

Linear Programming III: Duality Theory and Zero-Sum Games

Conditions for Optimality

Weak Duality

Theorem 1. *If \mathbf{x} is feasible in (P) and \mathbf{y} is feasible in (D) then $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$.*

Give an upper bound on maximum matching:

Give a lower bound on vertex cover:

Strong Duality

Theorem 2 (Strong Duality). *A pair of solutions $(\mathbf{x}^*, \mathbf{y}^*)$ are optimal for the primal and dual respectively if and only if $\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$.*

Proof. (\Rightarrow) Skip.

(\Leftarrow)

Complementary Slackness

Primal (P) :

$$\begin{aligned} & \max \quad \mathbf{c}^T \mathbf{x} \\ & \text{subject to} \quad \sum_i a_{ji} x_i \leq b_j \quad \forall j \quad (y_j) \\ & \quad \quad \quad x_i \geq 0 \quad \forall i \end{aligned}$$

Dual (D) :

$$\begin{aligned} & \min \quad \mathbf{b}^T \mathbf{y} \\ & \text{subject to} \quad \sum_i a_{ij} y_i \geq c_i \quad \forall i \quad (x_i) \\ & \quad \quad \quad y_j \geq 0 \quad \forall j \end{aligned}$$

Theorem 3 (Complementary Slackness). *A pair of solutions $(\mathbf{x}^*, \mathbf{y}^*)$ are optimal for the primal and dual respectively if and only if the following complementary slackness conditions (1) and (2) hold:*

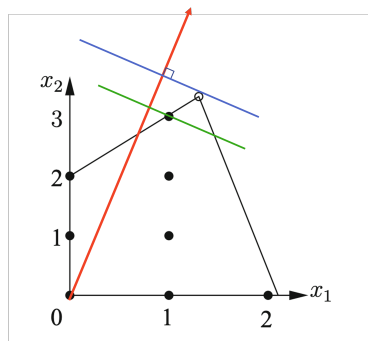
Proof.

Using Linear Programming for a Vertex Cover Approximation

$$\begin{array}{ll} \min & \sum_{i \in V} w_i x_i \\ \text{s.t.} & x_i + x_j \geq 1 \quad (i, j) \in E \\ & x_i \in [0, 1] \quad i \in V. \end{array}$$

Claim 1. Let S^* denote the optimal vertex cover of minimum weight, and let x^* denote the optimal solution to the Linear Program. Then $\sum_{i \in V} w_i x_i^* \leq w(S^*)$.

Proof. The vertex cover problem is equivalent to the integer program, whereas the linear program is a *relaxation*. Then there are simply more solutions allowed to the linear program, so the minimum can only be smaller. \square



$$\begin{array}{ll}\max & 4x_1 + x_2 \\ \text{subject to} & -x_1 + x_2 \leq 2 \\ & 8x_1 + 2x_2 \leq 17 \\ & x_1, x_2 \geq 0\end{array}$$

Claim 2. The set $S = \{i : x_i \geq 0.5\}$ is a vertex cover, and $w(S) \leq 2 \sum_{i \in V} w_i x_i^*$.

Zero-Sum Games and the Minimax Theorem

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

The Minimax Theorem

Theorem 4 (Minimax Theorem). *For every two-player zero-sum game \mathbf{A} ,*

$$\max_{\mathbf{x}} \left(\min_{\mathbf{y}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right) = \min_{\mathbf{y}} \left(\max_{\mathbf{x}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right). \quad (1)$$

From LP Duality to Minimax

Issues:

(1)

(2)

Observation:

$$\max_{\mathbf{x}} \left(\min_{\mathbf{y}} \mathbf{x}^T \mathbf{A} \mathbf{y} \right) = \max_{\mathbf{x}} \left(\min_{j=1}^n \mathbf{x}^T \mathbf{A} \mathbf{e}_j \right) \quad (2)$$

$$= \max_{\mathbf{x}} \left(\min_{j=1}^n \sum_{i=1}^m a_{ij} x_i \right) \quad (3)$$