# Lecture 6

## 1 Thresholds in random graphs

In Lecture 4 we proved that

$$\lim_{n\to\infty} \mathbb{P}\left(G(n,p) \text{ contains a triangle}\right) = \begin{cases} 0 & \text{if } p = o(1/n) \\ 1 & \text{if } p = \omega(1/n) \end{cases}$$

The following definitions and result extend this phenomenon to all monotone increasing graph properties.

**Definition 1.1.** *A function $p_0 = p_0(n)$ is called a* threshold *for the monotone increasing graph property $Q$ if*

$$\lim_{n\to\infty} \mathbb{P}\left(G(n,p) \in Q\right) = \begin{cases} 0 & \text{if } p = o(p_0) \\ 1 & \text{if } p = \omega(p_0) \end{cases}$$

In light of Definition 1.1 the result we proved in Lecture 4 shows that $1/n$ is a threshold for the property of containing a triangle. It is natural to ask which graph properties have a threshold? This question is answered by the following result.

**Theorem 1.2** (Bollobás and Thomason 1997)**.** *Every (non-trivial) monotone increasing graph property has a threshold.*

The existence of a threshold for a monotone increasing graph property $Q$ establishes a so-called *zero-one law* for this property. That is, it shows that for sufficiently large $n$, the probability that $G(n,p) \in Q$ is essentially 0 or 1, unless $p$ is in a small interval around the threshold. Sometimes (but not always) we can make this interval even smaller.

**Definition 1.3.** *A threshold $p_0$ for a monotone increasing graph property $Q$ is called* sharp *if for every $\varepsilon > 0$ it holds that*

$$\lim_{n\to\infty} \mathbb{P}\left(G(n,p) \in Q\right) = \begin{cases} 0 & \text{if } p \le (1-\varepsilon)p_0 \\ 1 & \text{if } p \ge (1+\varepsilon)p_0 \end{cases}$$

*If the threshold $p_0$ is not sharp, then it is called* coarse*.*

## 1.1 Minimum degree and connectivity of random graphs

**Theorem 1.4.** *$\ln n/n$ is a sharp threshold for $G(n,p)$ having positive minimum degree and for $G(n,p)$ being connected.*

*Proof.* Fix some $\varepsilon > 0$, let $n$ be a sufficiently large integer, and let $G \sim G(n,p)$. Since any connected graph has positive minimum degree, it suffices to prove the following two claims.

**(a)** If $p \leq (1-\varepsilon)\ln n/n$, then $\lim_{n\to\infty} \mathbb{P}(\delta(G) \geq 1) = 0$.

**(b)** If $p \geq (1+\varepsilon)\ln n/n$, then $\lim_{n\to\infty} \mathbb{P}(G \text{ is connected}) = 1$.

Starting with (a), assume that $p \leq (1-\varepsilon)\ln n/n$. For every $1 \leq j \leq n$ let $I_j$ be the indicator random variable for the event "$j$ is isolated in $G$". Let $X = \sum_{j=1}^n I_j$ denote the number of isolated vertices in $G$. Then

$$\mathbb{E}(X) = \sum_{j=1}^{n} \mathbb{E}(I_j) = \sum_{j=1}^{n} \mathbb{P}(I_j = 1) = n(1-p)^{n-1}.$$

Note that by our choice of $p$ it holds that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}(X) &= \lim_{n\to\infty} n(1-p)^{n-1} \geq \lim_{n\to\infty} ne^{-(p+p^2)(n-1)} \geq \lim_{n\to\infty} ne^{-\frac{(1-\varepsilon/2)\ln n}{n}\cdot n} \\ &= \lim_{n\to\infty} n^{1-(1-\varepsilon/2)} = \lim_{n\to\infty} n^{\varepsilon/2} = \infty, \end{aligned} \tag{1}$$

where the first inequality holds for sufficiently small $p$ by the Taylor expansion of $e^{-(x+x^2)}$.

Now, for every $1 \leq j \leq n$ it holds that

$$Var(I_j) = \mathbb{E}(I_j^2) - (\mathbb{E}(I_j))^2 \leq \mathbb{E}(I_j) = (1-p)^{n-1}.$$

Moreover, for every $1 \leq i < j \leq n$ it holds that

$$\begin{aligned} Cov(I_i, I_j) &= \mathbb{E}(I_i I_j) - \mathbb{E}(I_i)\mathbb{E}(I_j) = \mathbb{P}(I_i = 1, I_j = 1) - (1-p)^{2n-2} \\ &= (1-p)^{2n-3} - (1-p)^{2n-2} = p(1-p)^{2n-3}. \end{aligned}$$

Therefore

$$Var(X) = \sum_{j=1}^{n} Var(I_j) + 2\sum_{1\leq i<j\leq n} Cov(I_i, I_j) \leq n(1-p)^{n-1} + n^2 p(1-p)^{2n-3}.$$

We then have

$$\frac{Var(X)}{(\mathbb{E}(X))^2} \leq \frac{\mathbb{E}(X) + n^2p(1-p)^{2n-3}}{(\mathbb{E}(X))^2} = \frac{1}{\mathbb{E}(X)} + \frac{p}{1-p} = o(1),$$

where the last equality holds by the choice of $p$ and by (1). Applying the second moment method we conclude that

$$\lim_{n\to\infty} \mathbb{P}(\delta(G) \geq 1) = \lim_{n\to\infty} \mathbb{P}(X = 0) = 0.$$

Next, we prove (b). Assume that $p \geq (1+\varepsilon)\ln n/n$. If $G$ is disconnected, then its smallest connected component has $k$ vertices for some $1 \leq k \leq \lfloor n/2 \rfloor$. Hence

$$\begin{aligned}
\mathbb{P}\left(G \text{ is disconnected}\right) &\leq \sum_{k=1}^{n/2} \binom{n}{k} (1-p)^{k(n-k)} \\
&\leq \sum_{k=1}^{\sqrt{n}} \binom{n}{k} e^{-pk(n-k)} + \sum_{k=\sqrt{n}}^{n/2} \binom{n}{k} e^{-pk(n-k)}. \qquad (2)
\end{aligned}$$

We consider each of the two summands in (2) separately.

$$\begin{aligned}
\sum_{k=1}^{\sqrt{n}} \binom{n}{k} e^{-pk(n-k)} &\leq \sum_{k=1}^{\sqrt{n}} n^k e^{-pk(n-k)} = \sum_{k=1}^{\sqrt{n}} \left(ne^{-p(n-k)}\right)^k \\
&\leq \sum_{k=1}^{\sqrt{n}} \left(ne^{-\frac{(1+\varepsilon)\ln n}{n}\cdot(n-\sqrt{n})}\right)^k \leq \sum_{k=1}^{\sqrt{n}} \left(ne^{-(1+\varepsilon/2)\ln n}\right)^k \\
&= \sum_{k=1}^{\sqrt{n}} n^{-\varepsilon k/2} \leq \sum_{k=1}^{\infty} n^{-\varepsilon k/2} = \frac{n^{-\varepsilon/2}}{1-n^{-\varepsilon/2}}. \qquad (3)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sum_{k=\sqrt{n}}^{n/2} \binom{n}{k} e^{-pk(n-k)} &\leq \sum_{k=\sqrt{n}}^{n/2} \left(\frac{en}{k}\right)^k e^{-pk(n-k)} = \sum_{k=\sqrt{n}}^{n/2} \left(\frac{en}{k} \cdot e^{-p(n-k)}\right)^k \\
&\leq \sum_{k=\sqrt{n}}^{n/2} \left(\frac{en}{\sqrt{n}} \cdot e^{-\frac{(1+\varepsilon)\ln n}{n}\cdot(n-n/2)}\right)^k = \sum_{k=\sqrt{n}}^{n/2} \left(en^{1/2} \cdot e^{-(1/2+\varepsilon/2)\ln n}\right)^k \\
&= \sum_{k=\sqrt{n}}^{n/2} \left(en^{-\varepsilon/2}\right)^k \leq \sum_{k=1}^{\infty} \left(en^{-\varepsilon/2}\right)^k = \frac{en^{-\varepsilon/2}}{1-en^{-\varepsilon/2}}. \qquad (4)
\end{aligned}$$

Combining (2), (3) and (4) implies that

$$\lim_{n\to\infty} \mathbb{P}\left(G \text{ is disconnected}\right) \leq \lim_{n\to\infty} \frac{n^{-\varepsilon/2}}{1-n^{-\varepsilon/2}} + \lim_{n\to\infty} \frac{en^{-\varepsilon/2}}{1-en^{-\varepsilon/2}} = 0 + 0 = 0.$$

$\square$