

Revenue Maximization and Myersonian Virtual Welfare

Recap: For a single buyer will arrive with their private value v , for DSIC mechanisms:

- Maximize welfare ($\sum_i v_i x_i$): Always give the bidder the item, always give it away for free!
- Maximize revenue: Post a price that maximizes $\text{REV} = \max_r r \cdot [1 - F(r)]$.
- Critical bid: Given other bids \mathbf{b}_{-i} , bidder i 's *critical bid* is the minimum bid $b_i^* = \min\{b_i : x_i(b_i, \mathbf{b}_{-i}) = 1\}$ such that bidder i is allocated to. Then with \mathbf{b}_{-i} fixed, for all winning $v_i \geq b_i^*$, i 's payment $p_i(v_i, \mathbf{b}_{-i}) = b_i^*$ is their critical bid.
- The revelation principle says that it's without loss to focus only on truthful mechanisms.
- Payment is determined by the allocation:

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i}) dz$$

We want to maximize $\mathbb{E}_{\mathbf{v} \sim \mathbf{F}}[\sum_i p_i(\mathbf{v})]$.

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, \mathbf{v}_{-i})] =$$

where

$$\varphi_i(v_i) =$$

is the Myersonian virtual value and $(*)$ follows by switching the order of integration. Then

$$\text{REVENUE} = \mathbb{E}_{\mathbf{v} \sim \mathbf{F}}[\sum_i p_i(\mathbf{v})] =$$

$$= \text{VIRTUAL WELFARE}$$

Given this conclusion, how should we design our allocation rule x to maximize expected virtual welfare (expected revenue)?

When would this cause a problem with incentive-compatibility?

Definition 1. A distribution F is regular if the corresponding virtual valuation function $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.

Suppose we are in the single-item setting and all of the distributions are regular. What do the payments look like in the virtual-welfare-maximizing allocation?

For a fixed \mathbf{b}_{-i} , if i is the winner, then i 's payment is i 's critical bid, which is

Exercise: what about for k identical items?

Claim 1. A virtual welfare maximizing allocation x is monotone if and only if the virtual value functions are regular.

Exercise: Argue this.

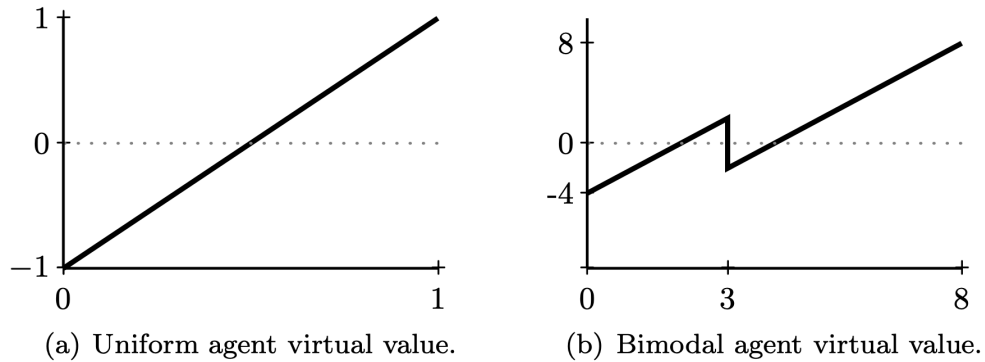


Figure 1: Virtual value functions $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ for the uniform and bimodal agent examples.

It will be helpful to keep the following two examples in mind:

- a. a uniform agent with $v \sim U[0, 1]$. Then $F(x) = x$ and $f(x) = 1$.
- b. a bimodal agent with

$$v \sim \begin{cases} U[0, 3] & w.p. \frac{3}{4} \\ U(3, 8] & w.p. \frac{1}{4} \end{cases} \quad \text{and} \quad f(v) = \begin{cases} \frac{3}{4} & v \in [0, 3] \\ \frac{1}{20} & v \in (3, 8] \end{cases}$$

Do the following:

- Calculate the virtual values for both examples.
- Are they regular? Are there any issues using the allocation that maximizes expected virtual welfare?
- What does that allocation actually look like?

Quantile Space and Ironing

In *value space*:

- an agent has value v .
- the fraction of the distribution with value above v is $1 - F(v)$.
- the revenue from posting a “take-it-or-leave-it” price of v is $v[1 - F(v)]$.

In *quantile space*: $q = 1 - F(v)$.

- an agent has value v .
- the fraction of the distribution with value above v is $q(v) = 1 - F(v)$.
- the revenue from posting a “take-it-or-leave-it” price of $v(q) = F^{-1}(1 - q)$ is $v(q) \cdot q$.

Example: Consider a distribution that is $U[\$0, \$10]$. Then the quantile 0.1 corresponds to \$9, where 10% of the population might have a higher value. We let $v(q)$ denote the corresponding value, so $v(0.1)$ is \$9.

Definition 2. The *quantile* of a single-dimensional agent with value $v \sim F$ is the measure with respect to F of stronger values, i.e., $q = 1 - F(v)$; the inverse demand curve maps an agent's quantile to her value, i.e., $v(q) = F^{-1}(1 - q)$.

Quantile Distribution: What distribution are quantiles drawn from? That is, what is the probability that an agent is in the top \hat{q} fraction of the distribution? For a distribution F , $\Pr_F[q \leq \hat{q}] =$ what?

Note: For everything we do today, we *could* stay in value space, (and sometimes we'll compare), but we'd have to normalize by the distribution using $f(v)$, which makes everything a bit messier and a bit trickier.

Example: For the example of a uniform agent where $F(z) = z$, the inverse demand curve is $v(q) = 1 - q$.

For an allocation rule $x(\cdot)$ in value space, we define an allocation rule in *quantile space* $y(\cdot)$:

$$y(q) = x(v(q)).$$

As $x(\cdot)$ is monotone weakly increasing, then $y(\cdot)$ is monotone *weakly decreasing*.

Definition 3. The *revenue curve* of a single-dimensional agent specified by $R(v) = v \cdot [1 - F(v)]$.

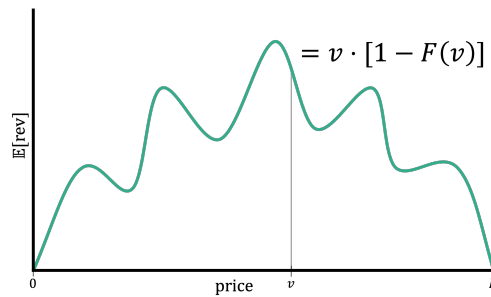


Figure 2: A revenue curve in *value space*.

Note: This is *only* the revenue that can be achieved by posting a single take-it-or-leave-it price. This does not capture the expected revenue of any given mechanism.

Definition 4. The *revenue curve* of a single-dimensional agent specified by inverse demand curve $v(\cdot)$:

Claim 2. Any allocation rule $y(\cdot)$ can be expressed as a distribution of posted prices.

Proof.

Claim 3. Any DSIC allocation rule $x(\cdot)$ can be expressed as a distribution of posted prices.

Claim 4. A distribution F is regular if and only if its corresponding revenue curve is concave.

Observe that $P'(q) = \varphi(v(q))$:

$$P'(q) = \frac{d}{dq} (q \cdot v(q)) = v(q) + qv'(q) = v - \frac{1 - F(v)}{f(v)} = \varphi(v(q)).$$

Thus $\Phi(q) = \int_0^q \varphi(\hat{q}) d\hat{q} = P(q)$.

To summarize: a distribution F is regular if and only if:

- its corresponding revenue curve *in quantile space* is concave.
- $\varphi(q)$ is strictly increasing.
- $f(v)\varphi(v)$ is strictly increasing. (Why?)

Definition 5. The *ironing procedure* for (non-monotone) virtual value function φ (in quantile space) is:

- (i) Define the cumulative virtual value function as
- (ii) Define ironed cumulative virtual value function

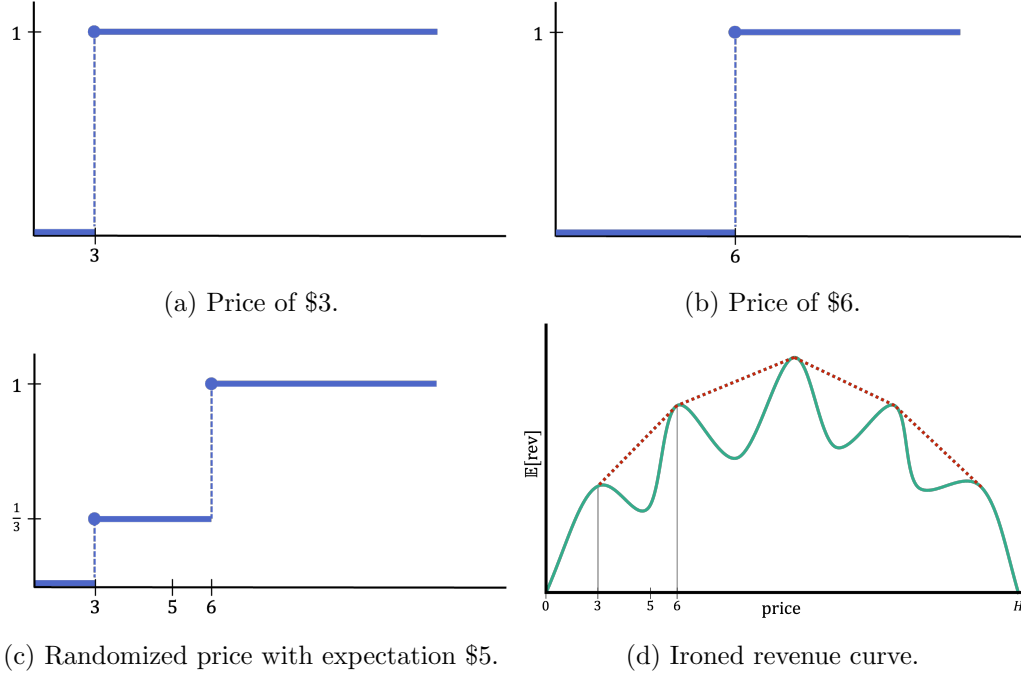


Figure 3: (a) An allocation rule for a take-it-or-leave-it price of \$3. (b) An allocation rule for a take-it-or-leave-it price of \$6. (c) An allocation that can be written $x(v) = 0$ for $v < 3$, $x(v) = \frac{1}{3}$ for $v \in [3, 6)$, and $x(v) = 1$ for $v \geq 6$. Alternatively, a randomized take-it-or-leave-it price that is \$3 with probability $\frac{1}{3}$ and \$6 with probability $\frac{2}{3}$, that is, $\$5 = \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 6$ in expectation. (d) The revenue curve in value space, including ironed intervals where convex combinations of prices can attain higher revenue than deterministic prices.

(iii) Define the ironed virtual value function as

Summary: Take the concave hull of the revenue curve in quantile space. Its derivative forms the ironed virtual values. (The derivatives of the original curve are the original virtual values.)

Theorem 1. *For any monotone allocation rule $y(\cdot)$ and any virtual value function $\varphi(\cdot)$, the expected virtual welfare of an agent is upper-bounded by her expected ironed virtual surplus, i.e.,*

$$\mathbb{E}[\varphi(q)y(q)] \leq \mathbb{E}[\bar{\varphi}(q)y(q)].$$

Furthermore, this inequality holds with equality if the allocation rule y satisfies $y'(q) = 0$ for all q where $\bar{\Phi}(q) > \Phi(q)$.

How do we modify this statement for value space?

Proof.

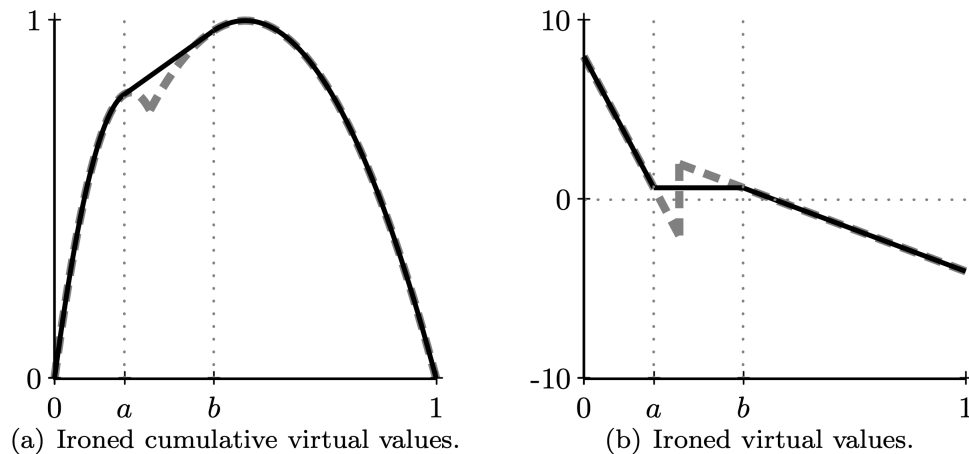


Figure 4: The bimodal agent's (ironed) revenue curve and virtual values in quantile space.

Claim 5. The expected revenue on the ironed revenue curve is attainable with a DSIC mechanism.

Example: How would you obtain the ironed revenue at \$5 instead of just $R(5)$?

Note: Recall that the expected revenue of *any mechanism*, not just a posted price, can be expressed by its virtual welfare. (We have now shown that you could decompose it into a distribution of posted prices and thus express the revenue that way, too, actually.)

What's the final mechanism?

For any ironed interval $[a, b]$, examine $\bar{\varphi}(v)$ for $v \in [a, b]$. Draw conclusions about $\bar{\varphi}(v)$ and $x(v)$. $P(q(v))$ is a straight line (linear) there, so $\bar{\varphi}(q(v))$ will be?

What does this imply for ironed-virtual-welfare-maximizing allocation in $[a, b]$?

Multiple Bidders

Imagine we have three bidders competing in a revenue-optimal auction for a single item. They are as follows:

- Bidder 1 is uniform. $F_1(v) = \frac{v-1}{H-1}$ on $[1, H]$.
- Bidder 2 is exponential. $F_2(v) = 1 - e^{-x}$ for $v \in (1, \infty)$.
- Bidder 3 is exponential. $F_3(v) = 1 - e^{-2x}$ for $v \in (1, \infty)$.

What does the optimal mechanism look like?

Definition 6. A *reserve price* r is a minimum price below which no buyer may be allocated the item. There may also be personalized reserve prices r_i where if $v_i < r_i$ then v_i will not be allocated to. Bidders above their reserves participate in the auction.