Practical session 3

Exercise 1 Show that for all integers $1 \leq a \leq n$, there exists a two-colouring of the edges of K_n with at most

$$\binom{n}{a} 2^{1-\binom{a}{2}}$$

monochromatic copies of K_a.

Solution

Consider a random two-colouring of the edges of K_n, that is, colour each edge red with probability $1/2$ or blue with probability $1/2$, where the choice of colour of an edge is independent of the choice for all other edges. Let X be the number of monochromatic copies of K_a. For every complete subgraph $G \subseteq K_n$ with a vertices, let

$$X_G = \begin{cases} 1 \text{ if } G \text{ is monochromatic} \\ 0 \text{ otherwise} \end{cases}$$

For every complete subgraph $G \subseteq K_n$ with a vertices we have

$$\Pr\left(X_G = 1\right) = \frac{2}{2^{\binom{a}{2}}}.$$

By linearity of expectation we then have

$$\mathbb{E}\left(X\right) = \sum_G \mathbb{E}\left(X_G\right) = \binom{n}{a} 2^{1-\binom{a}{2}}.$$

Therefore there exists a 2-coloring with at most

$$\binom{n}{a} 2^{1-\binom{a}{2}}$$

monochromatic copies of K_a.

Exercise 2 Let $G = (V, E)$ be a graph with n vertices and e edges. Then G contains a bipartite subgraph with at least $e/2$ edges.

Solution

Pick a random set of vertices $T \subseteq V$, where every vertex $x \in V$ is chosen to be in T with probability

$1/2$, independently of all other vertices. Call an edge xy "crossing" if *exactly one* of x, y is in T. Let X be the number of crossing edges. We write

$$X = \sum_{xy \in E} X_{xy},$$

where X_{xy} is the indicator random variable for the event that xy is a crossing edge. Then, for every edge $xy \in E$ it holds that

$$\begin{aligned}\mathbb{E}\left(X_{xy}\right) &= \Pr\left((x \in T \wedge y \notin T) \vee (x \notin T \wedge y \in T)\right) \\ &= \Pr\left(x \in T\right) \cdot \Pr\left(y \notin T\right) + \Pr\left(x \notin T\right) \cdot \Pr\left(y \in T\right) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2},\end{aligned}$$

where the second equality is due to the fact that the two events are disjoint, and every vertex is chosen to be in T independently of the other vertices. By linearity of expectation we obtain that $\mathbb{E}\left(X\right) = e/2$. Hence, there exists a set $T \subseteq V$ for which $X \geq e/2$, and the set of the corresponding crossing edges form a bipartite graph with parts T and $V \setminus T$.

Exercise 3 Let $G = (V, E)$ be a graph with n vertices and $nd/2$ edges, for some $d \geq 1$. Then $\alpha\left(G\right) \geq n/2d$, where $\alpha\left(G\right)$ is the size of a largest independent set in G.

Solution

Pick a random set of vertices $S \subseteq V$, where every vertex $x \in V$ is chosen to be in S with probability p, for some p to be determined later, independently of all other vertices. Let $X = |S|$ and let Y be the number of edges in $G[S]$ – the induced subgraph of G with vertex set S. For every edge $ij \in E$, let Y_{ij} be the indicator random variable for the event "$i, j \in S$"; note that we then have $Y = \sum_{ij \in E} Y_{ij}$. Since every vertex of V was chosen to be in S randomly with probability p and independently, it follows that for every edge $ij \in E$ we have

$$\mathbb{E}\left(Y_{ij}\right) = \Pr\left(Y_{ij} = 1\right) = \Pr\left(i, j \in S\right) = p^2.$$

By the linearity of expectation we then have

$$\mathbb{E}\left(Y\right) = \sum_{ij \in E} \mathbb{E}\left(Y_{ij}\right) = \frac{nd}{2} \cdot p^2.$$

Observe that $X \sim \mathrm{Bin}\left(n, p\right)$ and, in particular, $\mathbb{E}\left(X\right) = np$. It thus follows by the linearity of expectation that

$$\mathbb{E}\left(X - Y\right) = np - \frac{nd}{2} \cdot p^2. \tag{1}$$

A straightforward calculation shows that the right-hand side of (1) is maximized when $p = 1/d$. Substituting $p = 1/d$ then implies that

$$\mathbb{E}\left(X - Y\right) = \frac{n}{2d}.$$

Therefore, there exists a set $S \subseteq V$ for which the number of vertices in S minus the number of edges in $G[S]$ is at least $n/2d$. Deleting one endpoint (chosen arbitrarily) from each edge of $G[S]$ results in an independent set S^* with at least $n/2d$ vertices.