Lecture 11

1 Common continuous random variables

We will consider several continuous random variables whose distributions are commonly used.

1.1 The uniform distribution

For real numbers $a < b$, a continuous random variable X is said to have the uniform distribution over the interval $[a, b]$ if its probability density function is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Observe that this is indeed a probability density function as

$$\int_{-\infty}^{\infty} f_X(x)dx = \frac{1}{b-a} \int_a^b dx = 1.$$

We would now wish to determine the cumulative distribution function of X. Note first that for every $x < a$ it holds that $F_X(x) = \int_{-\infty}^{x} f_X(t)dt = 0$. Similarly, for every $x > b$ it holds that $F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_a^b f_X(t)dt = 1$. Finally, for every $a \leq x \leq b$ it holds that $F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \frac{1}{b-a} \int_a^x dt = \frac{x-a}{b-a}$. We conclude that

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$$

Example 1: Let X be a uniform random variable over the interval $[0, 1]$. Let $Y = aX + b$ for some real numbers $a > 0$ and b. We would like to determine the distribution of Y. Note that for every $y \in \mathbb{R}$ it holds that

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}(X \leq (y-b)/a) \\ &= \begin{cases} 0 & \text{if } (y-b)/a < 0 \\ \frac{y-b}{a} & \text{if } 0 \leq (y-b)/a \leq 1 \\ 1 & \text{if } (y-b)/a > 1 \end{cases} = \begin{cases} 0 & \text{if } y < b \\ \frac{y-b}{(a+b)-b} & \text{if } b \leq y \leq a+b \\ 1 & \text{if } y > a+b \end{cases} \end{aligned}$$

We conclude that Y is a uniform random variable over the interval $[b, a+b]$.

Next, we will calculate the expectation and variance of the random variable X which is uniform over $[a, b]$.

$$\begin{aligned}\mathbb{E}(X) &= \int_{-\infty}^{\infty} x f_X(x)dx = \frac{1}{b-a}\int_a^b x dx = \frac{1}{b-a}\cdot \frac{x^2}{2}\Big|_a^b \\ &= \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}.\end{aligned}$$

In order to calculate the variance of X we first calculate $\mathbb{E}(X^2)$.

$$\begin{aligned}\mathbb{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \frac{1}{b-a}\int_a^b x^2 dx = \frac{1}{b-a}\cdot \frac{x^3}{3}\Big|_a^b \\ &= \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}.\end{aligned}$$

Therefore

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{(b-a)^2}{12}.$$

1.2 The exponential distribution

For a real number $\lambda > 0$, a continuous random variable X is said to have the exponential distribution with parameter λ if its probability density function is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Observe that this is indeed a probability density function as

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^{\infty} \lambda e^{-\lambda x}dx = -e^{-\lambda x}|_0^{\infty} = 1.$$

We would now wish to determine the cumulative distribution function of X. Clearly, if $x < 0$, then $F_X(x) = \int_{-\infty}^{x} f_X(t)dt = 0$. On the other hand, if $x \geq 0$, then

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_0^x \lambda e^{-\lambda t}dt = -e^{-\lambda t}|_0^x = 1 - e^{-\lambda x}.$$

We conclude that

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

Example 2: Let X be an exponential random variable with parameter $\lambda > 0$. Let

$Y = cX$ for some real number $c > 0$. We would like to determine the distribution of Y. Since $c > 0$, for every $x \in \mathbb{R}$ it holds that $x < 0$ if and only if $cx < 0$. Therefore

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(cX \leq y) = \mathbb{P}(X \leq y/c) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - e^{-\lambda y/c} & \text{if } y \geq 0 \end{cases}$$

We conclude that Y is an exponential random variable with parameter λ/c.

Next, we will calculate the expectation and variance of X. Integrating by parts with $u = x$ and $v' = \lambda e^{-\lambda x}$ we obtain

$$\begin{aligned} \mathbb{E}(X) &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx = -xe^{-\lambda x}|_0^{\infty} - \int_0^{\infty} -e^{-\lambda x} dx \\ &= (0 - 0) + \frac{1}{\lambda} \int_0^{\infty} \lambda e^{-\lambda x} dx = \frac{1}{\lambda}. \end{aligned} \tag{1}$$

In order to calculate the variance of X we first calculate $\mathbb{E}(X^2)$. Integrating by parts with $u = x^2$ and $v' = \lambda e^{-\lambda x}$ and using (1) we obtain

$$\begin{aligned} \mathbb{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x} dx = -x^2 e^{-\lambda x}|_0^{\infty} - \int_0^{\infty} -2xe^{-\lambda x} dx \\ &= (0 - 0) + 2 \int_0^{\infty} xe^{-\lambda x} dx = \frac{2}{\lambda} \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}. \end{aligned}$$

Therefore

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

A non-negative random variable X is said to be *memoryless* if

$$\mathbb{P}(X > s + t \mid X > t) = \mathbb{P}(X > s) \text{ for all real numbers } s, t \geq 0.$$

Observe that

$$\mathbb{P}(X > s + t \mid X > t) = \frac{\mathbb{P}(X > s + t, X > t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > t)}$$

and thus the non-negative random variable X is memoryless if and only if

$$\mathbb{P}(X > s + t) = \mathbb{P}(X > s) \mathbb{P}(X > t) \text{ for all real numbers } s, t \geq 0. \tag{2}$$

The following result suggests that being memoryless is essentially the same as being exponentially distributed.

Proposition 1.1. *Let X be a non-negative continuous random variable. Then X is memoryless if and only if it is exponentially distributed.*

Proof. Assume first that X is exponentially distributed with parameter $\lambda > 0$. Then, for every $s, t \geq 0$ it holds that

$$\begin{aligned}\mathbb{P}\left(X > s\right)\mathbb{P}\left(X > t\right) &= (1 - F_X(s))(1 - F_X(t)) = e^{-\lambda s}e^{-\lambda t} = e^{-\lambda(s+t)} \\ &= 1 - F_X(s+t) = \mathbb{P}\left(X > s+t\right).\end{aligned}$$

Hence, X is memoryless by (2).

Let X be a memoryless non-negative continuous random variable. Let F_X be the cumulative distribution function of X and for every real number $x \geq 0$ let $g(x) = 1 - F_X(x)$. Since X is memoryless, it follows by (2) that

$$g(s+t) = g(s)g(t) \text{ holds for all real } s, t \geq 0. \tag{3}$$

We claim that $g(m/n) = g^m(1/n)$ for all positive integers m and n. We fix an arbitrary n and prove this by induction on m. The claim holds trivially for $m = 1$. Assume it holds for some $m \geq 1$. Then

$$g((m+1)/n) = g(m/n + 1/n) = g(m/n)g(1/n) = g^m(1/n)g(1/n) = g^{m+1}(1/n),$$

where the second equality holds by (3) and the third equality holds by the induction hypothesis.

Now, note that

$$g(1) = g(n/n) = g^n(1/n) \Longrightarrow g(1/n) = (g(1))^{1/n}$$

holds for any positive integer n. It then follows that

$$g(m/n) = g^m(1/n) = (g(1))^{m/n}$$

holds for all positive integers m and n. Observe that, since X is a continuous random variable, the function g is continuous. Therefore $g(x) = (g(1))^x$ holds for every real $x \geq 0$. Let $\lambda = -\ln(g(1))$; note that λ is well-defined and positive since $0 < g(1) < 1$[1]. We conclude that

$$g(x) = (g(1))^x = e^{x \ln g(1)} = e^{-\lambda x},$$

implying that X is distributed exponentially with parameter $\lambda > 0$. $\square$

[1]Since $g(1)$ is the probability of some event, we have $0 \leq g(1) \leq 1$. If $g(1) = 1$, then $g \equiv 1$ and thus $F_X \equiv 0$. This is of course a contradiction since $F_X(\infty) = 1$ must hold by the definition of a cumulative distribution function. Similarly, if $g(1) = 0$, then $F_X(x) = 0$ for every $x < 0$ and $F_X(x) = 1$ for every $x > 0$. This is a contradiction since F_X is a continuous function.