

Ascending Auctions

In *ascending auctions*, an auctioneer initializes prices for each item, iteratively raises the prices, and bidders decide which items to bid on in each round. Sometimes *activity rules* are enforced, e.g., once you drop out on an item, you can not bid on it again.

The most famous ascending auction is the single-item version, the English Auction.

The English Auction(ε):

- a. Initialize the item's price p_0 to
- b. The initial set S_0 of “active bidders” (willing to pay p_0 for the item) is
- c. For iteration $t = 1, 2, \dots$:
 - (a) Ask the set of active bidders S_{t-1} :
$$S_t =$$
 - (b) If $|S_t| \leq 1$:
 - (c) Otherwise, p_t

Benefits of using ascending auctions:

- Ascending auctions are easier for bidders.
- Less information leakage.
- Transparency.
- Potentially more seller revenue.
- When there are multiple items, the opportunity for “price discovery.”

What about k identical items? What should we do here?

The English Auction for k Identical Items:

Definition 1. In an ascending auction, *sincere bidding* means that a player answers all queries honestly.

Exercise: Prove the following claims.

Claim 1. In the k identical item setting, in an English auction, sincere bidding is a dominant strategy for every bidder (up to ε).

Claim 2. In the k identical item setting, if all bidders bid sincerely in an English auction, the welfare of the outcome is within $k\varepsilon$ of the maximum possible.

The English auction for k Identical Items terminates in v_{\max}/ε iterations.

Design process:

- a. As a sanity check, design a direct-revelation DSIC welfare-maximizing polytime mechanism.
- b. Implement this as an ascending auction.
- c. **(Truthfulness)** Check that it's EPIC.
- d. **(Performance)** Check that it still maximizes welfare under sincere bidding.
- e. **(Tractability)** Check that it terminates in a reasonable number of iterations.

Additive Valuations, Parallel Auctions

The Additive Setting: There are m non-identical items and n bidders where each bidder i has private valuation v_{ij} for each item j . Bidder i has an additive valuation for each set S , that is,

$$v_i(S) := \sum_{j \in S} v_{ij}.$$

Step 1: Recall: What is the welfare-optimal direct revelation mechanism here?

What's the analogous ascending implementation?

Is this DSIC?

Definition 2. A strategy profile $(\sigma_1, \dots, \sigma_n)$ is an *ex post Nash equilibrium (EPNE)* if, for every bidder i and valuation $v_i \in V_i$, the strategy $\sigma_i(v_i)$ is a best-response to every strategy profile $\sigma_{-i}(\mathbf{v}_{-i})$ with $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$.

In comparison, in a dominant-strategy equilibrium (DSE), for every bidder i and valuation v_i , the action $\sigma_i(v_i)$ is a best response to every action profile \mathbf{a}_{-i} of \mathbf{A}_{-i} , whether of the form $\sigma_{-i}(\mathbf{v}_{-i})$ or not.

Definition 3. A mechanism is *ex post incentive compatible (EPIC)* if sincere bidding is an ex post Nash equilibrium in which all bidders always receive nonnegative utility.

Claim 3. For n additive bidders with m heterogeneous items, in parallel English auctions, sincere bidding by all bidders is an ex post Nash equilibrium (up to $m\varepsilon$).

Unit Demand

The Unit-Demand Setting: There are m non-identical items and n bidders where each bidder i has private valuation v_{ij} for each item j . Bidder i is unit demand, that is, wants at most one item for any set S :

$$v_i(S) := \max_{j \in S} v_{ij}.$$

First, solve the direct-revelation problem. What do we observe about the welfare-maximizing allocation in the unit-demand setting?

Refresh yourself on what the VCG mechanism looks like. Then what does the analogous ascending auction look like?

Walrasian Equilibria in the Unit-Demand Setting

The Unit-Demand Setting: There are m non-identical items $[m]$ and n bidders where each bidder i has private valuation v_{ij} for each item j . Bidder i is unit demand, that is, wants at most one item for any set S :

$$v_i(S) := \max_{j \in S} v_{ij}.$$

Definition 4. In the unit-demand setting, a *Walrasian equilibrium* (or “competitive equilibrium”) is a price vector $\mathbf{q} \in \mathbb{R}^m$ defined on the items and a matching M of the bidders and items such that:

a. Each bidder i is matched to a favorite item $j \in \operatorname{argmax}\{v_{ij} - q_j\}_{j \in [m] \cup \{\emptyset\}}$. (WE1)

Equivalently, \mathbf{q} is an *envy-free pricing*.

b. An item $j \in [m]$ is unsold *only* if $q(j) = 0$. (WE2)

We call $D_i(\mathbf{q}) = \operatorname{argmax}\{v_{ij} - q_j\}_{j \in [m] \cup \{\emptyset\}}$ the *demand set* of i under prices \mathbf{q} .

Claim 4 (First Welfare Theorem). In the unit-demand setting, if (\mathbf{q}, M) is a Walrasian Equilibrium, then M is a welfare-maximizing allocation.

This essentially says “markets are efficient,” and there are many “First Welfare Theorems” each with this flavor. Exercise: Prove this.

What we'll now see that is the VCG allocation and payment *is* a WE, and in fact, is a lower bound on all WE for the unit-demand setting.

Recall the VCG payment in this setting:

$$p_i = \sum_{k \neq i} v_k(M^{-i}(k)) - \sum_{k \neq i} v_k(M(k))$$

where $M(k)$ is the item that k is allocated in the welfare-maximizing (maximum-weight) matching, and M^{-i} is the welfare-maximizing matching without bidder i .

Theorem 1 (VCG Payments Lower Bound WE). *In the unit-demand setting, let \mathbf{p} denote the induced item price vector of the truthful-revelation VCG outcome and \mathbf{q} a Walrasian price vector. Then $p(j) \leq q(j)$ for every item j .*

Proof. Let M denote the allocation computed by the VCG mechanism. Let M^{-i} denote a welfare-maximizing allocation among allocations that leave bidder i unmatched. The pair (\mathbf{q}, M) is a WE. (Why?) For every $k \neq i$, (WE1) of (\mathbf{q}, M) can be used to argue that k prefers $M(k)$ over $M^{-i}(k)$ at the prices \mathbf{q} :

and summing over all $k \neq i$ gives

where $Q = \sum_{j'} q(j')$, because $\sum_{k \neq i} q(M(k))$ sums over all of the items with non-zero q -prices except for the item matches to i (which we call j). Rearranging gives

where the equation follows from the definition of prices from the VCG mechanism. □