DS 574 Algorithmic Mechanism Design **Lecture #4 Worksheet**
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Revenue Maximization and Myersonian Virtual Welfare

Recap: For a single buyer will arrive with their private value v, for DSIC mechanisms:

- Maximize welfare ($\sum_i v_i x_i$): Always give the bidder the item, always give it away for free!
- Maximize revenue: Post a price that maximizes $\textsc{Rev} = \max_r r \cdot [1 - F(r)]$.
- Critical bid: For a deterministic mechanism, given other bids $\mathbf{b}_{-i}$, bidder i's *critical bid* is the minimum bid $b_i^* = \min\{b_i : x_i(b_i, \mathbf{b}_{-i}) = 1\}$ such that bidder i is allocated to. Then with $\mathbf{b}_{-i}$ fixed, for all winning $v_i \geq b_i^*$, i's payment $p_i(v_i, \mathbf{b}_{-i}) = b_i^*$ is their critical bid.
- The revelation principle says that it's without loss to focus only on truthful mechanisms.
- Payment is determined by the allocation:

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i})\, dz$$

We want to maximize $\mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})]$.

$$\mathbb{E}_{v_i\sim F_i}[p_i(v_i, \mathbf{v}_{-i})] =$$

where

$$\varphi_i(v_i) =$$

is the Myersonian virtual value and $(*)$ follows by switching the order of integration. Then

$$\textsc{Revenue} = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})] =$$

$= \text{VIRTUAL WELFARE}$

Given this conclusion, how should we design our allocation rule x to maximize expected virtual welfare (expected revenue)?

When would this cause a problem with incentive-compatibility?

Definition 1. A distribution F is regular if the corresponding virtual valuation function $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.

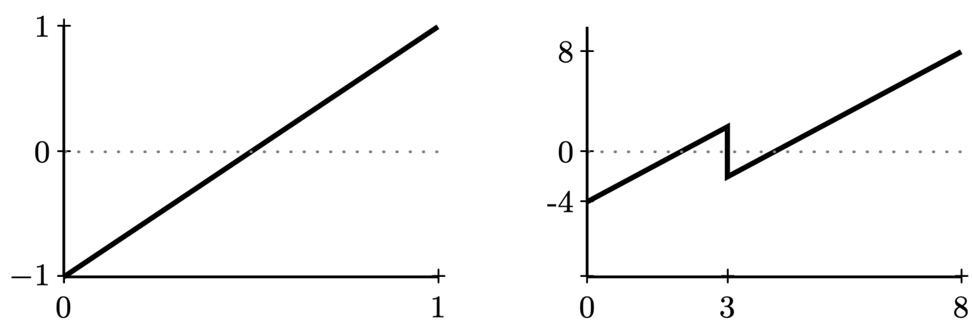
Suppose we are in the single-item setting and all of the distributions are regular. What do the payments look like in the virtual-welfare-maximizing allocation?

For a fixed $\mathbf{b}_{-i}$, if i is the winner, then i's payment is i's critical bid, which is

Exercise: what about for k identical items?

Claim 1. A virtual welfare maximizing allocation x is monotone if and only if the virtual value functions are regular.

Exercise: Argue this.



(a) Uniform agent virtual value. (b) Bimodal agent virtual value.

Figure 1: Virtual value functions $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ for the uniform and bimodal agent examples.

It will be helpful to keep the following two examples in mind:

a. a uniform agent with $v \sim U[0,1]$. Then $F(x) = x$ and $f(x) = 1$.

b. a bimodal agent with

$$v \sim \begin{cases} U[0,3] & w.p. \frac{3}{4} \\ U(3,8] & w.p. \frac{1}{4} \end{cases} \qquad \text{and} \qquad f(v) = \begin{cases} \frac{3}{4} & v \in [0,3] \\ \frac{1}{20} & v \in (3,8] \end{cases}$$

Do the following:

- Calculate the virtual values for both examples.

- Are they regular? Are there any issues using the allocation that maximizes expected virtual welfare?

- What does that allocation actually look like?

Quantile Space

In *value space*:

- an agent has value v.
- the fraction of the distribution with value above v is $1 - F(v)$.
- the revenue from posting a "take-it-or-leave-it" price of v is $v[1 - F(v)]$.

In *quantile space*: $q = 1 - F(v)$.

- an agent has value v.
- the fraction of the distribution with value above v is $q(v) = 1 - F(v)$.
- the revenue from posting a "take-it-or-leave-it" price of $v(q) = F^{-1}(1-q)$ is $v(q) \cdot q$.

Example: Consider a distribution that is $U[\$0, \$10]$. Then the quantile 0.1 corresponds to \$9, where 10% of the population might have a higher value. We let $v(q)$ denote the corresponding value, so $v(0.1)$ is \$9.

Definition 2. The *quantile* of a single-dimensional agent with value $v \sim F$ is the measure with respect to F of stronger values, i.e., $q = 1 - F(v)$; the inverse demand curve maps an agent's quantile to her value, i.e., $v(q) = F^{-1}(1-q)$.

Quantile Distribution: What distribution are quantiles drawn from? That is, what is the probability that an agent is in the top $\hat{q}$ fraction of the distribution? For a distribution F, $\Pr_F[q \leq \hat{q}] =$ what?

Note: For everything we do today, we *could* stay in value space, (and sometimes we'll compare), but we'd have to normalize by the distribution using $f(v)$, which makes everything a bit messier and a bit trickier.

Example: For the example of a uniform agent where $F(z) = z$, the inverse demand curve is $v(q) = 1 - q$.

For an allocation rule $x(\cdot)$ in value space, we define an allocation rule in *quantile space* $y(\cdot)$:

$$y(q) = x(v(q)).$$

As $x(\cdot)$ is monotone weakly increasing, then $y(\cdot)$ is monotone *weakly decreasing*.

Definition 3. The *revenue curve* of a single-dimensional agent specified by $R(v) = v \cdot [1 - F(v)]$.

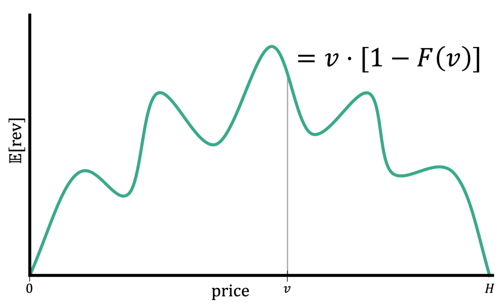


Figure 2: A revenue curve in *value space*.

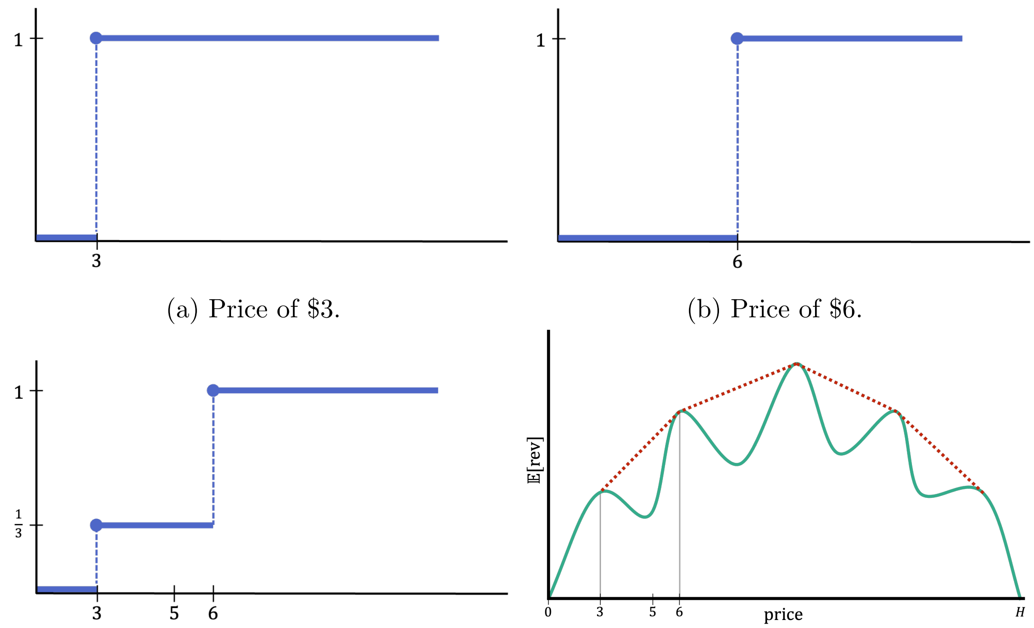
Note: This is *only* the revenue that can be achieved by posting a single take-it-or-leave-it price. This does not capture the expected revenue of any given mechanism.

Definition 4. The *revenue curve* of a single-dimensional agent specified by inverse demand curve $v(\cdot)$:

Claim 2. Any allocation rule $y(\cdot)$ can be expressed as a distribution of posted prices.

Proof.

Claim 3. Any DSIC allocation rule $x(\cdot)$ can be expressed as a distribution of posted prices.



(a) Price of \$3.

(b) Price of \$6.

(c) Randomized price with expectation \$5.

(d) Ironed revenue curve.

Figure 3: (a) An allocation rule for a take-it-or-leave-it price of \$3. (b) An allocation rule for a take-it-or-leave-it price of \$6. (c) An allocation that can be written $x(v) = 0$ for $v < 3$, $x(v) = \frac{1}{3}$ for $v \in [3, 6)$, and $x(v) = 1$ for $v \geq 6$. Alternatively, a randomized take-it-or-leave-it price that is \$3 with probability $\frac{1}{3}$ and \$6 with probability $\frac{2}{3}$, that is, $\$5 = \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 6$ in expectation. (d) The revenue curve in value space, including ironed intervals where convex combinations of prices can attain higher revenue than deterministic prices.