DS 320 Algorithms for Data Science | **Lecture #21 Worksheet**
Spring 2023 | Prof. Kira Goldner

Linear Programming IV: More Duality and the Minimax Theorem

The Dual of a Linear Program

To take the dual: Label each primal constraint with a new dual variable. In our new linear program, each dual constraint will correspond to a primal variable. For the left-hand side, count up the appearances of this constraint's primal variable (e.g., x_1) in each of the primal constraints and multiply them by the dual variable for those constraints. That is, if x_1 appears 5 times ($5x_1$) in constraint for y_1, then add $5y_1$ to x_1's constraint. Don't forget to include its appearance in the primal's objective function, but this will be the right-hand side of the constraint. Finally, the dual objective function is given by the right-hand side coefficients and their correspondence to the dual variables via the constraints in the primal.

The following is the normal form for a maximization problem primal and its primal:

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Example 3: Maximum Matching

Given a graph $G = (V, E)$ choose a maximum size matching—a set of edges S such that no vertex is covered by more than one edge.

Decision variables: x_e indicating whether edge e is in the matching.

Primal Linear Program:

$$\begin{aligned} \max \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \sum_{e: v \in e} x_e \leq 1 && \forall v \quad \text{(vertex matched at most once)} \\ & x_e \geq 0 && \forall e \quad \text{(non-negativity)} \end{aligned}$$

Taking the dual of the above primal, we get what linear program?

What problem is this?

Conditions for Optimality

Weak Duality

Theorem 1. *If* $\mathbf{x}$ *is feasible in (P) and* $\mathbf{y}$ *is feasible in (D) then* $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Give an upper bound on maximum matching:

Give a lower bound on vertex cover:

Strong Duality

Theorem 2 (Strong Duality)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if* $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$.

Proof. $(\Rightarrow)$ Skip.
$(\Leftarrow)$

Complementary Slackness

Primal (P):

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \sum_i a_{ji}x_i \leq b_j \quad \forall j \quad (y_j) \\ & x_i \geq 0 \quad \forall i \end{aligned}$$

Dual (D):

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \sum_i a_{ij}y_i \geq c_i \quad \forall i \quad (x_i) \\ & y_j \geq 0 \quad \forall j \end{aligned}$$

Theorem 3 (Complementary Slackness)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if the following complementary slackness conditions (1) and (2) hold:*

Proof.

Zero-Sum Games and the Minimax Theorem

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

The Minimax Theorem

Theorem 4 (Minimax Theorem)**.** *For every two-player zero-sum game* $\mathbf{A}$*,*

$$\max_{\mathbf{x}} \left(\min_{\mathbf{y}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right) = \min_{\mathbf{y}} \left(\max_{\mathbf{x}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right). \tag{1}$$

From LP Duality to Minimax

$$\max_{\mathbf{x}} \left(\min_{\mathbf{y}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right) = \max_{\mathbf{x}} \left(\min_{j=1}^{n} \mathbf{x}^T \mathbf{A} \mathbf{e}_j \right) \tag{2}$$

$$= \max_{\mathbf{x}} \left(\min_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_i \right) \tag{3}$$

$$\max v$$

subject to

$$\begin{aligned} v - \sum_{i=1}^{m} a_{ij} x_i \leq 0 &\quad \text{for all } j = 1, \ldots, n \\ \sum_{i=1}^{m} x_i = 1 & \\ x_1, \ldots, x_m \geq 0 &\quad \text{and} \quad v \in \mathbb{R}. \end{aligned}$$

$$\min w$$

subject to

$$\begin{aligned} w - \sum_{j=1}^{n} a_{ij} y_j &\geq 0 \quad \text{for all } i = 1, \ldots, m \\ \sum_{j=1}^{n} y_j &= 1 \\ y_1, \ldots, y_n &\geq 0 \quad \text{and} \quad w \in \mathbb{R}. \end{aligned}$$