Beyond Dominant-Strategy: Bayesian Settings

There are many reasons why we can't always require dominant strategies when design mechanisms.

- (1) Requiring such a strong concept might not be tractable.
- (2) Agents do not always have dominant strategies! What then?

We'll now introduce the Bayesian setting.

Suppose the valuation v_i of bidder i is drawn from a prior distribution F_i .

- CDF $F_i(x) = \Pr_{v_i \sim F_i} [v_i \le x].$
- PDF $f_i(x) = \frac{d}{dx}F_i(x)$.
- Joint distribution **F** or \vec{F} .

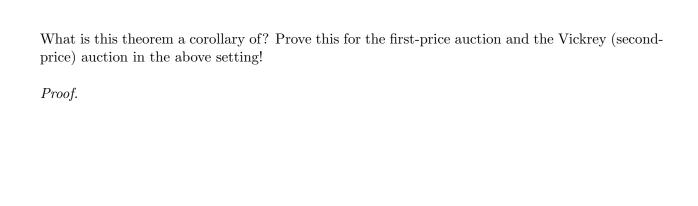
Unless otherwise noted, we assume that the prior distribution \mathbf{F} is *common knowledge* to all bidders and the mechanism designer (the seller).

Definition 1. A Bayes-Nash equilibrium (BNE) for a joint distribution \mathbf{F} is a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that for all i and v, $\sigma_i(v_i)$ is a best-response when other agents play $\sigma_{-i}(\mathbf{v}_{-i})$ when $\mathbf{v}_{-i} \sim \mathbf{F}_{-i} \mid_{v_i}$.

Claim 1. Consider two identically and independently drawn bidders from F = U[0,1]. It is a (symmetric) BNE for each bidder to bid $\sigma_i(v_i) = v_i/2$ in the first-price auction.

Proof.

Theorem 1 (Revenue Equivalence). The payment rule and revenue of a mechanism is uniquely determined by its allocation. Hence, any two mechanisms with the same allocation must earn the same expected revenue in Bayes-Nash Equilibrium.



Bayesian Settings

Using notions from the Bayesian setting and how bidders Bayesian update as they learn information, we define three stages of the auction:

- 1. ex ante:
- 2. interim:
- $3. \ ex \ post:$

Typically we discuss the ex post allocation and payment rules as a function of all of the values. However, in the Bayesian setting, to reason about BIC, it often makes sense to take in terms of interim allocation and payment rules which have the same information as bidder i before the auction is run.

Definition 2. We define the *interim* allocation and payment rules in expectation over the updated Bayesian prior given i's valuation:

$$x_i(v_i) =$$

and

$$p_i(v_i) =$$

Our definition of Bayesian Incentive-Compatibility then follows:

Definition 3. A mechanism with *interim* allocation rule x and *interim* payment rule p is Bayesian Incentive-Compatible (BIC) if

$$v_i x_i(v_i) - p_i(v_i) \ge v_i x_i(z) - p_i(z) \quad \forall i, v_i, z.$$

Virtual Welfare

Imagine a single buyer will arrive with their private value v. We want to design DSIC mechanisms.

What mechanism should you use to maximize welfare?

What should you do to maximize (expected) revenue?

Definition 4. In a deterministic mechanism, given other bids \mathbf{b}_{-i} , bidder *i*'s *critical bid* is the minimum bid $b_i^* = \min\{b_i : x_i(b_i, \mathbf{b}_{-i}) = 1\}$ such that bidder *i* is allocated to.

Then with \mathbf{b}_{-i} fixed, for all winning $v_i \geq b_i^*$, i's payment $p_i(v_i, \mathbf{b}_{-i}) = b_i^*$ is their critical bid.

What is winner i's critical bid in a single-item auction?

What about in the k identical item setting?