Practical session 11

Exercise 1 Let $X \sim \mathrm{U}([0,1])$ and let $Y = e^X$.

1. Find the cumulative distribution function and probability density function of Y.
2. Calculate $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

Solution

1. We will calculate the cumulative distribution function of Y using the cumulative distribution function of X. Since $X \sim \mathrm{U}([0,1])$ it follows that

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

For every $y \in [1, e]$, we then have

$$F_Y(y) = \Pr[Y \le y] = \Pr\left[e^X \le y\right] = \Pr[X \le \ln y] = F_X(\ln y) = \ln y,$$

where the third equality holds since e^x is an increasing function. The monotonicity of F_Y then implies that $F_Y(y) \le F_Y(1) = 0$ for every $y < 1$ and that $F_Y(y) \ge F_Y(e) = 1$ for every $y > e$. We conclude that

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 1 \\ \ln y & \text{if } 1 \le y \le e \\ 1 & \text{if } y > e \end{cases}$$

Differentiating F_Y then yields

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{1}{y} & \text{if } 1 \le y \le e \\ 0 & \text{otherwise} \end{cases}$$

2. We demonstrate two methods for calculating $\mathbb{E}(Y)$ and $\mathbb{E}(Y^2)$. The first method uses the definition of Y and the probability density function of X. We have that

$$\mathbb{E}(Y) = \mathbb{E}\left(e^X\right) = \int_{-\infty}^{\infty} e^x f_X(x)dx = \int_0^1 e^x dx = e^x\Big|_0^1 = e - 1,$$

and that

$$\mathbb{E}\left(Y^2\right) = \mathbb{E}\left(e^{2X}\right) = \int_{-\infty}^{\infty} e^{2x} f_X(x) dx = \int_0^1 e^{2x} dx = \frac{1}{2} \cdot e^{2x}\Big|_0^1 = \frac{e^2-1}{2}.$$

The second method uses the probability density function of Y which was calculated in part 1. of this exercise. It holds that

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_1^e y \cdot \frac{1}{y} dy = y\Big|_1^e = e - 1,$$

and that

$$\mathbb{E}\left(Y^2\right) = \int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_1^e y^2 \cdot \frac{1}{y} dy = \frac{y^2}{2}\Big|_1^e = \frac{e^2-1}{2}.$$

Either way, we conclude that $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = \frac{e^2-1}{2} - (e-1)^2 = \frac{-e^2+4e-3}{2}$.

Exercise 2 Let $X \sim \mathrm{Exp}(\lambda)$ be an exponential random variable with parameter $\lambda > 0$. Prove the following statements.

1. $\mathbb{E}(X^n) = \frac{n}{\lambda} \cdot \mathbb{E}\left(X^{n-1}\right)$ holds for every positive integer n.
2. $\mathbb{E}(X^n) = \frac{n!}{\lambda^n}$ holds for every positive integer n.

Solution

1. Recall that

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Integrating by parts with $u = x^n$ and $v' = \lambda e^{-\lambda x}$ then yields

$$\begin{aligned} \mathbb{E}(X^n) &= \int_{-\infty}^{\infty} x^n f_X(x) dx \\ &= \int_0^{\infty} x^n \cdot \lambda e^{-\lambda x} dx \\ &= x^n \cdot \left(-e^{-\lambda x}\right)\Big|_0^{\infty} - \int_0^{\infty} n x^{n-1} \cdot \left(-e^{-\lambda x}\right) dx \\ &= 0 - 0 + \frac{n}{\lambda} \cdot \int_0^{\infty} x^{n-1} \cdot \lambda e^{-\lambda x} dx \\ &= \frac{n}{\lambda} \cdot \mathbb{E}\left(X^{n-1}\right). \end{aligned}$$

2. We prove the claim by induction on n. For the induction basis, we take $n = 1$. As was proved in the lecture, it holds that $\mathbb{E}(X) = \frac{1}{\lambda}$, that is, the claim is true for $n = 1$. Next, we assume that the claim holds for $n - 1$ and prove it for n. By part 1. of this exercise, we have that

$$\mathbb{E}(X^n) = \frac{n}{\lambda} \cdot \mathbb{E}\left(X^{n-1}\right).$$

It then follows by the induction hypothesis that

$$\mathbb{E}\left(X^n\right) = \frac{n}{\lambda} \cdot \frac{(n-1)!}{\lambda^{n-1}} = \frac{n!}{\lambda^n},$$

as claimed.

Exercise 3 Let $U \sim \mathrm{U}[0,1]$ and let $X = -\ln(1-U)$. Prove that $X \sim \mathrm{Exp}\,(1)$.

Solution
Fix some $x \geq 0$. Then

$$\begin{aligned} F_X(x) &= \Pr\left[X \leq x\right] = \Pr\left[-\ln(1-U) \leq x\right] = \Pr\left[\frac{1}{1-U} \leq e^x\right] \\ &= \Pr\left[U \leq 1 - e^{-x}\right] = 1 - e^{-x}. \end{aligned}$$

It follows by the monotonicity of F_X that $F_X(x) \leq F_X(0) = 0$ holds for every $x < 0$. Hence

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0 \end{cases}$$

Recalling the cumulative distribution function of the exponential distribution, we conclude that $X \sim \mathrm{Exp}\,(1)$ as claimed.

Exercise 4 Let $X \sim \mathrm{U}(-1,1)$ and let $Y = X^2$.

1. Calculate the probability density function and cumulative distribution function of Y.
2. Calculate $\mathbb{E}\left(Y\right)$ and $\mathrm{Var}\left(Y\right)$.

Solution

1. Starting with the cumulative distribution function of Y, fix some $y \in [0,1)$. Then

$$\begin{aligned} F_Y(y) &= \Pr\left[Y \leq y\right] = \Pr\left[X^2 \leq y\right] = \Pr\left[-\sqrt{y} \leq X \leq \sqrt{y}\right] \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{\sqrt{y} - (-1)}{2} - \frac{-\sqrt{y} - (-1)}{2} = \sqrt{y}. \end{aligned}$$

It follows by the monotonicity of F_Y that $F_Y(y) \leq F_Y(0) = 0$ holds for every $y < 0$ and that $F_Y(y) = 1$ holds for every $y \geq 1$. Hence

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \sqrt{y} & \text{if } 0 \leq y < 1 \\ 1 & \text{if } y \geq 1 \end{cases}$$

Differentiating F_Y yields

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{2\sqrt{y}} & \text{if } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Starting with $\mathbb{E}(Y)$, it holds that.

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 y \cdot \frac{1}{2\sqrt{y}} dy = \frac{1}{2} \cdot \int_0^1 y^{1/2} dy = \frac{1}{2} \cdot \frac{y^{3/2}}{3/2}\Big|_0^1 = \frac{1}{3}.$$

Next, we calculate $\mathbb{E}(Y^2)$.

$$\mathbb{E}\left(Y^2\right) = \int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_0^1 y^2 \cdot \frac{1}{2\sqrt{y}} dy = \frac{1}{2} \cdot \int_0^1 y^{3/2} dy = \frac{1}{2} \cdot \frac{y^{5/2}}{5/2}\Big|_0^1 = \frac{1}{5}.$$

Therefore, $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$.