Recap/Big Picture

DS 574 LECTURE 10

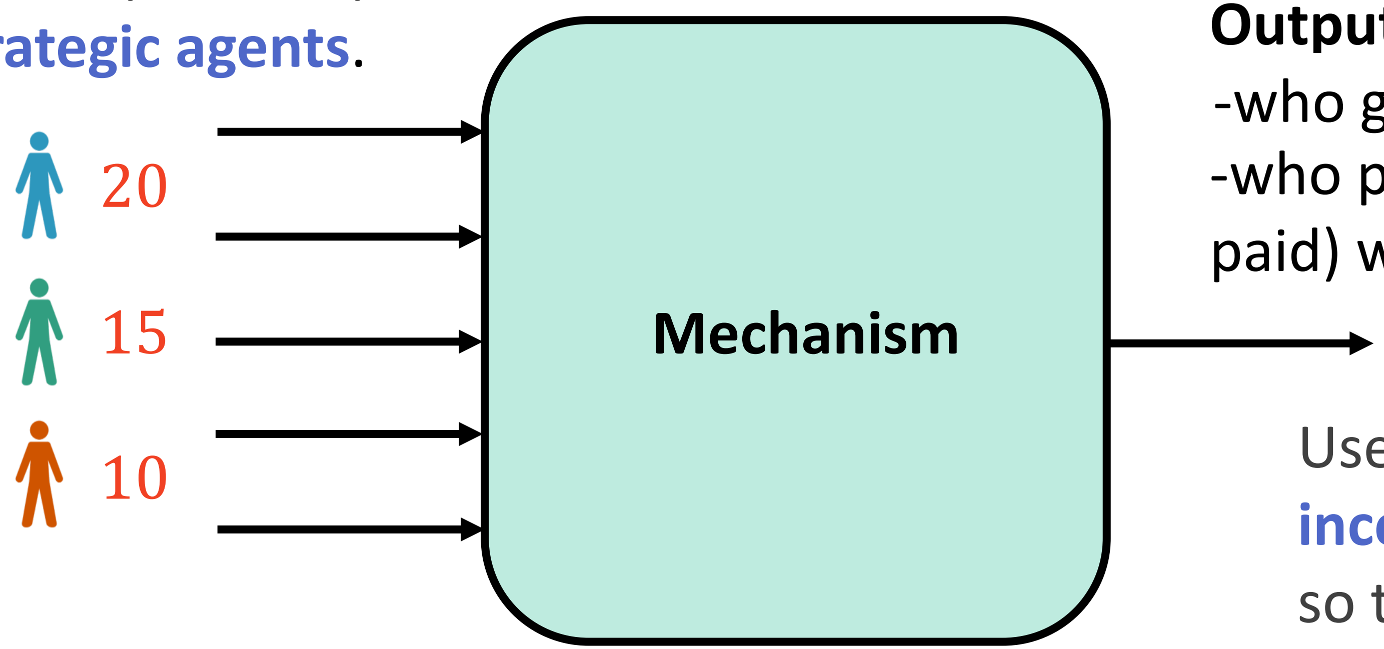
Econ→CS

Algorithmic problems

Economic concepts, arguments

Input:
Data reported by **strategic agents**.

Objective: Maximize buyer's value



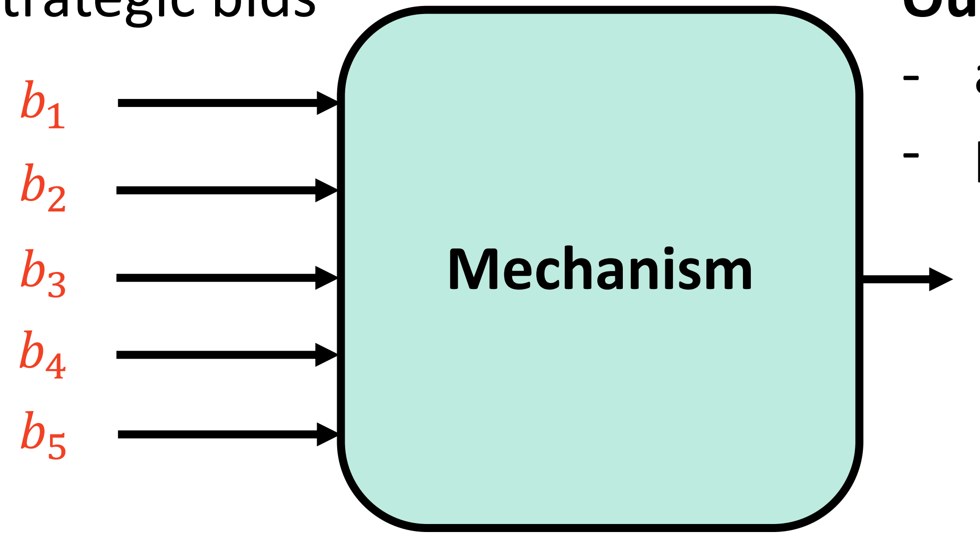
Output:
-who gets what
-who pays (gets paid) what

Use **game theory** to reason about **incentives** within the **algorithm** so that we can **guarantee** (approximate) optimality.

Maximize Social Welfare: 2nd Price

Objective: Maximize value of the allocation

Input: Strategic bids



Output:

- allocation — highest bidder — $x_1 = 1$
- payment — 2nd highest bid — b^2

2nd Price (Vickrey) Auction is DSIC:
maxes i's utility to have $b_i =$
v_i independent of all b_{-i}

value v_i
utility $v_i\ x_i(\boldsymbol{b}) - p_i(\boldsymbol{b})$
bid b_i

$\Rightarrow b_i = v_i\ \forall i$

$b_i > v_i$: if b^2 is in between, i wins and overpays
$b_i < v_i$: if b^2 is in between, i loses and gets 0 util instead of positive

Dominant Strategy Incentive Compatibility

More utility for bidding actual value:

$$\underline{v_i\, x_i(v_i, \boldsymbol{b}_{-i})} - \boxed{p_i(v_i, \boldsymbol{b}_{-i})} \geq v_i\, x_i(b_i, \boldsymbol{b}_{-i}) - p_i(b_i, \boldsymbol{b}_{-i}) \qquad \forall i, v_i, b_i, \boldsymbol{b}_{-i}$$

1) The allocation rule must be **monotone**, or this can't hold. implementable

Myerson's Lemma

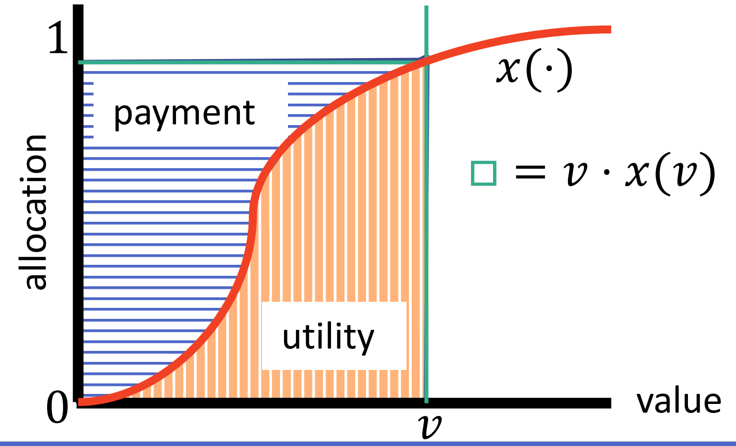
2) DSIC payments are completely determined by the allocation rule:

Payment Identity

$$\boxed{p_i(v_i, \boldsymbol{b}_{-i})} = \int_0^{v_i} z\, x_i'(z, \boldsymbol{b}_{-i})dz \qquad = \underline{v_i\, x_i(v_i, \boldsymbol{b}_{-i})} - \underline{\int_0^{v_i} x_i(z, \boldsymbol{b}_{-i})dz}$$

value v_i
utility $v_i\, x_i(\boldsymbol{b}) - p_i(\boldsymbol{b})$
bid b_i

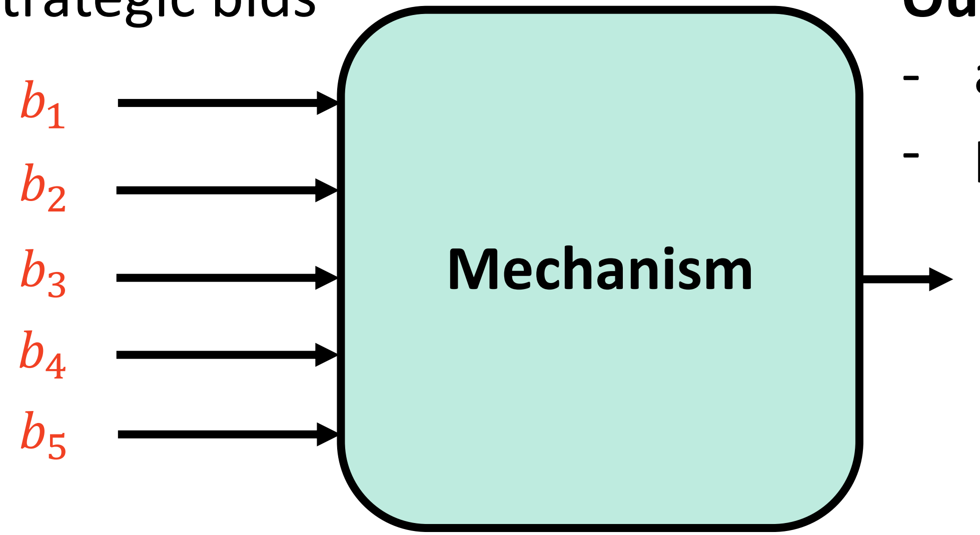
$$\Rightarrow b_i = v_i\ \forall i$$



Maximize Social Welfare: 1st Price

Objective: Maximize value of the allocation

Input: Strategic bids



Output:

- allocation highest bidder $x_1 = 1$
- payment own bid b^1

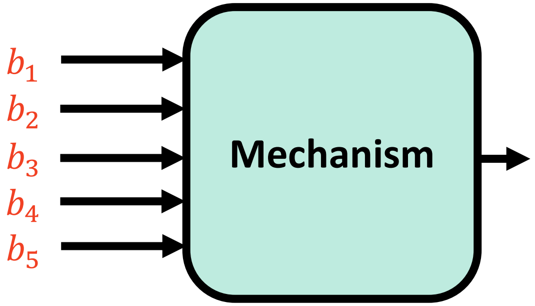
1st Price Auction is not DSIC:
$b_i = v_i$ means utility is 0,
better have $b_i < v_i$

value v_i
utility $v_i \, x_i(\boldsymbol{b})$ – $p_i(\boldsymbol{b})$
bid b_i

The Bayesian Setting: Stages

Each bidder i's value v_i is drawn from a distribution with CDF F_i and pdf f_i

- $F_1, \dots, F_n$ are common knowledge to all bidders and the auctioneer
- $F_i(x) = \Pr[v_i \le x]$
- $f_i(x) = \frac{d}{dx} F_i(x)$



value v_i
utility $v_i\, x_i(\boldsymbol{b}) - p_i(\boldsymbol{b})$
bid b_i

ex ante: no values are known. mechanism announced.

interim: i knows v_i, Bayesian updates given this
bidders submit bids

ex post: outcome announced. know $v_1, \dots, v_n$

needed:

- for bidders to reason about other bidders' behavior (BNE)
- for auctioneer to reason about objective in expectation

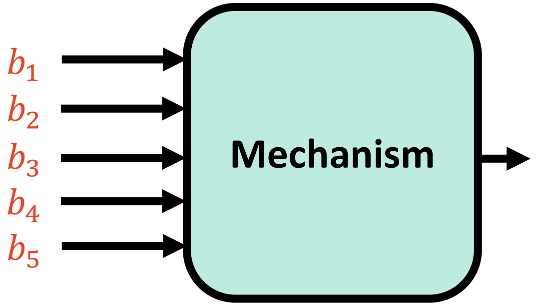
The Bayesian Setting: Incentive Compatibility

Each bidder i's value v_i is drawn from a known distribution F_i

BIC: $\mathbb{E}_{\boldsymbol{v}_{-i}}[v_i\, x_i(v_i, \boldsymbol{v}_{-i}) - p_i(v_i, \boldsymbol{v}_{-i})] \geq$
$\mathbb{E}_{\boldsymbol{v}_{-i}}[v_i\, x_i(b_i, \boldsymbol{v}_{-i}) - p_i(b_i, \boldsymbol{v}_{-i}))] \quad \forall i, v_i, b_i$

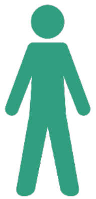
NOT $\forall \boldsymbol{b}_{-i}$ but in $\mathbb{E}_{\boldsymbol{v}_{-i}}$!

$$v_i \hat{x}_i(v_i) - \hat{p}_i(v_i) \geq v_i \hat{x}_i(b_i) - \hat{p}_i(b_i) \quad \forall i, v_i, b_i$$



interim: i knows v_i, Bayesian updates given this
bidders submit bids

$$\hat{x}_i(b_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[x_i(b_i, \boldsymbol{v}_{-i})] \qquad \hat{p}_i(b_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[p_i(b_i, \boldsymbol{v}_{-i})]$$



value v_i
utility $v_i\, x_i(\boldsymbol{b}) - p_i(\boldsymbol{b})$
bid b_i

ex post: outcome announced. know $v_1, \ldots, v_n$
$x_i(b_i, \boldsymbol{b}_{-i}) \qquad p_i(b_i, \boldsymbol{b}_{-i})$

DSIC: $v_i\, x_i(v_i, \boldsymbol{b}_{-i}) - p_i(v_i, \boldsymbol{b}_{-i}) \geq v_i\, x_i(b_i, \boldsymbol{b}_{-i}) - p_i(b_i, \boldsymbol{b}_{-i}) \quad \forall i, v_i, b_i, \boldsymbol{b}_{-i}$

Nash Equilibrium vs. Incentive-Compatibility

A mechanism is [concept] Incentive-Compatible if in the mechanism, truthful reporting is a [concept] Nash Equilibrium. (i.e. [concept] \in Dominant Strategy, Bayes-Nash, Ex Post*)

*sincere bidding may be required instead of truthful

BNE: Best-response strategies $\boldsymbol{\sigma}$ form a Bayes-Nash Equilibrium (BNE) in (x, p) when

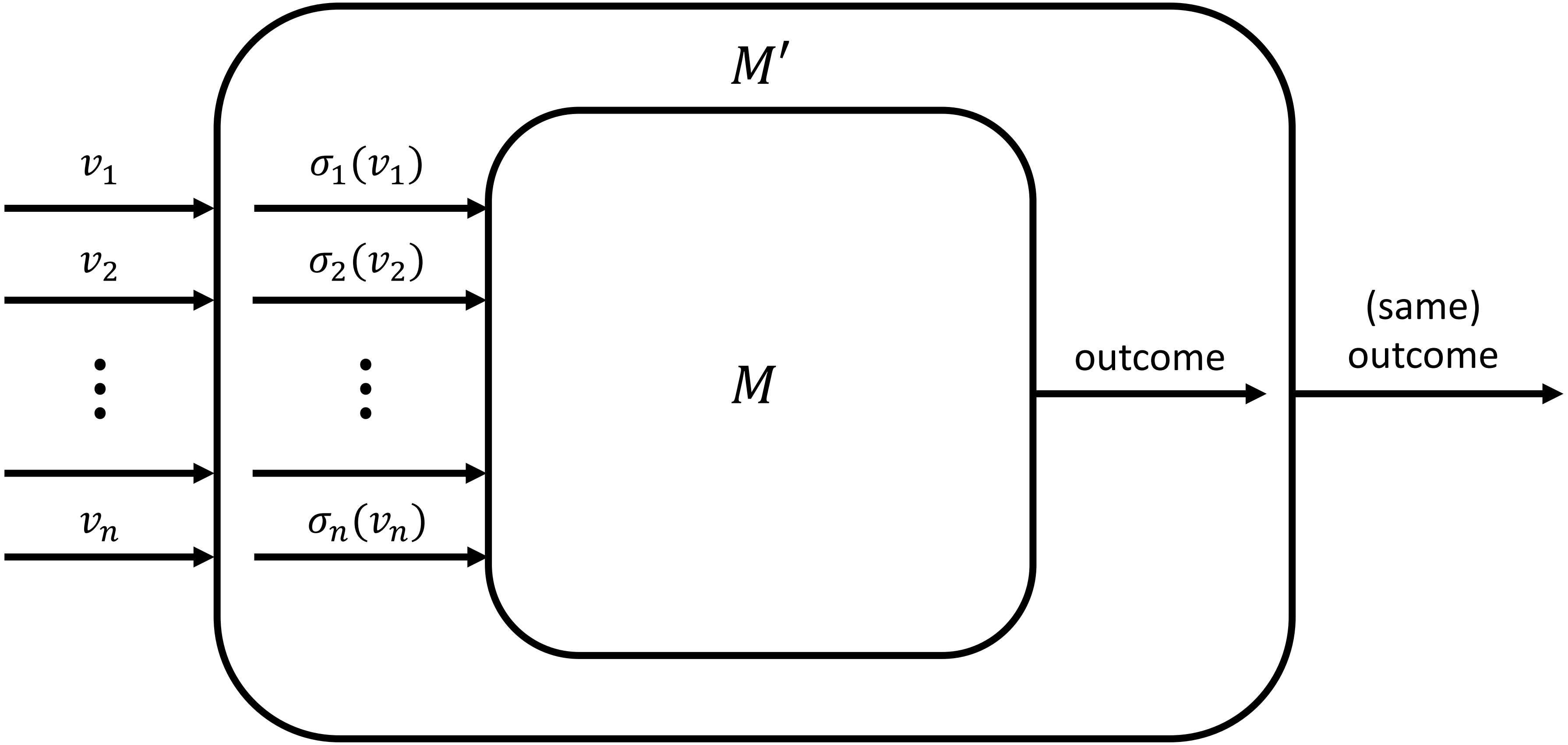
$$\mathbb{E}_{\boldsymbol{v}_{-i}}[v_i\, x_i(\sigma_i(v_i), \boldsymbol{\sigma}_{-i}(\boldsymbol{v}_{-i})) - p_i(\sigma_i(v_i), \boldsymbol{\sigma}_{-i}(\boldsymbol{v}_{-i}))] \geq$$
$$\mathbb{E}_{\boldsymbol{v}_{-i}}[v_i\, x_i(b_i, \boldsymbol{\sigma}_{-i}(\boldsymbol{v}_{-i})) - p_i(b_i, \boldsymbol{\sigma}_{-i}(\boldsymbol{v}_{-i}))] \qquad \forall i, v_i, b_i$$

BIC: A mechanism (x, p) is Bayesian Incentive-Compatible (BIC) when

$$\mathbb{E}_{\boldsymbol{v}_{-i}}[v_i\, x_i(v_i, \boldsymbol{v}_{-i}) - p_i(v_i, \boldsymbol{v}_{-i})] \geq \mathbb{E}_{\boldsymbol{v}_{-i}}[v_i\, x_i(b_i, \boldsymbol{v}_{-i}) - p_i(b_i, \boldsymbol{v}_{-i}))] \qquad \forall i, v_i, b_i$$

Revelation Principle + Revenue Equivalence

Revelation Principle: It is without loss to focus on [DS/B/EP]IC mechanisms.



Revenue Equivalence: Mechs w/ the same outcome have the same $\mathbb{E}[\text{Rev}]$.

Maximizing Revenue

How can we max revenue? Can't just charge v_i – not IC. Still need the payment identity.

Only DSIC if $\varphi_i(v_i)$ is monotone

Expected Revenue $= \mathbb{E}_v\left[\sum_i p_i(\boldsymbol{v})\right] = \mathbb{E}_v\left[\sum_i x_i(\boldsymbol{v})\varphi_i(v_i)\right] =$ Expected Virtual Welfare

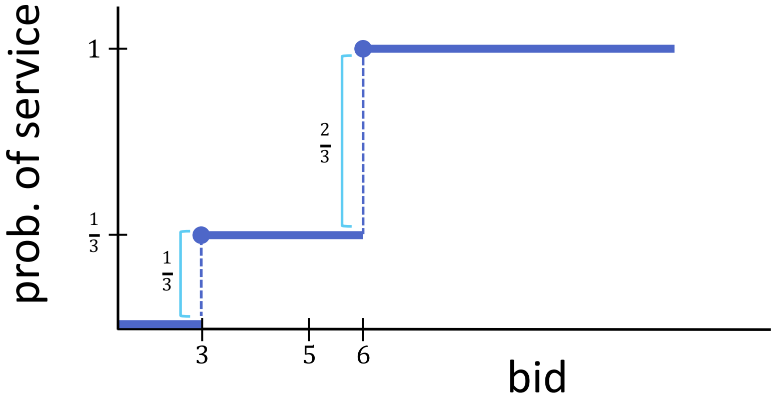
plug in the payment identity

To max rev, choose x to maximize this

For virtual value functions $$\varphi_i(v_i) = \frac{1 - F_i(v_i)}{f_i(v_i)}$$

How else can we express revenue?

Any allocation rule can be expressed as a distribution of prices.



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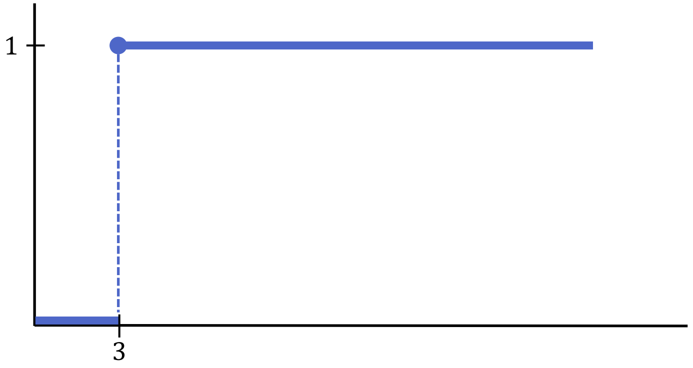
Menu

$(1,\ \frac{1}{3}\$3 + \frac{2}{3}\$6)$

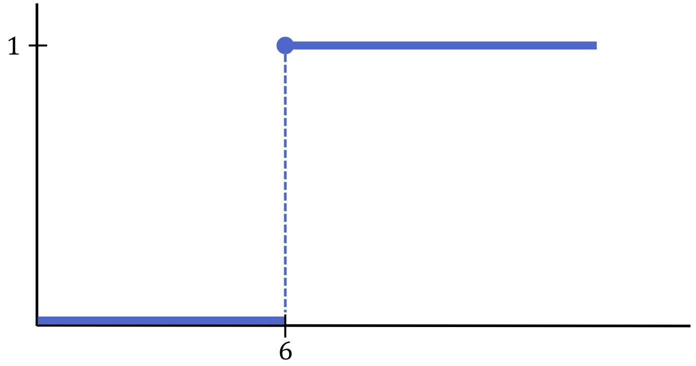
$(\frac{1}{3}\ ,\ \ \frac{1}{3}\$3)$

$(0\ ,\ \ 0)$

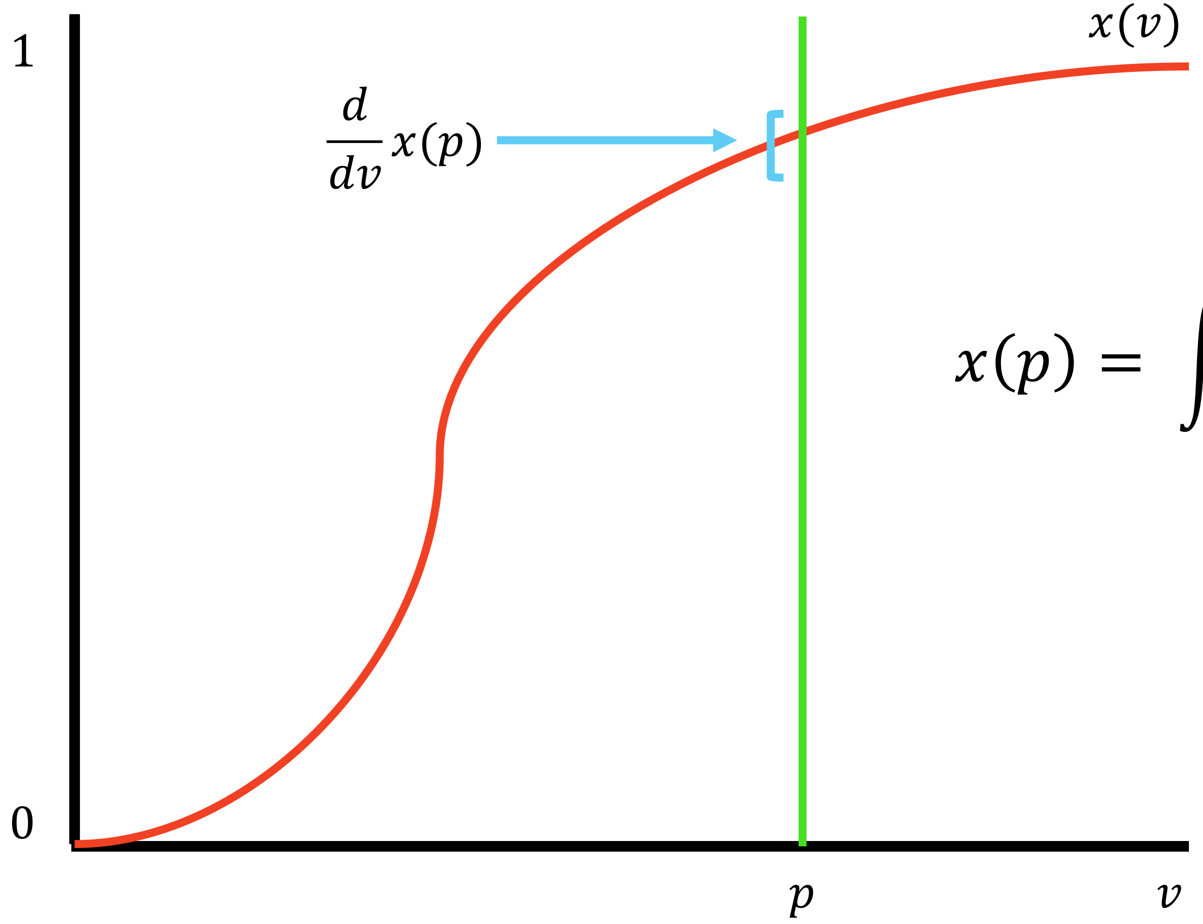
1/3



+ 2/3



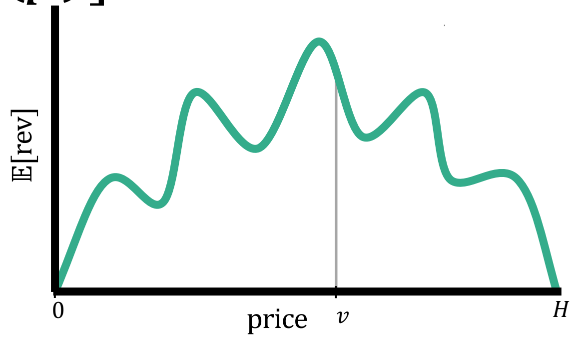
Any allocation is a distribution over prices



$$x(p) = \int_0^p \frac{d}{dv} x(v)dv$$

What is our revenue for a price p?

Single-bidder revenue curve $R(p) = p \cdot \Pr_v[v \geq p] = p \cdot [1 - F(p)]$

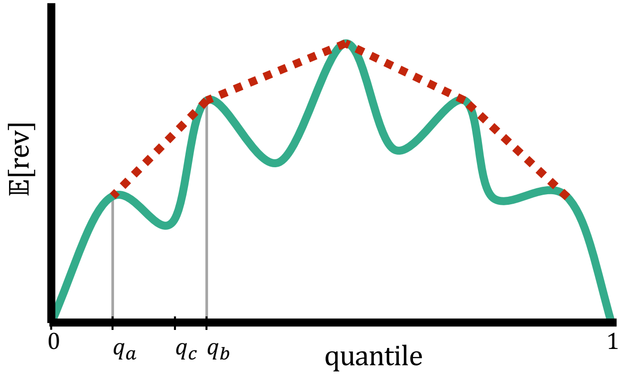


Moving to quantile space:

$$q = 1 - F(v) \qquad v(q) = F^{-1}(1 - q) \qquad q \sim U[0,1]$$

Single-bidder revenue curve in quantile space

$$P(q) = v(q) \cdot q$$



Happily, $$\frac{d}{dq} P(q) = \varphi(v(q))$$

We define $$\frac{d}{dq} \bar{P}(q) = \bar{\varphi}(v(q))$$ where is $\bar{P}(\cdot)$ the concave closure of $P(\cdot)$.

Maximizing Revenue

For virtual value functions $\varphi_i(v_i) = \frac{1 - F_i(v_i)}{f_i(v_i)}$

Only DSIC if $\varphi_i(v_i)$ is monotone

Expected Revenue $= \mathbb{E}_v\left[\sum_i p_i(\boldsymbol{v})\right] = \mathbb{E}_v\left[\sum_i x_i(\boldsymbol{v})\varphi_i(v_i)\right] =$ Expected Virtual Welfare

True by payment identity OR $\frac{d}{dq}P(q) = \varphi(v(q))$

To max rev, choose x to maximize this

$$= \mathbb{E}_v\left[\sum_i x_i(\boldsymbol{v})\bar{\varphi}_i(v_i)\right]$$

with $x = 0$ when $\bar{\varphi}_i \neq \varphi_i$

Multiparameter Social Welfare: VCG is DSIC

$$x \coloneqq \operatorname{argmax} \sum_j v_j(\, x_j(b_i, \boldsymbol{b}_{-i}))$$

More utility for bidding actual value:

$$\underline{v_i(\, x_i(v_i, \boldsymbol{b}_{-i}))} - \boxed{p_i(v_i, \boldsymbol{b}_{-i})} \geq v_i\,(x_i(b_i, \boldsymbol{b}_{-i})) - p_i(b_i, \boldsymbol{b}_{-i}) \qquad \forall i, v_i, b_i, \boldsymbol{b}_{-i}$$

i wants to max wrt $(v_i, \boldsymbol{b}_{-i})$

$$\boxed{p_i(b_i, \boldsymbol{b}_{-i})} = \sum_{j \neq i} v_j(\, x_j(0, \boldsymbol{b}_{-i})) - \underline{\sum_{j \neq i} v_j(\, x_j(b_i, \boldsymbol{b}_{-i}))}$$

max w/o i,
unrelated to i's bid

curr welf w/o i,
x is defined to max
wrt $\boldsymbol{b}$

value v_i
utility $v_i\, x_i(\boldsymbol{b}) - p_i(\boldsymbol{b})$
bid b_i