Practical 1

Exercise 1 For $1 \leq i \leq 4$, an experiment is performed on the ith day. Each experiment can either succeed or fail. For every $1 \leq i \leq 4$, let A_i be the event that the experiment on the ith day was a success, and let B_i be the event that the first day on which the experiment was a success was the ith day.

1. Are A_1, A_2, A_3, and A_4 disjoint? pairwise disjoint?
2. Are B_1, B_2, B_3, and B_4 disjoint? pairwise disjoint?

Solution

We can view the sample space Ω as the set of all binary strings of length 4, where, for $1 \leq i \leq 4$, 1 in the ith coordinate represents a success on the ith day, and 0 represent a failure.

1. The events are not disjoint, and in particular they are not pairwise disjoint, since

$$A_1 \cap A_2 \cap A_3 \cap A_4 = \{1111\}.$$

2. The events are pairwise disjoint, and in particular they are disjoint, since

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_1^c \cap A_2 \\
B_3 &= A_1^c \cap A_2^c \cap A_3 \\
B_4 &= A_1^c \cap A_2^c \cap A_3^c \cap A_4
\end{aligned}$$

Exercise 2 Let $(\Omega, \mathbb{P})$ be a probability space where $\Omega = \mathbb{N}$ and $\forall i \in \mathbb{N}$, $\mathbb{P}(i) = \frac{k}{3^i}$, for some constant k.

1. Determine k.
2. What is the probability that a random sample would be an even number?
3. What is the probability that a random sample would be an odd number?

Solution

1. Since $(\Omega, \mathbb{P})$ is a probability space, it follows from the definition that

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$$

Therefore

$$1 = \sum_{i=1}^{\infty} \mathbb{P}(i) = \sum_{i=1}^{\infty} \frac{k}{3^i} = \frac{\frac{k}{3}}{1 - \frac{1}{3}} = \frac{k}{2}.$$

We conclude that $k = 2$.

2. Let E be the event that an even number was sampled. Since the events $\{i\}$ and $\{j\}$ are disjoint for every $i \neq j$, it follows that

$$\mathbb{P}(E) = \sum_{i=1}^{\infty} \mathbb{P}(2i) = \sum_{i=1}^{\infty} \frac{2}{3^{2i}} = \sum_{i=1}^{\infty} \frac{2}{9^i} = \frac{\frac{2}{9}}{1 - \frac{1}{9}} = \frac{1}{4}.$$

3. The event that an odd number was sampled is E^c. Therefore

$$\mathbb{P}(E^c) = 1 - \mathbb{P}(E) = \frac{3}{4}.$$

Exercise 3 10 ambassadors are being arranged uniformly at random in a row. What is the probability that:

1. The Israeli ambassador is next to the Russian ambassador?

2. The Israeli ambassador is not next to the American ambassador?

3. The French ambassador is next to the Russian ambassador, and the Israeli ambassador is not next to the American ambassador?

Solution

1. Let IR be the event that the Israeli ambassador is next to the Russian ambassador. By viewing the two ambassadors as one, we infer that there are exactly $2! \cdot 9!$ possibilities in which the Israeli ambassador is next to the Russian ambassador. By the fact that the probability space is uniform, it follows that

$$\mathbb{P}(IR) = \frac{|IR|}{10!} = \frac{2! \cdot 9!}{10!} = \frac{1}{5}.$$

2. Let IA be the event that the Israeli ambassador is next to the American ambassador. Similarly to the previous part of the exercise, there are exactly $2! \cdot 9!$ possibilities in which the Israeli ambassador is next to the American ambassador. By the fact that the probability space is uniform, it follows that

$$\mathbb{P}(IA^c) = 1 - \frac{|IA|}{10!} = 1 - \frac{2! \cdot 9!}{10!} = \frac{4}{5}.$$

3. Let FR be the event that the French ambassador is next to the Russian ambassador. Our main goal is to calculate $|FR \cap IA^c|$. We first view the French ambassador and the Russian ambassador as one and then there are 2! ways for arranging them. Now, there are 9 ambassadors left (including the "double" ambassador). By a similar argument to the previous part of this exercise (recalling that the French ambassador and the Russian ambassador are considered as one), we obtain that there are $(9! - 2! \cdot 8!)$ ways for the Israeli ambassador and the American ambassador to not be next to each other. We conclude that

$$\mathbb{P}\left(FR \cap IA^c\right) = \frac{|FR \cap IA^c|}{10!} = \frac{2! \cdot (9! - 2! \cdot 8!)}{10!} = \frac{7}{45}.$$