CSC418 Assignment 1

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1 Question 1

1.1 (a)

By page 115 in the textbook, the matrix to rotate a point an angle ϕ counter-clockwise is

$$\mathbf{T} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Thus, the 2D transformation matrix to rotate counterclockwise by 90 deg is

$$\mathbf{T} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

1.2 (b)

Given points $\vec{p_0}$ and $\vec{p_1}$, the line $\vec{l}$ between them can be expressed as

$$\vec{l}(t) = \vec{p_0} + (\vec{p_1} - \vec{p_0})t \tag{1}$$

Note that the normal to this line, $\vec{n}$, must have x and y components such that $\vec{n} \cdot \vec{l}(t) = 0$.

1.3 (c)

If we are given the implicit form of a line $f(x, y) = 0$, then we can determine whether or not two points $\vec{p_1}, \vec{p_2}$ are on the same side of a line by calculating $f(\vec{p_1})$ and $f(\vec{p_2})$. If both values are greater than 0, or both values are less than 0, then they are on the same side of the line.

1.4 (d)

The idea for the algorithm is as follows:

- to calculate and store each edge in its implicit form (i.e. as equations $f_1(x, y), f_2(x, y), f_3(x, y)$)
- for each edge expressed as $f_i(x, y)$ of the triangle, use the intersection point of the other two edges of the triangle to figure out when a point is on that side of this edge, whether $f_i(x, y)$ is greater than 0 or less than 0 and store this information
- for any given point (x, y), calculate $f_i(x, y)$ for all edges
- if any of the line equations has the point exactly equal to 0, then the point is on the edge of the triangle
- if for all $f_i(x, y)$, the value of $f_i(x, y)$ has the same sign as the value attained in the initialization, then the point is inside the triangle

- otherwise, the point is outside the triangle

It can be expressed with the following pseudocode:

Algorithm 1.1: ISPOINTINSIDETRIANGLE(x, y)

comment: assume we have $\vec{p_0} = p0 = (x0, y0)$, $\vec{p_1} = p1 = (x1, y1)$, $\vec{p_2} = p2 = (x2, y2)$

comment: initialize

$lines[0] \leftarrow (p0, p1)$
$lines[1] \leftarrow (p1, p2)$
$lines[2] \leftarrow (p2, p0)$
$lineSigns = []$

for $i \leftarrow 0$ **to** 2
do $\begin{cases} otherLines \leftarrow [] \\ \textbf{for } j \leftarrow 0 \textbf{ to } 2 \\ \quad \textbf{do } \begin{cases} \textbf{if } j \neq i \\ \quad \textbf{then } otherLines \leftarrow otherLines + lines[j] \end{cases} \\ p \leftarrow getIntersectionPoint(otherLines[0], otherLines[1]) \\ lineSigns[i] \leftarrow calculateFunctionValueAtPoint(lines[i], p) \end{cases}$

$insideTriangle = True$
for $i \leftarrow 0$ **to** 2
do $\begin{cases} value \leftarrow calculateFunctionValueAtPoint(lines[i], (x, y)) \\ \textbf{if } value == 0 \\ \quad \textbf{then return } (ON\_EDGE) \\ \quad \textbf{else if not } sameSign(value, lineSigns[i]) \\ \quad \textbf{then return } (OUTSIDE\_TRIANGLE) \end{cases}$
return $(INSIDE\_TRIANGLE)$

2 Question 2

Let $\vec{d} = \begin{pmatrix} d_x \\ d_y \end{pmatrix}$, $\vec{p_0} = \begin{pmatrix} p_{0x} p_{0y} \end{pmatrix}$. Then any point $\vec{p} = (x, y)$ on the line $\vec{p}(t)$ can be expressed as

$$x = p_{0x} + td_x \tag{2}$$

and

$$y = p_{0y} + td_y \tag{3}$$

As well, the equation for the ellipse is

$$\frac{(x - c_x)^2}{a^2} + \frac{(y - c_y)^2}{b^2} = 1 \tag{4}$$

So, in order to find the point(s) where the line $\vec{p}(t)$ intersects the ellipse, we must find the points that simultaneously satisfy equations (2), (3) and (4). We can substitute in equations (2) and (3) into (4) to get:

$$\frac{((p_{0x} + td_x) - c_x)^2}{a^2} + \frac{((p_{0y} + td_y) - c_y)^2}{b^2} = 1 \tag{5}$$

We can expand to get:

$$\begin{aligned} &\frac{p_{0x}^2 + 2p_{0x}td_x - 2p_{0x}c_x - 2td_xc_x + t^2d_x^2 + c_x^2}{a^2} + \\ &\frac{p_{0y}^2 + 2p_{0y}td_y - 2p_{0y}c_y - 2td_yc_y + t^2d_y^2 + c_y^2}{b^2} = 1 \end{aligned} \tag{6}$$

We can simplify and factor the equation to get:

$$\begin{aligned} (\frac{d_x^2}{a^2} + \frac{d_y^2}{b^2})t^2 + (\frac{2p_{0x}d_x - 2d_xc_x}{a^2} + \frac{2p_{0y}d_y - 2d_yc_y}{b^2})t + \\ (\frac{p_{0x}^2 - 2p_{0x}c_x + c_x^2}{a^2} + \frac{p_{0y}^2 - 2p_{0y}c_y + c_y^2}{b^2} - 1) = 0 \end{aligned} \tag{7}$$

Equation (7) is in the form of a quadratic function $At^2 + Bt + C$, so we can use the quadratic equation to solve for t. First, note the following:

$$A = \frac{d_x^2}{a^2} + \frac{d_y^2}{b^2} \tag{8}$$

$$B = \frac{2p_{0x}d_x - 2d_xc_x}{a^2} + \frac{2p_{0y}d_y - 2d_yc_y}{b^2} \tag{9}$$

$$C = \frac{p_{0x}^2 - 2p_{0x}c_x + c_x^2}{a^2} + \frac{p_{0y}^2 - 2p_{0y}c_y + c_y^2}{b^2} - 1 \tag{10}$$

Solving for the discriminant, $B^2 - 4AC$ will let us know how many points of intersection there are. If the discriminant is less than 0, there are no points of intersection. If the discriminant is equal to 0, there is exactly one point of intersection. Otherwise, the discriminant is greater than 0, and there are two points of intersection.

To find the values of t where the line intersects the ellipse, use the quadratic equation:

$$t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{11}$$

We can substitute those of t into the equations for x and y, equations (2) and (3) in order to get the coordinates at which the line intersects the ellipse. We can write the pseudocode for this as follows:

Algorithm 2.1: GETINTERSECTIONPOINTS(A, B, C)

comment: assume we have $\vec{d} = (dx, dy)$, $\vec{p_0} = (x0, y0)$ as fields

comment: A, B, and C are pre-calculated and handed to this function

$discriminant \leftarrow B * B - 4 * A * C$
if $discriminant < 0$
  then $\{$**return** ( )
  else if $discriminant == 0$
  then $\begin{cases} t \leftarrow -B/(2 * A) \\ x \leftarrow x0 + t * dx \\ y \leftarrow y0 + t * dy \\ \textbf{return } ((x, y)) \end{cases}$
  else $\begin{cases} t1 \leftarrow (-B + sqrt(discriminant))/2 * A \\ t2 \leftarrow (-B - sqrt(discriminant))/2 * A \\ x1 \leftarrow x0 + t1 * dx \\ y1 \leftarrow y0 + t1 * dy \\ x2 \leftarrow x0 + t2 * dx \\ y2 \leftarrow y0 + t2 * dy \\ \textbf{return } ((x1, y1), (x2, y2)) \end{cases}$

3 Question 3

The general idea to reflect a point about an arbitrary line $\vec{v}$ from point $\vec{p_0}$ to $\vec{p_1}$ is to:

- get the matrix T_1 which translates the line $\vec{v}$ to begin at the origin
- get the matrix T_2 which rotates the line $\vec{v}$ to be parallel to the x axis
- reflect the point about the x axis (let the matrix representing this transformation be T_0)
- apply the inverse of T_1, followed by the inverse of T_2 in order to get the point reflected about $\vec{v}$ (i.e. the transformed point is equal to $T_2^{-1} T_1^{-1} T_0 \vec{x}$, where $\vec{x}$ is the original point)

T_0 reflects a point about the x axis. It simply negates all y-coordinates, so $T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

T_1 is a translation matrix, and in order to move the line $\vec{v}$ to the origin we can subtract $\vec{p_0}$ from it. Thus $T_1 = \begin{pmatrix} 1 & 0 & -p_{0x} \\ 0 & 1 & -p_{0y} \\ 0 & 0 & 1 \end{pmatrix}$.

Finally, the line $\vec{v}$ creates an angle θ with the x axis. The angle is equal to:

$$\theta = \arctan \frac{p_{1y} - p_{0y}}{p_{1x} - p_{0x}} \tag{12}$$

To rotate the line counter-clockwise to be parallel to the x-axis, use the following matrix: $T_2 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, substituting in the value for θ from (12).

4 Question 4

We want to find the matrix $H = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ such that for each pair of points $\vec{x}_k, \vec{x}'_k$, $H\vec{x}_k = \vec{x}'_k$. We can set i to 1. We must then solve the following equations:
For every $k \in [1, 4]$, where $\vec{x}_k = (x_k, y_k)$,

$$\begin{aligned} &ax_k + by_k + c - x'_k(gx_k + hy_k + 1) = 0 \\ &dx_k + ey_k + f - y'_k(gx_k + hy_k + 1) = 0 \end{aligned} \tag{13}$$

We then get the following equations:
Using $\vec{x}_1, \vec{x}'_1$:

$$\begin{aligned} &(a)(1) + (b)(0) + c - (1)((g)(1) + (h)(0) + 1) = 0 \\ &a + c - g - 1 = 0 \\ &(d)(1) + (e)(0) + f - (0)(...) = 0 \\ &d + f = 0 \end{aligned} \tag{14}$$

Using $\vec{x}_2, \vec{x}'_2$:

$$\begin{aligned} &(a)(3) + (b)(0) + c - (2)((g)(3) + (h)(0) + 1) = 0 \\ &3a + c - 6g - 2 = 0 \\ &(d)(3) + (e)(0) + f - (1)((g)(3) + (h)(0) + 1) = 0 \\ &3d + f - 3g - 1 = 0 \end{aligned} \tag{15}$$

Using $\vec{x}_3, \vec{x}'_3$:

$$\begin{aligned} &(a)(2) + (b)(1) + c - (2)((g)(2) + (h)(1) + 1) = 0 \\ &2a + b + c - 4g - 2h - 1 = 0 \\ &(d)(2) + (e)(1) + f - (2)((g)(2) + (h)(1) + 1) = 0 \\ &2d + e + f - 4g - 2h - 1 = 0 \end{aligned} \tag{16}$$

Using $\vec{x}_4, \vec{x}'_4$:

$$\begin{aligned} &(a)(0) + (b)(1) + c - (0)(...) = 0 \\ &b + c = 0 \\ &(d)(0) + (e)(1) + f - (2)((0)(g) + (h)(2) + 1) = 0 \\ &e + f - 2g - 4h - 2 = 0 \end{aligned} \tag{17}$$

The above 8 equations can be expressed as an 8x8 matrix system, $A\vec{a} = \vec{b}$, where $A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 & -3 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 2 & 1 & 1 & -4 & -2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 4 \end{pmatrix}$, $\vec{a} = \begin{pmatrix} \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \\ \text{e} \\ \text{f} \\ \text{g} \\ \text{h} \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.

The solution to this linear system is $\vec{a} = \begin{pmatrix} 11/8 \\ 5/8 \\ -5/8 \\ 7/8 \\ 15/8 \\ -7/8 \\ 1/4 \\ 3/8 \end{pmatrix}$.

Therefore, the homography that maps points $\vec{x}_i$ to points $\vec{x}_i'$ for $i \in [1, 4]$ is $H = \begin{pmatrix} 11/8 & 5/8 & -5/8 \\ 7/8 & 15/8 & -7/8 \\ 1/4 & 3/8 & 1 \end{pmatrix}$.

This matrix does not represent an affine transformation because the bottom row does not contain two zeros followed by a 1 and instead has actual values in place of the zeros. Doing transforms using this homography will thus not preserve parallelism in lines.

5 Question 5

I will be using the fact that if two transformations T_1 and T_2 commute, then applying T_1 then T_2 to a vector $\vec{v}$ has the same effect as applying T_2 then T_1 to the vector $\vec{v}$. I.e., $T_1 T_2 \vec{v} = T_2 T_1 \vec{v}$. The solutions below will be using the vector $\vec{v} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$.

5.1 (a)

Let the translation matrix be $T = \begin{pmatrix} 0 & 0 & t_x \\ 0 & 0 & t_y \\ 0 & 0 & 1 \end{pmatrix}$, and the uniform scale matrix be $S = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then:

$$S\vec{v} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} sx \\ sy \\ 1 \end{pmatrix} \tag{18}$$

and

$$TS\vec{v} = \begin{pmatrix} 0 & 0 & t_x \\ 0 & 0 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} sx \\ sy \\ 1 \end{pmatrix} = \begin{pmatrix} sx + t_x \\ sy + t_y \\ 1 \end{pmatrix} \tag{19}$$

As well,

$$T\vec{v} = \begin{pmatrix} 0 & 0 & t_x \\ 0 & 0 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix} \tag{20}$$

and

$$ST\vec{v} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix} = \begin{pmatrix} s(x + t_x) \\ s(y + t_y) \\ 1 \end{pmatrix} = \begin{pmatrix} sx + st_x \\ sy + st_y \\ 1 \end{pmatrix} \tag{21}$$

As we can see, unless $t_x, t_y = 0$ or $s = 1$, the final transformed vectors in (21) and (19) are not equal. Therefore, the transformations of translation and uniform scaling do not commute.

5.2 (b)

Let the shear matrix be $H = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and the scale matrix be $S = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then:

$$H\vec{v} = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + sy \\ y \\ 1 \end{pmatrix} \tag{22}$$

and

$$SH\vec{v} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x + sy \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x(x + sy) \\ s_y y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x x + s_x sy \\ s_y y \\ 1 \end{pmatrix} \tag{23}$$

As well,

$$S\vec{v} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \\ 1 \end{pmatrix} \tag{24}$$

and

$$HS\vec{v} = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_x x \\ s_y y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x x + s_y s y \\ s_y y \\ 1 \end{pmatrix} \tag{25}$$

The final transformed vectors in (25) and (23) are not equal, except when $s_x = s_y$, i.e. when scaling the x and y components uniformly. Therefore, the transforms shear with respect to the x axis and scaling do not commute.

5.3 (c)

Let the rotation matrix for an angle ϕ be $R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and the scale matrix be $S = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then:

$$R\vec{v} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \\ 1 \end{pmatrix} \tag{26}$$

and

$$\begin{aligned} SR\vec{v} &= \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} s_x(x\cos\phi - y\sin\phi) \\ s_y(x\sin\phi + y\cos\phi) \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} s_x x\cos\phi - s_x y\sin\phi \\ s_y x\sin\phi + s_y y\cos\phi \\ 1 \end{pmatrix} \end{aligned} \tag{27}$$

As well,

$$S\vec{v} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \\ 1 \end{pmatrix} \tag{28}$$

and

$$
\begin{aligned}
RS\vec{v} &= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_x x \\ s_y y \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} s_x x(\cos\phi - \sin\phi) \\ s_y y(\sin\phi + \cos\phi) \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} s_x x\cos\phi - s_x x\sin\phi \\ s_y y\sin\phi + s_y y\cos\phi \\ 1 \end{pmatrix}
\end{aligned} \tag{29}
$$

The equations in (27) and (29) are not equal. Therefore, scaling and rotation, even with the same fixed points, do not commute.

6 Question 6

6.1 (a)

$$x(t) = at \tag{30}$$

By (30), $t = x/a$. We can substitute this value into the equation for y:

$$y(t) = -\frac{1}{2}gt^2 + bt + h = -\frac{1}{2}g(x/a)^2 + b(x/a) + h \tag{31}$$

We can use equation (31) to find the tangent to the curve, $\frac{dy}{dx}$:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d(-\frac{1}{2}g(x/a)^2 + b(x/a) + h)}{dx} \\
&= \frac{d(-\frac{gx^2}{2a^2} + \frac{bx}{a} + h)}{dx} \\
&= -\frac{gx}{a^2} + \frac{b}{a} = -\frac{gt}{a} + \frac{b}{a}
\end{aligned} \tag{32}
$$

We can rearrange (31) to get an implicit formula for the curve:

$$f(x,y) = -\frac{1}{2}g(x/a)^2 + b(x/a) + h - y = 0 \tag{33}$$

We can use this formula to get the normal to the curve, which is equal to the gradient:

$$\nabla f(x,y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (-\frac{gx}{a^2} + \frac{b}{a}, -1) = (-\frac{gt}{a} + \frac{b}{a}, -1) \tag{34}$$

6.2 (b)

In order to find the time of impact t_1, we must solve $y(t) = -\frac{1}{2}gt^2 + bt + h = 0$ for t. We can use the quadratic equation, $t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, to do this. Note that $A = -\frac{g}{2}$, $B = b$ and $C = h$.

$$\begin{aligned} &\frac{-b \pm \sqrt{b^2 - 4(-\frac{g}{2})(h)}}{2(-\frac{g}{2})} \\ &= \frac{-b \pm \sqrt{b^2 + 2gh}}{-g} \\ t_a &= \frac{b - \sqrt{b^2 + 2gh}}{g}, t_b = \frac{b + \sqrt{b^2 + 2gh}}{g} \end{aligned} \tag{35}$$

Since $b^2 + 2gh > b^2$ for positive $2gh$, t_b is the correct value for t_1, the time of impact. (Otherwise t_1 would be a negative number)

We can find the location of impact by solving for $x(t_1)$:

$$x(t_1) = at_1 \tag{36}$$

Thus the location of impact is $(at_1, 0)$. The velocity at impact is the tangent evaluated at the time of impact:

$$velocity(t_1) = -\frac{gt_1}{a} + \frac{b}{a} \tag{37}$$

Thus the velocity at the time of impact was $(-\frac{gt_1}{a} + \frac{b}{a}, 0)$.