Supplemental Material: Amplitude sensing below the zero-point fluctuations with a two-dimensional trapped-ion mechanical oscillator

K. A. Gilmore,[1,2,*] J. G. Bohnet,[1] B. C. Sawyer,[3] J. W. Britton,[4] and J. J. Bollinger[1,†]

[1] *National Institute of Standards and Technology, Boulder, Colorado 80305, USA*
[2] *JILA and Department of Physics, University of Colorado, Boulder, Colorado, 80309, USA*
[3] *Georgia Tech Research Institute, Atlanta, Georgia 30332, USA*
[4] *U.S. Army Research Laboratory, Adelphi, Maryland 20783, USA*

INTRODUCTION

In this supplemental material we provide detailed derivations for a number of theoretical formulas and results of the main text. Specifically, in the first section we derive the shift in the spin transition frequency due to the coherent amplitude Z_c (Eq. (2) of the main text) from a more basic perspective. We also discuss in detail our modulation scheme. In the second section we derive the lineshape function used in Fig. 2 of the main text. In section 3 we describe the formalism used to determine the optimum signal-to-noise ratio for a measurement of Z_c^2. We used this optimum signal-to-noise ration to generate the theoretical curve in Fig. 4 of the main text. We also derive the sensitivity limits for phase-incoherent amplitude sensing, where the phase difference between the driven motion and the ODF randomly varies from one realization of the experiment to the next. We show how these limits depend on $\Gamma/(U/\hbar)$, δk, and N. Finally, in section 4 we consider the amplitude sensing limits assuming phase coherence between the spin-dependent force and the driven amplitude.

CONTENTS

1. SHIFT IN THE SPIN TRANSITION FREQUENCY, AND THE MODULATION SCHEME

Figure 1 shows the Carr-Purcell-Meiboom-Gill (CPMG) sequence used to apply the 1D traveling-wave potential. The interaction of the spin degree of freedom with the 1D traveling-wave potential is given by

$$\hat{H}_{ODF} = U\sum_i \sin(\delta k\cdot \hat{z}_i - \mu t + \phi)\hat{\sigma}_i^z = U\sum_i \sin(\delta k\cdot \hat{z}_i)\cos(\mu t - \phi)\hat{\sigma}_i^z - U\sum_i \cos(\delta k\cdot \hat{z}_i)\sin(\mu t-\phi)\hat{\sigma}_i^z. \quad (1)$$

Here we explicitly include a phase ϕ for the traveling-wave potential. Without loss of generality, we assumed $\phi = 0$ in the main text. If $\delta k\,\langle \hat{z}_i\rangle \ll 1$, then $\langle \cos(\delta k\cdot \hat{z}_i)\rangle \sim 1$, and the spin precession due to the second term will be bounded by $(U/\hbar)/\mu$.

Typically, $(U/\hbar)/\mu \ll 1$ and thus this term is ignored in most treatments. At low frequencies $\mu \leq U/\hbar$ this term could be important, but it may be canceled by advancing the phase of the ODF by $\Delta\phi = \mu(T+t_\pi)$ at each microwave $\pi-$pulse of the CPMG sequence (see Fig. 1). When $\mu/2\pi = (2n+1)/(2(T+t_\pi))$ for some integer n, $\Delta\varphi = \pi$ and we recover the quantum lock-in phase advance of [1]. This phase advance coherently accumulates spin precession from the first term of Eq. (1) when $\omega/2\pi = (2n+1)/(2(T+t_\pi))$. The term that survives our modulation scheme is

$$\hat{H}_{ODF} \simeq U\sum_i \sin(\delta k\cdot \hat{z}_i)\cos(\mu t - \phi)\hat{\sigma}_i^z. \quad (2)$$

We now impose a weak, classically driven COM motion of constant amplitude and phase $\hat{z}_i \to \hat{z}_i + Z_c\cos(\omega t + \delta)$. This can be thought of as the center of the Penning trap being moved by $\pm Z_c$ at a frequency ω far from the trap axial frequency ω_z. With $\delta k\, Z_c \ll 1$, we obtain

$$\hat{H}_{ODF} \simeq U\sum_i \left(\delta k\, Z_c\cos(\delta k\cdot \hat{z}_i)\cos(\omega t+\delta)\cos(\mu t-\phi) + \sin(\delta k\cdot \hat{z}_i)\cos(\mu t - \phi)\right)\hat{\sigma}_i^z. \quad (3)$$

The second term of Eq. (3) is the usual term that gives rise to spin-motion entanglement with the drumhead modes and to effective spin-spin interactions [2, 3]. We assume we can neglect this term because we tune μ far from any drumhead modes.

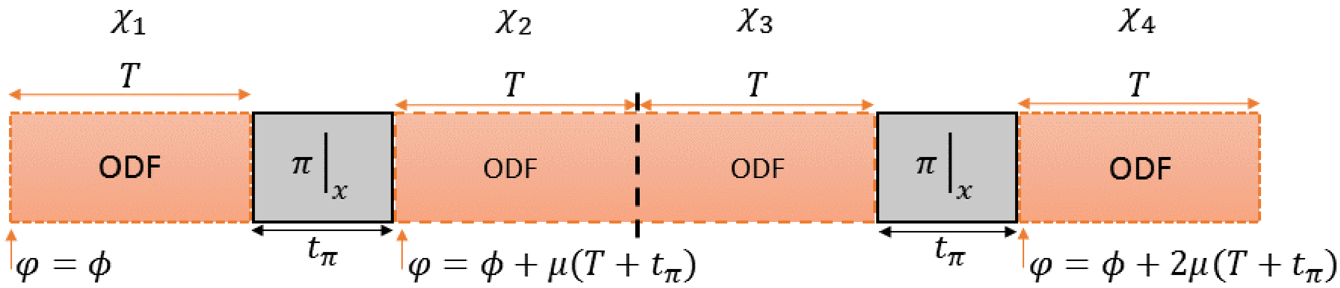

FIG. 1. $m = 2$ CPMG sequence with total ODF interaction time $4T$. φ is the phase of the ODF beatnote. The χ_i labels represent the periods over which the accumulated phase is considered in the text.

Deep in the Lamb-Dicke confinement regime, the $\cos(\delta k \cdot \hat{z}_i)$ factor in the first term of Eq. (3) equals one. Here we account for the possibility of not being deep in the Lamb-Dicke confinement regime. In this case, and assuming a thermal distribution of modes, $\langle \cos(\delta k \cdot \hat{z}_i)\rangle = \exp(-\delta k^2 \langle \hat{z}_i^2 \rangle /2)$. This factor is known as the Debye-Waller factor DWF. For our conditions all ions have approximately the same Debye-Waller factor, $DWF \approx 0.86$ [3].

With $\mu \sim \omega$, Eq. (3) can be written as

$$\hat{H}_{ODF} = (U \cdot \delta k \cdot DWF)\, \mathrm{Z_c} \cos((\omega - \mu)\mathrm{t} + \delta - \phi) \sum_{\mathrm{i}} \frac{\hat{\sigma}_{\mathrm{i}}^{\mathrm{z}}}{2}, \tag{4}$$

which is Eq. (2) of the main text with $F_0 = U \cdot \delta k \cdot DWF$.

2. LINESHAPE

To model the lineshape of the signal, it is necessary to account for the accumulated phase due to the spin-dependent ODF potential without making the simplification that $\omega = \mu$. This results in a characteristic response function for each sequence. For this Letter, we used an $m = 8$ CPMG sequence, as shown in Fig. 1 in the main text and also Fig. 1 of this supplemental material. In the following, we derive the lineshape of this sequence using the modulation discussed in the first section and assuming a delta function source at a frequency ω. This lineshape is used to generate the theory curves of Fig. 2 of the main text. In general, for a CPMG sequence it is necessary to calculate the phase evolution during $2m$ terms of length T, for a total interaction time of $2m$T. For simplicity, we first derive the lineshape for the $m = 2$ CPMG sequence (Fig. 1).

For a delta function source $Z_c \cos(\omega t + \delta)$, the spin precession accumulated in a general sequence like that shown in Fig. 1 is

$$\theta(\mu) = F_0\, Z_c \frac{2\sin\left(\frac{1}{2}(\omega - \mu)\, T\right)}{(\omega - \mu)} \chi(\mu, \omega), \tag{5}$$

where $\chi(\mu, \omega) = \sum_i \chi_i(\mu, \omega)$ is determined by the particular sequence used. In the case of the $m = 2$ CPMG sequence, the phase accumulated through four terms corresponding to four separate applications of the ODF (Fig. 1) must be considered:

$$\chi_1 = \cos\left[(\omega - \mu)\frac{T}{2} + \delta - \phi\right], \tag{6}$$

$$\chi_2 = -\cos\left[(\omega - \mu)\left(\frac{3T}{2} + t_\pi\right) + \delta - \phi + \mu(T + t_\pi)\right], \tag{7}$$

$$\chi_3 = -\cos\left[(\omega - \mu)\left(\frac{5T}{2} + t_\pi\right) + \delta - \phi + \mu(T + t_\pi)\right], \tag{8}$$

$$\chi_4 = \cos\left[(\omega-\mu)\left(\frac{7T}{2}+2t_\pi\right)+\delta-\phi+2\mu(T+t_\pi)\right]. \tag{9}$$

Note these terms now include a phase ϕ for the ODF interaction, which in the main text we set to zero with no loss of generality. Adding these terms up, pairwise:

$$\chi_1+\chi_2 = 2\sin\left(\frac{1}{2}\left[(\omega-\mu)(T+t_\pi)+\mu(T+t_\pi)\right]\right)\sin\left[(\omega-\mu)\left(T+\frac{t_\pi}{2}\right)+\delta-\phi+\frac{\mu(T+t_\pi)}{2}\right], \tag{10}$$

$$\chi_3+\chi_4 = -2\sin\left(\frac{1}{2}\left[(\omega-\mu)(T+t_\pi)+\mu(T+t_\pi)\right]\right)\sin\left[(\omega-\mu)\left(3T+\frac{3t_\pi}{2}\right)+\delta-\phi+\frac{3\mu(T+t_\pi)}{2}\right]. \tag{11}$$

Summing all four terms yields

$$\chi(\mu,\omega) = \sum_i \chi_i(\mu,\omega) = 2\sin\left(\frac{\omega}{2}(T+t_\pi)\right)\left[\sin(\xi+\delta-\phi)-\sin(3\xi+\delta-\phi)\right], \tag{12}$$

where $\xi = (\omega-\mu)(T+\frac{t_\pi}{2})+\frac{\mu(T+t_\pi)}{2} = \frac{1}{2}\left(\omega(T+t_\pi)+T(\omega-\mu)\right)$. Then, simplifying:

$$\chi(\mu,\omega) = 2\sin\left(\frac{\omega}{2}(T+t_\pi)\right)2\sin(-\xi)\cos(2\xi+\delta-\phi). \tag{13}$$

Using Eqs. 13 and 5,

$$\theta(\mu) = DWF\cdot\mathrm{U}\cdot\delta\mathrm{k}\cdot\mathrm{Z_c}\cdot\mathrm{T}\,\mathrm{sinc}\left(\frac{\mathrm{T}}{2}(\omega-\mu)\right)4\sin\left(\frac{\omega}{2}(\mathrm{T}+\mathrm{t}_\pi)\right)\sin(\xi)\cos(2\xi+\delta-\phi). \tag{14}$$

Since $4T=\tau$ for the $m=2$ CPMG, then

$$\theta(\mu) = \theta_{max}\,\mathrm{sinc}\left(\frac{T}{2}(\omega-\mu)\right)\sin\left(\frac{\omega}{2}(T+t_\pi)\right)\sin(\xi)\cos(2\xi+\delta-\phi), \tag{15}$$

where $\theta_{max}\equiv(F_0/\hbar)\,Z_c\,\tau$, the maximum precession angle on resonance as defined in the main text. Then, $\theta_{max}(\mu)$, defined as $\theta(\mu)=\theta_{max}(\mu)\cos(2\xi+\delta-\phi)$, is the μ-dependent generalization of θ_{max}. From Eq. 15, this is

$$\theta_{max}(\mu) = \theta_{max}\,\mathrm{sinc}\left(\frac{T}{2}(\omega-\mu)\right)\sin\left(\frac{\omega}{2}(T+t_\pi)\right)\sin(\xi). \tag{16}$$

For the $m=8$ CPMG sequence the same procedure is used, but now with 16 periods of accumulated phase. We obtain

$$\theta_{max}(\mu) = \theta_{max}\,\mathrm{sinc}\left(\frac{T}{2}(\omega-\mu)\right)\sin\left(\frac{\omega}{2}(T+t_\pi)\right)\sin(\xi)\cos(2\xi)\cos(4\xi). \tag{17}$$

As shown in the main text, the expression for population in $|\uparrow\rangle$ - now with a dependence on the ODF difference frequency μ - is

$$\langle P_\uparrow\rangle = \frac{1}{2}\left[1-e^{-\Gamma\tau}J_0(\theta_{max}(\mu))\right]. \tag{18}$$

Equations (17) and (18) are used to obtain the theoretical line shapes of Fig. (2) of the main text.

3. PHASE-INCOHERENT SENSING LIMITS

Here we derive Eq. (5) from the main text and provide additional mathematical background for the phase-incoherent experimental protocol, wherein the phase of the measured quadrature varies randomly from one realization of the experiment to the next. Following earlier discussions, the probability of measuring $|\uparrow\rangle$ at the end of the Ramsey sequence is

$$\langle P_\uparrow \rangle = \frac{1}{2}\left[1 - e^{-\Gamma\tau} J_0\left(\theta_{max}\right)\right] , \tag{19}$$

where $\langle\,\rangle$ denotes an average over many experimental trials and therefore over the random phase between the 1D traveling-wave potential and the classically driven COM motion, and

$$\theta_{max} = \left(F_0/\hbar\right) \cdot Z_c \cdot \tau . \tag{20}$$

Defining $G\left(\theta_{max}^2\right) \equiv \left(1 - J_0\left(\theta_{max}\right)\right)/2$ and denoting $\langle P_\uparrow\rangle_{bck} = \left[1 - e^{-\Gamma\tau}\right]/2$ as the probability of measuring $|\uparrow\rangle$ at the end of the sequence in the absence of a classically driven motion, θ_{max}^2 can be determined from a measurement of the difference $\langle P_\uparrow\rangle - \langle P_\uparrow\rangle_{bck}$ through

$$G\left(\theta_{max}^2\right) = e^{\Gamma\tau}\left(\langle P_\uparrow\rangle - \langle P_\uparrow\rangle_{bck}\right) . \tag{21}$$

The standard deviation $\delta\theta_{max}^2$ in estimating θ_{max}^2 is determined from the standard deviation $\sigma\left(P_\uparrow - P_{\uparrow,bck}\right)$ of the $\langle P_\uparrow\rangle - \langle P_\uparrow\rangle_{bck}$ difference measurements through

$$\delta\theta_{max}^2 = \frac{e^{\Gamma\tau}\sigma\left(\langle P_\uparrow\rangle - \langle P_\uparrow\rangle_{bck}\right)}{\frac{\mathrm{d}G\left(\theta_{max}^2\right)}{d\theta_{max}^2}} . \tag{22}$$

The signal-to-noise ratio of a measurement of θ_{max}^2 (and therefore Z_c^2) is $\theta_{max}^2/\delta\theta_{max}^2 = Z_c^2/\delta Z_c^2$. In general this signal-to-noise ratio depends on θ_{max}^2 and the experimental parameters $U\cdot\tau$, $\Gamma\cdot\tau$, δk, and N.

We use Eq. (22) to theoretically estimate $Z_c^2/\delta Z_c^2$ and the amplitude sensing limits. We assume the only sources of noise are projection noise in the measurement of the spin state and fluctuations in $P_\uparrow$ due to the random variation in the relative phase of the 1D traveling-wave potential and the driven COM motion. Experimentally this is obtained by collecting 10 photons for each $|\uparrow\rangle$ state, so photon counting shot noise can be neglected [3]. In this case $\sigma\left(P_\uparrow - P_{\uparrow,bck}\right) = \sqrt{\sigma_{P_\uparrow}^2 + \sigma_{P_{\uparrow,bck}}^2}$ where the relevant variances are

$$\sigma_{P_{\uparrow,bck}}^2 = \frac{1}{N}\langle P_\uparrow\rangle_{bck}\left(1 - \langle P_\uparrow\rangle_{bck}\right) = \frac{1}{4N}\left(1 - e^{-2\Gamma\tau}\right) \tag{23}$$

and

$$\sigma_{P_\uparrow}^2 = \sigma_\delta^2 + \frac{1}{N}\langle P_\uparrow\rangle\left(1 - \langle P_\uparrow\rangle\right) . \tag{24}$$

Here N is the number of spins. Equation (23) and the second term in Eq. (24) are projection noise. The variance

$$\sigma_\delta^2 = \left\langle P_\uparrow^2 - \langle P_\uparrow\rangle^2\right\rangle = \frac{e^{-2\Gamma\tau}}{8}\left(1 + J_0\left(2\theta_{max}\right) - 2J_0\left(\theta_{max}\right)^2\right) \tag{25}$$

is due to the random variation in the relative phase of the 1D traveling-wave and the driven COM motion. For our set-up, $DWF = \exp(-\delta\mathrm{k}^2\left\langle\hat{\mathrm{z}}_{\mathrm{i}}^2\right\rangle/2) = 0.86$ and $\delta k = 2\pi/\left(900\,\mathrm{nm}\right)$ are fixed, the decoherence Γ is a function of U, $\Gamma = \xi\left(U/\hbar\right)$ where $\xi = 1.156\times10^{-3}$, and $F_0 = DWF\cdot\mathrm{U}\cdot\delta\mathrm{k}$. For a given Z_c we use Eqs. (20) and (22)-(25) to find the optimum $Z_c^2/\delta Z_c^2$ as a function of $\left(U\tau\right)/\hbar$. This optimum value is the red dashed theoretical curve plotted in Fig. 4 of the main text.

The signal-to-noise $Z_c^2/\delta Z_c^2$ is optimized for relatively small values of θ_{max}^2 where $G\left(\theta_{max}^2\right) \approx \theta_{max}^2/8$ is a good approximation. This leads to some simplifications for Eqs. (21) and (22),

$$\theta_{max}^2 \approx 8e^{\Gamma\tau}\left(\langle P_\uparrow\rangle - \langle P_\uparrow\rangle_{bck}\right) \tag{26}$$

and

$$\delta\theta_{max}^2 \approx 8e^{\Gamma\tau}\sigma\left(P_\uparrow - P_{\uparrow,bck}\right) , \tag{27}$$

and to the following estimate for the signal-to-noise ratio of a single measurement,

$$\frac{\theta_{max}^2}{\delta\theta_{max}^2} = \frac{Z_c^2}{\delta Z_c^2} \approx \frac{\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}}{\sigma\left(P_\uparrow - P_{\uparrow,bck}\right)} . \tag{28}$$

Figure (4) of the main text uses Eq. (28), along with repeated measurements of $P_\uparrow - P_{\uparrow,bck}$, to experimentally determine the signal-to-noise ratio as a function of the imposed amplitude Z_c of the COM motion.

Finally we use Eqs. (20) and (22)-(25) to calculate the sensing limits for very small Z_c. For small Z_c the variance σ_δ^2 can be neglected compared to projection noise and $\sigma_{P_\uparrow}^2 \approx \sigma_{P_{\uparrow,bck}}^2$. In this case we obtain the following expression for the signal-to-noise ratio,

$$\frac{Z_c^2}{\delta Z_c^2} = \frac{\sqrt{N}}{4\sqrt{2}} \frac{DWF^2 \cdot \left(\delta \mathrm{k}\, \mathrm{Z_c}\right)^2 \left(\mathrm{U}\tau/\hbar\right)^2}{\sqrt{e^{2\xi U\tau/\hbar} - 1}} . \tag{29}$$

Equation (29) is maximized for $\xi U\tau \approx 1.9603$, resulting in

$$\left.\frac{Z_c^2}{\delta Z_c^2}\right|_{\text{limiting}} \approx 0.097 \frac{\sqrt{N}(DWF)^2(\delta \mathrm{k})^2}{\xi^2} Z_c^2 , \tag{30}$$

which is Eq. (5) of the main text. With $DWF = 0.86$, $\delta k = 2\pi/\left(900\,\mathrm{nm}\right)$, $\xi = 1.156 \times 10^{-3}$, and $N = 85$,

$$\left.\frac{Z_c^2}{\delta Z_c^2}\right|_{optimum} = \left[\frac{Z_c}{0.2\,\mathrm{nm}}\right]^2 . \tag{31}$$

For our set-up and available ODF power, $\xi U\tau/\hbar \approx 1.9603$ is realized for $\tau \approx 20\,\mathrm{ms}$. A measurement of the signal and a measurement of the background requires $\sim 60\,\mathrm{ms}$, allowing for 16 independent measurements of $P_\uparrow - P_{\uparrow,bck}$ in 1 s. The limiting sensitivity is approximately $\left(100\,\mathrm{pm}\right)^2$ in a 1 s measurement time, or $\left(100\,\mathrm{pm}\right)^2/\sqrt{\mathrm{Hz}}$. We note that the limiting sensitivity is determined by the ratio $\xi = \Gamma/\left(U/\hbar\right)$. In particular, the optimum value for Eq. (29) scales as $1/\xi^2$.

4. PHASE-COHERENT SENSING LIMITS

With appropriate care the phase of the 1D traveling-wave potential can be stable for long periods of time with respect to the ion trapping electrodes [4], enabling repeated phase-coherent sensing of the same quadrature of the COM motion $Z_c \cos(\omega t)$. In this case the same spin precession $\theta_{max} = DWF \cdot (\mathrm{U}/\hbar) \cdot \delta \mathrm{k}\, \mathrm{Z_c} \cdot \tau$ occurs for each experimental trial, which can be detected to first order in θ_{max} (or Z_c) in a Ramsey sequence with a $\pi/2$ phase shift between the two $\pi/2$-pulses. Assuming $\sin\left(\theta_{max}\right) \approx \theta_{max}$, appropriate for small amplitudes Z_c, the equivalent phase-coherent sensing expressions for Eqs. (26) and (27) are

$$\theta_{max} = 2e^{\Gamma\tau}\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right) \tag{32}$$

and

$$\delta\theta_{max} = 2e^{\Gamma\tau}\sigma\left(P_\uparrow - P_{\uparrow,bck}\right) . \tag{33}$$

For a Ramsey experiment with a $\pi/2$ phase shift, $\langle P_\uparrow \rangle_{bck} = 1/2$. If projection noise is the only source of noise, then for small Z_c, $\sigma_{P_\uparrow}^2 \approx \sigma_{P_{\uparrow,bck}}^2 = \frac{1}{N} \cdot \frac{1}{2} \cdot \frac{1}{2}$ and $\sigma\left(P_\uparrow - P_{\uparrow,bck}\right) \approx \frac{1}{\sqrt{2N}}$. The limiting signal-to-noise ratio $\theta_{max}/\delta\theta_{max}$ of a $\left(P_\uparrow - P_{\uparrow,bck}\right)$ measurement is

$$\frac{\theta_{max}}{\delta\theta_{max}} = \frac{Z_c}{\delta Z_c} = DWF \cdot \left(\delta \mathrm{k}\, \mathrm{Z_c}\right) \cdot \sqrt{\frac{\mathrm{N}}{2}} \cdot \frac{\left(\mathrm{U}\tau\right)}{\hbar} \mathrm{e}^{-\xi \mathrm{U}\tau/\hbar} . \tag{34}$$

Equation (34) is maximized for $\xi U\tau/\hbar = 1$. With $DWF = 0.86$, $\delta k = 2\pi/\left(900\,\mathrm{nm}\right)$, $\xi = 1.156 \times 10^{-3}$, and $N = 100$,

$$\left.\frac{Z_c}{\delta Z_c}\right|_{optimum} = \frac{Z_c}{0.074\,\mathrm{nm}} . \tag{35}$$

With 16 independent measurements of $\langle P_{\uparrow}\rangle - \langle P_{\uparrow}\rangle_{bck}$ in 1 s, this corresponds to a limiting sensitivity of $\sim (20\,\mathrm{pm})/\sqrt{\mathrm{Hz}}$. The optimum value for the signal-to-ratio of Eq. (34) scales as $1/\xi$. By employing spin-squeezed states that have been demonstrated in this system [3], $Z_c/\delta Z_c$ can be improved by another factor of 2.

Employing this technique to sense motion on resonance with the COM mode can lead to the detection of very weak forces and electric fields. The detection of a 20 pm amplitude resulting from a 100 ms coherent drive on the 1.57 MHz COM mode is sensitive to a force/ion of 5×10^{-5} yN, corresponding to an electric field of 0.35 nV/m.

* kevin.gilmore@colorado.edu
† john.bollinger@nist.gov

[1] S. Kotler, N. Akerman, Y. Glickman, A. Keselman, and R. Ozeri, Nature **473**, 61 (2011).
[2] J. W. Britton, B. C. Sawyer, A. C. Keith, C.-C. J. Wang, J. K. Freericks, H. Uys, M. J. Biercuk, and J. J. Bollinger, Nature **484**, 489 (2012) (see Supplementary Information).
[3] J. G. Bohnet, B. C. Sawyer, J. W. Britton, M. L. Wall, A. M. Rey, M. Foss-Feig, and J. J. Bollinger, Science **352**, 1297 (2016) (see Supplementary Information).
[4] D. B. Hume, C. W. Chou, D. R. Leibrandt, M. J. Thorpe, D. J. Wineland, and T. Rosenband, Physical Review Letters **107**, 243902 (2011).