Different Math Spaces for Machine Learning

1 Introduction

2 Definition of Different Mathematics Spaces

2.1 Vector Space

Definition A vector space is a non-empty set V equipped with two operations: *Vector Addition* "+" and *Scalar Multiplication* "$\cdot$" - which satisfy the two closure axioms as well as the eight vector space axioms :

- (**Closure under Vector Addition**) Given $\alpha, \beta \in$ V, $\alpha + \beta \in$ V.
- (**Closure under scalar multiplication**) Given $\alpha \in$ V and a scalar k, k$\alpha \in$ V.

For α, β, γ arbitrary vectors in V, and k,l arbitrary scalars in R,

- (**Commutativity**) $\alpha + \beta = \beta + \alpha$
- (**Associativity of vector addition**) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$
- (**Additive Identity**) For all α, $\mathbf{0} + \alpha = \alpha$
- (**Existence of additive inverse**) For any α, there exists a β such that $\alpha + \beta = 0$
- (**Scalar multiplication identity**) $1\alpha = \alpha$
- (**Associativity of scalar multiplication**) $k(l\alpha) = (kl)\alpha$
- (**Distributivity of scalar sums**) $(k + l)\alpha = k\alpha + l\alpha$

- (**Distributivity of vector sums**) $k(\alpha + \beta) = k\alpha + k\beta$

2.2 Normed Vector Space

Definition of Norm Let X be a vector space over K. A **norm** on X is a map$\|\cdot\| : X \to [0, \infty)$that satisfies the following three properties:

- (**Nonnegative**) For every vector x, $\|x\| \geq 0$
- (**Positive Definiteness**) For every vetor x, $\|x\|$ =0 if and only if x=0
- (**Positive Homogeneity**) For every vector x, and every scalar α, $\|\alpha x\| = |\alpha| \, \|x\|$
- (**Triangle Inequality**) For every vectors x and y, $\|x + y\| \leq \|x\| + \|y\|$

A **Normed Vector Space**is a pair $(X, \|\cdot\|)$, where X is a vector space and $\|\cdot\|$ is a norm on X. A **Banach Space** is a *complete* normed space $(X, \| \cdot \|)$.

2.3 Inner Product Space

Definition of Inner Product An **inner product** on V is a mapping that takes each ordered pair (u,v) of elements of V to a number $\langle u, v \rangle \in F$ $(\langle \cdot, \cdot \rangle : V \times V \to F)$and has the following properties:

- (**Positivity**) $\langle v, v \rangle \geq 0$ for all $v \in V$
- (**Definiteness**) $\langle v, v \rangle = 0$ if and only if v=0
- (**Homogeneity in the 1st argument**) $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ For all $\lambda \in F$ and all u,v $\in V$
- (**Additivity in the 1st argument**) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ For every u,v,w $\in$ V
- (**Conjugate Symmetry**) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ For every u,v $\in$ V

An **Inner Product Space** is a vector space V along with an inner product on V. A **Hilbert Space** is an inner product space that is complete with respect to the norm defined by the inner product

2.4 Topological Space

Definition of topology A **topology** on a nonempty set X is a collection of subsets of X, called *open set*,such that:

- the empty set $\emptyset$ and the set X are open
- the union of an arbitraty collection of open sets is open
- the intersection of a finite number of open sets is open

A subset A of X is a *closed set* if and only if its complement, $A^c = X\backslash A$, is open. More formally, a collection T of subsets of X is a topology on X if:

- $\emptyset, X \in T$
- if $G_\alpha \in T$ for $\alpha \in A$, then $\bigcup_{\alpha\in A} G_\alpha \in T$
- if $G_i \in T$ for i=1 to n, then $\bigcap_{i\in A} G_i \in T$

We call the pair (X,T) a **Topologically Space**;if T is clear from the context,then we often refer to X as a Topological Space.