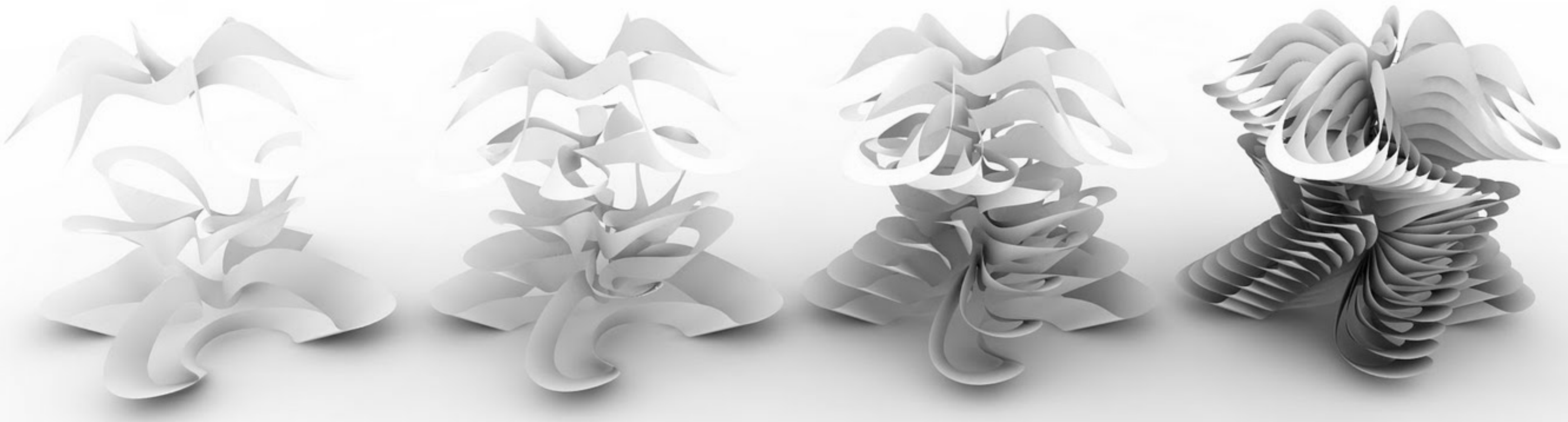

Parametric Curves

Introduction to Computer Graphics
CSE 533/333

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Representations

- Explicit
- Implicit
- Parametric

Explicit Representation

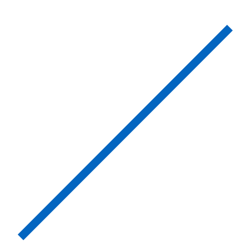
dependent variable represented in terms of independent variable.

For a curve in $\mathbb{R}^2, y = f(x)$

$$\mathbb{R}^3, y = f(x), z = g(x)$$

For a surface in $\mathbb{R}^3, z = f(x, y)$

$$y = mx + c \qquad\qquad z = mx + ny + c$$



Implicit Representation

An implicit representation is of the form $f(\mathbf{x}) = 0$

E.g.: Surfaces in $\mathbb{R}^3$

$$ax + by + cz = 0$$

$$x^2 + y^2 + z^2 - r^2 = 0$$

Curves in $\mathbb{R}^3$ can be represented by intersection of two surfaces:

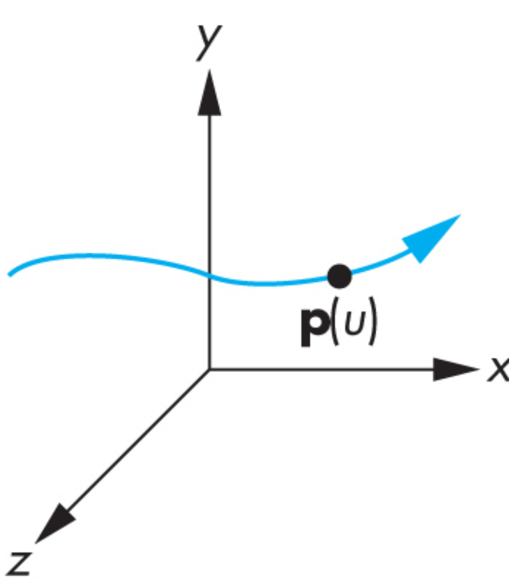
$$f(x, y, z) = 0$$

$$g(x, y, z) = 0$$

Parametric Form

Each spatial variable for points on the curve/surface expressed in terms of an independent variable *u* (the **parameter**)

Curve in $\mathbb{R}^3$	Surface in $\mathbb{R}^3$
$x = x(u)$	$x = x(u, v)$
$y = y(u)$	$y = y(u, v)$
$z = z(u)$	$z = z(u, v)$



Parameter usually defined over the range [0, 1]

Parameterizations

- Re-parameterization
- Arc-length parameterization

Re-Parameterization

- Given a function $\mathbf{f}(u), u \in [a, b]$, defining another function $\mathbf{f}_2(u) = \mathbf{f}(g(u))$ is called a **re-parameterization** of **f** (where, $g : \mathbb{R} \mapsto \mathbb{R}$)
- For a curve **f** with parameter $u \in [0, 1]$,

$$(x, y) = \mathbf{f}(u) = (u, u)$$

$$(x, y) = \mathbf{f}(u) = (u^2, u^2)$$

$$(x, y) = \mathbf{f}(u) = (u^5, u^5)$$

represent the same curve with different speeds

Arc-length Parameterization

- Arc-length distance along a curve from point $\mathbf{f}(0)$ to $\mathbf{f}(v)$:

$$s = \int_0^v \left| \frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t} \right| \, \mathrm{d}t$$

- s could be used as a *natural parameterisation* for $\mathbf{f}$
- For $\mathbf{f}(s)$ the magnitude of *tangent* is constant

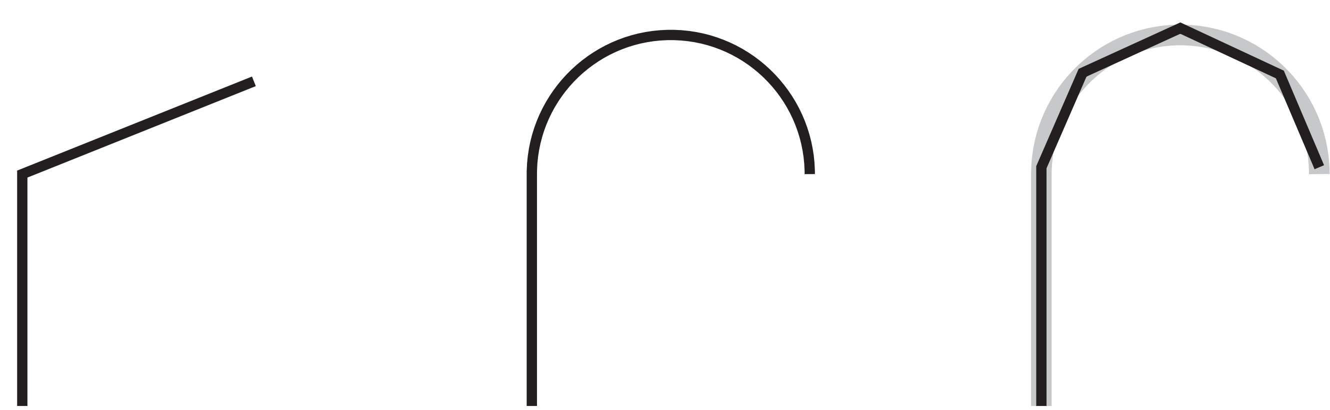
$$\left| \frac{\mathrm{d}\mathbf{f}(s)}{\mathrm{d}s} \right| = 1$$

Curve Representations

- Piecewise curves
  - Revisiting continuity
- Polynomial curves
- Piecewise polynomials
- Cubics
- Approximating curves

Piecewise Curves

- Representation for complex curves as multi-part simpler curves
- The pieces should join smoothly together

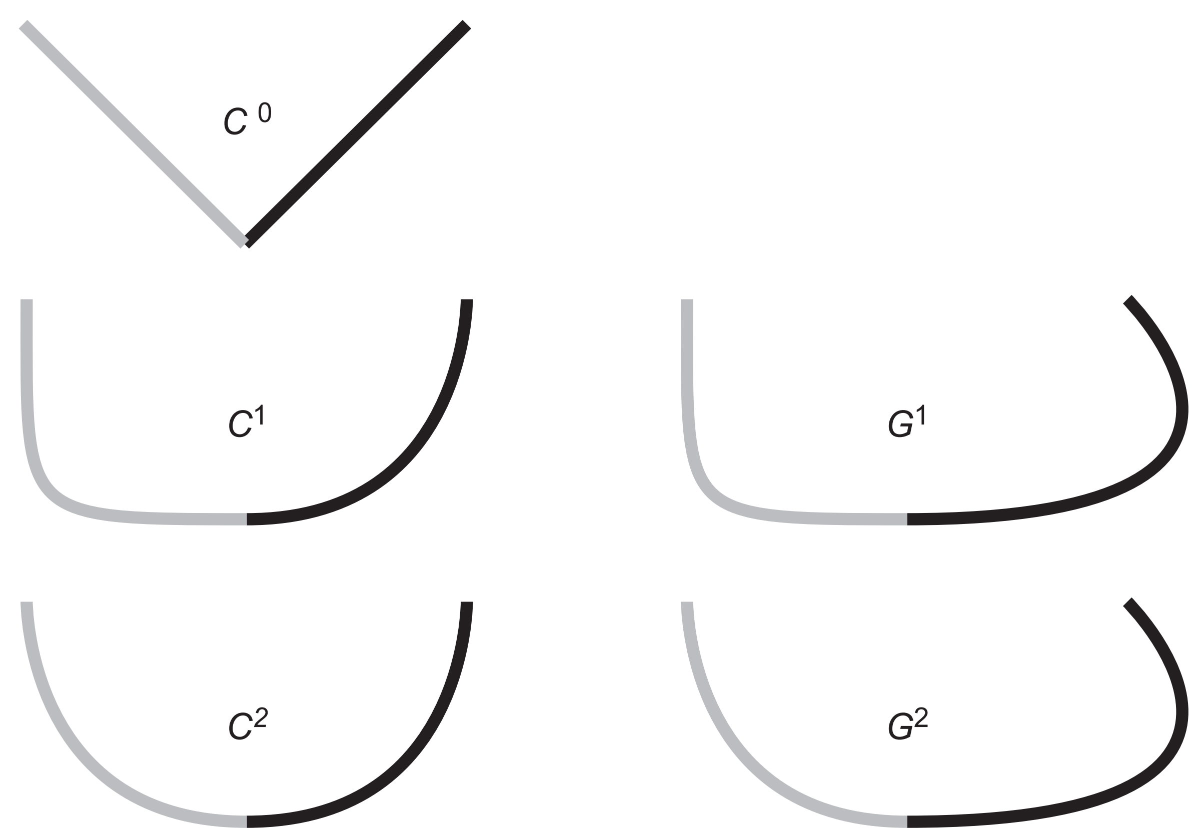


- Example:

$$\mathbf{f}(u) = \begin{cases} \mathbf{f}_1(2u), & \text{if } u \leq 0.5, \\ \mathbf{f}_2(2u-1), & \text{if } u > 0.5 \end{cases}$$

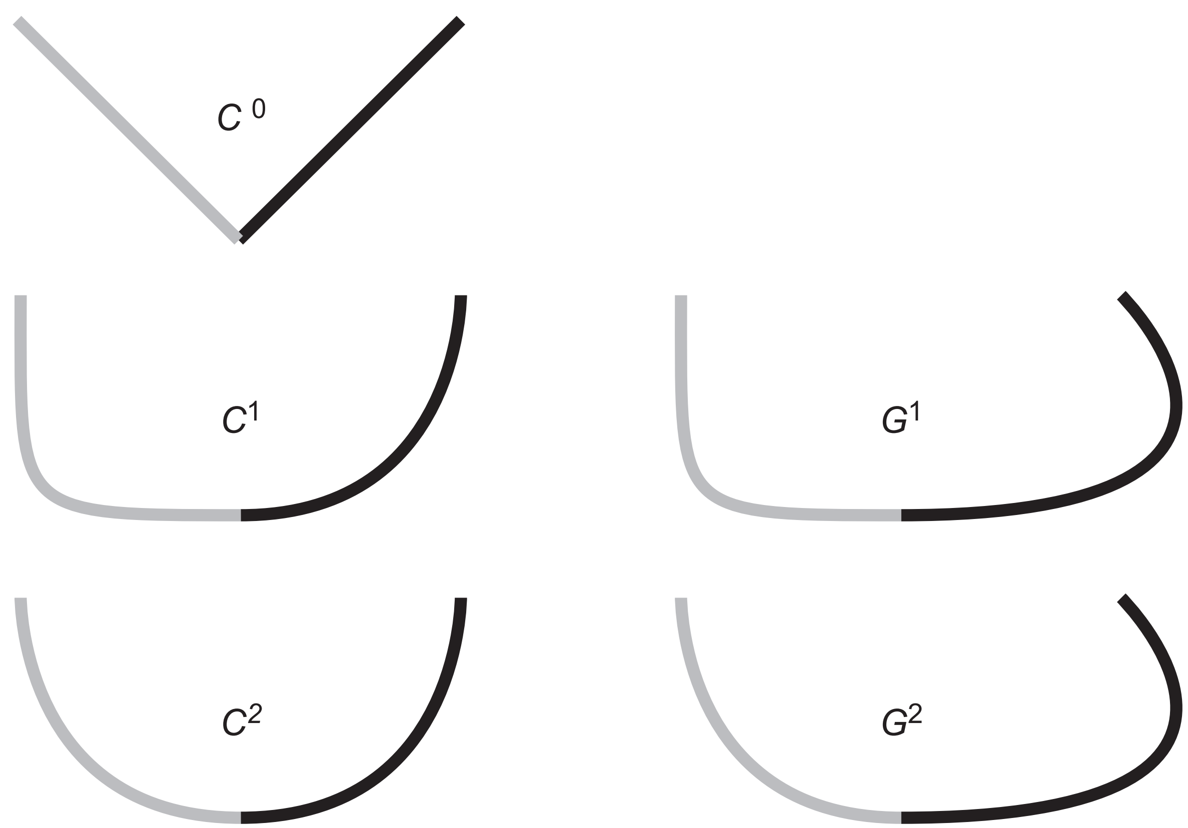
Continuity

- To measure smoothness, we look at the derivative
- A curve is C^n continuous if all the n derivatives match at a point from both sides of the curve



Continuity

- Geometric continuity, G^n, requires that the derivatives of the curve match when the different pieces are parameterised equivalently



Continuity

- Intuitively,
  - continuity looks at both magnitude and direction of derivatives
  - geometric continuity looks at only direction of derivatives
- C^1 continuity : $\mathbf{f}_1'(1) = \mathbf{f}_2'(0)$,

  G^1 continuity : $\mathbf{f}_1'(1) = k\,\mathbf{f}_2'(0), k \in \mathbb{R}$
- A C^n curve is also a G^n (except when parametric derivatives vanish)

Curve Representations

- Piecewise curves
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- Piecewise polynomials
- Cubics
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Polynomial Curves

- A polynomial function has the form

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n, a_n \neq 0$$

where a_i are coefficients and n is the degree of $f(t)$

- Canonical form of a polynomial curve

$$\mathbf{f}(t) = \sum_{i=0}^{n} \mathbf{a}_i t^i$$

Polynomial Curves

- The canonical form is generalised to

$$\mathbf{f}(t) = \sum_{i=0}^{n} \mathbf{c}_i b_i(t)$$

where $b_i(t)$ is a polynomial basis function

Line Segment

Polynomial Curves

- Line segment connecting $\mathbf{p}_0$ to $\mathbf{p}_1$

$$\begin{aligned}\mathbf{f}(u) &= \mathbf{p}_0 + (\mathbf{p}_1 - \mathbf{p}_0)u \\ &= \mathbf{a}_0 + u\mathbf{a}_i \equiv \mathbf{u} \cdot \mathbf{a}\end{aligned}$$

where $\mathbf{u} = [1 \; u]$

Line Segment

Polynomial Curves

- In general we can write:

$$\mathbf{p}_0 = \mathbf{f}(0) = [1 \;\; 0] \cdot [\mathbf{a}_0 \;\; \mathbf{a}_1]^T,$$

$$\mathbf{p}_1 = \mathbf{f}(1) = [1 \;\; 1] \cdot [\mathbf{a}_0 \;\; \mathbf{a}_1]^T$$

- In matrix form:

$$\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \end{bmatrix}$$

or

$$\mathbf{p} = \mathbf{C}\,\mathbf{a}$$

Basis Matrix

- Inversion of $\mathbf{C}$ yields: $\mathbf{f}(u) = \mathbf{u}\,\mathbf{B}\,\mathbf{p}$, where $\mathbf{B} = \mathbf{C}^{-1}$

Quadratic Curves

Polynomial Curves

- Same canonical form applies by letting n=2
- B is a 3x3 basis matrix obtained by solving a linear system of positional constraints
- Constraints on derivatives can as well be applied

$$\mathbf{f}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2,$$

$$\mathbf{f}'(u) = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}u} = \mathbf{a}_1 + 2\mathbf{a}_2 u,$$

$$\mathbf{f}''(u) = \frac{\mathrm{d}^2\mathbf{f}}{\mathrm{d}u^2} = 2\mathbf{a}_2$$

$$C = \begin{bmatrix} 1 & u & u^2 \\ 0 & 1 & 2u \\ 0 & 0 & 2 \end{bmatrix}$$

E.g.: evaluated at u = 0.5

Cubic Curves

Polynomial Curves

- Similar in form to linear and quadratic curves
- *Hermite* form is where positional and first derivative constraints are imposed at first (u = 0) and last points (u = 1)

$$\mathbf{p}_0 = \mathbf{f}(0)$$
$$\mathbf{p}_1 = \mathbf{f}'(0)$$
$$\mathbf{p}_2 = \mathbf{f}(1)$$
$$\mathbf{p}_3 = \mathbf{f}'(1)$$

$$\mathbf{B} = \mathbf{C}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix}$$

Blending Functions

Polynomial Curves

- Define a vector of functions, $\mathbf{b}(u) = \mathbf{u}\,\mathbf{B}$
- Elements of $\mathbf{b}(u)$ are called blending functions
- The control points can be blended linearly with these functions to get the curve

$$\mathbf{f}(u) = \sum_{i=0}^{n} \mathbf{b}_i(u)\mathbf{p}_i$$

- A nice abstraction
  Any degree curve can be represented as a linear combination of control points

Curve Representations

- Piecewise curves
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- Polynomial curves
- Piecewise polynomials
- Cubics
- Approximating curves

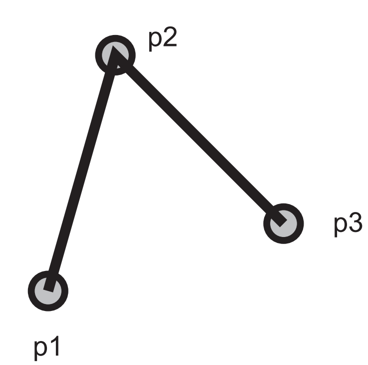
Piecewise Polynomials

- Every piece in a piecewise parametric curve is used over a certain parameter range
- Example:

$$\mathbf{f}(u) = \begin{cases} \mathbf{f}_1(2u) & \text{if } 0 \leq u \leq 0.5 \\ \mathbf{f}_2(2u-1) & \text{if } 0.5 \leq u \leq 1 \end{cases}$$

$\mathbf{f}_1, \mathbf{f}_2$ are functions for each of the two line segments

$$\mathbf{f}_1 = (1-u)\mathbf{p}_1 + u\mathbf{p}_2$$

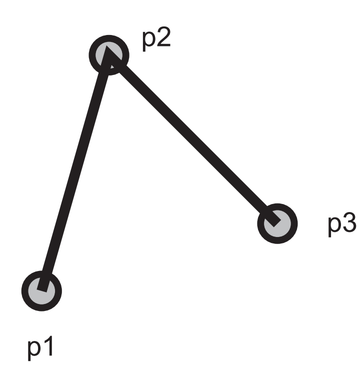


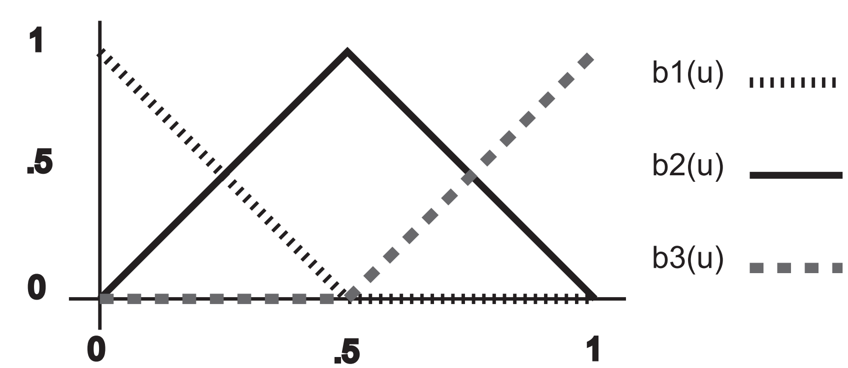
Piecewise Polynomials

- We can also write piecewise polynomials as weighted sum of basis functions
- Example:

$$\mathbf{f}(u) = \mathbf{p}_1 b_1(u) + \mathbf{p}_2 b_2(u) + \mathbf{p}_3 b_3(u)$$

$$b_1(u) = \begin{cases} 1 - 2u & \text{if } 0 \leq u \leq 0.5, \\ 0 & \text{otherwise} \end{cases}$$





Knots

Piecewise Polynomials

- In a piecewise function, *knots* are sites where a piece begins or ends
- E.g. : knots at $u = 0, 0.5,$ and 1 in previous example
- *Knot vector*: a vector of all knot values in ascending order

Curve Representations

- Piecewise curves
  - Revisiting continuity
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- Cubics
- Approximating curves

Cubics

- Allow for a C^2 continuity which is considered suitable for most visual tasks in graphics
- Provide minimum-curvature interpolants to a set of points
- Canonical form:

$$\mathbf{f}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3$$

Cubics

- Desirable properties

	B-splines	*Catmull-Rom Splines*	*Cardinal Splines*	*Natural cubics*
1. Piece-wise cubic	•	•	•	•
2. Interpolating curve		•	•	•
3. Curve has local control	•	•	•	
4. Curve has C^2 continuity	•			•

- Only three can be satisfied at any time.

Natural Cubic Splines

Cubics

- For one segment, parameterise by positions of its end-points and the first and second derivative at the beginning point

$$\begin{aligned} \mathbf{p}_0 &= \mathbf{f}(0) \\ \mathbf{p}_1 &= \mathbf{f}'(0) \\ \mathbf{p}_2 &= \mathbf{f}''(0) \\ \mathbf{p}_3 &= \mathbf{f}(1) \end{aligned}$$

$$\mathbf{B} = \mathbf{C}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ -1 & -1 & -0.5 & 1 \end{bmatrix}$$

- Not local, but C^2

Hermite Cubics

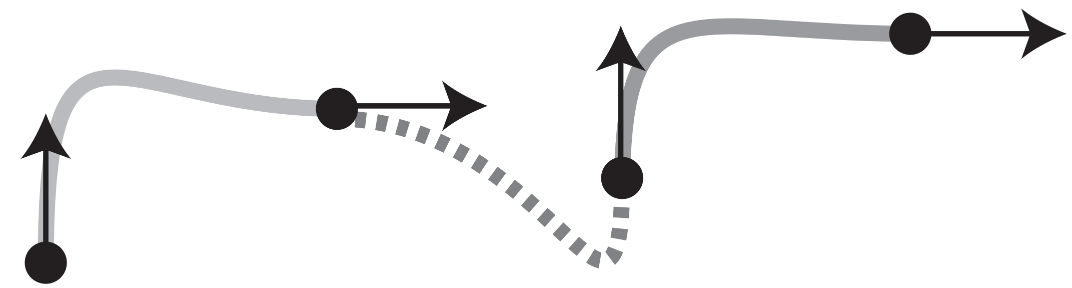
Cubics

- For one segment, parameterise by positions and the first derivative at the end-points

$$\begin{aligned}\mathbf{p}_0 &= \mathbf{f}(0)\\ \mathbf{p}_1 &= \mathbf{f}'(0)\\ \mathbf{p}_2 &= \mathbf{f}(1)\\ \mathbf{p}_3 &= \mathbf{f}'(1)\end{aligned}$$

$$\mathbf{B} = \mathbf{C}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix}$$

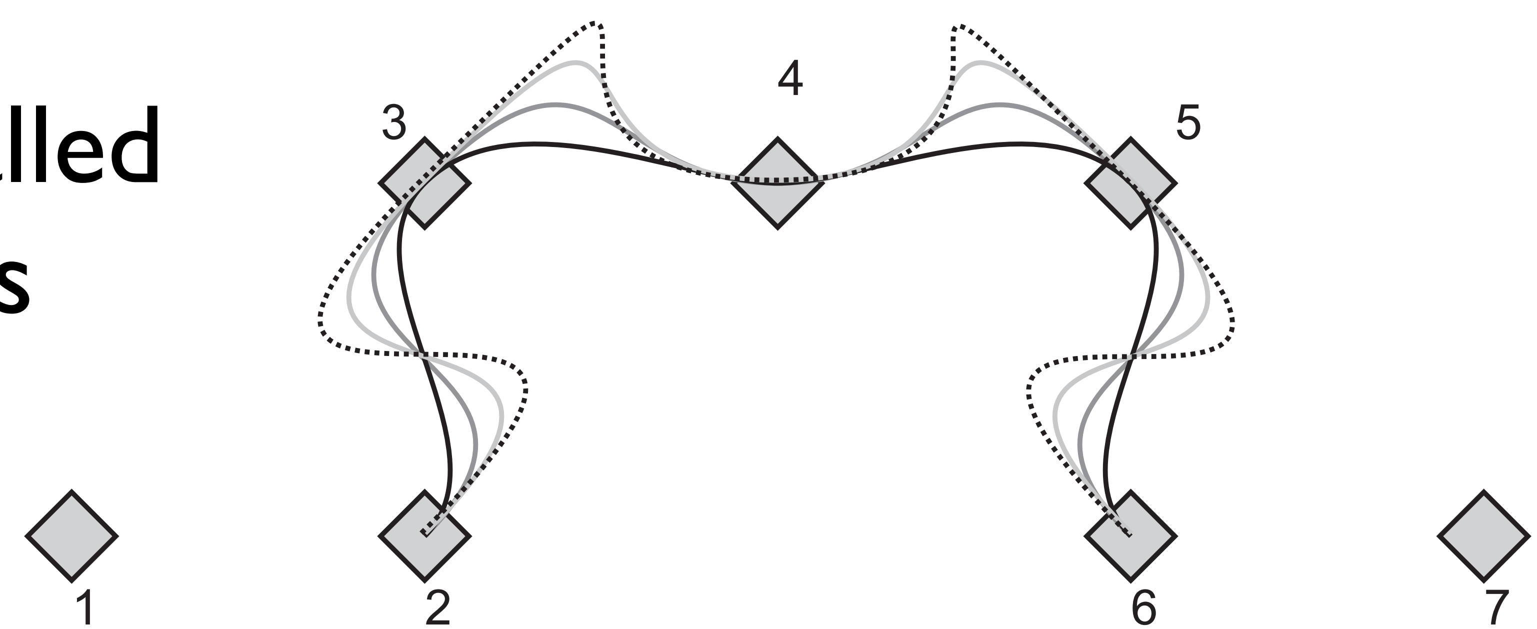
- Local, but only C^1



Cardinal Cubic Splines

Cubics

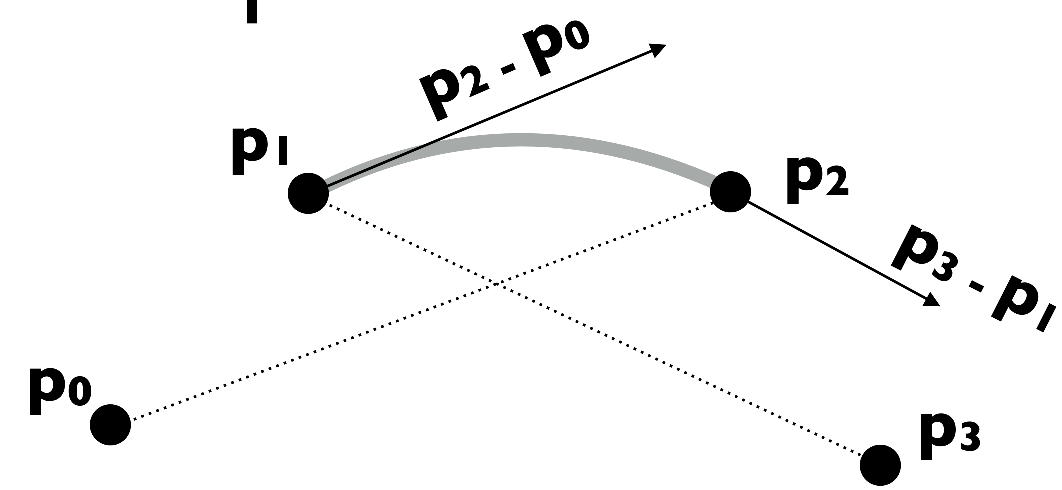
- Interpolate all but the first and last points
- *Tension* parameter to control tightness of the curve
  - $t \in [0, 1)$ controls how the curve bends towards the next control point
- $t = 0$ curves are called *Catmull-Rom* splines
- Local, but only C^1



Cardinal Cubic Splines

Cubics

- Each segment uses four control points: $i, i+1, i+2, i+3$



- Segment begins at second point $\mathbf{p}_1$ and ends at the third point $\mathbf{p}_2$
- Derivative at $\mathbf{p}_1$ is proportional to vector $\mathbf{p}_2 - \mathbf{p}_0$
- Derivative at $\mathbf{p}_2$ is proportional to vector $\mathbf{p}_3 - \mathbf{p}_1$

Cardinal Cubic Splines

Cubics

- Incorporate tension parameter by scaling derivatives by a factor (1 - *t*)/2

$$\mathbf{f}(0) = \mathbf{p}_2$$
$$\mathbf{f}(1) = \mathbf{p}_3$$
$$\mathbf{f}'(0) = \frac{1}{2}(1 - t)(\mathbf{p}_3 - \mathbf{p}_1)$$
$$\mathbf{f}'(1) = \frac{1}{2}(1 - t)(\mathbf{p}_4 - \mathbf{p}_2)$$

Cardinal Cubic Splines

Cubics

- Incorporate tension parameter by scaling derivatives by a factor (1 - *t*)/2

$$\mathbf{p}_0 = \mathbf{f}(1) - \frac{2}{1-t}\mathbf{f}'(0)$$

$$\mathbf{p}_1 = \mathbf{f}(0)$$

$$\mathbf{p}_2 = \mathbf{f}(1)$$

$$\mathbf{p}_3 = \mathbf{f}(0) + \frac{2}{1-t}\mathbf{f}'(1)$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -s & 0 & s & 0 \\ 2s & s-3 & 3-2s & -s \\ -s & 2-s & s-2 & s \end{bmatrix}$$

with s = (1-t)/2

Reading

- FCG: 15

ICG: Interactive Computer Graphics, E. Angel, and D. Shreiner, 6th ed.
FCG: Fundamentals of Computer Graphics, P. Shirley, M. Ashikhmin, and S. Marschner, 3rd ed.