Practical session 13

Exercise 1 Suppose that packages are loaded into a container with a total capacity of 1600 kg. Each package has a weight of Exp(3/5) kg, all weights being mutually independent. Use the Central Limit Theorem (CLT, for brevity) to estimate the probability that the cumulative weight of 900 packages is between 1425 to 1600.

Solution

For every $1 \leq i \leq 900$, let X_i denote the weight of the ith package. Let μ and σ^2 denote the expectation and variance of X_i, respectively. Note that $\mu = 5/3$ and $\sigma = 5/3$. For convenience, let $S = \sum_{i=1}^{900} X_i$ and let $Y = \frac{S-n\mu}{\sigma\sqrt{n}}$. It follows by the Central Limit Theorem that

$$\begin{aligned}
\Pr\left[1425 \leq S \leq 1600\right] &= \Pr\left[\frac{1425 - 900 \cdot 5/3}{\frac{5}{3} \cdot \sqrt{900}} \leq \frac{S - 900 \cdot 5/3}{\frac{5}{3} \cdot \sqrt{900}} \leq \frac{1600 - 900 \cdot 5/3}{\frac{5}{3} \cdot \sqrt{900}}\right] \\
&= \Pr\left[-1.5 \leq Y \leq 2\right] \\
&\approx \Phi\left(2\right) - \Phi\left(-1.5\right) \\
&\approx 0.9772 - 0.0668 \\
&= 0.9104.
\end{aligned}$$

Exercise 2 A fair die is rolled until the first time the sum of outcomes of all die rolls is 700 or larger; all die rolls are mutually independent. Use CLT to estimate the probability that

1. at least 210 rolls are required.
2. at most 190 rolls are required.
3. between 180 and 210 rolls, inclusive, are required.

Solution

For every positive integer n, let S_n be the sum of the outcomes of the first n die rolls (or of all die rolls if there were less than n). Then $\mathbb{E}\left(S_n\right) = 7n/2$. Moreover, since the die rolls are mutually independent, it holds that $\mathrm{Var}\left(S_n\right) = 35n/12$. Let $S_n^* = \frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}}$. Finally, let T denote the total number of die rolls. Our main observation, which we will use several times in the solution of this exercise, is that for every $\Pr\left[T \geq n+1\right] = \Pr\left[S_n \leq 699\right]$ holds for every positive integer n.

1. It follows by CLT that

$$\begin{aligned}
\Pr\left[S_{209} \leq 699\right] &= \Pr\left[S_{209}^{*} \leq \frac{699 - \frac{7\cdot 209}{2}}{\sqrt{\frac{35\cdot 209}{12}}}\right] = \Pr\left[S_{209}^{*} \leq \frac{-32.5}{\sqrt{\frac{7315}{12}}}\right] \\
&\approx \Phi\left(\frac{-32.5}{\sqrt{\frac{7315}{12}}}\right) \approx \Phi\left(-1.32\right) \\
&= 1 - \Phi\left(1.32\right) \approx 0.0934.
\end{aligned}$$

Therefore, $\Pr\left[T \geq 210\right] \approx 0.0934$.

2. Since $\Pr\left[T \leq 190\right] = 1 - \Pr\left[T > 190\right] = 1 - \Pr\left[S_{190} \leq 699\right]$, it follows that

$$\begin{aligned}
\Pr\left[T \leq 190\right] = 1 - \Pr\left[S_{190} \leq 699\right] &= 1 - \Pr\left[S_{190}^{*} \leq \frac{699 - \frac{7\cdot 190}{2}}{\sqrt{\frac{35\cdot 190}{12}}}\right] \\
&= 1 - \Pr\left[S_{190}^{*} \leq \frac{34}{\sqrt{\frac{3325}{6}}}\right] \approx 1 - \Phi\left(\frac{34}{\sqrt{\frac{3325}{6}}}\right) \\
&\approx 1 - \Phi\left(1.44\right) \approx 0.07493.
\end{aligned}$$

3. Observe that

$$\begin{aligned}
\Pr\left[180 \leq T \leq 210\right] &= \Pr\left[T \leq 210\right] - \Pr\left[T \leq 179\right] = \left(1 - \Pr\left[T > 210\right]\right) - \left(1 - \Pr\left[T > 179\right]\right) \\
&= \Pr\left[S_{179} \leq 699\right] - \Pr\left[S_{210} \leq 699\right].
\end{aligned}$$

Using CLT to estimate the first term, we obtain

$$\begin{aligned}
\Pr\left[S_{179} \leq 699\right] &= \Pr\left[S_{179}^{*} \leq \frac{699 - \frac{7\cdot 179}{2}}{\sqrt{\frac{35\cdot 179}{12}}}\right] = \Pr\left[S_{179}^{*} \leq \frac{72.5}{\sqrt{\frac{6265}{12}}}\right] \\
&\approx \Phi\left(\frac{72.5}{\sqrt{\frac{6265}{12}}}\right) \approx \Phi\left(3.17\right) \approx 0.99924.
\end{aligned}$$

Using CLT to estimate the second term, we obtain

$$\begin{aligned}
\Pr\left[S_{210} \leq 699\right] &= \Pr\left[S_{210}^{*} \leq \frac{699 - \frac{7\cdot 210}{2}}{\sqrt{\frac{35\cdot 210}{12}}}\right] = \Pr\left[S_{210}^{*} \leq -\frac{36\cdot\sqrt{2}}{35}\right] \\
&\approx \Phi\left(-\frac{36\cdot\sqrt{2}}{35}\right) \approx \Phi\left(-1.45\right) \\
&= 1 - \Phi\left(1.45\right) \approx 0.07353.
\end{aligned}$$

We conclude that

$$\Pr\left[180 \leq T \leq 210\right] = \Pr\left[S_{179} \leq 699\right] - \Pr\left[S_{210} \leq 699\right] \approx 0.99924 - 0.07353 = 0.92571.$$

Exercise 3 Let $X_1, \dots, X_{100} \sim \mathrm{Poi}\,(1)$ be independent random variables. Let $\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i$. Use CLT to estimate $\Pr\left[\bar{X} < 1.1\right]$.

Solution

Since $\mathbb{E}\,(X_i) = \mathrm{Var}\,(X_i) = 1$ for every $i \in \{1, \dots, 100\}$, it follows that $\mathbb{E}\,(X_i/100) = 1/100$ and $\mathrm{Var}\,(X_i/100) = 1/10000$ for every $i \in \{1, \dots, 100\}$. It thus follows by CLT that

$$\begin{aligned}\Pr\left[\bar{X} < 1.1\right] &= \Pr\left[\frac{\bar{X} - 100 \cdot 1/100}{\sqrt{1/10000} \cdot \sqrt{100}} < \frac{1.1 - 100 \cdot 1/100}{\sqrt{1/10000} \cdot \sqrt{100}}\right] = \Pr\left[\frac{\bar{X} - 1}{1/10} < 1\right] \\ &\approx \Phi(1) \approx 0.84134.\end{aligned}$$