Probability Theory 2 – Exercise sheet II

If you wish to submit your solutions to any of these questions, please hand them to your TA during the practical session held in the week starting on 14/11/2021. This deadline is strict!

1. Let m, n and t be positive integers. Prove that there exists a red/blue colouring of the edges of $K_{m,n}$ with at most $\binom{m}{t}\binom{n}{t}2^{1-t^2}$ monochromatic copies of $K_{t,t}$.

   Remark: $K_{m,n}$ is a complete bipartite graph. That is, its vertex set is $A \cup B$ such that $|A| = m$, $|B| = n$ and $A \cap B = \emptyset$, and its edge set is $\{xy : x \in A, y \in B\}$.

2. Let $n \geq k \geq 2$ and $m < 2^{k-1}$ be positive integers. Let $\{A_1, \ldots, A_m\}$ be a family of subsets of $\{1, \ldots, n\}$, where $|A_i| = k$ for every $1 \leq i \leq m$. Prove that there exists a colouring of the elements of $\{1, \ldots, n\}$ with 2 colours such that, for every $1 \leq i \leq m$, the set A_i contains elements of both colours.

3. Let $0 \leq p \leq 1$ be a real number and let k, ℓ and n be positive integers. Prove that if

   $$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell} (1-p)^{\binom{\ell}{2}} < 1,$$

   then $R(k, \ell) > n$.

   Remark: $R(k, \ell)$ is the "off-diagonal" Ramsey number, that is, it is the smallest integer n such that any *red/blue*-colouring of the edges of K_n yields a red K_k (i.e., a complete graph on k vertices such that all of its edges are coloured red) or a blue K_ℓ.

4. Prove that every graph G contains an independent set of size at least

   $$\sum_{v \in V(G)} \frac{1}{\deg_G(v) + 1}.$$

   Hint: Consider the vertices one by one according to a random permutation $\pi \in S_{|V(G)|}$ and build an independent set in the "obvious way".

5. Let $G = (V, E)$ be a bipartite graph on n vertices. For every $v \in V$ let $L(v) \subseteq \mathbb{N}$ be a set whose size is strictly larger than $\log_2 n$. Prove that there exists a function (usually called a colouring) $c : V \to \mathbb{N}$ which satisfies both of the following properties:

   (i) $c(v) \in L(v)$ for every $v \in V$.

   (ii) $c(u) \neq c(v)$ for every two vertices $u, v \in V$ for which $uv \in E$.