DS 574 Algorithmic Mechanism Design	Lecture #14
Fall 2023	Prof. Kira Goldner

Online Bipartite Matching [KVV '90, EFFS '21]

In the online bipartite matching setting [Karp, Vazirani, and Vazirani, 1990], there is a bipartite graph $G = (L \cup R, E)$ where are vertices are split into the left side, L, and the right side, R. Edges are unweighted, i.e., all have a weight of 1. We are in an online setting where we see R up front, but the vertices of L arrive online, and as each vertex arrives, we see which edges are incident to it from R. The objective is to match vertices in L to those in R, immediately and irrevocably as each vertex arrives, forming a matching M, such that we maximize the cardinality of the matching $|M|$ and compare well to the maximal offline matching.

In the original paper, Karp et al. [1990] show:

- For every deterministic algorithm, $|M| \leq n/2$.
- Choosing a random match for each vertex independently implies that $\mathbb{E}[|M|] \leq n/2$.
- RANKING (KVV): Choosing a global ranking π U.A.R. and matching according to π implies that $\mathbb{E}[|M|] \geq (1 - 1/e)n$.
- This is tight!

The first proof [Karp et al., 1990] was very complicated (and imprecise). Simplifications were given by Goel and Mehta [2008], Birnbaum and Mathieu [2008], Devanur, Jain, and Kleinberg [2013].

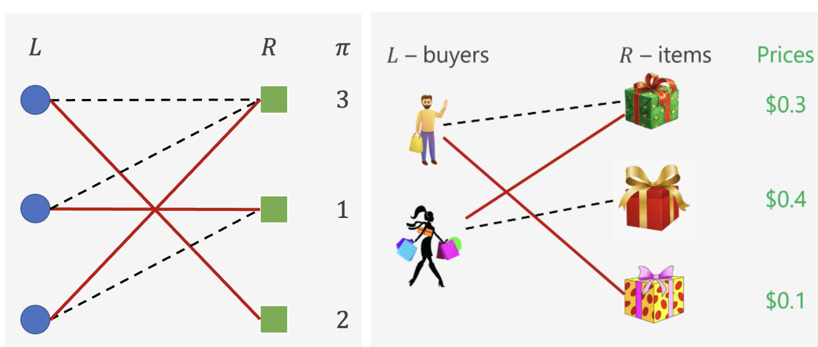

Figure 1: Online Bipartite Matching: RANKING and its economic interpretation.

Today, we'll look at an equivalent algorithm and simplified analysis due to Eden, Feldman, Fiat, and Segal [2021]. We'll use the following set-up: Let R be items and L buyers. They have value 1 or 0 for each item (depending on whether there is an edge). They are unit-demand (want one match). Then our algorithm is as follows.

- Choose item prices i.i.d. from some distribution F supported on $(0, 1)$ without point masses.

- Match buyers to the available item that maximizes utility $v - p$. That is, the lowest-priced item for which the buyer has an edge and is available.

Observations:

- For any F supported on $(0, 1)$ without point masses, choosing item prices i.i.d. from F is equivalent to choosing a UAR ranking π over items.
- $|M|$ is the *welfare* of the matching. Then we can rewrite it as seller's revenue + buyers' utility.

Lemma 1. *For F that samples $w \sim U[0, 1]$ and sets $p_j = e^{w-1}$, we have for every buyer i and item j such that (i, j) is an edge in M:*

$$\mathbb{E}[util_i + rev_j] \geq 1 - 1/e.$$

Corollary 1. $|M| \geq (1 - 1/e)n$.

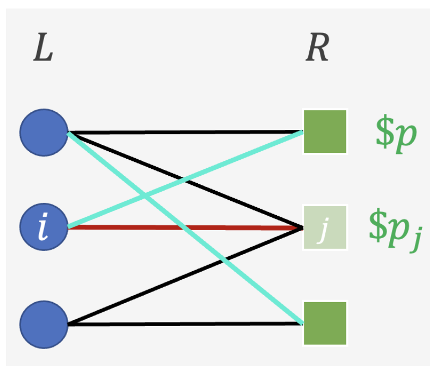

Figure 2: Economic analysis of RANKING.

Proof. Suppose (i, j) are matched in M^*. Let p be the price of the item that i takes in this event that item j is not in the market under the algorithm M.

We first observe that if $p_j < p$, then j is sold. Why? Either j is sold prior to i's arrival, or i prefers j to the item with price p.

Second, observe that i's utility is at least $1 - p$. Why? Adding an item only adds options for each agent, so utility cannot decrease.

$$\mathbb{E}[u_i + r_j] \geq 1 - p + \mathbb{E}_{p_j}[p_j \cdot \mathbb{1}[p_j < p]].$$

Recall that $p = e^{y-1}$ for some y. Then this is equal to

$$\begin{aligned}
&1 - p + \int_0^y e^{x-1}\,dx \\
&1 - p + e^{y-1} - 1/e \\
&1 - 1/e.
\end{aligned}$$

Then $\mathbb{E}[|M|] = \mathbb{E}[\sum_i u_i + \sum_j r_j] \geq (1 - 1/e)n$. $\square$

Conclusions:

- Taking the economic perspective can be very useful in algorithm analysis.
- Decomposing value into revenue + utility is a powerful tool, e.g., [Feldman, Gravin, and Lucier, 2015, Dütting, Feldman, Kesselheim, and Lucier, 2017, Ehsani, Hajiaghayi, Kesselheim, and Singla, 2018].

The Bulow-Klemperer Result

One famous result takes the form of resource augmentation.

Theorem 2 (Bulow Klemperer '96)**.** *For i.i.d. regular single-item environments, the expected revenue of the second-price auction with $n+1$ agents is at least that of the optimal auction with n agents.*

Let's talk about what this theorem is saying. Instead of finding the optimal auction tailored to a distribution F for n agents, you can use the Vickrey auction, which requires no prior knowledge of the distribution, so long as we require one extra bidder, regardless of the n that we start with, and earn more revenue than optimal. We do have two strong assumptions here (aside from being in the single-item environment):

- Bidders are i.i.d.—every bidder's value is drawn from F, and independently at that.
- F is a regular distribution. That is, $v - \frac{1-F(v)}{f(v)}$ is monotone non-decreasing.

This result **does not hold** without these assumptions. However, it is a *very strong* result, should our setting meet these assumptions.

Proof. First we claim that in the i.i.d. setting, the Vickrey auction earns the most revenue of all mechanism that *must allocate the item*. To maximize expected revenue, we know it is equivalent to maximize virtual welfare. If we *must* allocate the item in every case, then we should allocate the item to the bidder with the highest virtual value *even* when the virtual value is negative. Because we are in the i.i.d. setting *and* F is regular so $\varphi(\cdot)$ is monotone, virtual value functions are identical, so the bidder with the highest virtual value is identical to the bidder with the highest value. That is, the allocation rule is to *always* allocate to the highest bidder. This is precisely the allocation rule of the Vickrey auction.

Now, we compare the revenue of the Vickrey auction on $n+1$ bidders to another auction that always allocates the item, and there earns at most as much revenue—call this mechanism M. This mechanism runs the revenue-optimal mechanism for n bidders on the first n bidders $1, \ldots, n$. If the item is not allocated in that mechanism, it is then allocated to bidder $n+1$, so the item is always allocated, and is designed for $n+1$ bidders.

Then clearly

$$\textsc{opt}(n, F) \leq \textsc{Rev}_M(n+1, F) \leq \textsc{Rev}_{\text{Vickrey}}(n+1, F).$$

□

For more recent and complex Bulow-Klemperer style or "competition complexity" results, some examples include Eden, Feldman, Friedler, Talgam-Cohen, and Weinberg [2017], Babaioff, Goldner, and Gonczarowski [2020], Feldman, Friedler, and Rubinstein [2018], Beyhaghi and Weinberg [2019].

The Single Sample Mechanism

Can't recruit extra buyers? Instead, we can just exclude one. This is what the single sample result says.

Theorem 3. *Given a random sample from a bidder's distribution, posting it as a take-it-or-leave-it price gives a $\frac{1}{2}$-approximation to the optimal revenue.*

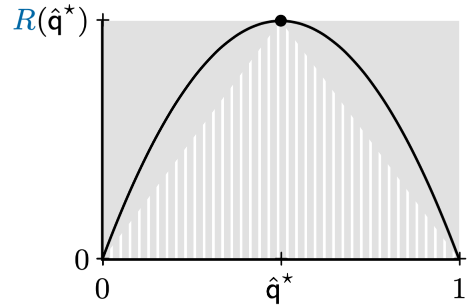

Figure 3: Geometric intuition for a posted-price from a single sample.

Proof. In quantile space! A randomly sampled value corresponds to a randomly sampled *quantile*, sampled uniformly $q \sim U[0,1]$ independent of the bidder's distribution. The revenue from this mechanism (call it M) is precisely $\textsc{Rev}(M) = \mathbb{E}_{q\sim U[0,1]}[R(q)] = \int_0^1 R(q)\,dq$ where R is the price-posting revenue curve in quantile space. This is exactly the area under the $R(\cdot)$.

What does depend on the bidder's distribution is the *optimal* quantile to sell to at a posted price, some q^*. The optimal single-bidder revenue that we aim to approximate is $\textsc{opt} = R(q^*)$. This is exactly the area of the rectangle with a height of $R(q^*)$ (the highest height of the curve) and the full width of the curve from 0 to 1 (a width of 1)—$R(q^*) \cdot 1$.

Now notice that the area under the curve contains the triangle with corners at $(0,0)$, $(0,1)$ and $(R(q^*),1)$. Hence this triangle must have area $R(q^*)/2$, that is, $\textsc{opt}/2$, contained in the area under of the curve, which is equal to

$$\textsc{Rev}(M) \geq \textsc{opt}/2.$$

□

It turns out, using a single sample from the buyers' distribution to set reserve prices and running VCG is a good approximation to the optimal mechanism. See Hartline chapter 5 for more.

Interested in these sort of sample complexity results? A good foundational result is Morgenstern and Roughgarden [2016], and Cole and Roughgarden [2014] then Morgenstern and Roughgarden [2015] after that. A more recent result that also contains an introduction surveying other results is Guo, Huang, and Zhang [2019].

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