

Beyond Dominant-Strategy: Bayesian Settings

There are many reasons why we can't always require dominant strategies when design mechanisms.

- (1) Requiring such a strong concept might not be tractable.
- (2) Agents do not always have dominant strategies! What then?

We'll now introduce the Bayesian setting.

Suppose the valuation v_i of bidder i is drawn from a prior distribution F_i .

- CDF $F_i(x) = \Pr_{v_i \sim F_i}[v_i \leq x]$.
- PDF $f_i(x) = \frac{d}{dx} F_i(x)$.
- Joint distribution \mathbf{F} or \vec{F} .

Unless otherwise noted, we assume that the prior distribution \mathbf{F} is *common knowledge* to all bidders and the mechanism designer (the seller).

Definition 1. A *Bayes-Nash equilibrium (BNE)* for a joint distribution \mathbf{F} is a strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$ such that for all i and v , $\sigma_i(v_i)$ is a best-response when other agents play $\sigma_{-i}(\mathbf{v}_{-i})$ when $\mathbf{v}_{-i} \sim \mathbf{F}_{-i} \mid v_i$.

Claim 1. Consider two identically and independently drawn bidders from $F = U[0, 1]$. It is a (symmetric) BNE for each bidder to bid $\sigma_i(v_i) = v_i/2$ in the first-price auction.

Proof.

Theorem 1 (Revenue Equivalence). *The payment rule and revenue of a mechanism is uniquely determined by its allocation. Hence, any two mechanisms with the same allocation must earn the same expected revenue in Bayes-Nash Equilibrium.*

What is this theorem a corollary of? Prove this for the first-price auction and the Vickrey (second-price) auction in the above setting!

Proof.

Bayesian Settings

Using notions from the Bayesian setting and how bidders Bayesian update as they learn information, we define three stages of the auction:

1. *ex ante*:

2. *interim*:

3. *ex post*:

Typically we discuss the *ex post* allocation and payment rules as a function of all of the values. However, in the Bayesian setting, to reason about BIC, it often makes sense to take in terms of *interim* allocation and payment rules which have the same information as bidder i before the auction is run.

Definition 2. We define the *interim* allocation and payment rules in expectation over the updated Bayesian prior given i 's valuation:

$$x_i(v_i) =$$

and

$$p_i(v_i) =$$

Our definition of Bayesian Incentive-Compatibility then follows:

Definition 3. A mechanism with *interim* allocation rule x and *interim* payment rule p is Bayesian Incentive-Compatible (BIC) if

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(z) - p_i(z) \quad \forall i, v_i, z.$$

Virtual Welfare

Imagine a single buyer will arrive with their private value v . We want to design DSIC mechanisms.

What mechanism should you use to maximize *welfare*?

What should you do to maximize (expected) revenue?

Definition 4. In a deterministic mechanism, given other bids \mathbf{b}_{-i} , bidder i 's *critical bid* is the minimum bid $b_i^* = \min\{b_i : x_i(b_i, \mathbf{b}_{-i}) = 1\}$ such that bidder i is allocated to.

Then with \mathbf{b}_{-i} fixed, for all winning $v_i \geq b_i^*$, i 's payment $p_i(v_i, \mathbf{b}_{-i}) = b_i^*$ is their critical bid.

What is winner i 's critical bid in a single-item auction?

What about in the k identical item setting?