Probability Theory 2
Proposed solution of moed aleph exam 2022

1. **(a)** If f is indeed a probability density function, then $\int_{-\infty}^{\infty} f(x)dx = 1$. Solving this equation yields

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 (ax+b)dx = a\int_0^1 xdx + b\int_0^1 dx = ax^2/2\big|_0^1 + bx\big|_0^1 = a/2 + b,$$

implying that $a + 2b = 2$.
Since, moreover, $\mathbb{E}(X) = 7/12$, it follows that

$$\begin{aligned} 7/12 &= \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 (ax^2+bx)dx = a\int_0^1 x^2dx + b\int_0^1 xdx \\ &= ax^3/3\big|_0^1 + bx^2/2\big|_0^1 = a/3 + b/2, \end{aligned}$$

implying that $2a + 3b = 7/2$.
Solving these two linear equations yields $a = 1$ and $b = 1/2$. Finally, observe that $f(x) = x + 1/2$ is non-negative for every $0 \leq x \leq 1$, implying that f is indeed a density function.

(b) Let F_X denote the cumulative distribution function of X. Observe that if $t < 0$, then $F_X(t) = \mathbb{P}(X \leq t) = 0$, and if $t > 1$, then $F_X(t) = \mathbb{P}(X \leq t) = 1$. Fix some $0 \leq t \leq 1$. Then

$$F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^{t} f(x)dx = \int_0^t xdx + 1/2 \cdot \int_0^t dx = x^2/2\big|_0^t + x/2\big|_0^t = (t^2+t)/2.$$

We conclude that

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ (t^2+t)/2 & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}$$

(c) Recall (see Proposition 1.1 in Lecture 10) that if X is a random variable with probability density function f and $g : \mathbb{R} \to \mathbb{R}$ is a function, then $\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$. Hence, for every positive integer n it holds that

$$\begin{aligned} \mathbb{E}(X^n) &= \int_{-\infty}^{\infty} x^n f(x)dx = \int_0^1 \left(x^{n+1} + x^n/2\right)dx = \int_0^1 x^{n+1}dx + 1/2 \cdot \int_0^1 x^n dx \\ &= x^{n+2}/(n+2)\big|_0^1 + x^{n+1}/(2n+2)\big|_0^1 = \frac{1}{n+2} + \frac{1}{2n+2}. \end{aligned}$$

2. We will present a randomized algorithm and then prove that it meets all the requirements of the question.

Algorithm:

(i) For every $1 \leq i \leq 10000$, draw three vertices $x_i, y_i, z_i \in V(G)$ uniformly at random, all random draws being mutually independent.

(ii) For every $1 \leq i \leq 10000$, check whether x_i, y_i, z_i form a triangle in G; if such an i exists, return x_i, y_i, z_i (say, for the smallest such i), otherwise return "did not find a triangle".

First, it is evident that the running time of the algorithm is constant as we make 30000 random vertex draws and then check the existence of 30000 edges in G.

Next, we prove the (randomized) correctness of the algorithm. Since G admits at least $n^3/100$ triangles, for every $1 \leq i \leq 10000$ the probability that x_i, y_i, z_i form a triangle in G is at least $1/100$. Since, moreover, all random vertex draws are mutually independent, the probability that, for every $1 \leq i \leq 10000$, the vertices x_i, y_i, z_i do not form a triangle in G is at most

$$(1 - 1/100)^{10000} \leq e^{-10000/100} = e^{-100} < 2^{-100},$$

where the first inequality holds since $1 - x \leq e^{-x}$ for every $x \in \mathbb{R}$.

3. **(a)** Let $G \sim G\left(n, \frac{3 \ln n}{4n}\right)$ and let $X = |E(G)|$. Observe that $X \sim \text{Bin}\left(\binom{n}{2}, \frac{3 \ln n}{4n}\right)$ and so, in particular, $\mathbb{E}(X) = 3(n-1)\ln n/8$. Hence

$$\mathbb{P}(X \geq n \ln n) \leq \mathbb{P}(X \geq 2\mathbb{E}(X)) \leq e^{-\mathbb{E}(X)/3} \leq e^{-(n-1)\ln n/8},$$

where the first inequality above holds by Chernoff's inequality (Theorem 1.5(c) from Lecture 1). We conclude that

$$\lim_{n \to \infty} \mathbb{P}(X \geq n \ln n) = 0.$$

(b) Let $G \sim G(n, p)$, where $p = \frac{3 \ln n}{4n}$. For every two distinct vertices $u, v \in V(G)$ let $\mathcal{E}_{uv}$ denote the event "uv is an isolated edge in G". Note that $\mathbb{P}(\mathcal{E}_{uv}) = p(1-p)^{2(n-2)}$ holds for every $u, v \in V(G)$. A union bound argument then shows that

$$\begin{aligned}
\mathbb{P}(\text{there exists an isolated edge in } G) &= \mathbb{P}\left(\bigcup_{u,v \in V(G)} \mathcal{E}_{uv}\right) \leq \sum_{u,v \in V(G)} \mathbb{P}(\mathcal{E}_{uv}) \\
&= \binom{n}{2} p(1-p)^{2(n-2)} \leq n^2 \cdot \frac{3 \ln n}{4n} \cdot e^{-2np} \\
&\leq n \ln n \cdot e^{-3 \ln n/2} < n^{-0.4}.
\end{aligned}$$

We conclude that

$$\lim_{n \to \infty} \mathbb{P}(\text{there exists an isolated edge in } G) = 0.$$