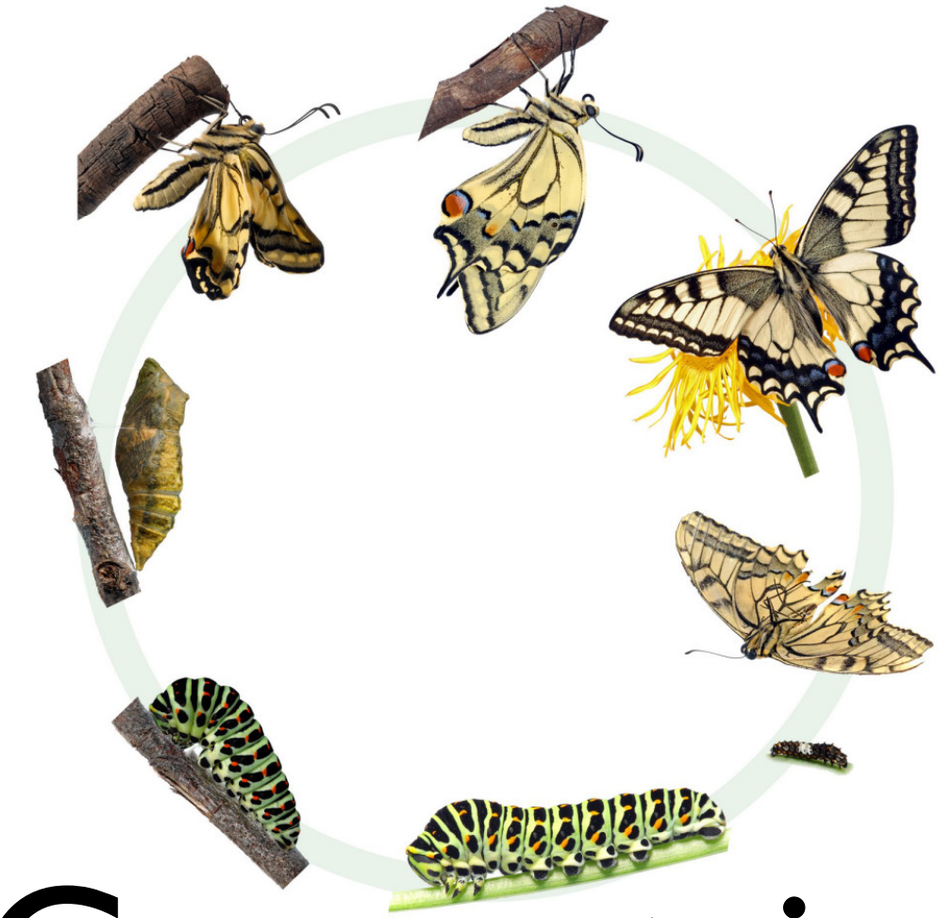


Geometric Transformations

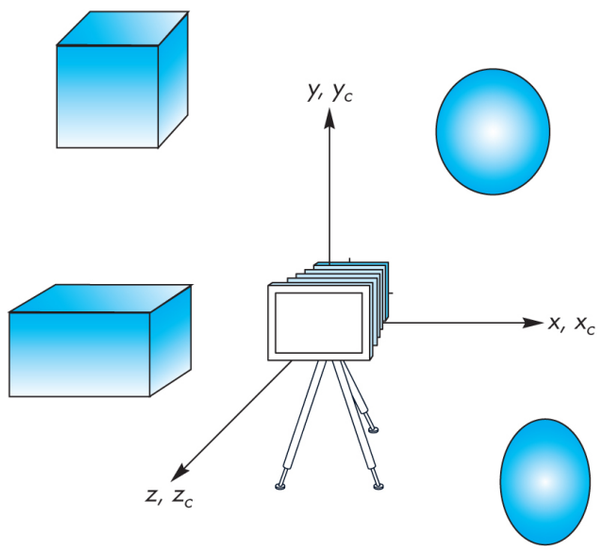
Introduction to Computer Graphics
CSE 533/333

Ojaswa Sharma

Transformation Matrices

- Fundamental to computer graphics
  - Object modelling
  - Viewing
  - Projecting

Source: http://pcwieczkowski.pl

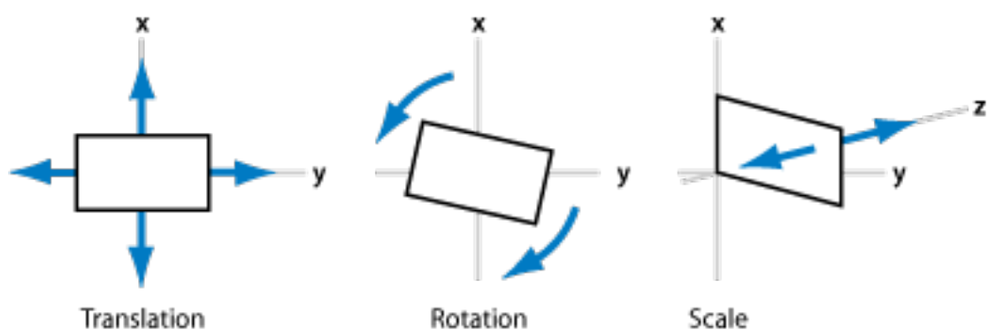


Source: http://en.wikipedia.org/wiki/Anamorphosis

Using Transformations

Object

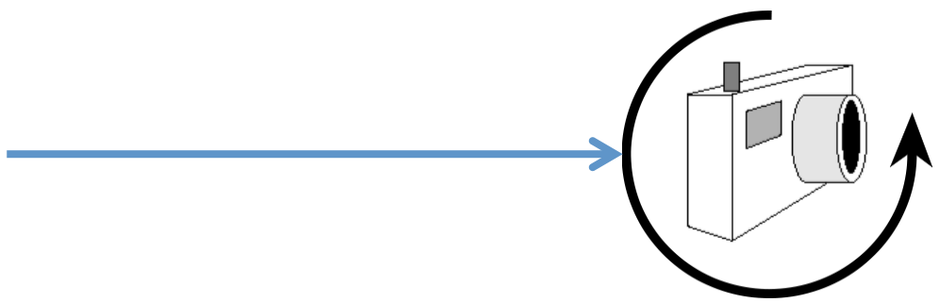
- Objects in a scene are collection of points
- Objects have location, orientation and size
- Correspond to transformations: Translation (***T***), Rotation (***R***), and Scaling (***S***)



Using Transformations

Camera

- Scene has a camera/view-point/eye that views the scene
- The camera has a location and orientation
- Correspond to transformations: Translation (***T***), Rotation (***R***)



Linear Transformations

- Can be represented as invertible (non-singular) matrices
- In 2D, can be represented by 2x2 matrices:

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- The basic transform looks like:

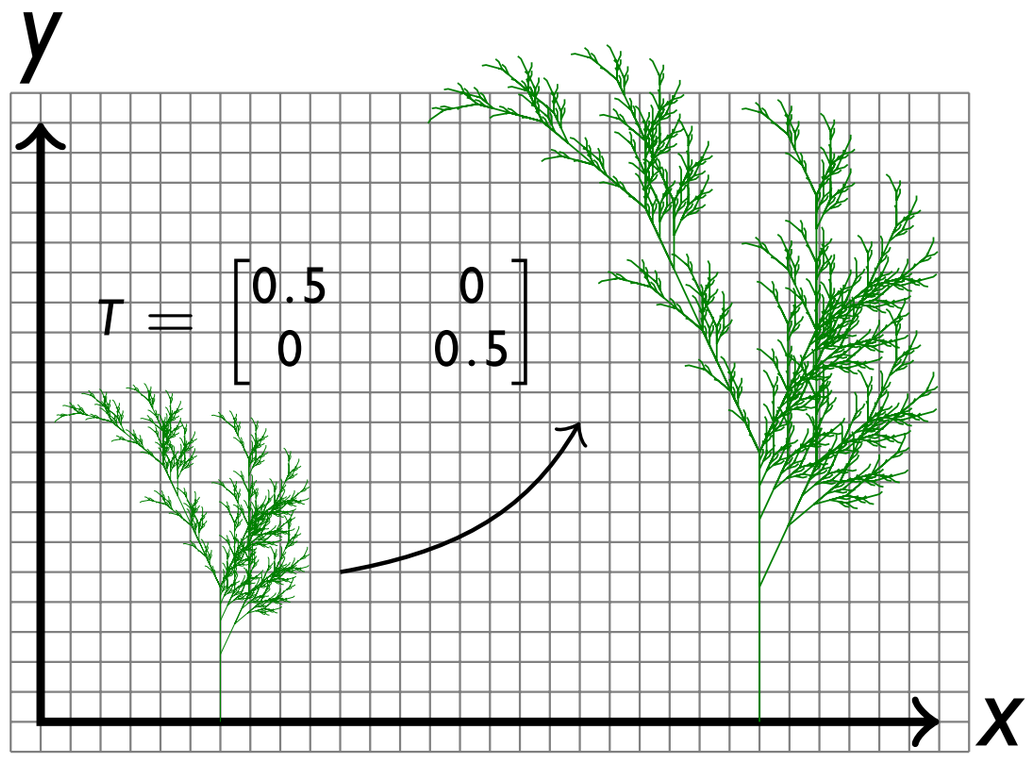
$$Tp = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

2D Transformations

Scaling (2D)

- Non-uniform scaling

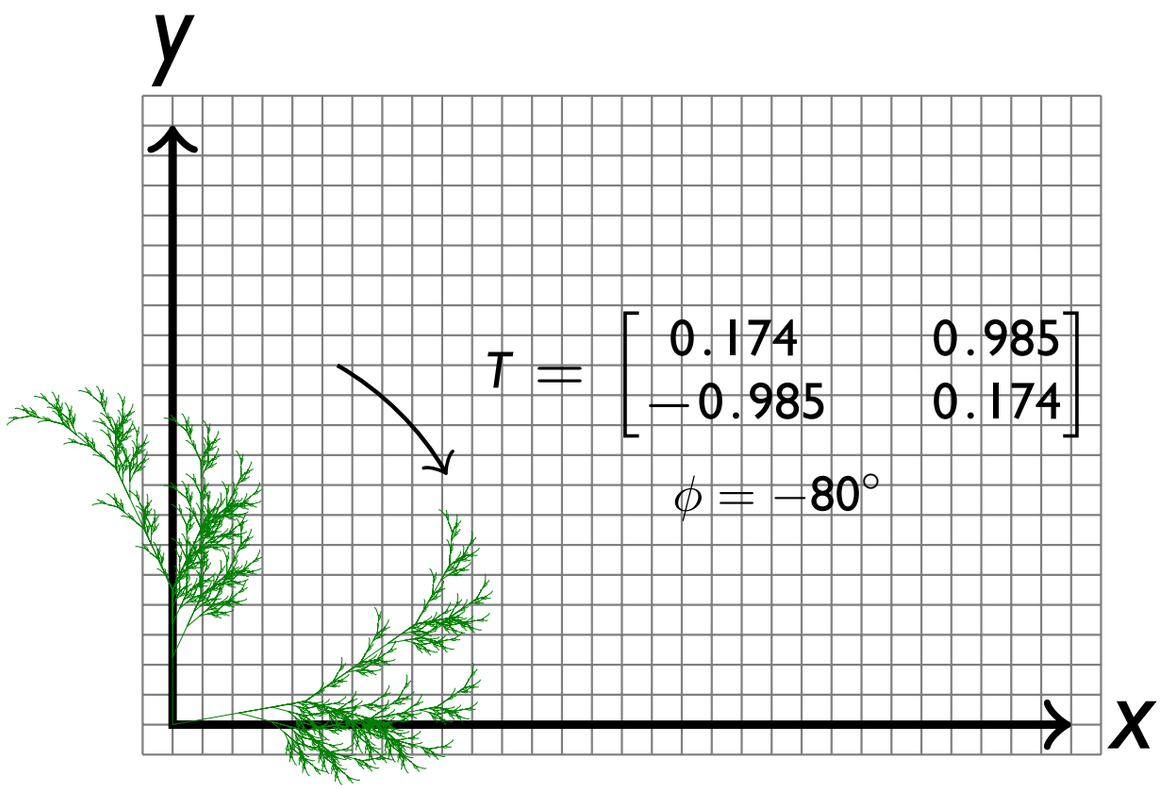
$$\text{scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



Rotation (2D)

- Rotation about origin

$$\textbf{rotate}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$



Translation (2D)

- Possible to write in a 2x2 matrix?
  - No!
- Translation is not a linear transform

(use linearity properties to show)

Non-linearity of Translation

- Linearity conditions:

$$L(p+q) = L(p) + L(q)$$
$$L(ap) = aL(p)$$

- For translation:

$$T(p) = p + t$$
$$T(p+q) = p + q + t \neq (p+t) + (q+t) = T(p) + T(q),$$
$$T(ap) = ap + t \neq a(p+t) = aT(p)$$

Translation (2D)

- Use **Homogeneous coordinates**
  - Add an additional dimension, the *w*-axis, and an extra coordinate, the *w*-component.
  - Effectively the hyperspace for embedding 2D space
- Clever trick to incorporate translations into a single matrix of transformations

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix}$$

Translation (2D)

- The transformation now looks like

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11}x + m_{12}y + x_t \\ m_{21}x + m_{22}y + y_t \\ 1 \end{bmatrix}$$

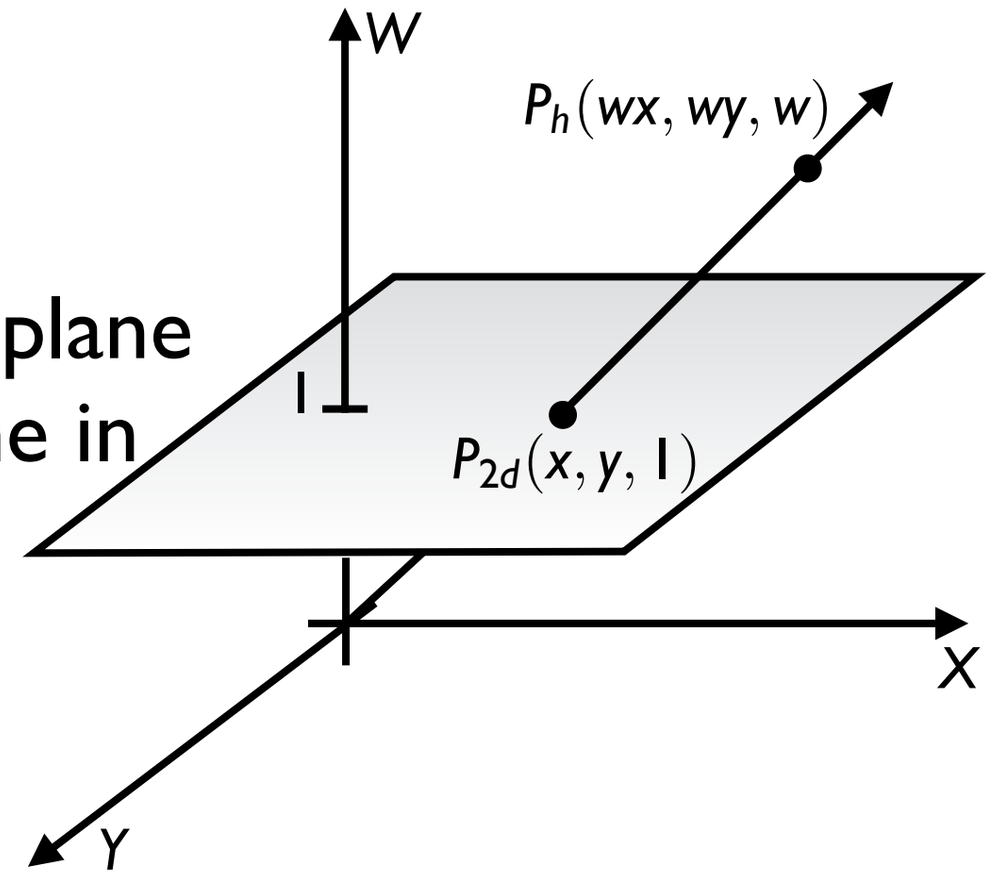
- For positions this is fine, but vectors (or displacements) cannot be translated.
  - Make the third coordinate zero!

$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ is a location, while $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is a vector/direction

Homogeneous Coordinates

- Allow expressing all three 2D transformations as 3x3 matrices
- We start with point P_{2d} on the XY plane and map it to bring it to the w-plane in the hyperspace

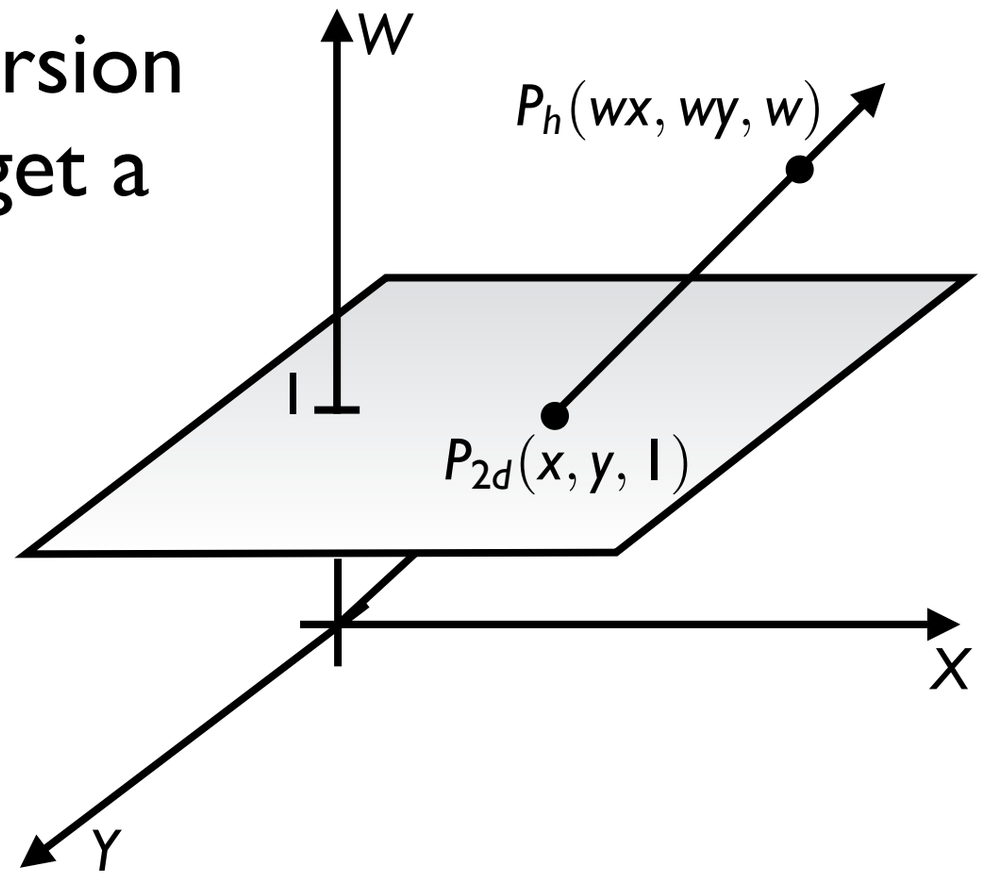
$$P_{2d}(x, y) \mapsto P_h(wx, wy, w), w \neq 0$$



- The resulting (x', y') coordinates in our new point P_h are different from the original (x, y), since $x' = wx, y' = wy$

Homogeneous Coordinates

- We can now apply homogenised version of our transformation matrices to get a new point in the hyperspace
- To get the corresponding 2D point back, we divide all the three coordinates by the *w*-component
- In effect, we want our 3x3 transformation matrix to map points

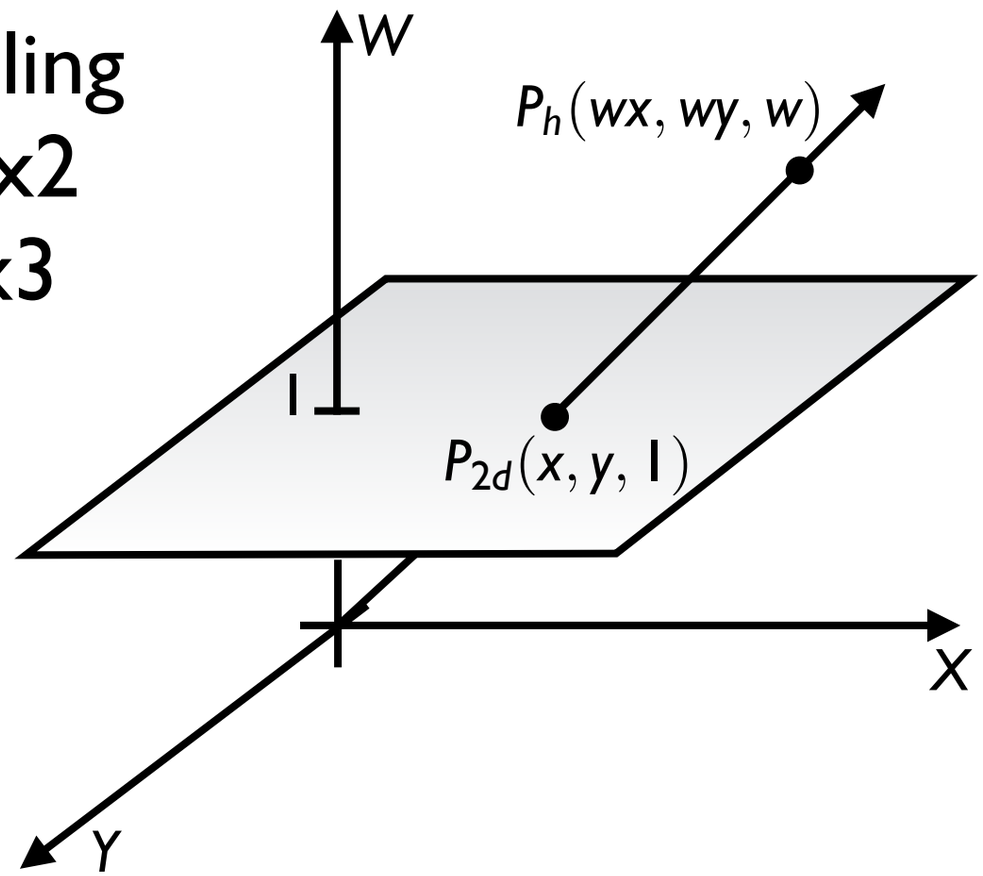


$$v = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \text{ to points } v' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- For linear transformations (i.e., scaling and rotation), embed the existing 2x2 matrix in the upper-left of a new 3x3 matrix:

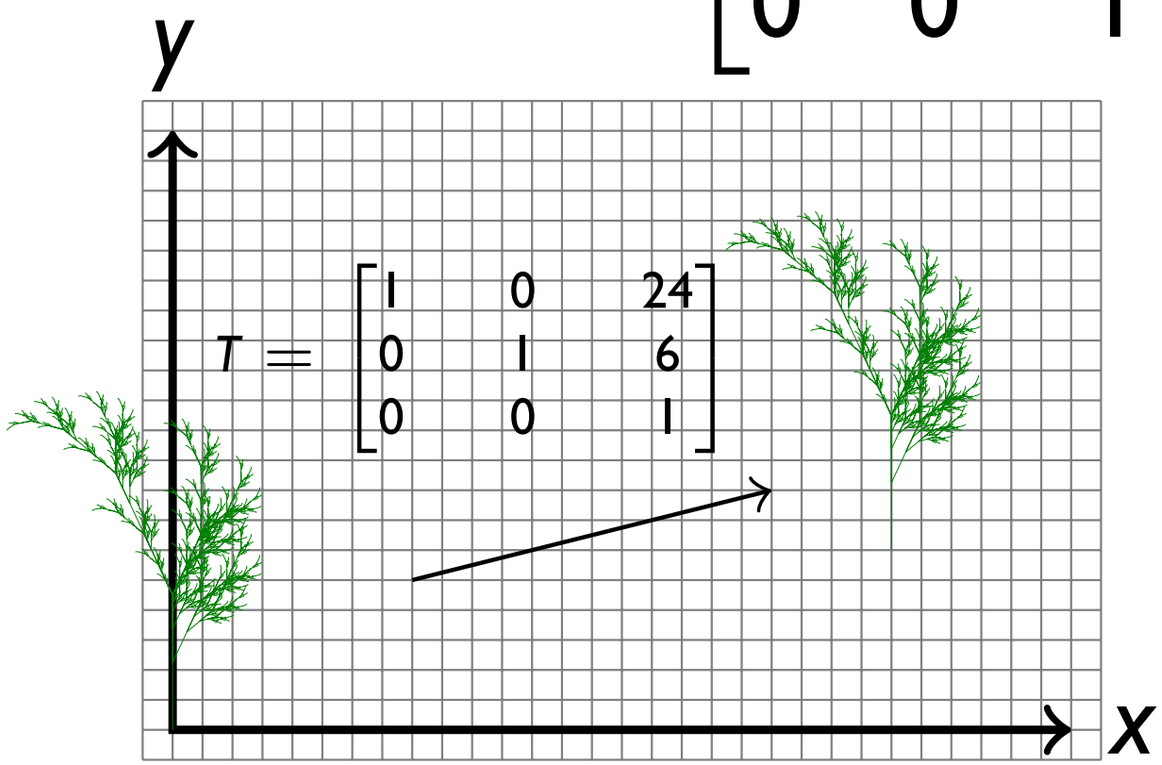
$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Translation (2D)

- Translation matrix can now be represented by embedding the translation vector in the right column

$$\text{translate}(dx, dy) = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$$

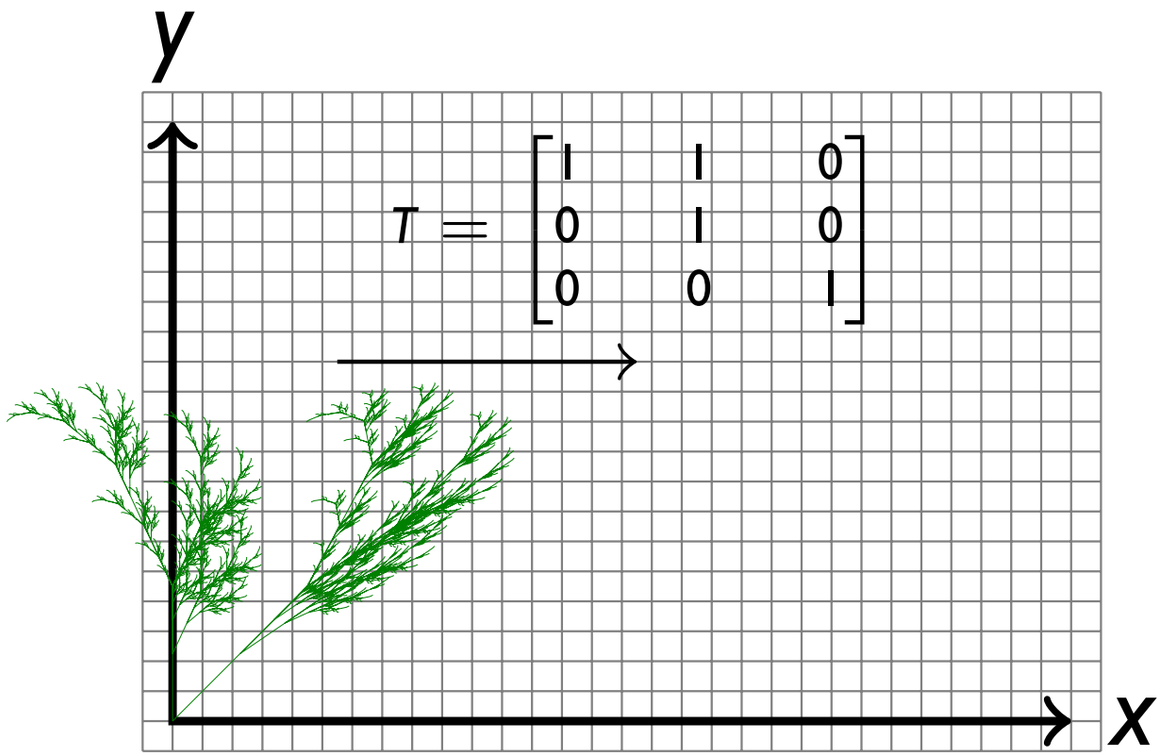


Shearing (2D)

- Skew an object to the side

$$\text{shear}_x = \begin{bmatrix} 1 & \tan\phi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{shear}_y = \begin{bmatrix} 1 & 0 & 0 \\ \tan\phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ϕ is angle with the axis



Transformations Homogenised

Transformation	Matrix
Scaling	$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Rotation	$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Translation	$\begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$

Note: These transformations are called **affine** transformations.

Inverse Transformations

Transformation	Matrix Inverse
Scaling	$\begin{bmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Rotation	$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Translation	$\begin{bmatrix} 1 & 0 & -dx \\ 0 & 1 & -dy \\ 0 & 0 & 1 \end{bmatrix}$

Composition of Transformations

- Transformation is a function; by associativity, we can compose functions

$$(f \circ g)(x) \equiv f(g(x))$$

- Given transformations M_1, M_2 and input vertex v, our composition is equivalent to

$$v' = M_1 M_2 v$$

Composition of Transformations

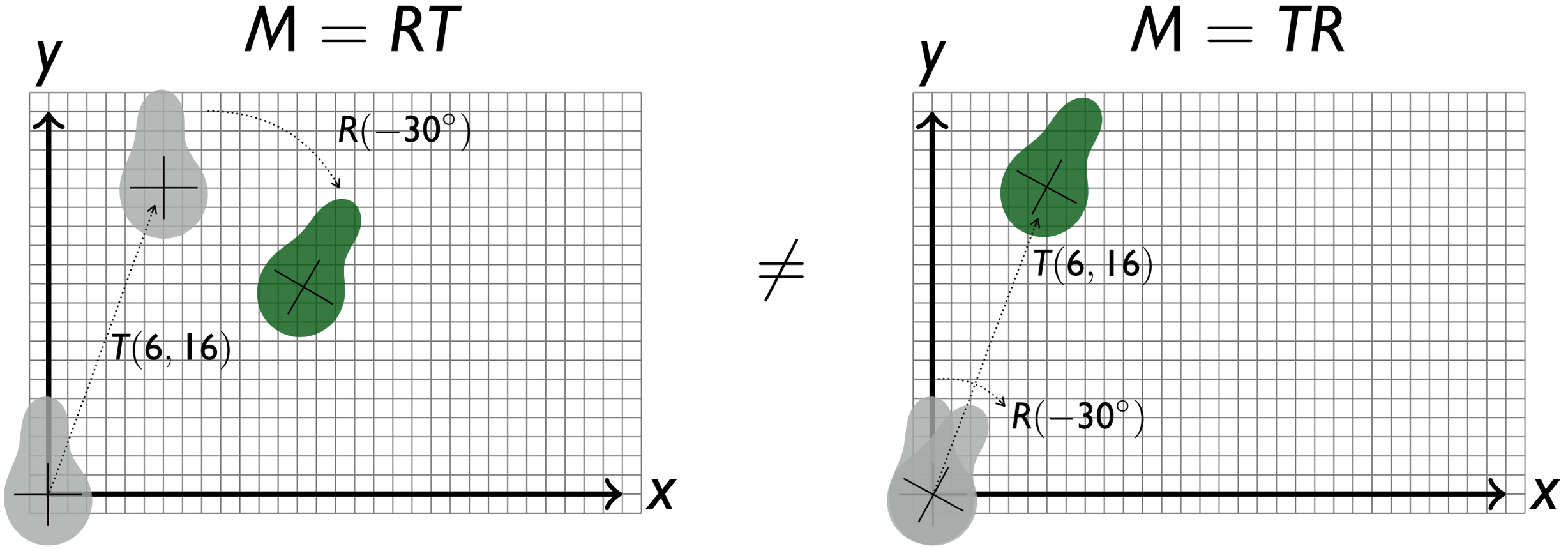
- Transformations can be combined to form a more complex transformation
- E.g.: $M = TRS$, which scales a point, then rotates it, and then translates it:

$$\underbrace{\begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{S}$$

- Note: matrices are applied in sequence right to left.

Composition of Transformations

- Order of transformations matters!
- Matrix multiplication is NOT commutative



Transformation About Arbitrary Point

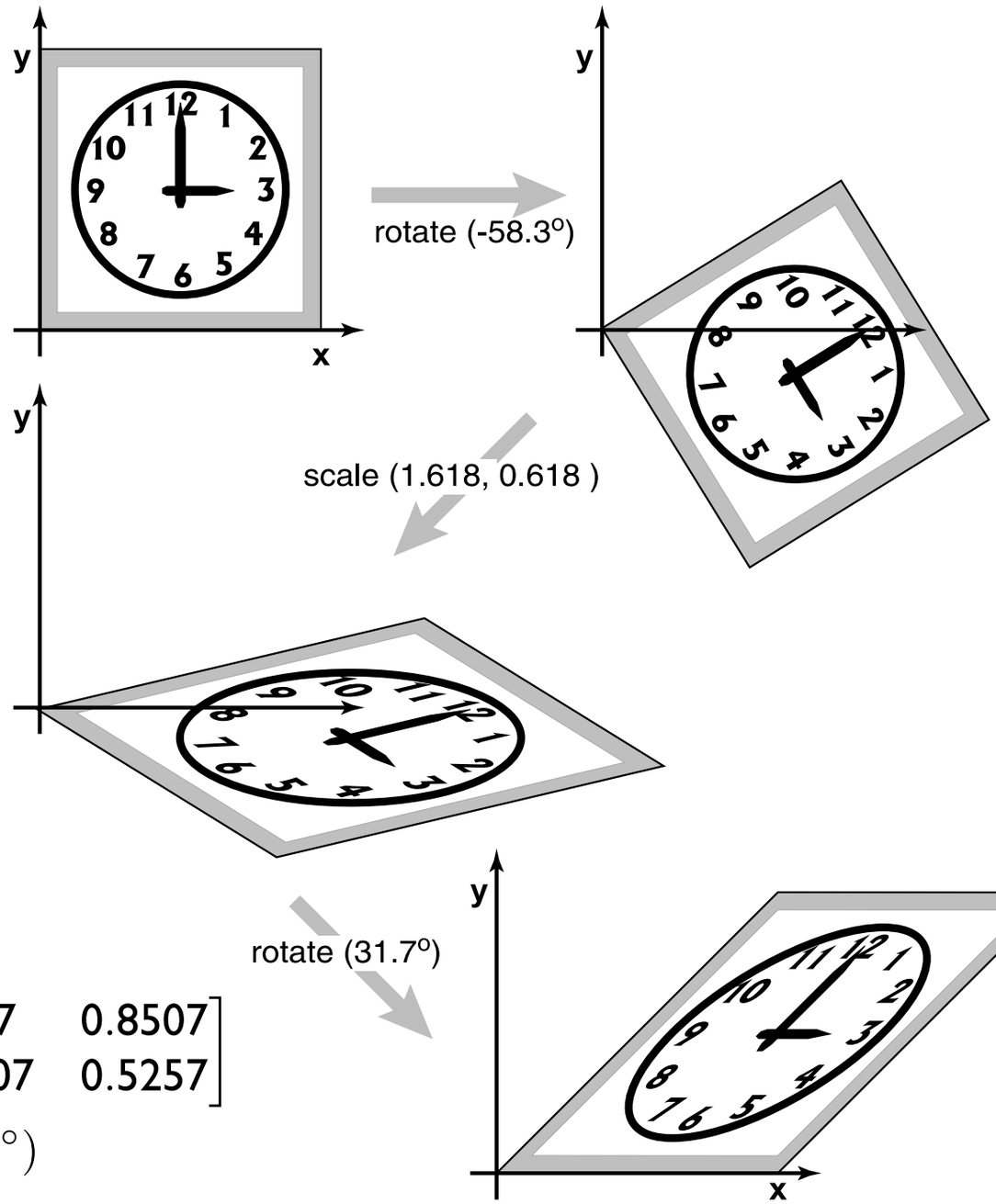
- Previous matrices rotate and scale about origin.
- Using composition we can build a scale and rotation about arbitrary point (or *pivot*) p

$$R_p(\phi) = T(p_x, p_y)R(\phi)T(-p_x, -p_y)$$
$$S_p(s_x, s_y) = T(p_x, p_y)S(s_x, s_y)T(-p_x, -p_y)$$

- In general, translate pivot to origin first, apply rotation/scaling, then translate back

Decomposition of A Transformation

- "Undo" a composition of transformations
- Any 2D matrix can be decomposed into a product of rotation, scale, rotation
- E.g.: SVD of a shear matrix



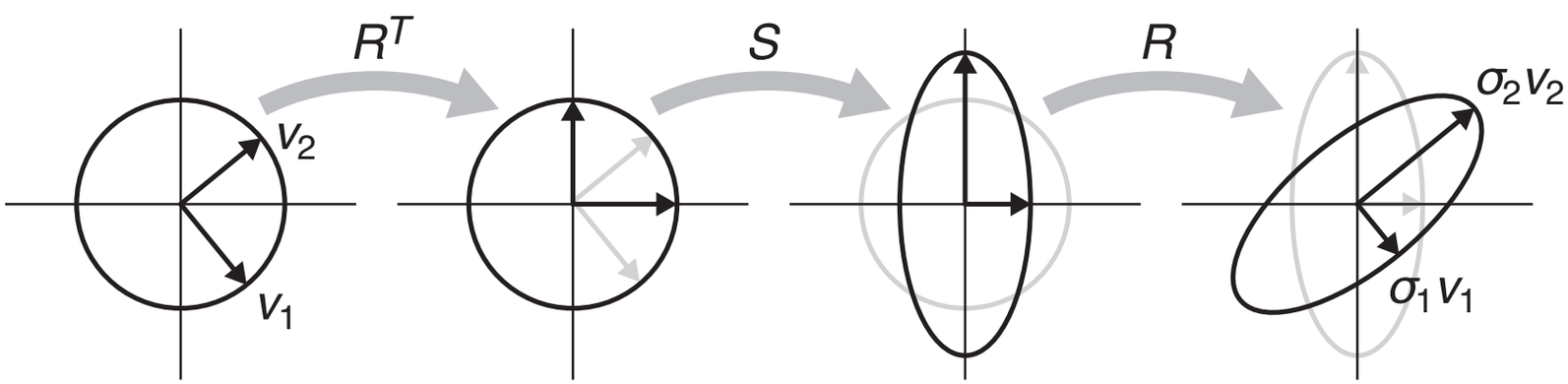
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8507 \end{bmatrix} \begin{bmatrix} 1.618 & 0 \\ 0 & 1.618 \end{bmatrix} \begin{bmatrix} 0.5257 & 0.8507 \\ -0.8507 & 0.5257 \end{bmatrix}$$

$$= \text{rotate}(31.7^\circ)\text{scale}(1.618, 0.618)\text{rotate}(-58.3^\circ)$$

Symmetric Eigenvalue Decomposition

- Eigenvalue decomposition: $A = RSR^T$
- A: symmetric, R: is orthogonal & S: is diagonal
- Interpretation: (simple scale in arbitrary direction)
  1. Rotate v_1 and v_2 to the X and Y axes (transform by R^T)
  2. Scale in x and y by (λ_1, λ_2) (transform by S)
  3. Rotate the X and Y axes back to v_1 and v_2 (transform by R)

$$R = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}, S = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$



Symmetric Eigenvalue Decomposition

- Analytically reversing the diagonalization process

$$\begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} =$$

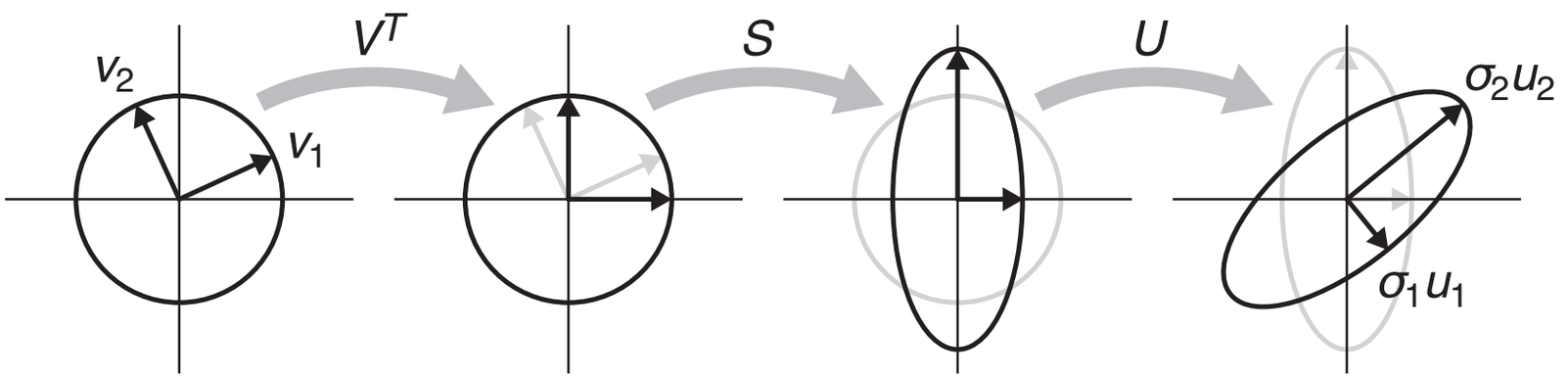
$$\begin{bmatrix} \lambda_1 \cos^2(\phi) + \lambda_2 \sin^2(\phi) & (\lambda_2 - \lambda_1)\cos(\phi)\sin(\phi) \\ (\lambda_2 - \lambda_1)\cos(\phi)\sin(\phi) & \lambda_2 \cos^2(\phi) + \lambda_1 \sin^2(\phi) \end{bmatrix}$$

- This tells us that, symmetric matrices are just scaling operations - albeit potentially nonuniform and non-axis aligned ones.

Singular Value Decomposition (SVD)

- Singular Value Decomposition: $A = USV^T$
- A: non-symmetric, U and V: orthogonal & S: diagonal
- Interpretation:
  1. Rotate v_1 and v_2 to the X and Y axes (transform by V^T)
  2. Scale in x and y by (σ_1, σ_2) (transform by S)
  3. Rotate the X and Y axes to u_1 and u_2 (transform by U)

$$U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}, V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}, S = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$



Paeth Decomposition

- Applies only to rotations
- Uses shears to represent non-zero rotations

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & \frac{\cos\phi - 1}{\sin\phi} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sin\phi & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{\cos\phi - 1}{\sin\phi} \\ 0 & 1 \end{bmatrix}$$

- Very useful in raster image rotations
  - Image shears can be implemented efficiently using pixel translations

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} i + sj \\ j \end{bmatrix}$$

(i, j) is a pixel position

3D Transformations

3D Transformations

Transfor	Matrix
Scaling	$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Rotation	$\underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{rotate}_z(\phi)}, \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{rotate}_x(\phi)}, \underbrace{\begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{rotate}_y(\phi)}$
Translation	$\begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3D Transformations

- Rotation is considerably more complicated in 3D than in 2D due to more possible axes of rotation
- Any 3D linear transformation matrix can be decomposed using SVD into rotation, scale and rotation
- Any symmetric 3D linear transformation matrix can be decomposed using Eigen decomposition into rotation, scale and inverse rotation
- A 3D rotation can be decomposed into a product of 3D shear matrices

Arbitrary 3D Rotations

- Given three mutually orthogonal unit vectors

$$u = x_u\hat{i} + y_u\hat{j} + z_u\hat{k}$$

$$v = x_v i + y_v j + z_v k$$

$$w = x_w\hat{i} + y_w\hat{j} + z_w\hat{k}$$

- $R_{uvw} = \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{bmatrix}$, takes the *uvw* basis to corresponding cartesian basis via rotation.

Arbitrary 3D Rotations

- In order to rotate about an arbitrary vector *a*:
  1. Form an orthonormal *uvw* basis with *w* = *a*
  2. Rotate *uvw* basis to canonical basis *xyz*
  3. Rotate about the z-axis by ϕ
  4. Rotate the canonical basis back to *uvw* basis

$$R_a(\phi) = \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{bmatrix}$$

Constructing a Basis from a Single Vector

- No unique basis, but we just need any one (robust)

1. Make w a unit vector: $w = \frac{a}{\|a\|}$

2. Choose any vector t not collinear with w, and build u as:

$$u = \frac{t \times w}{\|t \times w\|}$$

3. Complete the basis by computing v as:

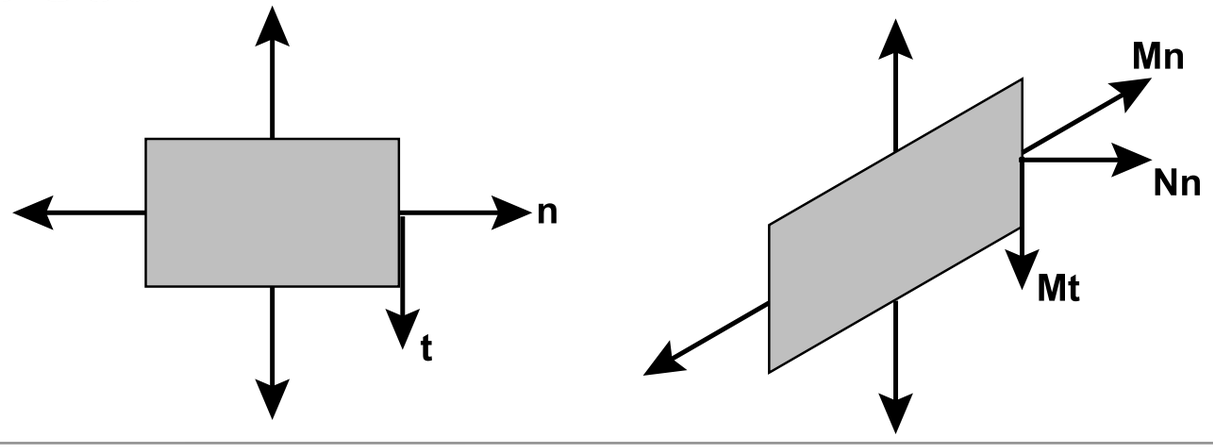
$$v = w \times u$$

A simple procedure to find t significantly different from w is:

1. Start with t equal to w
2. Change the smallest magnitude component of t to 1

Transforming Normals

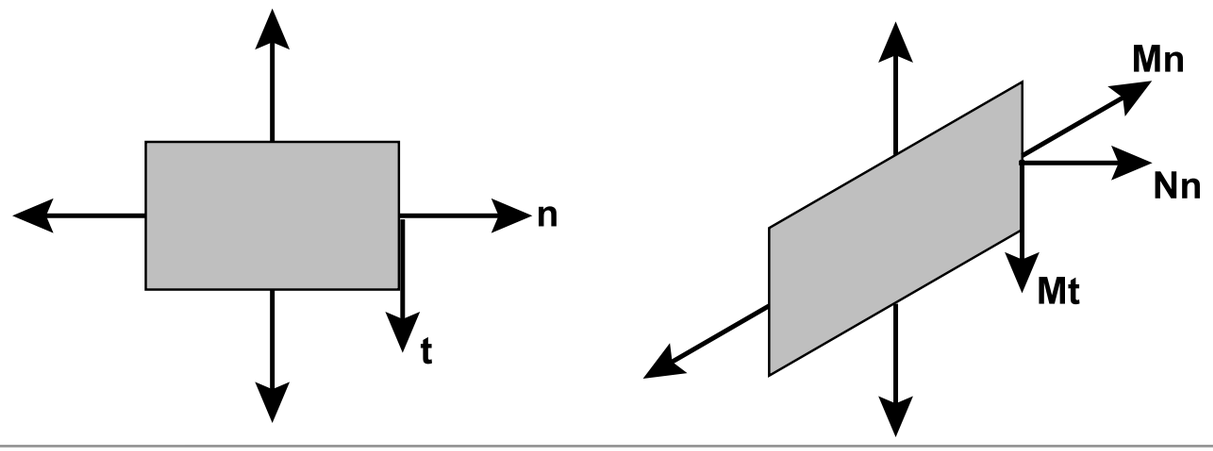
- Apart from representing positions, 3D vectors can also represent surface *normals*
- Surface normal vectors are orthogonal to the tangent plane of the surface at a point
- Unfortunately, these do not transform, the way surface does!



Transforming Normals

- Let the surface points be transformed by matrix *M*, and let *N* be the suitable matrix for the normal *n*
- Tangent *t* and normal *n* are orthogonal: $n^T t = 0$

$$N = \left(M^{-1}\right)^T$$



Reading

- FCG: 6

ICG: Interactive Computer Graphics, E. Angel, and D. Shreiner, 6th ed.
FCG: Fundamentals of Computer Graphics, P. Shirley, M. Ashikhmin, and S. Marschner, 3rd ed.