Monge-D'Alembert Circle Theorem

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In this article we study about monge's circle theorem

Monge's Circle Theorem (also known as **Monge-D'Alembert Circle Theorem**) is a result related to similitudes of circles. Let's first state a few key definitions.

§1 Homothetic Centers

We have already seen that when two circles are tangent, their point of tangency is a point of homothetic transformation that maps one circle onto another. Infact we can generalise this. We can state that for a pair circle (even when they are not tangent), we can find a point and a suitable homothety that maps one circle onto another.

Definition 1.1. Consider two circles ω_1 and ω_2 with centers O_1 and O_2 and radii r_1 and r_2. Then

1. the **exsimilicenter** of ω_1 and ω_2 is the point E on the line O_1O_2 that satisfies
$$\frac{\overline{EO_1}}{\overline{EO_2}} = \frac{r_1}{r_2}$$
2. the **insimilicenter** of ω_1 and ω_2 is the point I lying on the line O_1O_2 that satisfies
$$\frac{\overline{IO_1}}{\overline{IO_2}} = -\frac{r_1}{r_2}$$

Immediately from the definition of these, we have that

Proposition 1.2

For two circles ω_1 and ω_2, their **insimilicenter** and **exsimilicenter** are the centers of homothetic transformations that map ω_1 to ω_2.

Proof. Follows from identifying similar triangles and using the SAS similarity criterion for proving the homothety. □

Infact, it is possible to construct these centers for pair of circles that lie outside each other.

Proposition 1.3

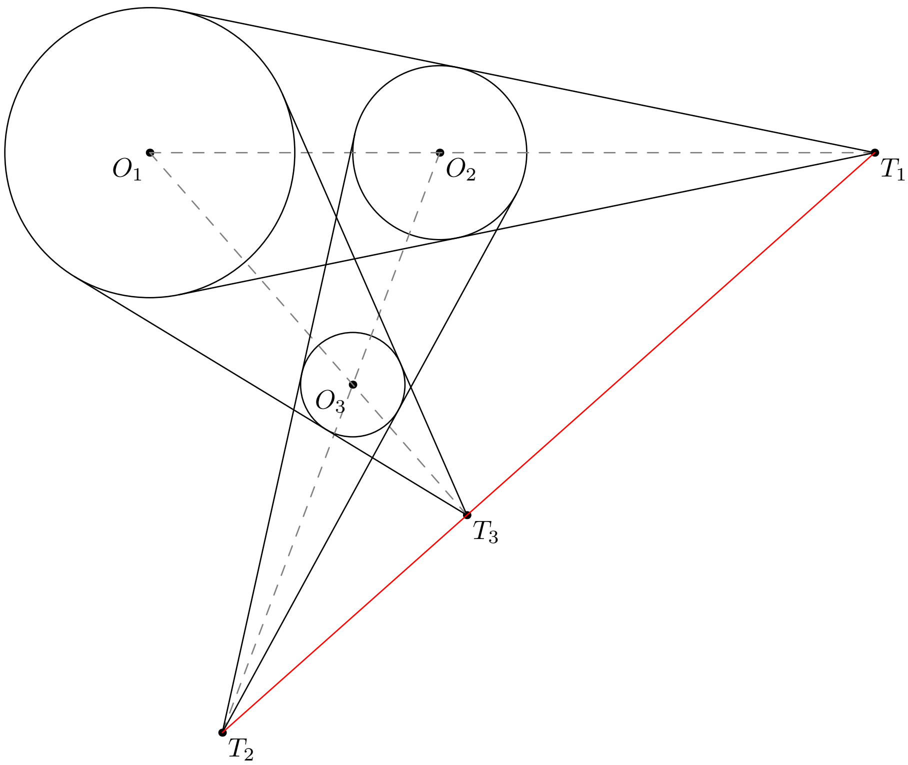
For two circles ω_1 and ω_2 that are not contained within each other, the intersection of common internal tangents and common external tangents are the **insimilicenter** and **exsimilicenter** of ω_1 and ω_2.

The proof again follows from similar triangles, so we shall omit it here. Something even more counter-intuitive is that these centers exist even for circles contained in one another.

§2 Statement of Monge's Circle Theorem

Theorem 2.1 (Monge's Circle Theorem)

For three distinct circles, their pairwise exsimilicenters are collinear.



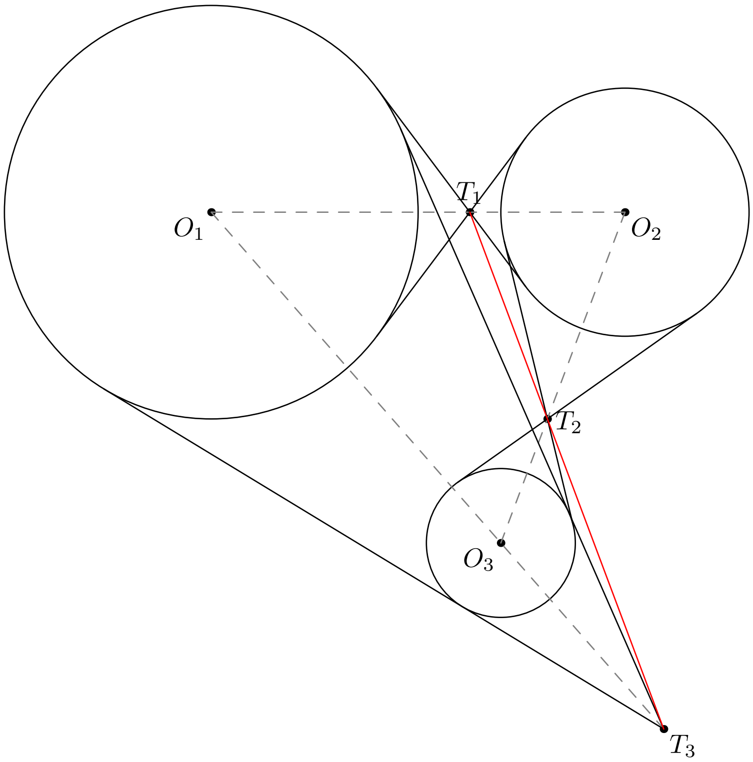
Proof. Let the centers of the circles ω_1, ω_2 and ω_3 be at O_1, O_2 and O_3, and let the pairwise exsimilicenters of $\{\odot(O_1), \odot(O_2)\}$, $\{\odot(O_2), \odot(O_3)\}$ and $\{\odot(O_3), \odot(O_1)\}$ be T_1, T_2 and T_3. We want to show that T_1, T_2 and T_3 are collinear, and to prove this we shall use **Menelaus' Theorem**. Applying Menelaus' Theorem on $\triangle O_1O_2O_3$ we get

$$\frac{\overline{O_1T_1}}{\overline{T_1O_2}} \cdot \frac{\overline{O_2T_2}}{\overline{T_2O_3}} \cdot \frac{\overline{O_3T_3}}{\overline{T_3O_1}} = \frac{r_1}{r_2} \cdot \frac{r_2}{r_3} \cdot \frac{r_3}{r_1} = 1$$

which implies that T_1, T_2 and T_3 are collinear. $\square$

Theorem 2.2 (Extension of Monge's Circle Theorem)

For three distinct circles, the exsimilicenter of a pair of circles is collinear with the insimilicenters of the other two pairs of circles.



Proof. Let the centers of the circles ω_1, ω_2 and ω_3 be at O_1, O_2 and O_3, and let the pairwise insimilicenters of $\{\odot(O_1), \odot(O_2)\}$, $\{\odot(O_2), \odot(O_3)\}$ and exsimilicenter of $\{\odot(O_3), \odot(O_1)\}$ be T_1, T_2 and T_3. We want to show that T_1, T_2 and T_3 are collinear, and to prove this we shall use **Menelaus' Theorem**. Applying Menelaus' Theorem on $\triangle O_1O_2O_3$ we get

$$\frac{\overline{O_1T_3}}{\overline{T_3O_3}} \cdot \frac{\overline{O_3T_2}}{\overline{T_2O_2}} \cdot \frac{\overline{O_2T_1}}{\overline{T_1O_1}} = \frac{r_1}{r_3} \cdot \frac{r_3}{r_2} \cdot \frac{r_2}{r_1} = 1$$

which implies that T_1, T_2 and T_3 are collinear. □

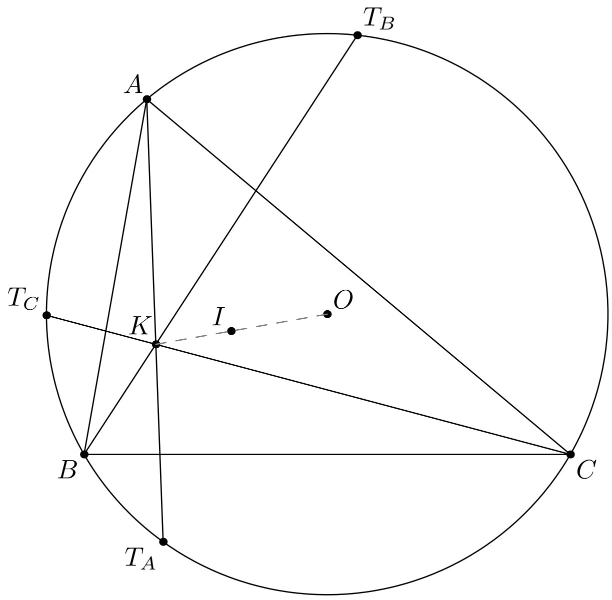
Now let's take a look at some applications of this theorem.

§3 Exsimilicenter of $\odot(I)$ and $\odot(O)$

Theorem 3.1 (Exsimilicenter of incenter and circumcenter)

Suppose I is the incenter and O is the circumcenter of $\triangle ABC$. Let T_A, T_B and T_C be the A-mixtilinear, B-mixtilinear and C-mixtilinear incircle touchpoints with the circumcircle. Then $\overline{AT_A}$, $\overline{BT_B}$ and $\overline{CT_C}$ are concurrent and the point of concurrency lies on OI.

Proof. Let ω_A, ω_B and ω_C be the A-mixtilinear, B-mixtilinear and C-mixtilinear incircles of $\triangle ABC$. Let K be the exsimilicenter of the incircle and circumcircle of $\triangle ABC$. Since



A is the exsimilicenter of the incircle and ω_A and T_A is the exsimilicenter of ω_A and the circumcircle $\implies$ Applying monge's circle theorem on ω_A, incircle and circumcircle, we get that K lies on $\overline{AT_A}$. Similarly, applying monge's circle theorem on ω_B, incircle and circumcircle $\implies$ $K \in \overline{BT_B}$ and on ω_C, incircle and circumcircle $\implies$ $K \in \overline{CT_C}$. Therefore $\overline{AT_A}$, $\overline{BT_B}$ and $\overline{CT_C}$ are concurrent at the exsimilicenter of the incircle and circumcircle, which is K and lies on OI by definition. □

§4 Examples

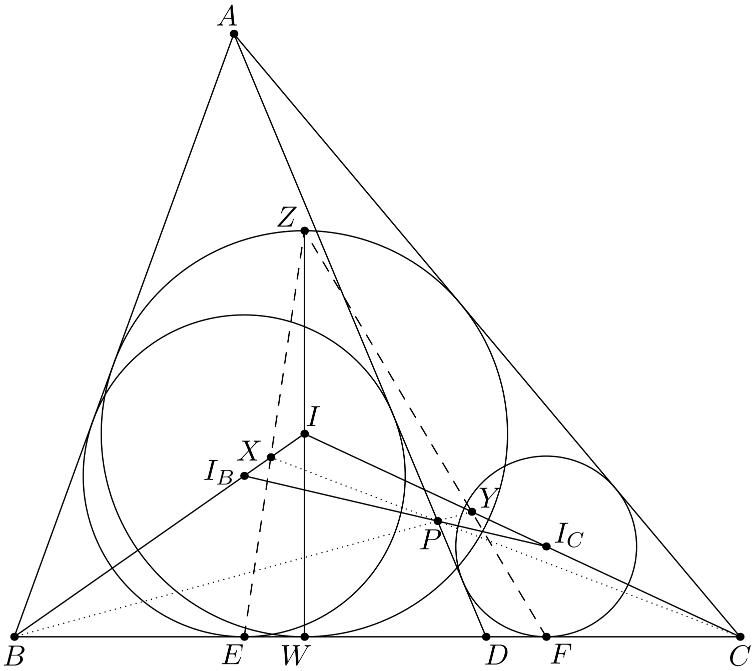
Problem 4.1 (USA TSTST 2017)

Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

Proof. Suppose the incircle touches $\overline{BC}$ at W and let Z be the diametrically opposite point of W in the incircle.

Claim 4.2. X lies on $\overline{EZ}$.

Proof. Observe that P is the insimilicenter of ω_B and ω_C. This is because $\overline{AD}$ is the internal common tangent of ω_B and ω_C and P lies on $\overline{I_BI_C}$. Furthermore, C is the exsimilicenter of ω_C and $\odot(I)$. This is because, $\overline{AC}$ and $\overline{BC}$ are external common tangents to ω_C and $\odot(I)$. Therefore, the insimilicenter of ω_B and $\odot(I)$ must lie on the line CP. This is precisely point X because X lies on $\overline{II_B}$. Since Z is the diametrically opposite point to the point of tangency of the external common tangent $\implies$ X lies on $\overline{EZ}$. □



Similarly, we can show that Y is the insimilicenter of $\odot(I)$ and $\omega_C \implies Y$ lies on $\overline{ZF}$ $\implies \overline{EX}$ and $\overline{YF}$ meet on the incircle at point Z. $\square$

Problem 4.3 (IMO Shortlist 2007)

Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

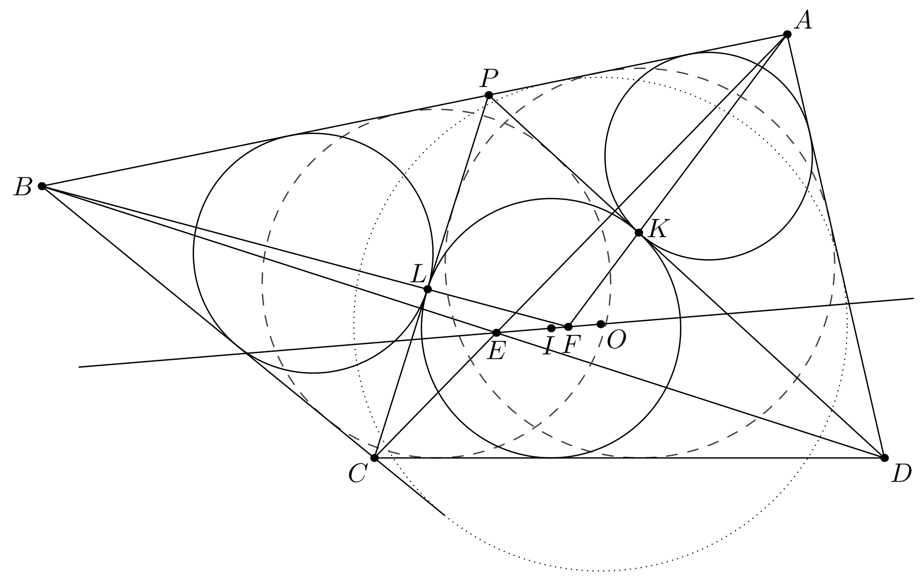
Proof. Let ω_A and ω_B be the incircles of $\triangle APD$ and $\triangle BPC$. Since

$$\frac{\overline{PC}+\overline{PD}-\overline{CD}}{2}=\overline{PK}=\frac{\overline{PA}+\overline{PD}-\overline{AD}}{2}$$

which implies that $\overline{PC}+\overline{AD}=\overline{PA}+\overline{CD}$. Therefore by pitot's theorem, the quadrilateral $PADC$ is circumscribed and has an incircle. Similarly, $PBCD$ has is circumscribed and has an incircle too. Let Ω_A be the inscribed circle of $PACD$ and Ω_B be the inscribed circle of $PBCD$. Let Ω be the circle tangent to the sides $\overline{AB}$, $\overline{BC}$ and $\overline{AD}$ and let O be the center of Ω.

Claim 4.4. F is the insimilicenter of ω and Ω.

Proof. Applying Monge's Circle Theorem on triplets of circles ω_A, ω and Ω, and ω_B, ω and Ω, we get that the insimilicenter of ω and Ω must lie on BL and AK. This is because A is the exsimilicenter of ω_A and Ω and K is the insimilicenter of ω_A and ω. Similarly, B is the exsimilicenter of ω_B and Ω and L is the insimilicenter of ω_B and ω $\implies$ F is the insimilicenter and must lie on OI. $\square$



Claim 4.5. E is the exsimilicenter of ω and Ω.

Proof. Applying Monge's Circle Theorem on triplets of circles ω, Ω_A and Ω, and ω, Ω_B and Ω, we get that E lies on BD and AC. This is because C is the exsimilicenter of ω and Ω_A an A is the exsimilicenter of Ω and Ω_A. Similarly, D is the exsimilicenter of ω and Ω_B and B is the exsimilicenter of Ω and Ω_B $\implies$ E is the exsimilicenter and must lie on OI. $\square$

This implies the points E, F, I and O are collinear. $\square$

§5 Exercises

Exercise 5.1. Let k_1 and k_2 be two given circles. Consider all circles k externally tangent to both of them and denote the tangency points by T_1 and T_2 respectively. Prove that line T_1T_2 passes through a fixed point.

Exercise 5.2. Given a triangle ABC, let Γ_A be a circle tangent to AB and AC, let Γ_B be a circle tangent to sides BA and BC, and let Γ_C be a circle tangent to CA and CB. Suppose $\Gamma_A, \Gamma_B, \Gamma_C$ are all tangents to one another . Let E be tangency point between Γ_C and Γ_A and let F be the tangency point between Γ_A and Γ_B. Prove that the lines BF and CE concur on the $A-$internal angle bisector of ΔABC.

Exercise 5.3 (Romania TST 2007)**.** Let ABC be a triangle, and ω_a, ω_b, ω_c be circles inside ABC, that are tangent (externally) one to each other, such that ω_a is tangent to AB and AC, ω_b is tangent to BA and BC, and ω_c is tangent to CA and CB. Let D be the common point of ω_b and ω_c, E the common point of ω_c and ω_a, and F the common point of ω_a and ω_b. Show that the lines AD, BE and CF have a common point.

Exercise 5.4 (China 2013)**.** Let non-intersecting circles $\omega_1, \omega_2, \omega_3$ all be internally tangent to a circle Ω at points A, B, C respectively. Let lines l_1 and l_2 and l_3 be common external tangents to circles ω_2, ω_3 and ω_3, ω_1 and ω_1, ω_2 and let $X = l_1 \cap l_3, Y = l_3 \cap l_1$, and $Z = l_1 \cap l_2$. Prove that lines AX, BY, CZ concur om line IO where I is the incenter of XYZ and O is the center of Ω

§6 Practice Problems

Exercise 6.1 (ELMO Shortlist 2011)**.** Let ABC be a triangle. Draw circles ω_A, ω_B, and ω_C such that ω_A is tangent to AB and AC, and ω_B and ω_C are defined similarly. Let P_A be the insimilicenter of ω_B and ω_C. Define P_B and P_C similarly. Prove that AP_A, BP_B, and CP_C are concurrent.

Exercise 6.2. Let k_1, k_2 be two circles and let ω be a circle externally tangent to both k_1 and k_2 at A, B respectively. Let Ω be a circle orthogonal to both k_1 and k_2 and let C be one of the intersections of Ω and k_1 and let D be one of the intersections of Ω and k_2. Then the exsimilicenter X of k_1 and k_2 is on radical axis of ω and Ω.

Exercise 6.3. Circles k_1 and k_2 are tangent to one of their common external tangents at T_1 and T_2 respectively. A circle k is externally tangent to k_1 and k_2 at points L_2, L_2 respectively. Prove that lines L_1T_1 and L_2T_2 concur on k.

Exercise 6.4 (RMM 2010)**.** Let $A_1A_2A_3A_4$ be a quadrilateral with no pair of parallel sides. For each $i = 1, 2, 3, 4$, define ω_1 to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i, A_iA_{i+1}$ and $A_{i+1}A_{i+2}$ (indices are considered modulo 4 so $A_0 = A_4, A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with the side A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.

Exercise 6.5 (RMM Shortlist 2019)**.** Let Ω be the circumcircle of an acute-angled triangle ABC. A point D is chosen on the internal bisector of $\angle ACB$ so that the points D and C are separated by AB. A circle ω centered at D is tangent to the segment AB at E. The tangents to ω through C meet the segment AB at K and L, where K lies on the segment AL. A circle Ω_1 is tangent to the segments AL, CL, and also to Ω at point M. Similarly, a circle Ω_2 is tangent to the segments BK, CK, and also to Ω at point N. The lines LM and KN meet at P. Prove that $\angle KCE = \angle LCP$.

Exercise 6.6 (Turkey 2022)**.** We have three circles w_1, w_2 and Γ at the same side of line l such that w_1 and w_2 are tangent to l at K and L and to Γ at M and N, respectively. We know that w_1 and w_2 do not intersect and they are not in the same size. A circle passing through K and L intersect Γ at A and B. Let R and S be the reflections of M and N with respect to l. Prove that A, B, R, S are concyclic.

Exercise 6.7 (Russia 2013)**.** Let ω be the incircle of the triangle ABC and with centre I. Let Γ be the circumcircle of the triangle AIB. Circles ω and Γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC.

Exercise 6.8 (RMM Shortlist 2019)**.** A quadrilateral $ABCD$ is circumscribed about a circle with center I. A point $P \neq I$ is chosen inside $ABCD$ so that the triangles PAB, PBC, PCD, and PDA have equal perimeters. A circle Γ centered at P meets the rays PA, PB, PC, and PD at A_1, B_1, C_1, and D_1, respectively. Prove that the lines PI, A_1C_1, and B_1D_1 are concurrent.