Math 465: Introduction to Combinatorics

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These slides will be posted on Canvas.

Course management

I am teaching Math 465 section 3 this Fall
Location: East Hall B737
Time: Tuesday and Thursday, 2:30-3:50 PM.

The best way to communicate with me is by email.

Most of the course management will occur on Canvas.

Office hours will be conducted in my office, EH 3827:

- Tuesday/Thursdasy 3:50–5:20 PM

Grading

The grade will be based on:

- homework (45%),
- quizzes (15%), and
- two 1.5-hour exams (20% each).

This course will not be graded on a curve, i.e., there are not a set percentage of each grade to be given out.

Final grade cutoffs:

- total score of 90% guarantees the final grade of A or higher;
- total score of 80% guarantees the final grade of B or higher;
- total score of 70% guarantees the final grade of C or higher;
- total score of 60% guarantees the final grade of D.

Exams and quizzes

Exams (20% each)

There will be two exams, each covering one half of the course.
These exams will not be cumulative.

Quizzes (15% total)

There will be three quizzes, administered via Canvas:

- at the end of next week, on Friday-Saturday, January 17–18;
- at the end of the week after, on Friday-Saturday, January 24–25;
- in the first week after spring break.

The goal of the first two quizzes is to help you decide whether this course is a good match for you.

Homework

There will be approximately 11 problem sets, posted on Canvas.

Homework #1 will be released on Tuesday, Jan 14. It will be due on Jan 20 at 11:59pm.

Homework answers have to be justified. No justification—no credit.

Collaboration in small groups (up to four people total in a group) is allowed—but each case of collaboration has to be explicitly acknowledged on every homework submission.

Homework submission and grading

Homework submission and grading will be done via Gradescope.

On each problem set, only 5 problems will be graded.

Late homework will not be accepted.

The lowest homework score will be dropped in the final calculation.

Gradescope has a built-in mechanism for regrade requests.

Any questions about the administrative aspects of the course?

Content overview

This course introduces the fundamental notions, techniques, and theorems of enumerative combinatorics and graph theory.

Level: undergraduate. Next level courses: Math 565, Math 566.

Recommended texts (neither is required):

- M. Bóna, *A walk through combinatorics*, 4^{th} ed., World Scientific, 2016-17.
- S. Shahriari, *Invitation to Combinatorics*, Cambridge University Press, 2022.

We will roughly cover Part II (Enumerative Combinatorics) and Part III (Graph Theory) of Bóna's textbook, plus some other topics.

We will not follow either textbook too closely. The lectures and the books will often provide slightly different approaches.

Proofs

This is a proof-based course. Students are expected to understand rigorous mathematical proofs, and perhaps even more importantly, supply their own proofs when asked.

Some specific prerequisites:

- proofs by induction,
- proofs by contradiction,
- basic set theory terminology (intersection, union, etc.),
- basic function terminology (one-to-one, onto, composition, etc.).

If you are not familiar with Mathematical Induction, read either [Bóna, Chapter 2] or [Shahriari, Sections 1.1–1.2].

Problem-solving challenges

In past years, many students ended up dropping out of Math 465.

Why? Much of the content is quite elementary.

However, many problems are "one of a kind," with no off-the-shelf algorithmic solution.

The first two quizzes will give you some idea of where you stand.

How do I prepare for quizzes/exams/homework?

Use problems in the textbooks to practice!

Also, come to office hours.

Questions?

Basic counting principles [Chapter 3]

First half of the course: enumerative combinatorics.

We will study the basic techniques of counting, or enumeration.

Two versions of counting

(1) Counting concrete objects;

(2) Counting choices/possibilities.

Examples

(1) counting beer bottles in a party store;

(2) buying plane tickets for a given itinerary.

The golden rule

The golden rule of counting

Count everything exactly once.

That is: count everything, and don't count anything more than once.

As we develop more sophisticated methods of counting, it will be important to make mental checks that everything is accounted for, and nothing is overcounted.

The Addition Principle (=Summation Rule)

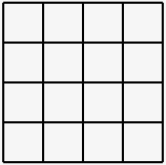
Two incarnations of the Addition Principle:

(1) splitting a set into subsets;

(2) splitting choices into cases.

Problem

How many squares can be found in this picture?



Solution

$16 + 9 + 4 + 1 = \mathbf{30}$.

(We shall later solve this for the $n \times n$ grid.)

Unordered set partitions

Problem

In how many ways can a 4-element set be partitioned into nonempty subsets (=blocks)?

Example: distributing students among Zoom breakout rooms.

Solution

There are 4 cases, depending on the number of blocks:

- 1 block: $abcd$
- 2 blocks: $abc|d \quad abd|c \quad acd|b \quad bcd|a \quad ab|cd \quad ac|bd \quad ad|bc$
- 3 blocks: $ab|c|d \quad ac|b|d \quad ad|b|c \quad bc|a|d \quad bd|a|c \quad cd|a|b$
- 4 blocks: $a|b|c|d$

Total: $1+7+6+1=15$.

Exercise: Solve the analogous problem for a 5-element set.

The Multiplication Principle (=Product Rule)

Problem

Jimmy John's **Slim Roast Beef Sandwich** can be ordered:

- on French bread, wheat bread, or without bread;
- with regular, extra, or small amount of roast beef;
- with regular, extra, small, or **no** amount of avocado spread;
- with regular, extra, small, or **no** amount of provolone cheese;
- with, without, or with extra Dijon mustard;
- with or without tomato.

How many ways are there to order the *Slim Roast Beef Sandwich*?

Solution

$3 \cdot 3 \cdot 4 \cdot 4 \cdot 3 \cdot 2 = 864.$

The Multiplication Principle, continued

Problem

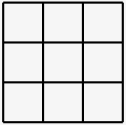
How many divisors does the number 600 have?

Solution

Since $600 = 2^3 \cdot 3 \cdot 5^2$, the answer is $4 \cdot 2 \cdot 3 = 24$.

Problem

How many rectangles are there in this picture?



Solution

$(3 + 2 + 1) \times (3 + 2 + 1) = \mathbf{36}$.

Counting permutations

Theorem

An n-element set has $n!$ permutations.

$\boxed{n=3}$	123	132	213	231	312	321

Proof #1: selecting the entries, left to right

n choices for the 1^{st} entry, then $n-1$ choices for the 2^{nd} entry, etc.

Proof #2: inserting the entries, in increasing order

$n=3$	123	132	213	231	312	321
$n=4$	1234	1324	2134	2314	3124	3214
	1243	1342	2143	2341	3142	3241
	1423	1432	2413	2431	3412	3421
	4123	4132	4213	4231	4312	4321

Permutations of subsets

Problem

Given 34 people, how many ways are there for at most 4 of them to wait in line?

Solution

Let us divide the set of possibilities into *cases*:

- The line has 0 people. There is 1 way this can happen.
- The line has 1 person. There are 34 ways this can happen.
- The line has 2 people. There are $34 \cdot 33 = 1122$ ways.
- The line has 3 people. There are $34 \cdot 33 \cdot 32 = 35904$ ways.
- The line has 4 people. There are $34 \cdot 33 \cdot 32 \cdot 31 = 1113024$ ways.

Hence the total number of ways is

$$1 + 34 + 1122 + 35904 + 1113024 = \mathbf{1150085}.$$

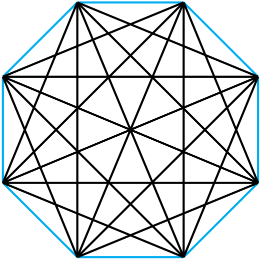
Note that we used both Addition and Multiplication Principles.

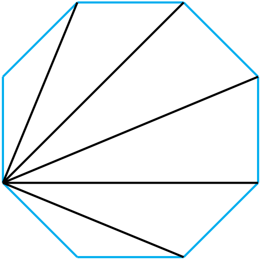
The Division Principle

The Division Principle is used in situations where each object of interest is counted several times (a fixed number).

Problem

How many diagonals are there in a convex n-gon?





Solution

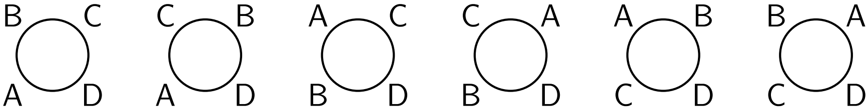
For each of the n vertices, there are $n-3$ diagonals connecting to it. Each diagonal is counted twice. **Answer**: $n(n-3)/2$.

Cyclic permutations

Problem

How many ways are there to arrange n distinct objects in a circle, up to rotation?

Example: seating a party of n people around a round table.



Answer: $(n-1)!$.

Exercise

Given n beads of different colors, how many circular necklaces are there that use all of these beads?

The Subtraction Principle

The Subtraction Principle is used in situations where it is more practical to count the objects of interest together with some other objects, and then subtract the number of those extra objects.

Problem

How many ways are there to place 5 people named

Amy, Bernie, Cory, Donald, and Elizabeth

on the stage so that Amy and Bernie are <u>not</u> adjacent to each other?

Solution

$5! - 4! \cdot 2 = 72.$

Counting words

An alphabet is a finite set of symbols (letters).

A word in an alphabet is a string (i.e., finite sequence) of letters.

Theorem

The number of k-letter words in an n-letter alphabet is n^k.

$k = 2$, $n = 3$	AA	AB	AC	BA	BB	BC	CA	CB	CC
$k = 3$, $n = 2$	AAA	AAB	ABA	ABB	BAA	BAB	BBA	BBB	

Subsets and binary strings

Theorem

A k-element set has 2^k subsets.

$$\boxed{k=3} \quad \varnothing \quad \{A\} \quad \{B\} \quad \{C\} \quad \{A,B\} \quad \{A,C\} \quad \{B,C\} \quad \{A,B,C\}$$

Proof

The subsets of a k-element set are encoded by binary strings of length k, i.e., by k-letter words in a 2-letter alphabet:

$\varnothing$	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
000	100	010	001	110	101	011	111

The number of such words is 2^k.

Boolean functions

Problem

How many Boolean functions in k variables are there?

Solution

Each of the k input bits can take value 0 or 1. There are 2^k possible input binary strings. A Boolean function can send each of them to either 0 or 1. So the total number of possibilities is 2^{2^k}.

Exercise

Count microchips with k binary inputs and ℓ binary outputs.
Two microchips are considered identical if they compute the same function $\{\text{binary strings of length } k\} \to \{\text{binary strings of length } \ell\}$.

For $\ell = 1$, you should recover the previous answer.