problem 1

Newton Method for Non linear system

Solution

we have,

$$J|_{x_0}(x^1 - x^0) = -r^0 \quad \text{--- (i)}$$

where,

$$J|_{x_0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$r_0 = \begin{bmatrix} f_1(x_0) \\ f_2(x_0) \\ \vdots \\ f_n(x_0) \end{bmatrix}$$, x_0 is a vector of initial guess

Now, $x^1 - x^0 = \Delta^1$ --- (ii)

From (i) and (ii)

$$J|_{x_0}(\Delta^1) = -r^0$$

or, $\Delta^1 = -\left[J|_{x_0}^{-1} \cdot r^0\right]$

$$\boxed{\text{and } x^1 = \Delta^1 + x_0}$$

Now,

Given. system:

$$\begin{cases} x+y^2+z-1=0 \\ x^2+y+z-6=0 \\ x+y+z^2-4=0 \end{cases} \quad , \quad x^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(1) setting up the Jacobian metsix ; for this case the ~~deri~~ partial derivatives are

$$\frac{\partial f_1}{\partial x} = 1 \qquad \frac{\partial f_1}{\partial y} = 2y \quad , \quad \frac{\partial f_1}{\partial z} = 1$$

$$\frac{\partial f_2}{\partial x} = 2x \qquad \frac{\partial f_2}{\partial y} = 1 \quad , \quad \frac{\partial f_2}{\partial z} = 1$$

$$\frac{\partial f_3}{\partial x} = 1 \quad , \quad \frac{\partial f_3}{\partial y} = 1 \quad , \quad \frac{\partial f_3}{\partial z} = 2z$$

Hence, the Jacobian matsix ~~is~~,

$$J|_{x_0} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad J|_{x_0}^{-1} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \end{bmatrix}$$

$$r_0 = f(x_0) = \begin{bmatrix} -4 \\ -6 \\ -4 \end{bmatrix}$$

Now

we have,

$$J|_{x_0} \Delta^1 = -r^0$$

or $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$$

where $\Delta^1 = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$

Now, solving the above system:

$\delta_1 + \delta_3 = 4$ — (i)

$\delta_2 + \delta_3 = 6$ — (ii)

$\delta_1 + \delta_2 = 4$ — (iii)

now, From (i) and (ii), we get

$\delta_1 + 6 - \delta_2 = 4$ — (iv)

$\delta_1 - \delta_2 = -2$

From (iii) and (iv)

$\delta_1 + \delta_1 + 2 = 4$

or, $2\delta_1 = 2$

or $\delta_1 = 1$

Again

Solving the system, we get,

$1 - \delta_2 = -2$

or $-\delta_2 = -3$

or, $\delta_2 = -3$

And

$1 + \delta_3 = 4$

or $\delta_3 = 3$

then $\Delta^1 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$

and we also have,

$x^1 - x^0 = \Delta^1$

or, $x^1 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$

and our new r^1 is

$r^1 = f(x^1) = \begin{bmatrix} 9 \\ 1 \\ 9 \end{bmatrix}$

Now

Another method

$$\Delta^1 = -J\big|_{x_0}^{-1} \cdot r_0$$

$$= \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} -2 \\ -6 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

Hence,

$$x_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

and ~~at~~ our new r' is

$$r' = f(x') = \begin{bmatrix} 1+9+3-4 \\ 1+3+3-6 \\ 1+3+9-4 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 9 \end{bmatrix}$$

Problem 5

Consider the data point set $\{x_k, y_k\}_{k=1,2 \dots n}$ with

$x > 0$.

Now, finding the fit ~~for~~ the form

$$y = Cx^D$$ to the above data set.

using linearization.

Now, our error vector is

$$e_k = f(x_k) - y_k$$

where $f(x_k) = Cx_k^D$

Hence, $e_k = Cx_k^D - y_k$

$$e_k^2 = (Cx_k^D - y_k)^2$$

Hence, The error to be minimized is

$$\|e\|_2^2 = \sum_{k=1}^{n} (Cx_k^D - y_k)^2$$

So, for minimum to occur, we have to perform following operation,

$$\frac{\partial \|e\|_2^2}{\partial C} = 0 \quad \text{and}$$

$$\frac{\partial \|e\|_2^2}{\partial D} = 0$$

Now,
However, solving above system will create non-linear system of equation, so we have to apply another approach to the above system
we will take log on both side:

$$y_k = Cx^D, \quad C > 0$$

or, $\log y_k = \log(Cx^D)$

or, $\log y_k = \log C + D \log x$

Let, $\log(y_k) = Y$

$\log C = B$

$\log x = X$

where X and Y are two new variables
and B and D, ~~are~~ ~~and~~ C are constants.
Now, the linearization is

$$Y_k = DX + B \quad \text{where } B = \log C$$

Thus, what is needed now is to fit the transformed set of data points

$$\{(x_k, y_k)\} \rightarrow \{(x_k, \log y_k)\}$$

$$= \{X_k \; Y_k\}$$

to the straight line ~~Ø~~ $DX + B$

Now.

$$\|e\|_2^2(D, B) = \sum_{k=1}^{n} \left(D\log x + \log c - \log y_k\right)^2,$$