An Analysis of Quantum Mechanics using Real, Quaternion, and Octonion Probability Amplitudes

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Abstract: Quantum mechanics has traditionally relied on complex-valued probability amplitudes. In this paper, I examine why real-valued probability amplitudes are not sufficient to describe reality, and why a quantum mechanical theory based on quaternion- or octonion-valued probability amplitudes would be unphysical.

I. INTRODUCTION

This paper will extend the "standard textbook" mathematical formalism of complex-valued quantum mechanics[1]. Every system can be described by a quantum state, which is represented as a vector in abstract n-dimensional Hilbert space equipped with an inner product between vectors and their conjugate transposes, where n represents the number of eigenstates of an observable that describes the system. A basis that spans the space of possible state vectors must be an eigenstate of that observable. Quantum states that are not basis states are superpositions of basis states, and are represented by a sum of probability amplitudes each multiplied by a basis state vector. Valid quantum states must be represented by state vectors, often called kets kets, that are normalized, meaning that the sum of the square of the probability amplitudes for each basis state must be one. Measured quantum states collapse to an eigenstate of the observable being measured, and the corresponding eigenvalues must be real-valued. The probability that an arbitrary state $|\psi\rangle$ is measured in an eigenstate $|a\rangle$ follows the Born probability rule: $P_a = |\langle a|\psi\rangle|^2$, where $|\psi\rangle$ is the ket that represents the state ψ, $|a\rangle$ is the ket that represents the state a, and $\langle a|$ a bra, the conjugate transpose of $|a\rangle$.

II. REAL-VALUED PROBABILITY AMPLITUDES

To illustrate the issue with real-valued probability amplitude quantum mechanics, or real QM, consider a particle with spin one-half. Measure the particle as spinning in a particular direction parallel to one axis. If you measure the direction of the particle's spin parallel to a different axis, there's a 50% chance of measuring it spinning in either direction[2]. The eigenvalues of each spin operator are $\pm\frac{\hbar}{2} = \pm 1$ in a conveniently-chosen units system. To fully describe a spin one-half particle, there must exist three different operators with different eigenvectors that all have these and only these eigenvalues. Potential operators and their corresponding eigenvectors must be of the following form: Therefore, we can derive the following results, assuming real-valued probability amplitudes:

$$\langle +|0\rangle = \langle +|1\rangle = \langle -|0\rangle = \langle -|1\rangle = \pm\frac{1}{\sqrt{2}} \tag{2.1}$$

$$\langle +|\uparrow\rangle = \langle +|\downarrow\rangle = \langle -|\uparrow\rangle = \langle -|\downarrow\rangle = \pm\frac{1}{\sqrt{2}} \tag{2.2}$$

$$\langle 0|\uparrow\rangle = \langle 0|\downarrow\rangle = \langle 1|\uparrow\rangle = \langle 1|\downarrow\rangle = \pm\frac{1}{\sqrt{2}} \tag{2.3}$$

where $|+\rangle$ and $|-\rangle$, $|0\rangle$ and $|1\rangle$, and $|\uparrow\rangle$ and $|\downarrow\rangle$ are three different sets of eigenkets, each corresponding to one spin operator.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} \tag{2.4}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} -p \\ -q \end{bmatrix} \tag{2.5}$$

An examination of (2.1) and some educated guessing yields two potential pairs of eigenkets:

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} |0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} |1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{2.6}$$

Substituting these sets of eigenkets into (2.4) and (2.5) yields two spin operators

$$S_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{and } S_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.7}$$

respectively. To find $|\uparrow\rangle$ and $|\downarrow\rangle$, write them as $\begin{bmatrix} m & n \end{bmatrix}^T$ and $\begin{bmatrix} p & q \end{bmatrix}^T$ respectively. By (2.2), we have that $m = \pm\frac{1}{\sqrt{2}}$ and that $n = \pm\frac{1}{\sqrt{2}}$. However, by (2.3), $m + n = 1$, so a contradiction is reached. It is impossible to find m and n using real-valued probability amplitudes alone, and so real QM cannot fully describe reality.

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[1] McIntyre, David H (2012). *Quantum Mechanics: A Paradigms Approach.* Pearson Addison-Wesley, ISBN 0-321-76579-6.

[2] Section 1.1.2, Page 7 from McIntyre

III. QUATERNIONS

Quaternions are an extension of the complex numbers. Every quaternion is of the form[3]

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \tag{3.1}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit quaternions. The conjugate of the above quaternion is

$$q^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} \tag{3.2}$$

Quaternions add componentwise:

$$q + r = a + e + (b + f)\mathbf{i} + (c + g)\mathbf{j} + (d + h)\mathbf{k} \tag{3.3}$$

Quaternion addition is commutative and associative. Quaternions multiply according to the following rules:

$$\mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}, \mathbf{ik} = -\mathbf{j}, \mathbf{ji} = -\mathbf{k}, \mathbf{kj} = -\mathbf{i} \tag{3.4}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \tag{3.5}$$

We can show simply but work-intensively using a brute force method that

$$(qr)^* = r^* q^*$$

but for the sake of brevity I will exclude the proof. Quaternion multiplication is distributive and associative but not commutative. (These results are trivial from the definitions of quaternion addition and multiplication.) Quaternion matrix multiplication is associative; to see why this is true, note that

$$((AB)\,C)_{mn} = \sum_l \left(\sum_p A_{mp} B_{pl} \right) C_{ln} \tag{3.6}$$

$$(A\,(BC))_{mn} = \sum_l A_{ml} \left(\sum_p B_{lp} C_{pn} \right) \tag{3.7}$$

We can apply the distributivity and then the associativity of quaternion multiplication to convert (3.5) and (3.6) to

$$((AB)\,C)_{mn} = \sum_l \sum_p A_{mp} B_{pl} C_{ln} \tag{3.8}$$

$$(A\,(BC))_{mn} = \sum_l \sum_p A_{ml} B_{lp} C_{pn} \tag{3.9}$$

respectively, using a left-to-right multiplication convention. Interchanging the index labels l and p in (3.9), which is valid thanks to the commutativity of quaternion addition, completes the proof.

[3] Weisstein, Eric W. "Quaternion." From Mathworld, a Wolfram Web Resource.

IV. QUATERNION-VALUED PROBABILITY AMPLITUDES

To see why quaternion-valued probability amplitudes lead to an unphysical quantum mechanics, consider measurement operators. As in complex QM, measurement operators must satisfy[4]

$$H^\dagger = H \text{ and } H_{mn} = H^*_{nm} \tag{4.1}$$

to have real eigenvalues; matrices that satisfy this property are called Hermitian matrices. The tensor product of two Hermitian operators, an operator applied to a composite which is defined as applying the first operator to the first component system and the second operator to the second component system, is defined as

$$(G \otimes H)_{mn} = G_{wy} H_{xz} \tag{4.2}$$

where G and H are Hermitian matrices and the integer indices w, x, y, and z are defined to satisfy

$$m = wd + x \text{ and } n = yd + z \tag{4.3}$$

where d is the dimension of each matrix, which represents the number of eigenvalues of the quantity being measured. By the definition of a Hermitian matrix (4.1), we must have

$$(G \otimes H)_{mn} = (G \otimes H)^\dagger_{mn} = (G \otimes H)^*_{nm} \tag{4.4}$$

Using (4.2), (4.4) may be expressed as

$$(G_{yw} H_{zx})^* = H^*_{zx} G^*_{yw} = H_{xz} G_{wy} \tag{4.5}$$

The only way for $H_{xz} G_{wy} = G_{wy} H_{xz}$ is for multiplication to be commutative, which is only true if $G_{wy},\ H_{xz} \in \mathbb{C}$. Unfortunately, quaternion quantum mechanics fails on this front.

V. OCTONIONS

The octonions are an extension of the quaternions[5]. We can conveniently define an octionion as an ordered pair of quaternions[6]

$$o = (a,\ b) \tag{5.1}$$

Octonions defined in this manner add component-wise

$$o + p = (a + c,\ b + d) \tag{5.2}$$

[4] Section 2.2.2, Page 44 from McIntyre

[5] Weisstein, Eric W. "Octonion." From Mathworld, a Wolfram Web Resource.

[6] Macher, A. "Hypercomplex Numbers Package." From Wolfram Community.

It's simple to prove from this definition that octonion addition is commutative and associative. Octonions multiply according to the formula

$$op = (ac - d^*b,\ da + bc^*) \tag{5.3}$$

and from this definition it's easy to find that octonion multiplication is distributive but neither associative nor commutative. However, octonion multiplication does satisfy the alternative property, which means

$$(xx)\,y = x\,(xy) \tag{5.4}$$

$$(yx)\,x = y\,(xx) \tag{5.5}$$

$$(xy)\,x = x\,(yx) \tag{5.6}$$

using the left-to-right multiplication convention. The conjugate of an octonion is defined as

$$o^* = (a^*,\ -b) \tag{5.7}$$

and it is once again trivial to show the conjugate of the products of two octonions obeys

$$(op)^* = p^*o^* \tag{5.8}$$

Octonion matrix multiplication is clearly not commutative, and it is also not associative. To see why this is the case, note that

$$((AB)\,C)_{mn} = \sum_l \left(\sum_p A_{mp}B_{pl}\right) C_{ln} \tag{5.9}$$

$$(A\,(BC))_{mn} = \sum_l A_{ml} \left(\sum_p B_{lp}C_{pn}\right) \tag{5.10}$$

We can apply the distributivity of octonion multiplication to convert (3.5) and (3.6) to

$$((AB)\,C)_{mn} = \sum_l \sum_p (A_{mp}B_{pl})\,C_{ln} \tag{5.11}$$

$$(A\,(BC))_{mn} = \sum_l \sum_p A_{ml}\,(B_{lp}C_{pn}) \tag{5.12}$$

but we know (5.11) and (5.12) are not necessarily equal because octonion multiplication is not associative. However, we may show using a similar process that octonion matrix multiplication is alternative,

VI. OCTONION-VALUED PROBABILITY AMPLITUDES

Octonion QM suffers from the same problem as quaternion QM, namely that the tensor product of Hermitian matrices is non-Hermitian, We cannot measure the state of a compound system in quaternion or octonion QM, which does not correspond to reality. To revisit the proof with octonions, note that

$$(G \otimes H)_{mn} = (G \otimes H)^\dagger_{mn} = (G \otimes H)^*_{nm} \tag{6.1}$$

which may be expressed as

$$(G_{yw}H_{zx})^* = H^*_{zx}G^*_{yw} = H_{xz}G_{wy} \tag{6.2}$$

Again, the only way for $H_{xz}G_{wy} = G_{wy}H_{xz}$ is for multiplication to be commutative, which is only true if $G_{wy},\ H_{xz} \in \mathbb{C}$ and not true for general octonion probability amplitudes.