# Divide & Conquer III: Integer and Matrix Multiplication

**Theorem 1** (The Master Theorem)**.** *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the non-negative integers by the recurrence*

$$T(n) = a\,T(n/b) + f(n),$$

*where we interpret $n/b$ as $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then*

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } f(n) = O(n^{\log_b a - \varepsilon}) \text{ for a constant } \varepsilon > 0 \\ \Theta(n^{\log_b a} \log_2 n) & \text{if } f(n) = \Theta(n^{\log_b a}) \\ \Theta(f(n)) & \text{if } f(n) = \Omega(n^{\log_b a + \varepsilon}) \text{ for a constant } \varepsilon > 0 \text{ and} \\ & \quad af(n/b) \leq cf(n) \text{ for a constant } c < 1 \text{ and for all sufficiently large } n. \end{cases}$$

### The Problem: Integer Multiplication

Your input for the *integer multiplication* problem is two $n$-digit numbers $x$ and $y$. The goal is to output their product, $x \cdot y$.

What's the **naïve algorithm** and what's its **running time**?

### Step 1: Define your recursive subproblem.

One idea is to split each number into two parts: $x = 10^{n/2}a + b$ and $y = 10^{n/2}c + d$. Then

$$xy = 10^n ac + 10^{n/2}(ad + bc) + bd.$$

Additions and multiplications by powers of 10 (just shifts) are linear-time, so this reduces the problem to smaller multiplication problems:

$$T(n) = a\,T(n/b) + \Theta(f(n)). \qquad \text{What are } a, b, \text{ and } f(n)?$$

Which gives what running time?

### The Speed Up:

We actually only need to make *three* recursive calls: $ac$, $bd$, and $(a+b)(c+d)$.

### Step 2: Combine the solutions to your subproblems.

Show why this is enough.

Then:

$$T(n) = a\,T(n/b) + O(f(n)) \qquad \text{What are } a,\ b, \text{ and } f(n)?$$

$$\implies$$

## Matrix Multiplication: Strassen's Algorithm

### The Problem: Matrix Multiplication

Your input for the *matrix multiplication* problem is two $n \times n$ matrices $A$ and $B$. The goal is to output their product, $C = AB$. Recall that the $ik^{\text{th}}$ entry of $C$ is given by $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

What's the **running time** of the **naïve algorithm** here and why?

### Step 1: Define your recursive subproblem.

We divide each matrix into four submatrices.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{bmatrix}$$

What running time does this give us?

$$T(n) = a\,T(n/b) + \Theta(f(n)) \qquad \text{What are } a,\ b, \text{ and } f(n)?$$

$$\implies$$

**The Speed Up:**

We compute only the following products.

- $P1 = A(F - H)$
- $P2 = (A + B)H$
- $P3 = (C + D)E$
- $P4 = D(G - E)$
- $P5 = (A + D)(E + H)$
- $P6 = (B - D)(G + H)$
- $P7 = (A - C)(E + F)$

## Step 2: Combine the solutions to your subproblems.

Show why this is enough.

Then the running time:

$$T(n) = a\,T(n/b) + \Theta(f(n)) \qquad \text{What are } a,\ b, \text{ and } f(n)?$$

$$\Longrightarrow$$