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| **DS 320 Algorithms for Data Science** | **Lecture #19 Worksheet** |
| **Spring 2023** | Prof. Kira Goldner |

# Linear Programming II: Algorithms, Problems, and Duality

## What Does Linear Programming Buy Us?

**a.** We know efficient algorithms exist (and have a nice theory behind them).

**b.** We can relate problems to one another through relaxations, duality.

**c.** It gives us techniques for approximation.

# Linear Programming Algorithms

**a.** Simplex

**b.** Ellipsoid

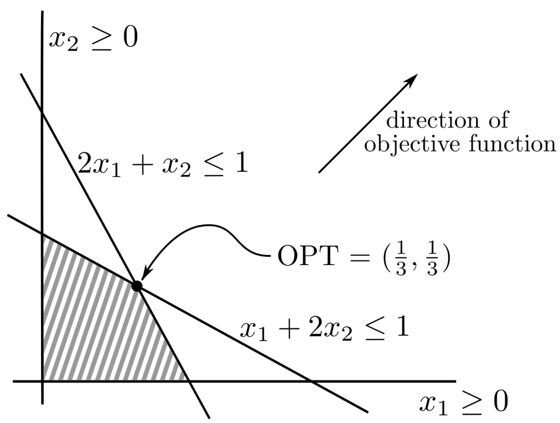



Figure 1: A toy example of a linear program.

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 \geq 0 \\ & x_2 \geq 0 \\ & 2x_1 + x_2 \leq 1 \\ & x_1 + 2x_2 \leq 1. \end{aligned}$$

# Writing Problems We Know as Linear Programs

## Independent Set

Given a graph $G = (V, E)$, each vertex $i$ has weight $w_i$, find a maximum weighted *independent set*. $S$ is an independent set if it does not contain both $i$ and $j$ for $(i, j) \in E$.

**a.** *Decision variables:* What are we try to solve for?

**b.** *Constraints:*

**c.** *Objective function:*

The linear program:

Integer programs vs. linear relaxations:

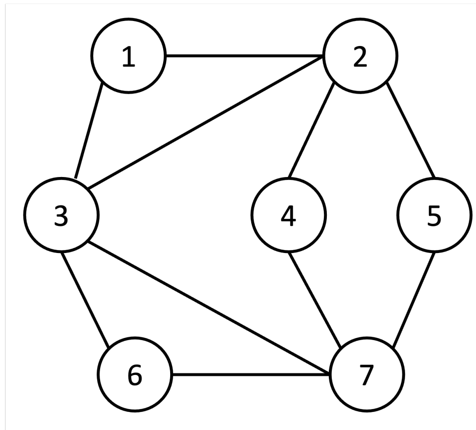
## Knapsack

Given $n$ items, each item $i$ with value $v_i$ and weight $w_i$, select a set $S$ that contains maximum value but has total weight of at most $W$.

### The Vertex Cover Problem

Given a graph $G = (V, E)$, we say that a set of nodes $S \subseteq V$ is a *vertex cover* if every edge $e = (i, j) \in E$ has at least one endpoint $i$ or $j$ in $S$. Our goal is to find a *minimum* vertex cover.

Given a graph $G$ and a number $k$, does $G$ contain a vertex cover of size at most $k$?




In this graph, the *minimum* vertex cover is

This is the same graph from last time when we discussed Independent Set. Do we notice any relationship? **Are there any implications of this?**

### Vertex Cover as a Linear Program

**a.** *Decision variables:* What are we try to solve for?

**b.** *Constraints:*

**c.** *Objective function:*

Vertex Cover as a Linear Program:

**Claim 1.** Let $S^*$ denote the optimal vertex cover of minimum weight, and let $x^*$ denote the optimal solution to the Linear Program. Then $\sum_{i \in V} w_i x_i^* \leq w(S^*)$.

**Claim 2.** The set $S = \{i : x_i \geq 0.5\}$ is a vertex cover, and $w(S) \leq 2 \sum_{i \in V} w_i x_i^*$.

## The Dual of a Linear Program

Every linear program has a *dual* linear program. We call the original linear program the *primal.* There are a bunch of amazing properties that come from LP duality.

Consider the following maximization problem. We want to find the dual linear program. A maximization problem's dual is a minimization problem.

Primal:

$$
\begin{aligned}
\max \quad & 8x_1 + 15x_2 + 3x_3 & \\
\text{subject to} \quad & 5x_1 + 4x_2 + 2x_3 \leq 0.6 & (y_1) \\
& 7x_1 + 2x_2 + 1x_3 \leq 0.35 & (y_2) \\
& x_1, x_2, x_3 \geq 0 & \text{(non-negativity)}
\end{aligned}
$$

Dual:

The following is the normal form for a maximization problem primal and its primal:

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \mathbf{y}^T\mathbf{b} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \end{aligned}$$

For the above example:

$\mathbf{A} =$ $\qquad\qquad$ $\mathbf{b} =$ $\qquad\qquad$ $\mathbf{c} =$

## Example 3: Maximum Matching

Given a graph $G = (V, E)$ choose a maximum size matching—a set of edges $S$ such that no vertex is covered by more than one edge.

Decision variables:

Linear Program:

Taking the dual of the above primal, we get what linear program?

What problem is this?