Practical session 2

Exercise 1 An n-vertex graph G is generated as follows: every pair of vertices of G is connected by an edge with probability $p = n^{-2/5}$, independently of all other pairs. Prove that the probability that the maximum degree of G is at least $n^{2/3}$ tends to 0 as n tends to infinity.

Solution

Consider an arbitrary fixed vertex $v \in V(G)$. Observe that $\deg_G(v) \sim \mathrm{Bin}(n-1, p)$. Therefore

$$\mathbb{E}(\deg_G(v)) = p(n-1) \leq n^{3/5} \leq \frac{1}{2} n^{2/3}, \tag{1}$$

where the last inequality holds for sufficiently large n (since we only care about the limit of some probability when n tends to infinity, we can assume that n is as large as we want it to be).

Let $t = \frac{1}{2} n^{2/3}$. Then

$$\begin{aligned} \Pr\left(\deg_G(v) \geq n^{2/3}\right) &\leq \Pr(\deg(v) \geq p(n-1) + t) \\ &\leq e^{-\frac{2t^2}{n-1}} \\ &= e^{-\frac{n^{4/3}}{2(n-1)}} \\ &\leq e^{-n^{1/3}/2}, \end{aligned}$$

where the first inequality is due to (1) and the second inequality holds by Chernoff's bound. Applying a union bound, we conclude that the probability that there exists a vertex whose degree in G is at least $n^{2/3}$ is at most $n \cdot e^{-n^{1/3}/2}$. Clearly this probability tends to 0 as n tends to infinity.

Exercise 2 Let $X_1, X_2, \ldots$ be an infinite sequence of mutually independent and identically distributed discrete random variables, with probability mass function $p(\cdot)$ (that is, $\Pr(X_i = x) = p(x)$ for every $i \in \mathbb{N}$ and every x in the support of X_i). Let $\mu = \mathbb{E}(X_1)$, let $H = \mathbb{E}(-\log_2 p(X_1))$, and let $\varepsilon > 0$ be a real number. For every $n \in \mathbb{N}$, let $X^n = (X_1, \ldots, X_n)$ and let $p_n(\cdot)$ be the probability mass function of X^n. Define the following sets

$$\begin{aligned} A^n &= \left\{ x^n : \left| \frac{1}{n} \log_2 p_n(x^n) + H \right| < \varepsilon \right\}, \\ B^n &= \left\{ (x_1, \ldots, x_n) : \left| \frac{1}{n} \sum_{i=1}^{n} x_i - \mu \right| < \varepsilon \right\}. \end{aligned}$$

Prove that

1. $\lim_{n\to\infty} \Pr\left(X^n \in A^n\right) = 1$.
2. $\lim_{n\to\infty} \Pr\left(X^n \in A^n \cap B^n\right) = 1$.
3. $|A^n \cap B^n| \le 2^{n(H+\varepsilon)}$.
4. $|A^n \cap B^n| \ge (1-\varepsilon)2^{n(H-\varepsilon)}$.

Solution

1. Note that

   $$\mathbb{E}(-\log_2 p_n\left(X^n\right)) = \mathbb{E}(-\log_2\left(p\left(X_1\right)\cdot\ldots\cdot p\left(X_n\right)\right)) = \mathbb{E}\left(-\sum_{i=1}^{n}\log_2 p(X_i)\right) = nH,$$

   where the first equality holds since the random variables $X_1, \ldots, X_n$ are mutually independent and the last equality holds by the linearity of expectation and by the definition of H. It thus follows by the weak law of large numbers that

   $$\lim_{n\to\infty} \Pr\left(\left|-\frac{1}{n}\log_2 p_n(X^n) - H\right| < \varepsilon\right) = \lim_{n\to\infty} \Pr\left(\left|-\frac{1}{n}\sum_{i=1}^{n}\log_2 p(X_i) - H\right| < \varepsilon\right) = 1.$$

   This concludes the proof of 1. as

   $$\Pr\left(\left|\frac{1}{n}\log_2 p_n(X^n) + H\right| < \varepsilon\right) = \Pr\left(X^n \in A^n\right).$$

2. Fix some $\varepsilon' > 0$. By part 1 of this exercise, there exists an N_1 such that for every integer $n > N_1$ it holds that $\Pr\left(X^n \in A^n\right) > 1 - \varepsilon'/2$. Similarly, by the weak law of large numbers, it holds that

   $$\lim_{n\to\infty} \Pr\left(X^n \in B^n\right) = \lim_{n\to\infty} \Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| < \varepsilon\right) = 1.$$

   Therefore, there exists an N_2 such that for every integer $n > N_2$ it holds that $\Pr\left(X^n \in B^n\right) > 1 - \varepsilon'/2$. Thus, for every integer $n > \max\{N_1, N_2\}$, it holds that

   $$\begin{aligned}\Pr\left(X^n \in A^n \cap B^n\right) &= \Pr\left(X^n \in A^n\right) + \Pr\left(X^n \in B^n\right) - \Pr\left(X^n \in A^n \cup B^n\right)\\ &> (1-\varepsilon'/2) + (1-\varepsilon'/2) - 1\\ &= 1 - \varepsilon'.\end{aligned}$$

   We conclude that $\lim_{n\to\infty} \Pr\left(X^n \in A^n \cap B^n\right) = 1$ as claimed.

3. Observe that

   $$A^n = \left\{x^n : \left|\frac{1}{n}\log_2 p_n\left(x^n\right) + H\right| < \varepsilon\right\} = \left\{x^n : 2^{-n(H+\varepsilon)} < p_n\left(x^n\right) < 2^{-n(H-\varepsilon)}\right\}.$$

Therefore

$$\begin{aligned}
1 &= \sum_{x^n} p_n(x^n) \\
&\geq \sum_{x^n \in A^n \cap B^n} p_n(x^n) \\
&\geq \sum_{x^n \in A^n \cap B^n} 2^{-n(H+\varepsilon)} \\
&= |A^n \cap B^n| 2^{-n(H+\varepsilon)},
\end{aligned}$$

implying that $|A^n \cap B^n| \leq 2^{n(H+\varepsilon)}$.

4. By part 2 of this exercise, there exists an N such that for every integer $n > N$, it holds that

$$\Pr\left(X^n \in A^n \cap B^n\right) \geq 1 - \varepsilon.$$

Therefore

$$\begin{aligned}
1 - \varepsilon &\leq \Pr\left(X^n \in A^n \cap B^n\right) \\
&= \sum_{x^n \in A^n \cap B^n} p_n(x^n) \\
&\leq \sum_{x^n \in A^n \cap B^n} 2^{-n(H-\varepsilon)} \\
&= |A^n \cap B^n| 2^{-n(H-\varepsilon)},
\end{aligned}$$

implying that $|A^n \cap B^n| \geq (1-\varepsilon) 2^{n(H-\varepsilon)}$.