

Linearity of Power of a Point

MMUKUL KHEDEKAR

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In this article, we will learn about an advanced technique that exploits the linearity of the power of a point function.

§1 Background

Let's reiterate over the definition of the power of a point function.

Definition 1.1. For a given circle ω centered at O and radius r , and a point P , the **power of a point** is defined as

$$\text{Pow}_\omega(P) = \overline{OP}^2 - r^2$$

The big claim here is that the difference of power of a point computed against two circles is *linear*. In mathematics, when we talk about **linear functions**, we mean functions that satisfy the following conditions

$$\begin{aligned} f(x + y) &= f(x) + f(y) \\ f(\alpha x) &= \alpha f(x) \end{aligned}$$

In general, these conditions can be combined and written as

$$f(\alpha x + (1 - \alpha)x) = \alpha f(x) + (1 - \alpha)f(x)$$

for some real number α .

§2 Proving Linearity

Theorem 2.1 (Linearity of Power of a Point)

Define a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$f(P) = \text{Pow}_{\omega_1}(P) - \text{Pow}_{\omega_2}(P)$$

for two fixed circles ω_1 and ω_2 . Then f is linear.

Proof. Suppose the centers of the circles ω_1 and ω_2 are O_1 and O_2 , and their radii are r_1 and r_2 respectively. Pick two points A and B and choose point C on \overline{AB} such that

$$\frac{\overline{AC}}{\overline{CB}} = \frac{k}{1-k}$$

We would like to show that

$$f(C) = kf(A) + (1 - k)f(B)$$

Simplifying the left hand side expression

$$\begin{aligned} f(C) &= \text{Pow}_{\omega_1}(C) - \text{Pow}_{\omega_2}(C) \\ &= (\overline{O_1C}^2 - r_1^2) - (\overline{O_2C}^2 - r_2^2) \end{aligned}$$

Applying Stewart's Theorem on $\triangle ABO_1$ and $\triangle ABO_2$,

$$\begin{aligned} &= (\overline{O_1C}^2 - r_1^2) - (\overline{O_2C}^2 - r_2^2) \\ &= \left(-k(1-k)\overline{AB}^2 + ((1-k)\overline{O_1B}^2 + k\overline{O_1A}^2) \right) \\ &\quad - \left(-k(1-k)\overline{AB}^2 + ((1-k)\overline{O_2B}^2 + k\overline{O_2A}^2) \right) + r_2^2 - r_1^2 \\ &= k(\overline{O_1A}^2 - \overline{O_2A}^2) + (1-k)(\overline{O_1B}^2 - \overline{O_2B}^2) + r_2^2 - r_1^2 \\ &= k(\overline{O_1A}^2 - r_1^2) - k(\overline{O_2A}^2 - r_2^2) \\ &\quad + (1-k)(\overline{O_1B}^2 - r_1^2) - (1-k)(\overline{O_2B}^2 - r_2^2) \\ &= kf(A) + (1 - k)f(B) \end{aligned}$$

thus proving the linearity. \square

§3 Applications: Radical Axis

Linearity of power of a point turns out to be a good criteria to determine if a point lies on the radical axis of two circles. Since the radical axis is the locus of points having equal power, it essentially means that it's the locus of zeros of f .

Problem 3.1 (USA 2020)

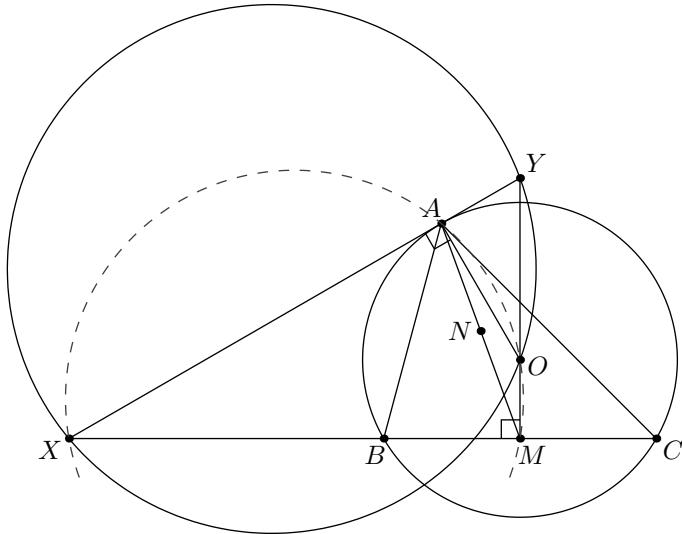
Let O and Γ denote the circumcenter and circumcircle, respectively, of scalene $\triangle ABC$. Furthermore, let M be the midpoint of side \overline{BC} . The tangent to Γ at A intersects BC and OM at points X and Y , respectively. If the circumcircle of $\triangle OXY$ intersects Γ at two distinct points P and Q , prove that PQ bisects \overline{AM} .

Proof. Suppose N is the midpoint of \overline{AM} . We would like to show that N lies on the radical axis of the circles $\odot(ABC)$ and $\odot(OXY)$. Suppose f is defined as

$$f(P) = \text{Pow}_{\odot(ABC)}(P) - \text{Pow}_{\odot(OXY)}(P)$$

Then we would like to show that

$$f(N) = \frac{1}{2}(f(A) + f(M)) = 0$$



Keeping in mind that the power of a point function is negative for points inside the circle,

$$\begin{aligned}
 f(A) + f(M) &= (0 - (-\overline{AX} \cdot \overline{AY})) + (-\overline{BM} \cdot \overline{MC} - \overline{OM} \cdot \overline{MY}) \\
 &= \overline{AX} \cdot \overline{AY} - \frac{1}{4}\overline{BC}^2 - \overline{OM} \cdot \overline{MY} \\
 &= \overline{AX} \cdot \overline{AY} + \overline{OM}^2 - \overline{OB}^2 - \overline{OM} \cdot \overline{MY} \\
 &= \overline{AX} \cdot \overline{AY} - \overline{OA}^2 - \overline{OM} \cdot \overline{OY} \\
 &= \overline{AX} \cdot \overline{AY} + \overline{AY}^2 - \overline{OY}^2 - \overline{OM} \cdot \overline{OY} \\
 &= \overline{AY} \cdot \overline{XY} - \overline{OY} \cdot \overline{MY}
 \end{aligned}$$

Since $\angle OAX = \angle OMX = 90^\circ \implies AOMX$ is a cyclic quadrilateral. Therefore,

$$\overline{AY} \cdot \overline{XY} = \overline{YO} \cdot \overline{YM} \implies f(A) + f(M) = 0$$

hence $f(N) = 0$, implying that the midpoint of \overline{AM} lies on the radical axis \overline{PQ} . \square

§3.1 Exercises

Exercise 3.2 (Taiwan TST 2016). Let O be the circumcenter of triangle ABC , and ω be the circumcircle of triangle BOC . Line AO intersects with circle ω again at the point G . Let M be the midpoint of side BC , and the perpendicular bisector of BC meets circle ω at the points O and N . Prove that the midpoint of the segment AN lies on the radical axis of the circumcircle of triangle OMG , and the circle whose diameter is AO .

Exercise 3.3. Given $\triangle ABC$ inscribed in $\odot(O)$. Let M be the midpoint of \overline{BC} , H be the projection of A onto \overline{BC} . Suppose \overline{OH} meets \overline{AM} at P . Prove that P lies on the radical axis of $\odot(BOC)$ and the nine-point circle of $\triangle ABC$.

Exercise 3.4 (USA TSTST 2016). Let ABC be a triangle with incenter I , and whose incircle is tangent to \overline{BC} , \overline{CA} , \overline{AB} at D , E , F , respectively. Let K be the foot of the altitude from D to \overline{EF} . Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2 , while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2 . Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of \overline{DK} .

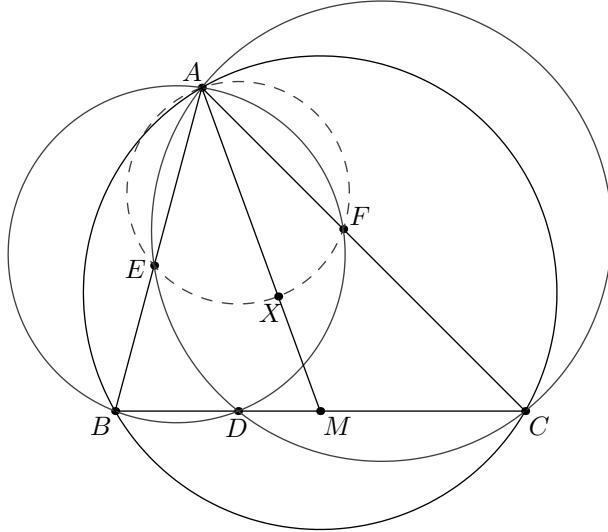
Exercise 3.5. Let I be the incenter and O be the circumcenter of $\triangle ABC$. Let M be the midpoint of \overline{BC} . Suppose $\odot(I)$ touches \overline{BC} at D and M' is the reflection of M over \overline{AI} . Then $\overline{DM}' \perp \overline{OI}$.

§4 Applications: Identifying Fixed Points & Locus

Often we would like to show that a circle passes through a fixed point. There are several ways to prove such results. One of the ways is to pick a fixed point on the secant through the desired fixed point and show that the power of point is constant.

Problem 4.1 (ELMO Shortlist 2013)

In $\triangle ABC$, a point D lies on line BC . The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A , and that this point lies on the median from A to BC .



Proof. If we can show that the power of M with respect to $\odot(AEF)$ is constant, then that would mean $\odot(AEF)$ passes through a fixed point on \overline{AM} . Suppose $\overline{AM} \cap \odot(AEF) = X$. Define f as

$$f(P) = \text{Pow}_{\odot(ABC)}(P) - \text{Pow}_{\odot(AEF)}(P)$$

Using the fact that f is linear, we get

$$\begin{aligned} f(M) &= \frac{1}{2}(f(B) + f(C)) \\ &= \frac{1}{2}((0 - \overline{BE} \cdot \overline{AB}) + (0 - \overline{CF} \cdot \overline{AC})) \\ &= -\frac{1}{2}(\overline{BE} \cdot \overline{AB} + \overline{CF} \cdot \overline{AC}) \\ &= -\frac{1}{2}(\overline{BD} \cdot \overline{BC} + \overline{CD} \cdot \overline{BC}) \\ &= -\frac{1}{2}\overline{BC}^2 \end{aligned}$$

Since,

$$f(M) = \text{Pow}_{\odot(ABC)}(M) - \text{Pow}_{\odot(AEF)}(M)$$

$$\begin{aligned} &= -\overline{BM} \cdot \overline{MC} - \text{Pow}_{\odot(AEF)}(M) \\ &= -\frac{1}{2}\overline{BC}^2 \end{aligned}$$

Therefore,

$$\text{Pow}_{\odot(AEF)} = \frac{1}{4}\overline{BC}^2$$

which is a constant. This shows that the product $\overline{AM} \cdot \overline{MX}$ is a constant and thus $\odot(AEF)$ passes through the fixed point X . \square

§4.1 Exercises

Exercise 4.2 (USA Math Olympiad 2015). Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment \overline{PQ} . Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that \overline{XT} is perpendicular to \overline{AX} . Let M denote the midpoint of chord \overline{ST} . As X varies on segment \overline{PQ} , show that M moves along a circle.

Exercise 4.3 (USA TST 2012). In acute triangle ABC , $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC . Points D and E lie on sides AB and AC , respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC , the circumcircle of triangle ADE passes through a fixed point other than A .

Exercise 4.4 (USA TST 2021). Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ , in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$. Let X be a variable point on the arc V_1V_2 of Γ not containing A or B . Line XA meets line U_1V_1 at C , while line XB meets line U_2V_2 at D . Let O and ρ denote the circumcenter and circumradius of $\triangle XCD$, respectively. Prove there exists a fixed point K and a real number c , independent of X , for which $OK^2 - \rho^2 = c$ always holds regardless of the choice of X .

Exercise 4.5 (IMO 2019). Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA , and AB at D, E , and F , respectively. The line through D perpendicular to EF meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q . Prove that lines DI and PQ meet on the line through A perpendicular to AI .