Sensitive amplitude detection (for θ_{max})

October 28, 2016

Precession due to axial COM oscillation

From Bohnet supplementary material (S2) the spin-dependent ODF potential can be written,

$$H_{ODF} = U \sum_i \sin\left[\delta k \cdot z_i - \mu t + \phi\right] \sigma_i^z \ .$$

This assumes alignment of the 1D optical lattice with the single plane. Note that there is a mistake in the line before (S2) where U should be defined as $U \equiv \frac{1}{2} U_\uparrow$. We desire an expression for U in terms of the measured AC Stark shift for vertical (or horizontal) polarization. This is most easily done by following the supplementary material from the Britton Nature manuscript. This manuscript assumes linear polarization that is rotated by an angle ϕ_p from vertical polarization. From Equation (3) the Stark shift of the qubit transition can be written

$$\Delta_{acss} = \left(A_\uparrow - A_\downarrow\right) \cos^2\left(\phi_p\right) + \left(B_\uparrow - B_\downarrow\right) \sin^2\left(\phi_p\right)$$

where $\left(A_\uparrow - A_\downarrow\right)$ is the AC Stark shift of a single beam when the polarization is rotated to vertical and $\left(B_\uparrow - B_\downarrow\right)$ is the AC Stark shift of a single beam when rotated to horizontal polarization. $\Delta_{acss} = 0$ implies $\left(A_\uparrow - A_\downarrow\right) \cos^2\left(\phi_p\right) = -\left(B_\uparrow - B_\downarrow\right) \sin^2\left(\phi_p\right)$. Equation (4) of the Britton supplementary material shows that the spatially dependent AC Stark shift from both beams can be written

$$\left[\left(A_\uparrow - A_\downarrow\right) \cos^2\left(\phi_p\right) - \left(B_\uparrow - B_\downarrow\right) \sin^2\left(\phi_p\right)\right] \cdot 2\cos\left(\delta k \cdot z_i - \mu t\right)$$

$$= 2\left(A_\uparrow - A_\downarrow\right) \cos^2\left(\phi_p\right) \cdot 2\cos\left(\delta k \cdot z_i - \mu t\right) \ .$$

Writing this AC Stark shift in terms of Pauli spin matrices gives

$$H = 2\left(A_\uparrow - A_\downarrow\right) \cos^2\left(\phi_p\right) \cdot \cos\left(\delta k \cdot z_i - \mu t\right) \sigma_i^z \ .$$

It follows that U in the above expression for H_{ODF} is given by $U = 2\left(A_\uparrow - A_\downarrow\right) \cos^2\left(\phi_p\right)$. For vertical and horizontal polarizations U is given by $2\left(A_\uparrow - A_\downarrow\right)$ and $2\left(B_\uparrow - B_\downarrow\right)$ respectively.

We can write H_{ODF} as the sum of two terms,

$$H_{ODF} = U \sum_i \sin\left[\delta k \cdot z_i - \mu t + \phi\right] \sigma_i^z$$

$$\begin{array}{rcl} = & U \sum_i \sin\left(\delta k \cdot z_i\right) cos\left(\mu t + \phi\right) \sigma_i^z & \leftarrow term\ 1 \\ & -U \sum_i \cos\left(\delta k \cdot z_i\right) \sin\left(\mu t + \phi\right) \sigma_i^z & \leftarrow term\ 2\ . \end{array}$$

Term 1 is the term we are interested in, although at low frequencies μ, term 2 can also be important. Consider a classically driven motion of constant amplitude and phase (for any given shot of the experiment),

$$z_i \to z_i + z_0 \cos\left(\omega t + \delta\right)\ .$$

With the assumption that $\delta k \cdot z_0 \ll 1$, term 1 can be written

$$\begin{array}{rcl} term1 & = & U \sum_i \sin\left(\delta k \cdot z_i\right) \cos(\mu t + \phi) \sigma_z^i \\ & & +U \sum_i \cos\left(\delta k \cdot z_i\right) \cdot \delta k \cdot z_0 \cos\left(\omega t + \delta\right) \cos\left(\mu t + \phi\right) \sigma_i^z\ . \end{array}$$

The term on the 1st line is the usual term that gives rise to spin-motion entanglement and induces a spin-spin interaction. We assume we can neglect this term because we tune μ far from any modes. Term 1 then is approximately given by

$$term\ 1 \simeq U\, \delta k \cdot z_0 \cos\left(\omega t + \phi\right) \cos\left(\mu t + \phi\right) \sum_i \cos\left(\delta k \cdot z_i\right) \sigma_i^z\ .$$

We do experiments with a thermal distribution of mode energies. Assuming the thermal distribution of the modes and spins are not correlated, $\left\langle \cos\left(\delta k \cdot z_i\right) \sigma_i^z \right\rangle = \left\langle \cos\left(\delta k \cdot z_i\right) \right\rangle \sigma_i^z$. One can show

$$\begin{array}{rcl} \left\langle \cos\left(\delta k \cdot z_i\right) \right\rangle & = & \exp\left[-\delta k^2 \left\langle z_i^2 \right\rangle /2\right] \\ & \equiv & \exp\left[-\delta k^2 \left\langle z_i^2 \right\rangle /2\right] \end{array}$$

where the Debye-Waller factor $DWF_i \equiv \exp\left[-\delta k^2 \left\langle z_i^2 \right\rangle /2\right]$. From the Bohnet supplementary material for rotation frequencies of 180 kHz and less, DWF_i has a weak dependence of i, $DWF_i \simeq 0.86$ near the center of a 127 ion array and $DWF_i \simeq 0.88$ near the edge of the array. This is for a temperature $T = 0.5$ mK for all the modes. Doubling the temperature squares the Debye-Waller factor. We will assume that $DWF_i = DWF$ is independent of temperature.

With the identity $\cos\left(\omega t + \delta\right) \cos\left(\mu t + \phi\right) = \frac{1}{2} \cos\left[\left(\omega - \mu\right) t + \delta - \phi\right] + \frac{1}{2} \cos\left[\left(\omega + \mu\right) t + \delta + \phi\right]$ and neglecting the rapidly oscillating term which will average to zero for interaction times long compared to $2\pi/(\omega + \mu)$, term 1 is given by

$$term\ 1 = DWF \cdot U \cdot \delta k \cdot z_0 \cos\left[\left(\omega - \mu\right) t + \delta - \phi\right] \sum_i \sigma_i^z /2\ .$$

The difference in energy $\Delta(t)$ between spin-up and spin-down for each ion is simply

$$\Delta(t) = DWF \cdot U \cdot \delta k \cdot z_0 \cos\left[\left(\omega - \mu\right) t + \delta - \phi\right]\ .$$

The precession angle θ is calculated by integrating $\Delta(t)$ over the over the time interval T during which the ODF interaction is applied. The maximum precession $\Delta \cdot T$ is obtained for $\omega = \mu$,

$$\theta = \Delta \cdot T = DWF \cdot U \cdot \delta k \cdot z_0 \cdot T \cdot \cos\left[\delta - \phi\right] \equiv \theta_{max} \cos\left(\delta - \phi\right) \ .$$

The maximum precession angle occurs for $\delta = \phi$, and is

$$\theta_{max} \equiv DWF \cdot U \cdot \delta k \cdot z_0 \cdot T \ .$$

Here T is the total length of the of the time the ODF beam are on. For example, for the $2-\pi$ pulse CPMG sequence with individual 1 ms precession times, $T = 4$ ms. A measurement of the classical amplitude z_0 is equivalent to a measurement of θ_{max}.

Incoherent signal size and background

Here we consider the size of the signal for incoherent detection and on-resonance ($\mu = \omega$) drives. If a Bloch vector undergoes precession by an angle θ, then at the end of the sequence (when we rotate a Bloch vector with no precession to the South pole) the component along the South pole has length $\cos\theta$. The probability of measuring $|\uparrow\rangle$ is

$$P_\uparrow = \frac{1}{2}\left[1 - e^{-\Gamma T}\cos\theta\right] \ .$$

Here $\theta = \theta_{max}\cos\left(\delta - \phi\right)$. T as before is the total length of time (i.e. the sum of all the arm times) the ODF beams are on. $\Gamma = \frac{1}{2}\left(\Gamma_{el} + \Gamma_{ram}\right)$ is the spontaneous decay rate where $\Gamma_{el}\,(\Gamma_{ram})$ are the elastic (Raman) scattering decoherence rates as defined in the Uys PRL. For long sequences or low ODF power there will also be a contribution due to dephasing from magnetic field fluctuations. This can be modeled by a factor $\exp\left(-\phi_{rms}^2/2\right)$ that multiplies the $e^{-\Gamma T}$ factor. Here we neglect magnetic field fluctuations as it appears the highest sensitivities will occur when the background is limited by spontaneous scattering from the ODF laser beams. We want to calculate the average (over many measurements) $\langle\cos\left(\theta\right)\rangle$. If the distribution of θ measurements was Gaussian, then one can show $\langle\cos\theta\rangle = \exp\left(-\left\langle\theta^2\right\rangle/2\right)$. However the θ measurements are not normally distributed, and the average can be calculated explicitly. With $\phi' \equiv \delta - \phi$, we want to calculate $\langle\cos\left(\theta\right)\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi' \cos\left[\theta_{max}\cos\left(\phi'\right)\right]$. With the identity (see Wikipedia page on Bessel functions)

$$e^{i\theta_{max}\cos\phi'} = \sum_{n=-\infty}^{+\infty} i^n J_n\left(\theta_{max}\right) e^{in\phi'} \ ,$$

where $J_n(\theta_{max})$ is the Bessel function of integer order n, one can show

$$\langle\cos\left(\theta\right)\rangle = J_0\left(\theta_{max}\right)$$

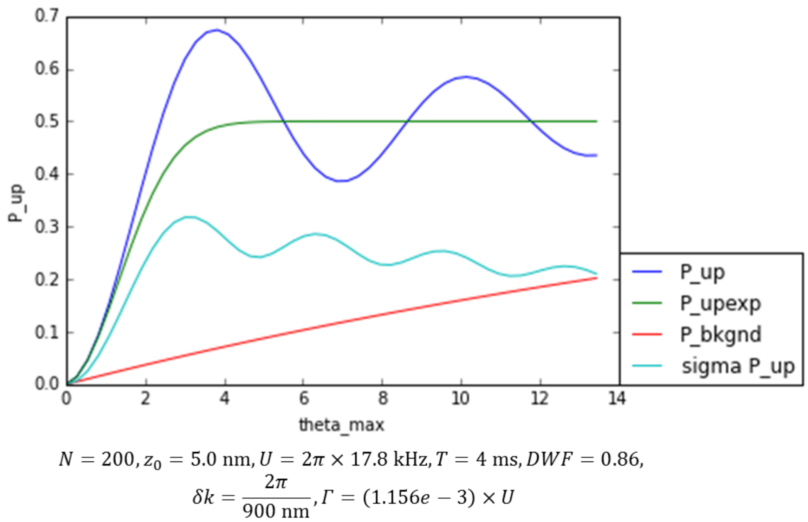

Figure 1: Calculation of the incoherent signal on resonance with a narrow frequency drive. The parameters are indicated on the figure. U sets the largest θ_{max}, and calculations are done with an AC Stark shift $f \times U$ with f varying from 0 to 1.

and

$$\langle P_\uparrow \rangle = \frac{1}{2}\left[1 - e^{-\Gamma T} J_0\left(\theta_{max}\right)\right] \ .$$

For small θ_{max}, $J_0\left(\theta_{max}\right) \simeq 1 - \theta_{max}^2/4$, so small angle expansion the same as before. For large θ_{max}, $J_0\left(\theta_{max}\right) \simeq \left(1/\sqrt{\theta_{max}}\right) \cdot \cos\left(\theta_{max} + \cdots\right)$, so it appears that the approach to saturation will go more slowly than with $\langle \cos\theta \rangle = \exp\left(-\left\langle \theta^2 \right\rangle/2\right)$.

We can also calculate the variance in the measurements due to the shot-to-shot fluctuations in $\cos\left(\theta\right)$. With some straight forward algebra one can show

$$\sigma_{P_\uparrow}^2 = \frac{1}{2\pi}\int_0^{2\pi} d\phi' \left(P_\uparrow - \langle P_\uparrow \rangle\right)^2 = \frac{e^{-2\Gamma T}}{8}\left(1 + J_0\left(2\theta_{max}\right) - 2J_0(\theta_{max})^2\right) \ . \quad (1)$$

Figure 1 shows an example calculation of $\langle P_\uparrow \rangle$ for both the Bessel function evaluation and the approximate exponential treatment of saturation, $\langle P_{bkgnd} \rangle$, and $\sigma_{P_\uparrow}$ for $DWF = 0.86$, $U = 2\pi \times 17.8$ kHz, $\delta k = 2\pi/\left(900\,\text{nm}\right)$, $z_0 = 5\,\text{nm}$, $T = 4\,\text{ms}$.

Signal-to-noise

The goal is to determine a value for θ_{max} from which z_0 can be determined. To do this we look at the difference between $\langle P_\uparrow \rangle = \frac{1}{2}\left[1 - e^{-\Gamma T} J_0\left(\theta_{max}\right)\right]$ and $\langle P_\uparrow \rangle_{bkgnd} = \frac{1}{2}\left[1 - e^{-\Gamma T}\right]$. Solving for $J_0\left(\theta_{max}\right)$, one obtains,

$$J_0\left(\theta_{max}\right) = 1 - 2e^{\Gamma T}\left\{\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bkgnd}\right\} .$$

Once $J_0\left(\theta_{max}\right)$ is determined, the Bessel function is inverted to determine θ_{max}. (This inversion is not unique for large θ_{max}, but usually we will be operating with smaller θ_{max} for optimum sensitivity.) The uncertainty $\delta J\left(\theta_{max}\right)$ in the estimate of $J\left(\theta_{max}\right)$ will contain contributions from $\sigma_{P_\uparrow}$ (Eq. 1) and projection noise from the $\langle P_\uparrow \rangle$ and $\langle P_\uparrow \rangle_{bkgnd}$ measurements,

$$\begin{aligned}
\delta\langle P_\uparrow \rangle &= \frac{1}{\sqrt{N}}\sqrt{\langle P_\uparrow \rangle\left(1 - \langle P_\uparrow \rangle\right)} \\
\delta\langle P_\uparrow \rangle_{bkgnd} &= \frac{1}{\sqrt{N}}\sqrt{\langle P_\uparrow \rangle_{bkgnd}\left(1 - \langle P_\uparrow \rangle_{bkgnd}\right)} = \frac{1}{\sqrt{N}}\sqrt{\frac{1}{4}\left[1 - e^{-2\Gamma T}\right]} .
\end{aligned}$$

We model

$$\delta J_0\left(\theta_{max}\right) = 2e^{\Gamma T}\cdot\sqrt{\sigma_{P_\uparrow}^2 + \delta\langle P_\uparrow \rangle^2 + \delta\langle P_\uparrow \rangle_{bkgnd}^2} .$$

We now estimate the uncertainty $\delta\theta_{max}$ in the θ_{max} estimate from

$$\delta\theta_{max} \equiv \delta J_0\left(\theta_{max}\right) / \left(\frac{dJ_0\left(\theta_{max}\right)}{d\theta_{max}}\right) .$$

Figure 2 shows a calculation of the signal-to-noise $S/N|_{single} = \theta_{max}/\delta\theta_{max}$ for the same parameters as Fig. 1. This is the effective single-to-noise for a single measurement. One expects the signal-to-noise ratio for n measurements to scale (i.e improve) as $\sqrt{n}$. $S/N \sim 2$ is obtained for $1 < \theta_{max} < 1.5$.

Figure 3 shows calculations for $z_0 = 1\,\mathrm{nm}$ and $T = 6\,\mathrm{ms}$, but otherwise the parameters the same as Fig. 1. Once again for $1 < \theta_{max} < 2$, S/N of ~ 2 is obtained.

Figure 4 shows $\theta_{max}/\delta\theta_{max}$ for $z_0 = 0.3\,\mathrm{nm}$. The best S/N is obtained at $\theta_{max} \simeq 1.5$. In order to obtain this angle, $T = 20\,\mathrm{ms}$. The background is now 80% of $\langle P_\uparrow \rangle$, but the calculation that an S/N close to 2 is still possible. The S/N drops to 0.5 with $z_0 = 0.1\,\mathrm{nm}$.

Optimum operating conditions

The condition for optimally detecting a classical amplitude z_0 depends on the size of z_0. The main parameter that changes is the sequence ODF on-time T. Table 1 lists the optimum operating conditions for different z_0. A basic conclusion is that it should be possible to detect a $z_0 = 0.1$ nm amplitude in 1 s by using long T.

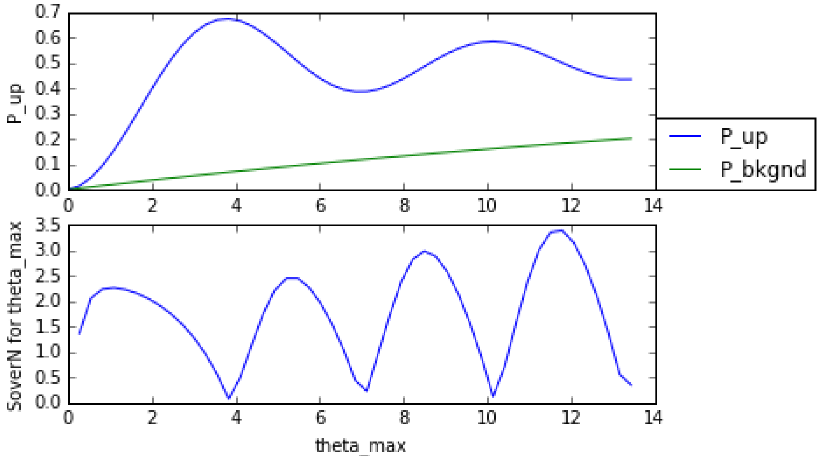

Figure 2: Estimated single measurement $\theta_{max}/\delta\theta_{max}$ for the parameters of Fig. 1. In addition to the signal-to-noise plot, $\langle P_{\uparrow}\rangle$ and $\langle P_{\uparrow}\rangle_{bkgnd}$ are shown.

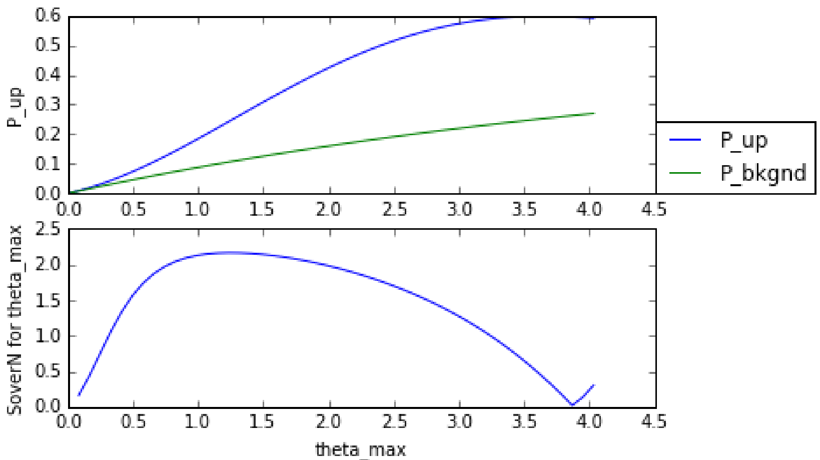

Figure 3: Single measurement $\theta_{max}/\delta\theta_{max}$ for $z_0 = 1$ nm and $T = 6$ ms and otherwise all parameters the same as Fig. 1.

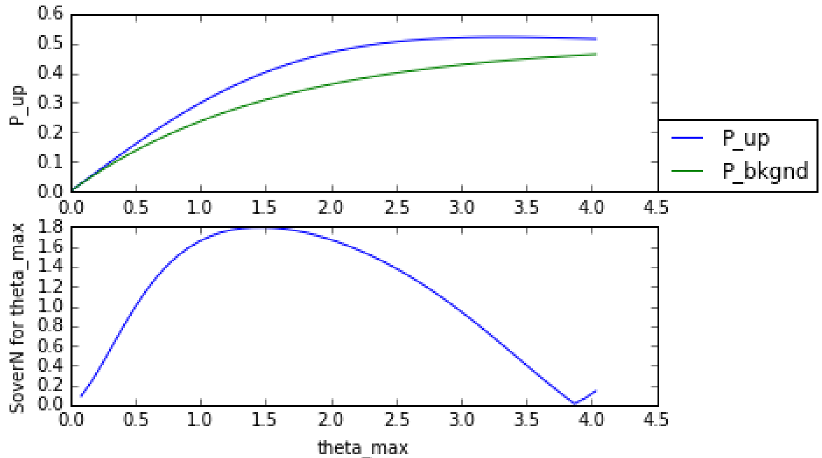

Figure 4: Single measurement $\theta_{max}/\delta\theta_{max}$ for $z_0 = 0.3$ nm and $T = 20$ ms. All other parameters are the same as Fig. 1.

z_0 (nm)	θ_{max} for best S/N	T required for optimum θ_{max}	S/N	n (number of measurements in 1 s)	S/N (1 s)	$z_0/(S/N$ (1 s))
5	1.0	0.6 ms	2.3	94	22.3	0.22
2	1.2	2 ms	2.2	83	20.0	0.1
1	1.25	4 ms	2.2	71	18.5	0.05
0.5	1.4	8.5 ms	2.0	54	14.7	0.034
0.2	1.4	20 ms	1.4	33	8.0	0.025
0.1	0.9	26 ms	0.56	27	2.9	0.034
0.05	0.5	30 ms	0.16	25	0.8	0.063

Table 1: List of the optimum operating conditions for different classical amplitudes z_0. The optimum operating condition is defined as that which gives the best S/N for a single measurement. For generating the conditions, in particular T, a value for $U = 2\pi \times 8.9\,\mathrm{kHz}$ is assumed. $n = (1\,\mathrm{s})\,/\,(T + 10\,\mathrm{ms})$. θ_{max} is in radians. A ratio $\Gamma/U = 1.156e-3$ is assumed, appropriate for experimental conditions when the AC Stark shift of a single beam is nulled. The number if ions is assumed to be 200.

z_0 (nm)	θ_{max}	T (ms)	S/N	S/N (1 s)	$z_0/(S/N$ (1 s)) (nm)
5		4	0.7	5.9	0.85
2		4	1.6	13.5	0.15
1	1.25	4	2.2	18.5	0.054
0.7		4	2.0	16.9	0.041
0.5		4	1.6	13.5	0.037
0.4		4	1.2	10.1	0.040
0.3		4	0.8	6.7	0.045
0.2		4	0.37	3.1	0.065
0.1		4	0.1	0.84	0.12

Table 2: List of the S/N obtained for different z_0, but with T optimized for $z_0 = 1$ nm. θ_{max} varies linearly with z_0. $n = 71$ experiments in 1 second is assumed.

The final column for $z_0/(S/N$ (1 s)) in Table 1 is the uncertainty of a 1 second measurement of z_0. However, it should be interpreted as the uncertainty on a measurement of an amplitude of approximately the same size as given in column 1, not the sensitivity for measuring a much smaller amplitude for the given T of the row. It is interesting to choose the conditions for a given row, say for $z_0 = 1$ nm, and ask what the sensitivity is for measuring smaller amplitudes at a total measurement time of 1 s. This is done in Table 2 where $T = 4$ ms is chosen to optimize the sensitivity for detecting $z_0 = 1$ nm. The $T = 4$ ms sequence still looks useful for detecting a $z_0 = 0.2$ nm amplitude, although with S/N reduced by a factor of 2 to 3.

Scaling with ion number N

The scaling of the amplitude sensitivity with the ion number N appears to be soft. In the regime where $\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bkgnd}$ is large, then the contribution to the noise $\delta J_0(\theta_{max})$ is dominated by $\sigma_{P_\uparrow}$. Both $\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bkgnd}$ and $\sigma_{P_\uparrow}$ are independent of N, so the S/N of the measurement will be relatively independent of N. In this regime, $S/N \sim$2 to 2.5. As the amplitude z_0 decreases, eventually $\langle P_\uparrow \rangle - \langle P_\uparrow \rangle_{bkgnd}$ will be small compared to $\langle P_\uparrow \rangle_{bkgnd}$. Projection noise becomes important and the S/N ratio starts dropping. The condition where $\sigma_{P_\uparrow}$ is approximately equal to projection noise,

$$\sigma_{P_\uparrow} \approx \sqrt{2} \cdot \delta \langle P_\uparrow \rangle_{bkgnd} = \frac{\sqrt{2}}{\sqrt{N}} \sqrt{\frac{1}{4}\left[1 - e^{-2\Gamma T}\right]}, \tag{2}$$

sets an “ease of measurement” sensitivity limit. Classical amplitudes z_0 for which $\sigma_{P_\uparrow}$ is smaller than projection noise require long integration times. For small θ_{max}, I calculate $\sigma_{P_\uparrow} = \frac{\sqrt{3}}{16} e^{-\Gamma T} \theta_{max}^2$, which gives a sensitivity limit

$$\theta_{max} \geqq \frac{1}{N^{1/4}} \cdot e^{\Gamma T/2} \left(\frac{1}{6}\left[1 - e^{-2\Gamma T}\right]\right)^{1/4}.$$

The scaling with N goes as $N^{-1/4}$. Note that with long integration times in principle one can detect θ_{max} smaller than the above inequality. For this regime the S/N will scale as $N^{-1/4}$. This scaling derives from the fact that our measured signal is actually proportional to θ_{max}^2, and projection noise on this signal scales as $N^{-1/2}$.

Averaging down with incoherent sensing

Suppose we want to detect a phase coherent signal imbedded in broad background noise. The power of lock-in or phase coherent detection is that one can eventually detect the phase coherent signal that is buried in the broad, continuous background noise. Is this possible with incoherent sensing (or decoherence spectroscopy, as Roee called it)? I do not think so, but this requires looking more closely at the spectral response function for an incoherent sensing sequence.