Assignment 1

If you wish to submit your solutions to any of these questions, please send them via email to your TA by 04/04/2021. This deadline is strict!

Exercise 1 Two fair dice are rolled. We then output the sum of the results mod 6. Prove that the corresponding probability space is uniform.

Exercise 2 A computer samples a binary string of length n uniformly at random (i.e., the probability space is uniform over all binary strings of length n).

1. For every $0 \leq k \leq n$, calculate the probability that there are exactly k 1's in the chosen string?
2. Let E be the event that the number of 1's in the chosen string is even, and let O be the event that the number of 1's in the chosen string is odd. Prove that $\mathbb{P}(E) = \mathbb{P}(O)$.
3. Prove that $\mathbb{P}(E) = \frac{1}{2}$.

Exercise 3 A fair coin is tossed repeatedly until the second time the outcome is heads.

1. Describe the sample space for this experiment.
2. For some $k \in \mathbb{N}$, calculate the probability that the coin will be tossed exactly k times.
3. Prove that the experiment forms a probability space.

Exercise 4 Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space $(\Omega, \mathbb{P})$. Prove that

1. $$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}(A_i^c).$$
2. $$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - n + 1.$$

Exercise 5 Four men go to a party and hang their coats in a closet. Upon leaving, each of the four men picks one of these four coats uniformly at random. Calculate the probability that none of them picks the coat they came with to the party.

Exercise 6 We are given n different bins and $k \geq n$ balls. Each ball is thrown into a bin chosen uniformly at random. Calculate the probability that no bin is empty if

1. The balls are different.
2. The balls are identical.