Team notebook

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1 DP

1.1 LineContainer

```
bool Q;
struct Line {
  mutable ll k, m, p;
  bool operator<(const Line& o) const { return Q ?
      p < o.p : k < o.k; }
};
struct LineContainer : multiset<Line> {
  // ( for doubles , use inf = 1/.0 , div (a, b) =
      a/b)
  const ll inf = LLONG_MAX;
  ll div(ll a, ll b) { // floored division
    return a / b - ((a  b) < 0 && a % b);
  }
  bool isect(iterator x, iterator y) {
    if (y == end()) {
      x->p = inf;
      return false;
    }
    if (x->k == y->k)
      x->p = x->m > y->m ? inf : -inf;
    else
      x->p = div(y->m - x->m, x->k - y->k);
    return x->p >= y->p;
  }
  void add(ll k, ll m) {
    auto z = insert({k, m, 0}), y = z++, x = y;
    while (isect(y, z)) z = erase(z);
    if (x != begin() && isect(--x, y)) isect(x, y =
        erase(y));
    while ((y = x) != begin() && (--x)->p >= y->p)
        isect(x, erase(y));
  }
  ll query(ll x) {
    assert(!empty());
    Q = 1;
    auto l = *lower_bound({0, 0, x});
    Q = 0;
    return l.k * x + l.m;
  }
};
```

2 DS

2.1 Fenwick2D

```
struct Fenwick2D {
#define gb(x) (x) & -(x)
  vector<vector<int>> nodes;
  vector<vector<int>> bit;
  int sx;
  void init(int _sx) {
    sx = _sx;
    nodes.resize(sx + 1);
    bit.resize(sx + 1);
  }
  void init_nodes() {
    for (int i = 1; i <= sx; ++i) {
      sort(all(nodes[i]));
      nodes[i].resize(unique(all(nodes[i])) -
          nodes[i].begin());
      bit[i].resize(sz(nodes[i]) + 1);
    }
  }
  void fake_update(int x, int y) {
    for (; x <= sx; x += gb(x))
        nodes[x].push_back(y);
  }
  void fake_get(int x, int y) {
    for (; x > 0; x -= gb(x)) nodes[x].push_back(y);
  }
  void update(int x, int yy, int val) {
    for (; x <= sx; x += gb(x))
      for (int y = lower_bound(all(nodes[x]), yy) -
          nodes[x].begin() + 1;
          y <= sz(nodes[x]); y += gb(y))
        bit[x][y] = max(bit[x][y], val);
  }
  int get(int x, int yy) {
    int res = 0;
    for (; x > 0; x -= gb(x))
      for (int y = upper_bound(all(nodes[x]), yy) -
          nodes[x].begin(); y > 0;
          y -= gb(y))
        res = max(res, bit[x][y]);
    return res;
  }
};
```

2.2 ImplicitTreap

```
// Implicit Treap
// Tested: https://oj.vnoi.info/problem/sqrt_b
struct Treap {
  ll val;
  int prior, size;
  ll sum;
  Treap *left, *right;
  Treap(ll val)
      : val(val), prior(rng()), size(1), sum(val),
          left(NULL), right(NULL){};
};
int size(Treap *t) { return t == NULL ? 0 :
    t->size; }
void down(Treap *t) {
  // do lazy propagation here
}
void refine(Treap *t) {
  if (t == NULL) return;

  t->size = 1;
  t->sum = t->val;
  if (t->left != NULL) {
    t->size += t->left->size;
    t->sum += t->left->sum;
  }
  if (t->right != NULL) {
    t->size += t->right->size;
    t->sum += t->right->sum;
  }
}
void split(Treap *t, Treap *&left, Treap *&right,
    int val) {
  if (t == NULL) return void(left = right = NULL);
  down(t);
  if (size(t->left) < val) {
    split(t->right, t->right, right, val -
        size(t->left) - 1);
    left = t;
  } else {
    split(t->left, left, t->left, val);
    right = t;
  }
  refine(t);
}
void merge(Treap *&t, Treap *left, Treap *right) {
  if (left == NULL) {
    t = right;
    return;
  }
  if (right == NULL) {
    t = left;
    return;
  }

  down(left);
  down(right);
  if (left->prior < right->prior) {
```

```
    merge(left->right, left->right, right);
    t = left;
  } else {
    merge(right->left, left, right->left);
    t = right;
  }
  refine(t);
}
array<Treap *, 2> split(Treap *root, int val) {
  array<Treap *, 2> t;
  split(root, t[0], t[1], val);
  return t;
}
array<Treap *, 3> split(Treap *root, int l, int r) {
  array<Treap *, 3> t;
  Treap *tmp;

  split(root, t[0], t[1], l - 1);
  tmp = t[1];
  split(tmp, t[1], t[2], r - l + 1);

  return t;
}
Treap *root;
```

2.3 MeldableHeap

```
mt19937 gen(0x94949);
template<typename T>
struct Node {
        Node *l, *r;
        T v;
        Node(T x): l(0), r(0), v(x){}
};
template<typename T>
Node<T>* Meld(Node<T>* A, Node<T>* B) {
        if(!A) return B; if(!B) return A;
        if(B->v < A->v) swap(A, B);
        if(gen()&1) A->l = Meld(A->l, B);
        else A->r = Meld(A->r, B);
        return A;
}
template<typename T>
struct Heap {
        Node<T> *r; int s;
        Heap(): r(0), s(0){}
        void push(T x) {
                r = Meld(new Node<T>(x), r);
                ++s;
        }
        int size(){ return s; }
        bool empty(){ return s == 0;}
        T top(){ return r->v; }
        void pop() {
                Node<T>* p = r;
                r = Meld(r->l, r->r);
                delete p;
                --s;
        }
        void Meld(Heap x) {
                s += x->s;
                r = Meld(r, x->r);
        }
};
```

2.4 OrderStatisticTree

```
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>
using namespace __gnu_pbds;
template <class T>
using Tree =
    tree<T, null_type, less<T>, rb_tree_tag,
        tree_order_statistics_node_update>;
void example() {
  Tree<int> t, t2;
  t.insert(8);
  auto it = t.insert(10).first;
  assert(it == t.lower_bound(9));
  assert(t.order_of_key(10) == 1);
  assert(t.order_of_key(11) == 2);
  assert(*t.find_by_order(0) == 8);
  t.join(t2); // assuming T < T2 or T > T2, merge
      t2 into t
}
```

2.5 PalindromeTree

```
template <int MAXC = 26>
struct PalindromicTree {
  PalindromicTree(const string& str) :
      _sz(str.size() + 5), next(_sz,
      vector<int>(MAXC, 0)), link(_sz, 0),
      qlink(_sz, 0), cnt(_sz, 0), right_id(_sz,
      0), len(_sz, 0), s(_sz, 0) {
    init();
    for (int i = 0; i < (int)str.size(); ++i) {
      add(str[i], i);
    }
    count();
  }
  int _sz;

  // returns vector of (left, right, frequency)
  vector<tuple<int, int, int>> get_palindromes() {
    vector<tuple<int, int, int>> res;
    dfs(0, res);
    dfs(1, res);
    return res;
  }

  void dfs(int u, vector<tuple<int, int, int>>&
      res) {
    if (u > 1) { // u = 0 and u = 1 are two empty
        nodes
      res.emplace_back(right_id[u] - len[u] + 1,
          right_id[u], cnt[u]);
    }
    for (int i = 0; i < MAXC; ++i) {
      if (next[u][i]) dfs(next[u][i], res);
    }
  }

  int last, n, p;
  vector<vector<int>> next, dlink;
  vector<int> link, qlink, cnt, right_id, len, s;

  int newnode(int l, int right) {
    len[p] = l;
    right_id[p] = right;
    return p++;
  }
  void init() {
    p = 0;
    newnode(0, -1), newnode(-1, -1);
    n = last = 0;
    s[n] = -1, link[0] = 1;
  }
  int getlink(int x) {
    while (s[n - len[x] - 1] != s[n]) {
      if (s[n - len[link[x]] - 1] == s[n])
        x = link[x];
      else
        x = qlink[x];
    }
    return x;
  }
  void add(char c, int right) {
    c -= 'a';
    s[++n] = c;
    int cur = getlink(last);
    if (!next[cur][(int)c]) {
      int now = newnode(len[cur] + 2, right);
      link[now] = next[getlink(link[cur])][(int)c];
      next[cur][(int)c] = now;
      if (s[n - len[link[now]]] == s[n -
          len[link[link[now]]]]) {
```

```
        qlink[now] = qlink[link[now]];
      } else {
        qlink[now] = link[link[now]];
      }
    }
    last = next[cur][(int)c];
    cnt[last]++;
  }
  void count() {
    for (int i = p - 1; i >= 0; i--) {
      cnt[link[i]] += cnt[i];
    }
  }
};
```

2.6 RMQ

```
ll a[N], st[LG + 1][N];

void pre() {
  for (int i = 1; i <= n; ++i) st[0][i] = a[i];
  for (int j = 1; j <= LG; ++j)
    for (int i = 1; i + (1 << j) - 1 <= n; ++i)
      st[j][i] = __gcd(st[j - 1][i], st[j - 1][i +
          (1 << (j - 1))]);
}

ll query(int l, int r) {
  int k = __lg(r - l + 1);
  return __gcd(st[k][l], st[k][r - (1 << k) + 1]);
}
```

2.7 RMQ

```
// RMQ O(1): 1-indexed
// remember to change the constants, types
using ll = long long;
#define For(i, j, k) for ( int i = (j) ; i <= (k) ;
    i++ )
#define Fol(i, j, k) for ( int i = (j) ; i >= (k) ;
    i-- )
namespace RMQ
{
        using T = int; constexpr int N = 2e6 + 6; //
            change this
        inline bool cmp(T x, T y) { return x < y; }
            // change to '>' to query max
        inline T calc(T x, T y) { return cmp(x, y) ?
            x : y; }
        T val[N], pre[N], st[__lg(( N >> 5 ) + 9) +
            1][( N >> 5 ) + 9]; unsigned f[N];
        __attribute__((target("bmi" ))) inline int
            lb(unsigned x) { return
            __builtin_ctz(x) ; }
        __attribute__((target("lzcnt"))) inline int
            hb(unsigned x) { return
            __builtin_clz(x) ^ 31; }
        inline void build(int n, T *a)
        {
                int m = ( n - 1 ) >> 5, o = hb(m +
                    1), stk[33]; copy(a + 1, a + n +
                    1, val);
                For(i, 0, n - 1) pre[i] = i & 31 ?
                    calc(pre[i - 1], val[i]) :
                    val[i];
                For(i, 0, m) st[0][i] = pre[min(n -
                    1, i << 5 | 31)];
                For(i, 1, o) For(j, 0, m + 1 - ( 1
                    << i ))
                    st[i][j] = calc(st[i - 1][j],
                        st[i - 1][j + ( 1 << ( i
                        - 1 ) )]);
                For(i, 0, n - 1)
                        if ( i & 31 )
                        {
                                f[i] = f[i - 1];
                                while ( o &&
                                    !cmp(val[stk[o]],
                                    val[i]) ) f[i] &=
                                    ~( 1u << (
                                    stk[o--] & 31 ) );
                                f[i] |= 1u << ( (
                                    stk[++o] = i ) &
                                    31 );
                        }
                        else f[i] = 1u << ( ( stk[o =
                            1] = i ) & 31 );
        }
        inline T qry(int l, int r)
        {
                if ( ( --l >> 5 ) == ( --r >> 5 ) )
                    return val[l + lb(f[r] >> ( l &
                    31 ))];
                T z = calc(pre[r], val[l + lb(f[l |
                    31] >> ( l & 31 ))]);
                if ( ( l = ( l >> 5 ) + 1 ) == ( r
                    >>= 5 ) ) return z;
                int t = hb(r - l); return calc(z,
                    calc(st[t][l], st[t][r - ( 1 <<
                    t )]));
        }
}
// build: RMQ::build(n, a), a is an array (not a
    vector!)
// query: RMQ::qry(l, r)
```

2.8 SegmentTree

```
struct Tree {
  typedef int T;
  static constexpr T unit = INT_MIN;
  T f(T a, T b) { return max(a, b); } // (any
      associative fn)
  vector<T> s;
  int n;
  Tree(int n = 0, T def = unit) : s(2 * n, def),
      n(n) {}
  void update(int pos, T val) {
    for (s[pos += n] = val; pos /= 2;) s[pos] =
        f(s[pos * 2], s[pos * 2 + 1]);
  }
  T query(int b, int e) { // query [b , e)
    T ra = unit, rb = unit;
    for (b += n, e += n; b < e; b /= 2, e /= 2) {
      if (b % 2) ra = f(ra, s[b++]);
      if (e % 2) rb = f(s[--e], rb);
    }
    return f(ra, rb);
  }
};
```

2.9 SegTree2N

```
const int N = 1e5; // limit for array size
int n; // array size
int t[2 * N];
void build() { // build the tree
  for (int i = n - 1; i > 0; --i) t[i] = t[i<<1] +
      t[i<<1|1];
}
void modify(int p, int value) { // set value at
    position p
  for (t[p += n] = value; p > 1; p >>= 1) t[p>>1] =
      t[p] + t[p^1];
}
int query(int l, int r) { // sum on interval [l, r)
  int res = 0;
  for (l += n, r += n; l < r; l >>= 1, r >>= 1) {
    if (l&1) res += t[l++];
    if (r&1) res += t[--r];
  }
  return res;
}
```

3 Geometry

3.1 AngleBisector

```
// bisector vector of <abc
PT angle_bisector(PT &a, PT &b, PT &c){
    PT p = a - b, q = c - b;
    return p + q * sqrt(dot(p, p) / dot(q, q));
}
```

3.2 Centroid

```
// centroid of a (possibly non-convex) polygon,
// assuming that the coordinates are listed in a
     clockwise or
// counterclockwise fashion. Note that the centroid
     is often known as
// the "center of gravity" or "center of mass".
PT centroid(vector<PT> &p) {
    int n = p.size(); PT c(0, 0);
    double sum = 0;
    for (int i = 0; i < n; i++) sum += cross(p[i],
         p[(i + 1) % n]);
    double scale = 3.0 * sum;
    for (int i = 0; i < n; i++) {
        int j = (i + 1) % n;
        c = c + (p[i] + p[j]) * cross(p[i], p[j]);
    }
    return c / scale;
}
```

3.3 Circle

```
struct circle {
    PT p; double r;
    circle() {}
    circle(PT _p, double _r): p(_p), r(_r) {};
    // center (x, y) and radius r
    circle(double x, double y, double _r): p(PT(x,
         y)), r(_r) {};
    // circumcircle of a triangle
    // the three points must be unique
    circle(PT a, PT b, PT c) {
        b = (a + b) * 0.5;
        c = (a + c) * 0.5;
        line_line_intersection(b, b + rotatecw90(a -
             b), c, c + rotatecw90(a - c), p);
        r = dist(a, p);
    }
    // inscribed circle of a triangle
    circle(PT a, PT b, PT c, bool t) {
        line u, v;
        double m = atan2(b.y - a.y, b.x - a.x), n =
             atan2(c.y - a.y, c.x - a.x);
        u.a = a;
        u.b = u.a + (PT(cos((n + m)/2.0), sin((n +
             m)/2.0)));
        v.a = b;
        m = atan2(a.y - b.y, a.x - b.x), n =
             atan2(c.y - b.y, c.x - b.x);
        v.b = v.a + (PT(cos((n + m)/2.0), sin((n +
             m)/2.0)));
        line_line_intersection(u.a, u.b, v.a, v.b,
             p);
        r = dist_from_point_to_seg(a, b, p);
    }
    bool operator == (circle v) { return p == v.p
         && sign(r - v.r) == 0; }
    double area() { return PI * r * r; }
    double circumference() { return 2.0 * PI * r; }
};
//0 if outside, 1 if on circumference, 2 if inside
     circle
int circle_point_relation(PT p, double r, PT b) {
    double d = dist(p, b);
    if (sign(d - r) < 0) return 2;
    if (sign(d - r) == 0) return 1;
    return 0;
}
// 0 if outside, 1 if on circumference, 2 if inside
     circle
int circle_line_relation(PT p, double r, PT a, PT
     b) {
    double d = dist_from_point_to_line(a, b, p);
    if (sign(d - r) < 0) return 2;
    if (sign(d - r) == 0) return 1;
    return 0;
}
//compute intersection of line through points a and
     b with
//circle centered at c with radius r > 0
vector<PT> circle_line_intersection(PT c, double r,
     PT a, PT b) {
    vector<PT> ret;
    b = b - a; a = a - c;
    double A = dot(b, b), B = dot(a, b);
    double C = dot(a, a) - r * r, D = B * B - A * C;
    if (D < -eps) return ret;
    ret.push_back(c + a + b * (-B + sqrt(D + eps))
         / A);
    if (D > eps) ret.push_back(c + a + b * (-B -
         sqrt(D)) / A);
    return ret;
}
//5 - outside and do not intersect
//4 - intersect outside in one point
//3 - intersect in 2 points
//2 - intersect inside in one point
//1 - inside and do not intersect
int circle_circle_relation(PT a, double r, PT b,
     double R) {
    double d = dist(a, b);
    if (sign(d - r - R) > 0) return 5;
    if (sign(d - r - R) == 0) return 4;
    double l = fabs(r - R);
    if (sign(d - r - R) < 0 && sign(d - l) > 0)
         return 3;
    if (sign(d - l) == 0) return 2;
    if (sign(d - l) < 0) return 1;
    assert(0); return -1;
}
vector<PT> circle_circle_intersection(PT a, double
     r, PT b, double R) {
    if (a == b && sign(r - R) == 0) return
         {PT(1e18, 1e18)};
    vector<PT> ret;
    double d = sqrt(dist2(a, b));
    if (d > r + R || d + min(r, R) < max(r, R))
         return ret;
    double x = (d * d - R * R + r * r) / (2 * d);
    double y = sqrt(r * r - x * x);
    PT v = (b - a) / d;
    ret.push_back(a + v * x + rotateccw90(v) * y);
    if (y > 0) ret.push_back(a + v * x -
         rotateccw90(v) * y);
    return ret;
}
// returns two circle c1, c2 through points a, b
     and of radius r
// 0 if there is no such circle, 1 if one circle, 2
     if two circle
int get_circle(PT a, PT b, double r, circle &c1,
     circle &c2) {
    vector<PT> v = circle_circle_intersection(a, r,
         b, r);
    int t = v.size();
    if (!t) return 0;
    c1.p = v[0], c1.r = r;
    if (t == 2) c2.p = v[1], c2.r = r;
    return t;
}
// returns two circle c1, c2 which is tangent to
     line u, goes through
// point q and has radius r1; 0 for no circle, 1 if
     c1 = c2 , 2 if c1 != c2
int get_circle(line u, PT q, double r1, circle &c1,
     circle &c2) {
    double d = dist_from_point_to_line(u.a, u.b, q);
    if (sign(d - r1 * 2.0) > 0) return 0;
```

```
    if (sign(d) == 0) {
        cout << u.v.x << ' ' << u.v.y << '\n';
        c1.p = q + rotateccw90(u.v).truncate(r1);
        c2.p = q + rotatecw90(u.v).truncate(r1);
        c1.r = c2.r = r1;
        return 2;
    }
    line u1 = line(u.a +
         rotateccw90(u.v).truncate(r1), u.b +
         rotateccw90(u.v).truncate(r1));
    line u2 = line(u.a +
         rotatecw90(u.v).truncate(r1), u.b +
         rotatecw90(u.v).truncate(r1));
    circle cc = circle(q, r1);
    PT p1, p2; vector<PT> v;
    v = circle_line_intersection(q, r1, u1.a, u1.b);
    if (!v.size()) v = circle_line_intersection(q,
         r1, u2.a, u2.b);
    v.push_back(v[0]);
    p1 = v[0], p2 = v[1];
    c1 = circle(p1, r1);
    if (p1 == p2) {
        c2 = c1;
        return 1;
    }
    c2 = circle(p2, r1);
    return 2;
}
// returns area of intersection between two circles
double circle_circle_area(PT a, double r1, PT b,
     double r2) {
    double d = (a - b).norm();
    if(r1 + r2 < d + eps) return 0;
    if(r1 + d < r2 + eps) return PI * r1 * r1;
    if(r2 + d < r1 + eps) return PI * r2 * r2;
    double theta_1 = acos((r1 * r1 + d * d - r2 *
         r2) / (2 * r1 * d)),
        theta_2 = acos((r2 * r2 + d * d - r1 *
             r1)/(2 * r2 * d));
    return r1 * r1 * (theta_1 - sin(2 *
         theta_1)/2.) + r2 * r2 * (theta_2 - sin(2
         * theta_2)/2.);
}
// tangent lines from point q to the circle
int tangent_lines_from_point(PT p, double r, PT q,
     line &u, line &v) {
    int x = sign(dist2(p, q) - r * r);
    if (x < 0) return 0; // point in cricle
    if (x == 0) { // point on circle
        u = line(q, q + rotateccw90(q - p));
        v = u;
        return 1;
    }
    double d = dist(p, q);
    double l = r * r / d;
    double h = sqrt(r * r - l * l);
    u = line(q, p + ((q - p).truncate(l) +
         (rotateccw90(q - p).truncate(h))));
    v = line(q, p + ((q - p).truncate(l) +
         (rotatecw90(q - p).truncate(h))));
    return 2;
}
// returns outer tangents line of two circles
// if inner == 1 it returns inner tangent lines
int tangents_lines_from_circle(PT c1, double r1, PT
     c2, double r2, bool inner, line &u, line &v) {
    if (inner) r2 = -r2;
    PT d = c2 - c1;
    double dr = r1 - r2, d2 = d.norm2(), h2 = d2 -
         dr * dr;
    if (d2 == 0 || h2 < 0) {
        assert(h2 != 0);
        return 0;
    }
    vector<pair<PT, PT>>out;
    for (int tmp: {- 1, 1}) {
        PT v = (d * dr + rotateccw90(d) * sqrt(h2) *
             tmp) / d2;
        out.push_back({c1 + v * r1, c2 + v * r2});
    }
    u = line(out[0].first, out[0].second);
    if (out.size() == 2) v = line(out[1].first,
         out[1].second);
    return 1 + (h2 > 0);
}
//O(n^2 log n)
struct CircleUnion {
    int n;
    double x[2020], y[2020], r[2020];
    int covered[2020];
    vector<pair<double, double> > seg, cover;
    double arc, pol;
    inline int sign(double x) {return x < -eps ? -1
         : x > eps;}
    inline int sign(double x, double y) {return
         sign(x - y);}
    inline double SQ(const double x) {return x * x;}
    inline double dist(double x1, double y1, double
         x2, double y2) {return sqrt(SQ(x1 - x2) +
         SQ(y1 - y2));}
    inline double angle(double A, double B, double
         C) {
        double val = (SQ(A) + SQ(B) - SQ(C)) / (2 *
             A * B);
        if (val < -1) val = -1;
        if (val > +1) val = +1;
        return acos(val);
    }
    CircleUnion() {
        n = 0;
        seg.clear(), cover.clear();
        arc = pol = 0;
    }
    void init() {
        n = 0;
        seg.clear(), cover.clear();
        arc = pol = 0;
    }
    void add(double xx, double yy, double rr) {
        x[n] = xx, y[n] = yy, r[n] = rr, covered[n]
             = 0, n++;
    }
    void getarea(int i, double lef, double rig) {
        arc += 0.5 * r[i] * r[i] * (rig - lef -
             sin(rig - lef));
        double x1 = x[i] + r[i] * cos(lef), y1 =
             y[i] + r[i] * sin(lef);
        double x2 = x[i] + r[i] * cos(rig), y2 =
             y[i] + r[i] * sin(rig);
        pol += x1 * y2 - x2 * y1;
    }
    double solve() {
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < i; j++) {
                if (!sign(x[i] - x[j]) && !sign(y[i]
                     - y[j]) && !sign(r[i] - r[j])) {
                    r[i] = 0.0;
                    break;
                }
            }
        }
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < n; j++) {
                if (i != j && sign(r[j] - r[i]) >= 0
                     && sign(dist(x[i], y[i], x[j],
                     y[j]) - (r[j] - r[i])) <= 0) {
                    covered[i] = 1;
                    break;
                }
            }
        }
        for (int i = 0; i < n; i++) {
            if (sign(r[i]) && !covered[i]) {
                seg.clear();
                for (int j = 0; j < n; j++) {
                    if (i != j) {
                        double d = dist(x[i], y[i],
                             x[j], y[j]);
                        if (sign(d - (r[j] + r[i]))
                             >= 0 || sign(d -
                             abs(r[j] - r[i])) <= 0) {
                            continue;
                        }
                        double alpha = atan2(y[j] -
                             y[i], x[j] - x[i]);
```

```
                double beta = angle(r[i], d,
                    r[j]);
                pair<double, double>
                    tmp(alpha - beta, alpha
                    + beta);
                if (sign(tmp.first) <= 0 &&
                    sign(tmp.second) <= 0) {
                    seg.push_back(pair<double,
                        double>(2 * PI +
                        tmp.first, 2 * PI +
                        tmp.second));
                }
                else if (sign(tmp.first) < 0)
                    {
                    seg.push_back(pair<double,
                        double>(2 * PI +
                        tmp.first, 2 * PI));
                    seg.push_back(pair<double,
                        double>(0,
                        tmp.second));
                }
                else {
                    seg.push_back(tmp);
                }
            }
        }
        sort(seg.begin(), seg.end());
        double rig = 0;
        for (vector<pair<double, double>
             >::iterator iter = seg.begin();
             iter != seg.end(); iter++) {
            if (sign(rig - iter->first) >= 0)
                {
                rig = max(rig, iter->second);
            }
            else {
                getarea(i, rig, iter->first);
                rig = iter->second;
            }
        }
        if (!sign(rig)) {
            arc += r[i] * r[i] * PI;
        }
        else {
            getarea(i, rig, 2 * PI);
        }
    }
  }
  return pol / 2.0 + arc;
 }
} CU;
```

3.4 ClosestPair

```
typedef Point<ll> P;
pair<P, P> closest(vector<P> v) {
  assert(sz(v) > 1);
  set<P> S;
  sort(all(v), [](P a, P b) { return a.y < b.y; });
  pair<ll, pair<P, P>> ret{LLONG_MAX, {P(), P()}};
  int j = 0;
  for (P p : v) {
    P d{1 + (ll)sqrt(ret.first), 0};
    while (v[j].y <= p.y - d.x) S.erase(v[j++]);
    auto lo = S.lower_bound(p - d), hi =
         S.upper_bound(p + d);
    for (; lo != hi; ++lo) ret = min(ret, {(*lo -
         p).dist2(), {*lo, p}});
    S.insert(p);
  }
  return ret.second;
```

3.5 ConvexPolygon

```
vector<PT> convex_hull(vector<PT> &p) {
    if (p.size() <= 1) return p;
    vector<PT> v = p;
    sort(v.begin(), v.end());
    vector<PT> up, dn;
    for (auto& p : v) {
        while (up.size() > 1 &&
             orientation(up[up.size() - 2],
             up.back(), p) >= 0) {
            up.pop_back();
        }
        while (dn.size() > 1 &&
             orientation(dn[dn.size() - 2],
             dn.back(), p) <= 0) {
            dn.pop_back();
        }
        up.push_back(p);
        dn.push_back(p);
    }
    v = dn;
    if (v.size() > 1) v.pop_back();
    reverse(up.begin(), up.end());
    up.pop_back();
    for (auto& p : up) {
        v.push_back(p);
    }
    if (v.size() == 2 && v[0] == v[1]) v.pop_back();
    return v;
}
 //checks if convex or not
bool is_convex(vector<PT> &p) {
    bool s[3]; s[0] = s[1] = s[2] = 0;
    int n = p.size();
    for (int i = 0; i < n; i++) {
        int j = (i + 1) % n;
        int k = (j + 1) % n;
        s[sign(cross(p[j] - p[i], p[k] - p[i])) + 1]
             = 1;
        if (s[0] && s[2]) return 0;
    }
    return 1;
}
// -1 if strictly inside, 0 if on the polygon, 1 if
     strictly outside
// it must be strictly convex, otherwise make it
     strictly convex first
int is_point_in_convex(vector<PT> &p, const PT& x)
     { // O(log n)
    int n = p.size(); assert(n >= 3);
    int a = orientation(p[0], p[1], x), b =
         orientation(p[0], p[n - 1], x);
    if (a < 0 || b > 0) return 1;
    int l = 1, r = n - 1;
    while (l + 1 < r) {
        int mid = l + r >> 1;
        if (orientation(p[0], p[mid], x) >= 0) l =
             mid;
        else r = mid;
    }
    int k = orientation(p[l], p[r], x);
    if (k <= 0) return -k;
    if (l == 1 && a == 0) return 0;
    if (r == n - 1 && b == 0) return 0;
    return -1;
}
```

3.6 ExtremeVertex

```
// id of the vertex having maximum dot product with
     z
// polygon must need to be convex
// top - upper right vertex
// for minimum dot prouct negate z and return
     -dot(z, p[id])
int extreme_vertex(vector<PT> &p, const PT &z,
     const int top) { // O(log n)
    int n = p.size();
    if (n == 1) return 0;
    double ans = dot(p[0], z); int id = 0;
    if (dot(p[top], z) > ans) ans = dot(p[top], z),
         id = top;
    int l = 1, r = top - 1;
    while (l < r) {
        int mid = l + r >> 1;
```

```
        if (dot(p[mid + 1], z) >= dot(p[mid], z)) l
            = mid + 1;
        else r = mid;
    }
    if (dot(p[l], z) > ans) ans = dot(p[l], z), id
        = l;
    l = top + 1, r = n - 1;
    while (l < r) {
        int mid = l + r >> 1;
        if (dot(p[(mid + 1) % n], z) >= dot(p[mid],
            z)) l = mid + 1;
        else r = mid;
    }
    l %= n;
    if (dot(p[l], z) > ans) ans = dot(p[l], z), id
        = l;
    return id;
}
```

3.7 GeometricMedian

```
// it returns a point such that the sum of distances
// from that point to all points in p is minimum
// O(n log^2 MX)
PT geometric_median(vector<PT> p) {
    auto tot_dist = [&](PT z) {
        double res = 0;
        for (int i = 0; i < p.size(); i++) res +=
            dist(p[i], z);
        return res;
    };
    auto findY = [&](double x) {
        double yl = -1e5, yr = 1e5;
        for (int i = 0; i < 60; i++) {
            double ym1 = yl + (yr - yl) / 3;
            double ym2 = yr - (yr - yl) / 3;
            double d1 = tot_dist(PT(x, ym1));
            double d2 = tot_dist(PT(x, ym2));
            if (d1 < d2) yr = ym2;
            else yl = ym1;
        }
        return pair<double, double> (yl,
            tot_dist(PT(x, yl)));
    };
    double xl = -1e5, xr = 1e5;
    for (int i = 0; i < 60; i++) {
        double xm1 = xl + (xr - xl) / 3;
        double xm2 = xr - (xr - xl) / 3;
        double y1, d1, y2, d2;
        auto z = findY(xm1); y1 = z.first; d1 =
            z.second;
        z = findY(xm2); y2 = z.first; d2 = z.second;
        if (d1 < d2) xr = xm2;
        else xl = xm1;
    }
    return {xl, findY(xl).first };
}
```

3.8 GeometryTemplate

```
const long double PI = acos(-1);
struct Vector {
  using type = long long;
  type x, y;
  Vector operator-(const Vector &other) const {
    return {x - other.x, y - other.y};
  }
  type operator*(const Vector &other) const {
    return x * other.y - other.x * y;
  }
  type operator%(const Vector &other) const {
    return x * other.x + y * other.y;
  }
  bool operator==(const Vector &other) const {
    return x == other.x and y == other.y;
  }
  bool operator!=(const Vector &other) const {
      return !(*this == other); }
  friend type cross(const Vector &A, const Vector
      &B, const Vector &C) {
    return (B - A) * (C - A);
  }
  friend type dist(Vector A) { return A.x * A.x +
      A.y * A.y; }
  friend type dot(const Vector &A, const Vector &B,
      const Vector &C) {
    Vector u = (B - A), v = (C - A);
    return u % v;
  }
  friend istream &operator>>(istream &is, Vector
      &V) {
    is >> V.x >> V.y;
    return is;
  }
  friend ostream &operator<<(ostream &os, Vector
      &V) {
    os << V.x << ' ' << V.y;
    return os;
  }
  friend double angle(const Vector &A, const Vector
      &B, const Vector &C) {
    double x = dot(B, A, C) / sqrt(dist(A - B) *
        dist(C - B));
    return acos(min(1.0, max(-1.0, x))) * 180.0 /
        PI;
  }
};
using Point = Vector;
const Point origin = {0, 0};

long double area(Point A, Point B, Point C) {
  long double res =
      cross(origin, A, B) + cross(origin, B, C) +
          cross(origin, C, A);
  return abs(res) / 2.0;
}
```

3.9 HalfPlane

```
// contains all points p such that: cross(b - a, p
    - a) >= 0
struct HP {
    PT a, b;
    HP() {}
    HP(PT a, PT b) : a(a), b(b) {}
    HP(const HP& rhs) : a(rhs.a), b(rhs.b) {}
    int operator < (const HP& rhs) const {
        PT p = b - a;
        PT q = rhs.b - rhs.a;
        int fp = (p.y < 0 || (p.y == 0 && p.x < 0));
        int fq = (q.y < 0 || (q.y == 0 && q.x < 0));
        if (fp != fq) return fp == 0;
        if (cross(p, q)) return cross(p, q) > 0;
        return cross(p, rhs.b - a) < 0;
    }
    PT line_line_intersection(PT a, PT b, PT c, PT
        d) {
        b = b - a; d = c - d; c = c - a;
        return a + b * cross(c, d) / cross(b, d);
    }
    PT intersection(const HP &v) {
        return line_line_intersection(a, b, v.a,
            v.b);
    }
};
int check(HP a, HP b, HP c) {
    return cross(a.b - a.a, b.intersection(c) -
        a.a) > -eps; //-eps to include polygons of
        zero area (straight lines, points)
}
// consider half-plane of counter-clockwise side of
    each line
// if lines are not bounded add infinity rectangle
// returns a convex polygon, a point can occur
    multiple times though
```

```
// complexity: O(n log(n))
vector<PT> half_plane_intersection(vector<HP> h) {
   sort(h.begin(), h.end());
   vector<HP> tmp;
   for (int i = 0; i < h.size(); i++) {
      if (!i || cross(h[i].b - h[i].a, h[i - 1].b
          - h[i - 1].a)) {
         tmp.push_back(h[i]);
      }
   }
   h = tmp;
   vector<HP> q(h.size() + 10);
   int qh = 0, qe = 0;
   for (int i = 0; i < h.size(); i++) {
      while (qe - qh > 1 && !check(h[i], q[qe -
          2], q[qe - 1])) qe--;
      while (qe - qh > 1 && !check(h[i], q[qh],
          q[qh + 1])) qh++;
      q[qe++] = h[i];
   }
   while (qe - qh > 2 && !check(q[qh], q[qe - 2],
       q[qe - 1])) qe--;
   while (qe - qh > 2 && !check(q[qe - 1], q[qh],
       q[qh + 1])) qh++;
   vector<HP> res;
   for (int i = qh; i < qe; i++)
      res.push_back(q[i]);
   vector<PT> hull;
   if (res.size() > 2) {
      for (int i = 0; i < res.size(); i++) {
         hull.push_back(res[i].intersection(res[(i
              + 1) % ((int)res.size())]));
      }
   }
   return hull;
}
```

3.10 IsPoint

```
// -1 if strictly inside, 0 if on the polygon, 1 if
    strictly outside
int is_point_in_triangle(PT a, PT b, PT c, PT p) {
   if (sign(cross(b - a,c - a)) < 0) swap(b, c);
   int c1 = sign(cross(b - a,p - a));
   int c2 = sign(cross(c - b,p - b));
   int c3 = sign(cross(a - c,p - c));
   if (c1<0 || c2<0 || c3 < 0) return 1;
   if (c1 + c2 + c3 != 3) return 0;
   return -1;
}
bool is_point_on_polygon(vector<PT> &p, const PT&
    z) {
   int n = p.size();
   for (int i = 0; i < n; i++) {
      if (is_point_on_seg(p[i], p[(i + 1) % n],
          z)) return 1;
   }
   return 0;
}

// returns 1e9 if the point is on the polygon
int winding_number(vector<PT> &p, const PT& z) { //
    O(n)
   if (is_point_on_polygon(p, z)) return 1e9;
   int n = p.size(), ans = 0;
   for (int i = 0; i < n; ++i) {
      int j = (i + 1) % n;
      bool below = p[i].y < z.y;
      if (below != (p[j].y < z.y)) {
         auto orient = orientation(z, p[j], p[i]);
         if (orient == 0) return 0;
         if (below == (orient > 0)) ans += below
             ? 1 : -1;
      }
   }
   return ans;
}
// -1 if strictly inside, 0 if on the polygon, 1 if
    strictly outside
int is_point_in_polygon(vector<PT> &p, const PT& z)
    { // O(n)
   int k = winding_number(p, z);
   return k == 1e9 ? 0 : k == 0 ? 1 : -1;
}
```

3.11 Line

```
struct line {
   PT a, b; // goes through points a and b
   PT v; double c; //line form: direction vec
       [cross] (x, y) = c
   line() {}
   //direction vector v and offset c
   line(PT v, double c) : v(v), c(c) {
      auto p = get_points();
      a = p.first; b = p.second;
   }
   // equation ax + by + c = 0
   line(double _a, double _b, double _c) : v({_b,
       -_a}), c(-_c) {
      auto p = get_points();
      a = p.first; b = p.second;
   }
   // goes through points p and q
   line(PT p, PT q) : v(q - p), c(cross(v, p)),
       a(p), b(q) {}
      pair<PT, PT> get_points() { //extract any
          two points from this line
      PT p, q; double a = -v.y, b = v.x; // ax +
          by = c
      if (sign(a) == 0) {
         p = PT(0, c / b);
         q = PT(1, c / b);
      }
      else if (sign(b) == 0) {
         p = PT(c / a, 0);
         q = PT(c / a, 1);
      }
      else {
         p = PT(0, c / b);
         q = PT(1, (c - a) / b);
      }
      return {p, q};
      }
   //ax + by + c = 0
   array<double, 3> get_abc() {
      double a = -v.y, b = v.x;
      return {a, b, c};
   }
   // 1 if on the left, -1 if on the right, 0 if
       on the line
   int side(PT p) { return sign(cross(v, p) - c); }
   // line that is perpendicular to this and goes
       through point p
   line perpendicular_through(PT p) { return {p, p
       + perp(v)}; }
   // translate the line by vector t i.e. shifting
       it by vector t
   line translate(PT t) { return {v, c + cross(v,
       t)}; }
   // compare two points by their orthogonal
       projection on this line
   // a projection point comes before another if
       it comes first according to vector v
   bool cmp_by_projection(PT p, PT q) { return
       dot(v, p) < dot(v, q); }
   line shift_left(double d) {
      PT z = v.perp().truncate(d);
      return line(a + z, b + z);
   }
};
```

3.12 LineLineIntersection

```
// intersection point between ab and cd assuming
    unique intersection exists
bool line_line_intersection(PT a, PT b, PT c, PT d,
    PT &ans) {
    double a1 = a.y - b.y, b1 = b.x - a.x, c1 =
        cross(a, b);
    double a2 = c.y - d.y, b2 = d.x - c.x, c2 =
        cross(c, d);
    double det = a1 * b2 - a2 * b1;
    if (det == 0) return 0;
    ans = PT((b1 * c2 - b2 * c1) / det, (c1 * a2 -
        a1 * c2) / det);
    return 1;
}
```

3.13 MaximumCircleCover

```
// find a circle of radius r that contains as many
    points as possible
// O(n^2 log n);
double maximum_circle_cover(vector<PT> p, double r,
    circle &c) {
    int n = p.size();
    int ans = 0;
    int id = 0; double th = 0;
    for (int i = 0; i < n; ++i) {
        // maximum circle cover when the circle goes
            through this point
        vector<pair<double, int>> events = {{-PI,
            +1}, {PI, -1}};
        for (int j = 0; j < n; ++j) {
            if (j == i) continue;
            double d = dist(p[i], p[j]);
            if (d > r * 2) continue;
            double dir = (p[j] - p[i]).arg();
            double ang = acos(d / 2 / r);
            double st = dir - ang, ed = dir + ang;
            if (st > PI) st -= PI * 2;
            if (st <= -PI) st += PI * 2;
            if (ed > PI) ed -= PI * 2;
            if (ed <= -PI) ed += PI * 2;
            events.push_back({st - eps, +1}); //
                take care of precisions!
            events.push_back({ed, -1});
            if (st > ed) {
                events.push_back({-PI, +1});
                events.push_back({+PI, -1});
            }
        }
        sort(events.begin(), events.end());
        int cnt = 0;
        for (auto &&e: events) {
            cnt += e.second;
            if (cnt > ans) {
                ans = cnt;
                id = i; th = e.first;
            }
        }
    }
    PT w = PT(p[id].x + r * cos(th), p[id].y + r *
        sin(th));
    c = circle(w, r); //best_circle
    return ans;
}
```

3.14 MaximumInscribedCircle

```
// radius of the maximum inscribed circle in a
    convex polygon
double maximum_inscribed_circle(vector<PT> p) {
    int n = p.size();
    if (n <= 2) return 0;
    double l = 0, r = 20000;
    while (r - l > eps) {
        double mid = (l + r) * 0.5;
        vector<HP> h;
        const int L = 1e9;
        h.push_back(HP(PT(-L, -L), PT(L, -L)));
        h.push_back(HP(PT(L, -L), PT(L, L)));
        h.push_back(HP(PT(L, L), PT(-L, L)));
        h.push_back(HP(PT(-L, L), PT(-L, -L)));
        for (int i = 0; i < n; i++) {
            PT z = (p[(i + 1) % n] - p[i]).perp();
            z = z.truncate(mid);
            PT y = p[i] + z, q = p[(i + 1) % n] + z;
            h.push_back(HP(p[i] + z, p[(i + 1) % n]
                + z));
        }
        vector<PT> nw = half_plane_intersection(h);
        if (!nw.empty()) l = mid;
        else r = mid;
    }
    return l;
}
```

3.15 MinimumEnclosingCircle

```
// given n points, find the minimum enclosing
    circle of the points
// call convex_hull() before this for faster
    solution
// expected O(n)
circle minimum_enclosing_circle(vector<PT> &p) {
    random_shuffle(p.begin(), p.end());
    int n = p.size();
    circle c(p[0], 0);
    for (int i = 1; i < n; i++) {
        if (sign(dist(c.p, p[i]) - c.r) > 0) {
            c = circle(p[i], 0);
            for (int j = 0; j < i; j++) {
                if (sign(dist(c.p, p[j]) - c.r) > 0)
                    {
                    c = circle((p[i] + p[j]) / 2,
                        dist(p[i], p[j]) / 2);
                    for (int k = 0; k < j; k++) {
                        if (sign(dist(c.p, p[k]) -
                            c.r) > 0) {
                            c = circle(p[i], p[j],
                                p[k]);
                        }
                    }
                }
            }
        }
    }
    return c;
}
```

3.16 MinimumEnclosingRectangle

```
// minimum perimeter
double minimum_enclosing_rectangle(vector<PT> &p) {
    int n = p.size();
    if (n <= 2) return perimeter(p);
    int mndot = 0; double tmp = dot(p[1] - p[0],
        p[0]);
    for (int i = 1; i < n; i++) {
        if (dot(p[1] - p[0], p[i]) <= tmp) {
            tmp = dot(p[1] - p[0], p[i]);
            mndot = i;
        }
    }
    double ans = inf;
    int i = 0, j = 1, mxdot = 1;
    while (i < n) {
        PT cur = p[(i + 1) % n] - p[i];
        while (cross(cur, p[(j + 1) % n] - p[j]) >=
            0) j = (j + 1) % n;
        while (dot(p[(mxdot + 1) % n], cur) >=
            dot(p[mxdot], cur)) mxdot = (mxdot + 1)
            % n;
        while (dot(p[(mndot + 1) % n], cur) <=
            dot(p[mndot], cur)) mndot = (mndot + 1)
            % n;
```

```
        ans = min(ans, 2.0 * ((dot(p[mxdot], cur) /
            cur.norm() - dot(p[mndot], cur) /
            cur.norm()) +
            dist_from_point_to_line(p[i], p[(i + 1)
            % n], p[j])));
        i++;
    }
    return ans;
}
```

3.17 MinkowskiSum

```
// a and b are strictly convex polygons of DISTINCT
    points
// returns a convex hull of their minkowski sum
    with distinct points
vector<PT> minkowski_sum(vector<PT> &a, vector<PT>
    &b) {
    int n = (int)a.size(), m = (int)b.size();
    int i = 0, j = 0; //assuming a[i] and b[j] both
        are (left, bottom)-most points
    vector<PT> c;
    c.push_back(a[i] + b[j]);
    while (1) {
        PT p1 = a[i] + b[(j + 1) % m];
        PT p2 = a[(i + 1) % n] + b[j];
        int t = orientation(c.back(), p1, p2);
        if (t >= 0) j = (j + 1) % m;
        if (t <= 0) i = (i + 1) % n, p1 = p2;
        if (t == 0) p1 = a[i] + b[j];
        if (p1 == c[0]) break;
        c.push_back(p1);
    }
    return c;
}
```

3.18 MonotoneChain

```
// warning: different template
vector<Point> convex_hull(vector<Point> p, int n){
    sort(p.begin(), p.end(), [](const Point &A,
        const Point &B){
        return A.x != B.x ? A.x < B.x : A.y < B.y;
    });
    Point st = p[0], en = p[n - 1];
    vector<Point> up = {p[0]};
    vector<Point> down = {p[0]};
    for(int i = 1; i < n; ++i){
        // upper hull
        if(i == n - 1 or cross(st, p[i], en) < 0){
            while((int)up.size() >= 2 and
                cross(up[up.size() - 2], up.back(),
                p[i]) >= 0)
                up.pop_back();
            up.push_back(p[i]);
        }
        // lower hull
        if(i == n - 1 or cross(st, p[i], en) > 0){
            while((int)down.size() >= 2 and
                cross(down[down.size() - 2],
                down.back(), p[i]) <= 0)
                down.pop_back();
            down.push_back(p[i]);
        }
    }
    p.clear();
    for(int i = 0; i < (int)up.size(); ++i)
        p.push_back(up[i]);
    for(int i = down.size() - 2; i >= 1; --i)
        p.push_back(down[i]);
    // return hull in clockwise order
    return p;
}
```

3.19 Point2D

```
const double inf = 1e100;
const double eps = 1e-9;
const double PI = acos((double)-1.0);
int sign(double x) { return (x > eps) - (x < -eps);
    }
struct PT {
    double x, y;
    PT() { x = 0, y = 0; }
    PT(double x, double y) : x(x), y(y) {}
    PT(const PT &p) : x(p.x), y(p.y) {}
    PT operator + (const PT &a) const { return PT(x
        + a.x, y + a.y); }
    PT operator - (const PT &a) const { return PT(x
        - a.x, y - a.y); }
    PT operator * (const double a) const { return
        PT(x * a, y * a); }
    friend PT operator * (const double &a, const PT
        &b) { return PT(a * b.x, a * b.y); }
    PT operator / (const double a) const { return
        PT(x / a, y / a); }
    bool operator == (PT a) const { return sign(a.x
        - x) == 0 && sign(a.y - y) == 0; }
    bool operator != (PT a) const { return !(*this
        == a); }
    bool operator < (PT a) const { return sign(a.x
        - x) == 0 ? y < a.y : x < a.x; }
    bool operator > (PT a) const { return sign(a.x
        - x) == 0 ? y > a.y : x > a.x; }
    double norm() { return sqrt(x * x + y * y); }
    double norm2() { return x * x + y * y; }
    PT perp() { return PT(-y, x); }
    double arg() { return atan2(y, x); }
    PT truncate(double r) { // returns a vector
        with norm r and having same direction
        double k = norm();
        if (!sign(k)) return *this;
        r /= k;
        return PT(x * r, y * r);
    }
};
inline double dot(PT a, PT b) { return a.x * b.x +
    a.y * b.y; }
inline double dist2(PT a, PT b) { return dot(a - b,
    a - b); }
inline double dist(PT a, PT b) { return sqrt(dot(a
    - b, a - b)); }
inline double cross(PT a, PT b) { return a.x * b.y
    - a.y * b.x; }
inline double cross2(PT a, PT b, PT c) { return
    cross(b - a, c - a); }
inline int orientation(PT a, PT b, PT c) { return
    sign(cross(b - a, c - a)); }
PT perp(PT a) { return PT(-a.y, a.x); }
PT rotateccw90(PT a) { return PT(-a.y, a.x); }
PT rotatecw90(PT a) { return PT(a.y, -a.x); }
PT rotateccw(PT a, double t) { return PT(a.x *
    cos(t) - a.y * sin(t), a.x * sin(t) + a.y *
    cos(t)); }
PT rotatecw(PT a, double t) { return PT(a.x *
    cos(t) + a.y * sin(t), -a.x * sin(t) + a.y *
    cos(t)); }
double SQ(double x) { return x * x; }
double rad_to_deg(double r) { return (r * 180.0 /
    PI); }
double deg_to_rad(double d) { return (d * PI /
    180.0); }
double get_angle(PT a, PT b) {
    double costheta = dot(a, b) / a.norm() /
        b.norm();
    return acos(max((double)-1.0, min((double)1.0,
        costheta)));
}
bool is_point_in_angle(PT b, PT a, PT c, PT p) { //
    does point p lie in angle <bac
    assert(orientation(a, b, c) != 0);
    if (orientation(a, c, b) < 0) swap(b, c);
    return orientation(a, c, p) >= 0 &&
        orientation(a, b, p) <= 0;
}
```

3.20 PointInsideHull

```
bool on_segment(const Point &A, const Point &B,
     const Point &C) { return cross(A, B, C) == 0
     and dot(C, A, B) <= 0; }

bool check(vector<Point> &hull, Point &a) {
  int n = sz(hull);
  if (n == 1) return hull[0] == a;
  if (n == 2) return on_segment(hull[0], hull[1],
       a);
  if (cross(hull[0], hull[1], a) > 0) return 0;
  if (cross(hull[n - 1], hull[0], a) >= 0) return
       on_segment(hull[n - 1], hull[0], a);
  int l = 2, r = n - 1, ans = -1;
  while (l <= r) {
    int mid = (l + r) / 2;
    if (cross(hull[0], hull[mid], a) >= 0) {
      ans = mid;
      r = mid - 1;
    } else
      l = mid + 1;
  }
  debug(hull[0], hull[ans - 1], hull[ans], a, ans);
  return cross(hull[ans - 1], hull[ans], a) < 0 or
       on_segment(hull[ans - 1], hull[ans], a);
}
```

3.21 PointPolygonTangents

```
pair<PT, PT> convex_line_intersection(vector<PT>
     &p, PT a, PT b) {
    return {{0, 0}, {0, 0}};
}

pair<PT, int> point_poly_tangent(vector<PT> &p, PT
     Q, int dir, int l, int r) {
    while (r - l > 1) {
        int mid = (l + r) >> 1;
        bool pvs = orientation(Q, p[mid], p[mid -
             1]) != -dir;
        bool nxt = orientation(Q, p[mid], p[mid +
             1]) != -dir;
        if (pvs && nxt) return {p[mid], mid};
        if (!(pvs || nxt)) {
            auto p1 = point_poly_tangent(p, Q, dir,
                 mid + 1, r);
            auto p2 = point_poly_tangent(p, Q, dir,
                 l, mid - 1);
            return orientation(Q, p1.first,
                 p2.first) == dir ? p1 : p2;
        }
        if (!pvs) {
            if (orientation(Q, p[mid], p[l]) == dir)
                 r = mid - 1;
            else if (orientation(Q, p[l], p[r]) ==
                 dir) r = mid - 1;
            else l = mid + 1;
        }
        if (!nxt) {
            if (orientation(Q, p[mid], p[l]) == dir)
                 l = mid + 1;
            else if (orientation(Q, p[l], p[r]) ==
                 dir) r = mid - 1;
            else l = mid + 1;
        }
    }
    pair<PT, int> ret = {p[l], l};
    for (int i = l + 1; i <= r; i++) ret =
         orientation(Q, ret.first, p[i]) != dir ?
         make_pair(p[i], i) : ret;
    return ret;
}
// (cw, ccw) tangents from a point that is outside
     this convex polygon
// returns indexes of the points
pair<int, int>
     tangents_from_point_to_polygon(vector<PT> &p,
     PT Q){
    int cw = point_poly_tangent(p, Q, 1, 0,
         (int)p.size() - 1).second;
    int ccw = point_poly_tangent(p, Q, -1, 0,
         (int)p.size() - 1).second;
    return make_pair(cw, ccw);
}
```

3.22 PolarSort

```
bool half(PT p) {
    return p.y > 0.0 || (p.y == 0.0 && p.x < 0.0);
}
void polar_sort(vector<PT> &v) { // sort points in
     counterclockwise
    sort(v.begin(), v.end(), [](PT a,PT b) {
        return make_tuple(half(a), 0.0, a.norm2()) <
             make_tuple(half(b), cross(a, b),
             b.norm2());
    });
}
void polar_sort(vector<PT> &v, PT o) { // sort
     points in counterclockwise with respect to
     point o
    sort(v.begin(), v.end(), [&](PT a,PT b) {
        return make_tuple(half(a - o), 0.0, (a -
             o).norm2()) < make_tuple(half(b - o),
             cross(a - o, b - o), (b - o).norm2());
    });
}
```

3.23 PolygonCircleIntersection

```
// intersection between a simple polygon and a
     circle
double polygon_circle_intersection(vector<PT> &v,
     PT p, double r) {
    int n = v.size();
    double ans = 0.00;
    PT org = {0, 0};
    for(int i = 0; i < n; i++) {
        int x = orientation(p, v[i], v[(i + 1) % n]);
        if(x == 0) continue;
        double area =
             triangle_circle_intersection(org, r,
             v[i] - p, v[(i + 1) % n] - p);
        if (x < 0) ans -= area;
        else ans += area;
    }
    return abs(ans);
}
```

3.24 PolygonCut

```
// returns a vector with the vertices of a polygon
     with everything
// to the left of the line going from a to b cut
     away.
vector<PT> cut(vector<PT> &p, PT a, PT b) {
    vector<PT> ans;
    int n = (int)p.size();
    for (int i = 0; i < n; i++) {
        double c1 = cross(b - a, p[i] - a);
        double c2 = cross(b - a, p[(i + 1) % n] - a);
        if (sign(c1) >= 0) ans.push_back(p[i]);
        if (sign(c1 * c2) < 0) {
            if (!is_parallel(p[i], p[(i + 1) % n],
                 a, b)) {
                PT tmp; line_line_intersection(p[i],
                     p[(i + 1) % n], a, b, tmp);
                ans.push_back(tmp);
            }
        }
    }
    return ans;
```

```
}
```

3.25 PolygonDiameter

```
// Maximum distance of 2 points
double diameter(vector<PT> &p) {
    int n = (int)p.size();
    if (n == 1) return 0;
    if (n == 2) return dist(p[0], p[1]);
    double ans = 0;
    int i = 0, j = 1;
    while (i < n) {
        while (cross(p[(i + 1) % n] - p[i], p[(j +
             1) % n] - p[j]) >= 0) {
            ans = max(ans, dist2(p[i], p[j]));
            j = (j + 1) % n;
        }
        ans = max(ans, dist2(p[i], p[j]));
        i++;
    }
    return sqrt(ans);
}
```

3.26 PolygonDistances

```
// minimum distance from a point to a convex polygon
// it assumes point lie strictly outside the polygon
double dist_from_point_to_polygon(vector<PT> &p, PT
     z) {
    double ans = inf;
    int n = p.size();
    if (n <= 3) {
        for(int i = 0; i < n; i++) ans = min(ans,
             dist_from_point_to_seg(p[i], p[(i + 1)
             % n], z));
        return ans;
    }
    auto [r, l] = tangents_from_point_to_polygon(p,
         z);
    if(l > r) r += n;
    while (l < r) {
        int mid = (l + r) >> 1;
        double left = dist2(p[mid % n], z), right=
             dist2(p[(mid + 1) % n], z);
        ans = min({ans, left, right});
        if(left < right) r = mid;
        else l = mid + 1;
    }
    ans = sqrt(ans);
    ans = min(ans, dist_from_point_to_seg(p[l % n],
         p[(l + 1) % n], z));
    ans = min(ans, dist_from_point_to_seg(p[l % n],
         p[(l - 1 + n) % n], z));
    return ans;
}
// minimum distance from convex polygon p to line ab
// returns 0 is it intersects with the polygon
// top - upper right vertex
double dist_from_polygon_to_line(vector<PT> &p, PT
     a, PT b, int top) { //O(log n)
    PT orth = (b - a).perp();
    if (orientation(a, b, p[0]) > 0) orth = (a -
         b).perp();
    int id = extreme_vertex(p, orth, top);
    if (dot(p[id] - a, orth) > 0) return 0.0; //if
         orth and a are in the same half of the
         line, then poly and line intersects
    return dist_from_point_to_line(a, b, p[id]);
         //does not intersect
}
// minimum distance from a convex polygon to
     another convex polygon
// the polygon doesnot overlap or touch
// tested in https://toph.co/p/the-wall
double dist_from_polygon_to_polygon(vector<PT> &p1,
     vector<PT> &p2) { // O(n log n)
    double ans = inf;
    for (int i = 0; i < p1.size(); i++) {
        ans = min(ans,
             dist_from_point_to_polygon(p2, p1[i]));
    }
    for (int i = 0; i < p2.size(); i++) {
        ans = min(ans,
             dist_from_point_to_polygon(p1, p2[i]));
    }
    return ans;
}
// maximum distance from a convex polygon to
     another convex polygon
double
     maximum_dist_from_polygon_to_polygon(vector<PT>
     &u, vector<PT> &v){ //O(n)
    int n = (int)u.size(), m = (int)v.size();
    double ans = 0;
    if (n < 3 || m < 3) {
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < m; j++) ans =
                 max(ans, dist2(u[i], v[j]));
        }
        return sqrt(ans);
    }
    if (u[0].x > v[0].x) swap(n, m), swap(u, v);
    int i = 0, j = 0, step = n + m + 10;
    while (j + 1 < m && v[j].x < v[j + 1].x) j++ ;
    while (step--) {
        if (cross(u[(i + 1)%n] - u[i], v[(j + 1)%m]
             - v[j]) >= 0) j = (j + 1) % m;
        else i = (i + 1) % n;
        ans = max(ans, dist2(u[i], v[j]));
    }
    return sqrt(ans);
}
```

3.27 PolygonLineIntersection

```
// not necessarily convex, boundary is included in
     the intersection
// returns total intersected length
double polygon_line_intersection(vector<PT> p, PT
     a, PT b) {
    int n = p.size();
    p.push_back(p[0]);
    line l = line(a, b);
    double ans = 0.0;
    vector< pair<double, int> > vec;
    for (int i = 0; i < n; i++) {
        int s1 = sign(cross(b - a, p[i] - a));
        int s2 = sign(cross(b - a, p[i+1] - a));
        if (s1 == s2) continue;
        line t = line(p[i], p[i + 1]);
        PT inter = (t.v * l.c - l.v * t.c) /
             cross(l.v, t.v);
        double tmp = dot(inter, l.v);
        int f;
        if (s1 > s2) f = s1 && s2 ? 2 : 1;
        else f = s1 && s2 ? -2 : -1;
        vec.push_back(make_pair(tmp, f));
    }
    sort(vec.begin(), vec.end());
    for (int i = 0, j = 0; i + 1 < (int)vec.size();
         i++){
        j += vec[i].second;
        if (j) ans += vec[i + 1].first -
             vec[i].first;
    }
    ans = ans / sqrt(dot(l.v, l.v));
    p.pop_back();
    return ans;
}
```

3.28 PolygonUnion

```
// calculates the area of the union of n polygons
     (not necessarily convex).
```

```
// the points within each polygon must be given in
     CCW order.
// complexity: O(N^2), where N is the total number
     of points
double rat(PT a, PT b, PT p) {
        return !sign(a.x - b.x) ? (p.y - a.y) / (b.y
             - a.y) : (p.x - a.x) / (b.x - a.x);
 };
double polygon_union(vector<vector<PT>> &p) {
    int n = p.size();
    double ans=0;
    for(int i = 0; i < n; ++i) {
        for (int v = 0; v < (int)p[i].size(); ++v) {
            PT a = p[i][v], b = p[i][(v + 1) %
                 p[i].size()];
            vector<pair<double, int>> segs;
            segs.emplace_back(0, 0),
                 segs.emplace_back(1, 0);
            for(int j = 0; j < n; ++j) {
                if(i != j) {
                    for(size_t u = 0; u <
                         p[j].size(); ++u) {
                        PT c = p[j][u], d = p[j][(u +
                             1) % p[j].size()];
                        int sc = sign(cross(b - a, c
                             - a)), sd = sign(cross(b
                             - a, d - a));
                        if(!sc && !sd) {
                            if(sign(dot(b - a, d - c))
                                 > 0 && i > j) {
                                segs.emplace_back(rat(a,
                                     b, c), 1),
                                     segs.emplace_back(rat(a,
                                     b, d), -1);
                            }
                        }
                        else {
                            double sa = cross(d - c, a
                                 - c), sb = cross(d -
                                 c, b - c);
                            if(sc >= 0 && sd < 0)
                                 segs.emplace_back(sa
                                 / (sa - sb), 1);
                            else if(sc < 0 && sd >= 0)
                                 segs.emplace_back(sa
                                 / (sa - sb), -1);
                        }
                    }
                }
            }
            sort(segs.begin(), segs.end());
            double pre = min(max(segs[0].first,
                 0.0), 1.0), now, sum = 0;
            int cnt = segs[0].second;
            for(int j = 1; j < segs.size(); ++j) {
                now = min(max(segs[j].first, 0.0),
                     1.0);
                if (!cnt) sum += now - pre;
                cnt += segs[j].second;
                pre = now;
            }
            ans += cross(a, b) * sum;
        }
    }
    return ans * 0.5;
}
```

3.29 PolygonWidth

```
// Maximum distance between 2 points IN the polygon
double width(vector<PT> &p) {
    int n = (int)p.size();
    if (n <= 2) return 0;
    double ans = inf;
    int i = 0, j = 1;
    while (i < n) {
        while (cross(p[(i + 1) % n] - p[i], p[(j +
             1) % n] - p[j]) >= 0) j = (j + 1) % n;
        ans = min(ans, dist_from_point_to_line(p[i],
             p[(i + 1) % n], p[j]));
        i++;
    }
    return ans;
}
```

3.30 Ray

```
// minimum distance from point c to ray (starting
     point a and direction vector b)
double dist_from_point_to_ray(PT a, PT b, PT c) {
    b = a + b;
    double r = dot(c - a, b - a);
    if (r < 0.0) return dist(c, a);
    return dist_from_point_to_line(a, b, c);
}
// starting point as and direction vector ad
bool ray_ray_intersection(PT as, PT ad, PT bs, PT
     bd) {
    double dx = bs.x - as.x, dy = bs.y - as.y;
    double det = bd.x * ad.y - bd.y * ad.x;
    if (fabs(det) < eps) return 0;
    double u = (dy * bd.x - dx * bd.y) / det;
    double v = (dy * ad.x - dx * ad.y) / det;
    if (sign(u) >= 0 && sign(v) >= 0) return 1;
    else return 0;
}
double ray_ray_distance(PT as, PT ad, PT bs, PT bd)
     {
    if (ray_ray_intersection(as, ad, bs, bd))
         return 0.0;
    double ans = dist_from_point_to_ray(as, ad, bs);
    ans = min(ans, dist_from_point_to_ray(bs, bd,
         as));
    return ans;
}
```

3.31 Segment

```
// returns true if point p is on line segment ab
bool is_point_on_seg(PT a, PT b, PT p) {
    if (fabs(cross(p - b, a - b)) < eps) {
        if (p.x < min(a.x, b.x) || p.x > max(a.x,
             b.x)) return false;
        if (p.y < min(a.y, b.y) || p.y > max(a.y,
             b.y)) return false;
        return true;
    }
    return false;
}
// minimum distance point from point c to segment
     ab that lies on segment ab
PT project_from_point_to_seg(PT a, PT b, PT c) {
    double r = dist2(a, b);
    if (sign(r) == 0) return a;
    r = dot(c - a, b - a) / r;
    if (r < 0) return a;
    if (r > 1) return b;
    return a + (b - a) * r;
}
// minimum distance from point c to segment ab
double dist_from_point_to_seg(PT a, PT b, PT c) {
    return dist(c, project_from_point_to_seg(a, b,
         c));
}
// intersection point between segment ab and
     segment cd assuming unique intersection exists
bool seg_seg_intersection(PT a, PT b, PT c, PT d,
     PT &ans) {
    double oa = cross2(c, d, a), ob = cross2(c, d,
         b);
    double oc = cross2(a, b, c), od = cross2(a, b,
         d);
    if (oa * ob < 0 && oc * od < 0){
        ans = (a * ob - b * oa) / (ob - oa);
        return 1;
    }
    else return 0;
```

```
}
// intersection point between segment ab and
     segment cd assuming unique intersection may
     not exists
// se.size()==0 means no intersection
// se.size()==1 means one intersection
// se.size()==2 means range intersection
set<PT> seg_seg_intersection_inside(PT a, PT b, PT
     c, PT d) {
    PT ans;
    if (seg_seg_intersection(a, b, c, d, ans))
         return {ans};
    set<PT> se;
    if (is_point_on_seg(c, d, a)) se.insert(a);
    if (is_point_on_seg(c, d, b)) se.insert(b);
    if (is_point_on_seg(a, b, c)) se.insert(c);
    if (is_point_on_seg(a, b, d)) se.insert(d);
    return se;
}
// intersection between segment ab and line cd
// 0 if do not intersect, 1 if proper intersect, 2
     if segment intersect
int seg_line_relation(PT a, PT b, PT c, PT d) {
    double p = cross2(c, d, a);
    double q = cross2(c, d, b);
    if (sign(p) == 0 && sign(q) == 0) return 2;
    else if (p * q < 0) return 1;
    else return 0;
}
// intersection between segament ab and line cd
     assuming unique intersection exists
bool seg_line_intersection(PT a, PT b, PT c, PT d,
     PT &ans) {
    bool k = seg_line_relation(a, b, c, d);
    assert(k != 2);
    if (k) line_line_intersection(a, b, c, d, ans);
    return k;
}
// minimum distance from segment ab to segment cd
double dist_from_seg_to_seg(PT a, PT b, PT c, PT d)
     {
    PT dummy;
    if (seg_seg_intersection(a, b, c, d, dummy))
         return 0.0;
    else return min({dist_from_point_to_seg(a, b,
        c), dist_from_point_to_seg(a, b, d),
        dist_from_point_to_seg(c, d, a),
             dist_from_point_to_seg(c, d, b)});
}
```

3.32 SmallestEnclosingCircle

```
double eps = 1e-9;
using Point = complex<double>;
struct Circle{ Point p; double r; };
double dist(Point p, Point q){ return abs(p-q); }
double area2(Point p, Point q){ return
     (conj(p)*q).imag();}
bool in(const Circle& c, Point p){ return dist(c.p,
     p) < c.r + eps; }
Circle INVAL = Circle{Point(0, 0), -1};
Circle mCC(Point a, Point b, Point c){
        b -= a; c -= a;
        double d = 2*(conj(b)*c).imag();
             if(abs(d)<eps) return INVAL;
        Point ans = (c*norm(b) - b*norm(c)) *
             Point(0, -1) / d;
        return Circle{a + ans, abs(ans)};
}
Circle solve(vector<Point> p) {
        mt19937 gen(0x94949); shuffle(p.begin(),
             p.end(), gen);
        Circle c = INVAL;
        for(int i=0; i<p.size(); ++i) if(c.r<0
             ||!in(c, p[i])){
                c = Circle{p[i], 0};
                for(int j=0; j<=i; ++j) if(!in(c,
                     p[j])){
                        Circle ans{(p[i]+p[j])*0.5,
                             dist(p[i], p[j])*0.5};
                        if(c.r == 0) {c = ans;
                             continue;}
                        Circle l, r; l = r = INVAL;
                        Point pq = p[j]-p[i];
                        for(int k=0; k<=j; ++k)
                             if(!in(ans, p[k])) {
                                double a2 = area2(pq,
                                     p[k]-p[i]);
                                Circle c = mCC(p[i],
                                     p[j], p[k]);
                                if(c.r<0) continue;
                                else if(a2 > 0 &&
                                     (l.r<0||area2(pq,
                                     c.p-p[i]) >
                                     area2(pq,
                                     l.p-p[i]))) l = c;
                                else if(a2 < 0 &&
                                     (r.r<0||area2(pq,
                                     c.p-p[i]) <
                                     area2(pq,
                                     r.p-p[i]))) r = c;
                        }
                        if(l.r<0&&r.r<0) c = ans;
                        else if(l.r<0) c = r;
                        else if(r.r<0) c = l;
                        else c = l.r<=r.r?l:r;
                }
        }
        return c;
}
```

3.33 TriangleCircleIntersection

```
// system should be translated from circle center
double triangle_circle_intersection(PT c, double r,
     PT a, PT b) {
    double sd1 = dist2(c, a), sd2 = dist2(c, b);
    if(sd1 > sd2) swap(a, b), swap(sd1, sd2);
    double sd = dist2(a, b);
    double d1 = sqrtl(sd1), d2 = sqrtl(sd2), d =
         sqrt(sd);
    double x = abs(sd2 - sd - sd1) / (2 * d);
    double h = sqrtl(sd1 - x * x);
    if(r >= d2) return h * d / 2;
    double area = 0;
    if(sd + sd1 < sd2) {
        if(r < d1) area = r * r * (acos(h / d2) -
             acos(h / d1)) / 2;
        else {
            area = r * r * ( acos(h / d2) - acos(h /
                 r)) / 2;
            double y = sqrtl(r * r - h * h);
            area += h * (y - x) / 2;
        }
    }
    else {
        if(r < h) area = r * r * (acos(h / d2) +
             acos(h / d1)) / 2;
        else {
            area += r * r * (acos(h / d2) - acos(h /
                 r)) / 2;
            double y = sqrtl(r * r - h * h);
            area += h * y / 2;
            if(r < d1) {
                area += r * r * (acos(h / d1) -
                     acos(h / r)) / 2;
                area += h * y / 2;
            }
            else area += h * x / 2;
        }
    }
    return area;
}
```

3.34 Utilities

```
double perimeter(vector<PT> &p) {
```

```
    double ans=0; int n = p.size();
    for (int i = 0; i < n; i++) ans += dist(p[i],
        p[(i + 1) % n]);
    return ans;
}
double area(vector<PT> &p) {
    double ans = 0; int n = p.size();
    for (int i = 0; i < n; i++) ans += cross(p[i],
        p[(i + 1) % n]);
    return fabs(ans) * 0.5;
}
double area_of_triangle(PT a, PT b, PT c) {
    return fabs(cross(b - a, c - a) * 0.5);
}
// 0 if cw, 1 if ccw
bool get_direction(vector<PT> &p) {
    double ans = 0; int n = p.size();
    for (int i = 0; i < n; i++) ans += cross(p[i],
        p[(i + 1) % n]);
    if (sign(ans) > 0) return 1;
    return 0;
}
// find a point from a through b with distance d
PT point_along_line(PT a, PT b, double d) {
    assert(a != b);
    return a + (((b - a) / (b - a).norm()) * d);
}
// projection point c onto line through a and b
    assuming a != b
PT project_from_point_to_line(PT a, PT b, PT c) {
    return a + (b - a) * dot(c - a, b - a) / (b -
        a).norm2();
}
// reflection point c onto line through a and b
    assuming a != b
PT reflection_from_point_to_line(PT a, PT b, PT c) {
    PT p = project_from_point_to_line(a,b,c);
    return p + p - c;
}
// minimum distance from point c to line through a
    and b
double dist_from_point_to_line(PT a, PT b, PT c) {
    return fabs(cross(b - a, c - a) / (b -
        a).norm());
}
// 0 if not parallel, 1 if parallel, 2 if collinear
int is_parallel(PT a, PT b, PT c, PT d) {
    double k = fabs(cross(b - a, d - c));
    if (k < eps){
        if (fabs(cross(a - b, a - c)) < eps &&
            fabs(cross(c - d, c - a)) < eps) return
            2;
        else return 1;
    }
    else return 0;
}
// check if two lines are same
bool are_lines_same(PT a, PT b, PT c, PT d) {
    if (fabs(cross(a - c, c - d)) < eps &&
        fabs(cross(b - c, c - d)) < eps) return
        true;
    return false;
}

// 1 if point is ccw to the line, 2 if point is cw
    to the line, 3 if point is on the line
int point_line_relation(PT a, PT b, PT p) {
    int c = sign(cross(p - a, b - a));
    if (c < 0) return 1;
    if (c > 0) return 2;
    return 3;
}
```

4 Graph

4.1 2pac

```
struct TwoSatSolver {
  int n_vars;
  int n_vertices;
  vector<vector<int>> adj, adj_t;
  vector<bool> used;
  vector<int> order, comp;
  vector<bool> assignment;
  TwoSatSolver(int _n_vars)
      : n_vars(_n_vars),
        n_vertices(2 * n_vars),
        adj(n_vertices),
        adj_t(n_vertices),
        used(n_vertices),
        order(),
        comp(n_vertices, -1),
        assignment(n_vars) {
    order.reserve(n_vertices);
  }
  void dfs1(int v) {
    used[v] = true;
    for (int u : adj[v]) {
      if (!used[u]) dfs1(u);
    }
    order.push_back(v);
  }
  void dfs2(int v, int cl) {
    comp[v] = cl;
    for (int u : adj_t[v]) {
      if (comp[u] == -1) dfs2(u, cl);
    }
  }
  bool solve_2SAT() {
    order.clear();
    used.assign(n_vertices, false);
    for (int i = 0; i < n_vertices; ++i) {
      if (!used[i]) dfs1(i);
    }

    comp.assign(n_vertices, -1);
    for (int i = 0, j = 0; i < n_vertices; ++i) {
      int v = order[n_vertices - i - 1];
      if (comp[v] == -1) dfs2(v, j++);
    }

    assignment.assign(n_vars, false);
    for (int i = 0; i < n_vertices; i += 2) {
      if (comp[i] == comp[i + 1]) return false;
      assignment[i / 2] = comp[i] > comp[i + 1];
    }
    return true;
  }
  void add_disjunction(int a, bool na, int b, bool
      nb) {
    // na and nb signify whether a and b are to be
        negated
    a = 2 * a ^ na;
    b = 2 * b ^ nb;
    int neg_a = a ^ 1;
    int neg_b = b ^ 1;
    adj[neg_a].push_back(b);
    adj[neg_b].push_back(a);
    adj_t[b].push_back(neg_a);
    adj_t[a].push_back(neg_b);
  }
  static void example_usage() {
    TwoSatSolver solver(3);                 // a,
        b, c
    solver.add_disjunction(0, false, 1, true); //
        a  v  not b
    solver.add_disjunction(0, true, 1, true); //
        not a v not b
    solver.add_disjunction(1, false, 2, false); //
        b  v     c
    solver.add_disjunction(0, false, 0, false); //
        a  v     a
    assert(solver.solve_2SAT() == true);
    auto expected = vector<bool>(True, False, True);
    assert(solver.assignment == expected);
  }
};
```

4.2 BiconnectedComponents

```
struct BiconnectedComponent {
  vector<int> low, num, s;
  vector<vector<int> > components;
  int counter;

  BiconnectedComponent() : low(n, -1), num(n, -1),
      counter(0) {
    for (int i = 0; i < n; i++)
      if (num[i] < 0) dfs(i, 1);
  }

  void dfs(int x, int isRoot) {
    low[x] = num[x] = ++counter;
    if (g[x].empty()) {
      components.push_back(vector<int>(1, x));
      return;
    }
    s.push_back(x);

    for (int i = 0; i < (int)g[x].size(); i++) {
      int y = g[x][i];
      if (num[y] > -1)
        low[x] = min(low[x], num[y]);
      else {
        dfs(y, 0);
        low[x] = min(low[x], low[y]);

        if (isRoot || low[y] >= num[x]) {
          components.push_back(vector<int>(1, x));
          while (1) {
            int u = s.back();
            s.pop_back();
            components.back().push_back(u);
            if (u == y) break;
          }
        }
      }
    }
  }
};
```

4.3 Dinic

```
const ll INF = 1e18;
struct Dinic {
  const static bool SCALING = false; // scaling =
      EV log(max C) with larger constant
  ll lim = 1;
  struct Edge {
    int u, v;
    ll cap, flow;
  };
  int n, s, t;
  vector<int> level, ptr;
  vector<Edge> e;
  vector<vector<int>> g;
  Dinic(int _n) : n(_n), level(_n), ptr(_n), g(_n) {
    e.clear();
    for (int i = 0; i < n; ++i) {
      ptr[i] = 0;
      g[i].clear();
    }
  }
  void add_edge(int u, int v, ll c) {
    debug(u, v, c);
    g[u].push_back(sz(e));
    e.push_back({u, v, c, 0});
    g[v].push_back(sz(e));
    e.push_back({v, u, 0, 0});
  }
  ll get_max_flow(int _s, int _t) {
    s = _s, t = _t;
    ll flow = 0;
    for (lim = SCALING ? (1 << 30) : 1; lim > 0;
        lim >>= 1) {
      while (1) {
        if (!bfs()) break;
        fill(all(ptr), 0);
        while (ll pushed = dfs(s, INF)) flow +=
            pushed;
      }
    }
    return flow;
  }
 private:
  bool bfs() {
    queue<int> q;
    q.push(s);
    fill(all(level), -1);
    level[s] = 0;
    while (!q.empty()) {
      int u = q.front();
      q.pop();
      for (int id : g[u]) {
        if (e[id].cap - e[id].flow < 1) continue;
        if (level[e[id].v] != -1) continue;
        if (SCALING and e[id].cap - e[id].flow <
            lim) continue;
        level[e[id].v] = level[u] + 1;
        q.push(e[id].v);
      }
    }
    return level[t] != -1;
  }
  ll dfs(int u, ll flow) {
    if (!flow) return 0;
    if (u == t) return flow;
    for (; ptr[u] < sz(g[u]); ++ptr[u]) {
      int id = g[u][ptr[u]], to = e[id].v;
      if (level[to] != level[u] + 1) continue;
      ll pushed = dfs(to, min(flow, e[id].cap -
          e[id].flow));
      if (pushed) {
        e[id].flow += pushed;
        e[id ^ 1].flow -= pushed;
        return pushed;
      }
    }
    return 0;
  }
};
```

4.4 EulerPath

```
struct EulerUndirected {
  EulerUndirected(int _n) : n(_n), m(0), adj(_n),
      deg(_n, 0) {}
  void add_edge(int u, int v) {
    adj[u].push_front(Edge(v));
    auto it1 = adj[u].begin();
    adj[v].push_front(Edge(u));
    auto it2 = adj[v].begin();

    it1->rev = it2;
    it2->rev = it1;

    ++deg[u];
    ++deg[v];
    ++m;
  }
  std::pair<bool, std::vector<int>> solve() {
    int cntOdd = 0;
    int start = -1;
    for (int i = 0; i < n; i++) {
      if (deg[i] % 2) {
        ++cntOdd;
        if (cntOdd > 2) return {false, {}};

        if (start < 0) start = i;
      }
    }
    // no odd vertex -> start from any vertex with
        positive degree
    if (start < 0) {
      for (int i = 0; i < n; i++) {
        if (deg[i]) {
          start = i;
          break;
        }
```

```
    }
    if (start < 0) {
      // no edge -> empty path
      return {true, {}};
    }
  }

  std::vector<int> path;
  find_path(start, path);

  if (m + 1 != static_cast<int>(path.size())) {
    return {false, {}};
  }

  return {true, path};
 }
 struct Edge {
  int to;
  std::list<Edge>::iterator rev;

  Edge(int _to) : to(_to) {}
 };
 // private:
 int n, m;
 std::vector<std::list<Edge>> adj;
 std::vector<int> deg;

 void find_path(int v, std::vector<int>& path) {
  while (adj[v].size() > 0) {
    int next = adj[v].front().to;
    adj[next].erase(adj[v].front().rev);
    adj[v].pop_front();
    find_path(next, path);
  }
  path.push_back(v);
 }
};
```

4.5 EulerPathDirected

```
struct EulerDirected {
 EulerDirected(int _n) : n(_n), adj(n), in_deg(n,
      0), out_deg(n, 0) {}
 void add_edge(int u, int v) { // directed edge
  assert(0 <= u && u < n);
  assert(0 <= v && v < n);
  adj[u].push_front(v);
  in_deg[v]++;
  out_deg[u]++;
 }
 std::pair<bool, std::vector<int>> solve() {
  int start = -1, last = -1;
  for (int i = 0; i < n; i++) {
   // for all u, |in_deg(u) - out_deg(u)| <= 1
   if (std::abs(in_deg[i] - out_deg[i]) > 1)
       return {false, {}};

   if (out_deg[i] > in_deg[i]) {
    // At most 1 vertex with out_deg[u] -
         in_deg[u] = 1 (start vertex)
    if (start >= 0) return {false, {}};
    start = i;
   }

   if (in_deg[i] > out_deg[i]) {
    // At most 1 vertex with in_deg[u] -
         out_deg[u] = 1 (last vertex)
    if (last >= 0) return {false, {}};
    last = i;
   }
  }

  // can start at any vertex with degree > 0
  if (start < 0) {
   for (int i = 0; i < n; i++) {
    if (in_deg[i]) {
     start = i;
     break;
    }
   }
   // no start vertex --> all vertices have
       degree == 0
   if (start < 0) return {true, {}};
  }
  std::vector<int> path;
  find_path(start, path);
  std::reverse(path.begin(), path.end());

  // check that we visited all vertices with
      degree > 0
  std::vector<bool> visited(n, false);
  for (int u : path) visited[u] = true;

  for (int u = 0; u < n; u++) {
   if (in_deg[u] && !visited[u]) {
    return {false, {}};
   }
  }

  return {true, path};
 }
private:
 int n;
 std::vector<std::list<int>> adj;
 std::vector<int> in_deg, out_deg;

 void find_path(int v, std::vector<int>& path) {
  while (adj[v].size() > 0) {
   int next = adj[v].front();
   adj[v].pop_front();
   find_path(next, path);
  }
  path.push_back(v);
 }
};
```

4.6 GeneralMatching

```
const int MAXN = 2020 + 1;
struct GM { // 1-based Vertex index
 int vis[MAXN], par[MAXN], orig[MAXN],
     match[MAXN], aux[MAXN], t, N;
 vector<int> conn[MAXN];
 queue<int> Q;
 void addEdge(int u, int v) {
  conn[u].push_back(v);
  conn[v].push_back(u);
 }
 void init(int n) {
  N = n;
  t = 0;
  for (int i = 0; i <= n; ++i) {
   conn[i].clear();
   match[i] = aux[i] = par[i] = 0;
  }
 }
 void augment(int u, int v) {
  int pv = v, nv;
  do {
   pv = par[v];
   nv = match[pv];
   match[v] = pv;
   match[pv] = v;
   v = nv;
  } while (u != pv);
 }
 int lca(int v, int w) {
  ++t;
  while (true) {
   if (v) {
    if (aux[v] == t) return v;
    aux[v] = t;
    v = orig[par[match[v]]];
   }
   swap(v, w);
  }
 }
 void blossom(int v, int w, int a) {
  while (orig[v] != a) {
```

```
      par[v] = w;
      w = match[v];
      if (vis[w] == 1) Q.push(w), vis[w] = 0;
      orig[v] = orig[w] = a;
      v = par[w];
    }
  }
  bool bfs(int u) {
    fill(vis + 1, vis + 1 + N, -1);
    iota(orig + 1, orig + N + 1, 1);
    Q = queue<int>();
    Q.push(u);
    vis[u] = 0;
    while (!Q.empty()) {
      int v = Q.front();
      Q.pop();
      for (int x : conn[v]) {
        if (vis[x] == -1) {
          par[x] = v;
          vis[x] = 1;
          if (!match[x]) return augment(u, x), true;
          Q.push(match[x]);
          vis[match[x]] = 0;
        } else if (vis[x] == 0 && orig[v] !=
             orig[x]) {
          int a = lca(orig[v], orig[x]);
          blossom(x, v, a);
          blossom(v, x, a);
        }
      }
    }
    return false;
  }
  int Match() {
    int ans = 0;
    // find random matching (not necessary,
         constant improvement)
    vector<int> V(N - 1);
    iota(V.begin(), V.end(), 1);
    shuffle(V.begin(), V.end(), mt19937(0x94949));
    for (auto x : V)
      if (!match[x]) {
        for (auto y : conn[x])
          if (!match[y]) {
            match[x] = y, match[y] = x;
            ++ans;
            break;
          }
      }
    for (int i = 1; i <= N; ++i)
      if (!match[i] && bfs(i)) ++ans;
    return ans;
  }
};
```

4.7 GlobalMinCut

```
pair<int, vi> GetMinCut(vector<vi>& weights) {
  int N = sz(weights);
  vi used(N), cut, best_cut;
  int best_weight = -1;
  for (int phase = N - 1; phase >= 0; phase--) {
    vi w = weights[0], added = used;
    int prev, k = 0;
    rep(i, 0, phase) {
      prev = k;
      k = -1;
      rep(j, 1, N) if (!added[j] && (k == -1 ||
           w[j] > w[k])) k = j;
      if (i == phase - 1) {
        rep(j, 0, N) weights[prev][j] +=
             weights[k][j];
        rep(j, 0, N) weights[j][prev] =
             weights[prev][j];
        used[k] = true;
        cut.push_back(k);
        if (best_weight == -1 || w[k] < best_weight)
             {
          best_cut = cut;
          best_weight = w[k];
        }
      } else {
        rep(j, 0, N) w[j] += weights[k][j];
        added[k] = true;
      }
    }
  }
  return {best_weight, best_cut};
}
```

4.8 HopcroftKarp

```
struct maximum_bipartite_matching {
  int n, m;
  vector<int> matchX, matchY, dist;
  vector<vector<int>> g;
  int matched;
  maximum_bipartite_matching(int _n, int _m)
      : n(_n), m(_m), matchX(n + 1, -1), matchY(m +
           1, -1), dist(n + 1, -1), g(n + 1),
           matched(0) {}

  void add_edge(int u, int v) { g[u].push_back(v); }
  void bfs() {
    queue<int> q;
    for (int i = 0; i < n; ++i) {
      if (matchX[i] == -1)
        q.push(i), dist[i] = 0;
      else
        dist[i] = -1;
    }
    while (!q.empty()) {
      int u = q.front();
      q.pop();
      for (int v : g[u]) {
        if (matchY[v] != -1 and dist[matchY[v]] ==
             -1) {
          dist[matchY[v]] = dist[u] + 1;
          q.push(matchY[v]);
        }
      }
    }
  }
  bool dfs(int u) {
    for (int v : g[u]) {
      if (matchY[v] == -1) {
        matchX[u] = v, matchY[v] = u;
        return 1;
      }
    }
    for (int v : g[u]) {
      if (dist[matchY[v]] == dist[u] + 1 and
           dfs(matchY[v])) {
        matchX[u] = v, matchY[v] = u;
        return 1;
      }
    }
    return 0;
  }

  void match() {
    while (1) {
      bfs();
      int augment = 0;
      for (int i = 0; i < n; ++i)
        if (matchX[i] == -1) augment += dfs(i);
      if (!augment) break;
      matched += augment;
    }
  }
  vector<pii> get_edges() {
    vector<pii> res;
    for (int i = 0; i < n; ++i)
      if (matchX[i] != -1) res.push_back({i,
           matchX[i]});
    return res;
  }
};
```

4.9 KhopCau

```
#include <bits/stdc++.h>

using namespace std;

const int maxN = 10010;

int n, m;
bool joint[maxN];
int timeDfs = 0, bridge = 0;
int low[maxN], num[maxN];
vector <int> g[maxN];

void dfs(int u, int pre) {
  int child = 0; // So luong con truc tiep cua dinh
      u trong cy DFS
  num[u] = low[u] = ++timeDfs;
  for (int v : g[u]) {
    if (v == pre) continue;
    if (!num[v]) {
      dfs(v, u);
      low[u] = min(low[u], low[v]);
      if (low[v] == num[v]) bridge++;
      child++;
      if (u == pre) { // Neu u l dinh goc cua cy DFS
        if (child > 1) joint[u] = true;
      }
      else if (low[v] >= num[u]) joint[u] = true;
    }
    else low[u] = min(low[u], num[v]);
  }
}

int main() {
  cin >> n >> m;
  for (int i = 1; i <= m; i++) {
    int u, v;
    cin >> u >> v;
    g[u].push_back(v);
    g[v].push_back(u);
  }
  for (int i = 1; i <= n; i++)
    if (!num[i]) dfs(i, i);

  int cntJoint = 0;
  for (int i = 1; i <= n; i++) cntJoint += joint[i];

  cout << cntJoint << ' ' << bridge;
}
```

4.10 MCMF

```
#include <bits/extc++.h>
const ll INF = numeric_limits<ll>::max() / 4;
typedef vector<ll> VL;
struct MCMF {
  int N;
  vector<vi> ed, red;
  vector<VL> cap, flow, cost;
  vi seen;
  VL dist, pi;
  vector<pii> par;
  MCMF(int N) : N(N), ed(N), red(N), cap(N, VL(N)),
      flow(cap), cost(cap), seen(N), dist(N),
      pi(N), par(N) {}
  void addEdge(int from, int to, ll cap, ll cost) {
    this->cap[from][to] = cap;
    this->cost[from][to] = cost;
    ed[from].push_back(to);
    red[to].push_back(from);
  }
  void path(int s) {
    fill(all(seen), 0);
    fill(all(dist), INF);
    dist[s] = 0;
    ll di;
    __gnu_pbds::priority_queue<pair<ll, int>> q;
    vector<decltype(q)::point_iterator> its(N);
    q.push({0, s});
    auto relax = [&](int i, ll cap, ll cost, int
        dir) {
      ll val = di - pi[i] + cost;
      if (cap && val < dist[i]) {
        dist[i] = val;
        par[i] = {s, dir};
        if (its[i] == q.end())
          its[i] = q.push({-dist[i], i});
        else
          q.modify(its[i], {-dist[i], i});
      }
    };
    while (!q.empty()) {
      s = q.top().second;
      q.pop();
      seen[s] = 1;
      di = dist[s] + pi[s];
      trav(i, ed[s]) if (!seen[i]) relax(i,
          cap[s][i] - flow[s][i], cost[s][i], 1);
      trav(i, red[s]) if (!seen[i]) relax(i,
          flow[i][s], -cost[i][s], 0);
    }
    rep(i, 0, N) pi[i] = min(pi[i] + dist[i], INF);
  }
  pair<ll, ll> maxflow(int s, int t) {
    ll totflow = 0, totcost = 0;
    while (path(s), seen[t]) {
      ll fl = INF;
      for (int p, r, x = t; tie(p, r) = par[x], x
          != s; x = p) fl = min(fl, r ? cap[p][x] -
          flow[p][x] : flow[x][p]);
      totflow += fl;
      for (int p, r, x = t; tie(p, r) = par[x], x
          != s; x = p)
        if (r)
          flow[p][x] += fl;
        else
          flow[x][p] -= fl;
    }
    rep(i, 0, N) rep(j, 0, N) totcost += cost[i][j]
        * flow[i][j];
    return {totflow, totcost};
  }
  // I f some costs can be negative , call this
      before maxflow:
  void setpi(int s) { // (otherwise , leave this
      out)
    fill(all(pi), INF);
    pi[s] = 0;
    int it = N, ch = 1;
    ll v;
    while (ch-- && it--) rep(i, 0, N) if (pi[i] !=
        INF) trav(to, ed[i]) if (cap[i][to]) if
        ((v = pi[i] + cost[i][to]) < pi[to])
        pi[to] = v, ch = 1;
    assert(it >= 0); // negative cost cycle
  }
};
```

4.11 spfa

```
#include<bits/stdc++.h>
typedef pair<int, int> ii;
const int MaxN = 1e5 + 5;
const int Inf = 1e9;
vector<vector<ii>> AdjList;
int Dist[MaxN];
int Cnt[MaxN];
bool inqueue[MaxN];
int S;
int N;
queue<int> q;

bool spfa() {
    for(int i = 1 ; i <= N ; i++) {
        Dist[i] = Inf;
        Cnt[i] = 0;
        inqueue[i] = false;
    }
```

```
    Dist[S] = 0;
    q.push(S);
    inqueue[S] = true;
    while(!q.empty()) {
        int u = q.front();
        q.pop();
        inqueue[u] = false;

        for (ii tmp: AdjList[u]) {
            int v = tmp.first;
            int w = tmp.second;

            if (Dist[u] + w < Dist[v]) {
                Dist[v] = Dist[u] + w;
                if (!inqueue[v]) {
                    q.push(v);
                    inqueue[v] = true;
                    Cnt[v]++;
                    if (Cnt[v] > N)
                        return false;
                }
            }
        }
    }
    return true;
}
```

4.12 StronglyConnected

```
struct DirectedDfs {
  vector<vector<int>> g;
  int n;
  vector<int> num, low, current, S;
  int counter;
  vector<int> comp_ids;
  vector<vector<int>> scc;

  DirectedDfs(const vector<vector<int>>& _g)
      : g(_g),
        n(g.size()),
        num(n, -1),
        low(n, 0),
        current(n, 0),
        counter(0),
        comp_ids(n, -1) {
    for (int i = 0; i < n; i++) {
      if (num[i] == -1) dfs(i);
    }
  }

  void dfs(int u) {
    low[u] = num[u] = counter++;
    S.push_back(u);
    current[u] = 1;
    for (auto v : g[u]) {
      if (num[v] == -1) dfs(v);
      if (current[v]) low[u] = min(low[u], low[v]);
    }
    if (low[u] == num[u]) {
      scc.push_back(vector<int>());
      while (1) {
        int v = S.back();
        S.pop_back();
        current[v] = 0;
        scc.back().push_back(v);
        comp_ids[v] = ((int)scc.size()) - 1;
        if (u == v) break;
      }
    }
  }

  // build DAG of strongly connected components
  // Returns: adjacency list of DAG
  std::vector<std::vector<int>> build_scc_dag() {
    std::vector<std::vector<int>> dag(scc.size());
    for (int u = 0; u < n; u++) {
      int x = comp_ids[u];
      for (int v : g[u]) {
        int y = comp_ids[v];
        if (x != y) {
          dag[x].push_back(y);
        }
      }
    }
    return dag;
  }
};
```

4.13 TopoSort

```
std::pair<bool, std::vector<int>> topo_sort(const
     std::vector<std::vector<int>>& g) {
  int n = g.size();
  // init in_deg
  std::vector<int> in_deg(n, 0);
  for (int u = 0; u < n; u++) {
    for (int v : g[u]) {
      in_deg[v]++;
    }
  }

  // find topo order
  std::vector<int> res;
  std::queue<int> qu;
  for (int u = 0; u < n; u++) {
    if (in_deg[u] == 0) {
      qu.push(u);
    }
  }

  while (!qu.empty()) {
    int u = qu.front();
    qu.pop();
    res.push_back(u);
    for (int v : g[u]) {
      in_deg[v]--;
      if (in_deg[v] == 0) {
        qu.push(v);
      }
    }
  }

  if ((int)res.size() < n) {
    return {false, {}};
  }
  return {true, res};
}
```

5 Math

5.1 Euclid

```
ll gcd(ll a, ll b) { return __gcd(a, b); }
ll euclid(ll a, ll b, ll &x, ll &y) {
  if (b) {
    ll d = euclid(b, a % b, y, x);
    return y -= a / b * x, d;
  }
  return x = 1, y = 0, a;
}
```

5.2 Factorization

```
inline long long qpow(long long a, int b) {
  long long ans = 1;
  while (b) {
    if (b & 1) ans = ans * a % mod;
    a = a * a % mod;
    b >>= 1;
  }
  return ans;
}
```

```
inline long long rv(int x) { return qpow(x, mod -
    2) % mod; }
bool is_prime(long long n) {
  if (n <= 1) return false;
  for (int a : {2, 3, 5, 13, 19, 73, 193, 407521,
       299210837}) {
    if (n == a) return true;
    if (n % a == 0) return false;
  }
  long long d = n - 1;
  while (!(d & 1)) d >>= 1;
  for (int a : {2, 325, 9375, 28178, 450775,
       9780504, 1795265022}) {
    long long t = d, y = ipow(a, t, n);
    while (t != n - 1 && y != 1 && y != n - 1) y =
        mul(y, y, n), t <<= 1;
    if (y != n - 1 && !(t & 1)) return false;
  }
  return true;
}

long long pollard(long n) {
  auto f = [n](long x) { return mul(x, x, n) + 1; };
  long long x = 0, y = 0, t = 0, prd = 2, i = 1, q;
  while (t++ % 40 || gcd(prd, n) == 1) {
    if (x == y) x = ++i, y = f(x);
    if ((q = mul(prd, max(x, y) - min(x, y), n)))
        prd = q;
    x = f(x), y = f(f(y));
  }
  return gcd(prd, n);
}
vector<long long> factor(long n) {
  if (n == 1) return {};
  if (is_prime(n)) return {n};
  long x = pollard(n);
  auto l = factor(x), r = factor(n / x);
  l.insert(l.end(), r.begin(), r.end());
  return l;
}
```

5.3 FFT

```
using ld = double;
// Can use std::complex<ld> instead to make code
    shorter (but it will be slightly slower)
struct Complex {
  ld x[2];

  Complex() { x[0] = x[1] = 0.0; }
  Complex(ld a) { x[0] = a; }
  Complex(ld a, ld b) {
    x[0] = a;
    x[1] = b;
  }
  Complex(const std::complex<ld>& c) {
    x[0] = c.real();
    x[1] = c.imag();
  }

  Complex conj() const { return Complex(x[0],
      -x[1]); }

  Complex operator+(const Complex& c) const {
    return Complex{
        x[0] + c.x[0],
        x[1] + c.x[1],
    };
  }
  Complex operator-(const Complex& c) const {
    return Complex{
        x[0] - c.x[0],
        x[1] - c.x[1],
    };
  }
  Complex operator*(const Complex& c) const {
      return Complex(x[0] * c.x[0] - x[1] *
      c.x[1], x[0] * c.x[1] + x[1] * c.x[0]); }

  Complex& operator+=(const Complex& c) { return
      *this = *this + c; }
  Complex& operator-=(const Complex& c) { return
      *this = *this - c; }
  Complex& operator*=(const Complex& c) { return
      *this = *this * c; }
};
void fft(vector<Complex>& a) {
  int n = a.size();
  int L = 31 - __builtin_clz(n);
  static vector<Complex> R(2, 1);
  static vector<Complex> rt(2, 1);
  for (static int k = 2; k < n; k *= 2) {
    R.resize(n);
    rt.resize(n);
    auto x = Complex(polar(ld(1.0), acos(ld(-1.0))
        / k));
    for (int i = k; i < 2 * k; ++i) {
      rt[i] = R[i] = i & 1 ? R[i / 2] * x : R[i /
          2];
    }
  }
  vector<int> rev(n);
  for (int i = 0; i < n; ++i) rev[i] = (rev[i / 2]
      | (i & 1) << L) / 2;
  for (int i = 0; i < n; ++i)
    if (i < rev[i]) swap(a[i], a[rev[i]]);
  for (int k = 1; k < n; k *= 2) {
    for (int i = 0; i < n; i += 2 * k) {
      for (int j = 0; j < k; ++j) {
        auto x = (ld*)&rt[j + k].x, y = (ld*)&a[i +
            j + k].x;
        Complex z(x[0] * y[0] - x[1] * y[1], x[0] *
            y[1] + x[1] * y[0]);
        a[i + j + k] = a[i + j] - z;
        a[i + j] += z;
      }
    }
  }
}
vector<ld> multiply(const vector<ld>& a, const
    vector<ld>& b) {
  if (a.empty() || b.empty()) return {};
  vector<ld> res(a.size() + b.size() - 1);
  int L = 32 - __builtin_clz(res.size()), n = 1 <<
      L;
  vector<Complex> in(n), out(n);

  for (size_t i = 0; i < a.size(); ++i) in[i].x[0]
      = a[i];
  for (size_t i = 0; i < b.size(); ++i) in[i].x[1]
      = b[i];

  fft(in);
  for (Complex& x : in) x *= x;

  for (int i = 0; i < n; ++i) out[i] = in[-i & (n -
      1)] - in[i].conj();
  fft(out);

  for (size_t i = 0; i < res.size(); ++i) res[i] =
      out[i].x[1] / (4 * n);
  return res;
}
long long my_round(ld x) {
  if (x < 0) return -my_round(-x);
  return (long long)(x + 1e-2);
}
vector<long long> multiply(const vector<int>& a,
    const vector<int>& b) {
  vector<ld> ad(a.begin(), a.end());
  vector<ld> bd(b.begin(), b.end());
  auto rd = multiply(ad, bd);
  vector<long long> res(rd.size());
  for (int i = 0; i < (int)res.size(); ++i) {
    res[i] = my_round(rd[i]);
  }
  return res;
}
```

5.4 Interpolate

```
const int mod = 1e9 + 7;
const int N = 1e6 + 6;

long long inv[N], po[N], pre[N], suf[N], dakdak[N];
long long ans, num;

inline long long qpow(long long a, int b) {
  long long ans = 1;
  while (b) {
    if (b & 1) ans = ans * a % mod;
    a = a * a % mod;
    b >>= 1;
  }
  return ans;
}
inline long long rv(int x) { return qpow(x, mod -
     2) % mod; }
void prec() {
  inv[0] = 1;
  for (int i = 1; i <= k + 1; ++i) {
    inv[i] = (1LL * inv[i - 1] * rv(i)) % mod;
    po[i] = (po[i - 1] + qpow(i, k)) % mod;
  }
  for (int i = 1; i <= k + 1; ++i) {
    dakdak[i] = (inv[i] * inv[k + 1 - i]) % mod;
  }
}
inline long long interpolate(int x, int k, bool bf
     = false) {
  if (k == 0) return x;
  if (x <= k + 1 || bf) {
    return po[x];
  }
  pre[0] = x;
  suf[k + 1] = x - (k + 1);
  for (int i = 1; i <= k; i++) pre[i] = (pre[i - 1]
       * (x - i)) % mod;
  for (int i = k; i >= 1; i--) suf[i] = (suf[i + 1]
       * (x - i)) % mod;
  ans = 0;
  for (int i = 0; i <= k + 1; i++) {
    if (i == 0)
      num = suf[1];
    else if (i == k + 1)
      num = pre[k];
    else
      num = (pre[i - 1] * suf[i + 1]) % mod; //
          numerator

    if ((i + k) & 1)
      ans = (ans + ((po[i] * num % mod) *
          dakdak[i])) % mod;
    else
      ans = (ans - ((po[i] * num % mod) *
          dakdak[i])) % mod;

    ans = (ans + mod) % mod;
  }
  return ans;
}
```

5.5 Lucas

```
ll lucas(ll n, ll m, int p, vi& fact, vi& invfact) {
  ll c = 1;
  while (n || m) {
    ll a = n % p, b = m % p;
    if (a < b) return 0;
    c = c * fact[a] % p * invfact[b] % p *
        invfact[a - b] % p;
    n /= p;
    m /= p;
  }
  return c;
}
```

5.6 Matrix

```
template <class T, int N>
struct Matrix {
  typedef Matrix M;
  array<array<T, N>, N> d{};
  M operator*(const M& m) const {
    M a;
    rep(i, 0, N) rep(j, 0, N) rep(k, 0, N)
        a.d[i][j] += d[i][k] * m.d[k][j];
    return a;
  }
  vector<T> operator*(const vector<T>& vec) const {
    vector<T> ret(N);
    rep(i, 0, N) rep(j, 0, N) ret[i] += d[i][j] *
        vec[j];
    return ret;
  }
  M operator(ll p) const {
    assert(p >= 0);
    M a, b(*this);
    rep(i, 0, N) a.d[i][i] = 1;
    while (p) {
      if (p & 1) a = a * b;
      b = b * b;
      p >>= 1;
    }
    return a;
  }
};
```

5.7 MillerRabin

```
inline uint64_t mod_mult64(uint64_t a, uint64_t b,
     uint64_t m) { return __int128_t(a) * b % m; }
uint64_t mod_pow64(uint64_t a, uint64_t b, uint64_t
     m) {
  uint64_t ret = (m > 1);
  for (;;) {
    if (b & 1) ret = mod_mult64(ret, a, m);
    if (!(b >>= 1)) return ret;
    a = mod_mult64(a, a, m);
  }
}

// Works for all primes p < 2^64
bool is_prime(uint64_t n) {
  if (n <= 3) return (n >= 2);
  static const uint64_t small[] = {
      2, 3, 5, 7, 11, 13, 17, 19, 23, 29,
          31, 37, 41, 43, 47, 53, 59, 61, 67,
          71, 73, 79, 83,
      89, 97, 101, 103, 107, 109, 113, 127, 131,
          137, 139, 149, 151, 157, 163, 167, 173,
          179, 181, 191, 193, 197, 199,
  };
  for (size_t i = 0; i < sizeof(small) /
       sizeof(uint64_t); ++i) {
    if (n % small[i] == 0) return n == small[i];
  }

  // Makes use of the known bounds for Miller-Rabin
      pseudoprimes.
  static const uint64_t millerrabin[] = {
      2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
  };
  static const uint64_t A014233[] = {
      // From OEIS.
      2047LL, 1373653LL, 25326001LL, 3215031751LL,
          2152302898747LL, 3474749660383LL,
          341550071728321LL, 341550071728321LL,
          3825123056546413051LL,
          3825123056546413051LL,
          3825123056546413051LL, 0,
  };
  uint64_t s = n - 1, r = 0;
  while (s % 2 == 0) {
    s /= 2;
```

```
    r++;
  }
  for (size_t i = 0, j; i < sizeof(millerrabin) /
      sizeof(uint64_t); i++) {
    uint64_t md = mod_pow64(millerrabin[i], s, n);
    if (md != 1) {
      for (j = 1; j < r; j++) {
        if (md == n - 1) break;
        md = mod_mult64(md, md, n);
      }
      if (md != n - 1) return false;
    }
    if (n < A014233[i]) return true;
  }
  return true;
}
```

5.8 Mobius

```
mobius[1] = 1;
for (int i = 2; i < N; ++i) {
  --mobius[i];
  for (int j = i + i; j < N; j += i) mobius[j] -=
      mobius[i];
}
```

5.9 ModInverse

```
const ll mod = 1000000007, LIM = 200000;
ll* inv = new ll[LIM] - 1; inv[1] = 1;
for(ll i = 2; i < LIM; ++i) inv[i] = mod - (mod /
    i) * inv[mod % i] % mod;
```

5.10 ModMulLL

```
typedef unsigned long long ull;
const int bits = 10; // i f a l l numbers are less
     than 2^k , set bits = 64k
const ull po = 1 << bits;
ull mod_mul(ull a, ull b, ull &c) {
  ull x = a * (b & (po - 1)) % c;
  while ((b >>= bits) > 0) {
    a = (a << bits) % c;
    x += (a * (b & (po - 1))) % c;
  }
  return x % c;
}
ull mod_pow(ull a, ull b, ull mod) {
  if (b == 0) return 1;
  ull res = mod_pow(a, b / 2, mod);
  res = mod_mul(res, res, mod);
  if (b & 1) return mod_mul(res, a, mod);
  return res;
}
```

5.11 ModularArithmetic

```
const ll mod = 17; // change to something else
struct Mod {
  ll x;
  Mod(ll xx) : x(xx) {}
  Mod operator+(Mod b) { return Mod((x + b.x) %
      mod); }
  Mod operator-(Mod b) { return Mod((x - b.x + mod)
      % mod); }
  Mod operator*(Mod b) { return Mod((x * b.x) %
      mod); }
  Mod operator/(Mod b) { return *this * invert(b); }
  Mod invert(Mod a) {
    ll x, y, g = euclid(a.x, mod, x, y);
    assert(g == 1);
    return Mod((x + mod) % mod);
  }
  Mod operator (ll e) {
    if (!e) return Mod(1);
    Mod r = *this (e / 2);
    r = r * r;
    return e & 1 ? *this * r : r;
  }
};
```

5.12 Notes

5.12.1 Cycles

Let $g_S(n)$ be the number of n-permutations whose cycle lengths all belong to the set S. Then

$$\sum_{n=0}^{\infty} g_S(n)\frac{x^n}{n!} = \exp\left(\sum_{n\in S}\frac{x^n}{n}\right)$$

5.12.2 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1)+D(n-2)) = nD(n-1)+(-1)^n = \left\lfloor \frac{n!}{e} \right\rceil$$

5.12.3 Burnside's lemma

Given a group G of symmetries and a set X, the number of elements of X *up to symmetry* equals

$$\frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where X^g are the elements fixed by g $(g.x = x)$.

If $f(n)$ counts "configurations" (of some sort) of length n, we can ignore rotational symmetry using $G = Z_n$ to get

$$g(n) = \frac{1}{n}\sum_{k=0}^{n-1} f(\gcd(n,k)) = \frac{1}{n}\sum_{k|n} f(k)\phi(n/k).$$

5.12.4 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1,\ p(n) = \sum_{k\in Z\setminus\{0\}} (-1)^{k+1} p(n - k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	∼2e5	∼2e8

5.12.5 Lucas' Theorem

Let n, m be non-negative integers and p a prime. Write $n = n_k p^k + \ldots + n_1 p + n_0$ and $m = m_k p^k + \ldots + m_1 p + m_0$. Then $\binom{n}{m} \equiv \prod_{i=0}^{k}\binom{n_i}{m_i} \pmod{p}$.

5.12.6 Bernoulli numbers

EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t-1}$ (FFT-able). $B[0,\dots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots]$

Sums of powers:

$$\sum_{i=1}^{n} n^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty} f(i) = \int_m^\infty f(x)dx - \sum_{k=1}^{\infty}\frac{B_k}{k!}f^{(k-1)}(m)$$

$$\approx \int_m^\infty f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

5.12.7 Stirling numbers of the first kind

Number of permutations on n items with k cycles.

$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k),\ c(0,0) = 1$
$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\dots(x+n-1)$

$c(8,k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$
$c(n,2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \dots$

5.12.8 Eulerian numbers

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$

$$E(n,0) = E(n,n-1) = 1$$

$$E(n,k) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+1-j)^n$$

5.12.9 Stirling numbers of the second kind

Partitions of n distinct elements into exactly k groups.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

$$S(n,1) = S(n,n) = 1$$

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

5.12.10 Bell numbers

Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$. For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod p$$

5.12.11 Labeled unrooted trees

on n vertices: n^{n-2}
on k existing trees of size n_i: $n_1 n_2 \cdots n_k n^{k-2}$
with degrees d_i: $(n-2)!/((d_1-1)!\cdots(d_n-1)!)$

5.12.12 Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1,\ C_{n+1} = \frac{2(2n+1)}{n+2}C_n,\ C_{n+1} = \sum C_i C_{n-i}$$

$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$

- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.

5.12.13 Hockey Stick Identity

$$\sum_{k=r}^{n}\binom{k}{r} = \binom{n+1}{r+1}$$

5.13 NTT

```
/* NTT with modulo 998244353
notes:
NTT with mod m
g is any primitive root modulo m (g = 3 works well
    for 998244353)
n divides m - 1 evenly
wn = g^((m - 1) / n)
https://codeforces.com/blog/entry/75326
*/

const int N = 1 << 21;
const ll mod = 998244353;
const ll g = 3;

int rev[N];
ll w[N], iw[N], wt[N], inv_n;

ll binpow(ll a, ll b) {
  ll res = 1;
  for (; b; b >>= 1, a = (1ll * a * a) % mod)
    if (b & 1) res = (1ll * res * a) % mod;
  return res;
}

void precalc(int lg) {
  int n = 1 << lg;
  inv_n = binpow(n, mod - 2);
  for (int i = 0; i < n; ++i) {
    rev[i] = 0;
    for (int j = 0; j < lg; ++j)
      if (i & (1 << j)) rev[i] |= (1 << (lg - j -
          1));
  }
  ll wn = binpow(g, (mod - 1) / n);
  w[0] = 1;
  for (int i = 1; i < n; ++i) w[i] = (1ll * w[i -
      1] * wn) % mod;
  ll iwn = binpow(wn, mod - 2);
  iw[0] = 1;
  for (int i = 1; i < n; ++i) iw[i] = (1ll * iw[i -
      1] * iwn) % mod;
}

void ntt(vector<ll> &a, int lg, bool inv = 0) {
  int n = (1 << lg);
  for (int i = 0; i < n; ++i)
    if (i < rev[i]) swap(a[i], a[rev[i]]);
  for (int len = 2; len <= n; len <<= 1) {
    int d = n / len;
    for (int j = 0; j < (len >> 1); ++j) wt[j] =
        (inv ? iw[d * j] : w[d * j]);
    for (int i = 0; i < n; i += len) {
```

```
      for (int j = 0; j < (len >> 1); ++j) {
        ll x = a[i + j], y = (1ll * a[i + j + (len
            >> 1)] * wt[j]) % mod;
        a[i + j] = (x + y) % mod;
        a[i + j + (len >> 1)] = (x - y + mod) % mod;
      }
    }
  }

  if (inv)
    for (int i = 0; i < n; ++i) a[i] = (1ll * a[i]
        * inv_n) % mod;
}

vector<ll> multiply(vector<ll> a, vector<ll> b) {
  int n = 1, lg = 0;
  int na = sz(a), nb = sz(b);
  while (n < na + nb) n <<= 1, ++lg;
  precalc(lg);
  a.resize(n);
  b.resize(n);
  ntt(a, lg);
  ntt(b, lg);
  for (int i = 0; i < n; ++i) a[i] = (1ll * a[i] *
      b[i]) % mod;
  ntt(a, lg, 1);
  vector<ll> c;
  for (int i = 0; i < na + nb - 1; ++i)
      c.push_back(a[i]);

  // while(!c.empty() and c.back() == 0)
  //    c.pop_back();

  return c;
}
```

5.14 PhiFunction

```
const int LIM = 5000000;
int phi[LIM];
void calculatePhi() {
  rep(i, 0, LIM) phi[i] = i & 1 ? i : i / 2;
  for (int i = 3; i < LIM; i += 2)
    if (phi[i] == i)
      for (int j = i; j < LIM; j += i) (phi[j] /=
          i) *= i - 1;
}
```

5.15 PollardFactorize

```
using ll = long long;
using ull = unsigned long long;
using ld = long double;
ll mult(ll x, ll y, ll md) {
  ull q = (ld)x * y / md;
  ll res = ((ull)x * y - q * md);
  if (res >= md) res -= md;
  if (res < 0) res += md;
  return res;
}

ll powMod(ll x, ll p, ll md) {
  if (p == 0) return 1;
  if (p & 1) return mult(x, powMod(x, p - 1, md),
      md);
  return powMod(mult(x, x, md), p / 2, md);
}

bool checkMillerRabin(ll x, ll md, ll s, int k) {
  x = powMod(x, s, md);
  if (x == 1) return true;
  while (k--) {
    if (x == md - 1) return true;
    x = mult(x, x, md);
    if (x == 1) return false;
  }
  return false;
}
bool isPrime(ll x) {
  if (x == 2 || x == 3 || x == 5 || x == 7) return
      true;
  if (x % 2 == 0 || x % 3 == 0 || x % 5 == 0 || x %
      7 == 0) return false;
  if (x < 121) return x > 1;
  ll s = x - 1;
  int k = 0;
  while (s % 2 == 0) {
    s >>= 1;
    k++;
  }
  if (x < 1LL << 32) {
    for (ll z : {2, 7, 61}) {
      if (!checkMillerRabin(z, x, s, k)) return
          false;
    }
  } else {
    for (ll z : {2, 325, 9375, 28178, 450775,
        9780504, 1795265022}) {
      if (!checkMillerRabin(z, x, s, k)) return
          false;
    }
  }
  return true;
}
ll gcd(ll x, ll y) { return y == 0 ? x : gcd(y, x %
    y); }

void pollard(ll x, vector<ll> &ans) {
  if (isPrime(x)) {
    ans.push_back(x);
    return;
  }
  ll c = 1;
  while (true) {
    c = 1 + get_rand(x - 1);
    auto f = [&](ll y) {
      ll res = mult(y, y, x) + c;
      if (res >= x) res -= x;
      return res;
    };
    ll y = 2;
    int B = 100;
    int len = 1;
    ll g = 1;
    while (g == 1) {
      ll z = y;
      for (int i = 0; i < len; i++) {
        z = f(z);
      }
      ll zs = -1;
      int lft = len;
      while (g == 1 && lft > 0) {
        zs = z;
        ll p = 1;
        for (int i = 0; i < B && i < lft; i++) {
          p = mult(p, abs(z - y), x);
          z = f(z);
        }
        g = gcd(p, x);
        lft -= B;
      }
      if (g == 1) {
        y = z;
        len <<= 1;
        continue;
      }
      if (g == x) {
        g = 1;
        z = zs;
        while (g == 1) {
          g = gcd(abs(z - y), x);
          z = f(z);
        }
      }
      if (g == x) break;
      assert(g != 1);
      pollard(g, ans);
      pollard(x / g, ans);
      return;
```

```
    }
  }
}
// return list of all prime factors of x (can have
     duplicates)
vector<ll> factorize(ll x) {
  vector<ll> ans;
  for (ll p : {2, 3, 5, 7, 11, 13, 17, 19}) {
    while (x % p == 0) {
      x /= p;
      ans.push_back(p);
    }
  }
  if (x != 1) {
    pollard(x, ans);
  }
  sort(ans.begin(), ans.end());
  return ans;
}
// return pairs of (p, k) where x = product(p^k)
vector<pair<ll, int>> factorize_pk(ll x) {
  auto ps = factorize(x);
  ll last = -1, cnt = 0;
  vector<pair<ll, int>> res;
  for (auto p : ps) {
    if (p == last)
      ++cnt;
    else {
      if (last > 0) res.emplace_back(last, cnt);
      last = p;
      cnt = 1;
    }
  }
  if (cnt > 0) {
    res.emplace_back(last, cnt);
  }
  return res;
}
vector<ll> get_all_divisors(ll n) {
  auto pks = factorize_pk(n);

  vector<ll> res;
  function<void(int, ll)> gen = [&](int i, ll prod)
      {
    if (i == static_cast<int>(pks.size())) {
      res.push_back(prod);
      return;
    }

    ll cur_power = 1;
    for (int cur = 0; cur <= pks[i].second; ++cur) {
      gen(i + 1, prod * cur_power);
      cur_power *= pks[i].first;
    }
  };
  gen(0, 1LL);
  sort(res.begin(), res.end());
  return res;
}
```

5.16 PrimitiveRoot

```
// Primitive root of modulo n is integer g iff for
     all a < n & gcd(a, n) == 1, there exist k: g^k
     = a mod n
// k is called discrete log of a (in case P is
     prime, can find in O(sqrt(P)) by noting that
     (P-1) is divisible by k)
//
// Exist if:
// - n is 1, 2, 4
// - n = p^k for odd prime p
// - n = 2*p^k for odd prime p
int powmod (int a, int b, int p) {
    int res = 1;
    while (b)
        if (b & 1)
            res = int (res * 1ll * a % p), --b;
        else
            a = int (a * 1ll * a % p), b >>= 1;
    return res;
}

int generator (int p) {
    vector<int> fact;
    int phi = p-1, n = phi;
    for (int i=2; i*i<=n; ++i)
        if (n % i == 0) {
            fact.push_back (i);
            while (n % i == 0)
                n /= i;
        }
    if (n > 1)
        fact.push_back (n);

    for (int res=2; res<=p; ++res) {
        bool ok = true;
        for (size_t i=0; i<fact.size() && ok; ++i)
            ok &= powmod (res, phi / fact[i], p) !=
                 1;
        if (ok) return res;
    }
    return -1;
}
```

5.17 TwoSat

```
struct TwoSatSolver {
  TwoSatSolver(int _n_vars) : n_vars(_n_vars), g(2
      * n_vars) {}

  void x_or_y_constraint(bool is_x_true, int x,
      bool is_y_true, int y) {
    assert(x >= 0 && x < n_vars);
    assert(y >= 0 && y < n_vars);
    if (!is_x_true) x += n_vars;
    if (!is_y_true) y += n_vars;
    // x || y
    // !x -> y
    // !y -> x
    g[(x + n_vars) % (2 * n_vars)].push_back(y);
    g[(y + n_vars) % (2 * n_vars)].push_back(x);
  }

  // Returns:
  // If no solution -> returns {false, {}}
  // If has solution -> returns {true, solution}
  //    where |solution| = n_vars, solution = true /
      false
  pair<bool, vector<bool>> solve() {
    DirectedDfs tree(g);
    vector<bool> solution(n_vars);
    for (int i = 0; i < n_vars; i++) {
      if (tree.comp_ids[i] == tree.comp_ids[i +
          n_vars]) {
        return {false, {}};
      }
      // Note that reverse(tree.scc) is topo sorted
      solution[i] = tree.comp_ids[i] <
          tree.comp_ids[i + n_vars];
    }
    return {true, solution};
  }

  // number of variables
  int n_vars;
  // vertex 0 -> n_vars - 1: Ai is true
  // vertex n_vars -> 2*n_vars - 1: Ai is false
  vector<vector<int>> g;
};
```

5.18 XorBasis

```
struct Basis {
  const int LGX = 19;
  vector<int> a;
  Basis() : a(LGX + 1, 0) {}
```

```
  void add(int x) {
    for (int i = LGX; i >= 0; --i) {
      if (x & (1 << i)) {
        if (a[i])
          x ^= a[i];
        else {
          a[i] = x;
          break;
        }
      }
    }
  }
  void add(Basis o) {
    for (int i = LGX; i >= 0; --i) add(o.a[i]);
  }
  bool is_spannable(int x) {
    for (int i = LGX; i >= 0; --i)
      if (x & (1 << i)) x ^= a[i];
    return (x == 0);
  }
};
```

6 String

6.1 AhoCorasick

```
template <int MAXC = 26> struct AhoCorasick {
        vector<array<int, MAXC>> C;
        vector<int> F;
        vector<vector<int>> FG;
        vector<bool> E;

        int node() {
                int r = C.size();
                E.push_back(0);
                F.push_back(-1);
                C.emplace_back();
                fill(C.back().begin(),
                     C.back().end(), -1);
                return r;
        }
        int ctrans(int n, int c) {
                if (C[n][c] == -1) C[n][c] = node();
                return C[n][c];
        }
        int ftrans(int n, int c) const {
                while (n && C[n][c] == -1) n = F[n];
                return C[n][c] != -1 ? C[n][c] : 0;
        }
        AhoCorasick(vector<vector<int>> P) {
                node();
                for (int i = 0; i < (int)P.size();
                     i++) {
                        int n = 0;
                        for (int c : P[i]) n =
                             ctrans(n, c);
                        E[n] = 1;
                }
                queue<int> Q;
                F[0] = 0;
                for (int c : C[0]) if (c != -1)
                     Q.push(c), F[c] = 0;
                while (!Q.empty()) {
                        int n = Q.front(); Q.pop();
                        for (int c = 0; c < MAXC;
                             ++c) if (C[n][c] != -1) {
                                int f = F[n];
                                while (f && C[f][c] ==
                                     -1) f = F[f];
                                F[C[n][c]] = C[f][c]
                                     != -1 ? C[f][c] :
                                     0;
                                Q.emplace(C[n][c]);
                        }
                }
                FG.resize(F.size());
                for (int i = 1; i < (int)F.size();
                     i++) {
                        FG[F[i]].push_back(i);
                        if (E[i]) Q.push(i);
                }
                while (!Q.empty()) {
                        int n = Q.front();
                        Q.pop();
                        for (int f : FG[n]) E[f] = 1,
                             Q.push(f);
                }
        }
        bool check(vector<int> V) {
                if (E[0]) return 1;
                int n = 0;
                for (int c : V) {
                        n = ftrans(n, c);
                        if (E[n]) return 1;
                }
                return 0;
        }
};
```

6.2 KMP

```
// prefix function: *length* of longest prefix
     which is also suffix:
//   pi[i] = max(k: s[0..k-1] == s[i-(k-1)..i]
//
// KMP {{{
template <typename Container>
std::vector<int> prefix_function(const Container&
     s) {
  int n = s.size();
  std::vector<int> pi(n);
  for (int i = 1; i < n; ++i) {
    int j = pi[i - 1];
    while (j > 0 && s[i] != s[j]) j = pi[j - 1];
    if (s[i] == s[j]) ++j;
    pi[i] = j;
  }
  return pi;
}

// Tested: https://oj.vnoi.info/problem/substr
// Return all positions (0-based) that pattern
     'pat' appears in 'text'
std::vector<int> kmp(const std::string& pat, const
     std::string& text) {
  auto pi = prefix_function(pat + '\0' + text);
  std::vector<int> res;
  for (size_t i = pi.size() - text.size(); i <
      pi.size(); ++i) {
    if (pi[i] == (int)pat.size()) {
      res.push_back(i - 2 * pat.size());
    }
  }
  return res;
}

// Tested: https://oj.vnoi.info/problem/icpc22_mt_b
// Returns cnt[i] = # occurrences of prefix of
     length-i
// NOTE: cnt[0] = n+1 (0-length prefix appears n+1
     times)
std::vector<int> prefix_occurrences(const string&
     s) {
  int n = s.size();
  auto pi = prefix_function(s);
  std::vector<int> res(n + 1);
  for (int i = 0; i < n; ++i) res[pi[i]]++;
  for (int i = n - 1; i > 0; --i) res[pi[i - 1]] +=
      res[i];
  for (int i = 0; i <= n; ++i) res[i]++;
  return res;
}
```

6.3 Manacher

```
vector<int> manacher_odd(string s) {
    int n = s.size();
    s = "$" + s + "^";
    vector<int> p(n + 2);
    int l = 1, r = 1;
    for(int i = 1; i <= n; i++) {
        p[i] = max(0, min(r - i, p[l + (r - i)]));
        while(s[i - p[i]] == s[i + p[i]]) {
            p[i]++;
        }
        if(i + p[i] > r) {
            l = i - p[i], r = i + p[i];
        }
    }
    return vector<int>(begin(p) + 1, end(p) - 1);
}
vector<int> manacher(string s) {
    string t;
    for(auto c: s) {
        t += string("#") + c;
    }
    auto res = manacher_odd(t + "#");
    return vector<int>(begin(res) + 1, end(res) -
         1);
}
```

6.4 StringHashing

```
int power(long long n, long long k, const int mod) {
  int ans = 1 % mod;
  n %= mod;
  if (n < 0) n += mod;
  while (k) {
    if (k & 1) ans = (long long) ans * n % mod;
    n = (long long) n * n % mod;
    k >>= 1;
  }
  return ans;
}

const int MOD1 = 127657753, MOD2 = 987654319;
const int p1 = 137, p2 = 277;
int ip1, ip2;
pair<int, int> pw[N], ipw[N];
void prec() {
  pw[0] = {1, 1};
  for (int i = 1; i < N; i++) {
    pw[i].first = 1LL * pw[i - 1].first * p1 % MOD1;
    pw[i].second = 1LL * pw[i - 1].second * p2 %
         MOD2;
  }
  ip1 = power(p1, MOD1 - 2, MOD1);
  ip2 = power(p2, MOD2 - 2, MOD2);
  ipw[0] = {1, 1};
  for (int i = 1; i < N; i++) {
    ipw[i].first = 1LL * ipw[i - 1].first * ip1 %
         MOD1;
    ipw[i].second = 1LL * ipw[i - 1].second * ip2 %
         MOD2;
  }

}
struct Hashing {
  int n;
  string s; // 0 - indexed
  vector<pair<int, int>> hs; // 1 - indexed
  Hashing() {}
  Hashing(string _s) {
    n = _s.size();
    s = _s;
    hs.emplace_back(0, 0);
    for (int i = 0; i < n; i++) {
      pair<int, int> p;
      p.first = (hs[i].first + 1LL * pw[i].first *
           s[i] % MOD1) % MOD1;
      p.second = (hs[i].second + 1LL * pw[i].second
           * s[i] % MOD2) % MOD2;
      hs.push_back(p);
    }
  }
  pair<int, int> get_hash(int l, int r) { // 1 -
       indexed
    assert(1 <= l && l <= r && r <= n);
    pair<int, int> ans;
    ans.first = (hs[r].first - hs[l - 1].first +
         MOD1) * 1LL * ipw[l - 1].first % MOD1;
    ans.second = (hs[r].second - hs[l - 1].second +
         MOD2) * 1LL * ipw[l - 1].second % MOD2;
    return ans;
  }
  pair<int, int> get_hash() {
    return get_hash(1, n);
  }
};
```

6.5 SuffixArray

```
vector<int> SA(const vector<int>& s, int upper) {
        int n=s.size();
        if (n == 0) return {};
        if (n == 1) return {0};
        if (n == 2) {
                if (s[0] < s[1]) return {0, 1};
                else return {1, 0};
        }
        vector<int> sa(n), sum_l(upper+1),
             sum_s(upper+1);
        vector<bool> ls(n);
        for (int i=n-2; i>=0; i--)
                ls[i]=(s[i] == s[i+1]) ? ls[i+1] :
                     (s[i] < s[i+1]);
        for (int i = 0; i<n; i++)
                if (!ls[i]) sum_s[s[i]]++;
                else sum_l[s[i]+1]++;
        for (int i=0; i<=upper; i++) {
                sum_s[i] += sum_l[i];
                if (i < upper) sum_l[i+1] +=
                     sum_s[i];
        }
        auto induce=[&](const vector<int>& lms) {
                fill(sa.begin(), sa.end(), -1);
                vector<int> buf(upper+1);
                copy(sum_s.begin(), sum_s.end(),
                     buf.begin());
                for (auto d : lms) {
                        if (d == n) continue;
                        sa[buf[s[d]]++] = d;
                }
                copy(sum_l.begin(), sum_l.end(),
                     buf.begin());
                sa[buf[s[n-1]]++] = n-1;
                for (int i=0; i<n; i++) {
                        int v=sa[i];
                        if (v>=1 && !ls[v-1])
                                sa[buf[s[v-1]]++] = v-1;
                }
                copy(sum_l.begin(), sum_l.end(),
                     buf.begin());
                for (int i=n-1; i>=0; i--) {
                        int v=sa[i];
                        if (v>=1 && ls[v-1])
                                sa[--buf[s[v-1]+1]] =
                                v-1;
                }
        };
        vector<int> lms_map(n+1, -1), lms;
        int m=0;
        for (int i=1; i<n; i++) if (!ls[i-1] &&
             ls[i]) {
                lms_map[i]=m++;
                lms.push_back(i);
        }
        induce(lms);
        if (m) {
                vector<int> sorted_lms, rec_s(m);
                for (int v : sa) if (lms_map[v] !=
                     -1) sorted_lms.push_back(v);
                int rec_upper=0;
                rec_s[lms_map[sorted_lms[0]]]=0;
```

```
            for (int i=1; i<m; i++) {
                  int l=sorted_lms[i-1],
                      r=sorted_lms[i];
                  int end_l = (lms_map[l]+1 <
                      m) ? lms[lms_map[l]+1] :
                      n;
                  int end_r = (lms_map[r]+1 <
                      m) ? lms[lms_map[r]+1] :
                      n;
                  bool same=true;
                  if (end_l-l != end_r-r)
                      same=false;
                  else {
                        while (l < end_l) {
                              if (s[l] !=
                                  s[r])
                                  break;
                              l++, r++;
                        }
                        if (l == n || s[l] !=
                            s[r]) same=false;
                  }
                  if (!same) rec_upper++;
                  rec_s[lms_map[sorted_lms[i]]]=rec_upper;
            }
            auto rec_sa = SA(rec_s, rec_upper);
            for (int i=0; i<m; i++)
                sorted_lms[i] = lms[rec_sa[i]];
            induce(sorted_lms);
      }
      return sa;
}

vector<int> lcp_array(const vector<int>& s, const
    vector<int>& sa) {
      int n=int(s.size());
      assert(n>=1);
      vector<int> rnk(n), lcp(n-1);
      for (int i=0; i<n; i++) rnk[sa[i]] = i;
      int h=0;
      for (int i=0; i<n; i++) {
            if (h > 0) h--;
            if (rnk[i] == 0) continue;
            int j=sa[rnk[i]-1];
            for (; j+h < n && i+h < n; h++)
                  if (s[j+h] != s[i+h]) break;
            lcp[rnk[i]-1]=h;
      }
      return lcp;
}
```

6.6 Z

```
vector<int> zfunc(const string& s) {
  int n = (int)s.length();
  vector<int> z(n);
  z[0] = n;
  for (int i = 1, l = 0, r = 0; i < n; ++i) {
    if (i <= r) z[i] = min(r - i + 1, z[i - l]);
    while (i + z[i] < n && s[z[i]] == s[i + z[i]])
        ++z[i];
    if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
  }
  return z;
}
```

7 Utilities

7.1 FastInput

```
inline char gc() { // l ike getchar ()
  static char buf[1 << 16];
  static size_t bc, be;
  if (bc >= be) {
    buf[0] = 0, bc = 0;
    be = fread(buf, 1, sizeof(buf), stdin);
  }
  return buf[bc++]; // returns 0 on EOF
}
int readInt() {
  int a, c;
  while ((a = gc()) < 40)
    ;
  if (a == '-') return -readInt();
  while ((c = gc()) >= 48) a = a * 10 + c - 480;
  return a - 48;
}
```

7.2 multivec

```
template<int D, typename T> struct Vec : public
    vector<Vec<D - 1, T>> { template<typename...
    Args> Vec(int n = 0, Args... args) : vector <
    Vec < D - 1, T >> (n, Vec < D - 1, T >
    (args...)) {} };
template<typename T> struct Vec<1, T> : public
    vector<T> { Vec(int n = 0, const T &val = T())
    : vector<T>(n, val) {} };
```

7.3 template

```
#include "bits/stdc++.h"
using namespace std;

#ifdef LOCAL
#include "debug.h"
#else
#define debug(...)
#endif

using ll = long long;
using pii = pair<int, int>;

#define F first
#define S second
#define sz(x) (int)((x).size())
#define all(x) (x).begin(), (x).end()

mt19937_64
    rng(chrono::steady_clock::now().time_since_epoch().cou
ll get_rand(ll l, ll r) {
   assert(l <= r);
   return uniform_int_distribution<ll> (l, r)(rng);
}

void solve(){

}

int32_t main() {
   cin.tie(nullptr)->sync_with_stdio(0);
   #define task "troll"
   if(fopen(task".inp", "r")){
      freopen(task".inp", "r", stdin);
      freopen(task".out", "w", stdout);
   }
   int test = 1;
//   cin >> test;
   for(int i = 1; i <= test; ++i){
//        cout << "Case #" << i << ": ";
      solve();
   }
   #ifdef LOCAL
      cerr << "\n[Time]: " << 1000.0 * clock() /
          CLOCKS_PER_SEC << " ms.\n";
   #endif
   return 0;
}
```