Probability Theory 2
Proposed solution of moed aleph exam 2022b

1. **(a)** Since f is a probability density function, it follows that

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^{1/2} cxdx + \int_{1/2}^{1} c(1-x)dx = cx^2/2\Big|_0^{1/2} + (cx - cx^2/2)\Big|_{1/2}^{1}$$
$$= (c/8 - 0) + [(c - c/2) - (c/2 - c/8)] = c/4.$$

Solving the latter equation yields $c = 4$; observe that for this value of c, the function f is non-negative.

(b) Fix an arbitrary positive integer k. It follows by a theorem that was proved in the lectures that

$$\begin{aligned}\mathbb{E}\left(X^k\right) &= \int_{-\infty}^{\infty} x^k f(x)dx = \int_0^{1/2} 4x^{k+1}dx + \int_{1/2}^{1} 4x^k(1-x)dx \\ &= \frac{4x^{k+2}}{k+2}\Big|_0^{1/2} + \left(\frac{4x^{k+1}}{k+1} - \frac{4x^{k+2}}{k+2}\right)\Big|_{1/2}^{1} \\ &= \frac{1}{2^k(k+2)} + \frac{4}{k+1} - \frac{4}{k+2} - \frac{1}{2^{k-1}(k+1)} + \frac{1}{2^k(k+2)} \\ &= \frac{1}{(k+1)(k+2)}\left(4 - \frac{1}{2^{k-1}}\right).\end{aligned}$$

2. Fix an arbitrary tournament $T = ([n], E)$. Draw a permutation $\pi \in S_n$ uniformly at random. For every directed edge $(i,j) \in E$, let $X_{(i,j)}$ be the indicator random variable for the event "π agrees with (i,j)". Then

$$\mathbb{E}(X_{(i,j)}) = \mathbb{P}(X_{(i,j)} = 1) = \mathbb{P}(\pi(i) < \pi(j)) = \frac{\binom{n}{2}(n-2)!}{n!} = 1/2$$

for every $(i,j) \in E$. Let $X_\pi = \sum_{(i,j)\in E} X_{(i,j)}$; then X_π is the number of directed edges of T with which π agrees. It follows by the linearity of expectation that $\mathbb{E}(X_\pi) = \sum_{(i,j)\in E} \mathbb{E}(X_{(i,j)}) = |E|/2$. Therefore, there exists a permutation $\sigma \in S_n$ such that $X_\sigma \geq |E|/2$.

3. Let $G \sim \mathbb{G}(n, 1/4)$ and fix an arbitrary $i \in [n]$. Note that $\deg_G(i) \sim \text{Bin}(n-1, 1/4)$ and so, in particular, $\mathbb{E}(\deg_G(i)) = (n-1)/4$. Therefore

$$\begin{aligned}\mathbb{P}(\deg_G(i) > 0.26n \text{ or } \deg_G(i) < 0.24n) &< \mathbb{P}(|\deg_G(i) - \mathbb{E}(\deg_G(i))| \geq 0.05n) \\ &\leq 2e^{-\frac{2(n/20)^2}{n-1}} \leq 2e^{-n/200}\end{aligned}$$

where the second inequality holds by Chernoff's bound. A union bound over all vertices of G then shows that the probability that there exists a vertex whose degree in G is larger than $0.26n$ or smaller than $0.24n$ is at most $2ne^{-n/200} = o(1)$.

4. For every $1 \leq i \leq 10000$, let X_i be the indicator random variable for the event "the outcome of the ith coin toss is HEADS", that is, $X_i = 1$ if the outcome of the ith coin toss is HEADS and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{10000} X_i$. For every $1 \leq i \leq 10000$, it holds that $\mathbb{E}(X_i) = \mathbb{P}(X_i = 1) = 1/2$ and $\mathrm{Var}(X_i) = \mathbb{E}\left(X_i^2\right) - (\mathbb{E}(X_i))^2 = 1/2 - 1/4 = 1/4$. It thus follows by the Central Limit Theorem that

$$\begin{aligned}
\mathbb{P}(4800 \leq X \leq 5100) &= \mathbb{P}\left(\frac{4800 - 10000 \cdot 1/2}{\sqrt{1/4} \cdot \sqrt{10000}} \leq \frac{X - 10000 \cdot 1/2}{\sqrt{1/4} \cdot \sqrt{10000}} \leq \frac{5100 - 10000 \cdot 1/2}{\sqrt{1/4} \cdot \sqrt{10000}}\right) \\
&= \mathbb{P}\left(-4 \leq \frac{X - 5000}{50} \leq 2\right) \\
&\approx \frac{1}{\sqrt{2\pi}} \int_{-4}^{2} e^{-x^2/2} dx.
\end{aligned}$$

We conclude that $a = -4$ and $b = 2$ satisfy the desired property.