Early Disposition, Master Thesis

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1 Binary NANDs

Definition 1. A NAND-clause is **binary** if it consists of only one or two atoms[1].

Definition 2. A NAND-clause is **binary-derivable** if it is provable in our refutation system using only other binary clauses.

Definition 3. A NAND-clause is **purely binary** if it is both binary and binary-derivable.

The main goal in this section will be to answer the following question: Given a purely binary NAND-clause $\overline{ab}$, how are the nodes a and b structurally related in the corresponding graph? In order to answer this question, let us first get some intuition behind this peculiar subgroup of clauses. Here are some immediate observations:

Lemma 1. *All axiomatic NAND-clauses are purely binary.*

Proof. The axiomatic NAND-clauses require no proofs, so they are vacuously binary-derivable. Since they are all binary, they are by definition purely binary. □

Since these axioms corresponds to *edges* in the graph, we can already conclude that whatever graph relation we are looking for, it needs to hold for adjacent nodes (i.e. nodes with an edge between them).

Lemma 2. *Given any instance of the* (*Rneg*)*-rule, the clause in the conclusion is binary-derivable if and only if all the clauses in the premise are purely binary.*

Proof. (⇒) Given an instance of the (*Rneg*)-rule with the conclusion clause $\overline{A}$, suppose one of the clauses $\overline{B}$ in the premise is *not* purely binary. Then either $\overline{B}$ itself is not binary, or the proof of $\overline{B}$ contains a non-binary clause $\overline{X}$. Since both $\overline{B}$ and $\overline{X}$ are contained in the proof of $\overline{A}$, we get that $\overline{A}$ can not be binary derived.
(⇐) Given a similar instance as above with the conclusion clause $\overline{A}$, suppose $\overline{A}$ is not binary-derivable. Then there exists a non-binary clause $\overline{Y}$ in the proof of $\overline{A}$. $\overline{Y}$ is either *in* the premise, or it is in the proof of a clause $\overline{C}$ in the premise. Either way, there will be a clause in the premise that is not purely binary. □

Even though the condition above ensures the binary derivability of the conclusion, it is however not strong enough to guarantee the purely binary property. The following proof shows this point:

$$\dfrac{\dfrac{\cdots}{\Gamma \vdash \overline{ab}} \qquad \dfrac{\cdots}{\Gamma \vdash \overline{cd}} \qquad \dfrac{\cdots}{\Gamma \vdash \overline{ef}}}{\Gamma \vdash \overline{ace}}\; bdf$$

[1] atoms?

$\overline{ace}$ is obviously not purely binary in the proof above, seeing it consists of 3 atoms. The assumption that the three clauses $\overline{ab}$, $\overline{cd}$ and $\overline{ef}$ are purely binary does not change this fact.

Since the condition in the preceding lemma coincides with the notion of binary derivability, we can simply add the binary condition to it in order to get a condition for purely binary clauses:

Corollary 3. *Given any instance of the (Rneg)-rule, the clause in the conclusion is purely binary if and only if it is binary and all the clauses in the premise are purely binary.*

Using these observations, we can now define our purely binary NAND-clauses inductively:

Definition 4. A NAND-clause $\overline{X}$ is purely binary iff:

- (Base Case): $\overline{X}$ is an axiom.
- (Inductive Case): $\overline{X}$ is binary and is the conclusion of a rule with only purely binary clauses in the premise.

TODO: Explain binary conclusion in terms of premise.

TODO: Give a corresponding graph structural definition based on the inductive definition of a purely binary clause.

TODO: Give graph examples.

2 Main Theorem

Theorem 4. *Any binary NAND $\overline{ab}$ proved in the refutation system using only binary NANDs corresponds to one of the following vels in the graph model: $V(a,a)$, $V(a,b)$ or $V(b,b)$.*

2.1 Proof Outline

We will prove this fact by induction on the (length of the) proof.

2.1.1 Base case

The shortest proofs of binary nands are the axioms themselves. The axiomatic NANDs in the refutation system corresponds to simple edges in the graph, which are vels by the base definition.

2.1.2 Inductive step

The refutation system consists of a single rule. We want to show that any application of this rule, that complies with our restrictions, will preserve the property stated in the theorem.

Since we are restricting ourselves to the proofs consisting purely of NANDs that are binary, we need only look at the applications where this holds. In our case this means looking only at the applications where the NANDs in both the premise and the conclusion are binary.

In order for such an application to have a binary result, we need to have an instance of the following situation (with J, K and L being disjoint, and J and K nonempty):

$$\frac{\{\Gamma \vdash \overline{ax_i} \mid i \in J\} \quad \{\Gamma \vdash \overline{bx_i} \mid i \in K\} \quad \{\Gamma \vdash \overline{x_i} \mid i \in L\} \quad \Gamma \vdash \{x_i \mid i \in J \cup K \cup L\}}{\overline{ab}}$$

If and only if[2] your premise is on the above form, the result will be binary.

Since we assume that our NAND in the conclusion is derived using only binary NANDs, we can immediately assume the same thing for all the NANDs in the premise. Our induction hypothesis thus tells us that for all the NANDs in the premise, since they are all binary and derived from binary NANDS only, they correspond to vels in the graph.

Now, in order to continue, recall first that for an OR $\{x_1 x_2 \dots x_n\}$ in our system, we have that some x_i

[2] Probably in need of some more justification.

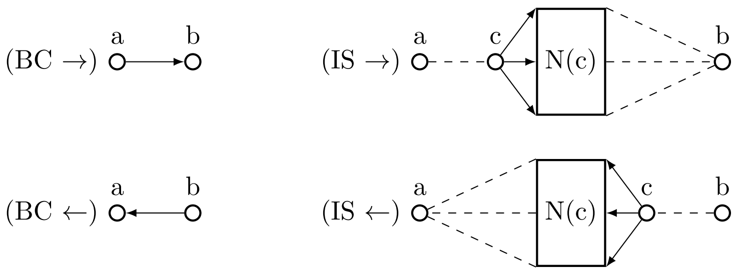
2.2 New definitions

We define the concept of a vel inductively:

Definition 5. We say that two nodes a, b are connected by a **vel** ($V(a, b)$) if:

- (BC $\rightarrow$) $b \in N(a)$.
- (BC $\leftarrow$) $a \in N(b)$.
- (IS $\rightarrow$) $\exists c : [V(a, c) \wedge (\forall x \in N(c) : (V(x, a) \vee V(x, x) \vee V(x, b))) \wedge (\exists x \in N(c) : V(x, b))]$.
- (IS $\leftarrow$) $\exists c : [V(b, c) \wedge (\forall x \in N(c) : (V(x, a) \vee V(x, x) \vee V(x, b))) \wedge (\exists x \in N(c) : V(x, b))]$.

We can illustrate the different cases with the following figures (with dashed lines representing vels):



Note that both the base case and the inductive step is given in symmetric pairs, making V a symmetric relation: $V(a, b) \Leftrightarrow V(b, a)$.

Definition 6. A path is a directed version of the vel. Its definition consists of (BC $\rightarrow$) and (IS $\rightarrow$) from above.

Lemma 5. *If there is a vel between the two nodes a and b and b has no outgoing edges contained in the vel, then the vel is a path going from a to b.*

Assuming a vel between two nodes, we overload the notation and let $V(a, b)$ be the set of edges contained in the vel. This is obviously only well-defined when the vel in question actually exists. We can now write the above lemma formally as:

$$V(a, b) \cap N(b) = \emptyset \quad \Rightarrow \quad P(a, b)$$

Proof. TODO $\square$

We will show that paths can be constructed in several ways, and we will define a more general way to construct vels using paths. This will lead to

Lemma 6. *If there is a vel between two nodes a and b such that all neighbors of a contained*[3] *in the vel also has vels to b, then there is a vel between b and b.*

$$((\forall c \in V(a, b) \cap N(a))(V(c, b))) \Rightarrow V(b, b)$$

This is a property purely on the vel structure, so we prove it using structural induction on the vel $V(a, b)$.

[3]We need a precise notion of what it means for an edge/node to be contained in a vel

Proof. (BC $\rightarrow$) In this case, the only neighbor of a contained in the vel is b. Since we assume that all neighbors of a has a vel to be, we immediately get $V(b, b)$.

(BC $\leftarrow$) In this case a has no neighbors, so our statement becomes vacuously true.

(IS $\rightarrow$) Referring to our illustration of this situation, our assumption gives us $V(a, b)$ and $V(c, b)$. By the definition of a vel, we have that for each $x \in N(c)$, either $V(x, b)$, $V(x, x)$ or $V(x, a)$. Since we assume $V(c, b)$, we also get that there is a node x_i in $N(c)$ such that for all $y \in N(x_i)$, we get that either $V(y, b)$, $V(y, y)$ or $V(y,)$. $\square$

3 Introduction

4 Definitions and Observations

4.1 Graphs and Paths

We start by writing out some basic definitions from graph theory.

Definition 7. A **graph** $\boldsymbol{G} = (G, N)$ is a tuple where the first element G is a set representing the nodes in the graph, while the second element N is a set representing the edges (neighbours). N is usually represented as a subset of $G \times G$ where $(u, v) \in N$ iff there is an edge from node u to node v in the graph.

Another way to think of N is as a function $N : G \to \mathcal{P}(G)$ where $N(x)$ returns the set of all out-neighbours of x.

Definition 8. A **path** is a sequence of nodes $(x_0, x_1, x_2, \ldots, x_n)$ from the graph in which all nodes (except possibly the first and last) are distinct, such that for any two consequtive nodes x_i, x_{i+1}, we have that $(x_i, x_{i+1}) \in N$. In this case we say that there is a path from x_0 to x_n.

For every x in the graph, we have the unique empty path (x) of length 0. This path is distinct from the loop (x, x).

4.2 Proof-specific relations

Definition 9. Given a node x and a collection of nodes Y, we have that $\boldsymbol{E(x, Y)}$ holds iff x has outgoing edges targeting exactly the nodes in Y[4].

Definition 10. Given two nodes x, y we have that $\boldsymbol{P(x, y)}$ holds iff there exists a path between x and y. We will use $\boldsymbol{P_o}$ and $\boldsymbol{P_e}$ to denote paths of odd and even lengths, respectively.

The text will sometimes use the notion of *path lengths*. This will not refer to the actual length of the path, but whether or not it is of even or odd length.

Definition 11. A path $P(x_0, x_n)$ is **fully trimmed** (denoted $\boldsymbol{P^f(x_0, x_n)}$) if no nodes in the path branches off elsewhere. This means that for every pair of consecutive nodes x_i, x_{i+1} in the path, we need that $E(x_i, \{x_{i+1}\})$.

Definition 12. A path $P(x_0, x_n)$ is **evenly trimmed** (denoted $\boldsymbol{P^e(x_0, x_n)}$) if every evenly indexed node is non-branching. Said differently, for every *even* $i < n$ we need $E(x_i, \{x_{i+1}\})$ to hold.

Definition 13. A path $P(x_0, x_n)$ is **oddly trimmed** (denoted $\boldsymbol{P^o(x_0, x_n)}$) if for every *odd* $i < n$, $E(x_i, \{x_{i+1}\})$ holds.

We will usually denote paths with both length and trim in combination, i.e. $P^e_o(x, y)$. This is solely to simplify the proofs and is nothing more than an abbreviation for $P^e(x, y) \wedge P_o(x, y)$, meaning an evenly trimmed odd path from x to y.

Definition 14. There is a **vel** between two nodes x and y (denoted $\boldsymbol{V(x, y)}$) if and only if there exists a node c such that $P^o(x, c)$ and $P^o(y, c)$ and the sum of the two path lengths is odd. In formal terms we have that

$$V(x, y) \Leftrightarrow (P^o_o(x, c) \wedge P^o_e(y, c)) \vee (P^o_e(x, c) \wedge P^o_o(y, c))$$

[4]This definition has to be changed

By introducing a bit more notation, we will be able to write this formal defintion of a vel in a nicer way. By generalizing over the actual lengths and trims of paths we can make statements like:

$$P_x(a, b) \wedge P_x(b, c) \Rightarrow P_e(a, c) \quad \text{where} \quad x \in \{e, o\}$$

This lets us argue about paths of equal length (even or odd) in one statement instead of explicitly describing both cases. The statements above tells us that the concatenation of two paths of equal length (even or odd) results in an even path[5].

Similarly, we can argue about paths of different lengths by introducing $\overline{x}$ as a piece of notation such that $\overline{o} = e$ and $\overline{e} = o$. We can now make statements like

$$P_x(a, b) \wedge P_{\overline{x}}(b, c) \Rightarrow P_o(a, c)$$

telling us that concatenating two paths of different lengths (even or odd) results in an odd path. We are now able to write our definition of a vel in a nice, shorter way[6]:

$$V(a, b) \Leftrightarrow \exists x(P_x^o(a, c) \wedge P_{\overline{x}}^o(b, c))$$

4.3 Immediate observations

Lemma 7. $P^f(x, y) \Rightarrow P^o(x, y) \wedge P^e(x, y)$

This implication should be easy to accept; a fully trimmed path has branching restrictions on all its nodes, both the odd and the even ones. It is thus both oddly and evenly trimmed.

The opposite implication is actually also true, allthough not as obvious, since it would seem that $P^o(x, y)$ could refer to a path partly disjoint from $P^e(x, y)$ and there would thus be no single path both oddly and evenly trimmed. We will later show that this can't actually be the case.

Lemma 8. $P_e(x, x)$ *for every* $x \in G_v$

This will denote the empty path mentioned earlier. The path is even simply because 0 is even.

Lemma 9. $E(x, \{y\}) \Rightarrow P_o^f(x, y)$

An edge is another trivial example of a path. The path is fully trimmed since x does not branch off. In the more general case of $E(x, Y)$, we have that $P_o^o(x, y)$ holds for every $y \in Y$.

Lemma 10. $P_o^o(x, y) \Rightarrow V(x, y)$

A vel needs two paths of even and odd length, respectively. Since the even path can be empty, we get that any odd path is a vel by itself.

5 Path composition

Suppose we have two paths; one from a to b and one from b to c. We can now obtain[7] a path going from a to c by first traversing from a to b using the first path, then traversing from b to c using the second path. This operation is called **path composition** and the resulting path

[5] Rewrite section. Use P+E examples instead of P+P

[6] Does the definition below need an existencial quantifier?

[7] We don't really create anything here. Is *have* a better wording?

is called a **composite path**. Our notation captures the concept of path composition in the following way:

$$P(a,b) \wedge P(b,c) \Rightarrow P(a,c)$$

In this case, we say that $P(a,b)$ is composed with $P(b,c)$.

Let's extend this statement in order to describe the change in length. We observe that when composing two even – or two odd – paths, the resulting path will always be even, while composing one path of even length with one of odd length, the resulting path will be odd. Introducing length to the equation thus gives us two cases:

$$P_x(a,b) \wedge P_x(b,c) \Rightarrow P_e(a,c)$$
$$P_x(a,b) \wedge P_{\overline{x}}(b,c) \Rightarrow P_o(a,c)$$

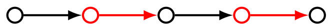
The next step will be to extend these statements even further to also capture the behavior of trimming under composition. In order to do this, let's first make some observations:

First, unlike with the length property, we do not have any guarantee that a composite path has a trim at all. We might need to set certain restrictions to our two original paths in order to get a properly trimmed (either evenly or oddly) path as a result after composition. It should at least be obvious that both original paths should be trimmed in some way.

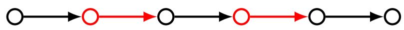
Secondly, no matter what path you compose a trimmed path with, the resulting composite path will always be of the same trim, if trimmed at all. This means that given a path $P^e(a,b)$ and a path $P(b,c)$, the composite path from a to c will never be oddly trimmed. Equivalently, a composition of $P^o(a,b)$ and $P(b,c)$ will never result in $P^e(a,c)$.

Now, let's look at some examples. The below figure shows instances of the four possible combinations of trims and lengths of a path. All nodes x and edges (x,y) such that $E(x,\{y\})$ are depicted in red[8] and will from here on be referred to as *non-branching* nodes and *unique* edges, respectively[9].

P_e^e 

P_e^o 

P_o^e 

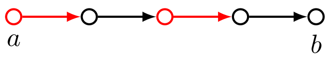
P_o^o 

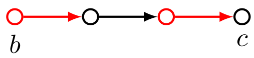
Now comes the question: What kind of paths can we put at the end of each of these different paths such that the resulting composite path would be properly trimmed (either evenly or oddly)?

Let us start by looking at evenly trimmed even paths (P_e^e). The first edge in a path of this type is obviously always unique, but since the path is of even length, we also know that its *last* edge does not have this uniqueness property. In order to preserve the even trim under composition, the first edge of the consecutive path therefore has to be unique. This means that given $P_e^e(a,b)$ and $P(b,c)$, the composite path is evenly trimmed only when $P(b,c)$ is evenly trimmed.

The following figures examplifies this:

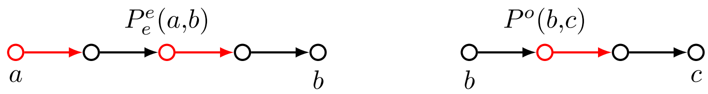
$P_e^e(a,b)$ $P^e(b,c)$





[8] Should I avoid using colors in the paper?
[9] Overkill?

Composing the two paths above will give an evenly trimmed path from a to b.



Composing the two paths above will *not* give an evenly trimmed path from a to b, since it will contain two consecutive edges without the uniqueness property.

The first thing to notice with these paths is that the last edges of P_e^o and P_o^e are both unique, while the last edges of P_e^e and P_o^o are not. It should be clear that this property holds for all instances of these four path types[10], not only the examples given above. It should therefore suffice to show the properties using our example paths only.

Notice how the last edge of these paths is never unique (red). This means that in order to preserve the even trim, the first edge of the consecutive path needs to be a unique one, i.e. the path has to be evenly trimmed.[11]

[10]Should this be further justified?

[11]This section is not done. I feel like I have already been talking too much about obvious concepts.

6 Rules

6.1 Specific Rules

6.1.1 Path Shortening Rules (S)

$$\frac{P_o^o(a,c), E(a,B)}{\bigvee_{b\in B} P_e^e(b,c)}\ \text{S1} \qquad \frac{P_o^e(a,c)}{P_e^o(b,c), E(a,\{b\})}\ \text{S2}$$

$$\frac{P_e^o(a,c), E(a,B)}{a=c \qquad \bigvee_{b\in B} P_o^e(b,c)}\ \text{S3} \qquad \frac{P_e^e(a,c)}{a=c \qquad P_o^o(b,c), E(a,\{b\})}\ \text{S4}$$

6.1.2 Path Composition Rules (C)

$$\frac{P_o^o(a,b), P^e(b,c)}{P^o(a,c)}\ \text{C1} \qquad \frac{P_o^e(a,b), P^o(b,c)}{P^e(a,c)}\ \text{C2}$$

$$\frac{P_e^o(a,b), P^o(b,c)}{P^o(a,c)}\ \text{C3} \qquad \frac{P_e^e(a,b), P^e(b,c)}{P^e(a,c)}\ \text{C4}$$

6.1.3 Brading Rules (B)

$$\frac{P_o^e(a,b), P_o^o(a,c)}{P_e^e(b,c) \qquad P_e^e(c,b)}\ \text{B1} \qquad \frac{P_o^e(a,b), P_e^o(a,c)}{P_o^e(b,c) \qquad P_o^e(c,b)}\ \text{B2}$$

$$\frac{P_e^e(a,b), P_o^o(a,c)}{P_o^o(b,c) \qquad P_o^o(c,b)}\ \text{B3} \qquad \frac{P_e^e(a,b), P_e^o(a,c)}{P_e^o(b,c) \qquad P_e^e(c,b)}\ \text{B4}$$

TODO: All of these will need justification.

6.2 General Rules

We can use variables to bring down the number of rules. There are several ways of doing this, but this way will show to be the appropriate for our intentions.

6.2.1 General Path Composition (GC)

$$\frac{P_x^o(a,b), P^{\overline{x}}(b,c)}{P^o(a,c)}\ \text{GC1} \qquad \frac{P_x^e(a,b), P^x(b,c)}{P^e(a,c)}\ \text{GC2}$$

Notice that GC1 covers C1 and C3 while GC2 covers C2 and C4.

6.2.2 General Brading (GB)

$$\frac{P_e^e(a,b), P_x^o(a,c)}{P_x^o(b,c) \qquad P_x^x(c,b)}\ \text{GB1} \qquad \frac{P_o^e(a,b), P_x^o(a,c)}{P_{\overline{x}}^e(b,c) \qquad P_{\overline{x}}^x(c,b)}\ \text{GB2}$$

GB1 covers B3 and B4 while GB2 covers B1 and B2.

7 Vel composition

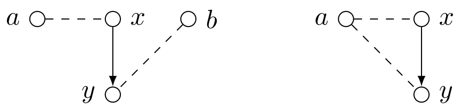
Theorem 11. *Given a node x and a nonempty set Y such that $E(x,Y)$, then for all nodes a,b such that $V(a,x)$ and either $V(a,y)$ or $V(b,y)$ for all $y \in Y$ we have that either $V(a,a)$ or $V(a,b)$.*

This general property will later be proven directly, but we will start by giving an exhaustive proof of the base case where $Y = \{y\}$. This will give us a good intuition of how the general proof will look like.

7.1 Proving the base case

In the base case we have a node x such that $E(x,\{y\})$ and nodes a and b such that $V(a,x)$ and either $V(a,y)$ or $V(b,y)$. We will show that these conditions implies either $V(a,a)$ or $V(a,b)$.

The figure below illustrates our two possible situations, where a solid arrow represents an edge while dashed lines represents vels:



At this point, we have two cases to cover: The one where there is a vel between a and y, and the case with a vel between b and y. The following proof will cover the second case, but it turns out that the first case will have a perfectly equivalent proof. We will come back to this later, and we will actively use this fact in the general proof.

Now that we have established our assumption, $V(a,x), V(b,y), E(x,\{y\})$, we will use our earlier definition of a vel to expand the assumption into four cases:

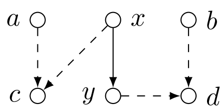
(i) $P_o^o(a,c), P_e^o(x,c), P_o^o(b,d), P_e^o(y,d), E(x,\{y\})$

(ii) $P_o^o(a,c), P_e^o(x,c), P_e^o(b,d), P_o^o(y,d), E(x,\{y\})$

(iii) $P_e^o(a,c), P_o^o(x,c), P_o^o(b,d), P_e^o(y,d), E(x,\{y\})$

(iv) $P_e^o(a,c), P_o^o(x,c), P_e^o(b,d), P_o^o(y,d), E(x,\{y\})$

The figure bellow illustrates how these paths are arranged:



Since each vel in a statements expands into the choice between two pairs of paths when we apply our vel definition, we get that any statement containing n vels expands into 2^n cases.

To avoid this explosion in later proofs, we will be using the general definition of a vel. This removes the case explosion all together and leaves us with a constant number of cases regardless of the original number of vels.

7.1.1 Proof, base case i

$$\dfrac{P^o_e(x,c), E(x,\{y\})}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{E(x,\{y\}), x=c}{E(c,\{y\})}\;(=)}{P^o_e(y,d), P^f_o(c,y)}\;\text{L3}}{P^o_o(a,c), P^e_o(c,d)}\;\text{C2}}{P^o_o(b,d), P^o_e(a,d)}\;\text{C1}}{V_o(a,b)}\;\text{def} \qquad \dfrac{P^e_o(y,c), P^o_e(y,d)}{\dfrac{\dfrac{P^o_o(a,c), P^e_o(c,d)}{P^o_o(b,d), P^o_e(a,d)}\;\text{C1}}{V_o(a,b)}\;\text{def} \qquad \dfrac{\dfrac{P^o_o(b,d), P^e_o(d,c)}{P^o_o(a,c), P^o_e(b,c)}\;\text{C1}}{V_o(a,b)}\;\text{def}}\;\text{B2}}\;\text{S3}$$

7.1.2 Proof, base case ii

$$\dfrac{P^o_e(x,c), E(x,\{y\})}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{E(x,\{y\}), x=c}{E(c,\{y\})}\;(=)}{P^o_o(y,d), P^f_o(c,y)}\;\text{L3}}{P^o_o(a,c), P^e_e(c,d)}\;\text{C2}}{P^o_e(b,d), P^o_o(a,d)}\;\text{C1}}{V_o(a,b)}\;\text{def} \qquad \dfrac{P^e_o(y,c), P^o_o(y,d)}{\dfrac{\dfrac{P^o_o(a,c), P^e_e(c,d)}{P^o_e(b,d), P^o_o(a,d)}\;\text{C1}}{V_o(a,b)}\;\text{def} \qquad \dfrac{\dfrac{P^o_e(b,d) P^o_e(d,c)}{P^o_o(a,c), P^o_e(b,c)}\;\text{C3}}{V_o(a,b)}\;\text{def}}\;\text{B1}}\;\text{S3}$$

7.1.3 Proof, base case iii

$$\dfrac{\dfrac{P^o_o(x,c), E(x,\{y\})}{P^e_e(y,c), P^o_e(y,d)}\;\text{S1}}{\dfrac{\dfrac{P^o_e(a,c), P^o_e(c,d)}{P^o_o(b,d), P^o_e(a,d)}\;\text{C3}}{V_o(a,b)}\;\text{def} \qquad \dfrac{\dfrac{P^o_o(b,d), P^e_e(d,c)}{P^o_e(a,c), P^o_o(b,c)}\;\text{C1}}{V_o(a,b)}\;\text{def}}\;\text{B4}$$

7.1.4 Proof, base case iv

$$\dfrac{\dfrac{P^o_o(x,c), E(x,\{y\})}{P^e_e(y,c), P^o_o(y,d)}\;\text{S1}}{\dfrac{\dfrac{P^o_e(a,c), P^o_o(c,d)}{P^o_e(b,d), P^o_o(a,d)}\;\text{C3}}{V_o(a,b)}\;\text{def} \qquad \dfrac{\dfrac{P^o_e(b,d), P^o_o(d,c)}{P^o_e(a,c), P^o_o(b,c)}\;\text{C3}}{V_o(a,b)}\;\text{def}}\;\text{B3}$$

7.1.5 Proving the generalized cases

As mentioned earlier, we can use our alternative definition of vels to avoid having to deal with an exponential number of cases. Using this alternative definition we are able to cut the number down to one:

$$P^o_{x_1}(a,c), P^o_{\overline{x_1}}(x,c), P^o_{x_2}(b,d), P^o_{\overline{x_2}}(y,d), E(x,\{y\})$$

The problem now is that our rules – the S-rules specifically – don't operate on this level of abstraction. Since we only use our S-rules in the first step of our proofs, where we deal with the vel between a and x, we will keep the case distinction when we expand that vel. This means that we finally are left with two cases:

(i) $P_o^o(a,c), P_e^o(x,c), P_x^o(b,d), P_{\bar{x}}^o(y,d), E(x,\{y\})$

(ii) $P_e^o(a,c), P_o^o(x,c), P_x^o(b,d), P_{\bar{x}}^o(y,d), E(x,\{y\})$

7.1.6 Proof abstract base case i

$$\dfrac{P_e^o(x,c), E(x,\{y\})}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{E(x,\{y\}), x=c}{E(c,\{y\})}\,(=)}{P_{\bar{x}}^o(y,d), P_o^f(c,y)}\,\mathrm{L3}}{P_o^o(a,c), P_x^e(c,d)}\,\mathrm{GC2}}{P_x^o(b,d), P_{\bar{x}}^o(a,d)}\,\mathrm{GC1}}{V_o(a,b)}\,\mathrm{def} \qquad \dfrac{P_o^e(y,c), P_{\bar{x}}^o(y,d)}{\dfrac{\dfrac{P_o^o(a,c), P_x^e(c,d)}{P_x^o(b,d), P_{\bar{x}}^o(a,d)}\,\mathrm{GC1}}{V_o(a,b)}\,\mathrm{def} \qquad \dfrac{\dfrac{P_x^o(b,d) P_x^x(d,c)}{P_o^o(a,c), P_e^o(b,c)}\,\mathrm{GC1}}{V_o(a,b)}\,\mathrm{def}}\,\mathrm{GB3}}\,\mathrm{S3}$$

7.1.7 Proof, abstract base case ii

$$\dfrac{\dfrac{P_o^o(x,c), E(x,\{y\})}{P_e^e(y,c), P_x^o(y,d)}\,\mathrm{S1}}{\dfrac{\dfrac{P_e^o(a,c), P_x^o(c,d)}{P_{\bar{x}}^o(b,d), P_x^o(a,d)}\,\mathrm{GC1}}{V_o(a,b)}\,\mathrm{def} \qquad \dfrac{\dfrac{P_{\bar{x}}^o(b,d), P_x^x(d,c)}{P_e^o(a,c), P_o^o(b,c)}\,\mathrm{GC1}}{V_o(a,b)}\,\mathrm{def}}\,\mathrm{GB1}$$

7.1.8 Conclusion, base case

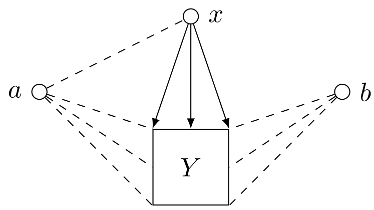
As mentioned earlier, the preceding proof is only for the case where we have $V(b,y)$. The theorem is also supposed to hold when we have $V(a,y)$. In that case we have this assumption: $V(a,x), V(a,y), E(x,y)$. The crucial thing to notice here, is that the proof for this latter case can be obtained simply by switching all instances of b with a in the proof. This will not violate any rules, and the proof will only change by resulting in $V(a,a)$ instead of $V(a,b)$.

We can thus conclude the following:

$$\frac{V(a,x), V(b,y), E(x,\{y\})}{V(a,b)} \qquad \frac{V(a,x), V(a,y), E(x,\{y\})}{V(a,a)}$$

7.2 Proving the general theroem

We will now look at the general version of our theorem, where the set of nodes Y can contain any number of nodes[12]. We thus have the following situation:



As earlier, arrows represent edges, while dashed lines represents vels. Y is now a set of nodes in which all elements are targeted by an edge from x and a vel from either a or b.

Using our notation, we get the following statement:

$$E(x,Y), V(a,x), \bigwedge_{y \in Y} V(a,y) \vee V(b,y)$$

As we discovered earlier, whether an element in Y is connected by a vel to a or b does not really matter for the proof. Therefore, in order to reduce the number of cases to prove, we will enumerate the elements in Y and let $v_i \in \{a,b\}$ denote the node conncted to $y_i \in Y$. By doing this, we get a simpler statement, both visually and with respect to the upcomming proof. We now have the following statement (with $y_i \in Y$):

$$E(x,Y), V(a,x), V(v_1,y_1), V(v_2,y_2), \ldots, V(v_n,y_n)$$

We will now expand this statement using our vel definition. Using the general definiton of vels, we will not get 2^n different cases, but we still want to distinguish between the two cases we get from $V(a,x)$. Our two cases are therefore:

(i) $E(x,Y), P^o_o(a,c_0), P^o_e(x,c_0), P^o_{\lambda_1}(v_1,c_1), P^o_{\overline{\lambda_1}}(y_1,c_1), \ldots, P^o_{\lambda_n}(v_n,c_n), P^o_{\overline{\lambda_n}}(y_n,c_n)$

(ii) $E(x,Y), P^o_e(a,c_0), P^o_o(x,c_0), P^o_{\lambda_1}(v_1,c_1), P^o_{\overline{\lambda_1}}(y_1,c_1), \ldots, P^o_{\lambda_n}(v_n,c_n), P^o_{\overline{\lambda_n}}(y_n,c_n)$

7.2.1 Proof, case i

$$\dfrac{P^o_e(x,c), E(x,\{y\})}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{E(x,\{y\}), x = c}{E(c,\{y\})}\,(=)}{P^o_{\overline{x}}(y,d), P^f_o(c,y)}\,\text{L3}}{P^o_o(a,c), P^e_x(c,d)}\,\text{GC2}}{P^o_x(b,d), P^o_{\overline{x}}(a,d)}\,\text{GC1}}{V_o(a,b)}\,\text{def} \qquad \dfrac{P^e_o(y,c), P^o_{\overline{x}}(y,d)}{\dfrac{\dfrac{P^o_o(a,c), P^e_x(c,d)}{P^o_x(b,d), P^o_{\overline{x}}(a,d)}\,\text{GC1}}{V_o(a,b)}\,\text{def} \qquad \dfrac{\dfrac{P^o_x(b,d) P^x_x(d,c)}{P^o_o(a,c), P^o_e(b,c)}\,\text{GC1}}{V_o(a,b)}\,\text{def}}\,\text{GB3}}\,\text{S3}$$

[12]is infinite neccessary here?

8 Infinite Case