Probability Theory 2 – Solutions II

1. Let m, n and t be positive integers. Prove that there exists a red/blue colouring of the edges of $K_{m,n}$ with at most $\binom{m}{t}\binom{n}{t}2^{1-t^2}$ monochromatic copies of $K_{t,t}$.

   Remark: $K_{m,n}$ is a complete bipartite graph. That is, its vertex set is $A \cup B$ such that $|A| = m$, $|B| = n$ and $A \cap B = \emptyset$, and its edge set is $\{xy : x \in A, y \in B\}$.

   Solution: Colour the edges of $K_{m,n}$ randomly as follows. For every edge of $K_{m,n}$ flip a fair coin, all coin flips being mutually independent. If the result of the coin flip for some edge is heads, colour this edge red, otherwise, colour it blue. There are $N := \binom{m}{t}\binom{n}{t}$ copies of $K_{t,t}$ in $K_{m,n}$ as any choice of t vertices from each of the two parts of the graph constitutes one such copy. Order them linearly in an arbitrary fashion. For every $1 \leq i \leq N$, let $X_i = 1$ if the ith copy is monochromatic (all red or all blue). Observe that $\mathbb{E}(X_i) = 2 \cdot 2^{-t^2} = 2^{1-t^2}$ (the first 2 is for choosing the colour and the term 2^{-t^2} is due to the fact that the coins are fair and all coin flips are independent). Let $X = \sum_{i=1}^{N} X_i$ and observe that X counts the number of monochromatic copies of $K_{t,t}$ in $K_{m,n}$. By linearity of expectation we have

   $$\mathbb{E}(X) = \sum_{i=1}^{N} \mathbb{E}(X_i) = \binom{m}{t}\binom{n}{t}2^{1-t^2}.$$

   We conclude that there exists a red/blue colouring of the edges of $K_{m,n}$ with at most $\binom{m}{t}\binom{n}{t}2^{1-t^2}$ monochromatic copies of $K_{t,t}$, as claimed.

2. Let $n \geq k \geq 2$ and $m < 2^{k-1}$ be positive integers. Let $\{A_1, \ldots, A_m\}$ be a family of subsets of $\{1, \ldots, n\}$, where $|A_i| = k$ for every $1 \leq i \leq m$. Prove that there exists a colouring of the elements of $\{1, \ldots, n\}$ with 2 colours such that, for every $1 \leq i \leq m$, the set A_i contains elements of both colours.

   Solution: A red/blue colouring of the elements of $\{1, \ldots, n\}$ is called *good* if A_i contains at least one red element and at least one blue element for every $1 \leq i \leq m$; otherwise it is called *bad*. Colour the elements of $\{1, \ldots, n\}$ randomly. That is, for every $1 \leq j \leq n$, flip a fair coin, where all coin flips are mutually independent. If the result of the coin flip for $1 \leq j \leq n$ is heads, then colour j red; otherwise, colour it blue. Denote the resulting colouring by c; our goal is to prove that the probability that c is good is strictly positive. For every $1 \leq i \leq m$, let M_i denote the event "A_i is monochromatic" (i.e., $c(j) =$ red for every $j \in A_i$ or $c(j) =$ blue for every $j \in A_i$). Then $\mathbb{P}(M_i) = 2 \cdot 2^{-k}$ for every $1 \leq i \leq m$. Hence

   $$\mathbb{P}(c \text{ is bad}) = \mathbb{P}\left(\bigcup_{i=1}^{m} M_i\right) \leq \sum_{i=1}^{m} \mathbb{P}(M_i) = m \cdot 2^{1-k} < 2^{k-1} \cdot 2^{1-k} = 1\,,$$

   where the first inequality follows by a union bound argument and the last inequality follows from our assumption that $m < 2^{k-1}$.

3. Let $0 \le p \le 1$ be a real number and let k, ℓ and n be positive integers. Prove that if

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell} (1-p)^{\binom{\ell}{2}} < 1,$$

then $R(k, \ell) > n$.

Remark: $R(k, \ell)$ is the "off-diagonal" Ramsey number, that is, it is the smallest integer n such that any *red/blue*-colouring of the edges of K_n yields a red K_k (i.e., a complete graph on k vertices such that all of its edges are coloured red) or a blue K_ℓ.

Solution: Colour the edges of K_n randomly as follows. Colour an edge red with probability p and blue with probability $1-p$. The colour of any edge is independent of the colour of any other edge. Given any copy of K_k in K_n, the probability that all of its edges are coloured red is $p^{\binom{k}{2}}$. Similarly, given any copy of K_ℓ in K_n, the probability that all of its edges are coloured blue is $(1-p)^{\binom{\ell}{2}}$. Since there are $\binom{n}{k}$ copies of K_k in K_n, we have

$$\mathbb{P}(\text{there exists a red copy of } K_k) \le \binom{n}{k} p^{\binom{k}{2}}.$$

Similarly, we have

$$\mathbb{P}(\text{there exists a blue copy of } K_\ell) \le \binom{n}{\ell} (1-p)^{\binom{\ell}{2}}.$$

Therefore

$$\mathbb{P}(\text{there exists a red copy of } K_k \text{ or a blue copy of } K_\ell) \le \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell} (1-p)^{\binom{\ell}{2}} < 1,$$

where the last inequality holds by assumption. We conclude that there exists a red/blue colouring of the edges of K_n with no red copy of K_k and no blue copy of K_ℓ, that is, $R(k, \ell) > n$ as claimed.

4. Prove that every graph G contains an independent set of size at least

$$\sum_{v \in V(G)} \frac{1}{\deg_G(v) + 1}.$$

Hint: Consider the vertices one by one according to a random permutation $\pi \in S_{|V(G)|}$ and build an independent set in the "obvious way".

Solution: Let $n := |V(G)|$ and let a permutation $\pi \in S_n$ be chosen uniformly at random. Let $u_1, u_2, \ldots, u_n$ be the corresponding ordering of $V(G)$, that is, $\pi(u_i) = i$ for every $1 \le i \le n$. Construct an independent set $I \subseteq V(G)$ in G greedily as follows. Initially, $I = \emptyset$ and $S = V(G)$. As long as $S \neq \emptyset$, let $1 \le i \le n$ be the smallest index for which $u_i \in S$, and let $I \leftarrow I \cup \{u_i\}$ and $S \leftarrow S \setminus (\{u_i\} \cup \{v \in V(G) : u_i v \in E(G)\})$. It is evident that I is indeed an independent set.

For every $1 \leq i \leq n$, let X_i be the indicator random variable for the event "$u_i \in I$". Clearly $|I| = \sum_{i=1}^{n} X_i$. It follows by the construction of I that if $i < j$ for every $1 \leq j \leq n$ such that $u_i u_j \in E(G)$, then $u_i \in I$. Hence, for every $1 \leq i \leq n$ it holds that

$$\mathbb{E}(X_i) = \mathbb{P}(X_i = 1) = \mathbb{P}(u_i \in I) \geq \frac{1}{\deg_G(u_i) + 1}\,.$$

It then follows by the linearity of expectation that

$$\mathbb{E}(|I|) = \sum_{i=1}^{n} \mathbb{E}(X_i) \geq \sum_{i=1}^{n} \frac{1}{\deg_G(u_i) + 1}\,.$$

We conclude that there exists an independent set I of G of size $|I| \geq \sum_{i=1}^{n} \frac{1}{\deg_G(u_i)+1}$ as claimed.

5. Let $G = (V, E)$ be a bipartite graph on n vertices. For every $v \in V$ let $L(v) \subseteq \mathbb{N}$ be a set whose size is strictly larger than $\log_2 n$. Prove that there exists a function (usually called a colouring) $c : V \to \mathbb{N}$ which satisfies both of the following properties:

   (i) $c(v) \in L(v)$ for every $v \in V$.

   (ii) $c(u) \neq c(v)$ for every two vertices $u, v \in V$ for which $uv \in E$.

   Solution: Let A and B denote the two parts of G. Let $L = \bigcup_{v \in V} L(v)$. For every $x \in L$ flip a fair coin, all coin flips being mutually independent. If the result of the coin flip for some $x \in L$ is heads, then delete x from $L(v)$ for every $v \in A$, and if it is tails, then delete x from $L(v)$ for every $v \in B$. Once this is done for every $x \in L$, denote the resulting set of v by $L'(v)$ for every $v \in V$. Now colour every $v \in V$ using an arbitrary colour from $L'(v)$. It is evident that any such colouring satisfies both (i) and (ii) above. The only problem is that perhaps $L'(v) = \emptyset$ for some $v \in V$. For every $v \in V$ let A_v denote the event "$L'(v) = \emptyset$". Since all coin flips are independent, it follows that $\mathbb{P}(A_v) = 2^{-|L(v)|} < 2^{-\log_2 n} = 1/n$. Hence

$$\mathbb{P}(\exists v \in V \text{ such that } L'(v) = \emptyset) \leq \sum_{v \in V} \mathbb{P}(A_v) < 1.$$

   We conclude that the desired colouring exists.