Method of Solving the Equation $xy + ax + by = c$.

$$\begin{aligned} xy + ax + by &= c \\ xy + ax + by + ab &= ab + c \\ x(y+a) + b(y+a) &= ab + c \\ (x+b)(y+a) &= ab + c \qquad (1) \end{aligned}$$

There are two possible cases:

Case 1:

If $ab + c = 0$, then $x + b = 0$ or $y + a = 0$.

If $x + b = 0$, then $x = -b, b \in \mathbb{Z}$ and y is an arbitrary integer.

If $y + a = 0$, then $y = -a, a \in \mathbb{Z}$ and x is an arbitrary integer.

Example: Solve $xy + x + y = -1$.

Solution: Here $a = 1, b = 1, c = -1$. Then $ab + c = 0$.

So the possible solutions are $(x, y) = (-1, y), y \in \mathbb{Z}$ and $(x, y) = (x, -1), x \in \mathbb{Z}$.

Case 2:

If $ab + c \neq 0$, then let $d_1, \cdots, d_n$ be the divisors of $ab + c$.

So $x + b = d_i$ for some $i \in \{1,2, \cdots, n\}$.

Thus (1) becomes

$$d_i(y + a) = ab + c$$

$$y = \frac{ab + c}{d_i} - a$$

In this case, the solutions are given as $(x, y) = \left(d_i - b, \frac{ab+c}{d_i} - a\right), i = 1,2, \cdots, n$.

Example: Solve $xy + 2x + 3y = 1$.

Solution: Here $a = 2, b = 3, c = 1$. Then $ab + c = 7 \neq 0$.

So let us collect all the divisors of $ab + c = 7$.

Take $d_1 = 1, d_2 = -1, d_3 = 7, d_4 = -7$.

d_i	$x = d_i - b$ $(x = d_i - 3)$	$y = \frac{ab + c}{d_i} - a$ $\left(y = \frac{7}{d_i} - 2\right)$
$d_1 = 1$	-2	5

$d_2 = -1$	-4	-9
$d_3 = 7$	4	-1
$d_4 = -7$	-10	-3

Thus the required solutions (x, y) are $(-2,5), (-4,-9), (4,-1), (-10,-3)$.