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EXERCISE 3: BASILAR MEMBRANE MODELS

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1. INTRODUCTION

1.1. Exercise overview

This exercise examines two phenomenological classes of basilar membrane (BM) models. They do not exactly describe the biomechanical properties of the cochlea, but they can model certain key aspects in simple ways. The first class is a simplified view of the biomechanical properties of the BM and the surrounding structures of the cochlea (*transmission-line* model). This is a macromechanical model that describes the BM velocity and forces (i.e., pressure difference between the cochlear duct scalas) as well as the fluid volume velocities in the scala vestibuli. This simplified one-dimensional model enables us to encompass the important underlying mechanical properties of the BM and the surrounding fluids, while the simplification allows it to be mathematically treatable within this exercise.

Transmission-line cochlear models can be thought of as a chain of independent (but coupled) mechanical resonators (filters), representing different segments along the cochlea. These filters are driven by forces that are transferred by the fluid in the scala vestibuli where the external driving force is applied at the stapes. In this exercise, this mechanical model is treated as an equivalent electrical circuit and formulated as a cascade of filters arranged in series. The velocity (represented by the electrical current) of a single segment along the cochlea can be thought of as representing the output of a BM filter.

Although a transmission-line model is powerful because of its realistic description of the underlying mechanical dynamics, the model is still quite expensive in terms of computational complexity. Therefore, a more functional approach is better suited when "only" an estimate of the filtering properties of the auditory system is required (e.g., the front-end for macroscopic signal-processing models of sound perception, such as speech intelligibility models).

The second class of models is parallel filter models, which mainly describe the frequency selectivity properties centered at a certain frequency. The well-established *gammatone* filter model will be investigated in this exercise. The gammatone filter was developed initially to describe the shape of the impulse response function of single auditory nerve neurons recorded from the cat [1]. Later, psychoacousticians adapted this filter to the human by adjusting its parameters using the results obtained from auditory filter bandwidth estimates in behavioral experiments [2, 3]. Therefore, the gammatone filter is not, *stricto sensu*, a model of a particular location of the BM, but rather a model of the frequency selectivity of the whole auditory system. However, it has been used as a proxy of peripheral filtering in many auditory models. The gammatone filter can be implemented as a cascade of recursive digital filters, facilitating real-time signal processing. If the input signal is applied in parallel to a group of gammatone filters, each tuned to a different frequency (referred to as a *filterbank*), the output approximates the cochlear delay between adjacent filters, as low-frequency filters have slower rise-times (larger time constants) than high-frequency filters. This property is used here to demonstrate the distribution of signal energy as a function of time along the cochlear partition (i.e., along the gammatone filterbank). The gammatone auditory filterbank is often argued to provide a reasonable trade-off between computational complexity and accuracy in simulating auditory peripheral filtering properties.

1.2. Exercise objectives

After completion of this lab exercise and the associated report it is expected that you should be able to:

- Derive the voltage and current transfer function of a single segment of the transmission-line model and evaluate the simulated results with respect to the forces and velocities in the cochlea.
- Link the mechanical transmission-line model to the electrical transmission-line model and relate the simulated results to BM dynamics and auditory filters.
- Describe the consequences of considering a nonlinearity in the BM model, and how this relates to the observed behavior of the cochlea.
- Implement a single gammatone filter in MATLAB and determine the relation between its impulse response and its centre frequency.
- Explain the differences in the excitation pattern of the frequencies in the gammatone filterbank in response to a rising chirp stimulus compared to a click stimulus.

- State one characteristic where the transmission-line model and the gammatone filterbank differ from each other with respect to the shape of the auditory filters.

1.3. Exercise setup

For the first part of this exercise you will use SIMULINK which can be launched from MATLAB. The files you need are located in `BMM\BMM_Simulink`. For the second part of this exercise, you will use the MATLAB scripts located in `BMM\BMM_Matlab`.

Launch MATLAB, go to the directory where you saved the BMM folder and type:

```
startup
```

If you use your own computer, be aware that the MATLAB and SIMULINK files were created using MATLAB 2018a. Older versions of MATLAB or missing installed toolboxes may lead to compatibility problems. In case of finding such compatibility issues, please update your MATLAB installation through the academic license provided by DTU here. Alternatively, you can use one of the computers in the databar to save all the SIMULINK models with compatibility for your MATLAB version.

2. PART 1: TRANSMISSION-LINE MODEL

In this exercise, you will build your own transmission-line model by using some electrical libraries in SIMULINK, a graphical block-diagramming environment for modeling dynamical systems. To do this, your tasks are:

- Derive analytical expressions of the cochlear mechanics (sec. 2.1).
- Simulate the transmission-line model with SIMULINK (sec. 2.2).
- Extend the transmission-line model by applying a non-linear cochlear model (sec. 2.3).

Mechanical equivalent circuit of the cochlea
A strongly simplified mechanical circuit of the cochlear mechanics is shown in Fig. 1. As a good first approximation, the scala vestibuli can be thought of a fluid-filled pipe. It consists of the distributed masses $m_{s_1}, m_{s_2}, ..., m_{s_N}$, to which a series of resonators are connected. Each of the resonators is meant to represent a segment of the BM. Resonator i consists of a mass m_i, a spring k_i and a damper b_i. The model is driven by a force F applied at the stapes. This results in forces being applied to each of the resonator segments and to the masses along the pipe. The force applied to a segment causes all elements of the segment to move at a certain velocity. Assuming

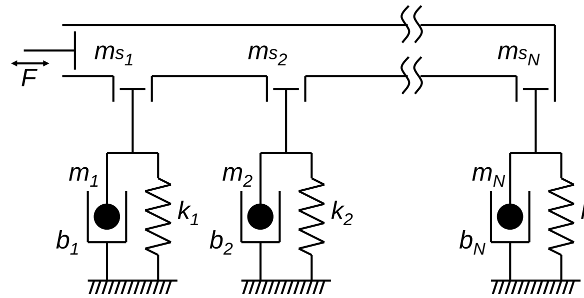

Fig. 1. Mechanical equivalent circuit of the cochlea. The driving force F is applied at the stapes. The scala vestibuli is modeled as a fluid-filled pipe and the BM as independent segments. The pressure (force) difference between the scala vestibuli and tympani, which drives the segments of the BM, is simplified as a force from the scala vestibuli only.

that F varies harmonically as $F_0 e^{j\omega t}$, the velocity v_i of element i can be obtained from the force F_i applied to it and its the mechanical impedance Z_i:

$$v_i = \frac{F_i}{Z_i} \ . \tag{1}$$

The mechanical impedance is frequency dependent and is defined for each element i as:

$$Z_i(s) = m_i s + \frac{k_i}{s} + b_i \ , \tag{2}$$

where s is the complex frequency variable $s = j\omega$. Similar equations can be solved for the whole model. The solution to each of these equations describes the velocity and forces at all segments along the BM.

Electrical-equivalent circuit of the cochlea
We use the first electro-mechanical analogy (voltage $\sim$ force) from the lecture. Table 1 shortly summarizes the relationship between mechanical and electrical elements. Using these relationships, the mechanical circuit is translated into the electrical-equivalent circuit shown in Fig. 2. All masses are replaced by inductors, L_i, the compliances of springs are replaced by capacitors, C_i, and the dampers are replaced by resistors, R_i. The parallel circuit of the mass–spring–damper system becomes a serial RLC element, because all elements that have the same velocity in a parallel mechanical circuit have to draw the same current in the electrical equivalent. The circuit is finished with a resistor R_L representing the load at the closing of the cochlear duct (or helicotrema).

2.1. Analytical analysis of a single segment

Before you start with SIMULINK, some analytical expressions should be derived. Consider the first cochlear segment (i.e., node between L_{s_1} and RLC branch with L_1, C_1 and R_1)

Element type	Electrical		Mechanical	
	Name	Impedance	Name	Impedance
Effort source	voltage [U]		force [F]	
Flow source	current [I]		velocity [v]	
Flow storage	inductor [L]	sL	mass [m]	ms
Effort storage	capacitor [C]	$1/(sC)$	spring [k]	k/s
Dissipator	resistor [R]	R	damper [b]	b

Table 1. Summary of equivalent elements in the first electro-mechanical analogy, where s represents the complex frequency variable $j\omega$.

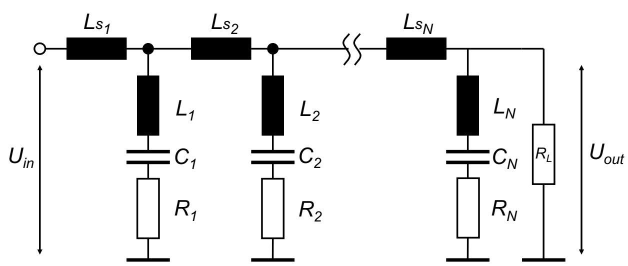

Fig. 2. Equivalent-electrical circuit for the mechanical model of the cochlea in Fig. 1.

in Fig. 2 when all other elements $L_{s_2}, L_2C_2R_2, \ldots, L_{s_N}$, $L_NC_NR_N$ are assumed to be absent.

2.1.1. *Questions*

- **[BACKGROUND]**: Derive the voltage transfer function $\left(\dfrac{U_{\text{out}}}{U_{\text{in}}}\right)$, where U_{out} is across the $L_1C_1R_1$ branch;
- **[BACKGROUND]**: Derive the current transfer function $\left(\dfrac{I}{U_{\text{in}}}\right)$.

The input voltage, U_{in}, is assumed to vary as $U_0e^{j\omega t}$. (Hint: Consider the impedances of L_{s_1} and the serial $L_1C_1R_1$ element. Assume $L_{s_1} \gg L_1$). From your results, answer the following questions:

- **[DISCUSS]**: What is the filter type of the voltage transfer function (low-pass, band-pass, high-pass)? HINT: Evaluate the transfer function for $\omega \to 0$ and $\omega \to \infty$.
- **[DISCUSS]**: What filter type describes the current as a function of frequency?
- **[DISCUSS]**: Compare the mechanical and the electrical equivalent circuits. Which forces and velocities in the cochlea can be associated with the voltage U_1 and the electrical current I_1 in the analyzed node?

2.2. Electrical circuit simulation

- **[IMPLEMENT]**: Start the circuit simulation by following the instructions below.
  1. Launch SIMULINK by typing `simulink` in the Command Window in MATLAB. Be patient, it can take a while to open.
  2. Click Open (top-left) and select `BMM_Simulink\-BMM_SingleElement.slx`.
     – The Single Element model contains a single RLC branch that you can use as a template to build the transmission-line model.
  3. Once the SIMULINK file is opened, a voltage and current spectrum analyzers as well as a time scope (`Scope Measurement LIN`) automatically open.
     – The voltage plots show the voltage U_1 measured with the `V probe` at the node between the RLC branch and the resistor R_L representing the closing of the cochlear duct.
     – The current plots show the current I_1 passing through the RLC branch measured with the `I probe`.
  4. To run the model, click on the green *play* button in the control panel (or click F5 in Windows or Cmd+T in Mac).
     – The traces in the spectrum analyzers represent the voltage and current transfer function of the analyzed node.
     – The time scope shows the temporal waveforms of the input voltage U_{in}, the measured voltage at the RLC branch U_1 and the current I_1 passing through the branch.
  5. Adjust the time and amplitude scales by using the scaling buttons (white box with arrows) in the time scope after running a simulation.
- **[IMPLEMENT]** Try two different stimuli.

  Two stimuli are ready to use: a pulse of 1 ms (default) and a chirp (sweep tone) ranging from 20 Hz to 10 kHz of a duration of 1-s. You can select one or the other by manually changing the switch with a left-button double click on it.

When using the pulse as stimulus, set the Simulation Stop Time (text panel in the control panel where the Run button is) to 0.05 s.[1] When using the chirp, set the Simulation Stop Time to 1-s.

- **[IMPLEMENT]** Assign values with `BMM_SingleElement__params.m`.

  Each model has a small associated MATLAB script that can be used to write and read data and run the model. Open the file `BMM_SingleElement__params.m` in MATLAB and explore it. You should realize how the script links to a given SIMULINK model, assigns a value to an element in the model and uses the data plotted in the time scope to be used in MATLAB. Use this integrated MATLAB- SIMULINK solution to plot the figures in MATLAB for your report.

- **[IMPLEMENT]** Build a full transmission-line model with `BMM_WholeCochlea.slx`. Here, assign different values for compliance and damping along the BM.

  1. Open the SIMULINK model named `BMM_WholeCochlea.slx`.
  2. Calculate the values for the capacitors and resistors that vary along the BM.
  3. Assign the values to the SIMULINK elements through the corresponding MATLAB script named `BMM_WholeCochlea__params.m`.

  If the parameters for all segments were kept the same, then we would assume that the BM has a constant mass, compliance and damping distribution. Instead, we should assume that the mass is constant but that compliance and damping vary with place along the BM, i.e., with segment number. According to literature (see lecture slides), the assumption of an exponentially increasing compliance along the BM seems reasonable. Therefore, the capacitors value should vary as a function of segment number i as:

  $$C_i = C_0 e^{(i-1)/2.8854} \quad i = 1...11 \ . \tag{3}$$

  C_0 is assumed to be 100 nF as in the template.

  In order to achieve a constant Q-value, Q, or loss-factor, δ, for all resonator segments, the resistors also have to vary. For a RLC serial circuit the loss-factor is:

  $$\delta = \frac{1}{Q} = R\sqrt{\frac{C}{L}} \ . \tag{4}$$

  You can calculate the values for R_i. Start with $R_1 = 150\,\Omega$ and $L_1 = 10\,\mathrm{mH}$, and assume that the loss-factor stays constant.

  These constraints result in filters with a constant Q-value, i.e., a constant center-frequency to bandwidth ratio, as it has been found in humans. However, the Q-value chosen here is certainly not realistic. The filters are too broad (the Q-value is too small) as compared to human auditory filters. This is a compromise for this exercise, since a more realistic transmission-line model would have 25 to 50 times more segments.

2.2.1. Questions

- **[RESULTS]** Analyze and compare the voltages and currents for segment 4, 6 and 8. (If you choose segments at either ends, you will notice that the discrete realization of the BM will create more and more problems, e.g., ripples in the transfer functions).
  - For investigating other branches, copy–paste the time scope `Scope Measurement LIN`, rename it and edit the MATLAB script `BMM_WholeCochlea__params.m` to read the `Time` and `Data` vectors to MATLAB.
  - To rename a scope, double click on it to open the scope window, then go to “Configuration properties” (gear icon) > “Logging” > “Variable name”
- **[RESULTS]** Plot the current magnitude response (proportional to BM velocity) as a function of frequency. For better visualization keep Y-axis linear. Use the pulse stimulus and plot the impulse response.
- **[RESULTS]** Plot the voltage transfer function. Use a log/log scale (Y-axis log) in this case.
- **[DISCUSS]** Which filter type corresponds to the voltage driving a segment and which filter type corresponds to the current drawn by a segment?
- **[DISCUSS]** Compare your results to the analysis of the single segment.
- **[DISCUSS]** Explain your understanding of the physical meaning for this morphological change in the compliance along the BM.

2.3. Non-linear cochlear model

As an extension of the transmission-line model, a simple modification can be applied in order to account for the cochlear non-linearities.[2] Before converting one of the cochlear segments into a non-linear branch, it can be useful to simulate the response of a linear cochlear segment as a function of input intensity.

First, connect the scope named `Scope_Measurement_NL` to the 6^{th} RLC branch. You must read out the

[1] In this case, when importing the signal to MATLAB the resolution can be too low. Please, increase the time to 1-s if you aim to plot the spectrum in MATLAB

[2] The model is a simplified version of [4]

current I_6 by connecting the scope to the `I_probe` box (as done in the previous linear section). Open the MATLAB script `BMM_WholeCochlea_NL__params.m` (which you should have edited to assign all the element values in all 11 branches), inspect it and run it. The segment simulation profile runs the model for different input levels.

- **[RESULTS]**: What do the curves look like? Does the filter shape depend on the input? HINT: compare the current measurements with the sensitivity measurements (I_6/U_{in}).

Second, come back to the SIMULINK model, open the subsystem box named `NL Part` and inspect it. It should look similar to the branch number 6 in Fig. 3:

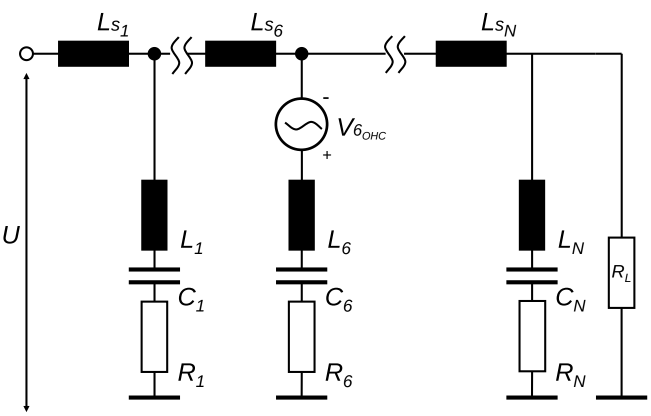

Fig. 3. Electrical model in Fig. 2 with a non-linear branch.

2.3.1. *Questions*

- **[DISCUSS]**: The component V_{OHC} is a dependent voltage source, so $V_{OHC} \cong (1 - U_{in})R_6 I_6$. What is V_{OHC} modeling?
- **[DISCUSS]**: What is the expected behavior of this cochlear partition at a low ($U_{in} << 1$), mid ($U_{in} = 1$) and high ($U_{in} >> 1$) input levels?
- **[OPTIONAL]**: Can you describe mathematically what happens in this branch when the stimulus is presented at its resonant frequency? How is this related to the cochlear non-linearities? HINT: At its resonant frequency, the impedance of the stiffness is nearly equal to that of the mass with opposite phase. These two impedances, therefore, cancel each other out.

Third, replace now the elements in branch number 6 (L_{s_6} and the RLC element with L_6, C_6 and R_6) by the `NL Part` box containing the non-linear branch (you have to connect it to the node coming from branch 5 and to the node going to branch 7). Connect directly the scope to the `Measurement I6` output in the `NL Part` box. It is not needed here to use the `I_probe` box. Go back to the MATLAB script `BMM_WholeCochlea_NL__params.m` and run it again.

- **[DISCUSS]**: What are the differences compared to the previous plot?
- **[DISCUSS]**: How can you relate the observed behavior to the cochlear physiology? HINT: Compare the sensitivity measurement (I_6/U_{in}) in the linear versus the non-linear system cochlea.
- **[RESULTS]**: Plot and describe input/output (I/O) relation for the cochlear segment 6 in the linear versus the non-linear case. HINT: Plot the maximum of the velocity transfer function (as a function of frequency, figure 1 in the MATLAB script `BMM_WholeCochlea_NL__params.m`) in the branch number 6 as a function of the input intensity.

3. PART 2: GAMMATONE-FILTER MODEL

The gammatone auditory filter at a certain centre frequency f_c can be described by its impulse response:

$$g(t) = a^{-1}\, t^{n-1} e^{-2\pi bt} \cos(2\pi f_c t + \phi)\ , \quad t > 0\ , \qquad (5)$$

where a is a normalization factor given by

$$a = \int_0^{\infty} t^{n-1} e^{-2\pi bt}\, dt. \qquad (6)$$

This function was introduced by [1] to characterize auditory nerve "impulse responses" from cat data [5]. The primary parameters of the filter are b and n:

- The parameter b largely determines the duration of the impulse response.
- The parameter n is the order of the filter and determines the slope of the skirts of the filter.
- The parameter ϕ determines the phase of the cosine function relative to the envelope $t^{n-1}e^{-2\pi bt}$. Here, ϕ is assumed to be zero.

When the order of the filter is in the range of 3-5, the shape of the magnitude characteristic of the gammatone filter is very similar to that of the "rounded exponential" filter, commonly used to represent the magnitude characteristic of the human auditory filter [2]. In [3], they have empirically quantified behavioral human auditory filter bandwidths by the means of an equivalent rectangular bandwidth (ERB) as a function of the center frequency f_c:

$$\mathrm{ERB} = 24.7 + 0.108\, f_c\ . \qquad (7)$$

Together, equations 5 and 7 define a gammatone auditory filterbank if one includes the common assumption that the filter center frequencies are distributed across frequency in proportion to their bandwidth. When the order of the filter is $n = 4$, b is 1.018 times the ERB and the 3-dB bandwidth of the gammatone filter is then 0.887 times the ERB. A more detailed discussion of the ERB and of the rounded exponential filter will be part of the next lecture and exercise "Frequency selectivity, masking, hearing impairment".

3.1. Analysis of a single gammatone filter

You should start by implementing a MATLAB function `gammaIR` that outputs a discrete-time realization of equation 5 combined with 7. The order of the filter should be $n = 4$ and $b = 1.018\cdot$ ERB. With $n = 4$, the integral in Eq. 6 becomes

$$a = \frac{6}{(-2\pi b)^4} \ . \tag{8}$$

Input variables should be the duration of the impulse response, the center frequency and the sampling rate.

3.1.1. Questions

- **[DISCUSS]** Compare the impulse responses (IRs) of filters at different center frequencies in the time domain (plot the `gammaIR` output vectors) and in the frequency domain (plot the respective magnitude spectra). Use a sampling rate of 32 kHz.
- **[DISCUSS]** How are the overall duration and the position of the maximum of the IRs qualitatively related to the center frequency?
- **[DISCUSS]** How are these values related to the bandwidth of the filters?
- **[RESULTS]** Next, derive an analytical formula for the time $t_{\max}(f_c)$, where the envelope of the gammatone IR has its maximum from Eq. 5. The result should prove the empirical relationship you have just established.
- **[DISCUSS]** Compare the magnitude frequency response of the gammatone filters to that of the transmission-line filters from the first part and discuss any differences (e.g. bandwidth, and filter slopes). If there are any differences, discuss how they can affect the perception of sound.

3.2. Analysis of a gammatone auditory filterbank

It is now clear that gammatone filters centered at different frequencies have different delays until the maximum of the excitation is observed. This delay can be thought of as an approximation of the cochlear delay for different frequencies along the BM. To analyze the approximated cochlear "excitation" in response to different stimuli as a function of frequency and time, you should use the MATLAB function `gammaFB`:

```
[out, cfs] = gammaFB(in, lowf, uppf, fs);
```

Type `help gammaFB` for more details. As an input stimulus, you should generate a delta pulse (use `dPulse`, the duration might be 125 ms) or a chirp stimulus (frequency sweep; use `genBMchirp`, where the duration is automatically 125 ms). The chirp stimulus is meant to compensate for BM delay based on a transmission-line model [6]. An easy way to plot the output of the different gammatone filters is provided by the MATLAB function `timeFreqDistribution`, which gives you a two-dimensional black-white image of the "excitation". The inputs are the outputs of `gammaFB`:

```
timeFreqDistribution(out, cfs, fs, range);
```

For details, type `help timeFreqDistribution` or `help timeFreqDistributionWaterfall`. Darker colors indicate more activity. Alternatively, you should have a look at the temporal waveforms in the different columns of `out`. For the pulse input and a certain center frequency, this should closely match the impulse response generated from your `gammaIR` function (`gammaFB` involves some simplifications for a faster implementation).

3.2.1. Questions

- **[DISCUSS]** Compare and discuss the time–frequency output pattern for the pulse and the BMchirp. The gammatone filters should range from 100 to 8000 Hz. (The BMchirp should have the same frequency range).
- **[DISCUSS]** Investigate the temporal waveform of the BMchirp. How are the different frequencies distributed in time?
- **[DISCUSS]** What does this mean for the distribution of excitation as a function of time in the different gammatone filters?

4. REFERENCES

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