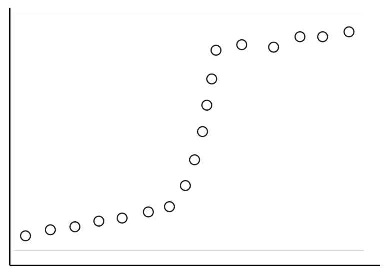
Dynamic Programming III: Segmented Least Squares

The Problem

We are given a set of points $\{p_1 = (x_1, y_2), p_2 = (x_2, y_2), \ldots, p_n = (x_n, y_n)\}$ sorted by x-coordinate. Our goal is to fit a (segmented) line to P with least squares error.



What is "error" here? We use square error (SSE) from any line we use. That is, if our line is determined by slope a and y-intercept b, then our SSE would be

$$SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

Using calculus, we can derive that this is minimized when we set

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2} \qquad \text{and} \qquad b = \frac{\sum_i y_i - a \sum_i x_i}{n}.$$

But what if we can use as many segments as we want, just with a penalty c for each additional segment? How should we decide on the number of segments, and on what the segments should look like?

Our goal is to partition P into some C contiguous segments with minimal least squares error when there is a penalty c for each segment.

Making the Key Observation

The last point p_n belongs to a single segment which must begin somewhere. Where does it begin? In each case, what does the optimal solution look like?

Step 1: The Subproblem

Let $\text{OPT}(i)$ denote the optimum solution for the points $p_1, \ldots, p_i$, and $\text{OPT}(0) = 0$. Let $e_{i,j}$ denote the minimum error of any line with respect to points $p_i, \ldots, p_j$ (i.e., find the line using the calculus solution up above).

Step 2: The Recurrence

If the last segment of the optimal partition is $p_i, \ldots, p_n$, then: $\textsc{opt}(n) = e_{i,n} + C + \textsc{opt}(i-1)$.

Hence

$$\textsc{opt}(j) = \min_{1 \leq i \leq j} \{e_{i,j} + C + \textsc{opt}(i-1)\}.$$

where we use the segment $p_i, \ldots, p_j$ if and only if $i \in$ argmax of the above.

Step 3: Prove that your recurrence is correct. In the optimal solution on j points, j must be in a segment that starts at the $i \in$ argmax of the above. $\textsc{opt}(i-1)$ gives the optimal segmented SSE for the first $i-1$ points and $e_{i,j}$ gives the optimal SSE for the segment from i to j, so adding these two error terms plus the penalty of C for using the one additional segment from i to j is the valid cost of this solution. If j instead was in a different segment that started at a different i', then for the same reasons, the cost of this solution would be $\textsc{opt}(i'-1) + C + e_{i',j}$, but this term did not minimize the above which is why it was not selected, hence it cannot have optimal (minimal) error. Hence the above recurrence is correct.

Step 4: State and prove your base cases. $\textsc{opt}(0) = 0$ is enough to get us off the ground. Notice that $\textsc{opt}(1)$ is then well-defined.

Step 5: State how to solve the original problem. This is once again $\textsc{opt}(n)$.

Step 6: The Algorithm

Algorithm 1 SegmentedLeastSquares($p_1, \ldots, p_n$)

> **Input:** Set of n points $p_1, \ldots p_n$.
> Initialize memo array M of length $n+1$ with $M[0] = 0$.
> **for** all pairs $i \leq j$ **do**
>   Compute $e_{i,j}$ for the segment $p_i, \ldots, p_j$
> **end for**
> **for** $j = 1, \ldots, n$ **do**
> $M[j] = \min_{1 \leq i \leq j}\{e_{i,j} + C + M[i-1]\}$
> **end for**
> **return** $M[n]$

Step 7: Running Time

The recurrence optimizes over n things, and we loop over n entries, so filling the memo takes $O(n^2)$ time. Solving for $e_{i,j}$ takes $O(n)$ time and there are $O(n^2)$ pairs of (i,j), so this takes $O(n^3)$ time, which is the dominant term for the algorithm.

Algorithm 2 FindSegments(j)

Input: Number of points n.

Initialize memo array M of length $n + 1$ with $M[0] = 0$.

if $j = 0$ **then**

  return $Null$

else

  Find an i that minimizes $e_{ij} + C + M[i - 1]$

  return the segment $\{p_i, \ldots, p_j\}$ and FindSegments($i - 1$)

end if