Bayesian Method for Bertrand Probability Paradox

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What is the probability of randomly selecting a chord in a circle with its length longer than a side of inscribed equilateral triangle? This probability problem, which has three obviously correct but contradicted results, leads to the famous Bertrand paradox. In fact, the three methods underlying the contradicted results (i.e. $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$) totally overlap in the circle area and the only difference is how to select chords randomly in the circle. Almost all the previous researches focus on the difference between the three different methods and the definition and selection of randomness, but they ignore the relations among them. In this article I calculated the mathematical relations among the three contradict results, and then proposed to solve this probability paradox by using Bayesian conditional probability method and validate the result via computer simulation.

Introduction

Draw an equilateral triangle inscribed in a circle and then draw a chord randomly in this circle. What is the probability of the chord is longer than the length of a side of inscribed equilateral triangle? Bertrand provided three solutions to calculate the probability which leads to three different results, but he failed to prove any one of solution is wrong. He concluded that the root of this paradox is that the "randomness" underlying the three solutions are different, and the paradox could dismiss once the random method was selected.

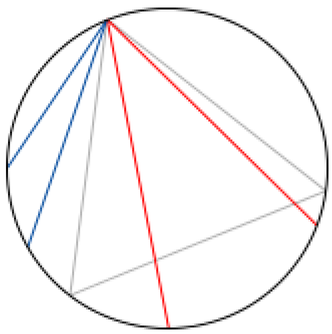
Many famous researchers (including Borel, Poincare, Kendall, Von Mises) from various filed paid their attention to this paradox and they also did not get the unique probability for this paradox. Most of them also thought the foundational issues of this paradox are the definition and selection of "randomness". So in almost all the text book or popular website for probability and statistics, the introduction of the Bertrand paradox still remains at the level of the definition and selection of "randomness" and never discuss at much deeper level.

In fact, whether there is a unique probability of randomly chosen chord longer than a side of inscribed equilateral triangle, pursued by many researchers in various field from probability field to physics field and even philosophical filed, are still not solved thoroughly. It is quite difficult and even impossible, within current logical and mathematical framework, to get the unique probability result. Jaynes proposed a well-posed solution to solve this paradox based on the principle of "maximum ignorance and got the unique result. However, Alon claimed that two other results could be also got based on Jaynes' principle. Rowbottom concluded that the Bertrand paradox is still not solved by using a vivid example of "cake cutting" in his paper.

Almost all the previous researches focus on the difference between the three different methods and the definition and selection of randomness, but they ignore the relations among them. In this paper, I firstly reviewed the three methods for Bertrand paradox, then demonstrated the relations among these three methods via mathematical formulas, and next proposed Bayesian conditional method to solve this paradox. We calculated the unique result for Bertrand paradox via mathematical formula using the proposed Bayesian method, and also validated this result through long-run computer simulation. Finally, we closed this paper with discussion.

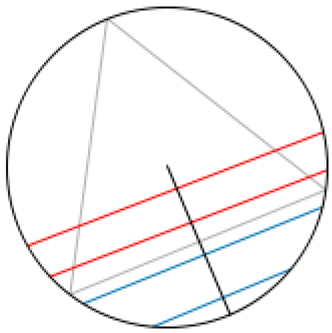
Method 1: The "random endpoints" method

Choose two points in the circle randomly and construct the chord by linking the two points. The probability of the randomly chosen chord longer than a side of inscribed equilateral triangle is $\frac{1}{3}$.



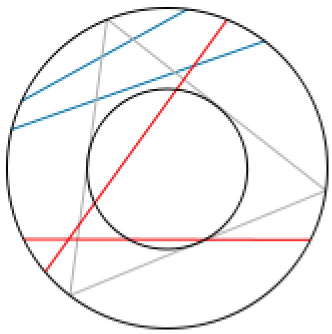
Method 2: The "random radius" method

Randomly choose the chord perpendicular to a pre-chose diameter in circle. The chord is longer than a side of inscribed equilateral triangle only when the intersection point is between the middle points of the two radiuses of this diameter, therefore the probability is $\frac{1}{2}$ based on this method.



Method 3: The "random midpoint" method

Randomly choose the chord which middle point is in the concentric circle with its diameter half of big circle's diameter. The probability is $\frac{1}{4}$ based on this method.



Bertrand Paradox

The above three results $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, all seems to be correct, which leads to the famous Bertrand paradox. In fact, these three different methods choose chords randomly from different probability spaces, i.e., periphery, diameter and concentric circle spaces. However, the chords chose from different methods which lengths are longer than a side of inscribed equilateral triangle fall in the completely same area.

The areas in which the chords chosen from “random endpoints”, “random radius” and “random midpoint” methods fall are represented as S_1, S_2, S_3. Below formulation can be known clearly as the chords fall in same area.

$$S_1 = S_2 = S_3.$$

Here, different probability spaces could be seen as different coordinate systems. The result $\frac{1}{3}$ is based on periphery coordinate system, the result $\frac{1}{2}$ based on the diameter coordinate system, and the $\frac{1}{4}$ based on concentric circle coordinate system. There are also three coordinate systems if we change the inscribed equilateral triangle to the inscribed equilateral square. Generalized speaking, there are three results if we randomly choose a chord which length is longer than a side of inscribed equilateral polygon (triangle and square are only special cases). Below are the generalized case on inscribed equilateral polygon.

Generalized case: Inscribed Equilateral Polygon (*n* sides)

What is the probability of randomly selecting a chord in a circle with its length longer than a side of inscribed equilateral polygon with n sides?

The results are respectively $\frac{n-2}{n}$, $sin(\frac{90(n-2)}{n})$, $sin^2(\frac{90(n-2)}{n})$ based on "random endpoint", "random radius" and "random midpoint" methods. Take the periphery of circle as the basic coordinate axis, the relation among periphery, diameter and concentric circle coordinate systems can be clearly stated. Take $x = \frac{n-2}{n}$as basic variable, y=$sin(\frac{90(n-2)}{n})$ can be represented via x formulated as y=$sin(90 * x)$, z=$sin^2(\frac{90(n-2)}{n})$ can be represented via x formulated as z=$sin^2(90 * x)$.

The results based on these systems are condition probabilities which are correlative and addictive. Then we adopt Bayesian condition probability method to solve the Bertrand paradox.

Bayesian method: condition probability

Based on three totally different methods, the results are correspondingly $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. Hence, we can get three conditional probabilities as following:

$p(\mathrm{x} > \sqrt{3} \mid M_i)$, *i=1,2,3*

Here, the probability of random selecting a chord in a circle with its length longer than a side of inscribed equilateral triangle based on the "random endpoints" method is formulated as below formula:

$$p(\mathrm{x} > \sqrt{3} \mid M_1)=\frac{1}{2}$$

The probability based on the "random radius" method is formulated as below formula:

$$p(\mathrm{x} > \sqrt{3} \mid M_2)=\frac{1}{3}$$

The probability based on the "random midpoint" method is formulated as below formula:

$$p(\mathrm{x} > \sqrt{3} \mid M_2)=\frac{1}{4}$$

Add the above three conditional probabilities, we get below formula:

$$\sum_{i=1}^{n} p(\mathrm{x} > \sqrt{3} \mid M_i) p(M_i) = \sum_{i=1}^{n} p(M_i \mid \mathrm{x} > \sqrt{3}) p(\mathrm{x} > \sqrt{3})$$

As the below formula equals to 1, the above formula at right side is left with

$p(\mathrm{x} > \sqrt{3})$.

$$\sum_{i=1}^{n} p(M_i \mid x > \sqrt{3}) = 1$$

Hence, the probability of random selecting a chord in a circle with its length longer than a side of inscribed equilateral triangle, without considering to choose which methods among the three methods, equals to the sum of three conditional probabilities based on above three methods multiplied by individual probability.

$$p(x > \sqrt{3}) = \sum_{i=1}^{n} p(x > \sqrt{3} \mid M_i) p(M_i)$$

Furtherly, the above formula can be written as below:

$$p(x > \sqrt{3}) = p(x > \sqrt{3} \mid M_1) * p(M_1) + p(x > \sqrt{3} \mid M_1) * p(M_1) + p(x > \sqrt{3} \mid M_1) * p(M_1)$$

Hence, the probability equals to $\frac{1}{2} * p(M_1) + \frac{1}{3} * p(M_1) + \frac{1}{4} * p(M_1)$ as below formula shown.

$$p(x > \sqrt{3}) = \frac{1}{2} * p(M_1) + \frac{1}{3} * p(M_2) + \frac{1}{4} * p(M_3)$$

Computer Simulation

The computer simulation is implemented in a statistical programming platform – R. This software can be downloaded freely in https://www.r-project.org/ . The simulation process is conducted via R programming which the R code is put in additional file with this paper. The simulation result indefinitely closes to $\frac{13}{36}$. The degree of close becomes much higher with the simulation times increasing.

Discussion

We can hardly indicate which method is not appropriate in calculating the Bertrand probability. In this situation, one feasible solving approach is to attaching probability to choice of the three contradict methods. Assume the probabilities of choosing three methods are $p(M_1), p(M_2), p(M_3)$, we can get the Bertrand probability using the Bayesian conditional method. If we regard the choices of three methods as indifference, the result is $\frac{13}{36}$, which is consistent with the result in paper titled "Solving the hard problem of Bertrand's paradox" written by Diederik Aerts. Compared with Diederik's solution, the advantage of my solution is we do not need to modify the problem of Bertrand paradox.

Conclusion

We get the unique probability for Bertrand paradox by using Bayesian conditional method and pass the validation via long-run computer simulation .

Acknowledgement

This paper was motivated by the speech about the "On Bertrand's Paradox" by Prof. Darrell Rowbottom at International Conference on Paradoxes, Logic and Philosophy, which this conference was hold at Department of Philosophy, Peking University on 10 Oct, 2016. I am indebted to Prof. Rowbottom for discussing with me on the Bertrand Paradox in details and to his paper titled "Bertrand's Paradox Revisited: Why Bertrand's 'Solutions' Are All Inapplicable" for much of the stimulation of my thoughts.