Optomechanical analogy in collective motional-spin interaction with state dependent force

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Interaction hamiltonian for a specific collective mode of the ion-plasma, described by position coordinate $\hat{z}$, with the internal spins, $\hat{\sigma}_i^{(z)}$, is given by,

$$\hat{H}_{int} = f \cdot \hat{z} \sum_{i=1}^{N} \hat{\sigma}_i^{(z)}. \tag{1}$$

Here, the interaction is expressed in the rotating-wave approximation, valid when the spin-dependent force is near-resonant to the specific mechanical mode under consideration. Denoting by

$$\hat{S}^{(a)} = \sum_{i=1}^{N} \hat{\sigma}_i^{(a)}, \qquad (a = x, y, z),$$

the collective spin operator, the above interaction hamiltonian describes a collective mechanical mode interacting with a collective spin, viz.

$$\hat{H}_{int} = f \cdot \hat{z}\hat{S}^{(z)}. \tag{2}$$

For a large number of spin-1/2's N, the operator $\hat{S}^{(z)}$ has eigenstates spanning the space

$$\mathcal{H}_N^{spin} = \{|J, -J\rangle, |J, J+1\rangle, \ldots, |J, J\rangle\}, \qquad J = \frac{N}{2}.$$

Unlike a single spin-1/2, for which the space $\mathcal{H}_2^{spin}$ is of dimension 2, the dimension of $\mathcal{H}_N^{spin}$ tends to infinity, as $N \to \infty$, with equally spaced levels, reminiscent of a harmonic oscillator. Thus one may think of $\hat{H}_{int}$ as a parametric interaction between two harmonic oscillators.

This can be formalized as follows. The conventional representation of the operators $\hat{S}^{(a)}$acts on the space $\mathcal{H}_N^{spin}$; however, they may be alternatively represented on the $(N+1)-$ dimensional Fock space,

$$\mathcal{H}_N^{Fock} = \{|0\rangle, \ldots, |N\rangle\},$$

with the set of identifications, $|J, J\rangle \to |0\rangle, \ldots, |J, -J\rangle \to |N\rangle$, between the states in the two spaces (this identification corresponds to the situation where spins aligned with the magnetic field is lowest energy). At the level of the operators, this identification is carried through, if the algebraic relations encoding the nature of the spin operators,

$$[\hat{S}^{(x)}, \hat{S}^{(y)}] = i\hat{S}^{(z)}, \qquad \text{(and cyclic permutations)}, \tag{3}$$

is mapped onto a set of operators in the Fock space. The so-called Holstein-Primakoff representation [see for example, Phys. Rev. A **12**, 2083 (1975)],

$$\hat{S}^{(z)} = N - \hat{a}^\dagger \hat{a}, \quad \hat{S}^{(x)} \pm i\hat{S}^{(y)} = (N - \hat{a}^\dagger \hat{a})^{1/2}\hat{a},$$

identically satisfies (3), if $[\hat{a}, \hat{a}^\dagger] = 1$. Inserting the above representation for $\hat{S}^{(z)}$ in (2), we get,

$$\hat{H}_{int} = f \cdot \hat{z}(N - \hat{a}^\dagger \hat{a}) = f \cdot \hat{z}\, \delta\hat{N}. \tag{4}$$

Here, $\delta\hat{N} = N - \hat{a}^\dagger \hat{a}$, is the operator representing the fluctuations in the number of spins. However, if interpreted as the number fluctuations of the transformed harmonic oscillator, this hamiltonian is precisely the same as the radiation pressure interaction in conventional cavity optomechanics, with the collective spin playing the role of the optical field; this analogy becomes exact for $N \to \infty$.

However, a finite N presumably alters the performance of this system compared to conventional optomechanics. The other difference is the ability to "easily" prepare states other than spin coherent states for measuring motion.