DS 574 Algorithmic Mechanism Design	**Lecture #2 Worksheet**
Fall 2025	Prof. Kira Goldner

Allocation and Payment Rules

Now, we formalize the concepts we've been using so far. A mechanism $M = (\mathbf{x}, \mathbf{p})$ is completely determined by its allocation rule $\mathbf{x}$ and payment rule $\mathbf{p}$.

Definition 1. An *allocation rule* x is a (potentially randomized) mapping from bidder actions (bids $\mathbf{b}$) to feasible outcomes in X.

In the single-item setting, what is the set of feasible outcomes X? We say $\mathbf{x} \in X$ where $\mathbf{x} = (x_1, \dots, x_n)$ and x_i denotes how much of the item bidder i gets.

- At most 1 item is allocated: $\sum_{i=1}^{n} x_i \leq 1$.
- A bidder is either allocated or isn't: $x_i \in \{0, 1\} \, \forall i$.

What does this mean for a potentially randomized allocation rule $\mathbf{x}(\mathbf{b})$?

Definition 2. A *payment rule* $\mathbf{p}(\mathbf{b}) \in \mathbb{R}^n$ is a mapping from bidder actions (bids $\mathbf{b}$) to (non-negative) real numbers where $p_i(\mathbf{b})$ is the amount that bidder i pays in the outcome $\mathbf{x}(\mathbf{b})$.

Now we can formalize quasilinear utility in terms of general allocation and payment rules.

Definition 3. For a mechanism $M = (\mathbf{x}, \mathbf{p})$, a bidder with *quasilinear utility* has utility

$$u_i(\mathbf{b}) = v_i \cdot x_i(\mathbf{b}) - p_i(\mathbf{b}).$$

We'll narrow our attention to payment rules that satisfy

$$p_i(\mathbf{b}) \in [0, b_i \cdot x_i(\mathbf{b})]$$

for every i and $\mathbf{b}$. The constraint that $p_i(\mathbf{b}) \geq 0$ is equivalent to prohibiting the seller from paying the bidders. The constraint that $p_i(\mathbf{b}) \leq b_i \cdot x_i(\mathbf{b})$ ensures that a truth-telling bidder receives nonnegative utility (do you see why?).

Again, our goal is to design DSIC mechanisms:

Definition 4. A mechanism is *dominant-strategy incentive-compatible (DSIC)* if it is a bidder's dominant strategy to bid their true value, i.e. it maximizes their utility, *no matter what* the other bidders do. That is,

$$u_i(v_i, \mathbf{b}_{-i}) \geq u_i(z, \mathbf{b}_{-i}) \quad \forall z, \mathbf{b}_{-i}.$$

Myerson's Lemma

We now come to two important definitions. Both articulate a property of allocation rules.

Definition 5 (Implementable Allocation Rule)**.** An allocation rule $\mathbf{x}$ is *implementable* if there is a payment rule $\mathbf{p}$ such the sealed-bid auction $(\mathbf{x}, \mathbf{p})$ is DSIC.

Definition 6 (Monotone Allocation Rule)**.** An allocation rule x for a single-parameter environment is *monotone* if for every bidder i and bids $\mathbf{b}_{-i}$ by the other bidders, the allocation $x_i(z, \mathbf{b}_{-i})$ to i is nondecreasing in its bid z.

That is, in a monotone allocation rule, bidding higher can only get you more stuff.

We state Myerson's Lemma in three parts; each is conceptually interesting and will be useful in later applications.

Theorem 1 (Myerson's Lemma Myerson [1981])**.** *Fix a single-parameter environment.*

(a)

(b)

(c)

Myerson's Lemma is the foundation on which we'll build most of our mechanism design theory. Let's review what it is saying.

Part (a): Finding an allocation rule that can be made DSIC (is implementable, Definition 5) seems confusing, but is actually equivalent to and just as easy as checking if the allocation is monotone (Definition 6).

Part (b): If an allocation rule *is* implementable (can be made to be DSIC), then there's no ambiguity in what the payment rule should be.

Part (c): There's a simple and explicit formula for this!

Proof of Myerson's Lemma (Theorem 1). As shorthand, write $x(z)$ and $p(z)$ for the allocation $x_i(z, \mathbf{b}_{-i})$ and payment $p_i(z, \mathbf{b}_{-i})$ of i when it bids z, respectively.

Suppose $(\mathbf{x}, \mathbf{p})$ is DSIC, and consider any $0 \leq y < z$. Because bidder i might well have private valuation z and can submit the false bid y if it wants, DSIC demands that

$$\underbrace{\hspace{4cm}}_{\text{utility of bidding } z \text{ given value } z} \geq \underbrace{\hspace{4cm}}_{\text{utility of bidding } y \text{ given value } z} \tag{1}$$

Similarly, since bidder i might well have the private valuation y and could submit the false bid z, $(\mathbf{x}, \mathbf{p})$ must satisfy

$$\underbrace{\hspace{4cm}}_{\text{utility of bidding } y \text{ given value } y} \geq \underbrace{\hspace{4cm}}_{\text{utility of bidding } z \text{ given value } y} \tag{2}$$

Rearranging inequalities (1) and (2) yields the following sandwich, bounding $p(y) - p(z)$ from below and above:

$$y \cdot [x(z) - x(y)] \leq p(z) - p(y) \leq z \cdot [x(z) - x(y)] \tag{3}$$

From here, we can conclude:

- $\mathbf{x}$ must be monotone.
- $p'(z) = z \cdot x'(z)$.

Why?

Assuming that $p(0) = 0$ and integrating then gives the payment identity

$$p_i(b_i, \mathbf{b}_{-i}) =$$

or alternatively, after integration by parts,

$$p_i(b_i, \mathbf{b}_{-i}) = \tag{4}$$

for every bidder i, bid b_i, and bids $\mathbf{b}_{-i}$ by the others.

Equation (3) tells us that this is the only payment rule that could possibly be DSIC. But does it in fact satisfy DSIC when x is monotone?

Bidder i's utility will then be

$$u_i(b_i, \mathbf{b}_{-i}) =$$

which is maximized when $b_i =$

independent of $\mathbf{b}_{-i}$, as desired. $\square$

Single-Parameter Environments

All of our definitions and Myerson's Lemma actually apply to a more general setting which we call *single-parameter environments*. The main idea here is that each bidder i only has a single piece of private information, like their value v_i, that needs to be elicited in order to run the mechanism. Here are some other examples of non-single-item yet single-parameter environments.

- **Single-item:** A seller has a single item to sell. The set of feasible outcomes X satisfy $\sum_{i=1}^n x_i \leq 1$ and $x_i \in \{0, 1\}$.
- k **identical items:** A seller has k identical items to sell and each buyer gets at most one. The set of feasible outcomes X satisfy:
- **Sponsored search:** There are k advertising slots, each with click-through-rate α_j. A buyer i gets value $v_i \cdot \alpha_j$ from winning the jth slot. The set of feasible outcomes X satisfy $\sum_{i=1}^n x_i \leq \sum_{j=1}^k \alpha_j$ and $x_i \in \{\alpha_j\}_{j=1}^k \cup \{0\}$ where $x_i = \alpha_j$ if bidder i is assigned the jth slot.

The Revelation Principle

So far, we've been investigating *Dominant-Strategy Incentive-Compatible (DSIC)* mechanisms. To be DSIC, this means that

(1) Every participant in the mechanism has a dominant strategy, no matter what their private valuation is.

(2) This dominant strategy is *direct revelation*, where the participant truthfully reports all of their private information to the mechanism.

There are mechanisms that satisfy (1) but not (2). Give an example:

For a formal definition of a direct revelation mechanism:

Definition 7. A mechanism is *direct revelation* if it is single-round, sealed-bid, and has action space equal to the type (value) space. That is, an agent can bid any type they might have, and an agent's action *is* bidding a type.

The Revelation Principle and the Irrelevance of Truthfulness

The Revelation Principle states that, given requirement (1), there is no need to relax requirement (2): it comes "for free."

Theorem 2 (Revelation Principle for DSIC Mechanisms)**.** *For every mechanism M in which every participant has a dominant strategy (no matter what their private information), there is an equivalent direct-revelation DSIC mechanism M'.*

Equivalent here means that as a function of the *valuation profile* (not bids), the allocation and payment $(x(\mathbf{v}), p(\mathbf{v}))$ are equivalent in both M and M'.

Proof.

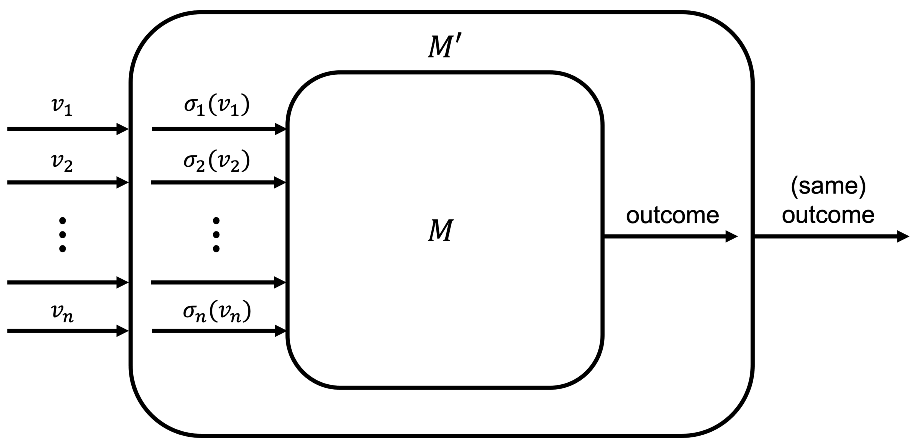

Figure 1: Proof of the Revelation Principle. Construction of the direct-revelation mechanism M', given a mechanism M with dominant strategies.

The takeaway from the Revelation Principle (Theorem 2) is that **it is without loss to design direct revelation mechanisms**. That is, you might as well require your mechanism to be

incentive-compatible.

Beyond Dominant-Strategy: Bayesian Settings

There are many reasons why we can't always require dominant strategies when design mechanisms.

(1) Requiring such a strong concept might not be tractable.

(2) Agents do not always have dominant strategies! What then?

We'll now introduce the Bayesian setting.

Suppose the valuation v_i of bidder i is drawn from a prior distribution F_i.

- CDF $F_i(x) = \Pr_{v_i \sim F_i}[v_i \leq x]$.
- PDF $f_i(x) = \frac{d}{dx} F_i(x)$.
- Joint distribution $\mathbf{F}$ or $\vec{F}$.

Unless otherwise noted, we assume that the prior distribution $\mathbf{F}$ is *common knowledge* to all bidders and the mechanism designer (the seller).

Definition 8. A *Bayes-Nash equilibrium (BNE)* for a joint distribution $\mathbf{F}$ is a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that for all i and v, $\sigma_i(v_i)$ is a best-response when other agents play $\sigma_{-i}(\mathbf{v}_{-i})$ when $\mathbf{v}_{-i} \sim \mathbf{F}_{-i} \mid_{v_i}$.

Claim 1. Consider two identically and independently drawn bidders from $F = U[0, 1]$. It is a (symmetric) BNE for each bidder to bid $\sigma_i(v_i) = v_i/2$ in the first-price auction.

Proof.

Theorem 3 (Revenue Equivalence)**.** *The payment rule and revenue of a mechanism is uniquely determined by its allocation. Hence, any two mechanisms with the same allocation must earn the same revenue.*

What is this theorem a corollary of? Prove this for the first-price auction and the Vickrey (second-price) auction in the above setting!

Proof.

Bayesian Settings

Using notions from the Bayesian setting and how bidders Bayesian update as they learn information, we define three stages of the auction:

1. *ex ante*: Before any information has been drawn; i only knows $\mathbf{F}$.
2. *interim*: Values v_i have been drawn; i only knows their own valuation, and thus the updated prior $\mathbf{F} \mid_{v_i}$.
3. *ex post*: The auction has run and concluded. All bidders know all $v_1, \ldots, v_n$.

Typically we discuss the *ex post* allocation and payment rules as a function of all of the values. However, in the Bayesian setting, to reason about BIC, it often makes sense to take in terms of *interim* allocation and payment rules which have the same information as bidder i before the auction is run.

Definition 9. We define the *interim* allocation and payment rules in expectation over the updated Bayesian prior given i's valuation:

$$x_i(v_i) = \Pr_{\mathbf{F}}[x_i(\mathbf{v}) = 1 \mid v_i] = \mathbb{E}_{\mathbf{F}}[x_i(\mathbf{v}) \mid v_i]$$

and

$$p_i(v_i) = \mathbb{E}_{\mathbf{F}}[p_i(\mathbf{v}) \mid v_i].$$

Our definition of Bayesian Incentive-Compatibility then follows:

Definition 10. A mechanism with *interim* allocation rule x and *interim* payment rule p is Bayesian Incentive-Compatible (BIC) if

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(z) - p_i(z) \quad \forall i, v_i, z.$$

References

Roger B. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981. URL `http://dx.doi.org/10.1287/moor.6.1.58`.