Part4- Voting

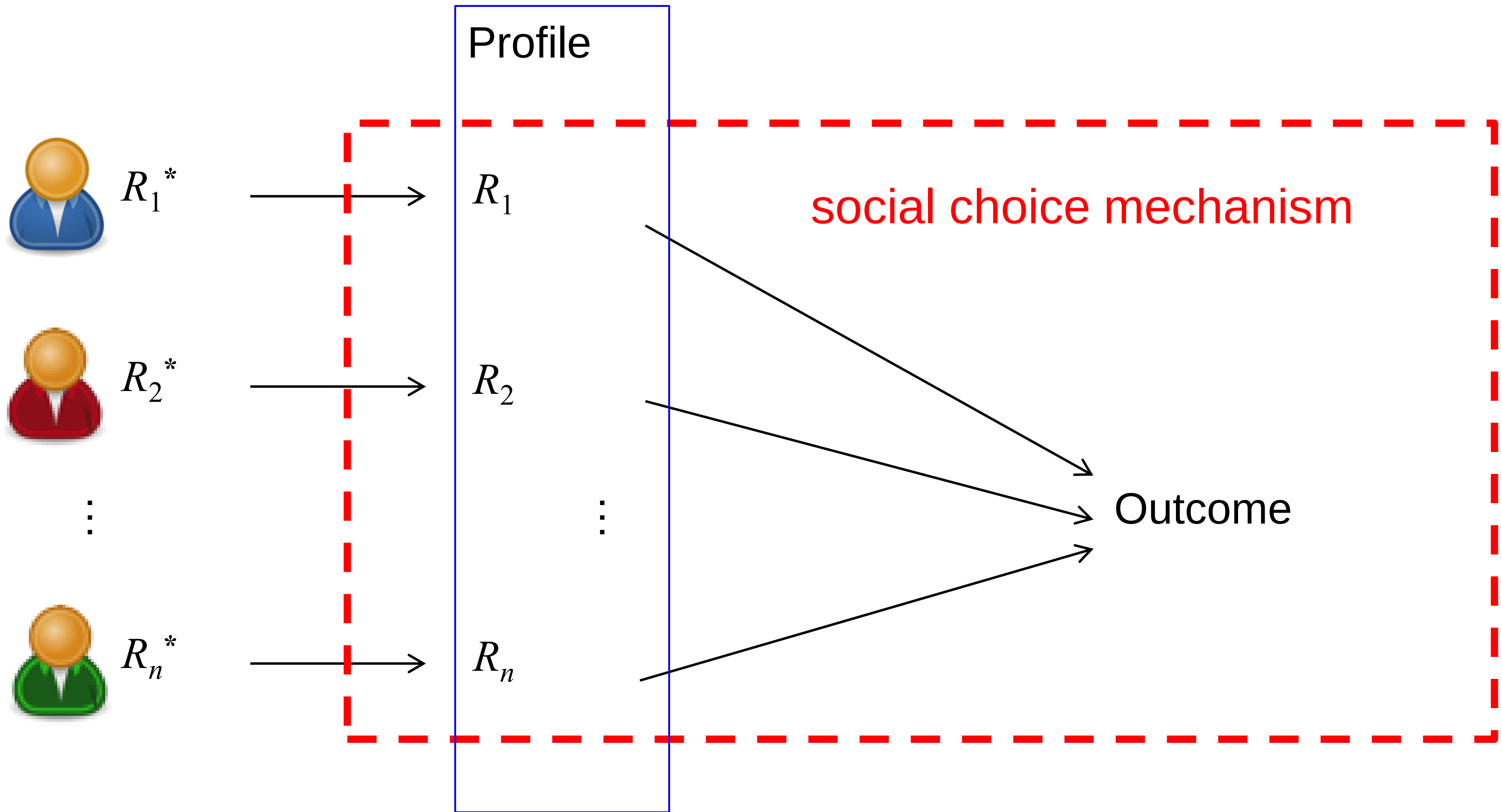
Based on slides by Lirong Xia, Ester David and Avinatan Hassidim

Social choice

"social choice is a theoretical framework for analysis of combining individual preferences, interests, or welfares to reach a collective decision or social welfare in some sense."

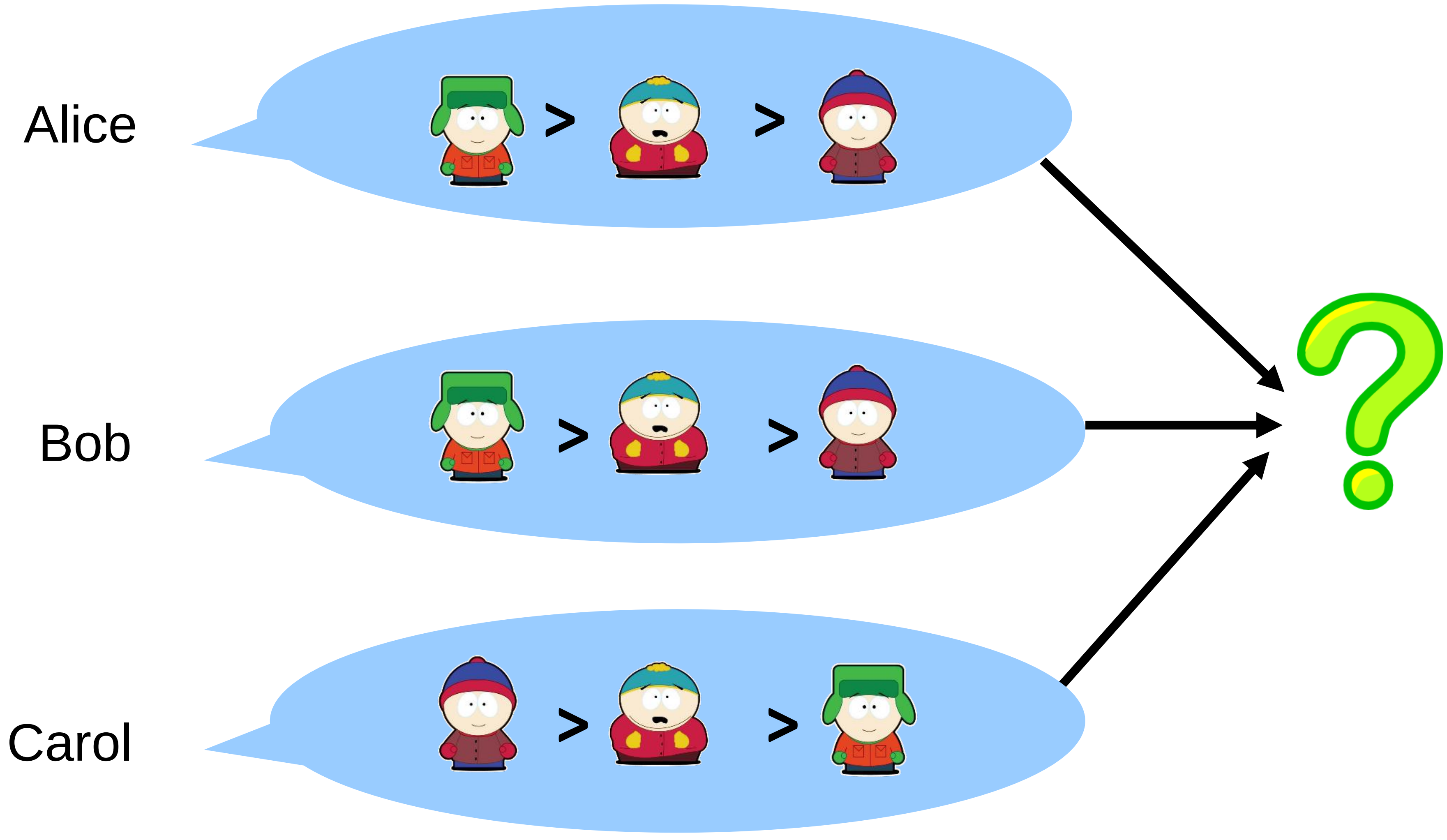
---Wikipedia Aug 26, 2013

Social choice problems

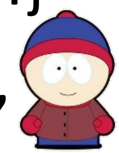


- Agents
- Alternatives
- Outcomes
- Preferences (true and reported)
- Social choice mechanism

Example: Political elections



Why is this social choice?

- Agents: {Alice, Bob, Carol}
- Alternatives: { , ,  }
- Outcomes: winners (alternatives)
- Preferences (vote): rankings over alternatives
- Mechanisms: voting rules

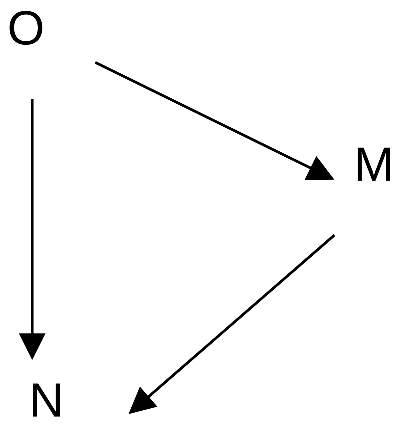
- Can vote over just about anything
  - political representatives, award nominees, where to go for dinner tonight, joint plans, allocations of tasks/resources, …
  - Also can consider other applications: e.g., aggregating search engines’ rankings into a single ranking

More formally

- Agents: n voters, N={1,…,n}
- Alternatives: m candidates,
  - A={a_1,…,a_m} or {a, b, c, d,…}
- Outcomes:
  - winners (alternatives): O=A. Social choice function
  - rankings over alternatives: O=Rankings(A). Social welfare function
- Preferences: R_j* and R_j are full rankings over A
  - Extensions include indifference and incompleteness
- Voting rule: a function that maps each profile to an outcome

Recall: binary relation

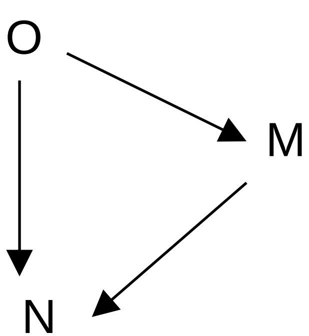
- Given a set of alternatives A
- A binary relation R is a subset of A×A
  - (a,b) ∈R means "a is preferred to b"
  - Also write $a >_R b$
- Example
  - A = {O, M, N}
  - R = {(O,M), (O,N), (M,N)}
- Graphical representation
  - Vertices are A
  - There is an edge a→b if and only if (a,b) ∈R



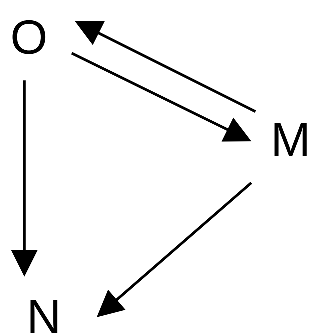
Linear orders - full rankings

Linear orders (rankings without ties): binary relations that satisfies

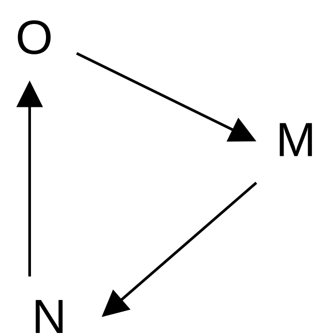
- Antisymmetry (no ties): $a >_R b$ and $b >_R a$ implies $a=b$
- Transitivity: $a >_R b$ and $b >_R c$ implies $a >_R c$
- Totality: for all a,b, one of $a >_R b$ or $b >_R a$ must hold



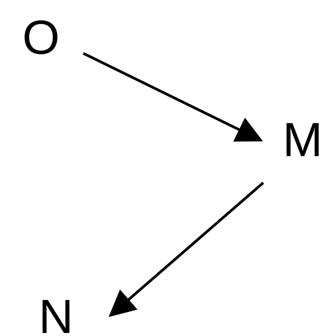
Yes



no,
Antisymmetry



no,
transitivity



no,
totality

Voting rules

- Majority rule: if a candidate is ranked first by most votes, that candidate should win
  - But what if there is no such candidate?

- Plurality: candidate with most votes wins
  - Otherwise known as “first past the post”

- Some (informal) criticisms
  - Ignores preferences other than favorite
  - Encourages voters to vote tactically
    - “*My candidate cannot win so I’ll vote for my second favorite*”

Is the winner indeed a “good” one?

$P = \{$ **a > b > c** ×4, **b > c > a** ×3

b > a > c ×2, **c > a > b** ×2 $\}$

Plurality(P)=b

But the majority of the voters prefer a to b (6 out of 11).

They also prefer a to c.

One possible generalization of Plurality

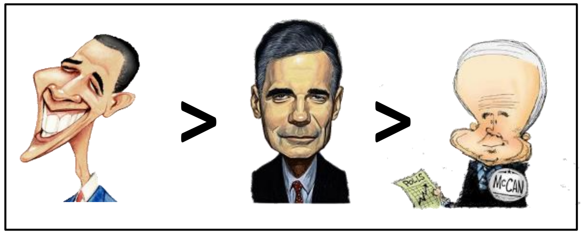
Jean Charles de Borda, 1733-1799

Borda: given m candidates

- ith ranked candidate score m-i
- Candidate with greatest sum of scores wins

Borda - example

$P=\{$



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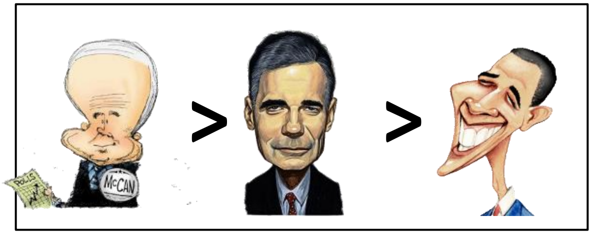
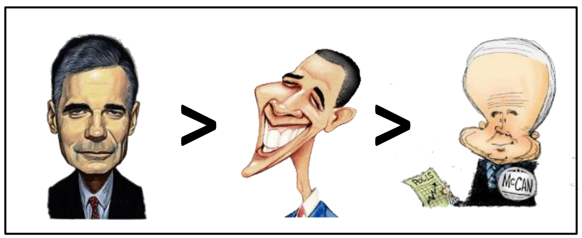
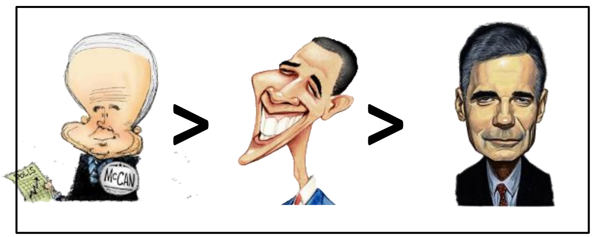


×3



×2,



×2 $\}$

Borda scores

: $4 \times 2 + 4 \times 1 = 12$

: $2 \times 2 + 7 \times 1 = 11$

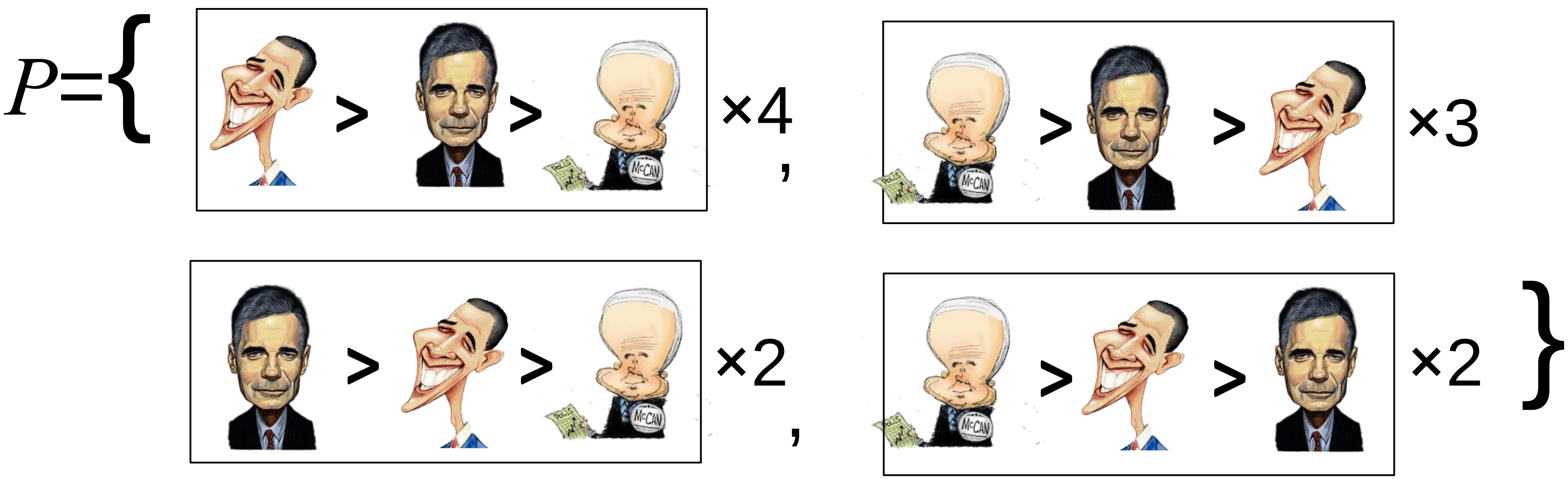
: $5 \times 2 = 10$

Borda(P)=

Positional scoring rules

- Characterized by a score vector $s_1,\ldots,s_m$ in non-increasing order
- For each vote R, the alternative ranked in the i-th position gets s_i points
- The alternative with the most total points is the winner
- Special cases
  - Borda: score vector (m-1, m-2, …,0)
  - k-approval: score vector $(\underbrace{1\ldots1}_{k}, 0\ldots0)$
  - Plurality: score vector (1, 0…0)
  - Veto: score vector (1...1, 0)

Example



Borda

Plurality
(1- approval)

Veto
(2-approval)

Is the winner indeed a "good" one?

$$P = \left\{ \boxed{a > b > c} \times 4,\ \boxed{b > c > a} \times 3\ \boxed{b > a > c} \times 2,\ \boxed{c > a > b} \times 2 \right\}$$

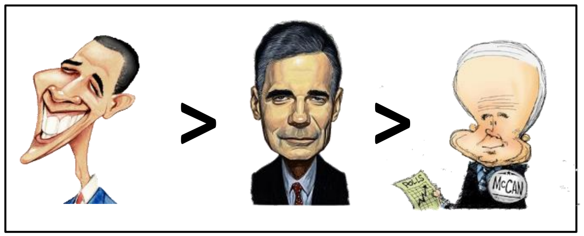
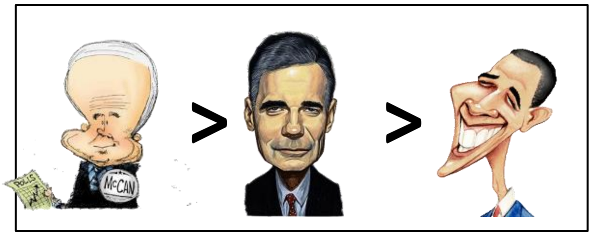
Borda(P)=b , Veto(P)=b

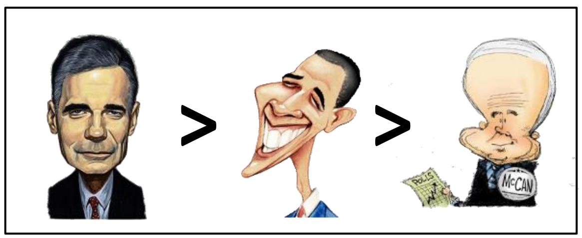
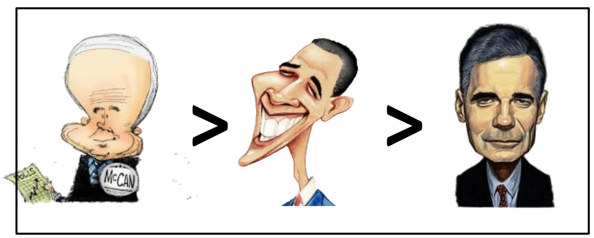
But the majority of the voters prefer a to b (6 out of 11).
They also prefer a to c.

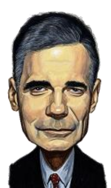
Another possible generalization of Plurality

- Plurality with runoff: the election has two rounds
  - First round, all alternatives except the two with the highest plurality scores drop out
  - Second round, the alternative preferred by more voters wins
- [used in France, Iran, North Carolina State]

Example: Plurality with runoff

$P=\{$  ×4,  ×3

 ×2,  ×2 $\}$

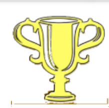
- First round:  drops out
- Second round:  defeats 

Different from Plurality!

Single transferable vote (STV)

- Also called instant run-off voting or alternative vote
- The election has m-1 rounds, in each round,
  - The alternative with the lowest plurality score drops out, and is removed from all votes
  - The last-remaining alternative is the winner
- [used in Australia and Ireland]

$a > b > c > d$	$d > a > b > c$	$c > d > a > b$	$b > c > d > a$
10	7	6	3

 a

Should we consider pairwise elections directly?

Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet (1743 – 1794)

- We saw several voting rules that are trying to generalize the concept of "majority" in different ways.
- Condorcet proposed another way, which relies on **pairwise elections.**
- a beats b in pairwise elections, if more voters prefer a over b.

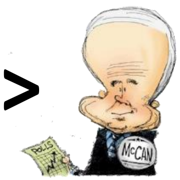
Should we consider pairwise elections directly?

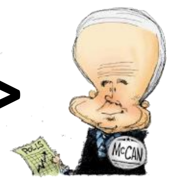
- We can define several voting rules based on the idea of pairwise elections. For example, the Copeland protocol.

- The Copeland score of an alternative is its total "pairwise wins" minus its "pairwise loses".

- The winner is the alternative with the highest Copeland score.

Example: Copeland

P={  >  >  ×4,  >  >  ×3

 >  >  ×2,  >  >  ×2 }

Copeland score:

  : 2

 : 0

 : -2

Which is best?

- So many voting rules to choose from ..
- How do we choose a rule from all of these rules?
- How do we know that there does not exist another, “perfect” rule?
- Let us look at some criteria that we would like our voting rule to satisfy
- The axiomatic approach (again…)

Fairness axioms

- Anonymity: names of the voters do not matter
  - Fairness for the voters
- Non-dictatorship: there is no dictator, whose top-ranked alternative is always the winner, no matter what the other votes are
  - Fairness for the voters
- Neutrality: names of the alternatives do not matter
  - Fairness for the alternatives

Monotonicity criteria

- Informally, monotonicity means that “ranking a candidate higher should help that candidate,” but there are multiple nonequivalent definitions

- A weak monotonicity requirement: if
  - candidate w wins for the current votes,
  - we then improve the position of w in some of the votes and leave everything else the same,

  then w should still win.

Weak monotonicity

- Does STV satisfy the weak monotonicity criterion?
  - 7 votes b > c > a
  - 7 votes a > b > c
  - 6 votes c > a > b
- c drops out first, its votes transfer to a, a wins
- But if 2 votes b > c > a change to a > b > c, b drops out first, its 5 votes transfer to c, and c wins.
- What about plurality with runoff?
- What about Copeland?

Strong monotonicity

- A strong monotonicity requirement: if
  - candidate w wins for the current votes,
  - we then change the votes in such a way that for each vote, if a candidate c was ranked below w originally, c is still ranked below w in the new vote
  - then w should still win.
- Note the other candidates can jump around in the vote, as long as they don't jump ahead of w

May's theorem (1952)

- Thm: With 2 candidates, a voting rule is anonymous, neutral and monotonic iff it is the plurality rule
  - Since these properties are uncontroversial, this about decides what to do with 2 candidates!
  - Proof: Plurality rule is clearly anonymous, neutral and monotonic
  - Other direction is more interesting
  - For simplicity, assume an odd number of voters

May’s theorem (1952)

- Thm: With 2 candidates, a voting rule is anonymous, neutral and monotonic iff it is the plurality rule

  – Proof: Anonymous and neutral implies only number of votes matters

  – Two cases:
    - N(A>B) = N(B>A)+1 and A wins.
      – By monotonicity, A wins whenever N(A>B) > N(B>A)
    - N(A>B) = N(B>A)+1 and B wins
      – Swap one vote A>B to B>A. By monotonicity, B still wins. But now N(B>A) = N(A>B)+1. By neutrality, A wins. This is a contradiction.

Weak Pareto efficiency criterion

- If all agents prefer *a* to *b*, the voting rule will never choose *b* to be the winner.
- Note: the voting rule does not have to choose *a*.
- However, if all votes rank *a* first, then *a* should win.
- This criterion is also called unanimity.
- Does Plurality satisfy weak Pareto efficiency?
- Does Copeland satisfy weak Pareto efficiency?

Condorcet criterion

- A candidate is a Condorcet winner if a beats any other candidate in a pairwise election.
- A voting rule is Condorcet consistent if the Condorcet winner is always selected.
- We already saw that Plurality and Borda are not Condorcet consistent

Condorcet paradox

$$P = \left\{ \boxed{a > b > c} \times 4, \quad \boxed{b > c > a} \times 3 \right.$$

$$\left. \boxed{b > a > c} \times 2, \quad \boxed{c > a > b} \times 2 \right\}$$

Plurality(P), Borda(P)=b

- Candidate a is the Condorcet winner.

Condorcet paradox

$$P = \left\{ \boxed{a > b > c} \times 4,\ \boxed{b > c > a} \times 3,\ \boxed{b > a > c} \times 2,\ \boxed{c > a > b} \times 4 \right\}$$

Plurality(P), Borda(P)=b

- Candidate a is the Condorcet winner.
- What is we add two voters that prefer c > a > b?
- Majority prefer a to b, and prefer b to c, and prefer c to a!

Condorcet criterion

- A voting rule is Condorcet consistent if the Condorcet winner is always selected, **when there is one**.

- If there is a Condorcet winner, then it is unique.

Condorcet criterion

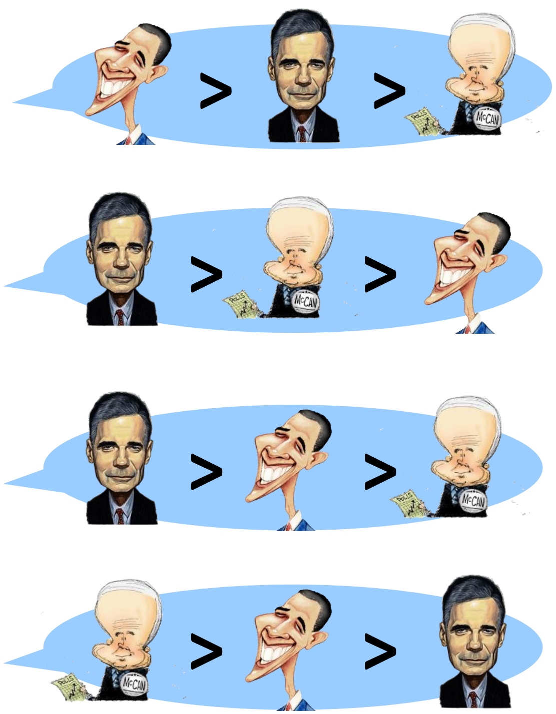
- Theorem (Fishburn-1974). No positional scoring rule with strict ordering of weights satisfies Condorcet criterion:
  - suppose $s_1 > s_2 > s_3$

3 Voters

2 Voters

1 Voter

1 Voter



 is the Condorcet winner

 $: 3s_1 + 2s_2 + 2s_3$

$\wedge$

 $: 3s_1 + 3s_2 + 1s_3$

CONTRADICTION

Majority criterion

- If a candidate is ranked first by most votes, that candidate should win.
  - Relationship to Condorcet criterion?

- Some rules do not even satisfy this
- E.g. Borda:
  - a > b > c > d > e
  - a > b > c > d > e
  - c > b > d > e > a
- a is the majority winner, but it does not win under Borda

Muller-Satterthwaite impossibility theorem [1977]

- Is Copeland the best voting rule?
- Theorem: Suppose there are at least 3 candidates. Then there exists no rule that simultaneously:
  - satisfies weak Pareto efficiency,
  - is non-dictatorial, and
  - is monotone (in the strong sense).

Social welfare function

- Let's look on our voting rules as social welfare functions.
- How to generalize our previous criteria:
  - Anonymity and neutrality: the same.
  - Non-dictatorship: there does not exist a voter such that the rule simply always copies that voter's ranking.
  - Weak Pareto efficiency → Pareto efficiency: if all votes rank *a* above *b*, then the rule should ranks *a* above *b*.

Independence of irrelevant alternatives

- Result between *a* and *b* only depends on the agents preferences between *a* and *b*.

- Formally, for two profiles $D_1 = (R_1, \ldots, R_n)$ and $D_2 = (R_1', \ldots, R_n')$ and any pair of alternatives a and b,
  - if for all voter j, the pairwise comparison between a and b in R_j is the same as that in R_j'
  - then the pairwise comparison between a and b are the same in $f(D_1)$ as in $f(D_2)$
  - even if voters' preferences between other pairs like *a* and *x*, *b* and *y*, or *x* and *y* change

Arrow’s impossibility theorem [1951]

- Theorem: Suppose there are at least 3 candidates. Then there exists no rule that simultaneously:
  - satisfies Pareto efficiency,
  - is non-dictatorial, and
  - satisfies independent of irrelevant alternatives.
- The idea: we have to break Condorcet cycles, but how we do this, inevitably leads to trouble
- A genius observation
  - Led to the Nobel prize in economics

Strategic behavior (of the agents)

- Manipulation: an agent (manipulator) casts a vote that does not represent her true preferences, to make herself better off

- A voting rule is strategy-proof if there is never a (beneficial) manipulation under this rule

- Do you think Plurality is strategy-proof?

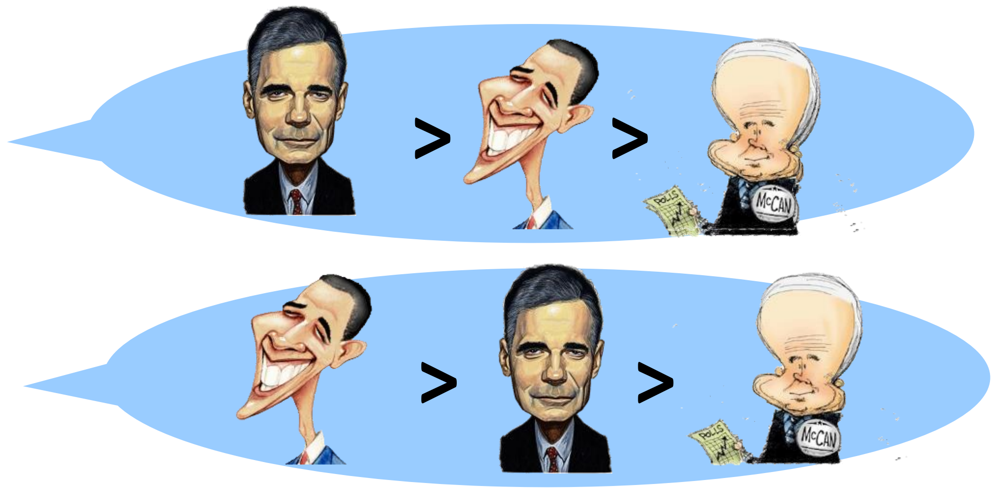
Manipulation under Plurality

- Suppose a voter prefers a > b > c
- Also suppose she knows that the other votes are
  - 2 times b > c > a
  - 2 times c > a > b
- Voting truthfully will lead to a tie between b and c
- She would be better off voting e.g. b > a > c, guaranteeing b wins

Manipulation under Borda

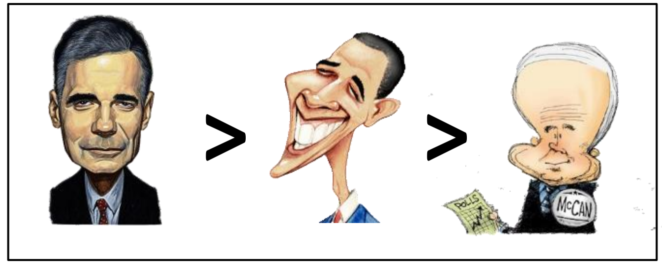
Alice

Bob



What if we change the tie-breaking mechanism?

Manipulation under STV



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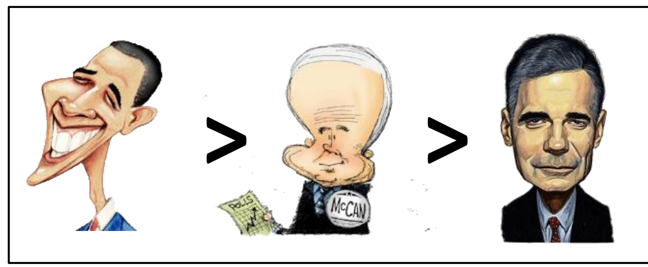

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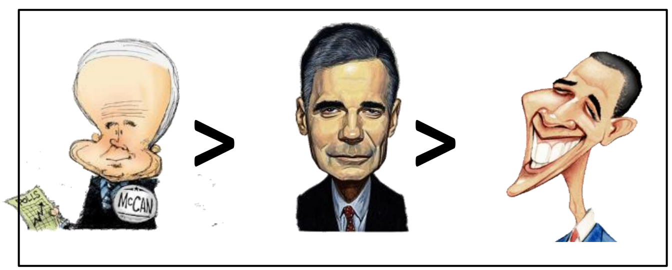

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N>O>M → O>N>M

Gibbard-Satterthwaite impossibility theorem

- Suppose there are at least 3 candidates
- There exists no rule that is simultaneously:
  - onto (for every candidate, there are some votes that would make that candidate win),
  - nondictatorial, and
  - nonmanipulable
- This is a powerful negative result

Computational perspective

- We first need to verify that the voting rules are not too complicated so that nobody can easily compute the winner.
- The winner determination problem:
  - Given: a voting rule f
  - Input: a preference profile P and an alternative c
  - input size: $nm\log m$
  - Output: is c the winner of f under P?

- We want a voting rule where the winner determination is in P.

Computational perspective

- The winner determination problem for all of the voting rules that we saw is in P. 

- There are some interesting and important voting rules where the winner determination problem is NP-hard. 
  - Dodgson rule
  - Kemeny rule

Dodgson voting rule

- We saw that there is not always a Condorcet winner.
- A Dodgson winner is a candidate who is "closest" to being a unique Condorcet winner.
- That is, the Dodgson score of a candidate, *a*, is the smallest number of sequential exchanges of adjacent candidates in preference orders such that after those exchanges *a* is a Condorcet winner.

Example: Dodgson voting rule

- 2 voters that vote for: $a > b > c$

- The Dodgson scores of a,b,c are 0,2,4 respectively.

- Now suppose we have 2 voters with
  - $a > b > c$
  - $b > a > c$

- The Dodgson scores of a,b,c are 1,1,4 respectively.

Kemeny voting rule

- The idea: create an overall ranking of the candidates that has as few *disagreements* as possible.
- Kendall tau distance
  - $K(R1, R2)$= # {different pairwise disagreements}
  - Also called bubble-sort distance, since it is also the number of swaps that the bubble sort algorithm would make to place one list in the same order as the other list

$$K(\, b \succ c \succ a \,,\, a \succ b \succ c \,) = \, ?$$

Kemeny voting rule

- Kemeny(P)=argmin$_W$ $\mathrm{K}(P,W)$=argmin$_W$ $\Sigma_{R\in \mathrm{P}}\mathrm{K}(R,W)$
- For single winner, choose the top-ranked alternative in Kemeny(P)
- E.G.,

$a \succ b \succ c \succ d$	$b \succ a \succ c \succ d$	$d \succ a \succ b \succ c$	$c \succ d \succ b \succ a$
1	1	2	2

- $\mathrm{K}(P, a \succ b \succ c \succ d)$=0+1+2*3+2*5=17

Computational perspective

- Gibbard-Satterthwaite only tells us that for some instances, successful manipulations exist.
- It does not say that these manipulations are always easy to find.
- If it is computationally too hard for a manipulator to compute a manipulation, she is best off voting truthfully
  - Similar as in cryptography

- For which common voting rules manipulation is computationally hard?

A formal computational problem

- The simplest version of the manipulation problem:
- CONSTRUCTIVE-MANIPULATION:
  - We are given a voting rule r, the (unweighted) votes of the other voters, and an alternative p.
  - We are asked if we can cast our (single) vote to make p win.
- E.g., for the Borda rule:
  - Voter 1 votes A > B > C
  - Voter 2 votes B > A > C
  - Voter 3 votes C > A > B
- Borda scores are now: A: 4, B: 3, C: 2
- Can we make B win?
- Answer: YES. Vote B > C > A (Borda scores: A: 4, B: 5, C: 3)

A formal computational problem

- We can also extend the single manipulation problem to a coalitional manipulation problem:
- CONSTRUCTIVE-COALITION-MANIPULATION:
  - We are given a voting rule r, the (unweighted) votes of the other voters, and an alternative p.
  - We are asked if we can cast k votes to make p win.

- Can be extended to a weighted version, and to a destructive version.

Results

- Plurality, Veto, Pluarlity with runoff: manipulation is easy.
- Copeland, Borda: easy for a single manipulator [BTT SCW-89], NP-C if there at least 2 manipulators [FHS AAMAS-08,10], [DKN+ AAAI-11] [BNW IJCAI-11].
- STV: NP-C even with a single manipulator! [BO SCW-91]

- But wait, is computational complexity a strong barrier?
  - NP-hardness is a worst-case concept

Manipulating Borda

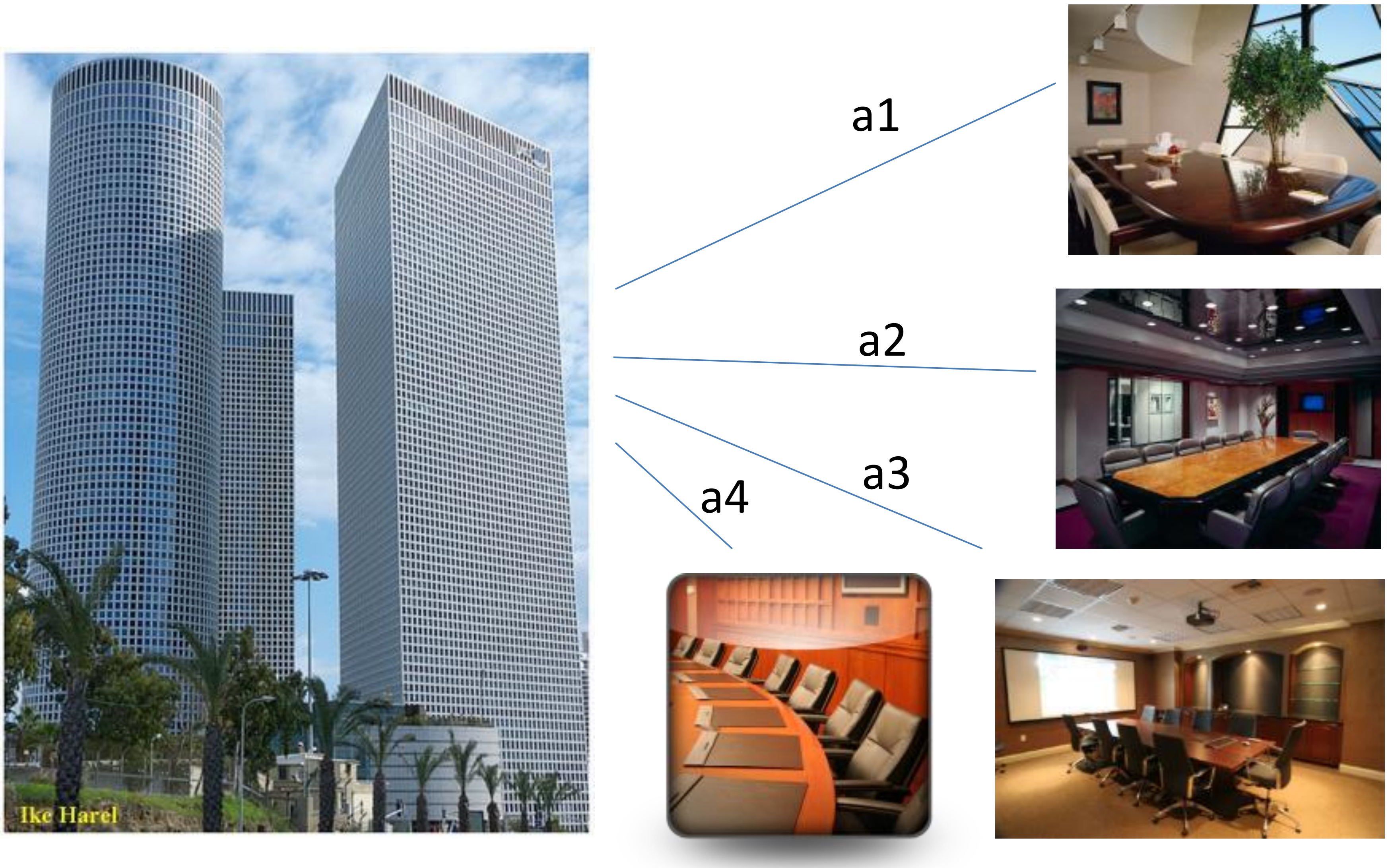
- Suppose you want to add k manipulators for Borda who would promote p
- Clearly, p should be ranked first in all of them
- If we could rank no one else, we would be happy
  - But we have to give points to other candidates
- So the goal is to give points to bad candidates who won't win

Greedy algorithm for UCM for Borda

- We want to promote p
- Each manipulator puts p on top
- Problem – we need to give points to other candidates..
  - Set the manipulators one by one
  - At each manipulator be greedy
- Thm [ZPR AIJ-09]: If exists a manipulation with k-1 manipulators this will succeed with k manipulators

Application: Reducing Energy Consumption

How to Choose a Meeting Room?



The Voting Process

a3 > a4 > a1 > a2

a4 > a2 > a1 > a3

a2 > a3 > a1 > a4

a1 > a2 > a3 > a4

The Voting Process

a3 > a4 > a1 > a2

a4 > a2 > a1 > a3

a2 > a3 > a1 > a4

a1 > a2 > a3 > a4

a1 > a2

The Voting Process

a3 > a4 > a1 > a2

a4 > a2 > a1 > a3

a2 > a3 > a1 > a4

a1 > a2 > a3 > a4

a3 > a1 > a4 > a2

?

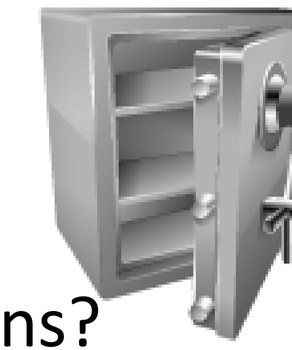
a1 > a2

The Persuasion Problems

- Given:
  - A set of alternatives
  - A set of voters with their preferences
  - A preferences list of the sender
- Is there a “good” set of suggestions?
- K-Persuasion: send at most k suggestions

Add Safety Requirement

- What if not all the voters accept the suggestions?



- Safe-Persuasion
  - Is there a "good" and safe set of suggestions?
- K-Safe-persuasion: send at most k suggestions

Persuasion ≠ Manipulation

- In coalitional manipulation
  - The manipulators always obey their suggestions
  - There is no requirement that they will benefit from it
  - How the manipulators attain full knowledge?
- In persuasion
  - Voters can accept or decline the sender's suggestions
  - Send suggestion only to voters that will benefit from it, and we add safety requirement
  - The sender is the election organizer

Complexity Results [HLK IJCAI-13]

	Persuasion	K-Persuasion	Safe-Persuasion	K-Safe-Persuasion
Plurality	P	P	P	P
Veto	P	P	P	P
K-Approval	P	NP-complete	NP-hard	NP-hard
Bucklin	P	NP-complete	NP-hard	NP-hard
Borda	NP-complete	NP-complete	NP-hard	NP-hard