DS 320 Algorithms for Data Science	**Lecture #20**
Spring 2023	Prof. Kira Goldner

Linear Programming III: Duality

Using Linear Programming for a Vertex Cover Approximation Algorithm

$$\begin{aligned} \min \sum_{i \in V} & w_i x_i & \\ \text{s.t.} \quad & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \in [0,1] & i \in V. \end{aligned}$$

Claim 1. Let S^* denote the optimal vertex cover of minimum weight, and let x^* denote the optimal solution to the Linear Program. Then $\sum_{i \in V} w_i x_i^* \leq w(S^*) = \text{OPT}$.

Proof. The vertex cover problem is equivalent to the integer program, whereas the linear program is a *relaxation.* Then there are simply more solutions allowed to the linear program, so the minimum can only be smaller. □

Claim 2. The set $S = \{i : x_i \geq 0.5\}$ is a vertex cover, and $w(S) \leq 2 \sum_{i \in V} w_i x_i^*$.

Proof. First, S is a vertex cover: for any edge $e = (i,j)$, at least one of i or j must be in S, because of our constraint $x_i + x_j \geq 1$, which forces at least one of these variables to be $\geq \frac{1}{2}$ and thus in S.

With respect to weight:

$$\sum_{i \in V} w_i x_i^* \geq \sum_{i \in S} w_i x_i^* \geq \frac{1}{2} \sum_{i \in S} w_i = \frac{1}{2} w(S).$$

□

Then our algorithm of running an LP and rounding it to give the vertex cover S is a 2-approximation to the optimal vertex cover S^*, as $w(S) \leq 2\, w(S^*)$ by Claims 1 and 2.

LP Duality

The Dual of a Linear Program

Every linear program has a *dual* linear program. We call the original linear program the *primal.* A maximization problem's dual is a minimization problem. There are a bunch of amazing properties that come from LP duality.

We have the following optimization problem: You're selling nutrients to the BU population and deciding what to price each macro at. The decision variables x_i will indicate the price per nutrient. The constraints indicate that these prices together cannot exceed the prices for the grains that you're extracting the nutrients from, since that's already the market price. The goal is to maximize your profits from a population that is buying exactly the nutrient diet of 8kg starch, 15kg proteins, and 3kg vitamins.

Primal:

$$\begin{array}{rlr} \max & 8x_1 + 15x_2 + 3x_3 & \\ \text{subject to} & 5x_1 + 4x_2 + 2x_3 \leq 0.6 & \text{(grain 1)} \quad (y_1) \\ & 7x_1 + 2x_2 + 1x_3 \leq 0.35 & \text{(grain 2)} \quad (y_2) \\ & x_1, x_2, x_3 \geq 0 & \text{(non-negativity)} \end{array}$$

Dual:

$$\begin{array}{rlr} \min & 0.6y_1 + 0.35y_2 & \\ \text{subject to} & 5y_1 + 7y_2 \geq 8 & \text{(starch)} \quad (x_1) \\ & 4y_1 + 2y_2 \geq 15 & \text{(proteins)} \quad (x_2) \\ & 2y_1 + 1y_2 \geq 3 & \text{(vitamins)} \quad (x_3) \\ & y_1, y_2 \geq 0 & \text{(non-negativity)} \end{array}$$

To take the dual: Label each primal constraint with a new dual variable. In our new linear program, each dual constraint will correspond to a primal variable. For the left-hand side, count up the appearances of this constraint's primal variable (e.g., x_1) in each of the primal constraints and multiply them by the dual variable for those constraints. That is, if x_1 appears 5 times ($5x_1$) in constraint for y_1, then add $5y_1$ to x_1's constraint. Don't forget to include its appearance in the primal's objective function, but this will be the right-hand side of the constraint. Finally, the dual objective function is given by the right-hand side coefficients and their correspondence to the dual variables via the constraints in the primal. (See above).

Sometimes, the dual can even be interpreted as a related problem. In fact, this dual can be interpreted as exactly our nutrition example from Lecture #18: BU has hired you to optimize nutrition for campus dining. There are two possible grains they can offer, grain 1 and grain 2, and each contains the macronutrients given in the table in Lecture #18, plus cost per kg for each of the grains. The nutrition requirement per day of starch, proteins, and vitamins is 8, 15, and 3 respectively. Determine how much of each grain to buy such that BU spends as little but meets its nutrition requirements.

The following is the normal form for a maximization problem primal and its primal:

$$\begin{array}{rl} \max & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array} \qquad \qquad \begin{array}{rl} \min & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \end{array}$$

For the above example:

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & 2 \\ 7 & 2 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0.6 \\ 0.35 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 8 \\ 15 \\ 3 \end{bmatrix}$$

Example 3: Maximum Matching

Given a graph $G = (V, E)$ choose a maximum size matching—a set of edges S such that no vertex is covered by more than one edge.

Decision variables: x_e indicating whether edge e is in the matching.

Primal Linear Program:

$$\begin{array}{rlrl}
\max & \sum_{e \in E} x_e & & \\
\text{subject to} & \sum_{e: v \in e} x_e \leq 1 & \forall v & \text{(vertex matched at most once)} \quad (y_v) \\
& x_e \geq 0 & \forall e & \text{(non-negativity)}
\end{array}$$

Taking the dual of the above primal, we get the following linear program:

$$\begin{array}{rlrl}
\min & \sum_{v \in V} y_v & & \\
\text{subject to} & \sum_{v \in e} y_v \geq 1 & \forall e & \text{(edge covered)} \quad (x_e) \\
& y_v \geq 0 & \forall v & \text{(non-negativity)}
\end{array}$$

What problem is this? (Fractional) Vertex Cover!

Conditions for Optimality

Weak Duality

Theorem 1 (Weak Duality). *If* $\mathbf{x}$ *is feasible in (P) and* $\mathbf{y}$ *is feasible in (D) then* $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Proof.

$$\mathbf{c}^T\mathbf{x} \overset{1}{\leq} (\mathbf{A}^T\mathbf{y})\mathbf{x} = \mathbf{y}^T\mathbf{A}\mathbf{x} \overset{2}{\leq} \mathbf{y}^T\mathbf{b} = \mathbf{b}^T\mathbf{y}.$$

Where (1) follows by the dual constraints $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$ and (2) follows by the primal constraints $\mathbf{A}\mathbf{x} \leq \mathbf{b}$. □

This theorem says that *any* feasible solution to the primal is a *lower bound* to *any* feasible solution to the dual, and likewise, any feasible solution to the dual is an *upper bound* to the primal.

That is, fractional vertex cover gives an upper bound on how large the (fractional) maximum matching can be, and likewise, fractional maximum matching gives a lower bound on how small the minimum (fractional) vertex cover can be.

Strong Duality

Strong duality states that everything in fact needs to hold with equality to be optimal.

Theorem 2 (Strong Duality)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if* $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$*.*

Proof. ($\Leftarrow$) The *if* direction is easy to see: we know that the dual gives an upper bound on the primal, so if these objectives are equal, then the primal objective that we are trying to maximize could not possible get any larger, as it's always *at most* the dual's objective. This is *as tight as possible.*

($\Rightarrow$) The *only if* direction is harder to prove, and we'll skip it for now. $\square$

Complementary Slackness

We rewrite the primal and dual with each constraint separated, and then formalize another condition for optimality called *complementary slackness*, which states that for each corresponding constraint and variable, at most one can be slack in an optimal solution.

Primal (P):

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \sum_i a_{ji}x_i \le b_j \quad \forall j \quad (y_j) \\ & x_i \ge 0 \quad \forall i \end{aligned}$$

Dual (D):

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \sum_i a_{ij}y_i \ge c_i \quad \forall i \quad (x_i) \\ & y_j \ge 0 \quad \forall j \end{aligned}$$

Theorem 3 (Complementary Slackness)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if the following complementary slackness conditions (1) and (2) hold:*

$$\sum_i a_{ji}x_i = b_j \quad \text{or} \quad y_j = 0 \qquad (1) \qquad\qquad \sum_i a_{ij}y_i = c_i \quad \text{or} \quad x_i = 0. \qquad (2)$$

Proof. ($\Rightarrow$) According to complementary slackness, by rearranging our constraint, either $\sum_i a_{ji}x_i - b_j = 0$ or $y_j = 0$. This ensures that the multiplied quantity $\left(\sum_i a_{ji}x_i - b_j\right)y_j = 0$, as *one* of the two terms on the left-hand side must be 0. Then multiplying out and rearranging gives that $y_j \sum_i a_{ji}x_i = y_j b_j$. This process with all rows gives the equality from complementary slackness that $\mathbf{y}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{b}$.

Similarly, using the condition that $\sum_i a_{ij}y_i = c_i$ or $x_i = 0$ gives that $\mathbf{c}^T\mathbf{x} = (\mathbf{A}^T\mathbf{y})\mathbf{x}$.

Then following our inequalities in the proof of weak duality, they now all hold with equality, so by Strong Duality, $(\mathbf{x}, \mathbf{y})$ are optimal solutions to the primal and dual.

$$\mathbf{c}^T\mathbf{x} \quad = \quad (\mathbf{A}^T\mathbf{y})\mathbf{x} \quad = \quad \mathbf{y}^T\mathbf{A}\mathbf{x} \quad = \quad \mathbf{y}^T\mathbf{b} \quad = \quad \mathbf{b}^T\mathbf{y}.$$

($\Leftarrow$) Similarly, if Strong Duality holds, the above inequalities hold with equality, in which case it must be that $y_j \sum_i a_{ji}x_i = y_j b_j$ for all j and $\sum_i a_{ij}y_i x_i = c_i x_i$ for all i, and hence that either $\sum_i a_{ji}x_i - b_j = 0$ or $y_j = 0$ for all j and that either $\sum_i a_{ij}y_i = c_i$ or $x_i = 0$ for all i. $\square$