# Lecture 3

# 1 Introduction to the Probabilistic Method

The probabilistic method, which has numerous applications, is based on the following obvious observation. If, when drawing an object at random from a finite universe, the probability of obtaining a certain specific object is strictly positive, then such an object exists. Indeed, imagine that a bin contains 10 balls. We draw one ball uniformly at random. What is the probability that this ball is red? Clearly, it is $r/10$, where $r$ is the number of red balls. If there are no red balls (i.e., $r = 0$), then this probability is 0. Equivalently, if the probability of drawing a red ball is not 0, then $r$ must be at least 1.

The strength of this method stems from the fact that one can now harness the power of probability theory. We will use several probabilistic tools later on, but in this lecture we will use only very basic probability theory. One simple tool we will use is the linearity of expectation and the fact that there is a point $\omega$ in the probability space for which $X(\omega) \leq \mathbb{E}(X)$ and a point for which $X(\omega) \geq \mathbb{E}(X)$.

We will now consider several examples of deterministic statements which can be proved by simple applications of the probabilistic method.

## 1.1 Lower bound on Ramsey numbers

For positive integers $k$ and $\ell$, the *Ramsey number* $R(k, \ell)$ is the smallest positive integer $n$ such that every graph on $n$ vertices contains a clique of size $k$ or an independent set of size $\ell$. It is not clear that $R(k, \ell)$ is finite for every $k$ and $\ell$; this fact was proved by Ramsey in 1929. The following result, whose proof is a straightforward application of the probabilistic method, yields essentially the best known lower bound on the diagonal Ramsey numbers.

**Theorem 1.1** (Erdős 1947)**.** *If* $\binom{n}{t} 2^{1-\binom{t}{2}} < 1$*, then* $R(t, t) > n$*.*

*Proof.* Call a graph *good* if it does not contain a clique of size $t$ nor an independent set of size $t$; otherwise it is called *bad*. Construct a graph $G$ with vertex set $[n]$ as follows. For every $1 \leq i < j \leq n$ flip a fair coin, where all coin flips are mutually independent. If the result of the coin flip for a pair $i < j$ is heads, then $ij$ is an edge of $G$; otherwise, it is not. Our goal is to show that the probability that the resulting graph $G$ is good is strictly positive. It would then follow by the probabilistic method that there exists a good graph on $n$ vertices, implying that $R(t, t) > n$. Let $S_1, \ldots, S_{\binom{n}{t}}$ be an enumeration of all the subsets of $[n]$ of size $t$. For every $1 \leq i \leq \binom{n}{t}$, let $A_i$ denote the event "$G[S_i]$ is a clique or an independent set".

For every $1 \leq i \leq \binom{n}{t}$ we have $Pr[A_i] = 2 \cdot 2^{-\binom{t}{2}}$. Hence

$$\mathbb{P}\left(G \text{ is bad}\right) = \mathbb{P}\left(\bigcup_{i=1}^{\binom{n}{t}} A_i\right) \leq \sum_{i=1}^{\binom{n}{t}} \mathbb{P}\left(A_i\right) = \binom{n}{t} 2^{1-\binom{t}{2}} < 1\,,$$

where the first inequality follows by a union bound argument and the last inequality follows by the assumption of the theorem. We conclude that $\mathbb{P}\left(G \text{ is good}\right) = 1 - \mathbb{P}\left(G \text{ is bad}\right) > 0$. □

**Corollary 1.2.** *$R(t,t) > \lfloor 2^{t/2} \rfloor$ holds for every $t \geq 4$.*

*Proof.* Let $n = \lfloor 2^{t/2} \rfloor$; by Theorem 1.1 it suffices to prove that $\binom{n}{t} 2^{1-\binom{t}{2}} < 1$. We have

$$\begin{aligned}
\binom{n}{t} 2^{1-\binom{t}{2}} &= \frac{n(n-1) \cdot \ldots \cdot (n-t+1)}{t!} 2^{1-\frac{t(t-1)}{2}} \\
&\leq \frac{n^t}{t!} 2^{1-\frac{t^2-t}{2}} \\
&\leq \frac{(2^{t/2})^t \cdot 2^{1-t^2/2+t/2}}{t!} \\
&= \frac{2^{t^2/2+1-t^2/2+t/2}}{t!} \\
&= \frac{2^{1+t/2}}{t!} \\
&\leq \frac{2^{t-1}}{t!} \\
&= \prod_{i=2}^{t} \frac{2}{i} \\
&< 1\,,
\end{aligned}$$

where the third inequality holds since $t \geq 4 \Rightarrow t - 1 \geq t/2 + 1$. □

## 1.2 Tournaments and Property $S_k$

A *tournament* on $n$ vertices is an orientation of the edges of the complete graph on $n$ vertices $K_n$. Let $T = (V, E)$ be a tournament, let $A \subseteq V$ and let $u \in V \setminus A$. We say that $u$ *dominates* $A$ if for every $v \in A$ the arc $uv$ is directed from $u$ to $v$ in $T$; we denote this by $\overrightarrow{uv} \in E$. Given a positive integer $k$, we say that $T$ satisfies *Property* $S_k$ if for every $A \subseteq V$ of size $k$ there is a vertex $u \in V \setminus A$ which dominates $A$.

**Theorem 1.3** (Erdős 1963)**.** *If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there is a tournament on $n$ vertices that satisfies Property $S_k$.*

*Proof.* Let $T = (V, E)$ be a random tournament on $n$ vertices, that is, $V = \{1, \ldots, n\}$ and for every $1 \le i < j \le n$ we flip a fair coin, where all coin flips are mutually independent, to determine whether $\overrightarrow{ij}$ is an arc of $T$ or not (in which case $\overleftarrow{ij} \in E$). Therefore $\overrightarrow{ij} \in E$ holds with probability $1/2$ and $\overleftarrow{ij} \in E$ holds with probability $1/2$. Fix some $A \subseteq V$ of size $k$ and some vertex $x \in V \setminus A$. The probability that $\overrightarrow{xa} \in E$ for every $a \in A$ is $2^{-k}$. Hence the probability that this does not hold is $1 - 2^{-k}$. Due to the independence of the coin flips, it follows that the probability that there is no vertex $x \in V \setminus A$ which dominates $A$ is $(1 - 2^{-k})^{n-k}$. Applying a union bound argument over all sets $A \subseteq V$ of size $k$ shows that the probability that $T$ does not satisfy $S_k$ is at most $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$. We conclude that there exists a tournament on $n$ vertices which satisfies $S_k$. □

## 1.3 Small dominating sets in graphs

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is called a *dominating* set of $G$ if for every $v \in V \setminus S$ there exists a vertex $u \in S$ such that $uv \in E$. Clearly $V$ itself is a dominating set. The main aim is to prove the existence of a small dominating set.

**Theorem 1.4.** *Let $G = (V, E)$ be a graph on $n$ vertices, with minimum degree $\delta > 1$. Then $G$ has a dominating set consisting of at most $\frac{1+\ln(\delta+1)}{\delta+1} n$ vertices.*

*Proof.* Let $p = \frac{\ln(\delta+1)}{\delta+1}$. Construct a random set $X$ as follows. Starting with $X = \emptyset$, for every $u \in V$, add $u$ to $X$ with probability $p$, where all random choices are mutually independent. Note that $|X| \sim \mathrm{Bin}\,(n, p)$ and thus, in particular, $\mathbb{E}(|X|) = np$. Let $Y$ denote the set of vertices of $V \setminus X$ that do not have a neighbour in $X$. For every $v \in V$, let $Y_v$ be the indicator random variable for the event "$v \in Y$". Then

$$\mathbb{E}(Y_v) = \mathbb{P}\,(Y_v = 1) = \mathbb{P}\,(\text{neither } v \text{ nor its neighbours are in } X) = (1-p)^{\deg_G(v)+1} \le (1-p)^{\delta+1}.$$

Let $S = X \cup Y$. Clearly, $S$ is a dominating set of $G$ and

$$\begin{aligned}\mathbb{E}(|S|) = \mathbb{E}(|X|) + \mathbb{E}(|Y|) &= np + \sum_{v \in V} \mathbb{E}(Y_v) \le np + n(1-p)^{\delta+1} \le np + ne^{-p(\delta+1)} \\ &= n\left(\frac{\ln(\delta+1)}{\delta+1} + e^{-\ln(\delta+1)}\right) = \frac{\ln(\delta+1)+1}{\delta+1} n.\end{aligned}$$

It follows that there exists a dominating set of $G$ of size at most $\frac{1+\ln(\delta+1)}{\delta+1} n$. □