Propositional Logic & Reasoning

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Knowledge Representation

- Expressing knowledge explicitly in a computer-tractable way
  - Knowledge Base: set of facts (or sentences)
  - These sentences are expressed in a (formal) language such as logic

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Why is it important?

- Core of AI
- Possibility of *automating* reasoning
- Reasoning: draw inferences from knowledge
  - answer queries
  - discover facts that follow from the knowledge base
  - decide what to do
  - etc.

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Logic in General

- Logics are formal languages for representing information such that conclusions can be drawn
- Syntax: Describes how to make sentences
- Semantics: How sentences relate to reality. The meaning of a sentence is not *intrinsic* to that sentence.
- Proof Theory: A set of rules for drawing conclusions (inferences, deductions).

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Why *formal* languages?

- Natural languages exhibit ambiguity.
  - Examples:
    - The boy saw a girl with a telescope
    - Our shoes are guaranteed to give you a fit
  - Ambiguity makes reasoning difficult / incomplete
- Formal languages promote rigour and thereby reduce possibility of human error.
- Formal languages help reduce implicit / unstated assumptions by removing *familiarity* with subject matter
- Formal languages help achieve generality due to possibility of finding *alternative interpretations* for sentences and arguments.

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Logical Arguments

- All humans have 2 eyes.
- Kishore is a human.
  - Therefore Kishore has 2 eyes.
- All humans have 4 eyes.
- Kishore is a human.
  - Therefore Kishore has 4 eyes.
- Both are (logically) valid arguments.
- Which statements are true / false ?

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Logical Arguments (contd)

- All humans have 2 eyes.
- Kishore has 2 eyes.
  - Therefore Kishore is a human.
- No human has 4 eyes.
- Kishore has 2 eyes.
  - Therefore Kishore is not human.
- Both are (logically) invalid arguments.
- Which statements are true / false ?

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From English to Propositional Formulae

- "it is not the case that the lectures are dull": $\neg D$ (alternatively "the lectures are not dull")
- "the lectures are dull and the text is readable": $D \wedge R$
- "either the lectures are dull or the text is readable": $D \vee R$
- "if the lectures are dull, then the text is not readable": $D \rightarrow R$
- "the lectures are dull if and only if (iif) the text is readable": $D \leftrightarrow R$
- "if the lectures are dull, then if the text is not readable, Kishore will not pass": $D \rightarrow (\neg R \rightarrow \neg P)$

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Propositional Logic

- Use letters to stand for "basic" propositions
- Complex sentences use operators for not, and, or, implies, iff.
- Brackets ( ) for grouping
  $(P \rightarrow (Q \rightarrow (\neg(R)))$ vs. $P \rightarrow (Q \rightarrow \neg R)$
- Omitting brackets
  - precedence from highest to lowest is: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
  - Binary operators are left associative (so $P \rightarrow Q \rightarrow R$ is $(P \rightarrow Q) \rightarrow R$)
- Questions:
  - Is $(P \vee Q) \vee R$ same as $P \vee (Q \vee R)$?
  - Is $(P \rightarrow Q) \rightarrow R$ same as $P \rightarrow (Q \rightarrow R)$?

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Semantics (Truth Tables)

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
True	True	False	True	True	True	True
True	False	False	False	True	False	False
False	True	True	False	True	True	False
False	False	True	False	False	True	True

- One row for each possible assignment of True/False to propositional variables
- **Important:** Above P and Q can be any sentence, including complex sentences

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Terminology

- A sentence is valid if it is True under all possible assignments of True/False to its propositional variables (e.g. $P \vee \neg P$).
- Valid sentences are also referred to as tautologies
- A sentence is satisfiable if and only if there is *some* assignment of True/False to its propositional variables for which the sentence is True
- A sentence is unsatisfiable if and only if it is not satisfiable (e.g. $P \wedge \neg P$).

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Semantics (Complex Sentences)

R	S	$\neg R$	$R \wedge S$	$\neg R \vee S$	$(R \wedge S) \rightarrow (\neg R \vee S)$
True	True	False	True	True	True
True	False	False	False	False	True
False	True	True	False	True	True
False	False	True	False	True	True

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Material Implication

- The only time $P \rightarrow Q$ evaluates to False is when P is True and Q is False
- If $P \rightarrow Q$ is True, then:
  - P is a sufficient condition for Q
  - Q is a necessary condition for P

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Exercises

Given: A and B are true; X and Y are false, determine truth values of:

- $\neg(A \vee X)$
- $A \vee (X \wedge Y)$
- $A \wedge (X \vee (B \wedge Y))$
- $[(A \wedge X) \vee \neg B] \wedge \neg[(A \wedge X) \vee \neg B]$
- $(P \wedge Q) \wedge (\neg A \vee X)$
- $[(X \wedge Y) \rightarrow \mathrm{A}] \rightarrow [X \rightarrow (Y \rightarrow \mathrm{A})]$

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Entailment

- $S \Rightarrow P$ — whenever all the formulae in the set S are True, P is True
- This is a *semantic* notion; it concerns the notion of *Truth*
- To determine if $S \Rightarrow P$ construct a truth table for S, P
  - $S \Rightarrow P$ if, in any row of the truth table where all formulae of S are true, P is also true
- A tautology is just the special case when S is the empty set (evaluates to False).

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Entailment Example

P	$P \rightarrow Q$	Q
True	True	True
True	False	False
False	True	True
False	True	False

Modus Ponens

Therefore, P, $P \rightarrow Q \Rightarrow Q$

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Exercises

Use truth tables to determine validity of:

- If it rains, Raju carries an umbrella. Raju is carrying an umbrella, therefore it will rain.
- If the weather is warm and the sky is clear, then either we go swimming or we go boating. It is not the case that if we do not go swimming, then the sky is not clear. Therefore, either the weather is warm or we go boating.

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Formal Proofs

- Intend to formally capture the notion of proof that is commonly applied in other fields (e.g. mathematics).
- A proof of a formula from a set of premises is a sequence of steps in which any step of the proof is:
  1. An axiom or premise
  2. A formula deduced from previous steps of the proof using some rule of inference
- The last step of the proof should deduce the formula we wish to prove.
- We say that S follows from (premises) P to denote that the set of formulae P "prove" the formula S.

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Soundness and Completeness

- A logic is sound if it preserves truth (i.e. if a set of premises are all true, any conclusion drawn from those premises *must* also be true).
- A logic is complete if it is capable of proving *any valid* consequence.
- A logic is decidable if there is a mechanical procedure (computer program) to prove *any* given consequence.

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Inference Rules

1. Modus Ponens: $P, P \rightarrow Q \Rightarrow Q$
2. Modus Tollens: $P \rightarrow Q, \neg Q \Rightarrow \neg P$
3. Hypothetical Syllogism: $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$
4. And-Elimination: $P_1 \wedge P_2 \wedge \dots \wedge P_n \Rightarrow P_i$
5. And-Introduction: $P_1, P_2, \dots, P_n \Rightarrow P_1 \wedge P_2 \wedge \dots \wedge P_n$
6. Or-Introduction: $P_i \Rightarrow P_1 \vee P_2 \vee \dots \vee P_n$
7. Double-Negation Elimination: $\neg\neg P \Rightarrow P$
8. Unit Resolution: $P \vee Q, \neg Q \Rightarrow P$
9. Resolution: $P \vee Q, \neg Q \vee R \Rightarrow P \vee R$

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Example Formal Proof

1. $A \vee (B \rightarrow D)$
2. $\neg C \rightarrow (D \rightarrow E)$
3. $A \rightarrow C$
4. $\neg C$ / $\therefore$ $B \rightarrow E$
5. $\neg A$ *3,4 (Modus Tollens)*
6. $B \rightarrow D$ *1,5 (Unit Resolution)*
7. $D \rightarrow E$ *2,4 (Modus Ponens)*
8. $B \rightarrow E$ *6,7 (Hypothetical Syllogism)*

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Exercises

Construct formal proof of validity for:

- If the investigation continues, then new evidence is brought to light. If new evidence is brought to light, then several leading citizens are implicated. If several leading citizens are implicated, then the newspapers stop publicizing the case. If continuation of the investigation implies that the newspapers stop publicizing the case, then the bringing to light of new evidence implies that the investigation continues. The investigation does not continue. Therefore, new evidence is not brought to light.
- *C*: The investigation continues. *N*: New evidence is brought to light. *I*: Several leading citizens are implicated. *S*: The newspapers stop publicizing the case.

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Solution

1. $C \rightarrow N$
2. $N \rightarrow I$
3. $I \rightarrow S$
4. $(C \rightarrow S) \rightarrow (N \rightarrow C)$
5. $\neg C$ / $\therefore$ $\neg N$
6. $C \rightarrow I$ *1,2 (Hypothetical Syllogism)*
7. $C \rightarrow S$ *6,3 (Hypothetical Syllogism)*
8. $N \rightarrow C$ *7,4 (Modus Ponens)*
9. $\neg N$ *8,5 (Modus Tollens)*

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Exercises (contd.)

- If I study, I make good grades. If I do not study, I enjoy myself. Therefore, either I make good grades or I enjoy myself.
- S, G, E

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Solution

1. $S \rightarrow G$
2. $\neg S \rightarrow E / \therefore G \vee E$
3. $\neg S \vee G$ 1
4. $\neg\neg S \vee E$ 2
5. $S \vee E$ 4 (DoubleNegationEliminate)
6. $G \vee E$ 3,5 (Resolution)

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Complete Proof Systems

- Truth Tables
- Inference Rules
  - 19 rules + Conditional Proof + Indirect Proof
  - *Method of Resolution*

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Resolution

- Better suited to computer implementation
- Generalizes to first-order logic
- The basis of Prolog's inference method
- To apply resolution, all formulae in the knowledge base and the query must be in clausal form

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Normal Forms

- A literal is a propositional letter or the negation of a propositional letter
- A clause is a disjunction of literals
- Conjunctive Normal Form (CNF) – a conjunction of clauses
  e.g. $(P \vee Q \vee \neg R) \wedge (\neg S \vee \neg R)$
- Disjunctive Normal Form (DNF) – a disjunction of conjunctions of literals
  e.g. $(P \wedge Q \wedge \neg R) \vee (\neg S \wedge \neg R)$
- Every propositional logic formula can be converted to CNF and DNF

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Conversion to CNF

- Eliminate $\leftrightarrow$ rewriting $P \leftrightarrow Q$ as $(P \rightarrow Q) \wedge (Q \rightarrow P)$
- Eliminate $\rightarrow$ rewriting $P \rightarrow Q$ as $\neg P \vee Q$
- Use *De Morgan's laws* to push $\neg$ inwards:
  - Rewrite $\neg(P \wedge Q)$ as $\neg P \vee \neg Q$
  - Rewrite $\neg(P \vee Q)$ as $\neg P \wedge \neg Q$
- Eliminate double negations: rewrite $\neg\neg P$ as P
- Use the *distributive laws* to get CNF:
  - Rewrite $(P \wedge Q) \vee R$ as $(P \vee R) \wedge (Q \vee R)$
  - Rewrite $(P \vee Q) \wedge R$ as $(P \wedge R) \vee (Q \wedge R)$
- Exercise: Convert $\neg(P \rightarrow (Q \wedge R))$ to CNF

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Solution

- $\neg(P \rightarrow (Q \wedge R))$
- $\neg(\neg P \vee (Q \wedge R))$
- $\neg\neg P \wedge \neg(Q \wedge R)$
- $\neg\neg P \wedge (\neg Q \vee \neg R)$
- $P \wedge (\neg Q \vee \neg R)$
- Two clauses: P, $\neg Q \vee \neg R$

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Resolution Rule of Inference

$$\frac{A \vee B \qquad \neg B \vee C}{A \vee C}$$

- where *B* is a propositional letter and *A* and *C* are clauses (possibly empty).
- $A \vee C$ is the resolvent of the two clauses.

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Applying Resolution

- How can we use the resolution rule? One way:
  - Convert knowledge base into clausal form
  - Repeatedly apply resolution rule to the resulting clauses
  - A query *A* follows from the knowledge base if and only if each of the clauses in the CNF of *A* can be derived using resolution
- There is a better way . . .

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Refutation Systems

- To show that *P* follows from *S* (i.e. $S \Rightarrow P$) using refutation, start with *S* and $\neg P$ in clausal form and derive a contradiction using resolution.
- The "empty clause □" (a clause with no literals) is unsatisfiable (always False) – a *contradiction*.
- So if the empty clause is derived using resolution, the original set of clauses is unsatisfiable.
- That is, if we can derive □ from the clausal forms of *S* and $\neg P$, these clauses can never be all True together.
- Hence whenever the clauses of *S* are all True, at least one clause from $\neg P$ must be False, i.e. $\neg P$ must be False and *P* must be True.
- Hence, by definition, $S \Rightarrow P$

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Applying Resolution Refutation

- Negate query to be proven.
- Convert knowledge base and negated conclusion into CNF and extract clauses.
- Repeatedly apply resolution until either the empty clause (contradiction) is derived or no more clauses can be derived.
- If the empty clause is derived, answer 'yes' (query follows from knowledge base), otherwise answer 'no' (query does not follow from knowledge base)

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Resolution: Example 1

$(G \vee H) \rightarrow (\neg J \wedge \neg K), G \Rightarrow \neg J$

Clausal form of $(G \vee H) \rightarrow (\neg J \wedge \neg K)$ is

$\{\neg G \vee \neg J, \neg H \vee \neg J, \neg G \vee \neg K, \neg H \vee \neg K\}$

1. $\neg G \vee \neg J$ [Premise]
2. $\neg H \vee \neg J$ [Premise]
3. $\neg G \vee \neg K$ [Premise]
4. $\neg H \vee \neg K$ [Premise]
5. G [Premise]
6. J [¬Conclusion]
7. $\neg G$ [1,6. Resolution]
8. □ [5,7. Resolution]

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Problems

- $P \rightarrow \neg Q, \neg Q \rightarrow R \Rightarrow P \rightarrow R$
- $\Rightarrow ((P \vee Q) \wedge \neg P) \rightarrow Q$

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Soundness and Completeness

- Resolution refutation is sound, i.e. it preserves truth (if a set of premises are all true, any conclusion drawn from those premises **will** also be true**).**
- Resolution refutation is complete, i.e. it is capable of proving all *valid* consequences of any knowledge base.
- Resolution refutation is decidable, i.e. there is an algorithm implementing resolution, which when asked whether $P \Rightarrow S$, can always answer ‘yes’ or ‘no’ (correctly).

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Heuristics in Applying Resolution

- Clause elimination — can disregard certain types of clauses
  - Pure clauses: contain literal L where $\neg L$ doesn’t appear elsewhere
  - Tautologies: clauses containing both L and $\neg L$
  - Subsumed clauses: another clause exists containing a subset of the literals
- Ordering strategies
  - Resolve unit clauses (only one literal) first
  - Start with query clauses
  - Aim to shorten clauses

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Conclusion

- We have now investigated one knowledge representation and reasoning formalism
- This means we can draw new conclusions from the knowledge we have: we can reason
- Have enough to build a knowledge-based agent
- However, propositional logic is a weak language; there are many things that cannot be expressed
- To express knowledge about objects, their properties and the relationships that exist between objects, we need a more expressive language: first-order logic

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