DS 574 Algorithmic Mechanism Design	**Lecture #6 Worksheet**
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Welfare Maximization in Multidimensional Settings

Multidimensional or *multi-parameter* environments are ones where we need to elicit more than one piece of information per bidder. The most common settings include m heterogenous (*different*) items and

- n unit-demand buyers; buyer i has value v_{ij} for item j but only wants at most 1 item. (You only want to buy 1 house!)
- n additive buyers: buyer i's value for set S is $\sum_{j \in S} v_{ij}$.
- n subadditive buyers for some subadditive functions
- n buyers who are k-demand: buyer i's value for a set of items S is $\max_{|S'|=k, S' \subseteq S} \sum_{j \in S'} v_{ij}$.
- n matroid-demand buyers for some matroid
- ...

With m heterogenous items, it's *possible* that our buyers could have different valuations for every single one of the 2^m bundles of items—that is why this general setting is referred to as *combinatorial auctions.*

Then how can we maximize welfare in this setting? How can we do so *tractably?* How can we even elicit preferences in a tractable way?

Theorem 1 (The Vickrey-Clarke-Groves (VCG) Mechanism)**.** *In every general mechanism design environment, there is a DSIC welfare-maximizing mechanism.*

Given bids $\mathbf{b}_1, \ldots, \mathbf{b}_n$ where each bid is indexed by the possible outcomes $\omega \in \Omega$, we define the welfare-maximizing allocation rule $\mathbf{x}$ by

Now that things are multidimensional, there's no more Myerson's Lemma! In multiple dimensions, what is monotonicity? What would the critical bid be?

Instead, we have bidders pay their *externality*—the loss of welfare caused due to i's participation:

$$p_i(\mathbf{b}) =$$

where $\omega^* = \mathbf{x}(\mathbf{b})$ is the outcome chosen when i *does* participate.

Claim 1. The VCG mechanism is DSIC.

What does the VCG mechanism look like for:

- bidders with additive valuations? $v_i(S) = \sum_{j \in S} v_{ij}$
- unit-demand bidders? $v_i(S) = \max_{j \in S} v_{ij}$

Exercise (optional): Prove that the payment $p_i(\mathbf{b})$ is always non-negative (and so the mechanism is IR).

Interdependent Values I

Thus far, we have been discussing private independent values. That is, each bidder i has private information $\mathbf{v}_i$ regarding their value for item i.

However, in many settings, the valuations may be correlated between buyers, depend on one another's information, or even be common.

The Interdependent Values Model [2]. Each bidder has a private *signal* s_i that is a piece of information about the item. Each buyer has a **public** *valuation function* $v_i(s_1, \cdots, s_n)$ that dictates how the buyer aggregates the information into a value for the item.

Assumptions on $v_i(\cdot)$:

- $v_i(\cdot)$ monotone in s_j for all i, j.
- $v_i(\cdot)$ is non-negative for all $\mathbf{s}$.

Example: Common Values [4]: The average of estimates $v_i(s_1, \ldots, s_n) = \frac{1}{n}\sum_i s_i \,\forall i$, or the wallet game $v_i(s_1, \ldots, s_n) = \sum_i s_i \,\forall i$.

Optimal Social Welfare

Mechanisms. How can we maximize social welfare in this setting, optimally? What does a mechanism even look like?

- Report:
- Calculate:
- Allocate to:

Incentive Compatibility. What conditions are necessary for maximizing social welfare optimally to be incentive-compatible? What definition of incentive-compatible are we going for?

DSIC? Why or why not?

Next best we can hope for is:

In this context that means:

Definition 1. Truth-telling is said to be [ ] if, for every bidder i, for every possible realization of the other bidders' signals $\mathbf{s}_{-i}$, and given that other bidders report their signals truthfully, then it is in bidder i's best interest to report their true signal.

Myerson in IDV. What is the analogue of Myerson's Lemma in the interdependent setting?

Theorem 2 (Myerson Analogue [3])**.** *Environment:*

(a) An allocation rule $\mathbf{x}$ *is [ ] if and only if*

(b) If $\mathbf{x}$ *is [ ], then there is a unique payment rule such that the sealed-bid mechanism* $(\mathbf{x}, \mathbf{p})$ *is [ ].*

(c) The payment rule in is given by:

$$\begin{aligned} p_i(\mathbf{s}) = x_i(\mathbf{s})v_i(\mathbf{s}) - \int_{v_i(0,\mathbf{s}_{-i})}^{v_i(s_i,\mathbf{s}_{-i})} x_i(v_i^{-1}(t \mid \mathbf{s}_{-i}), \mathbf{s}_{-i})dt \\ - \left[x_i(0,\mathbf{s}_{-i})v_i(0,\mathbf{s}_{-i}) - p_i(0,\mathbf{s}_{-i})\right]; \\ p_i(0,\mathbf{s}_{-i}) \leq x_i(0,\mathbf{s}_{-i})v_i(0,\mathbf{s}_{-i}). \end{aligned}$$

References

[1] Alon Eden, Michal Feldman, Amos Fiat, and Kira Goldner. Interdependent values without single-crossing. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, EC '18, pages 369–369, New York, NY, USA, 2018. ACM.

[2] Paul R Milgrom and Robert J Weber. A theory of auctions and competitive bidding. *Econometrica: Journal of the Econometric Society*, pages 1089–1122, 1982.

[3] Tim Roughgarden and Inbal Talgam-Cohen. Optimal and robust mechanism design with interdependent values. *ACM Trans. Econ. Comput.*, 4(3):18:1–18:34, June 2016.

[4] Robert B Wilson. Competitive bidding with asymmetric information. *Management Science*, 13(11):816–820, 1967.