DS 320 Algorithms for Data Science | **Lecture #19 Worksheet**
Spring 2024 | Prof. Kira Goldner

Linear Programming II: Algorithms, Problems, and Duality

What Does Linear Programming Buy Us?

a. We know efficient algorithms exist (and have a nice theory behind them).

b. We can relate problems to one another through relaxations, duality.

c. It gives us techniques for approximation.

Linear Programming Algorithms

a. Simplex

b. Ellipsoid

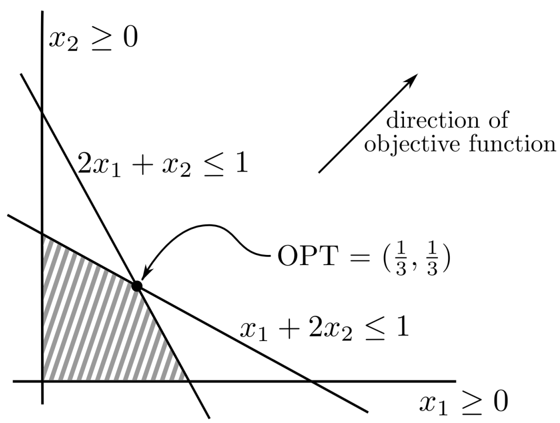


Figure 1: A toy example of a linear program.

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 \geq 0 \\
& x_2 \geq 0 \\
& 2x_1 + x_2 \leq 1 \\
& x_1 + 2x_2 \leq 1.
\end{aligned}
$$

Writing Problems We Know as Linear Programs

Independent Set

Recall from last lecture that we formulated the Independent set problem as a linear programming relaxation.

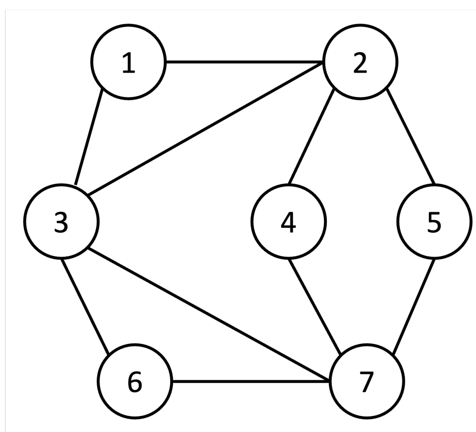
Given a graph $G = (V, E)$, each vertex i has weight w_i, find a maximum weighted *independent set*. S is an independent set if it does not contain both i and j for $(i, j) \in E$.

$$\begin{aligned} \max \sum_{i \in V} & w_i x_i & \\ \text{s.t.} \quad & x_i + x_j \leq 1 & (i, j) \in E \\ & 0 \leq x_i \leq 1 & i \in V. \end{aligned}$$

The Vertex Cover Problem

Given a graph $G = (V, E)$, we say that a set of nodes $S \subseteq V$ is a *vertex cover* if every edge $e = (i, j) \in E$ has at least one endpoint i or j in S. Our goal is to find a *minimum* vertex cover.

The decision version of the problem is: Given a graph G and a number k, does G contain a vertex cover of size at most k?



In this graph, the *minimum* vertex cover is

This is the same graph from last time when we discussed Independent Set. Do we notice any relationship? **Are there any implications of this?**

Vertex Cover as a Linear Program

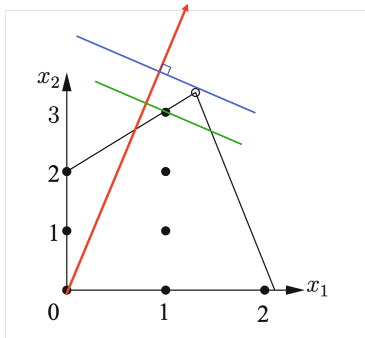
a. *Decision variables:* What are we try to solve for?

b. *Constraints:*

c. *Objective function:*

Vertex Cover as a Linear Program:

Claim 1. Let S^* denote the optimal vertex cover of minimum weight, and let x^* denote the optimal solution to the Linear Program. Then $\sum_{i \in V} w_i x_i^* \leq w(S^*)$.



$$
\begin{aligned}
\max \quad & 4x_1 + x_2 \\
\text{subject to} \quad & -x_1 + x_2 \leq 2 \\
& 8x_1 + 2x_2 \leq 17 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Figure 2: Left: The red arrow represents the objective function, with the green line tangent to the set of feasible integer solutions, indicating the optimal integral point, and the blue line tangent to the relaxed convex feasible set, indicating the best fractional point in the relaxation, with a larger objective function. Right: The linear program for the figure on the left.

Linear Programming Duality

The Dual of a Linear Program

Every linear program has a *dual* linear program. We call the original linear program the *primal.* A maximization problem's dual is a minimization problem. There are a bunch of amazing properties that come from LP duality.

To take the dual: Label each primal constraint with a new dual variable. In our new linear program, each dual constraint will correspond to a primal variable. For the left-hand side, count up the appearances of this constraint's primal variable (e.g., x_1) in each of the primal constraints and multiply them by the dual variable for those constraints. That is, if x_1 appears 5 times ($5x_1$) in constraint for y_1, then add $5y_1$ to x_1's constraint. Don't forget to include its appearance in the primal's objective function, but this will be the right-hand side of the constraint. Finally, the dual objective function is given by the right-hand side coefficients and their correspondence to the dual variables via the constraints in the primal.

Primal:

$$
\begin{aligned}
\max \quad & 8x_1 + 15x_2 + 3x_3 & \\
\text{subject to} \quad & 5x_1 + 4x_2 + 2x_3 \leq 0.6 & (y_1) \\
& 7x_1 + 2x_2 + 1x_3 \leq 0.35 & (y_2) \\
& x_1, x_2, x_3 \geq 0 & \text{(non-negativity)}
\end{aligned}
$$

Dual:

The following is the normal form for a maximization problem primal and its primal:

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \end{aligned}$$

For the above example:

$\mathbf{A} =$ $\qquad$ $\mathbf{b} =$ $\qquad$ $\mathbf{c} =$

Example 3: Maximum Matching

Given a graph $G = (V, E)$ choose a maximum size matching—a set of edges S such that no vertex is covered by more than one edge.

Decision variables:

Linear Program:

Taking the dual of the above primal, we get what linear program?

What problem is this?

Conditions for Optimality

Weak Duality

Theorem 1. *If* $\mathbf{x}$ *is feasible in (P) and* $\mathbf{y}$ *is feasible in (D) then* $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Proof.

Give an upper bound on maximum matching:

Give a lower bound on vertex cover:

Strong Duality

Theorem 2 (Strong Duality)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if* $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$.

Proof. ($\Rightarrow$) Skip.
($\Leftarrow$)

Complementary Slackness

Primal (P):

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \sum_i a_{ji}x_i \le b_j \quad \forall j \quad (y_j) \\ & x_i \ge 0 \quad \forall i \end{aligned}$$

Dual (D):

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \sum_i a_{ij}y_i \ge c_i \quad \forall i \quad (x_i) \\ & y_j \ge 0 \quad \forall j \end{aligned}$$

Theorem 3 (Complementary Slackness)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if the following complementary slackness conditions (1) and (2) hold:*

Proof.