DS 574 Algorithmic Mechanism Design	**Lecture #6 Worksheet**
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Welfare Maximization in Multidimensional Settings

Multidimensional or *multi-parameter* environments are ones where we need to elicit more than one piece of information per bidder. The most common settings include m heterogenous (*different*) items and

- n unit-demand buyers; buyer i has value v_{ij} for item j but only wants at most 1 item. (You only want to buy 1 house!)
- n additive buyers: buyer i's value for set S is $\sum_{j \in S} v_{ij}$.
- n subadditive buyers for some subadditive functions
- n buyers who are k-demand: buyer i's value for a set of items S is $\max_{|S'|=k, S' \subseteq S} \sum_{j \in S'} v_{ij}$.
- n matroid-demand buyers for some matroid
- $\ldots$

With m heterogenous items, it's *possible* that our buyers could have different valuations for every single one of the 2^m bundles of items—that is why this general setting is referred to as *combinatorial auctions.*

Then how can we maximize welfare in this setting? How can we do so *tractably?* How can we even elicit preferences in a tractable way?

Theorem 1 (The Vickrey-Clarke-Groves (VCG) Mechanism)**.** *In every general mechanism design environment, there is a DSIC welfare-maximizing mechanism.*

Given bids $\mathbf{b}_1, \ldots, \mathbf{b}_n$ where each bid is indexed by the possible outcomes $\omega \in \Omega$, we define the welfare-maximizing allocation rule $\mathbf{x}$ by

Now that things are multidimensional, there's no more Myerson's Lemma! In multiple dimensions, what is monotonicity? What would the critical bid be?

Instead, we have bidders pay their *externality*—the loss of welfare caused due to i's participation:

$$p_i(\mathbf{b}) =$$

where $\omega^* = \mathbf{x}(\mathbf{b})$ is the outcome chosen when i *does* participate.

Claim 1. The VCG mechanism is DSIC.

Exercise (optional): Prove that the payment $p_i(\mathbf{b})$ is always non-negative (and so the mechanism is IR).

Ascending Auctions

In *ascending auctions*, an auctioneer initializes prices for each item, iteratively raises the prices, and bidders decide which items to bid on in each round. Sometimes *activity rules* are enforced, e.g., once you drop out on an item, you can not bid on it again.

The most famous ascending auction is the single-item version, the English Auction.

The English Auction(ε):

a. Initialize the item's price p_0 to

b. The initial set S_0 of "active bidders" (willing to pay p_0 for the item) is

c. For iteration $t = 1, 2, \ldots,$:

(a) Ask the set of active bidders S_{t-1}:

$S_t =$

(b) If $|S_t| \leq 1$:

(c) Otherwise, p_t

Benefits of using ascending auctions:

- Ascending auctions are easier for bidders.
- Less information leakage.
- Transparency.
- Potentially more seller revenue.
- When there are multiple items, the opportunity for "price discovery."

What about k identical items? What should we do here?

The English Auction for k Identical Items:

Definition 1. In an ascending auction, *sincere bidding* means that a player answers all queries honestly.

Claim 2. In the k identical item setting, in an English auction, sincere bidding is a dominant strategy for every bidder (up to ε).

Claim 3. In the k identical item setting, if all bidders bid sincerely in an English auction, the welfare of the outcome is within $k\varepsilon$ of the maximum possible.

The English auction for k Identical Items terminates in $v_{\max}/\varepsilon$ iterations.

Design process:

a. As a sanity check, design a direct-revelation DSIC welfare-maximizing polytime mechanism.

b. Implement this as an ascending auction.

c. **(Truthfulness)** Check that its EPIC.

d. **(Performance)** Check that it still maximizes welfare under sincere bidding.

e. **(Tractability)** Check that it terminates in a reasonable number of iterations.

Additive Valuations, Parallel Auctions

The Additive Setting: There are m non-identical items and n bidders where each bidder i has private valuation v_{ij} for each item j. Bidder i has an additive valuation for each set S, that is,

$$v_i(S) := \sum_{j \in S} v_{ij}.$$

Step 1: What is the welfare-optimal direct revelation mechanism here?

What's the analogous ascending implementation?

Is this DSIC?

Definition 2. A strategy profile $(\sigma_1, \dots, \sigma_n)$ is an *ex post Nash equilibrium (EPNE)* if, for every bidder i and valuation $v_i \in V_i$, the strategy $\sigma_i(v_i)$ is a best-response to every strategy profile $\sigma_{-i}(\mathbf{v}_{-i})$ with $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$.

In comparison, in a dominant-strategy equilibrium (DSE), for every bidder i and valuation v_i, the action $\sigma_i(v_i)$ is a best response to every action profile $\mathbf{a}_{-i}$ of $\mathbf{A}_{-i}$, whether of the form $\sigma_{-i}(\mathbf{v}_{-i})$ or not.

Definition 3. A mechanism is *ex post incentive compatible (EPIC)* if sincere bidding is an ex post Nash equilibrium in which all bidders always receive nonnegative utility.

Claim 4. For n additive bidders with m heterogenous items, in parallel English auctions, sincere bidding by all bidders is an ex post Nash equilibrium (up to $m\varepsilon$).

Unit Demand

The Unit-Demand Setting: There are m non-identical items and n bidders where each bidder i has private valuation v_{ij} for each item j. Bidder i is unit demand, that is, wants at most one item for any set S:

$$v_i(S) := \max_{j \in S} v_{ij}.$$

First, solve the direct-revelation problem. What do we observe about the welfare-maximizing allocation in the unit-demand setting?

Refresh yourself on what the VCG mechanism looks like. Then what does the analogous ascending auction look like?