Probability Theory 2 – Exercise sheet I

If you wish to submit your solutions to any of these questions, please hand them to your TA during the practical session held in the week starting on 31/10/2021. This deadline is strict!

1. Let X be a random variable with finite expectation μ and let $k \geq 2$ be an even integer. Assume that $\mathbb{E}\left[(X-\mu)^k\right]$ exists and is finite. Prove that
$$Pr\left(|X-\mu| \geq t\mathbb{E}\left[(X-\mu)^k\right]^{1/k}\right) \leq t^{-k}$$
for every $t > 0$.

2. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables, each satisfying $Pr(X_i = 1) = Pr(X_i = -1) = 1/2$. Prove that
$$Pr\left(\sum_{i=1}^{n} X_i \leq t\right) \leq e^{-t^2/(2n)}$$
for every $t < 0$.

3. Let $X \sim Bin(n, 1/2)$ be a random variable. Use Exercise 2 to prove that
$$Pr\left(X \leq n/2 - t\right) \leq e^{-2t^2/n}$$
for every $t > 0$.

4. We construct two random subsets A and B of $\{1, \ldots, 1000\}$ as follows. For every $1 \leq i \leq 1000$ we flip two fair coins, all coin flips being mutually independent. We put i in A if and only if the first coin flipped for i resulted in heads and we put i in B if and only if the second coin flipped for i resulted in heads. Let $X = \sum_{a \in A} a - \sum_{b \in B} b$. Use Chernoff's inequality (any of the ones that were presented in class) to upper bound $Pr\left(X \geq 2\sqrt{1000^3}\right)$.

5. Let $0 \leq p_1, p_2, \ldots, p_n \leq 1$ be real numbers and let $p = (p_1 + \ldots + p_n)/n$. Let $X_1, X_2, \ldots, X_n$ be mutually independent random variables such that $Pr(X_i = 1) = p_i$ and $Pr(X_i = 0) = 1 - p_i$ for every $1 \leq i \leq n$. Prove that
$$\lim_{n \to \infty} Pr\left(\left|\frac{X_1 + \ldots + X_n}{n} - p\right| \geq \varepsilon\right) = 0$$
for every $\varepsilon > 0$.

6. For each of the following values of $\{X_n\}_{n=1}^{\infty}$ and X, decide whether $X_n \xrightarrow{p} X$ or not and whether $X_n \xrightarrow{a.s.} X$ or not.

   (a) $X \equiv 0$ and $\{X_n\}_{n=1}^{\infty}$ is a sequence of mutually independent random variables, such that

   $$X_n \sim \begin{cases} n, & 1/n^2 \\ 0, & 1 - 1/n^2 \end{cases}$$

   for every positive integer n.

   (b) $X \equiv 1$ and $\{X_n\}_{n=1}^{\infty}$ is a sequence of mutually independent random variables, such that $X_n \sim \text{Ber}\left(\frac{n}{n+1}\right)$ for every positive integer n.

   (c) $X, X_1, X_2, X_3, \ldots$ are mutually independent random variables, such that $X \sim \text{Ber}(1/2)$ and $X_n \sim \text{Ber}(1/2)$ for every positive integer n.