ELMO 2025 Shortlist Solution Notes

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A compilation of solutions for the ELMO 2025 Shortlist.

Contents

§1 Problems

§1.1 Algebra

Problem (A1)

Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for all positive integers m and n,

$$f^m(n) + f(mn) = f(m)f(n).$$

Note: $f^m(n) = \underbrace{f(f(\cdots f}_{m \text{ times}}(n)\cdots))$, that is, f applied m times to n.

§1.2 Combinatorics

§1.3 Geometry

Problem (G1)

Let $ABCD$ be a convex quadrilateral with $DA = AB = BC$. Let M be the midpoint of $\overline{AB}$, and let P be a point in the plane with $\angle PCA = \angle PDB = 90^\circ$. A circle centered at O is tangent to segments DA, AB, and BC. Prove that M, O, and P are collinear.

§1.4 Number Theory

§2 Solutions to Algebra

§2.1 ELMO 2025 Shortlist A1

Problem (A1)

Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for all positive integers m and n,

$$f^m(n) + f(mn) = f(m)f(n).$$

Note: $f^m(n) = \underbrace{f(f(\cdots f}_{m \text{ times}}(n)\cdots))$, that is, f applied m times to n.

§2.1.1 Solution 1 (Using Injectivity & Periodicity)

Let $P(m, n)$ denote the assertion,

$$f^m(n) + f(mn) = f(m)f(n)$$

Claim 2.1. $f \equiv 2$ and $f \equiv m + 1$ are the only solutions.

Proof. It's easy to see that these satisfy the assertion $P(m, n)$. We will show that these are the only solutions that satisfy.

From $P(1, 1)$, we have

$$f(1) + f(1) = f(1)f(1) \implies f(1) = 2$$

From $P(m, 1)$, we have

$$f^m(1) + f(m) = f(1)f(m) = 2f(m)$$

Therefore, $f^m(1) = f(m)$. We will now deal with the following two cases independently.

1. Suppose f is injective. Then,
$$f^{m+1}(1) = f(m + 1) \implies f(m + 1) = f(f(m)) \implies \boxed{f(m) = m + 1}$$
2. Suppose f is not injective. Then there exists a, b such that $a \neq b$ and $f(a) = f(b)$. Comparing $P(a, 1)$ and $P(b, 1)$,
$$f^a(1) = f(a) = f(b) = f^b(1)$$
Since, $f(m) = f^m(1)$. It follows that f only takes finite values and f is periodic. Suppose f achieves the largest value L at u. From $P(2, u)$,
$$f(f(u)) + f(2u) = f(2)f(u) \implies 2L \geq f(L) + f(2u) = f(2)L$$
Therefore $f(2) \leq 2$. If $f(2) = 1$, then we get that
$$f(m) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 2, & \text{if } m \text{ is odd.} \end{cases}$$
which is wrong since $P(2, 2) \implies f(f(2)) + f(4) = f(2)^2$, for which we get that its impossible. If $f(2) = 2$, then $f(m) = f^{m-1}(2) = 2 \implies \boxed{f(m) = 2}$

□

§3 Solutions to Combinatorics

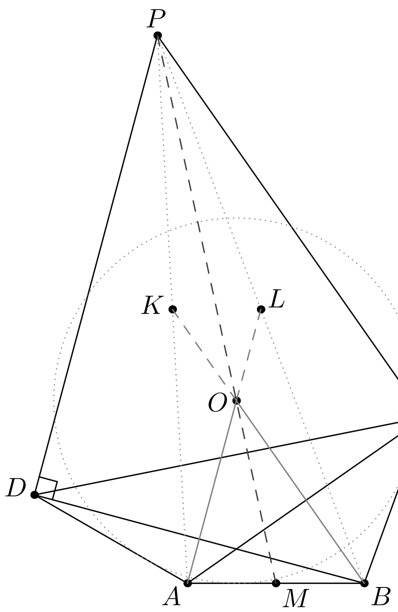
§4 Solutions to Geometry

§4.1 ELMO 2025 Shortlist G1

Problem (G1)

Let $ABCD$ be a convex quadrilateral with $DA = AB = BC$. Let M be the midpoint of $\overline{AB}$, and let P be a point in the plane with $\angle PCA = \angle PDB = 90°$. A circle centered at O is tangent to segments DA, AB, and BC. Prove that M, O, and P are collinear.

§4.1.1 Solution 1 (Using Centroid)



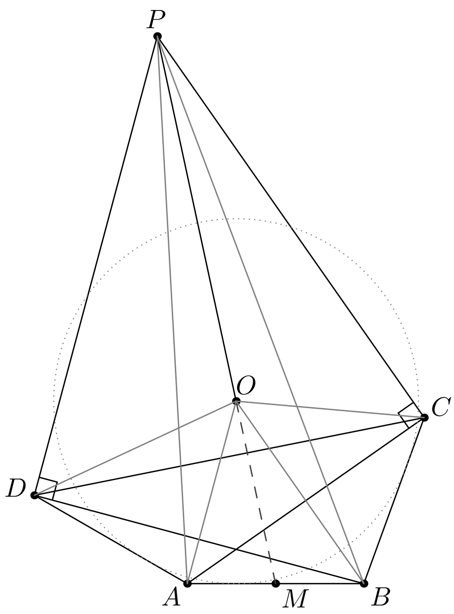
Proof. Let K and L be the midpoints of $\overline{AP}$ and $\overline{BP}$. Suppose ω is the circle tangent to sides $\overline{DA}$, $\overline{AB}$ and $\overline{BC}$.

Claim 4.1. L lies on line OA and K lies on line OB.

Proof. Since, $\overline{AB}$ and $\overline{BC}$ are equal in length and are tangents drawn from B to ω, we have that $\overline{OB}$ is the perpendicular bisector of $\overline{AC}$. Similarly, $\overline{OA}$ is the perpendicular bisector of $\overline{BD}$. Since $\angle ACP = 90° \implies \overline{OB} \parallel \overline{CP}$ and similarly, $\overline{OA} \parallel \overline{DP}$. Therefore by midpoint theorem in $\triangle ACP$ and $\triangle BDP$, we get that OA passes through L and OB passes through K. □

From Claim 1, we get that O is the centroid of $\triangle PAB$ and therefore, PO bisects $\overline{AB}$. □

§4.1.2 Solution 2 (Using Areas)



Claim 4.2. Line $\overline{OA} \parallel \overline{DP}$ and $\overline{OB} \parallel \overline{CP}$.

Proof. Proof. Same as Claim 1 □

Therefore,

$$\begin{aligned} \text{Area}\,(\triangle AOP) &= \text{Area}\,(\triangle AOD) = \text{Area}\,(\triangle AOB) = \text{Area}\,(\triangle BOC) \\ &= \text{Area}\,(\triangle BOP) \end{aligned}$$

Since, $\text{Area}\,(\triangle AOP) = \text{Area}\,(\triangle BOP) \implies PO$ bisects $\overline{AB}$ □

§5 Solutions to Number Theory