbb{P}(A) \cdot (1 - \mathbb{P}(B)) = \mathbb{P}(A) \cdot \mathbb{P}(B^c).$$

Exercise 7 For the following four equations, show that the correctness of any three of them, does not necessarily imply the fourth. That is, for each of the four equations, give an example of a probability space and three events $A$, $B$, and $C$ in this space, which will uphold the other three equations but not this one.

1. $$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$
2. $$\mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C).$$
3. $$\mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C).$$
4. $$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C).$$

Solution
We first show that the first three equations do not necessarily imply the fourth. Consider two independent tosses of a fair coin. Let $A$ be the event that the outcome of the first coin toss is

heads, let $B$ be the event that the outcome of the second coin toss is heads, and let $C$ be the event that both coin tosses have the same outcome. Straightforward calculations then show that

$$\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{2},$$

and

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A \cap B) = \mathbb{P}(A \cap C) = \mathbb{P}(A \cap B) = \frac{1}{4}.$$

Therefore, the first three equations hold but the fourth does not.

Next, we show that the last three equations do not necessarily imply the first one. By symmetry, this will imply the remaining two requirements (check it!). Consider the outcome of a roll of a fair die, that is, $\Omega = \{1, 2, \ldots, 6\}$ and $\mathbb{P}(i) = 1/6$ for every $1 \leq i \leq 6$. Let $A = \{1, 2, 3\}$, let $B = \{3, 4, 5\}$ and let $C = \{2, 3, 4, 6\}$. Then

$$\begin{aligned}
&\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}, \\
&\mathbb{P}(C) = \frac{2}{3}, \\
&\mathbb{P}(A \cap B) = \frac{1}{6}, \\
&\mathbb{P}(A \cap C) = \mathbb{P}(B \cap C) = \frac{1}{3}, \\
&\mathbb{P}(A \cap B \cap C) = \frac{1}{6}.
\end{aligned}$$

These results then imply that

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \frac{1}{6} \neq \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A) \cdot \mathbb{P}(B), \\
\mathbb{P}(A \cap C) &= \frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3} = \mathbb{P}(A) \cdot \mathbb{P}(C), \\
\mathbb{P}(B \cap C) &= \frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3} = \mathbb{P}(B) \cdot \mathbb{P}(C), \\
\mathbb{P}(A \cap B \cap C) &= \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C).
\end{aligned}$$