Q no 3 problem 3

Gauss Quadrature

For $N = 2$

we have,

$$I = \int_{-1}^{1}\int_{x^2-1}^{1-x^2} (x^2 - xy + y^2)\, dy\, dx$$

Solving

$$= \int_{-1}^{1} g(x) \cdot dx = w_1 g(x_1) + w_2 g(x_2)$$

Solving analytically, we have,

$$x_1 = \frac{1}{\sqrt{3}}, \quad x_2 = -\frac{1}{\sqrt{3}}$$

$$g(x_1) = \int_{-2/3}^{2/3} \left(\frac{1}{3} - \frac{1}{\sqrt{3}} y + y^2\right) \cdot dy$$

$$= \left[\frac{1}{3} y - \frac{1}{\sqrt{3}} \cdot \frac{y^2}{2} + \frac{y^3}{3}\right]_{-2/3}^{2/3}$$

$$= \left[\frac{1}{3} \cdot \frac{2}{3} - \frac{1}{\sqrt{3}} \cdot \frac{4}{18} + \frac{8}{27 \times 3}\right] - \left[-\frac{2}{9} - \frac{1}{\sqrt{3}} \cdot \frac{4}{18} - \frac{8}{27 \times 3}\right]$$

$$= 0.1927 - 0.4493$$

$$= 0.6420$$

Again

$X_2 = -\frac{1}{\sqrt{3}}$

$$g(x_2) = \int_{-2/3}^{2/3} \left(\frac{1}{3} + \frac{1}{\sqrt{3}}y + y^2\right) dy$$

$$\Rightarrow \left[\frac{1}{3}y + \frac{1}{2\sqrt{3}}y^2 + \frac{y^3}{3}\right]_{-2/3}^{2/3}$$

$$\Rightarrow \left[\frac{1}{3}\times\frac{2}{3} + \frac{1}{2\sqrt{3}}\cdot\left(\frac{2}{3}\right)^2 + \frac{\left(2/3\right)^3}{3}\right] - \left[-\frac{2}{9} + \frac{1}{2\sqrt{3}}\cdot\left(\frac{2}{3}\right)^2 - \frac{\left(\frac{2}{3}\right)^3}{3}\right]$$

$= 0.4493 + \cancel{0.00}\ 0.1927$

$= 0.6420$

Hence,

$$I = \int_{-1}^{1} g(x_0)\, dx$$

Hence and

$$I = \int_{-1}^{1} g(x) = w_1 g(x_1) + w_2 g(x_2) \quad \text{wher } w_1 = w_2 = 1$$

$= 0.6420 \times 2$

$= 1.2840$

And

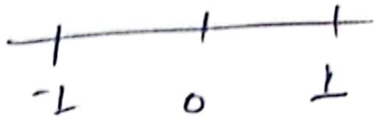
$\Rightarrow$

problem 3

Using Simpson Method, one panel

$$I = \int_{-1}^{1} g(x)\cdot dx$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right]$$



solving analytically,

Now,

$$f_0 = f(0) \cong \int_{0}^{0} (1+y+y^2)\cdot dy$$

$$= 0$$

$$f_1 = f(0) \cong \int_{-1}^{1} (y^2)\cdot dy$$

$$= \left.\frac{y^3}{3}\right|_{-1}^{1}$$

$$= \frac{1}{3} - \left(-\frac{1}{3}\right)$$

$$= \boxed{\frac{2}{3}}$$

Using Simpson's in y

or, $\frac{h}{3}\left[f(-1) + 4f_0 + f_1\right]$, $h = \frac{1+1}{2} = 1$

$\therefore \frac{1}{3}\left[1 + 0 + 1\right] \Rightarrow \frac{2}{3} = 0.6667$

$$f_1 = f(1) = \int_0^{0} (y^2) \cdot dy$$

$$\Rightarrow 0$$

Hence,

$$I = \frac{h}{3}\left[f_0 + 4 f_1 + f_2\right]$$

$$= \frac{1}{3}\left[0 + 4 \cdot \frac{2}{3} + 0\right]$$

$$\Rightarrow \frac{8}{9}$$

Hence, By Simpson's method,

$$\int_{-1}^{1}\int_{x^2-1}^{1-x^2} (x^2 - xy + y^2)\, dy\, dx = 8/9 = 0.8889$$

problem **5**

Romberg Integration

compute $R_{3,3}$

$$I = \int_0^2 x^2 e^{-x^2} dx$$

we have

$$R_{K,J} = \frac{4^{J-1} R_{K,J-1} - R_{K-1,J-1}}{4^{J-1} - 1} \quad , J = 2, 3 - -$$

Also, At first, we have,

$$R_{1,1} = \frac{b-a}{2}[f(a) + f(b)] \quad , f(x) = x^2 e^{-x^2}$$

$$= \frac{2-0}{2}\left[0 + 2^2 e^{-4}\right]$$

$$\approx 0.0735$$

Again

$$R_{2,1} = \frac{1}{2}\left[R_{1,1} + h_1 f(a + h_2)\right] = \left[R_{K,1}\right] \quad K = 2$$

Also, $h_1 = \frac{b-a}{h_1} = \frac{2-0}{2^{K-1}} = \frac{2-0}{2^{1-1}} \Rightarrow 2$

$$h_2 = \frac{b-a}{h_2} = \frac{2-0}{2^{2-1}} \Rightarrow 1$$

$$R_{2,1} = \frac{1}{2}\left[0.0735 + 2f(1)\right]$$

$$\Rightarrow \frac{1}{2}\left[0.0735 + 2 \times \cancel{0}1 \cdot e^{-1}\right]$$

$$\Rightarrow 0.4046$$

$$R_{2,2} = \frac{4R_{2,1} - R_{1,1}}{3}$$

$$= \frac{4 \times 0.4096 - 0.0735}{3}$$

$$\approx 0.5150$$

$$R_{3,1} = \frac{1}{2}\left[R_{2,1} + h_2\left[f(a+h_3) + f(a+3h_3)\right]\right]$$

Now, $h_3 = \frac{b-a}{n_3} = \frac{b-a}{2^{3-1}} \Rightarrow \frac{2}{4} = \frac{1}{2}$

$$= \frac{1}{2}\left[0.4046 + 1\left[f\left(\tfrac{1}{2}\right) + f\left(\tfrac{3}{2}\right)\right]\right]$$

$$= \frac{1}{2}\left[0.4046 + \left[0.1947 + 0.2371\right]\right]$$

$$\Rightarrow 0.4181$$

$$R_{3,2} = \frac{4R_{3,1} - R_{2,1}}{3} \simeq \frac{4 \times 0.4181 - 0.4046}{3} = 0.4226$$

$$R_{3,3} = \frac{16R_{3,2} - R_{2,2}}{15} \approx \frac{16 \times 0.4226 - 0.5150}{15}$$

$$\Rightarrow 0.41644$$

Hence, $\boxed{R_{3,3} = 0.41644}$

Problem 2 :

$$\int_0^{2\pi} (x-1)\cos^2 x \, dx$$

solution

Since we have to calculate the integra in the interval from -1 to 1 for Gauss Quadrature rule,

so, we can ~~the~~ use change of variable defined by linear map as follows

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}\cdot dx$$

Now, we have,

$$a=0, \; b=2\pi$$

$$= \int_{-1}^{1} f\left(\frac{2\pi}{2}x + \frac{2\pi}{2}\right)\frac{2\pi}{2}\,dx$$

$$= \int_{-1}^{1} f(\pi x + \pi)\,\pi \cdot dx$$

Now, we have, $\int_{-1}^{1} f(\pi x + \pi)\cdot dx = w_1 f(x_1) + w_2 f(x_2)$

where $w_1 = w_2 = 1$, & $x_1 = \frac{1}{\sqrt{3}}$, $x_2 = -\frac{1}{\sqrt{3}}$

so, solving for $f_1(x_1)$ & $f_1(x_2)$

we have, New $f_1(x) = \left[\left[(\pi x + \pi - 1)\cos^2(\pi x + \pi)\right] \times \pi\right]$

Now,

$$f_1(x_1) = f\left(\frac{1}{\sqrt{3}}\right)$$

$$\cancel{\frac{1}{2}\left(\left(\frac{\pi}{\sqrt{3}} + \pi\right)\pi - 1\right) \cdot \cos^2\left(\left(\frac{\pi}{\sqrt{3}} + \pi\right)\pi\right)\pi}$$

$$\Rightarrow \cancel{(15.5678 - 1) \cdot \cos^2(}$$

$$\Rightarrow \left[\left(\pi \cdot \frac{1}{\sqrt{3}} + \pi - 1\right)\cos^2\left(\pi \times \frac{1}{\sqrt{3}} + \pi\right)\right] \times \pi$$

$$\Rightarrow 0.7194$$

Again $f_1(x_2) = f\left(\frac{1}{\sqrt{3}}\right)$

$$\Rightarrow \left[\pi \cdot \left(-\frac{1}{\sqrt{3}}\right) + \pi - \right]\cos^2\left(\pi \times (-)\frac{1}{\sqrt{3}} + \pi\right)\right] \times \pi$$

$$= 0.0596$$

Hence, $I = w_1 f_1 + w_2 f_2$

$$\Rightarrow 0.7194 + 0.0596$$

$$\Rightarrow 0.7791$$

3

Again

for $n = 3$

we have,

$$I = w_1 f_1 + w_2 f_2 + w_3 f_3$$

$$f(x) = \left[(\pi x + \pi - 1)\cos^2(\pi x + \pi)\right]\pi$$

Now

Finding f_1, f_2, f_3, we already have

$w_1 = 0.\overline{55}$ & $x_1 = 0.77459667$

$w_2 = 0.\overline{88}$ $\quad x_2 = 0$

$w_3 = 0.\overline{55}$ $\quad x_3 = -0.77459667$

Hence,

$$f_1(x_1) = f_1(0.77459667)$$

$$= \left[\pi \times \sqrt{3/5} + \pi - 1\right]\cos^2\left(\pi \cdot \sqrt{3/5} + \pi\right)\Big] \times \pi$$

$$= 8.2927$$

$$f_1(x_2) = f_1(0) = 6.7280$$

$$f_2(x_3) = f_1(-\sqrt{3/5})$$

$$= \left[-\pi \sqrt{3/5} + \pi - 1\right]\cos^2\left(\pi \cdot \sqrt{3/5} + \pi\right] \times \pi$$

$$= -0.5290$$

Hence

$$I = \overset{\cancel{8.29}}{w_1 f_1(x_1)} + w_2 f_1(x_2) + w_3 f_1(x_3)$$

$$= 0.\overline{55} \cdot 8.2927 + \cancel{0}.\overline{80} \cdot 6.7280 + (+0.\overline{55}) \times (-0.5290)$$

$$\Rightarrow 10.2936$$