Lecture 9

1 Uncountable probability spaces

Recall from Probability Theory 1 that a (discrete) probability space is a pair $(\Omega, \mathbb{P})$, where Ω is a finite or countably infinite set called the *sample space*, and $\mathbb{P} : \Omega \to [0,1]$ is a function, called the *probability function*, which satisfies $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$. We then extended $\mathbb{P}$ to 2^Ω by defining $\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega)$ for every $A \subseteq \Omega$. One of the properties of this extended $\mathbb{P}$, called *finite additivity*, is that $\mathbb{P}(A_1 \cup \ldots \cup A_k) = \sum_{i=1}^{k} \mathbb{P}(A_i)$ holds for every positive integer k and pairwise disjoint sets $A_1, \ldots, A_k \subseteq \Omega$.

As a simple example consider the uniform distribution on $\{0,1\}$. In this case $\Omega = \{0,1\}$ and $\mathbb{P} : 2^\Omega \to [0,1]$ is a function satisfying $\mathbb{P}(\{0\}) = \mathbb{P}(\{1\})$. Since we also want to have

$$\mathbb{P}(\{0\}) + \mathbb{P}(\{1\}) = \mathbb{P}(\{0\} \cup \{1\}) = \mathbb{P}(\{0,1\}) = \mathbb{P}(\Omega) = 1,$$

it follows that $\mathbb{P}(\{0\}) = \mathbb{P}(\{1\}) = 1/2$.

Now, let us try to define an analogous distribution on the uncountably infinite set $[0,1]$. Let $\Omega = [0,1]$ and let $\mathbb{P} : 2^\Omega \to [0,1]$ be a function satisfying $\mathbb{P}(\{x\}) = p$ for every $x \in [0,1]$. It remains to determine the "correct" value of p. Assume first that $p > 0$. Let n be a positive integer such that $n > 1/p$. Then, assuming finite additivity, we have

$$\mathbb{P}(\{1, 1/2, 1/3, \ldots, 1/n\}) = \sum_{i=1}^{n} \mathbb{P}(\{1/i\}) = np > 1$$

which means that $\mathbb{P}$ is not a probability function. Assume then that $p = 0$. Finite additivity now implies that $\mathbb{P}(A) = 0$ for every finite set $A \subseteq [0,1]$. We will in fact require $\mathbb{P}$ to satisfy the stronger property of infinite countable additivity (usually referred to as σ-additivity). That is, for every (possibly infinite) countable set I and family of pairwise disjoint sets $\{A_i \subseteq [0,1] : i \in I\}$, we require $\mathbb{P}\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mathbb{P}(A_i)$. It will then imply that $\mathbb{P}(A) = 0$ for every countable set $A \subseteq [0,1]$. We cannot go further and expect uncountable additivity as this will imply

$$0 = \sum_{x \in [0,1]} \mathbb{P}(\{x\}) = \mathbb{P}([0,1]) = 1$$

which is an obvious contradiction.

Now that we have defined $\mathbb{P}$ for every $x \in [0,1]$, we would like, as in the discrete case, to extend the definition of $\mathbb{P}$ to all subsets of $\Omega = [0,1]$. What else should we require? It seems reasonable to expect $\mathbb{P}([0,1/2]) = 1/2$ to hold. We then expect $\mathbb{P}((1/2,1]) = 1 - \mathbb{P}([0,1/2]) = 1/2$ to hold as well. More generally, for every $0 \leq a \leq b \leq 1$ we require

$$\mathbb{P}([a,b]) = \mathbb{P}((a,b]) = \mathbb{P}([a,b)) = \mathbb{P}((a,b)) = b - a.$$

That is, we require the probability of a point x, chosen uniformly at random from $[0,1]$, to fall in a given line segment to be the length of that segment.

Finally, the probability that the random point x belongs to some set A should not be affected by its location in $[0,1]$. Namely, for a set $A \subseteq [0,1]$ and a real number r, let

$$A \oplus r = \{a + r : a \in A, a + r \leq 1\} \cup \{a + r - 1 : a \in A, a + r > 1\}.$$

Then $\mathbb{P}(A \oplus r) = \mathbb{P}(A)$ for every set $A \subseteq [0,1]$ and every real number $0 \leq r \leq 1$. We can now try to define a uniform distribution on $[0,1]$, but as the next result shows, we have already asked for too much.

Proposition 1.1. *There does not exist a function* $\mathbb{P} : 2^{[0,1]} \to [0,1]$ *which satisfies all of the following conditions:*

(1) $\mathbb{P}([a,b]) = \mathbb{P}((a,b]) = \mathbb{P}([a,b)) = \mathbb{P}((a,b)) = b - a$ *for every* $0 \leq a \leq b \leq 1$*;*

(2) $\mathbb{P}$ *is* σ*-additive;*

(3) $\mathbb{P}(A \oplus r) = \mathbb{P}(A)$ *for every set* $A \subseteq [0,1]$ *and every real number* $0 \leq r \leq 1$.

Proof. Suppose for a contradiction that $\mathbb{P} : 2^{[0,1]} \to [0,1]$ is a function which satisfies properties (1), (2) and (3) of Proposition 1.1. Define a relation $\sim$ on $[0,1]$ as follows: $\forall x, y \in [0,1]$ $x \sim y$ if and only if $y - x$ is rational. Observe that $\sim$ is an equivalence relation. Let $H \subseteq [0,1]$ be a set consisting of one element from each equivalence class of $\sim$ (such a set H exists by the axiom of choice). For convenience assume that $0 \notin H$ (if $0 \in H$, then replace it with $1/2$).

Claim 1.2. $\{H \oplus r : r \in \mathbb{Q} \cap [0,1)\}$ *is a partition of* $(0,1]$.

Proof. We need to prove that for every $x \in (0,1]$ there is a unique $r \in \mathbb{Q} \cap [0,1)$ such that $x \in H \oplus r$. We begin by proving that at least one set $H \oplus r$ containing x exists. Since $\sim$ is an equivalence relation, it partitions the elements of $[0,1]$ into equivalence classes. Let A denote the equivalence class of x and let a be the unique element in $A \cap H$. Then $x \sim a$, that is, $x - a = r$ for some $r \in \mathbb{Q} \cap (-1,1)$. If $r \geq 0$, then $x \in H \oplus r$ and if $r < 0$, then $x \in H \oplus (r+1)$.

Suppose now for a contradiction that there are real numbers $0 \leq r_1 < r_2 < 1$ such that $x \in H \oplus r_1$ and $x \in H \oplus r_2$. Then, there are real numbers $a, b \in H$ such that $x \in \{a + r_1, a + r_1 - 1\} \cap \{b + r_2, b + r_2 - 1\}$. It thus follows that $a - b \in \mathbb{Q}$ and so a and b belong to the same equivalence class of $\sim$. It then follows by the construction of H that $a = b$. Since, moreover, $r_1 < r_2$, it must hold that $a + r_1 = a + r_2 - 1$. However, we then have $r_2 = r_1 + 1$ which is not possible since $r_1, r_2 \in [0,1)$. □

Now, it follows by Claim 1.2 and by Property (2) that

$$\mathbb{P}((0,1]) = \sum_{r \in \mathbb{Q} \cap [0,1)} \mathbb{P}(H \oplus r).$$

Since, moreover, $\mathbb{P}(H \oplus r) = \mathbb{P}(H)$ holds for every $r \in \mathbb{Q} \cap [0,1)$ by Property (3), it follows that

$$1 = \mathbb{P}((0,1]) = \sum_{r \in \mathbb{Q} \cap [0,1)} \mathbb{P}(H),$$

where the first equality holds by Property (1). This is a contradiction as a countably infinite sum of the same non-negative quantity can only equal 0 or ∞. $\square$

Remark 1.3. *Note that in the proof of Proposition 1.1 we used Property (1) only in order to justify the equality $1 = \mathbb{P}((0,1])$. Indeed, while requiring $\mathbb{P}$ to satisfy properties (1), (2) and (3) is natural, a stronger form of Proposition 1.1 is true (but harder to prove). Namely, there does not exist a function $\mathbb{P} : 2^{[0,1]} \to \mathbb{R}$ which satisfies all of the following conditions:*

(a) $0 < \mathbb{P}([0,1]) < \infty$;

(b) $\mathbb{P}$ *is σ-additive;*

(c) $\mathbb{P}(\{x\}) = 0$ *for every* $x \in [0,1]$.

What can we do if we cannot even define the uniform distribution on $[0,1]$? We have to give up at least one of the requirements made in the statement of Proposition 1.1. We will give up the requirement that "every event has a probability", that is, the requirement that $\mathbb{P}$ is defined for every $A \subseteq [0,1]$.

Definition 1.4. *A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where*

(i) Ω *is a non-empty set called the* sample space*;*

(ii) $\mathcal{F} \subseteq 2^{\Omega}$ *is a σ-algebra, that is, $\mathcal{F}$ contains $\emptyset$ and is closed under the formation of complements, countable unions and countable intersections;*

(iii) $\mathbb{P} : \mathcal{F} \to [0,1]$ *is a* probability function *which is σ-additive and satisfies $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$.*

Remark 1.5. *When Ω is finite or countably infinite, we can take $\mathcal{F} = 2^{\Omega}$. Therefore, we often denote the corresponding probability space by $(\Omega, \mathbb{P})$.*