DS 574 Algorithmic Mechanism Design	**Lecture #15**
Fall 2022	Prof. Kira Goldner

Prophet Inequalities

You're at a casino gambling, and are offered to play the following game. Items will arrive one-by-one. As an item arrives, you see its value. You may only take a single item, and once you take an item, the game ends. A priori, you know the *distribution* of each item. At some point there will be a red item with the red distribution of values, and at some point there will be a blue item with the blue distribution of values, and so forth. However, you do not know the order of items (it is adversarial), and you do not know the exact values of the items (they are drawn from their specific distributions). Your goal is to come up with an algorithm that competes with the *prophet* who is all knowing, so knows the realization of values and the arrival order.

That is, n items will arrive in adversarial order. Item i (which is a label, not necessarily the order) has value v_i drawn from known distribution F_i. Your goal is to determine an algorithm $\textsc{Alg}$ such that the value you get from gambling competes with the prophet who always gets $\max_i v_i$. However, your competition is over the randomness of the values that are drawn, so you only have to compete with $\textsc{opt} = \mathbb{E}_{\mathbf{v}}[\max_i v_i]$.

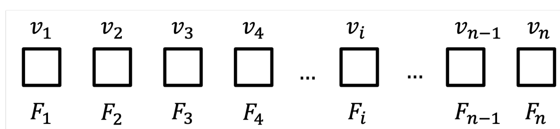

Figure 1: The prophet inequality problem.

To summarize:

- Goal: Pick one item; maximize its value.
- Gambler knows distribution for each item.
- Order is adversarial.
- Inspect each item online (see v_i) and irrevocably decide whether to take or pass forever.
- Compete with $\textsc{opt} = \mathbb{E}_{\mathbf{v}}[\max_i v_i]$.

The Prophet Inequality problem was posed by [Samuel-Cahn '84], with the original solution and analysis that we'll see by [Krengel Sucheston '78] and [Garling]. It was brought to Algorithmic Mechanism Design by [Hajiaghayi Kleinberg Sandholm '07], and a new analysis for this case was developed by [Kleinberg Weinberg '12].

Prove the following.

Theorem 1. *There is a threshold algorithm* $\textsc{Alg}$ *such that when the gambler takes an item if and only if its value is above* T*,* $\textsc{Alg} \geq \frac{1}{2}\textsc{opt}$*.*

Determine what threshold T to use and prove this statement using the following steps:

1. Divide what the algorithm yields from an item (in expectation) into exactly the threshold and the surplus above the threshold.
2. Lower bound your surplus term.
3. Set your threshold in order to combine like-terms and have $\textsc{opt}$ pop out.

Note: Can you find two different thresholds that give this same approximation?

Proof. We consider two different ways to set the threshold, starting a proof of what our algorithm obtains using the framework above. Let p denote the probability that *some* (at least one) $v_i \geq T$ for $i \in [n]$.

We will set the threshold T such that either (1) $T = \frac{1}{2}\mathbb{E}[\max_i v_i]$, or (2) such that $p = \frac{1}{2}$.

$$\begin{aligned}
\textsc{Alg} &= \Pr[\text{any above}]T + \sum_i \Pr[\text{all } j < i \text{ below}] \cdot \mathbb{E}[(v_i - T)^+] \\
&\geq pT + (1-p)\sum_i \mathbb{E}[(v_i - T)^+] \\
&\geq pT + (1-p)(\mathbb{E}[\max_i v_i] - T) && (*)
\end{aligned}$$

$$\text{using (1)} \qquad \begin{aligned}
(*) &\geq p\left(\frac{1}{2}\mathbb{E}[\max_i v_i]\right) + (1-p)\left(\frac{1}{2}\mathbb{E}[\max_i v_i]\right) \\
&= \frac{1}{2}\textsc{opt}.
\end{aligned}$$

$$\text{using (2)} \qquad \begin{aligned}
(*) &= \frac{1}{2}T + \frac{1}{2}\left(\mathbb{E}[\max_i v_i] - T\right) \\
&= \frac{1}{2}\textsc{opt}.
\end{aligned}$$

Hence a threshold algorithm set using (1) or (2) produces a $\frac{1}{2}$-approximation to the prophet (a $\frac{1}{2}$-competitive ratio). $\square$

Exercise: You could see this as a mechanism for a buyer to maximize social welfare. Could you design a mechanism to maximize revenue using the prophet inequality?

[Hint: Use virtual values.]

See Roughgarden Twenty Lectures (364A) Lecture 6 Section 3 for a formal treatment on how to do this.