Lecture 12

1 Normal random variables

A continuous random variable X has the normal distribution (or is normally distributed) with parameters μ and σ^2, where $\sigma > 0$, if its probability density function is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

for every $x \in \mathbb{R}$. We will first prove that f_X is indeed a probability density function. First, using the substitution $y = (x-\mu)/\sigma$ which implies $\frac{dy}{dx} = \frac{1}{\sigma}$, we see that

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy.$$

Hence, it suffices to prove that $\int_{-\infty}^{\infty} e^{-y^2/2} dy = \sqrt{2\pi}$. Let $I = \int_{-\infty}^{\infty} e^{-y^2/2} dy$. Then

$$I^2 = \int_{-\infty}^{\infty} e^{-y^2/2} dy \int_{-\infty}^{\infty} e^{-x^2/2} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(y^2+x^2)/2} dy dx.$$

In order to calculate this double integral we change the variables x and y to polar coordinates, that is, we use the substitutions $x = r\cos\theta$ and $y = r\sin\theta$ which imply $dydx = rd\theta dr$. Hence

$$I^2 = \int_0^{\infty} \int_0^{2\pi} e^{-r^2/2} r d\theta dr = 2\pi \int_0^{\infty} r e^{-r^2/2} dr = -2\pi e^{-r^2/2}|_0^{\infty} = 2\pi.$$

We conclude that $I = \sqrt{2\pi}$ as claimed.

Claim 1.1. *Let X be normally distributed with parameters μ and σ^2. Let $Y = aX + b$ for some real numbers $a > 0$ and b. Then Y is normally distributed with parameters $a\mu + b$ and $a^2\sigma^2$.*

Proof. Let F_Y denote the cumulative distribution function of Y. Then, for every $x \in \mathbb{R}$, it holds that

$$F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}(aX + b \leq x) = \mathbb{P}\left(X \leq \frac{x-b}{a}\right) = F_X\left(\frac{x-b}{a}\right).$$

Differentiating yields

$$\begin{aligned} f_Y(x) &= \frac{1}{a} f_X\left(\frac{x-b}{a}\right) = \frac{1}{\sqrt{2\pi}a\sigma} e^{-((x-b)/a-\mu)^2/(2\sigma^2)} \\ &= \frac{1}{\sqrt{2\pi}a\sigma} e^{-(x-b-a\mu)^2/(2a^2\sigma^2)}. \end{aligned}$$

We conclude that Y is a normal random variable with parameters $a\mu + b$ and $a^2\sigma^2$. $\square$

The following result is an immediate but very useful consequence of Claim 1.1.

Corollary 1.2. *Let X be normally distributed with parameters μ and σ^2. Then $Y := (X - \mu)/\sigma$ is normally distributed with parameters 0 and 1.*

A random variable which is normally distributed with parameters 0 and 1 is said to have the *standard normal distribution.* Note that its probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

It is customary to denote the cumulative distribution function of a standard normal random variable by $\Phi(x)$, that is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

An important trait of the cumulative distribution function of a standard normal distribution is its symmetry with respect to zero. This is made precise by the following simple claim.

Claim 1.3. *For every $x \in \mathbb{R}$ it holds that $\Phi(-x) = 1 - \Phi(x)$.*

Proof. For every $x \in \mathbb{R}$ it holds that

$$\begin{aligned}
\Phi(x) + \Phi(-x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-t^2/2} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt + \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt = 1,
\end{aligned}$$

where the second equality holds by the substitution $y = -t$. $\square$

Now, if X is a normal random variable with parameters μ and σ^2, then $Y := (X - \mu)/\sigma$ is a standard normal random variable, and then

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = \mathbb{P}\left(Y \leq \frac{x - \mu}{\sigma}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right).$$

That is, it suffices to work with the probability density function of the standard normal distribution.

Example 1: Let X be a normal random variable with parameters μ and σ^2. We would like to determine a value a for which $\mathbb{P}(X \leq a) \approx 0.99$. It holds that

$$0.99 \approx \mathbb{P}(X \leq a) = \mathbb{P}\left(\frac{X - \mu}{\sigma} \leq \frac{a - \mu}{\sigma}\right) = \Phi\left(\frac{a - \mu}{\sigma}\right).$$

Looking at the table of the standard normal distribution, we see that $\Phi(2.33) \approx 0.9901$. Therefore, an approximate solution would be $(a - \mu)/\sigma = 2.33$ implying that $a = 2.33\sigma + \mu$.

Next, we will calculate the expectation and variance of a normal random variable X with parameters μ and σ^2. Let $Y = (X-\mu)/\sigma$. Then

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} x f_Y(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2}dx = -\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}\Big|_{-\infty}^{\infty} = 0.$$

Using the linearity of expectation we then have

$$\mathbb{E}(X) = \mathbb{E}(\sigma Y + \mu) = \sigma \mathbb{E}(Y) + \mu = \mu.$$

In order to calculate the variance of X we first calculate $\mathbb{E}(Y^2)$. Integrating by parts with $u = x$ and $v' = xe^{-x^2/2}$ yields

$$\begin{aligned}\mathbb{E}(Y^2) &= \int_{-\infty}^{\infty} x^2 f_Y(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}} \left[-xe^{-x^2/2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -e^{-x^2/2}dx \right] \\ &= 0 - 0 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}dx = 1.\end{aligned}$$

Therefore

$$Var(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 1 - 0 = 1.$$

We conclude that

$$Var(X) = Var(\sigma Y + \mu) = \sigma^2 Var(Y) = \sigma^2.$$

Example 2: Let X be a standard normal random variable and let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying

$$\lim_{x\to\infty} g(x) \cdot e^{-x^2/2} = \lim_{x\to -\infty} g(x) \cdot e^{-x^2/2} = 0. \tag{1}$$

(a) Prove that $\mathbb{E}(g'(X)) = \mathbb{E}(X \cdot g(X))$.

(b) Prove that $\mathbb{E}(X^{n+1}) = n\mathbb{E}(X^{n-1})$ for every positive integer n.

(c) Calculate $\mathbb{E}(X^4)$.

We solve each part separately.

(a) Integrating by parts with $u = g(x)$ and $v' = xe^{-x^2/2}$ yields

$$\begin{aligned}\mathbb{E}(X \cdot g(X)) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot g(x) \cdot e^{-x^2/2}dx \\ &= -\frac{1}{\sqrt{2\pi}} \cdot g(x) \cdot e^{-x^2/2}\Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -g'(x) \cdot e^{-x^2/2}dx \\ &= 0 - 0 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g'(x) \cdot e^{-x^2/2}dx \\ &= \mathbb{E}(g'(X)),\end{aligned}$$

where the penultimate equality holds by (1).

(b) Fix some positive integer n and let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^n$. It then follows by part (a) that

$$\mathbb{E}\left(X^{n+1}\right) = \mathbb{E}\left(X \cdot X^n\right) = \mathbb{E}\left(X \cdot g(X)\right) = \mathbb{E}\left(g'(X)\right) = \mathbb{E}\left(nX^{n-1}\right) = n\mathbb{E}\left(X^{n-1}\right).$$

(c) Using the fact that X is a standard normal random variable, it follows by part (b) that

$$\mathbb{E}\left(X^4\right) = 3 \cdot \mathbb{E}\left(X^2\right) = 3 \cdot 1 = 3.$$