IMO Shortlist 2010 N3

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§1 Problem

Problem (IMO Shortlist 2010 N3)
Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2 .$$

§2 Solution 1 (Using Legendre's Three-Square Theorem)

Proof. We begin with the first observation that bounds the degree of the polynomials f_i.

Claim 2.1. For any polynomial $f_i \in \mathbb{Q}[x]$, we will always have

$$\deg(f_i) \leq 1$$

Proof. Comparing the degrees of polynomials on both sides of the equation, we get

$$\begin{aligned}\max\left(\deg(f_1(x)^2), \ldots, \deg(f_n(x)^2)\right) &= 2\max\left(\deg(f_1), \ldots, \deg(f_n)\right) \\ &= \deg\left(x^2+7\right) = 2\end{aligned}$$

which implies that $\max(\deg(f_1), \ldots, \deg(f_n)) = 1$. Hence, $\deg(f_i) \leq 1$. $\square$

Lemma 2.2

For $r \in \mathbb{Q}$, with its simplest form $r = \frac{p}{q}$ for p, $q \in \mathbb{Z}$ and $q \neq 0$. If pq is of the form $(8x + 7)$, then the equation

$$x^2 + y^2 + z^2 = r$$

has no solutions in $\mathbb{Q}$.

Proof. Suppose for the sake of contradiction, there exists rational solutions to the equation

$$x^2 + y^2 + z^2 = r$$

This could be re-written as,

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 + \left(\frac{e}{f}\right)^2 = \frac{p}{q}$$
$$\iff (qadf)^2 + (qcbf)^2 + (qbde)^2 = pq(bdf)^2$$

Since the quadratic residues $\pmod 8$ are $\{0, 1, 4\}$, we get that $pq(bdf)^2$ is of the form $4^k\,(8\ell + 7)$ which has no integer solutions due to Legendre's three-square theorem, proving the claim. $\square$

We claim that $\boxed{n = 5}$ is the smallest number for which there exists such polynomials. Constructing the answer is easy. Consider,

$$(f_1, f_2, f_3, f_4, f_5) = (x, 1, 1, 1, 2)$$

squares of which add up to $x^2 + 7$. Now we shall show that this is the smallest n possible.

Claim 2.3. For $n \leq 4$, there exists no such polynomials $f_1, f_2, \ldots, f_n$ that satisfy

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2$$

Proof. Suppose there exist such polynomials f_i. Then we can write,

$$x^2 + 7 = (ax + e)^2 + (bx + f)^2 + (cx + g)^2 + (dx + h)^2$$

So, we want to prove the existence of integers that satisfy,

$$a^2 + b^2 + c^2 + d^2 = 1$$
$$ae + bf + cg + dh = 0$$
$$e^2 + f^2 + g^2 + h^2 = 7$$

Using Euler's four-square identity,

$$\left(a^2 + b^2 + c^2 + d^2\right)\left(e^2 + f^2 + g^2 + h^2\right)$$
$$= (ae + bf + cg + dh)^2 + (-af + be + ch - dg)^2$$
$$+ (-ag - bh + ce + df)^2 + (-ah + bg - cf + de)^2$$

we get that

$$7 = (-af + be + ch - dg)^2 + (-ag - bh + ce + df)^2 + (-ah + bg - cf + de)^2$$

which has no rational solutions due to the previously proven lemma. This proves that for $n \leq 4$, we have no solutions. $\square$

$\square$