

1 Counting (5 points)

How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

2 Counting (5 points)

Every student in a discrete mathematics course is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?

3 Counting (5 points)

A bowl contains 10 red balls and 10 blue balls. A woman removes balls at random without looking at them.

- a. How many balls must she remove to be sure of having at least three balls of the same color?
- b. How many balls must she remove to be sure of having at least three blue balls?

4 Counting (5 points)

Show that if there are 101 people of different heights standing in a line, it is possible to find 11 (not necessarily consecutive) people in the order they are standing in the line with heights that are either increasing or decreasing.

5 Counting (5 points)

Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

6 Counting (5 points)

Thirteen people on a softball team show up for a game.

- a. How many ways are there to choose 10 players to take the field?

- b. How many ways are there to assign the 10 positions by selecting players from the 13 who show up?
- c. Of the 13 people who show up, 3 are women. How many ways are there to choose 10 players if at least one of these players must be a woman?

7 Counting (5 points)

How many ways are there to seat six people around a circular table, where seatings are considered to be identical if they can be obtained from each other by rotating the table?

8 Counting (5 points)

Give a formula for the coefficient of x^k in the expansion of $(x + \frac{1}{x})^{100}$, where k is an integer.

9 Counting (5 points)

How many ways are there for a horse race with four horses to finish if ties are possible? (Note that since ties are allowed, any number of the four horses may tie.)

10 Counting (5 points)

In bridge, the 52 cards of a standard deck are dealt to four players. How many different ways are there to deal bridge hands to four players.?

11 Counting (5 points)

A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen?

Hint: Represent the books that are chosen by bars and the books that are not chosen by stars. Count the number of sequences of five bars and seven stars so that no two bars are adjacent.

12 Probability (5 points)

What is the probability that a five-card poker hand contains the ace of hearts?

13 Probability (5 points)

To play the Pennsylvania superlottery, a player selects 7 numbers out of the first 80 positive integers. What is the probability that a person wins the grand prize by picking 7 numbers that are among the 11 numbers selected by the Pennsylvania lottery commission?

14 Probability (5 points)

Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, who knows what is behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?

15 Probability (5 points)

Show that if E and F are independent events, then \bar{E} and \bar{F} are independent events.

16 Probability (5 points)

What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

17 Probability (5 points)

Let E be the event that a randomly generated bit string of length three contains an odd number of 1's, and let F be the event that the string starts with a 1. Are E and F independent? Explain.

18 Probability (5 points)

In a certain country, each time a baby is born, it is equally likely to be a boy or a girl, independent of any other births. The country's queen would like to see more girls, so she decrees that all couples producing children must continue to have babies until they have a girl, and then produce no more. What is the expected fraction of baby girls in the country?

19 Probability (5 points)

What is the expected value when a \$1 lottery ticket is bought in which the purchaser wins exactly \$10 million if the ticket contains the six winning numbers chosen from the set $\{1, 2, 3, \dots, 50\}$ and the purchaser wins nothing otherwise?

20 Probability (5 points)

Provide an example that shows that the variance of the sum of two random variables is not necessarily equal to the sum of their variances when the random variables are not independent.