| DS 574 Algorithmic Mechanism Design | Lecture #5 Worksheet |
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| **Fall 2023** | Prof. Kira Goldner |

# Single-Parameter Optimal Revenue (continued)

## Virtual Welfare Recap

- Maximize welfare ($\sum_i v_i x_i$): Always give the bidder the item, always give it away for free!
- Maximize revenue: Post a price that maximizes $\textsc{Rev} = \max_r r \cdot [1 - F(r)]$.

Using only the revelation principle and the payment identity $p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i})\, dz$, we proved the following:

$$\textsc{Revenue} = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i \varphi_i(v_i) x_i(\mathbf{v})] = \textsc{Virtual Welfare}$$

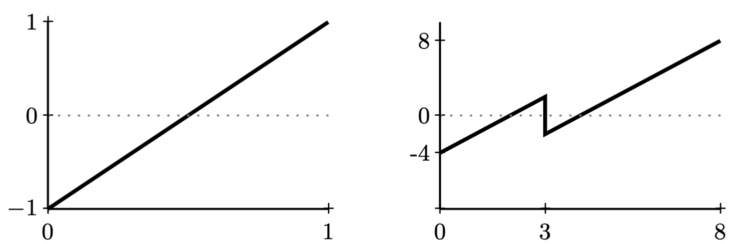
where

$$\varphi_i(v_i) = v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)}.$$

Then similarly to welfare, just give the item to the bidder with the highest (non-negative) *virtual* value! But this doesn't work when $\varphi(\cdot)$ isn't monotone, because then $x(\cdot)$ wouldn't be.

**Definition 1.** A distribution $F$ is regular if the corresponding virtual valuation function $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.

**Claim 1.** A virtual welfare maximizing allocation $x$ is monotone if and only if the virtual value functions are regular.




(a) Uniform agent virtual value. (b) Bimodal agent virtual value.

Figure 1: Virtual value functions $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ for the uniform and bimodal agent examples.

It will be helpful to keep the following two examples in mind:

**a.** a uniform agent with $v \sim U[0, 1]$. Then $F(x) = x$ and $f(x) = 1$. $\varphi(v) = 2v - 1$.

**b.** a bimodal agent with

$$v \sim \begin{cases} U[0,3] & w.p. \frac{3}{4} \\ U(3,8] & w.p. \frac{1}{4} \end{cases} \qquad \text{and} \qquad f(v) = \begin{cases} \frac{3}{4} & v \in [0,3] \\ \frac{1}{20} & v \in (3,8] \end{cases}$$

$$1 - F(v) = \begin{cases} \frac{1}{4} + \left(\frac{3-v}{3}\right) \cdot \frac{3}{4} & v \in [0,3] \\ \left(\frac{8-v}{5}\right) \cdot \frac{1}{4} & v \in (3,8] \end{cases} \qquad \text{so} \qquad \varphi(v) = \begin{cases} \frac{4}{3}(v-1) & v \in [0,3] \\ 2v - 8 & v \in (3,8] \end{cases}$$

## Quantile Space and Ironing

In *value space*:

- an agent has value $v$.
- the fraction of the distribution with value above $v$ is $1 - F(v)$.
- the revenue from posting a "take-it-or-leave-it" price of $v$ is $v[1 - F(v)]$.

In *quantile space*: $q = 1 - F(v)$.

- an agent has value $v$.
- the fraction of the distribution with value above $v$ is $q(v) = 1 - F(v)$.
- the revenue from posting a "take-it-or-leave-it" price of $v(q) = F^{-1}(1-q)$ is $v(q) \cdot q$.

Example: Consider a distribution that is $U[\$0, \$10]$. Then the quantile 0.1 corresponds to \$9, where 10% of the population might have a higher value. We let $v(q)$ denote the corresponding value, so $v(0.1)$ is \$9.

**Definition 2.** The *quantile* of a single-dimensional agent with value $v \sim F$ is the measure with respect to $F$ of stronger values, i.e., $q = 1 - F(v)$; the inverse demand curve maps an agent's quantile to her value, i.e., $v(q) = F^{-1}(1-q)$.

**Quantile Distribution:** What distribution are quantiles drawn from? That is, what is the probability that an agent is in the top $\hat{q}$ fraction of the distribution? For a distribution $F$, $\Pr_F[q \leq \hat{q}] =$ what?

Note: For everything we do today, we *could* stay in value space, (and sometimes we'll compare), but we'd have to normalize by the distribution using $f(v)$, which makes everything a bit messier and a bit trickier.

**Example:** For the example of a uniform agent where $F(z) = z$, the inverse demand curve is $v(q) = 1 - q$.

For an allocation rule $x(\cdot)$ in value space, we define an allocation rule in *quantile space* $y(\cdot)$:

$$y(q) = x(v(q)).$$

As $x(\cdot)$ is monotone weakly increasing, then $y(\cdot)$ is monotone *weakly decreasing.*

**Definition 3.** The *revenue curve* of a single-dimensional linear agent specified by inverse demand curve $v(\cdot)$:

**Claim 2.** Any allocation rule $y(\cdot)$ can be expressed as a distribution of posted prices.

*Proof.*

**Claim 3.** A distribution $F$ is regular if and only if its corresponding revenue curve is concave.

Observe that $P'(q) = \varphi(v(q))$:

$$P'(q) = \frac{d}{dq}\left(q \cdot v(q)\right) = v(q) + qv'(q) = v - \frac{1 - F(v)}{f(v)} = \varphi(v(q)).$$

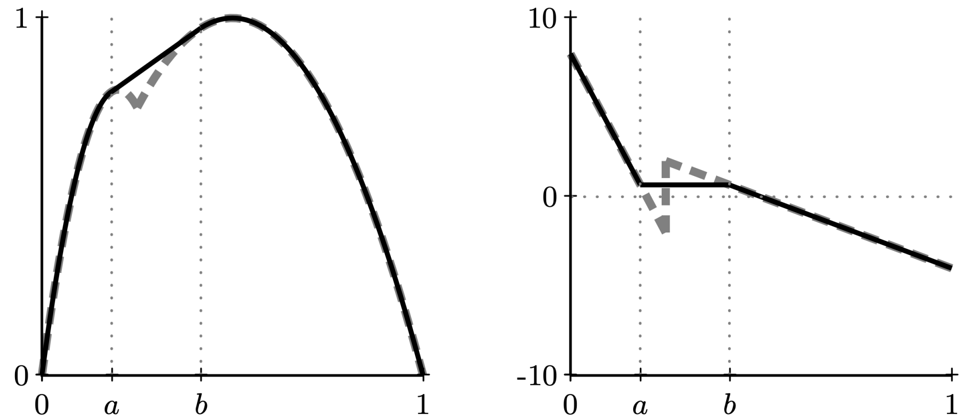
Thus $\Phi(q) = \int_0^q \varphi(\hat{q})\, d\hat{q} = P(q)$.

**Definition 4.** The *ironing procedure* for (non-monotone) virtual value function $\varphi$ (in quantile space) is:

(i) Define the cumulative virtual value function as

(ii) Define ironed cumulative virtual value function

(iii) Define the ironed virtual value function as




(a) Ironed cumulative virtual values. (b) Ironed virtual values.

Figure 2: The bimodal agent's (ironed) revenue curve and virtual values in quantile space.

**Theorem 1.** *For any monotone allocation rule $y(\cdot)$ and any virtual value function $\varphi(\cdot)$, the expected virtual welfare of an agent is upper-bounded by her expected ironed virtual surplus, i.e.,*

$$\mathbb{E}[\varphi(q)y(q)] \leq \mathbb{E}[\bar{\varphi}(q)y(q)].$$

*Furthermore, this inequality holds with equality if the allocation rule $y$ satisfies $y'(q) = 0$ for all $q$ where $\bar{\Phi}(q) > \Phi(q)$.*

*Proof.*

## Multiple Bidders

Imagine we have three bidders competing in a revenue-optimal auction for a single item. They are as follows:

- Bidder 1 is uniform. $F_1(v) = \frac{v-1}{H-1}$ on $[1, H]$.
- Bidder 2 is exponential. $F_2(v) = 1 - e^{-x}$ for $v \in (1, \infty)$.
- Bidder 2 is exponential. $F_3(v) = 1 - e^{-2x}$ for $v \in (1, \infty)$.

What does the optimal mechanism look like?

**Definition 5.** A *reserve price* $r$ is a minimum price below which no buyer may be allocated the item. There may also be personalized reserve prices $r_i$ where if $v_i < r_i$ then $v_i$ will not be allocated to. Bidders above their reserves participate in the auction.

## Now in Value Space

Now we'll repeat some of these results in value space. Remember, however, that while we can do everything we did in quantile space instead in value space, because quantile space is uniform *always* and value space is not, we'll need to normalize by the distribution in value space.

**Definition 6.** The *revenue curve* of a single-dimensional linear agent specified by $R(v) = v \cdot [1 - F(v)]$. Pointwise this equal to the revenue curve in quantile space: $R(v(q)) = P(q) = v(q) \cdot q$.

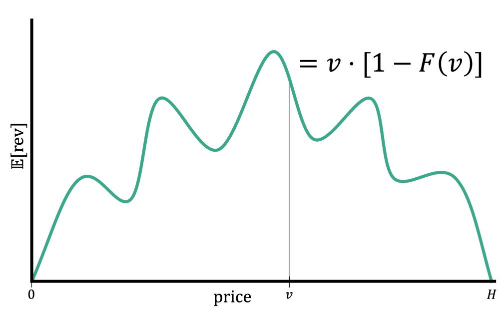



Figure 3: A revenue curve in *value space*.

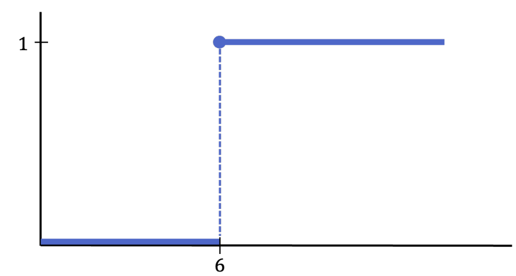
**Note:** This is *only* the revenue that can be achieved by posting a single take-it-or-leave-it price. This does not capture the expected revenue of any given mechanism.

**Claim 4.** A distribution $F$ is regular if and only if:

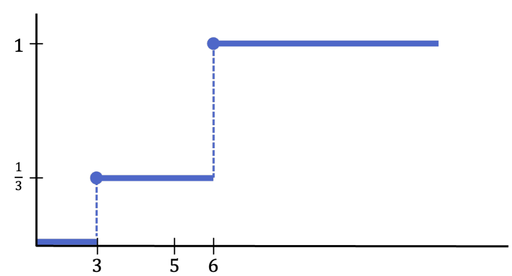
- its corresponding revenue curve *in quantile space* is concave.
- $\varphi(q)$ is strictly increasing.
- $f(v)\varphi(v)$ is strictly increasing. (Why?)

**Claim 5.** Any DSIC allocation rule $x(\cdot)$ can be expressed as a distribution of posted prices.

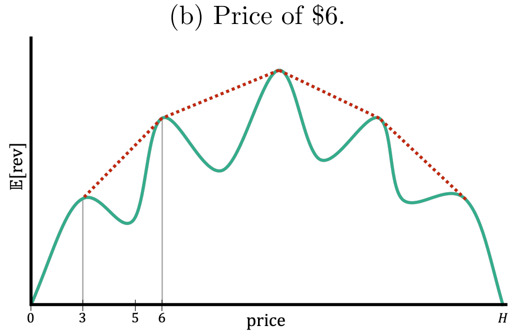

(a) Price of \$3.




(b) Price of \$6.




(c) Randomized price with expectation \$5.




(d) Ironed revenue curve.

Figure 4: (a) An allocation rule for a take-it-or-leave-it price of \$3. (b) An allocation rule for a take-it-or-leave-it price of \$6. (c) An allocation that can be written $x(v) = 0$ for $v < 3$, $x(v) = \frac{1}{3}$ for $v \in [3, 6)$, and $x(v) = 1$ for $v \geq 6$ . Alternatively, a randomized take-it-or-leave-it price that is \$3 with probability $\frac{1}{3}$ and \$6 with probability $\frac{2}{3}$, that is, $\$5 = \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 6$ in expectation. (d) The revenue curve in value space, including ironed intervals where convex combinations of prices can attain higher revenue than deterministic prices.

**Recall the ironing procedure:** Take the concave hull of the revenue curve in quantile space. Its derivative forms the ironed virtual values. (The derivatives of the original curve are the original virtual values.)

**Claim 6.** The expected revenue on the ironed revenue curve is attainable with a DSIC mechanism.

**Example:** How would you obtain the ironed revenue at \$5 instead of just $R(5)$?

**Note:** Recall that the expected revenue of *any mechanism*, not just a posted price, can be expressed by its virtual welfare. (We have now shown that you could decompose it into a distribution of posted prices and thus express the revenue that way, too, actually.)

**Theorem 2.** *For any monotone allocation rule $x(\cdot)$ and any virtual value function $\varphi(\cdot)$, the expected virtual welfare of an agent is upper-bounded by their expected ironed virtual welfare, i.e.,*

$$\mathbb{E}[\varphi(v)x(v)] \leq \mathbb{E}[\bar{\varphi}(v)x(v)].$$

*Furthermore, this inequality holds with equality if the allocation rule $x$ satisfies $x'(v) = 0$ for all $v$ where $\bar{\Phi}(v) > \Phi(v)$.*

What's the final mechanism?

For any ironed interval $[a, b]$, examine $\bar{\varphi}(v)$ for $v \in [a, b]$. Draw conclusions about $\bar{\varphi}(v)$ and $x(v)$. $P(q(v))$ is a straight line (linear) there, so $\bar{\varphi}(q(v))$ will be?

What does this imply for ironed-virtual-welfare-maximizing allocation in $[a, b]$?