Practical session 10

Exercise 1 Consider the function

$$f(x) = \begin{cases} 6x(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Show that f is a probability density function.

2. Let X be the continuous random variable whose probability density function is f. Calculate the following probabilities:

   (a) $\Pr(X \le -3)$,

   (b) $\Pr(0 \le X \le 1)$,

   (c) $\Pr(1/2 \le X \le 1)$.

Solution

1. First, observe that $f(x)$ is non-negative for every $x \in \mathbb{R}$. Indeed, $f(x) = 0$ for every $x \in \mathbb{R} \setminus [0,1]$, and for every $0 \le x \le 1$, it holds that $0 \le 1 - x \le 1$ and thus $f(x) \ge 0$. Now, it is evident by the definition of f that $F(x) = 0$ for every $x < 0$ and that $F(x) = 1$ for every $x > 1$. For every $x \in [0,1]$ it holds that

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{0}^{x} 6t(1-t)dt \\ &= \int_{0}^{x} 6tdt - \int_{0}^{x} 6t^2dt \\ &= 6 \cdot \frac{t^2}{2}\Big|_0^x - 6 \cdot \frac{t^3}{3}\Big|_0^x \\ &= 3x^2 - 2x^3. \end{aligned}$$

   In particular, $F(1) = 3 \cdot 1^2 - 2 \cdot 1^3 = 1$, implying that f is indeed a probability density function.

2. It follows from part 1. of this exercise that $F(x) = 3x^2 - 2x^3$ for every $x \in [0,1]$ and that $F(x) = 0$ for every $x < 0$. Therefore

(a) $\Pr(X \leq -3) = F(-3) = 0$.

(b) $\Pr(0 \leq X \leq 1) = F(1) - F(0) = 1 - 0 = 1$.

(c) $\Pr(1/2 \leq X \leq 1) = F(1) - F(1/2) = 1 - 3 \cdot (1/2)^2 + 2 \cdot (1/2)^3 = 1/2$.

Exercise 2 Fix $b \in \mathbb{R}^+$. Find all values of $a \in \mathbb{R}$ for which the function

$$\forall x \in \mathbb{R} \;\; f(x) = a \cdot e^{-|x|/b}$$

is a probability density function.

Solution
We first observe that $e^{-|x|/b} > 0$ for all $x \in \mathbb{R}$ and thus it suffices to find $a > 0$ for which

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

Note that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= a \cdot \int_{-\infty}^{\infty} e^{-|x|/b}dx \\
&= a \cdot \int_{-\infty}^{0} e^{x/b}dx + a \cdot \int_{0}^{\infty} e^{-x/b}dx \\
&= a \cdot be^{x/b}\Big|_{-\infty}^{0} - a \cdot be^{-x/b}\Big|_{0}^{\infty} \\
&= ab(1-0) - ab(0-1) \\
&= 2ab.
\end{aligned}$$

We conclude that $a = \frac{1}{2b}$.

Exercise 3 Let X be a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq 1/2 \\ -4x+4 & \text{if } 1/2 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Prove that f is indeed a probability density function and calculate its cumulative distribution function.

Solution
We first observe that $4x \geq 0$ for every $x \in [0, 1/2]$ and that $-4x + 4 \geq 0$ for every $x \in [1/2, 1]$.

Moreover

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_0^1 f(x)dx \\
&= \int_0^{1/2} 4xdx + \int_{1/2}^1 (-4x+4)dx \\
&= 4 \cdot \frac{x^2}{2}\Big|_0^{1/2} + 4\left(-\frac{x^2}{2}+x\right)\Big|_{1/2}^1 \\
&= \frac{1}{2} + 4\left(-\frac{1}{2}+1+\frac{1}{8}-\frac{1}{2}\right) \\
&= 1.
\end{aligned}
$$

We conclude that f is indeed a probability density function. Next, we calculate the cumulative distribution function of X. It is evident that $F_X(x) = 0$ for every $x < 0$ and that $F_X(x) = 1$ for every $x > 1$. Next, fix some $x \in [0, 1/2]$. Then

$$
\begin{aligned}
F_X(x) &= \int_0^x 4tdt \\
&= 2t^2\Big|_0^x \\
&= 2x^2.
\end{aligned}
$$

Finally, fix some $x \in (1/2, 1]$. Then

$$
\begin{aligned}
F(x) &= F(1/2) + \int_{1/2}^x (-4t+4)dt \\
&= 2 \cdot \left(\frac{1}{2}\right)^2 + \left(-2t^2+4t\right)\Big|_{1/2}^x \\
&= 1/2 + \left(-2x^2+4x+2\cdot\left(\frac{1}{2}\right)^2 - 4\cdot\frac{1}{2}\right) \\
&= -2x^2+4x-1.
\end{aligned}
$$

We conclude that

$$
F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \le x \le 1/2 \\ -2x^2+4x-1 & \text{if } 1/2 < x \le 1 \\ 1 & \text{if } x > 1 \end{cases}
$$

Exercise 4 Let X be a continuous random variable whose probability density function is given by

$$
f(x) = \begin{cases} C \cdot (x^2 - 3x + 2) & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}
$$

1. Find C.

2. Calculate $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

Solution

1. To find C, we solve the equation
$$\int_{-\infty}^{\infty} f(x)dx = 1$$
for C. It holds that
$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= C \cdot \int_{1}^{2} (x^2 - 3x + 2)dx \\ &= C \cdot \left(\frac{x^3}{3} - 3 \cdot \frac{x^2}{2} + 2x\right)\Bigg|_1^2 \\ &= C \cdot \left(\frac{8}{3} - 3 \cdot \frac{4}{2} + 2 \cdot 2 - \frac{1}{3} + 3 \cdot \frac{1}{2} - 2\right) \\ &= -\frac{C}{6}.\end{aligned}$$
Therefore, $C = -6$. Finally, observe that for $C = -6$, it holds that $f(x) = -6(x-1)(x-2)$ for every $1 \le x \le 2$, and thus $f(x) \ge 0$ for every $x \in \mathbb{R}$.

2. We first calculate $\mathbb{E}(X)$.
$$\begin{aligned}\mathbb{E}(X) &= \int_{-\infty}^{\infty} x f(x)dx \\ &= -6 \cdot \int_{1}^{2} x(x^2 - 3x + 2)dx \\ &= -6 \cdot \int_{1}^{2} (x^3 - 3x^2 + 2x)dx \\ &= -6 \cdot \left(\frac{x^4}{4} - 3 \cdot \frac{x^3}{3} + 2 \cdot \frac{x^2}{2}\right)\Bigg|_1^2 \\ &= -6 \cdot \left(\frac{16}{4} - 3 \cdot \frac{8}{3} + 2 \cdot \frac{4}{2} - \frac{1}{4} + 3 \cdot \frac{1}{3} - 2 \cdot \frac{1}{2}\right) \\ &= \frac{3}{2}.\end{aligned}$$

In order to calculate $\mathrm{Var}\left(X\right)$, we first calculate $\mathbb{E}\left(X^2\right)$. It holds that

$$\begin{aligned}
\mathbb{E}\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\
&= -6 \cdot \int_1^2 x^2(x^2 - 3x + 2)dx \\
&= -6 \cdot \int_1^2 (x^4 - 3x^3 + 2x^2)dx \\
&= -6 \cdot \left(\frac{x^5}{5} - 3 \cdot \frac{x^4}{4} + 2 \cdot \frac{x^3}{3}\right)\Bigg|_1^2 \\
&= -6 \cdot \left(\frac{32}{5} - 3 \cdot \frac{16}{4} + 2 \cdot \frac{8}{3} - \frac{1}{5} + 3 \cdot \frac{1}{4} - 2 \cdot \frac{1}{3}\right) \\
&= \frac{23}{10}.
\end{aligned}$$

We conclude that

$$\mathrm{Var}\left(X\right) = \mathbb{E}\left(X^2\right) - \left(\mathbb{E}\left(X\right)\right)^2 = \frac{23}{10} - \frac{9}{4} = \frac{1}{20}.$$