Practical session 12

Exercise 1 Let $X \sim N(4, 16)$. Express the following quantities using the cumulative distribution function of the standard normal distribution.

1. $\Pr[X > 0]$.
2. $\Pr[-4 < X < 16]$.
3. $\Pr[X > 8 \mid X > 4]$.

Solution

Let $Z = \frac{X-4}{4}$. Then, as was shown in the lecture, it holds that $Z \sim N(0, 1)$.

1. We have

$$\Pr[X > 0] = \Pr\left[\frac{X-4}{4} > \frac{0-4}{4}\right] = \Pr[Z > -1] = 1 - \Pr[Z \le -1] = 1 - \Phi(-1) = \Phi(1).$$

2. Observe that

$$\Pr[-4 < X < 16] = \Pr[X \le 16] - \Pr[X \le -4].$$

Next, it holds that

$$\Pr[X \le -4] = \Pr\left[\frac{X-4}{4} \le \frac{-4-4}{4}\right] = \Pr[Z \le -2] = \Phi(-2) = 1 - \Phi(2),$$

and that

$$\Pr[X \le 16] = \Pr\left[\frac{X-4}{4} \le \frac{16-4}{4}\right] = \Pr[Z \le 3] = \Phi(3).$$

We conclude that $\Pr[-4 < X < 16] = \Phi(2) + \Phi(3) - 1$.

3. It holds that

$$\Pr[X > 8] = \Pr\left[\frac{X-4}{4} > \frac{8-4}{4}\right] = \Pr[Z > 1] = 1 - \Pr[Z \le 1] = 1 - \Phi(1),$$

and that

$$\Pr[X > 4] = \Pr\left[\frac{X-4}{4} > \frac{4-4}{4}\right] = \Pr[Z > 0] = 1/2.$$

Therefore

$$\Pr[X > 8 \mid X > 4] = \frac{\Pr[X > 8 \wedge X > 4]}{\Pr[X > 4]} = \frac{\Pr[X > 8]}{\Pr[X > 4]} = 2 - 2\Phi(1).$$

Exercise 2 Let $X \sim N(0, \sigma^2)$ for some $\sigma > 0$. Calculate $\mathbb{E}(|X|)$.

Solution
Recall that the probability density function of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

Hence

$$\mathbb{E}(|X|) = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \, \mathrm{dx}.$$

Using the substitution $t = x/\sigma$ which implies $\frac{\mathrm{dt}}{\mathrm{dx}} = \frac{1}{\sigma}$, we obtain

$$\int_{-\infty}^{\infty} |x| \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \, \mathrm{dx} = \int_{-\infty}^{\infty} |\sigma t| \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{dt} = \sigma \cdot \int_{-\infty}^{\infty} |t| \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{dt}.$$

Moreover

$$\begin{aligned}
\int_{-\infty}^{\infty} |t| \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{dt} &= 2 \int_{0}^{\infty} t \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{dt} \\
&= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} t e^{-t^2/2} \, \mathrm{dt} \\
&= \sqrt{\frac{2}{\pi}} \cdot \left(-e^{-t^2/2}\right) \Big|_{0}^{\infty} \\
&= \sqrt{\frac{2}{\pi}},
\end{aligned}$$

where the first equality is due to the fact that the function $t \mapsto |t| \, e^{-t^2/2}$ is even. We conclude that $\mathbb{E}(|X|) = \sigma\sqrt{\frac{2}{\pi}}$.

Exercise 3 Let $Z \sim N(0, 1)$ and let $X = \sqrt{|Z|}$. Find the probability density function f_X of X.

Solution
Recall that the probability density function of Z is given by

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

We first calculate the cumulative distribution function of X.

$$\begin{aligned}
F_X(x) = \Pr[X \le x] = \Pr\left[\sqrt{|Z|} \le x\right] &= \Pr\left[|Z| \le x^2\right] \\
&= \Pr\left[-x^2 \le Z \le x^2\right] = \Pr\left[Z \le x^2\right] - \Pr\left[Z \le -x^2\right] \\
&= \Phi(x^2) - \Phi(-x^2) = 2\Phi(x^2) - 1.
\end{aligned}$$

Differentiating and using the fact that $\Phi'(x) = \varphi(x)$, we obtain

$$f_X(x) = F'_X(x) = 2\Phi'(x^2) \cdot 2x = 4x\varphi(x^2) = \sqrt{\frac{8}{\pi}} \cdot x e^{-x^4/2}.$$

Exercise 4 Let $Z \sim N(0,1)$. Prove that

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{x}{x^2+1} \cdot e^{-x^2/2} \leq \Pr[Z \geq x] \leq \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot e^{-x^2/2}$$

holds for every real $x > 0$.

Solution

We first prove the upper bound. It holds that

$$\begin{aligned}
\Pr[Z \geq x] &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, \mathrm{dt} \\
&\leq \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{t}{x} \cdot e^{-t^2/2}\, \mathrm{dt} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \left(-e^{-t^2/2}\right)\Big|_x^\infty \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot e^{-x^2/2},
\end{aligned}$$

where the inequality holds since $t \geq x > 0$.

Next, we prove the lower bound. Let

$$h(x) = \Pr[Z \geq x] - \frac{1}{\sqrt{2\pi}} \cdot \frac{x}{x^2+1} \cdot e^{-x^2/2}.$$

It thus suffices to prove that $h(x) \geq 0$ for every $x > 0$. This claim will follow from the following two observations.

1. $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \Pr[Z \geq x] - \lim_{x\to\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{x}{x^2+1} \cdot e^{-x^2/2} = 0 - 0 = 0;$

2. The function h is monotone decreasing in $[0, \infty)$ as

$$\begin{aligned}
h'(x) &= (1 - \Pr[Z \leq x])' - \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{x}{x^2+1} \cdot e^{-x^2/2}\right)' \\
&= -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} - \frac{1}{\sqrt{2\pi}} \left(\frac{x}{x^2+1} \cdot e^{-x^2/2} \cdot (-x) + e^{-x^2/2} \cdot \frac{1 \cdot (x^2+1) - x \cdot (2x)}{(x^2+1)^2}\right) \\
&= -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \left(1 - \frac{x^2}{x^2+1} - \frac{x^2-1}{(x^2+1)^2}\right) \\
&= -\frac{2}{\sqrt{2\pi}} \cdot \frac{1}{(x^2+1)^2} \cdot e^{-x^2/2} \\
&< 0.
\end{aligned}$$

Indeed, the above two properties of h imply that h decreases from some value towards 0, and thus $h(x) \geq 0$ for every $x \geq 0$.