Probability Theory 2 – Solutions I

1. Let X be a random variable with finite expectation μ and let $k \geq 2$ be an even integer. Assume that $\mathbb{E}\left[(X-\mu)^k\right]$ exists and is finite. Prove that
$$Pr\left(|X-\mu| \geq t\mathbb{E}\left[(X-\mu)^k\right]^{1/k}\right) \leq t^{-k}$$
for every $t > 0$.

   Solution: Since k is even, it follows that $|X-\mu| \geq a$ if and only if $(X-\mu)^k \geq a^k$, for any real number $a > 0$. Hence, it follows by Markov's inequality that
$$\begin{aligned} Pr\left(|X-\mu| \geq t \cdot \mathbb{E}\left[(x-\mu)^k\right]^{1/k}\right) &= Pr\left((X-\mu)^k \geq t^k \cdot \mathbb{E}[(x-\mu)^k]\right) \\ &\leq \frac{\mathbb{E}[(x-\mu)^k]}{t^k \mathbb{E}[(x-\mu)^k]} = \frac{1}{t^k}. \end{aligned}$$

2. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables, each satisfying $Pr(X_i = 1) = Pr(X_i = -1) = 1/2$. Prove that
$$Pr\left(\sum_{i=1}^{n} X_i \leq t\right) \leq e^{-t^2/(2n)}$$
for every $t < 0$.

   Solution: For every $1 \leq i \leq n$, let $Y_i = -X_i$ and let $k = -t$. Note that $Y_1, \ldots, Y_n$ and k satisfy the conditions of the Chernoff type bound which was proved in the lecture. Then
$$Pr\left(\sum_{i=1}^{n} X_i \leq t\right) = Pr\left(-\sum_{i=1}^{n} X_i \geq -t\right) = Pr\left(\sum_{i=1}^{n} Y_i \geq k\right) \leq e^{-k^2/(2n)} = e^{-t^2/(2n)}.$$

3. Let $X \sim Bin(n, 1/2)$ be a random variable. Use Exercise 2 to prove that
$$Pr\left(X \leq n/2 - t\right) \leq e^{-2t^2/n}$$
for every $t > 0$.

   Solution: We can write X as a sum of n independent Bernoulli random variables, namely, $X = \sum_{i=1}^{n} X_i$ where $Pr(X_i = 1) = Pr(X_i = 0) = 1/2$ for every $1 \leq i \leq n$. For every $1 \leq i \leq n$, let $Y_i = 2X_i - 1$. Observe that $Y_1, \ldots, Y_n$ are independent and that $Pr(Y_i = 1) = Pr(X_i = 1) = 1/2$ and $Pr(Y_i = -1) = Pr(X_i = 0) = 1/2$ for every $1 \leq i \leq n$. Hence
$$\begin{aligned} Pr\left(X \leq n/2 - t\right) &= Pr\left(\sum_{i=1}^{n} X_i \leq n/2 - t\right) = Pr\left(\sum_{i=1}^{n} \frac{Y_i + 1}{2} - \frac{n}{2} \leq -t\right) \\ &= Pr\left(\sum_{i=1}^{n} Y_i + n - n \leq -2t\right) = Pr\left(\sum_{i=1}^{n} Y_i \leq -2t\right) \\ &\leq e^{-(2t)^2/(2n)} = e^{-2t^2/n}, \end{aligned}$$

where the inequality holds by Exercise 2.

4. We construct two random subsets A and B of $\{1, \ldots, 1000\}$ as follows. For every $1 \leq i \leq 1000$ we flip two fair coins, all coin flips being mutually independent. We put i in A if and only if the first coin flipped for i resulted in heads and we put i in B if and only if the second coin flipped for i resulted in heads. Let $X = \sum_{a \in A} a - \sum_{b \in B} b$. Use Chernoff's inequality (any of the ones that were presented in class) to upper bound $Pr\left(X \geq 2\sqrt{1000^3}\right)$.

   Solution: We will use the following version of Chernoff's inequality that was stated in the lecture without proof.

   Theorem 1 *Let $X_1, \ldots, X_n$ be independent random variables such that $X_i \in [0,1]$ for every $1 \leq i \leq n$ and let $X = \sum_{i=1}^n X_i$. Then*

$$Pr\left(X \geq \mathbb{E}(X) + t\right) \leq e^{-2t^2/n}$$

   for every $t > 0$.

   For every $1 \leq i \leq 1000$ define the random variable X_i as follows: $X_i = 1$ if $i \in A \setminus B$, $X_i = -1$ if $i \in B \setminus A$, and $X_i = 0$ otherwise. Observe that $Pr(X_i = 1) = Pr(X_i = -1) = 1/4$ and $Pr(X_i = 0) = 1/2$; in particular, $\mathbb{E}(X_i) = 0$. Moreover, since all coin flips are independent and, for every $1 \leq i < j \leq 1000$, X_i and X_j rely on disjoint pairs of coin flips, it follows that $X_1, \ldots, X_{1000}$ are independent random variables. Now, for every $1 \leq i \leq 1000$, let $Y_i = (iX_i + 1000)/2000$. Observe that $Y_1, \ldots, Y_{1000}$ are independent, that $Y_i \in [0,1]$ for every $1 \leq i \leq 1000$, and that $X = \sum_{i=1}^{1000} iX_i = 2000\left(\sum_{i=1}^{1000} Y_i - 500\right)$. Note that

$$\mathbb{E}\left(\sum_{i=1}^{1000} Y_i\right) = \frac{1}{2000} \cdot \mathbb{E}\left(\sum_{i=1}^{1000} iX_i + 1000\right) = \frac{1}{2000} \cdot \sum_{i=1}^{1000} i \cdot \mathbb{E}(X_i) + 500 = 500.$$

   It follows that

$$\begin{aligned} Pr\left(X \geq 2\sqrt{1000^3}\right) &= Pr\left(\sum_{i=1}^{1000} Y_i - 500 \geq \sqrt{1000}\right) \\ &= Pr\left(\sum_{i=1}^{1000} Y_i \geq \mathbb{E}\left(\sum_{i=1}^{1000} Y_i\right) + \sqrt{1000}\right) \\ &\leq e^{-2(\sqrt{1000})^2/1000} = e^{-2}, \end{aligned}$$

   where the inequality holds by Theorem 1.

5. Let $0 \leq p_1, p_2, \ldots, p_n \leq 1$ be real numbers and let $p = (p_1 + \ldots + p_n)/n$. Let $X_1, X_2, \ldots, X_n$ be mutually independent random variables such that $Pr(X_i = 1) = p_i$ and $Pr(X_i = 0) = 1 - p_i$ for every $1 \leq i \leq n$. Prove that

$$\lim_{n \to \infty} Pr\left(\left|\frac{X_1 + \ldots + X_n}{n} - p\right| \geq \varepsilon\right) = 0$$

for every $\varepsilon > 0$.

Solution: Fix an arbitrary $1 \le i \le n$. Then

$$\mathbb{E}(X_i) = Pr(X_i = 1) = p_i$$

and

$$Var(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = p_i - p_i^2 \le 1.$$

Let $S_n = \frac{X_1 + \ldots + X_n}{n}$. Then

$$\mathbb{E}(S_n) = \mathbb{E}\left(\frac{\sum_{i=1}^n X_i}{n}\right) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}(X_i) = \frac{1}{n}\sum_{i=1}^n p_i = p$$

and

$$Var(S_n) = Var\left(\frac{\sum_{i=1}^n X_i}{n}\right) = \frac{1}{n^2}\sum_{i=1}^n Var(X_i) \le \frac{1}{n},$$

where the second equality holds since the X_i's are mutually independent.

Hence, it follows by Chebyshev's inequality that for any $\varepsilon > 0$

$$Pr\left(\left|\frac{X_1 + \ldots + X_n}{n} - p\right| \ge \varepsilon\right) = Pr\left(|S_n - \mathbb{E}(S_n)| \ge \varepsilon\right) \le \frac{Var(S_n)}{\varepsilon^2} \le \frac{1}{n\varepsilon^2}.$$

Hence

$$0 \le \lim_{n\to\infty} Pr\left(\left|\frac{X_1 + \ldots + X_n}{n} - p\right| \ge \varepsilon\right) \le \lim_{n\to\infty} \frac{1}{n\varepsilon^2} = 0$$

as claimed.

6. For each of the following values of $\{X_n\}_{n=1}^{\infty}$ and X, decide whether $X_n \stackrel{p}{\to} X$ or not and whether $X_n \stackrel{a.s.}{\to} X$ or not.

   (a) $X \equiv 0$ and $\{X_n\}_{n=1}^{\infty}$ is a sequence of mutually independent random variables, such that

   $$X_n \sim \begin{cases} n, & 1/n^2 \\ 0, & 1 - 1/n^2 \end{cases}$$

   for every positive integer n.

   (b) $X \equiv 1$ and $\{X_n\}_{n=1}^{\infty}$ is a sequence of mutually independent random variables, such that $X_n \sim \text{Ber}\left(\frac{n}{n+1}\right)$ for every positive integer n.

   (c) $X, X_1, X_2, X_3, \ldots$ are mutually independent random variables, such that $X \sim \text{Ber}(1/2)$ and $X_n \sim \text{Ber}(1/2)$ for every positive integer n.

Solution: We can skip some of the arguments in the solution by using the fact that if $X_n \stackrel{a.s.}{\to} X$, then $X_n \stackrel{p}{\to} X$. We do not use this fact in this model solution as it was stated in the lectures without proof.

(a) Fix any $\varepsilon > 0$. Then

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = \lim_{n\to\infty} \mathbb{P}(X_n = n) = \lim_{n\to\infty} 1/n^2 = 0,$$

implying that $X_n \xrightarrow{p} X$.

Next

$$\begin{aligned}\lim_{m\to\infty} \mathbb{P}(X_n = 0 \text{ for every } n \geq m) &= \lim_{m\to\infty} \prod_{n=m}^{\infty} (1 - 1/n^2) = \lim_{m\to\infty} \prod_{n=m}^{\infty} e^{-1/n^2} \\ &= \lim_{m\to\infty} e^{-\sum_{n=m}^{\infty} 1/n^2} = 1,\end{aligned}$$

where the second equality holds since $\lim_{x\to 0} \frac{e^{-x}}{1-x} = 1$ and the last equality holds since $\sum_{n=1}^{\infty} 1/n^2$ converges and thus $\lim_{m\to\infty} \sum_{n=m}^{\infty} 1/n^2 = 0$. We conclude that $X_n \xrightarrow{a.s.} X$.

(b) Fix any $\varepsilon > 0$. Then

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = \lim_{n\to\infty} \mathbb{P}(X_n = 0) = \lim_{n\to\infty} 1/(n+1) = 0,$$

implying that $X_n \xrightarrow{p} X$.

On the other hand, we will show that $X_n \overset{a.s.}{\not\to} X$. Suppose for a contradiction that $X_n \xrightarrow{a.s.} X$. It follows that for every $\varepsilon > 0$ there exists an integer m such that, with probability 1, for every $n \geq m$ it holds that $|X_n - 1| < \varepsilon$. However, for sufficiently small ε, the latter inequality holds if and only if $X_n = 1$. However

$$\begin{aligned}\lim_{m\to\infty} \mathbb{P}(X_n = 1 \text{ for every } n \geq m) &= \lim_{m\to\infty} \prod_{n=m}^{\infty} (1 - 1/(n+1)) = \lim_{m\to\infty} \prod_{n=m}^{\infty} e^{-1/(n+1)} \\ &= \lim_{m\to\infty} e^{-\sum_{n=m}^{\infty} 1/(n+1)} = 0,\end{aligned}$$

where the second equality holds since $\lim_{x\to 0} \frac{e^{-x}}{1-x} = 1$ and the last equality holds since $\sum_{n=1}^{\infty} 1/(n+1)$ diverges and thus $\lim_{m\to\infty} \sum_{n=m}^{\infty} 1/(n+1) = \infty$. We conclude that $X_n \overset{a.s.}{\not\to} X$.

(c) For any $0 < \varepsilon < 1$ and every $n \in \mathbb{N}$ it holds that

$$\mathbb{P}(|X_n - X| \geq \varepsilon) = \mathbb{P}(X_n \neq X) = \mathbb{P}(X_n = 0, X = 1) + \mathbb{P}(X_n = 1, X = 0) = 1/2.$$

Therefore $\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 1/2 > 0$ implying that $X_n \overset{p}{\not\to} X$.

Similarly,

$$\lim_{m\to\infty} \mathbb{P}(X_n = X \text{ for every } n \geq m) = \lim_{m\to\infty} \prod_{n=m}^{\infty} 1/2 = 0,$$

implying that $X_n \overset{a.s.}{\not\to} X$.