DS 574 Algorithmic Mechanism Design	**Lecture #14 Worksheet**
Fall 2025	Prof. Kira Goldner

Online Bipartite Matching [KVV '90, EFFS '21]

In the online bipartite matching setting [Karp, Vazirani, and Vazirani, 1990], there is a bipartite graph $G = (L \cup R, E)$ where are vertices are split into the left side, L, and the right side, R. Edges are unweighted, i.e., all have a weight of 1. We are in an online setting where we see R up front, but the vertices of L arrive online, and as each vertex arrives, we see which edges are incident to it from R. The objective is to match vertices in L to those in R, immediately and irrevocably as each vertex arrives, forming a matching M, such that we maximize the cardinality of the matching $|M|$ and compare well to the maximal offline matching.

In the original paper, Karp et al. [1990] show:

- For every deterministic algorithm, $|M| \leq n/2$.
- Choosing a random match for each vertex independently implies that $\mathbb{E}[|M|] \leq n/2$.
- RANKING (KVV): Choosing a global ranking π U.A.R. and matching according to π implies that $\mathbb{E}[|M|] \geq (1 - 1/e)n$.
- This is tight!

The first proof [Karp et al., 1990] was very complicated (and imprecise). Simplifications were given by Goel and Mehta [2008], Birnbaum and Mathieu [2008], Devanur, Jain, and Kleinberg [2013].

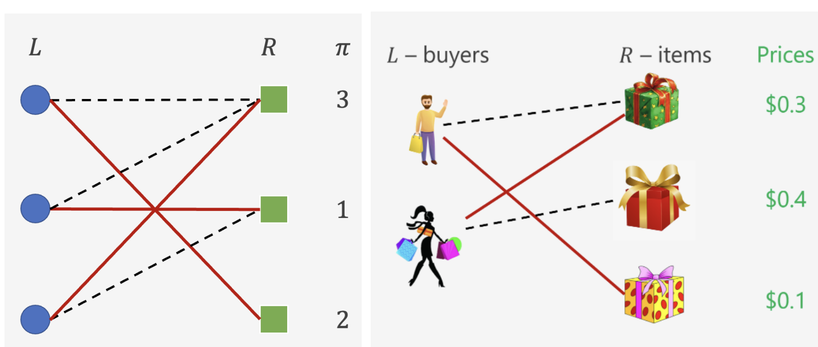

Figure 1: Online Bipartite Matching: RANKING and its economic interpretation.

Today, we'll look at an equivalent algorithm and simplified analysis due to Eden, Feldman, Fiat, and Segal [2021]. We'll use the following set-up: Let R be items and L buyers. They have value 1 or 0 for each item (depending on whether there is an edge). They are unit-demand (want one match). Then our algorithm is as follows.

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Observations:

- For any F supported on $(0,1)$ without pointmasses, choosing item prices i.i.d. from F is equivalent to:

- The *welfare* of the matching is:

  We can rewrite it as:

Lemma 1. *For F that samples $w \sim U[0,1]$ and sets $p_j = e^{w-1}$, we have for every buyer i and item j such that (i,j) is an edge in M:*

$$\mathbb{E}[util_i + rev_j] \geq 1 - 1/e.$$

Corollary 1. $|M| \geq$

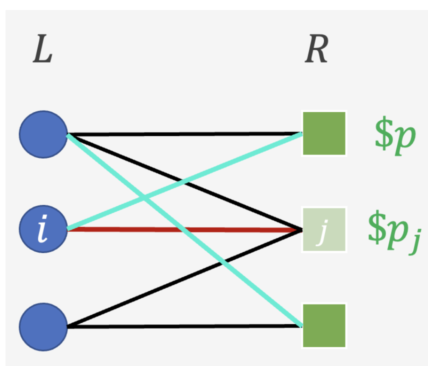

Figure 2: Economic analysis of RANKING.

Proof.

Conclusions:

- Taking the economic perspective can be very useful in algorithm analysis.
- Decomposing value into revenue + utility is a powerful tool, e.g., [Feldman, Gravin, and Lucier, 2015, Dütting, Feldman, Kesselheim, and Lucier, 2017, Ehsani, Hajiaghayi, Kesselheim, and Singla, 2018].

Robustness: Prior-Independence

"Prior-independent" results give us guarantees in the event that the designer *doesn't* know the distribution F from which the bidders' values are drawn. In this case, we assume that their values are still drawn from a prior distribution, as in the Bayesian setting, so there is some revenue-optimal mechanism $\textsc{opt}(F)$ that we wish to approximate, we just have to do so without knowing F.

The Bulow-Klemperer Result

One famous result takes the form of resource augmentation.

Theorem 2 (Bulow and Klemperer [1994])**.** *For i.i.d. regular single-item environments, the expected revenue of the second-price auction with* $n+1$ *agents is at least that of the optimal auction with* n *agents.*

Let's talk about what this theorem is saying. Instead of finding the optimal auction tailored to a distribution F for n agents, you can use the Vickrey auction, which requires no prior knowledge of the distribution, so long as:

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This result **does not hold** without these assumptions. However, it is a *very strong* result, should our setting meet these assumptions.

Proof.

The Single Sample Mechanism

Can't recruit extra buyers? Instead, we can just exclude one. This is what the single sample result says.

Theorem 3 (Dhangwatnotai, Roughgarden, and Yan [2015]). *Given a random sample from a bidder's distribution, posting it as a take-it-or-leave-it price gives a $\frac{1}{2}$-approximation to the optimal revenue.*

Figure 3: Geometric intuition for a posted-price from a single sample.

Proof. In quantile space!

It turns out, using a single sample from the buyers' distribution to set reserve prices and running VCG is a good approximation to the optimal mechanism. See Hartline chapter 5 for more.

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