Amplitude sensing with a trapped-ion mechanical oscillator - Supplemental Material

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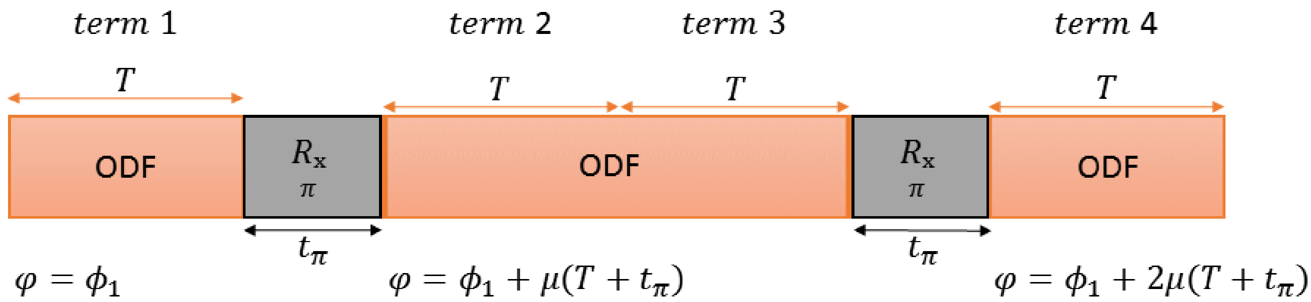

FIG. 1. $n = 2$ CPMG sequence with total ODF interaction time $4T$. φ is the phase of the ODF beatnote. The term labels represent the periods over which the accumulated phase is considered in the text.

Lineshape

To model the lineshape of the signal, it is necessary to account for the accumulated phase due to the spin-dependent ODF potential without making the simplification that $\omega = \mu$. This results in a characteristic response function for each sequence. For this Letter, we used an $n = 8$ CPMG sequence, as shown in Fig. 2 in the main text. In the following, we derive the lineshape of this sequence, seen in Fig. 2 (d). In general, for an n π pulse CPMG sequence it is necessary to calculate the phase evolution during $2n$ terms of length T, for a total interaction time of $2n$T. For simplicity, we first derive the lineshape for the $n = 2$ CPMG sequence (Fig. 1).

As shown in the main text, the expression for population in spin-up - now with a dependence on the ODF difference frequency μ - is:

$$\langle P_\uparrow \rangle = \frac{1}{2}\left[1 - e^{-\Gamma\tau} J_0(\theta_{max}(\mu))\right], \tag{1}$$

where $\theta(\mu)$ is the μ-dependent precession angle. In general,

$$\theta(\mu) = \int_0^\infty \theta(\mu, \omega) P(\omega) d\omega \tag{2}$$

where $\theta(\mu, \omega)$ is the response function of the sequence and $P(\omega)$ is the power spectral density. For a delta function source $Z_c \cos(\omega t + \delta)$,

$$\theta(\mu) = DWF \cdot U \cdot \delta k \cdot Z_c \frac{2\sin\left(\frac{1}{2}(\omega - \mu) T\right)}{(\omega - \mu)} f(\mu, \omega) \tag{3}$$

where $f(\mu, \omega)$ is determined by the phase accumulated through a particular sequence. In the case of the $n = 2$ CPMG sequence, the phase accumulated through 4 terms must be considered:

$$term\ 1 = \cos\left[(\omega - \mu)\frac{T}{2} + \delta + \phi_1\right], \tag{4}$$

$$term\ 2 = -\cos\left[(\omega - \mu)\left(\frac{3T}{2} + t_\pi\right) + \delta + \phi_1 + \mu(T + t_\pi)\right], \tag{5}$$

$$term\ 3 = -\cos\left[(\omega - \mu)\left(\frac{5T}{2} + t_\pi\right) + \delta + \phi_1 + \mu(T + t_\pi)\right], \tag{6}$$

$$term\ 4 = \cos\left[(\omega-\mu)\left(\frac{7T}{2}+2t_\pi\right)+\delta+\phi_1+2\mu(T+t_\pi)\right]. \tag{7}$$

Adding these terms up, pairwise:

$$term\ 1 + term\ 2 = 2\sin\left(\frac{1}{2}\left[(\omega-\mu)\left(T+t_\pi\right)+\mu(T+t_\pi)\right]\right)\sin\left[(\omega-\mu)\left(T+\frac{t_\pi}{2}\right)+\delta+\phi_1+\frac{\mu(T+t_\pi)}{2}\right], \tag{8}$$

$$term\ 3 + term\ 4 = -2\sin\left(\frac{1}{2}\left[(\omega-\mu)\left(T+t_\pi\right)+\mu(T+t_\pi)\right]\right)\sin\left[(\omega-\mu)\left(3T+\frac{3t_\pi}{2}\right)+\delta+\phi_1+\frac{3\mu(T+t_\pi)}{2}\right]. \tag{9}$$

$f(\mu,\omega)$ is the sum of all four terms:

$$f(\mu,\omega) = 2\sin\left(\frac{\omega}{2}\left(T+t_\pi\right)\right)\left[\sin\left(\xi+\delta+\phi\right)-\sin\left(3\xi+\delta+\phi\right)\right], \tag{10}$$

where $\xi = (\omega-\mu)(T+\frac{t_\pi}{2})+\frac{\mu(T+t_\pi)}{2} = \frac{1}{2}\left(\omega(T+t_\pi)+T\left(\omega-\mu\right)\right)$. Then, simplifying:

$$f(\mu,\omega) = 2\sin\left(\frac{\omega}{2}\left(T+t_\pi\right)\right)2\sin\left(-\xi\right)\cos\left(2\xi+\delta+\phi\right). \tag{11}$$

Using Eqs. 11 and 3,

$$\theta(\mu) = DWF\cdot U\cdot\delta k\cdot Z_c\cdot T\,\mathrm{sinc}\left(\frac{T}{2}\left(\omega-\mu\right)\right)4\sin\left(\frac{\omega}{2}\left(T+t_\pi\right)\right)\sin\left(\xi\right)\cos\left(2\xi+\delta+\phi\right). \tag{12}$$

Since $4T=\tau$ for the $n=2$ CPMG

$$\theta(\mu) = \theta_{max}\,\mathrm{sinc}\left(\frac{T}{2}\left(\omega-\mu\right)\right)\sin\left(\frac{\omega}{2}\left(T+t_\pi\right)\right)\sin\left(\xi\right)\cos\left(2\xi+\delta+\phi\right) \tag{13}$$

$\theta_{max}(\mu)$, defined as $\theta(\mu)=\theta_{max}(\mu)\cos\left(2\xi+\delta+\phi\right)$, is the μ-dependent generalization of θ_{max}. From Eq. 13, this is

$$\theta_{max}(\mu) = \theta_{max}\,\mathrm{sinc}\left(\frac{T}{2}\left(\omega-\mu\right)\right)\sin\left(\frac{\omega}{2}\left(T+t_\pi\right)\right)\sin\left(\xi\right). \tag{14}$$

For the $n=8$ CPMG sequence the same procedure is used, but now with 16 periods of accumulated phase. And so, the response function is given by:

$$\theta_{max}(\mu) = \theta_{max}\,\mathrm{sinc}\left(\frac{T}{2}\left(\omega-\mu\right)\right)\sin\left(\frac{\omega}{2}\left(T+t_\pi\right)\right)\sin(\xi)\cos(2\xi)\cos(4\xi). \tag{15}$$

Incoherent sensing limits

Following earlier discussions, the probability of measuring $|\uparrow\rangle$ at the end of the Ramsey sequence is

$$\langle P_\uparrow\rangle = \frac{1}{2}\left[1-e^{-\Gamma\tau}J_0\left(\theta_{max}\right)\right], \tag{16}$$

where $\langle\,\rangle$ denotes an average over many experimental trials and therefore over the random phase between the 1D optical lattice and the classically driven COM motion, and

$$\theta_{max} = DWF\cdot(U/\hbar)\cdot\delta k\,Z_c\cdot\tau\,. \tag{17}$$

Defining $F\left(\theta_{max}^2\right) \equiv \left(1 - J_0\left(\theta_{max}\right)\right)/2$ and denoting $\langle P_\uparrow \rangle_{bck} = \left[1 - e^{-\Gamma\tau}\right]/2$ as the probability of measuring $|\uparrow\rangle$ at the end of the sequence in the absence of a classically driven motion, θ_{max}^2 can be determined from a measurement of the difference $\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}$ through

$$F\left(\theta_{max}^2\right) = e^{\Gamma\tau}\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right) . \tag{18}$$

The standard deviation $\delta\theta_{max}^2$ in estimating θ_{max}^2 is determined from the standard deviation $\delta\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right)$ of the $\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}$ difference measurements through

$$\delta\theta_{max}^2 = \frac{e^{\Gamma\tau}\delta\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right)}{\frac{\mathrm{d}F\left(\theta_{max}^2\right)}{d\theta_{max}^2}} . \tag{19}$$

The signal-to-noise ratio of a measurement of θ_{max}^2 (and therefore Z_c^2) is then $\theta_{max}^2/\delta\theta_{max}^2$. In general this signal-to-noise ratio depends on θ_{max}^2 and the experimental parameters $U \cdot \tau$ and $\Gamma \cdot \tau$.

We use Eq. (19) to theoretically estimate $\theta_{max}^2/\delta\theta_{max}^2$ and the amplitude sensing limits. We assume the only sources of noise are projection noise in the measurement of the spin state and fluctuations in $P_\uparrow$ due to the random variation in the relative phase of the 1D optical lattice and the driven COM motion. In this case $\delta\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right) = \sqrt{\delta\langle P_\uparrow \rangle^2 + \delta\langle P_\uparrow \rangle_{bck}^2}$ where the relevant variances are

$$\delta\langle P_\uparrow \rangle_{bck}^2 = \frac{1}{N}\langle P_\uparrow \rangle_{bck}\left(1 - \langle P_\uparrow \rangle_{bck}\right) = \frac{1}{4N}\left(1 - e^{-2\Gamma\tau}\right) \tag{20}$$

and

$$\delta\langle P_\uparrow \rangle^2 = \sigma_\phi^2 + \frac{1}{N}\langle P_\uparrow \rangle\left(1 - \langle P_\uparrow \rangle\right) . \tag{21}$$

Here N is the number of spins. Equation (20) and the second term in Eq. (21) are projection noise. The variance

$$\sigma_\phi^2 = \left\langle P_\uparrow^2 - \langle P_\uparrow \rangle^2 \right\rangle = \frac{e^{-2\Gamma\tau}}{8}\left(1 + J_0\left(2\theta_{max}\right) - 2J_0\left(\theta_{max}\right)^2\right) \tag{22}$$

is due to the random variation in the relative phase of the 1D optical lattice and the driven COM motion. For our set-up, $DWF = 0.86$ and $\delta k = 2\pi/\left(900\,\mathrm{nm}\right)$ are fixed, and the decoherence Γ is a function of U, $\Gamma = \xi\left(U/\hbar\right)$ where $\xi = 1.156 \times 10^{-3}$. For a given Z_c we use Eqs. (17) and (19)-(22) to find the optimum $\theta_{max}^2/\delta\theta_{max}^2$ as a function of $\left(U\tau\right)/\hbar$. This optimum value is the theoretical curve plotted in Figure (amplitude sensing limits) of the main text.

One can show that $\theta_{max}^2/\delta\theta_{max}^2$ is optimized for relatively small values of θ_{max}^2 where $F\left(\theta_{max}^2\right) \approx \theta_{max}^2/8$ is a good approximation. This leads to some simplifications for Eqs. (18) and (19),

$$\theta_{max}^2 \approx 8e^{\Gamma\tau}\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right) \tag{23}$$

and

$$\delta\theta_{max}^2 \approx 8e^{\Gamma\tau}\delta\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right) , \tag{24}$$

and to the following estimate for the signal-to-noise ratio of a single measurement,

$$\frac{\theta_{max}^2}{\delta\theta_{max}^2} \approx \frac{\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}}{\delta\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right)} . \tag{25}$$

Figure (amplitude sensing limits) of the main text uses Eq. (25), along with repeated measurements of $\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}$, to experimentally determine the signal-to-noise ratio as a function of the amplitude Z_c of the COM motion.

Finally we use Eqs. (17) and (19)-(22) to calculate the sensing limits for very small Z_c. For small Z_c the variance σ_ϕ^2 can be neglected compared to projection noise and $\delta\langle P_\uparrow \rangle^2 \approx \delta\langle P_\uparrow \rangle_{bck}^2$. In this case we obtain the following expression for the signal-to-noise ratio,

$$\frac{\theta_{max}^2}{\delta\theta_{max}^2} = \frac{\sqrt{N}}{4\sqrt{2}}\frac{DWF^2 \cdot \left(\delta k\, Z_c\right)^2\left(U\tau/\hbar\right)^2}{\sqrt{e^{2\xi U\tau/\hbar} - 1}} . \tag{26}$$

Equation (26) is maximized for $\xi U \tau \approx 1.9603$. With $DWF = 0.86$, $\delta k = 2\pi/\left(900\,\mathrm{nm}\right)$, $\xi = 1.156 \times 10^{-3}$, and ~~$N = 100$,~~

$$\left.\frac{\theta^2_{max}}{\delta\theta^2_{max}}\right|_{optimum} = \left[\frac{Z_c}{0.196\,\mathrm{nm}}\right]^2 \tag{27}$$

For our set-up and available ODF power, $\xi U \tau/\hbar \approx 1.9603$ is realized for $\tau \approx 20$ ms. A measurement of the signal and a measurement of the background requires ~ 60 ms for 16 independent measurements of $\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}$ in 1 s. The limiting sensitivity is approximately $\left(100\,\mathrm{pm}\right)^2$ in a 1 s measurement time, or $\left(100\,\mathrm{pm}\right)^2/\sqrt{\mathrm{Hz}}$. We note that the limiting sensitivity is determined by the ratio $\xi = \Gamma/\left(U/\hbar\right)$. In particular, the optimum value for Eq. (26) scales as $1/\xi^2$.

Coherent sensing limits

With appropriate care the 1D optical lattice can be stable for long periods of time with respect to the ion trapping electrodes [Hume, PRL], enabling phase coherent sensing of a COM motion $Z_c \cos(\omega t)$. In this case the same spin precession $\theta_{max} = DWF \cdot (U/\hbar) \cdot \delta k\, Z_c \cdot \tau$ occurs for each experimental trial, which can be detected to first order in θ_{max} (or Z_c) in a Ramsey sequence with a $\pi/2$ phase shift between the two π-pulses. Assuming $\sin\left(\theta_{max}\right) \approx \theta_{max}$, appropriate for small amplitudes Z_c, the equivalent coherent sensing expressions for Eqs. (23) and (24) are

$$\theta_{max} = 2e^{\Gamma\tau}\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right) \tag{28}$$

and

$$\delta\theta_{max} = 2e^{\Gamma\tau}\delta\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right) . \tag{29}$$

For a Ramsey experiment with a $\pi/2$ phase shift, $\langle P_\uparrow \rangle_{bck} = 1/2$. If projection noise is the only source of noise, then for small Z_c, $\delta\langle P_\uparrow \rangle^2 \approx \delta\langle P_\uparrow \rangle^2_{bck} = \frac{1}{N}\cdot\frac{1}{2}\cdot\frac{1}{2}$ and $\delta\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right) \approx \frac{1}{\sqrt{2N}}$. The limiting signal-to-noise ratio $\theta_{max}/\delta\theta_{max}$ of a $\left(\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}\right)$ measurement is

$$\frac{\theta_{max}}{\delta\theta_{max}} = DWF \cdot \left(\delta k\, Z_c\right) \cdot \sqrt{\frac{N}{2}} \cdot \frac{(U\tau)}{\hbar} e^{-\xi U \tau/\hbar} . \tag{30}$$

Equation (30) is maximized for $\xi U \tau/\hbar = 1$. With $DWF = 0.86$, $\delta k = 2\pi/\left(900\,\mathrm{nm}\right)$, $\xi = 1.156 \times 10^{-3}$, and $N = 100$,

$$\left.\frac{\theta_{max}}{\delta\theta_{max}}\right|_{optimum} = \frac{Z_c}{0.074\,\mathrm{nm}} . \tag{31}$$

With 16 independent measurements of $\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bck}$ in 1 s, this corresponds to a limiting sensitivity of $\sim \left(20\,\mathrm{pm}\right)/\sqrt{\mathrm{Hz}}$. In particular the optimum value for the signal-to-ratio of Eq. (30) scales as $1/\xi$. Further, by employing spin-squeezed states that have been demonstrated in this system [1], $\theta_{max}/\delta\theta_{max}$ can be improved by another factor of 2.

Employing this technique to sense motion resonance with the COM mode can lead to the detection of very weak forces and electric fields. The detection of a 20 pm amplitude resulting from a 5 ms coherent drive on the 1.57 MHz COM mode (corresponds to a $Q \sim 10^4$) is sensitive to a force/ion of 10^{-3} yN corresponding to an electric field of 7 nV/m.

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[1] J. G. Bohnet, B. C. Sawyer, J. W. Britton, M. L. Wall, A. M. Rey, M. Foss-Feig, and J. J. Bollinger, arXiv e-prints , 1 (2015), arXiv:1512.03756.