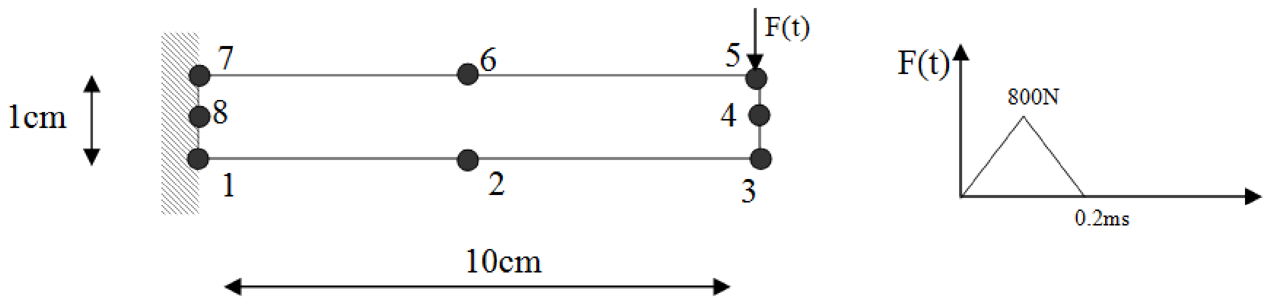

Figure 1: Five Node Isoparametric Element

Figure 1 shows the (8) node and (1) isoparametric element. The Global Stiffness Matrix can be fully assembled (Using E=70GPa, ν=0.25 and t=.01m) by combining all of the local stiffness matrices using:

$$[K]_e = \int_{-1}^{1}\int_{-1}^{1}[B]_r^T[D][B]_r|J|drds \tag{1}$$

Where,

$$[J] = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix}$$

$$X = [N][X\ coordinates]^T$$
$$Y = [N][Y\ coordinates]^T$$
$$N1 = .25(1-r)(1-s)(-r-s-1)$$
$$N2 = .5(1-s)(1+r)(1-r)$$
$$N3 = .25(1+r)(1-s)(r-s-1)$$
$$N4 = .5(1+r)(1+s)(1-s)$$
$$N5 = .25(1+r)(1+s)(r+s-1)$$
$$N6 = .5(1+s)(1+r)(1-r)$$
$$N7 = .25(1-r)(1+s)(-r+s-1)$$
$$N8 = .5(1-r)(1+s)(1-s)$$
$$X = [-5, 0, 5, 5, 5, 0, -5, -5] * .01$$
$$Y = [-.5, -.5, -.5, 0, .5, .5, .5, 0] * .01$$

$$[D] = \frac{Et}{1-\nu^2}\left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{array}\right]$$

$$[B]_r = \left[\begin{array}{ccccc} J^*_{11}\frac{\partial N_1}{\partial r} + J^*_{12}\frac{\partial N_1}{\partial s} & 0 & J^*_{11}\frac{\partial N_2}{\partial r} + J^*_{12}\frac{\partial N_2}{\partial s} & ... & 0 \\ 0 & J^*_{21}\frac{\partial N_1}{\partial r} + J^*_{22}\frac{\partial N_1}{\partial s} & 0 & ... & J^*_{21}\frac{\partial N_8}{\partial r} + J^*_{22}\frac{\partial N_8}{\partial s} \\ J^*_{21}\frac{\partial N_1}{\partial r} + J^*_{22}\frac{\partial N_1}{\partial s} & J^*_{11}\frac{\partial N_1}{\partial r} + J^*_{12}\frac{\partial N_1}{\partial s} & J^*_{21}\frac{\partial N_2}{\partial r} + J^*_{22}\frac{\partial N_2}{\partial s} & ... & J^*_{21}\frac{\partial N_8}{\partial r} + J^*_{22}\frac{\partial N_8}{\partial s} \end{array}\right]$$

With the Boundary Conditions:

$$U1 = V1 = U8 = V8 = 0$$

The Reduced Global Stiffness matrix becomes:

$$[K]_r = 1e10\left[\begin{array}{cccc} 0.1626 & 0 & ... & 0 \\ 0 & 0.4032 & ... & -0.3957 \\ 0.01203 & 0.01349 & ... & 0.005185 \\ 0.007259 & 0.0473 & ... & -0.05102 \\ 0 & -0.02074 & ... & 0.02074 \\ -0.02074 & 0 & ... & 0 \\ -0.022 & -0.005185 & ... & -0.01349 \\ -0.005185 & -0.05102 & ... & .0473 \\ -0.1427 & 0 & ... & 0 \\ 0 & -0.3957 & ... & 0.4032 \end{array}\right] N/m$$

Calculating the mass matrix:

$$[M]_e = \rho t \int_{-1}^{1}\int_{-1}^{1} [N]_r^T[N]|J|drds \tag{2}$$

$$[M]_r = \left[\begin{array}{ccccc} 0.004444 & 0 & -0.0008333 & ... & 0 \\ 0 & 0.004444 & 0 & ... & 0.002222 \\ -0.0008333 & 0 & 0.0008333 & ... & 0 \\ 0 & -0.0008333 & 0 & ... & -0.001111 \\ 0.002778 & 0 & -0.0008333 & ... & 0 \\ 0 & 0.002778 & 0 & ... & 0.002778 \\ -0.001111 & 0 & 0.0002778 & ... & 0 \\ 0 & -0.001111 & 0 & ... & -0.0008333 \\ 0.002222 & 0 & -0.001111 & ... & 0 \\ 0 & 0.002222 & 0 & ... & 0.004444 \end{array}\right] Kg$$

Solving the damping matrix:

$$[C]_e = \alpha[M] + \beta[K] \tag{3}$$

$$2\zeta_1\omega_1 = \alpha + \beta\omega_1^2$$

$$2\zeta_2\omega_2 = \alpha + \beta\omega_2^2$$

$$\zeta_1 = 0.01 \;\; \zeta_2 = 0.02$$

$$\omega_1 = 1057hz \;\; \omega_2 = 12935hz$$

$$[C]_r = 1e3 \begin{bmatrix} 0.773 & 0 & 0.0571 & ... & 0 \\ 0 & 1.92 & 0.0641 & ... & -1.89 \\ 0.0571 & 0.0641 & 0.789 & ... & 0.0246 \\ 0.0345 & 0.225 & -0.105 & ... & -0.243 \\ 3.11 \cdot 10^{-4} & -0.0986 & -1.19 & ... & 0.0986 \\ -0.0986 & 3.11 \cdot 10^{-4} & 0.0345 & ... & 3.11 \cdot 10^{-4} \\ -0.105 & -0.0246 & 0.421 & ... & -0.0641 \\ -0.0246 & -0.243 & 0.0037 & ... & 0.225 \\ -0.678 & 0 & -0.105 & ... & 0 \\ 0 & -1.89 & 0.0246 & ... & 1.92 \end{bmatrix} \frac{N-s}{m}$$

Using the Central Difference Method (with a time step = 1e-6) to solve:

$$(\frac{[M]}{\Delta t^2} + \frac{[C]}{2\Delta t})(u(t+\Delta t)) = F(t) - ([K] - \frac{2[M]}{\Delta t^2})u(t) - (\frac{[M]}{\Delta t^2} - \frac{[C]}{2\Delta t})(u(t-\Delta t)) \tag{4}$$

With the first two modes:

$$\phi = \begin{bmatrix} 0.7028 & 0.2452 \\ 3.683 & -8.086 \\ 1.316 & 3.855 \\ 13.91 & 11.04 \\ -1.33 \cdot 10^{-13} & -2.478 \cdot 10^{-12} \\ 13.89 & 10.98 \\ -1.316 & -3.855 \\ 13.91 & 11.04 \\ -0.7028 & -0.2452 \\ 3.683 & -8.086 \end{bmatrix}$$

The vertical displacement at node 5 was found to be:

$$\boxed{V \text{ at } 5 = 2.3mm}$$

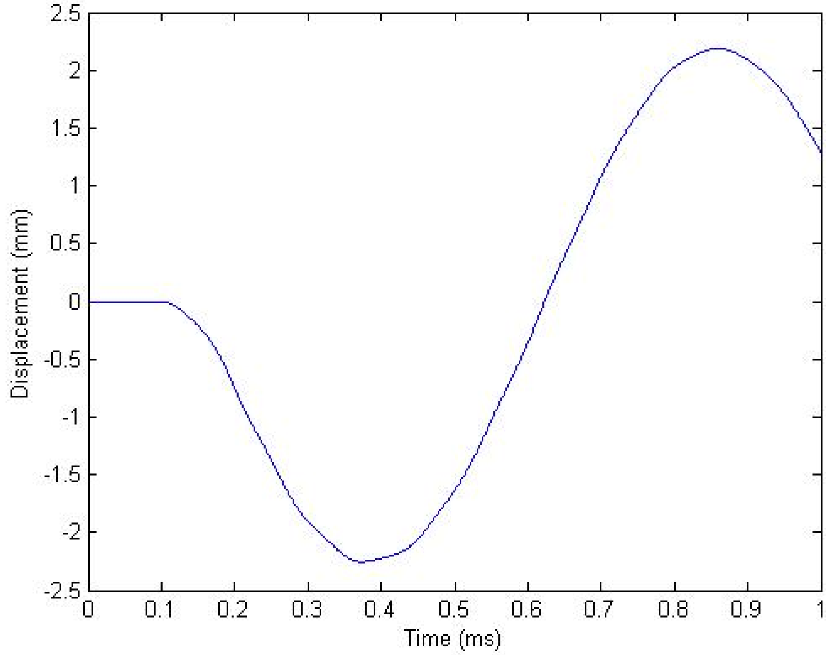

Figure 2: Displacement Time History (0 - 1ms)

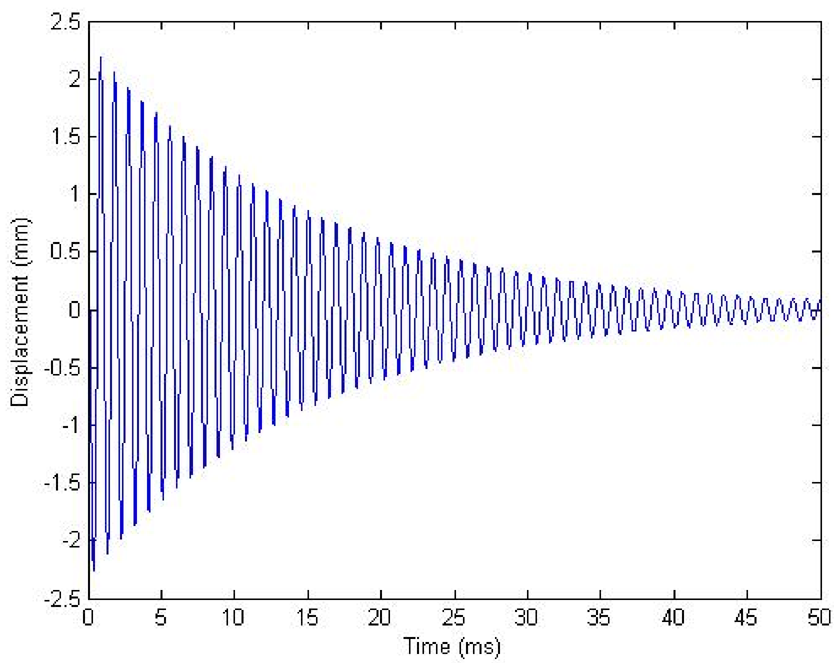

Figure 3: Displacement Time History (0 - 50ms)

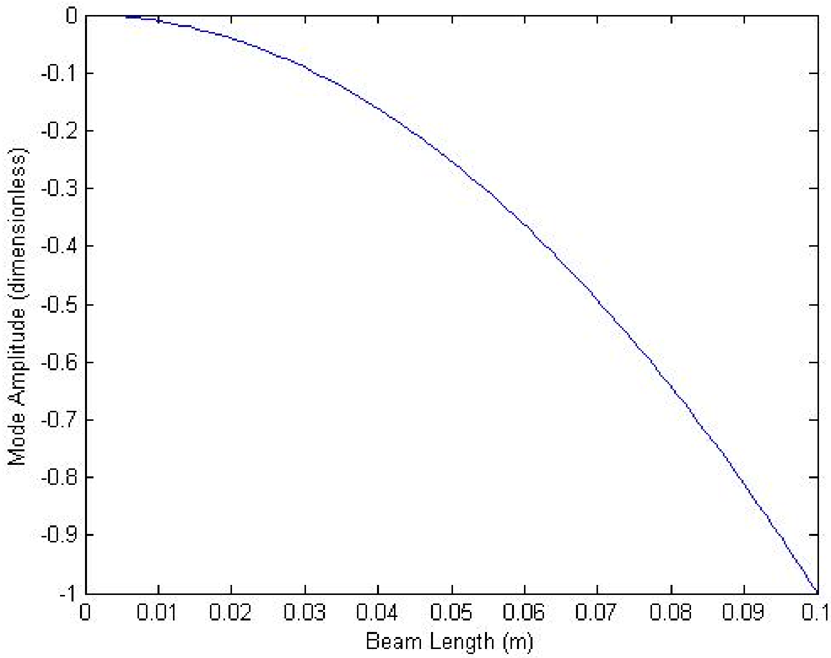

Figure 4: Modal shape for mode 1

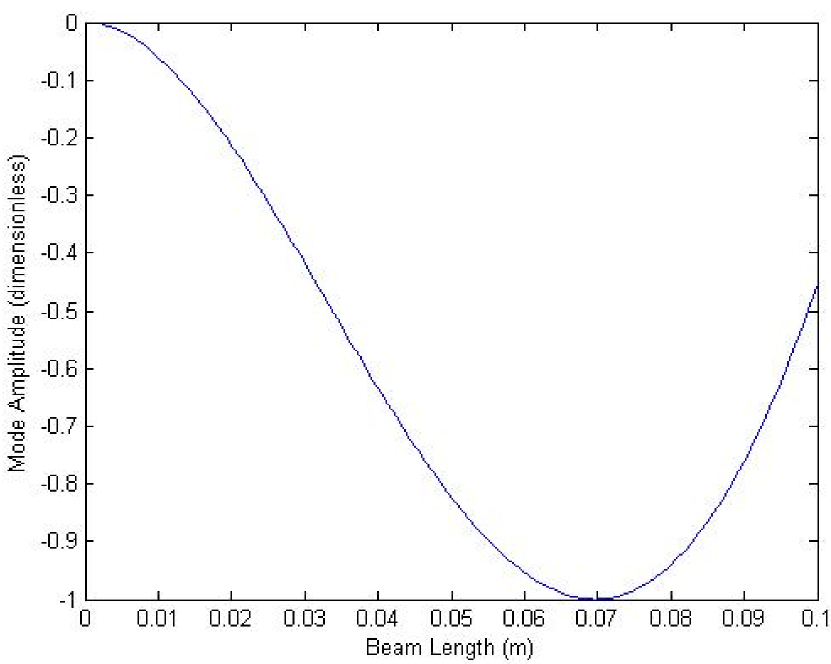

Figure 5: Modal shape for mode2