Probability Theory 2 – Exercise sheet III

If you wish to submit your solutions to any of these questions, please hand them to your TA during the practical session held in the week starting on 28/11/2021. This deadline is strict!

1. Let $G \sim G(n,p)$, where $p = 10 \ln n / n$. Let δ denote the minimum degree in G and let Δ denote the maximum degree in G. Prove the following two claims:
   (a) $\lim_{n\to\infty} \mathbb{P}(\delta \geq \ln n) = 1$.
   (b) $\lim_{n\to\infty} \mathbb{P}(\Delta \leq 20 \ln n) = 1$.

2. Prove that $n^{-2/3}$ is a threshold for the property of containing a copy of K_4 (the complete graph on 4 vertices). That is, prove the following two claims:
   (a) If $p = o(n^{-2/3})$, then $\lim_{n\to\infty} \mathbb{P}(G(n,p) \text{ contains a copy of } K_4) = 0$.
   (b) If $p = \omega(n^{-2/3})$, then $\lim_{n\to\infty} \mathbb{P}(G(n,p) \text{ contains a copy of } K_4) = 1$.

3. Let $n \geq k \geq 2$ and $1 \leq m \leq 2^{k-2}$ be integers. Let $\{A_1, \ldots, A_m\}$ be a family of subsets of $\{1, \ldots, n\}$, each of size k. Devise a randomized algorithm with the following properties:
   (a) Its input is the number n and the family $\{A_1, \ldots, A_m\}$.
   (b) Its output is a red/blue colouring of the elements of the set $\{1, \ldots, n\}$.
   (c) With probability at least $1 - 2^{-100}$, the colouring produced by the algorithm is such that, for every $1 \leq i \leq m$, the set A_i contains at least one red element and at least one blue element.
   (d) The running time of the algorithm is $O(n + km)$.

4. Let $\mathcal{F} \subseteq \mathbb{Z}_2^n$ be a family of binary vectors of length n. For any two binary vectors $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$, define the distance between them to be

$$dist(\bar{x}, \bar{y}) = |\{1 \leq i \leq n : x_i \neq y_i\}|.$$

   Assume that the distance between any two vectors in $\mathcal{F}$ is at least $n/10$. Devise a randomized algorithm with the following properties:
   (a) Its input are vectors $\bar{x}, \bar{y} \in \mathcal{F}$.
   (b) Its output is either $\bar{x} = \bar{y}$ or $\bar{x} \neq \bar{y}$.
   (c) If $\bar{x} = \bar{y}$, then the algorithm will output $\bar{x} = \bar{y}$.
   (d) If $\bar{x} \neq \bar{y}$, then the algorithm will output $\bar{x} \neq \bar{y}$ with probability at least $1 - 2^{-100}$.
   (e) The running time of the algorithm is constant (i.e., independent of n).