DS 574 Algorithmic Mechanism Design	**Lecture #7**
Fall 2023	Prof. Kira Goldner

Additive Review

The Additive Setting: There are m non-identical items and n bidders where each bidder i has private valuation v_{ij} for each item j. Bidder i has

$$v_i(S) := \sum_{j \in S} v_{ij}.$$

The welfare-optimal direct revelation mechanism: just handle each item separately—m Vickrey auctions!

Ascending implementation: Parallel English Auctions: Maintain a set of interested bidders for each item, and the auction for item j terminates when there's only one active bidder remaining, breaking ties arbitrarily.

Is this DSIC? No! One bidder can "threaten" another by doing something that doesn't maximize their own utility, motivating the need for another solution concept.

Definition 1. A strategy profile $(\sigma_1, \ldots, \sigma_n)$ is an *ex post Nash equilibrium (EPNE)* if, for every bidder i and valuation $v_i \in V_i$, the strategy $\sigma_i(v_i)$ is a best-response to every strategy profile $\sigma_{-i}(\mathbf{v}_{-i})$ with $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$.

In comparison, in a dominant-strategy equilibrium (DSE), for every bidder i and valuation v_i, the action $\sigma_i(v_i)$ is a best response to every action profile $\mathbf{a}_{-i}$ of $\mathbf{A}_{-i}$, whether of the form $\sigma_{-i}(\mathbf{v}_{-i})$ or not.

Definition 2. A mechanism is *ex post incentive compatible (EPIC)* if sincere bidding is an ex post Nash equilibrium in which all bidders always receive nonnegative utility.

Claim 1. For n additive bidders with m heterogenous items, in parallel English auctions, sincere bidding by all bidders is an ex post Nash equilibrium (up to $m\varepsilon$).

Unit Demand

The Unit-Demand Setting: There are m non-identical items and n bidders where each bidder i has private valuation v_{ij} for each item j. Bidder i is unit demand, that is, wants at most one item for any set S:

$$v_i(S) := \max_{j \in S} v_{ij}.$$

First, solve the direct-revelation problem. What do we observe about the welfare-maximizing allocation in the unit-demand setting? Each bidder gets at most one item. Each item is allocated to one bidder. If an "edge" (i, j) represents bidder i's value v_{ij} for item j, then want to choose the allocation that gives the maximum-weight bipartite matching. This problem can be solved in polynomial time!

Refresh yourself on what the VCG mechanism looks like. Then what does the analogous ascending auction look like?

1. Set a price q_j for each item j, initializing each price to 0.
2. Initially all bidders are unassigned.
3. while (TRUE):
   (a) Ask each bidder for a favorite item (or $\emptyset$) at the prices $\mathbf{q} + \varepsilon$, meaning an item $j \in D_i(\mathbf{q} + \varepsilon) := \text{argmax}_k\{v_{ik} - (q_k + \varepsilon)\}$. Treat this as a "bid" for item j.
   (b) If no unassigned bidder submits a bid, then halt with the current allocation and prices $\mathbf{q}$.
   (c) Otherwise, pick an arbitrary unassigned bidder i that bid for item j and assigned j to i.
      i. If item j was previously assigned to bidder i', mark i' as unassigned and increase the price q_j by ε.

So the ascending auction implementation essentially decreases demand (by raising prices) until supply is equal to demand, where "demand" is equal to a bidder's favorite item at the given prices. This is called the Crawford-Knoer (CK) Auction.

Walrasian Equilibria in the Unit-Demand Setting

The Unit-Demand Setting: There are m non-identical items U and n bidders where each bidder i has private valuation v_{ij} for each item j. Bidder i is unit demand, that is, wants at most one item for any set S:

$$v_i(S) := \max_{j \in S} v_{ij}.$$

Definition 3. In the unit-demand setting, a *Walrasian equilibrium* (or "competitive equilibrium") is a price vector $\mathbf{q} \in \mathbb{R}^m$ defined on the items and a matching M of the bidders and items such that:

1. Each bidder i is matched to a favorite item $j \in \text{argmax}\{v_{ij} - q_j\}_{j \in U \cup \{\emptyset\}}$. (WE1)

   Equivalently, $\mathbf{q}$ is an *envy-free pricing.*

2. An item $j \in U$ is unsold *only* if $q(j) = 0$. (WE2)

We call $D_i(\mathbf{q}) = \text{argmax}\{v_{ij} - q_j\}_{j \in U \cup \{\emptyset\}}$ the *demand set* of i under prices $\mathbf{q}$.

Claim 2 (First Welfare Theorem)**.** In the unit-demand setting, if $(\mathbf{q}, M)$ is a Walrasian Equilibrium, then M is a welfare-maximizing allocation.

This essentially says "markets are efficient," and there are many "First Welfare Theorems" each with this flavor. Exercise: Prove this.

What we'll now see that is the VCG allocation and payment *is* a WE, and in fact, is a lower bound on all WE for the unit-demand setting.

Recall the VCG payment in this setting:

$$p_i = \sum_{k \neq i} v_k(M^{-i}(k)) - \sum_{k \neq i} v_k(M(k))$$

where $M(k)$ is the item that k is allocated in the welfare-maximizing (maximum-weight) matching, and M^{-i} is the welfare-maximizing matching without bidder i.

Theorem 1 (VCG Payments Lower Bound WE)**.** *In the unit-demand setting, let* $\mathbf{p}$ *denote the induced item price vector of the truthful-revelation VCG outcome and* $\mathbf{q}$ *a Walrasian price vector. Then* $p(j) \leq q(j)$ *for every item* j*.*

Proof. Let M denote the allocation computed by the VCG mechanism. Let M^{-i} denote a welfare-maximizing allocation among allocations that leave bidder i unmatched. The pair $(\mathbf{q}, M)$ is a WE. (Why?) For every $k \neq i$, (WE1) of $(\mathbf{q}, M)$ can be used to argue that k prefers $M(k)$ over $M^{-i}(k)$ at the prices $\mathbf{q}$:

$$v_k(M(k)) - q(M(k)) \geq v_k(M^{-i}(k)) - q(M^{-i}(k)),$$

and summing over all $k \neq i$ gives

$$\sum_{k \neq i} v_k(M(k)) - \underbrace{\sum_{k \neq i} q(M(k))}_{= Q - q(j) \text{ by (WE2)}} \geq \sum_{k \neq i} v_k(M^{-i}(k)) - \underbrace{\sum_{k \neq i} q(M^{-i}(k))}_{\leq Q}$$

where $Q = \sum_{j'} q(j')$, because $\sum_{k \neq i} q(M(k))$ sums over all of the items with non-zero q-prices except for the item matches to i (which we call j). Rearranging gives

$$q(j) \geq \sum_{k \neq i} v_k(M^{-i}(k)) - \sum_{k \neq i} v_k(M(k)) = p(j),$$

where the equation follows from the definition of prices from the VCG mechanism. □

Theorem 2 (VCG Outcome is a WE)**.** *In the unit-demand setting, let* M *and* $\mathbf{p}$ *denote the allocation and induced item price vector of the truthful-revelation VCG outcome. Then* $(\mathbf{p}, M)$ *is a WE.*

Then in unit-demand settings, a WE is guaranteed to exists, there is a "smallest" WE, and the VCG outcome is precisely this smallest WE. We leave the proof as an exercise, but you may want to use the following lemma.

Lemma 1. *In the unit-demand setting, let* M *and* $\mathbf{p}$ *denote the allocation and induced item price vector of the truthful-revelation VCG outcome. For a good* $j \in U$*, let* M^{+j} *denote a welfare-maximizing allocation after adding a second copy* j' *of the good* j *(with* $v_{ij} = v_{ij'}$ *for every bidder* i*). Then*

$$p(j) = \sum_{k=1}^{n} v_k(M^{+j}) - \sum_{k=1}^{n} v_k(M).$$

The Crawford-Knoer Auction

1. Set a price q_j for each item j, initializing each price to 0.
2. Initially all bidders are unassigned.
3. while (TRUE):
   (a) Ask each bidder for a favorite item (or $\emptyset$) at the prices $\mathbf{q} + \varepsilon$, meaning an item $j \in D_i(\mathbf{q} + \varepsilon) := \text{argmax}_k\{v_{ik} - (q_k + \varepsilon)\}$. Treat this as a "bid" for item j.
   (b) If no unassigned bidder submits a bid, then halt with the current allocation and prices $\mathbf{q}$.
   (c) Otherwise, pick an arbitrary unassigned bidder i that bid for item j and assign j to i.
      i. If item j was previously assigned to bidder i', mark i' as unassigned and increase the price q_j by ε.

Observations:

- The price of item j starts at 0, and it takes someone bidding on item j (and each subsequent out-bid) to increase it by ε.
- After some bidder i bids for item j, they remain matched until j is outbid by another bidder—i cannot let go or stop bidding on j.
- Once j is bid on, it is forever more assigned to a bidder.
- Assuming sincere bidding, bidders i always honestly report an item from $D_i(q+\varepsilon)$ (or report that there none), then the CK auction will terminate in at most $mv_{\max}/\varepsilon$ iterations, $v_{\max}/\varepsilon$ per item, where $v_{\max} = \max_{i,j} v_{ij}$.

Analysis of the CK Auction

Theorem 3. *Up to ε terms, the outcome of the CK auction under sincere bidding is the VCG outcome under truthful revelation.*

Lemma 2. *If all bidders bid sincerely, then the CK auction terminates at an ε-WE (q, M).*

Corollary 4. *If all bidders bid sincerely, then the CK auction terminates with an allocation that has surplus within $m\varepsilon$ of the maximum possible.*

Note: Consistent vs. sincere bidding—i's possible actions:

1. Answer all queries honestly (with respect to v_i).
2. For some valuation $v_i' \neq v_i$, answer all queries as *if* its valuation was v_i'.
3. Answer queries in an arbitrary, possibly inconsistent, way. valuation.

(1) and (2) are consistent with respect to *some* valuation.

Proposition 5. *Let $\mathcal{A}$ be an iterative auction such that the sincere bidding outcome of $\mathcal{A}$ is the same as the truthful revelation outcome of the VCG mechanism. For every bidder i and valuation profile $\mathbf{v}$, if every player other than i bids sincerely, then sincere bidding is bidder i's best response* among consistent actions*.*

Theorem 6. *The CK auction is EPIC (up to error $2\varepsilon \cdot \min\{m, n\}$).*

See Tim Roughgarden's notes for an expansion on this section.

The Gross Substitutes Condition

A General Valuation Model

The most general welfare-maximization problem we'll consider in this course is the following.

- There is a set U of m non-identical goods.
- Each bidder $i = 1, 2, 3, \ldots, n$ has a private valuation $v_i(S)$ for each bundle $S \subseteq U$ of goods that it might receive.
  - Assumption #1: $v_i(\emptyset) = 0$.
  - Assumption #2: "free disposal," meaning the monotonicity condition that $v_i(S) \leq v_i(T)$ whenever $S \subseteq T$.

Generalized Walrasian Equilibrium: A Walrasian equilibrium (WE) is a nonnegative price vector $\mathbf{q}$ on the items and an allocation $(S_1, \cdots, S_n)$ such that:

(WE1) Each bidder i is matched to a favorite bundle

$$S \in \text{argmax}\{v_i(S) - \sum_{j \in S} q(j)\}_{S \subseteq U} = D_i(\mathbf{q}),$$

with the empty set $S = \emptyset$ is allowed.

(WE2) An item $j \in U$ is unsold only if $q(j) = 0$.

The Kelso-Crawford Auction

An extension of the CK auction where bidders can bid on more than one item at once, and can also bid for new items even if some items are already assigned to them. It remains impossible to withdraw from a bid.

Kelso-Crawford (KC) Auction:

1. Initialize the price of every item j to $q(j) = 0$.
2. For every bidder i, initialize the set S_i of items assigned to i to $\emptyset$.
3. while (TRUE):

(a) Ask each bidder for their favorite subset of items not assigned to them, given the items they already have and the current prices—an arbitrary set T_i in

$$\text{argmax}_{T \subseteq U \setminus S_i}\{v_i(S_i \cup T) - \mathbf{q}^{\varepsilon}(S_i \cup T)\},$$

where

$$\mathbf{q}^{\varepsilon}(S_i \cup T) = \sum_{j \in S_i} q(j) + \sum_{j \in T}(q(j) + \varepsilon).$$

(b) If $T_i = \emptyset$ for all bidders i, then halt with the current allocation $(S_1, \ldots, S_n)$ and prices $\mathbf{q}$.

(c) Otherwise, pick an arbitrary bidder i with $T_i \neq \emptyset$:

i. $S_i \leftarrow S_i \cup T_i$;

ii. for all $k \neq i$, $S_k \leftarrow S_k \setminus T_i$;

iii. for $j \in T_i$; $q(j) \leftarrow q(j) + \varepsilon$.

In the special case of unit-demand bidders, the KC auction is identical to the CK auction.

For bidders with general valuations, bidding sincerely in the KC auction can be a disaster.

Example: Suppose $U = \{L, R\}$—a left and right shoe. Bidder 1 is "single-minded" for the pair of shoes, that is, $v_1(\{L, R\}) = 3$ and otherwise $v_1 = 0$. Bidder 2 is unit-demand, only wanting a single shoe for an art project, with $v_2(\emptyset) = 0$ and otherwise $v_2 = 2$.

What happens in the KC auction? Bidder 1 will bid on both shoes, increasing by prices by ε, and bidder 2 will bid on the cheaper or otherwise an arbitrary shoe, taking turns incrementing the L and R prices by ε. This will continue until the prices for both shoes are 1.5, at which point bidder 1 will stop bidding as the pair costs too much. However, bidder 1 is stuck buying at least one shoe, since they cannot relinquish items and bidder 2 will not buy both. Then bidder 1 can either get -1.5 utility for a useless single shoe, or outbid bidder 2 for the second shoe when each shoe is priced at 2, for a total of 4, to get less negative utility of -1, which is preferable.

Note that a left and right shoe are *complements*—they have more value as a whole than the sum of their parts. This is typically a difficult setting to study, as opposed to *substitutes*, where the whole has *at most* the value of the sum of its parts.

The Gross Substitutes Condition

Definition 4. A valuation v_i defined on item set U satisfies the *gross substitutes (GS) condition* if and only if the following condition holds. For every price vector $\mathbf{p}$, every set $S \in D_i(\mathbf{p})$, and every price vector $\mathbf{q} \geq \mathbf{p}$, there is a set $T \subseteq U$ with

$$(S \setminus A) \cup T \in D_i(\mathbf{q}),$$

where $A = \{j : q(j) > p(j)\}$ is the set of items whose prices have increased (in $\mathbf{q}$ relative to $\mathbf{p}$).

Theorem 7. *If all bidders have gross substitutes valuations and bid sincerely, then the Kelso-Crawford auction terminates at a $m\varepsilon$-Walrasian equilibrium.*

Proof. Unsold items have price 0 because bidders only relinquish an item when outbid by another bidder, so an item goes unsold only if no bidder even bid on it, in which case its final price is 0.

With GS valuations, we claim the KC auction maintains the following invariant:

> for every bidder i, S_i is contained in a set of $D_i(\mathbf{q}^\varepsilon)$, where $\mathbf{q}^\varepsilon(j)$ equals $q(j)$ for $j \in S_i$ and $(q(j) + \varepsilon)$ for $j \notin S_i$.

That is, no bidder ever wants to withdraw its bids for the items it possesses. The base case, where $S_i = \emptyset$ for each i, is trivial.

For the inductive step, consider bidder i. If i is chosen to bid in this iteration, then the inductive hypothesis ensure that there is a set $T_i \subseteq U \setminus S_i$ such that $S_i \cup T_i \in D_i(\mathbf{q}^\varepsilon)$. Thus, after this iteration i will possess a set in $D_i(\mathbf{q}^\varepsilon)$. Otherwise, let A_i be the last set of items that i bid on. By the inductive hypothesis, B_i was a preferred bundle at the prices at the time. In the subsequent iterations, including the current one, the items reassigned from i to other bidders have had their price increased, while the prices of the items in S_i have stayed the same. By the definition of the GS condition, S_i belongs to a set of $D_i(\mathbf{q}^\varepsilon)$, completing the inductive step.

By the invariant and the KC auction's stopping rule, at termination $S_i \in D_i(\mathbf{q}^\varepsilon)$ for every bidder i. Since the final prices $\mathbf{q}$ differ from $\mathbf{q}^\varepsilon$ by at most ε on each good, the KC auction terminates with an $m\varepsilon$-WE. □

Taking the limit as $\varepsilon \to 0$ gives the following remarkable consequence of the KC auction.

Corollary 8. *If valuations $v_1, \cdots, v_n$ satisfy the gross substitutes condition, then there exists a Walrasian equilibrium.*

Proof Sketch. Consider a sequence of $\frac{1}{N}$-WE for $N = 1, 2, 3, \ldots$, which exists by Theorem 4.2. Some allocation $(S_1, \cdots, S_n)$ repeats infinitely often. The corresponding price vectors lies in a compact set, bounded by the valuations, so they have an accumulation point. This point, together with $(S_1, \cdots, S_n)$, is a Walrasian equilibrium. □

Thus far we've been taking for granted the existence of Walrasian equilibrium. In many cases, however, WE do not exist.

Example: In our same left-and-right-shoe example, no WE exists. By the First Welfare Theorem, a WE is welfare-maximizing, so in any WE, bidder 1 must be allocated both shoes. In order for this bundle to be in bidder 1's demand set, then the price of the shoes must be at most 3 (otherwise the bidder would prefer the empty set). However, this forces one of the shoes to be priced at less than 1.5, which is less than 2, so bidder 2 would have that shoe in their demand set and not be allocated it, meaning this is not a WE.

More generally, the gross substitutes condition is in some sense the frontier for the guaranteed existence of WE.

Theorem 9 (Gul-Stacchetti)**.** *If v_i is a valuation that does not satisfy the GS condition, there are unit-demand (and hence GS) valuations $\mathbf{v}_{-i}$ such that $\mathbf{v}$ admits no Walrasian equilibrium.*

An Impossibility Result

Can we get the VCG outcome with GS valuations using an ascending auction? Perhaps surprisingly, the answer is no.

Theorem 10 (Gul-Stacchetti)**.** *There is no ascending auction for which sincere bidding yields the VCG outcome for every profile of gross substitutes valuations.*