Practical 9

Exercise 1 Let $X \sim \mathrm{Poi}(\lambda)$, for some non-negative real number λ. Calculate $\mathbb{E}(X)$.

Solution
Recall that

$$\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

for every non-negative integer k. Therefore

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=0}^{\infty} k \cdot \mathbb{P}(X = k) \\
&= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\
&= \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!} \\
&= \lambda \cdot \sum_{m=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^m}{m!} \\
&= \lambda,
\end{aligned}$$

where the penultimate equality follows by the substitution $m = k - 1$ and the last equality follows since the sum of probabilities of a Poisson random variable over its support is 1.

Exercise 2 Let $X \sim \mathrm{Hyp}(N, D, n)$, for some $N, D, n \in \mathbb{N}$ satisfying $D, n \leq N$. Calculate $\mathbb{E}(X)$.

Solution
We will present two solutions to this exercise: the first via direct calculation and the second by expressing X as a sum of indicators.
First solution: Recall that

$$\mathbb{P}(X = k) = \frac{\binom{D}{k} \cdot \binom{N-D}{n-k}}{\binom{N}{n}}$$

for every integer k satisfying $\max\{0, n + D - N\} \leq k \leq D$.

Let S denote the support of X. Then

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k\in S} k\cdot \mathbb{P}(X=k) \\
&= \sum_{k\in S} k\cdot \frac{\binom{D}{k}\cdot\binom{N-D}{n-k}}{\binom{N}{n}} \\
&= \sum_{k\in S} k\cdot \frac{\frac{D}{k}\cdot\binom{D-1}{k-1}\cdot\binom{N-D}{n-k}}{\frac{N}{n}\cdot\binom{N-1}{n-1}} \\
&= n\cdot\frac{D}{N}\cdot\sum_{k\in S}\frac{\binom{D-1}{k-1}\cdot\binom{(N-1)-(D-1)}{(n-1)-(k-1)}}{\binom{N-1}{n-1}} \\
&= n\cdot\frac{D}{N},
\end{aligned}$$

where the third equality follows from the identity

$$\binom{n}{k} = \frac{n}{k}\cdot\binom{n-1}{k-1},$$

and the last equality follows since the sum of probabilities of a Hypergeometric random variable with parameters $N-1$, $D-1$ and $n-1$ over its support is 1.

Second solution: Consider the following experiment – a bin contains N balls, of which D are black. n balls are drawn from the bin uniformly at random and without replacement. We can view X as counting the number of black balls that were drawn from the bin. For $1\le j\le n$, let

$$I_j = \begin{cases} 1 & \text{the } j\text{th ball is black} \\ 0 & \text{otherwise} \end{cases}$$

Then $X=\sum_{j=1}^{n} I_j$. We will show that

$$\mathbb{P}(I_j=1) = \mathbb{P}(I_1=1) = \frac{D}{N}$$

holds for every $1\le j\le n$. By linearity of expectation this will then imply that

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{j=1}^{n} I_j\right) = \sum_{j=1}^{n}\mathbb{E}(I_j) = \sum_{j=1}^{n}\mathbb{P}(I_j) = n\cdot\frac{D}{N}.$$

Fix some $j\in\{1,2,\ldots,n\}$. Since the probability space is uniform, it is enough to show that $|\{I_j=1\}| = |\{I_1=1\}|$, which will be done by presenting a bijection between the two sets. Imagine that the ball are drawn from the bin one by one and that they are arranged in a row according to the order by which they were sampled. The bijection between $\{I_j=1\}$ and $\{I_1=1\}$ is defined by switching the locations of the first ball and the jth ball (this is clearly a bijection as it is equal to its inverse).

Exercise 3 A bin contains one black ball and one white ball. In each round, a ball is drawn from the bin uniformly at random and independently of all other rounds. If the drawn ball is white, it is returned to the bin with an additional white ball. If the ball is black, then the experiment is over. Let X be the total number of rounds in the above experiment.

1. Calculate the distribution of X, and show that the experiment will eventually stop with probability 1.

2. Calculate $\mathbb{E}(X)$.

3. Why is there no contradiction between the result of part 1 and the result of part 2?

Solution

1. Fix some positive integer k. Then the event $\{X = k\}$ holds if and only if we sample a white ball in the first $k-1$ rounds and then a black ball in the kth (and final) round. Hence

$$\mathbb{P}(X = k) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{k-1}{k} \cdot \frac{1}{k+1} = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

This implies that

$$\sum_{k=1}^{\infty} \mathbb{P}(X = k) = \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1} = 1.$$

Therefore, the probability that the experiment will continue indefinitely is

$$\mathbb{P}(\forall m \in \mathbb{N} \text{ there are more than } m \text{ rounds}) = \lim_{m \to \infty} \mathbb{P}(X > m) = 0.$$

2. We have

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} k \cdot \mathbb{P}(X = k) = \sum_{k=1}^{\infty} k \cdot \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty,$$

where the last equality follows from the divergence of the harmonic series.

3. Although the number of rounds per experiment is finite with probability 1, it can be arbitrarily large. If we take the average of the number of rounds over k such experiments, this average will (with high probability) be higher the greater k is. As k tends to infinity, so does the average of the number of rounds in the k experiments. Behind this boundless growth is the fact that every time an unlikely outcome is reached (a black ball was drawn even though there are many white balls in the bin), the number of rounds is so large that, when averaged with the number of rounds of more likely outcomes (the game ends after a few rounds), the average is skewed up.

Exercise 4 Let X be some random variable.

1. Prove that if $\mathbb{E}(X^2) < \infty$ then $\mathbb{E}(|X|) < \infty$.

2. Prove that $\mathbb{E}(X^2) < \infty$ if and only if $\mathbb{E}((X-a)^2) < \infty$ for all constants a.

3. Assume that $\mathbb{E}(X^2) < \infty$. Let m be the minimum of the function $f(k) = \mathbb{E}((X-k)^2)$. Prove that $\mathbb{E}(X)$ is the only value k for which $f(k) = m$.

Solution

1. Assume that $\mathbb{E}\left(X^2\right) < \infty$. Observe that for all $\omega \in \Omega$ it holds that $|X(\omega)| \leq X^2(\omega) + 1$. Thus

$$\mathbb{E}\left(|X|\right) = \sum_{\omega\in\Omega} |X(\omega)| \cdot \mathbb{P}(\omega) \leq \sum_{\omega\in\Omega} \left(X^2(\omega) + 1\right) \cdot \mathbb{P}(\omega) = \mathbb{E}\left(X^2 + 1\right) = \mathbb{E}\left(X^2\right) + 1 < \infty,$$

where the last equality is by linearity of expectation.

2. If $\mathbb{E}\left((X-a)^2\right) < \infty$ for all constants a, then it holds for $a = 0$ in particular. Assume that $\mathbb{E}\left(X^2\right) < \infty$. We show that $\mathbb{E}\left((X-a)^2\right) < \infty$ for all $a \in \mathbb{R}$. Since $\mathbb{E}\left(X^2\right) < \infty$ it follows that $\mathbb{E}\left(|X|\right) < \infty$, which implies that $\mathbb{E}(X)$ exists and finite. Therefore

$$\mathbb{E}\left((X-a)^2\right) = \mathbb{E}\left(X^2 - 2aX + a^2\right) = \mathbb{E}\left(X^2\right) - 2a\mathbb{E}(X) + a^2 < \infty,$$

where the last equality is by linearity of expectation.

3. By the previous part of the exercise it holds that

$$f(k) = \mathbb{E}\left((X-k)^2\right) = \mathbb{E}\left(X^2\right) - 2\mathbb{E}(X)\,k + k^2.$$

This is a parabola that is opening to the top, hence it has a global minimum at $k = -\frac{-2\mathbb{E}(X)}{2} = \mathbb{E}(X)$.