Probability Theory 2 – Solutions IV

1. Suppose that A is a Monte-Carlo randomized algorithm for some problem P, which, given any input, outputs a correct solution with probability at least $1/2$, and whose running time on inputs of size n is $f(n)$. Suppose also that, for any input x of A, the size of the output of $A(x)$ is not larger than the size of x. Finally, suppose that B is a deterministic algorithm which, given a potential solution for P of size m, verifies its correctness in time $g(m)$, where g is a non-decreasing function. Devise a Las-Vegas algorithm C for P whose expected running time is $O(f(n) + g(n))$.

   Solution: Given an input x, the algorithm C does the following:

   (a) Run $A(x)$ to obtain a potential solution y.

   (b) Run $B(y)$ to check if y is a correct solution.

   (c) If $B(y)$ outputs "YES", then output y. Otherwise, return to (a).

   Since C only outputs correct solutions (as they were verified by B), it is a Las-Vegas algorithm. Denote the size of x by n and the size of y by m; recall that $m \leq n$ by assumption. Then each iteration of the algorithm takes time $f(n) + g(m) \leq f(n) + g(n)$. The number of iterations is a geometric random variable X with parameter $1/2$. In particular

$$\mathbb{E}(X) \leq \sum_{i=1}^{\infty} i \cdot (1/2)^i < \sum_{i=0}^{\infty} (i+1) \cdot (1/2)^i = \frac{1}{(1-1/2)^2} = 4,$$

   where the first equality follows from the identity $\sum_{i=0}^{\infty}(i+1)x^i = \frac{1}{(1-x)^2}$ which holds for every real $-1 < x < 1$. We conclude that the expected running time of C is at most $4(f(n)+g(n))$.

2. For what values of the parameter C are the following functions probability density functions?

   (a) $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} C(4x - 2x^2) & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

   (b) $g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$g(x) = \begin{cases} Ce^{-x/100} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Solution: Recall that $h : \mathbb{R} \longrightarrow \mathbb{R}$ is a probability density function if and only if

$$\int_{-\infty}^{\infty} h(x)dx = 1.$$

   Hence

(a)
$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^2 C(4x - 2x^2)dx = C(2x^2 - 2x^3/3) \mid_0^2 = 8C/3.$$
It follows that $C = 3/8$.

(b)
$$1 = \int_{-\infty}^{\infty} g(x)dx = \int_0^{\infty} Ce^{-x/100}dx = -100Ce^{-x/100} \mid_0^{\infty} = 100C.$$
It follows that $C = 1/100$.

3. Let X be a random variable with probability density function f_X and let $Y = aX$ for some real number $a > 0$.

   (a) Find f_Y, the probability density function of Y, in terms of f_X.

   (b) Prove that $f_{-X}(x) = f_X(-x)$ for every $x \in \mathbb{R}$.

   (c) Prove that X and $-X$ have the same cumulative probability function if and only if $f_X(x) = f_X(-x)$ for every $x \in \mathbb{R}$.

Solution:

(a) Let F_X denote the cumulative probability function of X, that is, $F_X(x) = \mathbb{P}(X \leq x)$. Similarly, let F_Y denote the cumulative probability function of Y. Then
$$F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}(aX \leq x) = \mathbb{P}(X \leq x/a) = F_X(x/a).$$
We conclude that the probability density function of Y is
$$f_Y(x) = F_Y'(x) = F_X'(x/a) = f_X(x/a)/a.$$

(b) Fix some $x \in \mathbb{R}$. Then
$$F_{-X}(x) = \mathbb{P}(-X \leq x) = \mathbb{P}(X \geq -x) = \mathbb{P}(X > -x) = 1 - \mathbb{P}(X \leq -x) = 1 - F_X(-x),$$
where the third equality holds since $\mathbb{P}(X = x) = 0$ as X is a continuous random variable. Hence
$$f_{-X}(x) = F_{-X}'(x) = (1 - F_X(-x))' = f_X(-x).$$

(c) Assume first that X and $-X$ have the same cumulative probability function, i.e. $F_X(x) = F_{-X}(x)$ for every $x \in \mathbb{R}$. Hence, for every $x \in \mathbb{R}$ we have
$$f_X(-x) = f_{-X}(x) = F_{-X}'(x) = F_X'(x) = f_X(x),$$
where the first equality holds by part (b) of this question.
Now, assume that $f_X(x) = f_X(-x)$ holds for every $x \in \mathbb{R}$. Then, it follows by part (b) of this question that $f_{-X}(x) = f_X(-x) = f_X(x)$ for every $x \in \mathbb{R}$. Therefore, for every $x \in \mathbb{R}$ we have
$$F_X(x) = \int_{-\infty}^{x} f_X(x) = \int_{-\infty}^{x} f_{-X}(x) = F_{-X}(x).$$
That is, X and $-X$ have the same cumulative probability function.