DS 574 Algorithmic Mechanism Design **Lecture #3 Worksheet**
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Beyond Dominant-Strategy: Bayesian Settings

There are many reasons why we can't always require dominant strategies when design mechanisms.

(1) Requiring such a strong concept might not be tractable.

(2) Agents do not always have dominant strategies! What then?

We'll now introduce the Bayesian setting.

Suppose the valuation v_i of bidder i is drawn from a prior distribution F_i.

- CDF $F_i(x) = \Pr_{v_i \sim F_i}[v_i \leq x]$.
- PDF $f_i(x) = \frac{d}{dx} F_i(x)$.
- Joint distribution $\mathbf{F}$ or $\vec{F}$.

Unless otherwise noted, we assume that the prior distribution $\mathbf{F}$ is *common knowledge* to all bidders and the mechanism designer (the seller).

Definition 1. A *Bayes-Nash equilibrium (BNE)* for a joint distribution $\mathbf{F}$ is a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that for all i and v, $\sigma_i(v_i)$ is a best-response when other agents play $\sigma_{-i}(\mathbf{v}_{-i})$ when $\mathbf{v}_{-i} \sim \mathbf{F}_{-i} |_{v_i}$.

Claim 1. Consider two identically and independently drawn bidders from $F = U[0, 1]$. It is a (symmetric) BNE for each bidder to bid $\sigma_i(v_i) = v_i/2$ in the first-price auction.

Proof.

Theorem 1 (Revenue Equivalence)**.** *The payment rule and revenue of a mechanism is uniquely determined by its allocation. Hence, any two mechanisms with the same allocation must earn the same revenue.*

What is this theorem a corollary of? Prove this for the first-price auction and the Vickrey (second-price) auction in the above setting!

Proof.

Bayesian Settings

Using notions from the Bayesian setting and how bidders Bayesian update as they learn information, we define three stages of the auction:

1. *ex ante*:

2. *interim*:

3. *ex post*:

Typically we discuss the *ex post* allocation and payment rules as a function of all of the values. However, in the Bayesian setting, to reason about BIC, it often makes sense to take in terms of *interim* allocation and payment rules which have the same information as bidder i before the auction is run.

Definition 2. We define the *interim* allocation and payment rules in expectation over the updated Bayesian prior given i's valuation:

$$x_i(v_i) =$$

and

$$p_i(v_i) =$$

Our definition of Bayesian Incentive-Compatibility then follows:

Definition 3. A mechanism with *interim* allocation rule x and *interim* payment rule p is Bayesian Incentive-Compatible (BIC) if

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(z) - p_i(z) \quad \forall i, v_i, z.$$

Virtual Welfare

Imagine a single buyer will arrive with their private value v. We want to design DSIC mechanisms.

What mechanism should you use to maximize *welfare*?

What should you do to maximize (expected) revenue?

Definition 4. In a deterministic mechanism, given other bids $\mathbf{b}_{-i}$, bidder i's *critical bid* is the minimum bid $b_i^* = \min\{b_i : x_i(b_i, \mathbf{b}_{-i}) = 1\}$ such that bidder i is allocated to.

Then with $\mathbf{b}_{-i}$ fixed, for all winning $v_i \geq b_i^*$, i's payment $p_i(v_i, \mathbf{b}_{-i}) = b_i^*$ is their critical bid.

What is winner i's critical bid in a single-item auction?

What about in the k identical item setting?

Maximizing Expected Revenue

Recall:

- The revelation principle says that it's without loss to focus only on truthful mechanisms.
- Payment is determined by the allocation:

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i})\, dz$$

We want to maximize $\mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})]$.

$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, \mathbf{v}_{-i})] =$

where

$$\varphi_i(v_i) =$$

is the Myersonian virtual value and $(*)$ follows by switching the order of integration. Then

$$\textsc{Revenue} = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})] =$$

$$= \textsc{Virtual Welfare}$$

Given this conclusion, how should we design our allocation rule x to maximize expected virtual welfare (expected revenue)?

When would this cause a problem with incentive-compatibility?

Definition 5. A distribution F is regular if the corresponding virtual valuation function $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.

Suppose we are in the single-item setting and all of the distributions are regular. What do the payments look like in the virtual-welfare-maximizing allocation?

For a fixed $\mathbf{b}_{-i}$, if i is the winner, then i's payment is i's critical bid, which is

Exercise: what about for k identical items?

Claim 2. A virtual welfare maximizing allocation x is monotone if and only if the virtual value functions are regular.

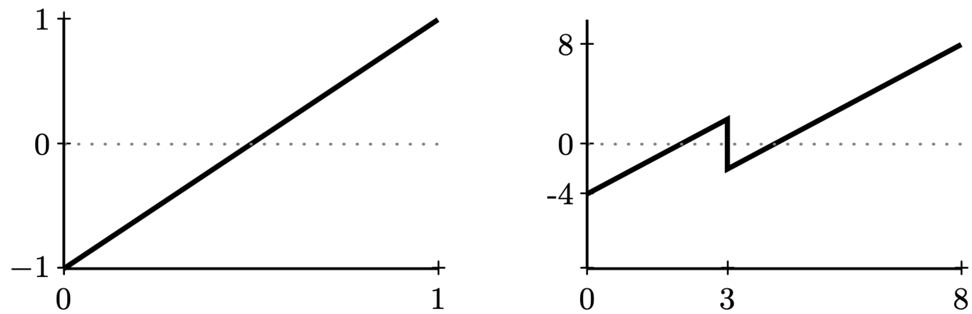
Exercise: Argue this.

It will be helpful to keep the following two examples in mind:

a. a uniform agent with $v \sim U[0,1]$. Then $F(x) = x$ and $f(x) = 1$.

b. a bimodal agent with

$$v \sim \begin{cases} U[0,3] & w.p. \frac{3}{4} \\ U(3,8] & w.p. \frac{1}{4} \end{cases} \qquad \text{and} \qquad f(v) = \begin{cases} \frac{3}{4} & v \in [0,3] \\ \frac{1}{20} & v \in (3,8] \end{cases}$$



(a) Uniform agent virtual value.

(b) Bimodal agent virtual value.

Figure 1: Virtual value functions $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ for the uniform and bimodal agent examples.

Do the following:

- Calculate the virtual values for both examples.

- Are they regular? Are there any issues using the allocation that maximizes expected virtual welfare?

- What does that allocation actually look like?