

Menelaus' Theorem

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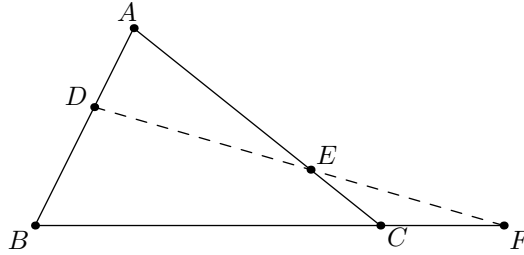
In this article, we introduce Menelaus' Theorem as a criterion for establishing collinearity using ratio of directed segments.

§1 Menelaus' Theorem

Theorem 1.1 (Menelaus' Theorem)

In a $\triangle ABC$, choose points D , E , F on lines AB , AC and BC . Then, the points D , E and F are collinear if and only if

$$\left(\frac{\overline{BF}}{\overline{FC}}\right) \cdot \left(\frac{\overline{CE}}{\overline{EA}}\right) \cdot \left(\frac{\overline{AD}}{\overline{DB}}\right) = -1$$



Remark 1.2. The negative sign appears because we are working with signed lengths. This convention ensures that the relation remains valid in every configuration. If we prefer to work only with magnitudes, the negative sign can be dropped and the statement will still hold true.

Proof. We will use the extended ratio lemma in $\triangle BFE$, $\triangle ABC$ and $\triangle AEB$ where segments EC , BE and ED act as the cevians, respectively. This gives us

$$\begin{aligned}\left(\frac{\overline{BF}}{\overline{FC}}\right) &= \left(\frac{\sin \angle BEF \sin \angle ECF}{\sin \angle CEF \sin \angle EBF}\right) \\ \left(\frac{\overline{CE}}{\overline{EA}}\right) &= \left(\frac{\sin \angle BAC \sin \angle EBF}{\sin \angle ABE \sin \angle BCA}\right) \\ \left(\frac{\overline{AD}}{\overline{DB}}\right) &= \left(\frac{\sin \angle AED \sin \angle ABE}{\sin \angle BEF \sin \angle BAC}\right)\end{aligned}$$

If we multiply these, the right hand side cancels out completely and results in 1, proving the result. \square

Exercise 1.3. There's another way to prove [Menelaus' Theorem](#). Draw a line ℓ parallel to line BC through A and extend \overline{EF} to meet ℓ at X . Do you see some similar triangles? Use them to prove the result.

Exercise 1.4. Use [Menelaus' Theorem](#) to prove [Ceva's Theorem](#).

§2 Examples

§2.1 Existence of the Centroid

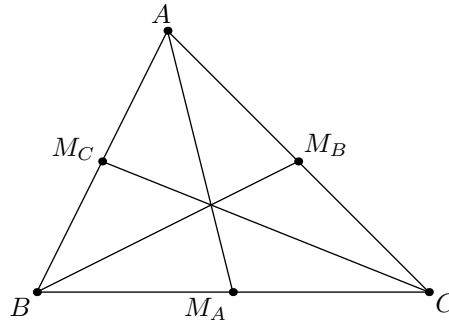
We will take a look at proving the *existence* of the centroid.

Definition 2.1. In a $\triangle ABC$, let M be the midpoint of segment \overline{BC} . Then, \overline{AM} is the A -median of the $\triangle ABC$.

The claim is that all the three medians of a triangle intersect at a point called the **Centroid** of the triangle.

Theorem 2.2 (Existence of the Centroid)

In a $\triangle ABC$, let M_A , M_B and M_C be the midpoints of the side \overline{BC} , \overline{CA} and \overline{AB} . Then the cevians AM_A , BM_B and CM_C are concurrent. The concurrency point is called the **Centroid** of $\triangle ABC$.



Proof. Using Ceva's Theorem, to prove the concurrency we only need to show that

$$\left(\frac{\overline{BM_A}}{\overline{M_AC}}\right) \cdot \left(\frac{\overline{CM_B}}{\overline{M_BA}}\right) \cdot \left(\frac{\overline{AM_C}}{\overline{M_CB}}\right) = 1$$

Since M_A , M_B and M_C are the midpoints of side \overline{BC} , \overline{CA} and \overline{AB} \implies

$$\overline{BM_A} = \overline{M_AC}$$

$$\overline{CM_B} = \overline{M_BA}$$

$$\overline{AM_C} = \overline{M_CB}$$

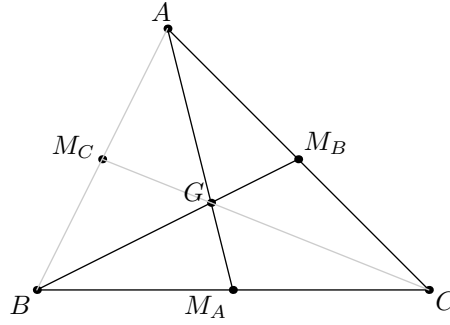
and the result follows. □

§2.2 Centroid Division

Theorem 2.3 (Centroid Division)

In a $\triangle ABC$, with G centroid. Let $\triangle M_A M_B M_C$ be the medial triangle of $\triangle ABC$. Then

$$\frac{\overline{AG}}{\overline{GM_A}} = \frac{\overline{BG}}{\overline{GM_B}} = \frac{\overline{CG}}{\overline{GM_C}} = 2$$



Proof. There are several ways to do this. Let's prove this using Menelaus' Theorem. Applying Menelaus' Theorem in $\triangle AM_A C$ where BM_B is the transversal, then we have

$$\left(\frac{\overline{CB}}{\overline{BM_A}} \right)^2 \cdot \left(\frac{\overline{M_A G}}{\overline{GA}} \right) \cdot \left(\frac{\overline{AM_B}}{\overline{M_B C}} \right) = 1 \implies \frac{\overline{AG}}{\overline{GM_A}} = 2$$

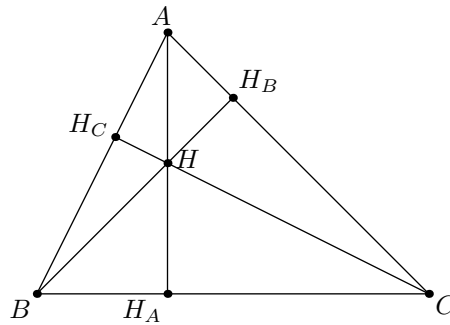
which implies the desired result. \square

§2.3 Existence of the Orthocenter

The orthocenter is another important triangle center that we shall see quite often. It is the intersection of the three altitudes of the triangle. Let's show that the three altitudes of the triangle are concurrent!

Theorem 2.4 (Existence of the Orthocenter)

In a $\triangle ABC$, let \overline{AD} , \overline{BE} and \overline{CF} be the three altitudes of the $\triangle ABC$. Then the altitudes \overline{AD} , \overline{BE} and \overline{CF} are all concurrent. The concurrency point is called the **Orthocenter** of $\triangle ABC$.



Proof. Again there are several ways to do this. Let's use the Trigonometric form of Ceva's Theorem to prove this. We have \overline{AD} , \overline{BE} and \overline{CF} are concurrent if and only if

$$\left(\frac{\sin \angle BAD}{\sin \angle DAC} \right) \cdot \left(\frac{\sin \angle CBE}{\sin \angle EBA} \right) \cdot \left(\frac{\sin \angle ACF}{\sin \angle FCB} \right) = 1$$

from the definitions of altitudes, we have that the above condition is equivalent to having

$$\left(\frac{\cos \angle B}{\cos \angle C} \right) \cdot \left(\frac{\cos \angle C}{\cos \angle A} \right) \cdot \left(\frac{\cos \angle A}{\cos \angle B} \right) = 1$$

which is true implying the concurrency. \square

An important property about the orthocenter is the fact that it is the isogonal conjugate of the circumcenter! Prove this fact.

Exercise 2.5. In a $\triangle ABC$, show that the orthocenter H is the **Isogonal Conjugate** of the circumcenter O .

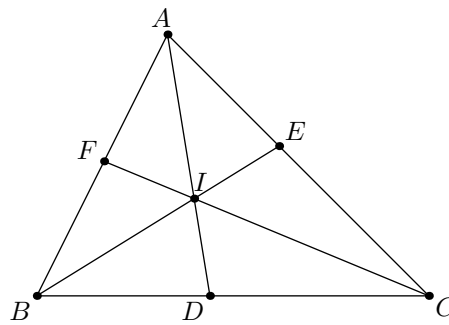
Just to mention, the line passing through points H and O is called the **Euler Line**. There are several interesting properties of this line as we shall see later on.

§2.4 Existence of Incenter

Incenter is yet another important triangle center that we need to study. It is the intersection of the three internal angle bisectors of the triangle. Let's prove that the three angle bisectors of the triangle are concurrent.

Theorem 2.6 (Existence of the Incenter)

In a $\triangle ABC$, let \overline{AD} , \overline{BE} and \overline{CF} be the three angle bisectors. Then these angle bisectors are all concurrent. The point of concurrency is called the **Incenter** of $\triangle ABC$.



Proof. Since these cevians are the internal angle bisectors, we know that $\angle BAD = \angle DAC$ and therefore we can use the Trigonometric form of Ceva's Theorem,

$$\left(\frac{\sin \angle BAD}{\sin \angle DAC} \right) \cdot \left(\frac{\sin \angle CBE}{\sin \angle EBA} \right) \cdot \left(\frac{\sin \angle ACF}{\sin \angle FCB} \right) = 1$$

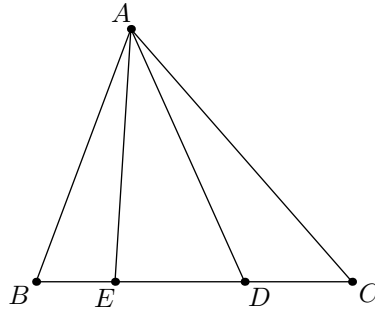
and since all the terms in the left hand side cancel out to 1, we have the desired concurrency. \square

§2.5 Isogonal Ratio Lemma

Problem 2.7

In $\triangle ABC$, let D and E be points on side \overline{BC} such that the cevians \overline{AD} and \overline{AE} are isogonal. Then

$$\left(\frac{\overline{BD}}{\overline{DC}}\right) \cdot \left(\frac{\overline{BE}}{\overline{EC}}\right) = \left(\frac{\overline{AB}}{\overline{AC}}\right)^2$$



Proof. Apply the Ratio Lemma on $\triangle ABC$ with cevian \overline{AD} and \overline{AE} . From these we get

$$\left(\frac{\overline{BD}}{\overline{DC}}\right) = \left(\frac{\overline{AB}}{\overline{AC}}\right) \cdot \left(\frac{\sin \angle BAD}{\sin \angle DAC}\right)$$

and similarly,

$$\left(\frac{\overline{BE}}{\overline{EC}}\right) = \left(\frac{\overline{AB}}{\overline{AC}}\right) \cdot \left(\frac{\sin \angle BAE}{\sin \angle EAC}\right)$$

However by the definition of the cevians being isogonal, we have $\angle BAD = \angle EAC$ and $\angle BAE = \angle DAC$. Therefore multiplying these equations, we get

$$\left(\frac{\overline{BD}}{\overline{DC}}\right) \cdot \left(\frac{\overline{BE}}{\overline{EC}}\right) = \left(\frac{\overline{AB}}{\overline{AC}}\right)^2$$

proving the result as desired. \square

§2.6 RMO 2025 P5

Problem 2.8 (KV RMO 2025 P5)

Let ABC be an acute-angled triangle with $\angle BAC = 60^\circ$ and $AB < BC < AC$. Let M, N be the midpoints of AB, AC respectively. Suppose BE, CF are altitudes, with E on CA and F on AB . Let X be the image of M under reflection in the midpoint of BF , and Y be the image of N under reflection in the midpoint of CE . Prove that XY bisects BC .

Proof. Let M_A be the midpoint of \overline{BC} . By definition, $\overline{BX} = \overline{MF}$ and $\overline{CY} = \overline{EN}$. So we just want to calculate \overline{MF} .

$$\overline{MF} = \overline{BM} - \overline{BF}$$

$$\begin{aligned}
&= \frac{c}{2} - a \cos B \\
&= 2R \left(\frac{\sin C}{2} - \frac{\sqrt{3}}{2} \cos B \right) \\
&= 2R \left(\sin C - \frac{\sqrt{3}}{2} \cos B - \frac{1}{2} \sin C \right) \\
&= 2R \left(\sin(B + 60^\circ) - \frac{\sqrt{3}}{2} \cos B - \frac{1}{2} \sin C \right) \\
&= 2R \left(\frac{1}{2} \sin B - \frac{1}{2} \sin C \right) \\
&= \frac{b - c}{2}
\end{aligned}$$

Therefore $\overline{BX} = \frac{b-c}{2} \implies \overline{AX} = \frac{b+c}{2}$. Similarly, $\overline{CY} = \frac{b-c}{2}$ and $\overline{AY} = \frac{b+c}{2}$. As a result, we have

$$\begin{aligned}
\left(\frac{\overline{AX}}{\overline{XB}} \right) \cdot \left(\frac{\overline{BM_A}}{\overline{M_A C}} \right) \cdot \left(\frac{\overline{CY}}{\overline{YA}} \right) &= \left(\frac{\frac{b+c}{2}}{\frac{b-c}{2}} \right) \cdot \left(\frac{\frac{b-c}{2}}{\frac{b+c}{2}} \right) \\
&= 1
\end{aligned}$$

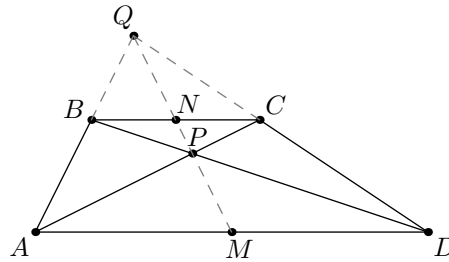
Hence by the converse of Menelaus' Theorem, the points X , M_A and Y are collinear $\implies \overline{XY}$ bisects \overline{BC} . \square

§2.7 Midpoints of Trapezium

There is a very beautiful result about trapeziums. Let's state and prove it.

Problem 2.9 (Concurrency on Midpoints of Trapezium)

Consider a trapezium $ABCD$ with side \overline{BC} parallel to side \overline{AD} . Suppose the diagonals of this trapezium intersect at point P . Let M and N be the midpoints of sides \overline{AD} and \overline{BC} . Then the points M , P and N are collinear.



Proof. Suppose we extend the sides AB and CD to meet at Q . Since $\overline{BC} \parallel \overline{AD}$, therefore

$$\left(\frac{\overline{QB}}{\overline{BA}} \right) = \left(\frac{\overline{QC}}{\overline{CD}} \right)$$

the motivation for the solution is that the terms in equality remind us of Ceva's Theorem to $\triangle QAD$. Infact it holds that,

$$\left(\frac{\overline{AM}}{\overline{MD}} \right) \cdot \left(\frac{\overline{DC}}{\overline{CQ}} \right) \cdot \left(\frac{\overline{QB}}{\overline{BA}} \right) = 1$$

because M is the midpoint of segment \overline{AD} . Hence by converse of Ceva's Theorem, we have that \overline{QM} , \overline{AC} and \overline{BD} are concurrent, but \overline{AC} and \overline{BD} intersect at point P , therefore \overline{QM} passes through P too. Hence PQ bisects \overline{AD} . Since $\triangle QBC \sim \triangle QAD \implies QP$ bisects \overline{BC} as well. \square

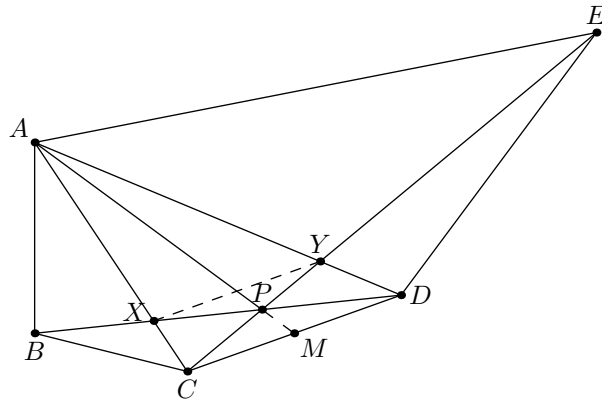
§2.8 IMO Shortlist 2006

Problem 2.10 (ISL 2006 G3)

Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P . Prove that the line AP bisects the side CD .



Proof. Suppose \overline{BD} intersects \overline{AC} at X and \overline{CE} intersects \overline{AD} at Y . If we want AP bisects \overline{CD} , then by Ceva's Theorem, we want $\overline{XY} \parallel \overline{BC}$. The angle condition implies that $\triangle ABC \sim \triangle ACD \sim \triangle ADE$. Hence $\triangle ABD \sim \triangle ACE$. Since \overline{AX} and \overline{AY} are angle bisectors of $\angle BAD$ and $\angle CAE \implies \triangle ABX \sim \triangle ACY$

$$\left(\frac{\overline{AB}}{\overline{AX}} \right) = \left(\frac{\overline{AC}}{\overline{AY}} \right)$$

Since $\triangle ABC \sim \triangle ACD$, we get

$$\left(\frac{\overline{AB}}{\overline{AC}} \right) = \left(\frac{\overline{AC}}{\overline{AD}} \right)$$

Comparing these two, we get

$$\left(\frac{\overline{AX}}{\overline{AY}} \right) = \left(\frac{\overline{AC}}{\overline{AD}} \right)$$

which implies that $\overline{XY} \parallel \overline{BC}$. \square

Remark 2.11. There is another solution that avoids Ceva's Theorem. We angle chase to show that $\odot(ABC)$ and $\odot(ADE)$ are tangent to \overline{CD} and the conclusion follows from the **Power of a Point** theorem.

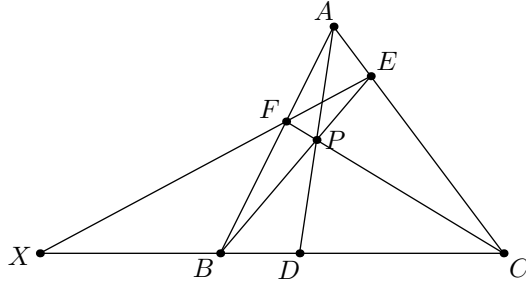
§2.9 Cevian Triangles

We'll discuss two popular results about cevian triangles here.

Problem 2.12 (Harmonic Bundles)

In a $\triangle ABC$, choose a point P inside the triangle and let $\triangle DEF$ be the cevian triangle of P . Let EF intersect BC at X . Then

$$\frac{\overline{BD}}{\overline{CD}} = \frac{\overline{BX}}{\overline{CX}}$$



To prove this, we apply both, Ceva's theorem and Menelaus' theorem on $\triangle ABC$.

Proof. Applying Menelaus' Theorem on $\triangle ABC$, we have

$$\left(\frac{\overline{CX}}{\overline{BX}}\right) \cdot \left(\frac{\overline{BF}}{\overline{FA}}\right) \cdot \left(\frac{\overline{AE}}{\overline{EC}}\right) = 1$$

Applying Ceva's Theorem on $\triangle ABC$, we have

$$\left(\frac{\overline{AE}}{\overline{EC}}\right) \cdot \left(\frac{\overline{CD}}{\overline{DB}}\right) \cdot \left(\frac{\overline{BF}}{\overline{FA}}\right) = 1$$

Dividing these two equations, we get our desired result. \square

Problem 2.13

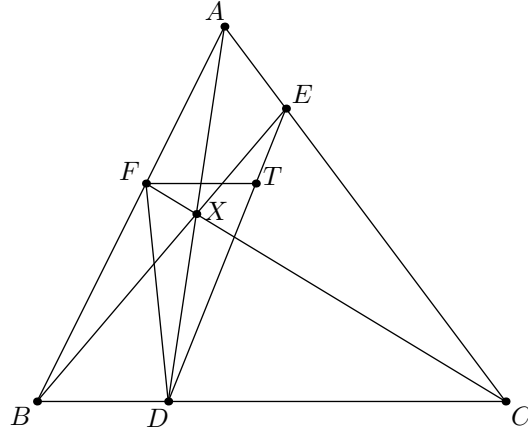
Let X be a point inside $\triangle ABC$ and $\triangle DEF$ be the cevian triangle of point X with respect to $\triangle ABC$. Draw a line parallel to \overline{BC} through F that intersects \overline{DE} at T . Show that \overline{AD} bisects \overline{FT} .

Proof. Let AX intersect \overline{FT} at M . Using the extended ratio lemma, we have

$$\begin{aligned} \left(\frac{\overline{FM}}{\overline{MT}}\right) &= \left(\frac{\sin \angle FDA \sin \angle DTF}{\sin \angle EDA \sin \angle DFT}\right) \\ &= \left(\frac{\sin \angle FDA \sin \angle EDC}{\sin \angle EDA \sin \angle FDB}\right) \end{aligned}$$

Applying ratio lemma in $\triangle ADB$ and $\triangle ADC$, we get

$$\left(\frac{\sin \angle FDA}{\sin \angle FDB}\right) = \left(\frac{\overline{AF}}{\overline{FB}}\right) \cdot \left(\frac{\overline{BD}}{\overline{AD}}\right)$$



and

$$\left(\frac{\sin \angle EDC}{\sin \angle EDA} \right) = \left(\frac{\overline{CE}}{\overline{EA}} \right) \cdot \left(\frac{\overline{AD}}{\overline{DC}} \right)$$

Using these, we can write

$$\begin{aligned} \left(\frac{\overline{FM}}{\overline{MT}} \right) &= \left(\frac{\overline{AF}}{\overline{FB}} \right) \cdot \left(\frac{\overline{BD}}{\overline{AD}} \right) \cdot \left(\frac{\overline{CE}}{\overline{EA}} \right) \cdot \left(\frac{\overline{AD}}{\overline{DC}} \right) \\ &= \left(\frac{\overline{AF}}{\overline{FB}} \right) \cdot \left(\frac{\overline{BD}}{\overline{DC}} \right) \cdot \left(\frac{\overline{CE}}{\overline{EA}} \right) \\ &= 1 \end{aligned}$$

Since the cevians \overline{AD} , \overline{BE} and \overline{CF} are concurrent $\implies \overline{FM} = \overline{MT}$. Hence AD bisects FT . \square

§2.10 Lemoine Line

An application of Menelaus' Theorem when the transversal lies completely outside the triangle. This is a nice trick worth demonstrating.

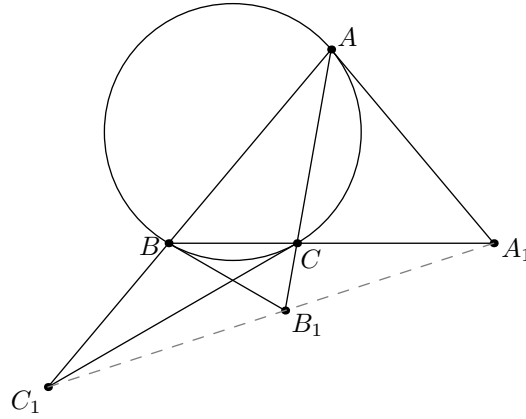
Problem 2.14

Let A_1 be the intersection of tangent at A to the circumcircle of a triangle $\triangle ABC$ with sideline BC . Similarly define B_1 , C_1 . Show that A_1 , B_1 and C_1 are collinear. This line is also known as the **Lemoine Line**.

It is not very obvious to spot the menelaus triangle in the diagram, because we are very habitual of seeing the transversal line cutting the triangle. However, Menelaus' Theorem is valid even if the transversal lies completely outside the triangle (which is the current case).

Proof. Apply Menelaus' Theorem in $\triangle ABC$ with points A_1 , B_1 and C_1 . These points are collinear if and only if the ratios of unsigned lengths

$$\left(\frac{\overline{BA_1}}{\overline{A_1C}} \right) \cdot \left(\frac{\overline{CB_1}}{\overline{B_1A}} \right) \cdot \left(\frac{\overline{AC_1}}{\overline{C_1B}} \right) = 1$$



We can compute the left hand side ratios using ratio lemma applied on $\triangle ABC$ for points A_1 , B_1 and C_1 .

$$\begin{aligned} \left(\frac{\overline{BA_1}}{\overline{A_1C}} \right) &= \left(\frac{\overline{AB}}{\overline{AC}} \right) \cdot \left(\frac{\sin \angle BAA_1}{\sin \angle CAA_1} \right) \\ \left(\frac{\overline{CB_1}}{\overline{B_1A}} \right) &= \left(\frac{\overline{BC}}{\overline{AB}} \right) \cdot \left(\frac{\sin \angle CBB_1}{\sin \angle ABB_1} \right) \\ \left(\frac{\overline{AC_1}}{\overline{C_1B}} \right) &= \left(\frac{\overline{AC}}{\overline{BC}} \right) \cdot \left(\frac{\sin \angle ACC_1}{\sin \angle BCC_1} \right) \end{aligned}$$

Let's multiply all these, we will have

$$\left(\frac{\overline{BA_1}}{\overline{A_1C}} \right) \cdot \left(\frac{\overline{CB_1}}{\overline{B_1A}} \right) \cdot \left(\frac{\overline{AC_1}}{\overline{C_1B}} \right) = \left(\frac{\sin \angle BAA_1}{\sin \angle CAA_1} \right) \cdot \left(\frac{\sin \angle CBB_1}{\sin \angle ABB_1} \right) \cdot \left(\frac{\sin \angle ACC_1}{\sin \angle BCC_1} \right)$$

However since AA_1 , BB_1 and CC_1 are tangents, we have

$$\sin \angle BAA_1 = \sin \angle A_1CA = \sin (180^\circ - \angle A_1CA) = \sin \angle ACB$$

and

$$\sin \angle CAA_1 = \sin \angle ABC$$

Similarly for all the other angles, and we eventually have

$$\begin{aligned} \left(\frac{\overline{BA_1}}{\overline{A_1C}} \right) \cdot \left(\frac{\overline{CB_1}}{\overline{B_1A}} \right) \cdot \left(\frac{\overline{AC_1}}{\overline{C_1B}} \right) &= \left(\frac{\sin \angle BAA_1}{\sin \angle CAA_1} \right) \cdot \left(\frac{\sin \angle CBB_1}{\sin \angle ABB_1} \right) \cdot \left(\frac{\sin \angle ACC_1}{\sin \angle BCC_1} \right) \\ &= \left(\frac{\sin \angle ACB}{\sin \angle ABC} \right) \cdot \left(\frac{\sin \angle BAC}{\sin \angle ACB} \right) \cdot \left(\frac{\sin \angle ABC}{\sin \angle BAC} \right) \\ &= 1 \end{aligned}$$

which proves that A_1 , B_1 and C_1 are collinear. □

Remark 2.15. This is a very straight forward application of [Pascal's Theorem](#). We will learn later how to use pascal's theorem.

§3 Practice Problems

Exercise 3.1 (USA Math Olympiad 2003). Let ABC be a triangle. A circle passing through A and B intersects the segments AC and BC at D and E , respectively. Lines AB and DE intersect at F , while line BD and CF intersect at M . Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Exercise 3.2 (Blanchet's Theorem). Let D be the foot of the altitude from A in triangle ABC and let M, N be points on the sides CA, AB such that the lines BM, CN intersect on AD . Prove that AD is the bisector of angle $\angle MDN$.

Exercise 3.3 (Cevian Nest Theorem). Let D, E, F be points on sides BC, CA, AB respectively of a triangle ABC . Also let X, Y, Z be points on sides EF, FD, DE respectively of triangle DEF . Consider the three triples of lines (AX, BY, CZ) , (AD, BE, CF) , and (DX, EY, FZ) . If any two of these triples are concurrent, the third one is as well.

Exercise 3.4 (Trilinear Polar of P). Let P be an interior point in $\triangle ABC$. Let AP, BP and CP intersect the opposite sides at A_1, B_1 and C_1 . Consider the intersections of BC with B_1C_1 , CA with C_1A_1 and AB with A_1B_1 and name them X, Y, Z . Prove, X, Y and Z are collinear

Exercise 3.5 (IMO Shortlist 2015). Let ABC be an acute triangle with orthocenter H . Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI . Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J . Prove that $IJ = AH$.

Exercise 3.6. Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB respectively of a triangle ABC . Denote by G_a, G_b, G_c are the centroids of triangles $AB_1C_1, BC_1A_1, CA_1B_1$, respectively. Prove that the lines AG_a, BG_b, CG_c are concurrent if and only if lines AA_1, BB_1, CC_1 are concurrent.

Exercise 3.7 (Van Aubel's Theorem). Let P be the point in the interior of $\triangle ABC$. Let AP, BP and CP intersect opposite sides of the triangle at A', B' and C' , respectively. Prove,

$$\frac{PA}{PA'} = \frac{C'A}{C'B} + \frac{B'A}{B'C}$$

Exercise 3.8. Let $\overline{AD}, \overline{BE}, \overline{CF}$ be concurrent cevians in a triangle, meeting at P . Prove that,

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1$$

Exercise 3.9 (USA Math Olympiad 2012). Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P . Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.

Exercise 3.10 (China 2012). $\triangle ABC$ is a scalene triangle, the incircle touches BC, CA, AB at A_1, B_1, C_1 respectively. Let A_2 be the symmetry point of A_1 about B_1C_1 and let B_2, C_2 be defined similarly. Let $AA_2 \cap BC = A_3, BB_2 \cap CA = B_3, CC_2 \cap AB = C_3$. Prove A_3, B_3, C_3 are collinear. Further, show that line $\overline{A_3B_3C_3}$ is the the **Euler Line** of $\triangle A_1B_1C_1$.