DS 574 Algorithmic Mechanism Design	Lecture #5 Worksheet
Fall 2025	Prof. Kira Goldner

Ironing for Single-Parameter Optimal Revenue

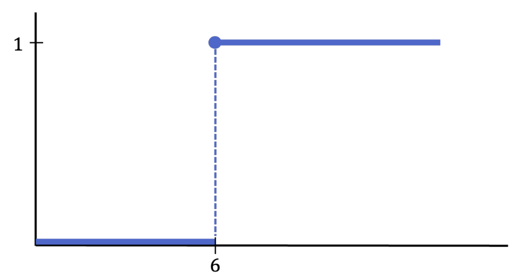
Recap

Myerson's theory for single-parameter revenue maximization says:

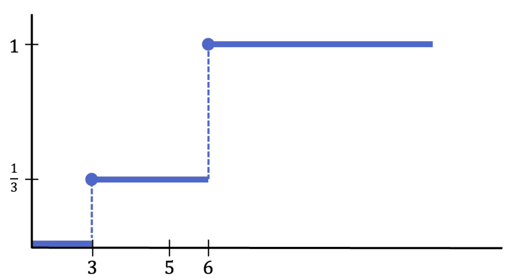
Price-posting revenue curves in

- Value space:
- Quantile Space:
  where

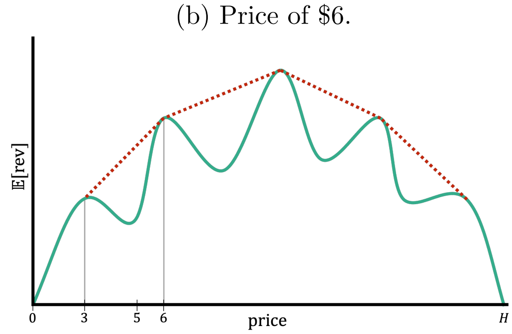
(a) Price of \$3.



(b) Price of \$6.



(c) Randomized price with expectation \$5.



(d) Ironed revenue curve.

Back to Quantile Space and Ironing

Claim 1. A distribution F is regular if and only if its corresponding revenue curve is concave.

Observe that $P'(q) = \varphi(v(q))$:

$$P'(q) = \frac{d}{dq}\left(q \cdot v(q)\right) = v(q) + qv'(q) = v - \frac{1 - F(v)}{f(v)} = \varphi(v(q)).$$

Thus $\Phi(q) = \int_0^q \varphi(\hat{q})\, d\hat{q} = P(q)$.

To summarize: a distribution F is regular if and only if:

- its corresponding revenue curve *in quantile space* is concave.
- $\varphi(q)$ is strictly increasing.
- $f(v)\varphi(v)$ is strictly increasing. (Why?)

Definition 1. The *ironing procedure* for (non-monotone) virtual value function φ (in quantile space) is:

(i) Define the cumulative virtual value function as

(ii) Define ironed cumulative virtual value function

(iii) Define the ironed virtual value function as

Summary: Take the concave hull of the revenue curve in quantile space. Its derivative forms the ironed virtual values. (The derivatives of the original curve are the original virtual values.)

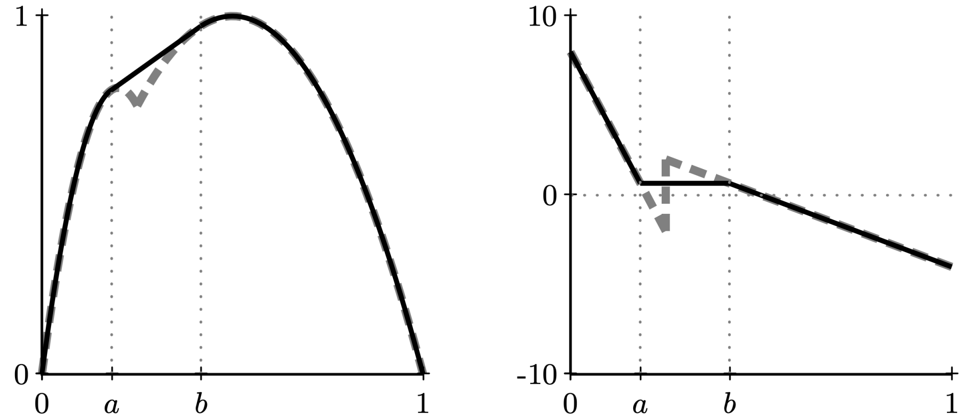
Theorem 1. *For any monotone allocation rule $y(\cdot)$ and any virtual value function $\varphi(\cdot)$, the expected virtual welfare of an agent is upper-bounded by her expected ironed virtual surplus, i.e.,*

$$\mathbb{E}[\varphi(q)y(q)] \leq \mathbb{E}[\bar{\varphi}(q)y(q)].$$

Furthermore, this inequality holds with equality if the allocation rule y satisfies $y'(q) = 0$ for all q where $\bar{\Phi}(q) > \Phi(q)$.

How do we modify this statement for value space?

Proof.



(a) Ironed cumulative virtual values.
(b) Ironed virtual values.

Figure 2: The bimodal agent's (ironed) revenue curve and virtual values in quantile space.

Claim 2. The expected revenue on the ironed revenue curve is attainable with a DSIC mechanism.

Example: How would you obtain the ironed revenue at \$5 instead of just $R(5)$?

Note: Recall that the expected revenue of *any mechanism*, not just a posted price, can be expressed by its virtual welfare. (We have now shown that you could decompose it into a distribution of posted prices and thus express the revenue that way, too, actually.)

What's the final mechanism?

For any ironed interval $[a, b]$, examine $\bar{\varphi}(v)$ for $v \in [a, b]$. Draw conclusions about $\bar{\varphi}(v)$ and $x(v)$. $P(q(v))$ is a straight line (linear) there, so $\bar{\varphi}(q(v))$ will be?

What does this imply for ironed-virtual-welfare-maximizing allocation in $[a, b]$?

Multiple Bidders

Imagine we have three bidders competing in a revenue-optimal auction for a single item. They are as follows:

- Bidder 1 is uniform. $F_1(v) = \frac{v-1}{H-1}$ on $[1, H]$.
- Bidder 2 is exponential. $F_2(v) = 1 - e^{-x}$ for $v \in (1, \infty)$.
- Bidder 2 is exponential. $F_3(v) = 1 - e^{-2x}$ for $v \in (1, \infty)$.

What does the optimal mechanism look like?

Definition 2. A *reserve price* r is a minimum price below which no buyer may be allocated the item. There may also be personalized reserve prices r_i where if $v_i < r_i$ then v_i will not be allocated to. Bidders above their reserves participate in the auction.

Welfare Maximization in Multidimensional Settings

Multidimensional or *multi-parameter* environments are ones where we need to elicit more than one piece of information per bidder. The most common settings include m heterogenous (*different*) items and

- n unit-demand buyers; buyer i has value v_{ij} for item j but only wants at most 1 item. (You only want to buy 1 house!)
- n additive buyers: buyer i's value for set S is $\sum_{j \in S} v_{ij}$.
- n subadditive buyers for some subadditive functions
- n buyers who are k-demand: buyer i's value for a set of items S is $\max_{|S'|=k, S' \subseteq S} \sum_{j \in S'} v_{ij}$.
- n matroid-demand buyers for some matroid
- ...

With m heterogenous items, it's *possible* that our buyers could have different valuations for every single one of the 2^m bundles of items—that is why this general setting is referred to as *combinatorial auctions.*

Then how can we maximize welfare in this setting? How can we do so *tractably?* How can we even elicit preferences in a tractable way?

Theorem 2 (The Vickrey-Clarke-Groves (VCG) Mechanism)**.** *In every general mechanism design environment, there is a DSIC welfare-maximizing mechanism.*

Given bids $\mathbf{b}_1, \ldots, \mathbf{b}_n$ where each bid is indexed by the possible outcomes $\omega \in \Omega$, we define the welfare-maximizing allocation rule $\mathbf{x}$ by

Now that things are multidimensional, there's no more Myerson's Lemma! In multiple dimensions, what is monotonicity? What would the critical bid be?

Instead, we have bidders pay their *externality*—the loss of welfare caused due to i's participation:

$$p_i(\mathbf{b}) =$$

where $\omega^* = \mathbf{x}(\mathbf{b})$ is the outcome chosen when i *does* participate.

Claim 3. The VCG mechanism is DSIC.

Exercise (optional): Prove that the payment $p_i(\mathbf{b})$ is always non-negative (and so the mechanism is IR).