Lecture Notes (29th Dec, 2025)

Mmukul Khedekar

29 December 2025

In this lecture we explore the incenter configuration and its extensions to the excenters.

§1 Ratio Lemma

Lemma 1.1 (Ratio Lemma)
In $\triangle ABC$, let D be a point on $\overline{BC}$. Then

$$\left(\frac{\overline{BD}}{\overline{DC}}\right) = \left(\frac{\overline{AB}}{\overline{AC}}\right) \cdot \left(\frac{\sin \angle BAD}{\sin \angle DAC}\right)$$

Proof. Applying the Law of Sines in $\triangle ABD$ gives

$$\left(\frac{\overline{BD}}{\overline{AB}}\right) = \left(\frac{\sin \angle BAD}{\sin \angle ADB}\right)$$

and in $\triangle ADC$,

$$\left(\frac{\overline{DC}}{\overline{AC}}\right) = \left(\frac{\sin \angle DAC}{\sin \angle ADC}\right)$$

Since $\angle ADB$ and $\angle ADC$ are supplementary, we have $\sin \angle ADB = \sin \angle ADC$. Dividing the two equalities yields the result. □

There are two popular variants of this lemma. They are not difficult to derive, but they appear frequently in complicated configurations, where using these forms makes computations significantly easier.

§1.1 Extended Ratio Lemma (Angle Form)

Lemma 1.2 (Extended Ratio Lemma (Angle Form))
In $\triangle ABC$, let D be a point on $\overline{BC}$. Then

$$\left(\frac{\overline{BD}}{\overline{DC}}\right) = \left(\frac{\sin \angle ACB}{\sin \angle ABC}\right) \cdot \left(\frac{\sin \angle BAD}{\sin \angle DAC}\right)$$

Proof. From the Law of Sines in $\triangle ABC$,

$$\left(\frac{\overline{AB}}{\overline{AC}}\right) = \left(\frac{\sin \angle ACB}{\sin \angle ABC}\right)$$

Substituting this expression into the Ratio Lemma gives the desired result. □

§1.2 Extended Ratio Lemma (Cyclic Quadrilaterals)

Lemma 1.3 (Extended Ratio Lemma (Cyclic Quadrilaterals))
Let $ABCD$ be a cyclic quadrilateral, and suppose its diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Then

$$\left(\frac{\overline{BE}}{\overline{DE}}\right) = \left(\frac{\overline{AB}}{\overline{AD}}\right) \cdot \left(\frac{\overline{BC}}{\overline{CD}}\right)$$

Although this identity looks quite different from the previous variants of the Ratio Lemma, it is essentially the same result in disguise. We now prove it.

Proof. Applying the Ratio Lemma in $\triangle ABD$, we obtain

$$\left(\frac{\overline{BE}}{\overline{DE}}\right) = \left(\frac{\overline{AB}}{\overline{AD}}\right) \cdot \left(\frac{\sin \angle BAC}{\sin \angle CAD}\right)$$

Since $ABCD$ is cyclic, we have $\angle BAC = \angle BDC$ and $\angle CAD = \angle CBD$. Applying the Law of Sines in $\triangle BCD$ yields

$$\begin{aligned}\left(\frac{\sin \angle BAC}{\sin \angle CAD}\right) &= \left(\frac{\sin \angle BDC}{\sin \angle CBD}\right)\\ &= \left(\frac{\overline{BC}}{\overline{CD}}\right)\end{aligned}$$

Substituting back into the earlier expression completes the proof. □

§2 Incenter & Excenters

Let's begin by defining the notion of an angle bisector.

> **Definition 2.1.** In $\triangle ABC$, if D is a point on $\overline{BC}$ such that $\angle BAD = \angle CAD$ then $\overline{AD}$ is the A-**angle bisector**.

When we speak of angle bisectors of an angle, we must differentiate between two possibilities: the **internal angle bisector** and the **external angle bisector**.

The internal bisector of $\angle BAC$ lies in the region between the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, dividing the angle into two equal parts. The external bisector, on the other hand, bisects the supplementary angle formed between the rays $\overrightarrow{AB}$ and $\overrightarrow{CA}$ (equivalently, between $\overrightarrow{BA}$ and $\overrightarrow{AC}$).

A crucial property that relates both of them is as follows.

Proposition 2.2
If ℓ_1 and ℓ_2 are the internal and external angle bisectors of an angle, then ℓ_1 is perpendicular to ℓ_2.

Consider the angle $\angle BAC$ and extend the ray $\overrightarrow{CA}$ to a point D. Then ℓ_2 is the internal angle bisector of $\angle BAD$.

$$\begin{aligned}\angle\left(\ell_2, \overrightarrow{AB}\right) &= \frac{1}{2}\angle BAD \\ &= \frac{1}{2}\left(180^\circ - \angle BAC\right) \\ &= 90^\circ - \frac{1}{2}\angle BAC \\ &= 90^\circ - \angle\left(\overrightarrow{AB}, \ell_1\right)\end{aligned}$$

which implies that $\ell_1 \perp \ell_2$.

Now we can define the **Incenter** and the **Excenters** of a triangle.

> **Definition 2.3.** In $\triangle ABC$, the three internal angle bisectors are concurrent at the **incenter**, usually denoted by I.

Clearly, a triangle has exactly one incenter. The excenters, however, are defined in a slightly different way.

> **Definition 2.4.** In $\triangle ABC$, the internal angle bisector of $\angle A$ and the external angle bisectors of $\angle B$ and $\angle C$ are concurrent. The point of concurrency is called the A-**excenter** of the triangle, and is usually denoted by I_A.

There are three excenters in a triangle, one opposite to each vertex. The existence of the incenter can be shown trivially via **Trigonometric Form** of **Ceva's Theorem**. We will establish the existence of the excenters in the following subsections.

§2.1 Incenter Angle Theorem

> **Theorem 2.5** (Incenter Angle Theorem)
> Let I be the incenter of $\triangle ABC$. Then
> $$\angle BIC = 90^\circ + \frac{1}{2}\angle A$$

Incenter configurations are often very convenient for angle chasing because of the angle bisector properties associated with the incenter.

$$\begin{aligned}\angle BIC &= 180^\circ - (\angle IBC + \angle ICB) \\ &= 180^\circ - \left(\tfrac{\angle B}{2} + \tfrac{\angle C}{2}\right) \\ &= 90^\circ + \tfrac{1}{2}\angle A\end{aligned}$$

This result appears far more often than one might expect, and it is a favourite trick in construction problems that encode this angle in numerical form.

§2.2 Angle Bisector Theorem

There are two popular variants of this theorem: one related to the internal angle bisector and the other to the external angle bisector.

Theorem 2.6 (Angle Bisector Theorem)

In $\triangle ABC$, let points D and E lie on line BC such that $\overline{AD}$ and $\overline{AE}$ are the internal and external angle bisectors of $\angle BAC$, respectively. Then

$$\frac{\overline{BD}}{\overline{CD}} = \frac{\overline{BE}}{\overline{CE}} = \frac{\overline{AB}}{\overline{AC}}$$

This result is not very difficult to prove; it follows immediately from the **Ratio Lemma**. What is more interesting is that there are now two distinct points on line BC that divide the segments to B and C in the same ratio. This phenomenon is closely related to Projective Geometry and Apollonian circles, as we shall see later on.

§2.3 Lengths related to Incenter

Proposition 2.7

In $\triangle ABC$, let D, E and F be points on sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, such that the cevians $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are the internal angle bisectors of $\angle A$, $\angle B$ and $\angle C$. Then

1. $\overline{AD} = \frac{2bc}{b+c}\cos(A/2)$
2. $\overline{BE} = \frac{2ca}{c+a}\cos(B/2)$
3. $\overline{CF} = \frac{2ab}{a+b}\cos(C/2)$

It is also worth mentioning that $\triangle DEF$ is called the **incentral triangle** of $\triangle ABC$ with respect to $\triangle ABC$.

To prove the proposition stated above, there are several possible approaches; however, the quickest one is to equate areas and use the sine formula for the area of a triangle. We have

$$\begin{aligned}
[\triangle ABC] &= [\triangle ABD] + [\triangle ADC] \\
\frac{1}{2}bc\sin A &= \frac{1}{2}b\cdot\overline{AD}\,\sin\left(\frac{A}{2}\right) + \frac{1}{2}c\cdot\overline{AD}\,\sin\left(\frac{A}{2}\right) \\
\overline{AD} &= \frac{2bc}{b+c}\cos\left(\frac{A}{2}\right),
\end{aligned}$$

which proves the proposition.

Let's add the circumcircle of $\triangle ABC$ to the picture, which reveals our next major result in this configuration.

§2.4 Incenter/Excenter Lemma

This is a collection of crucial propositions in this configuration that help us connect the big picture. Let us begin with the first proposition.

Proposition 2.8

In $\triangle ABC$, let I be the incenter. Suppose AI intersects $\odot(ABC)$ again at D. Then

$$\overline{DB} = \overline{DI} = \overline{DC}$$

or, D is the circumcenter of $\triangle BIC$.

In other words, D is the midpoint of the arc BC of the circumcircle $\odot(ABC)$ that does not contain A. Let us first show that $\triangle DBC$ is isosceles. As mentioned earlier, angle chasing is usually the first thing we should try in incenter configurations:

$$\angle DBC = \angle DAC = \angle DAB = \angle DCB$$

which indeed shows that $\triangle DBC$ is isosceles. Now we would like to show that $\triangle BDI$ and $\triangle CDI$ are isosceles. It is sufficient to prove only one of them, since the other follows immediately. Fortunately, we can compute $\angle BID$, which leads to the following angle chase:

$$\begin{aligned}
\angle BID &= 180^\circ - \angle AIB \\
&= 180^\circ - \left(90^\circ + \frac{1}{2}\angle C\right) \\
&= 90^\circ - \frac{1}{2}\angle C \\
&= \frac{1}{2}\angle A + \frac{1}{2}\angle B \\
&= \angle DBC + \angle IBC \\
&= \angle DBI
\end{aligned}$$

which proves that $\triangle DBI$ is isosceles as well, and hence the result follows. The next proposition ties the excenter to this diagram.

Proposition 2.9

In $\triangle ABC$, let I be the incenter, and suppose AI intersects $\odot(ABC)$ again at D. Let I' be the reflection of I across D. Then I' is the A-excenter of $\triangle ABC$.

By definition of I', the points A, I, and I' are collinear. Thus I' already lies on the A-angle bisector. It remains to show that I' also lies on the external angle bisectors of $\angle B$ and $\angle C$ in $\triangle ABC$.

Also $IBI'C$ is a cyclic quadrilateral, since it follows directly from the definition

$$\overline{DB} = \overline{DI} = \overline{DC} = \overline{DI'}$$

Hence D is the center of the circle passing through I, B, I', and C. Using this fact to angle chase, we obtain

$$\begin{aligned}
\angle I'BC &= \angle I'IC \\
&= 180^\circ - \angle AIC \\
&= 90^\circ - \frac{1}{2}\angle B
\end{aligned}$$

This implies that $\overline{I'B} \perp BI$, and therefore BI' is the external angle bisector of $\angle B$. Similarly, since $I'C \perp CI$, the line $I'C$ is the external angle bisector of $\angle C$. Hence I' is indeed the A-excenter of $\triangle ABC$.

For the sake of geometric terminology, the triangle formed by joining the excenters has a dedicated name.

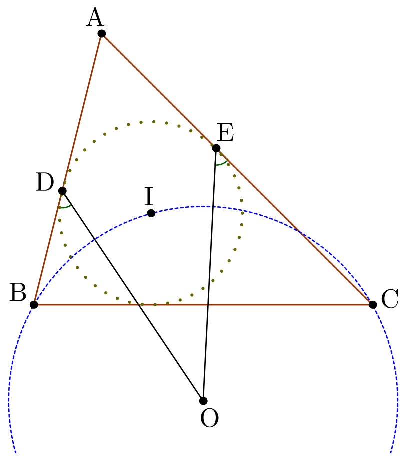
Definition 2.10. In $\triangle ABC$, let I_A, I_B, and I_C be the excenters opposite vertices A, B, and C, respectively. Then $\triangle I_A I_B I_C$ is called the **excentral triangle** of $\triangle ABC$.

Let's look at some examples to realise why this lemma is such a big deal.

§2.4.1 Examples

Problem 2.11 (China 2012)

As shown in the figure below, the in-circle of ABC is tangent to sides AB and AC at D and E respectively, and O is the circumcenter of BCI. Prove that $\angle ODB = \angle OEC$.



Proof. We claim that point O lies on line AI. This follows from angle chasing because

$$\begin{aligned}\angle BIO &= 90^\circ - \frac{1}{2}\angle BOI\\ &= 90^\circ - \angle BCI\\ &= 90^\circ - \frac{1}{2}\angle C\end{aligned}$$

Since, $\angle AIB = 90^\circ + \frac{1}{2}\angle C \implies \angle AIB + \angle BIO = 180^\circ$, which implies the collinearity of points A, I and O. Since $\overline{AD} = \overline{AE}$ (as they are tangents from point A to the incircle), by SAS congruence criterion we can show $\triangle DAO \cong \triangle EAO$. Therefore $\angle ADO = \angle AEO \implies \angle ODB = \angle OEC$, as desired. □

Problem 2.12 (IMO 2006)

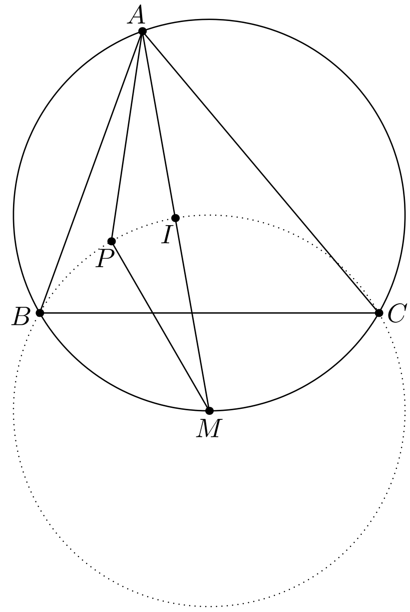
Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Proof. We claim that $BPIC$ is a cyclic quadrilateral. This can be shown using angle chasing

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB$$



$$\implies \angle B + \angle C = 2\left(\angle PBC + \angle PCB\right)$$
$$\implies 180^\circ - \angle A = 2\left(\angle PBC + \angle PCB\right)$$
$$\implies 90^\circ - \frac{1}{2}\angle A = \angle PBC + \angle PCB$$
$$\implies \angle BPC = 90^\circ + \frac{1}{2}\angle A = \angle BIC$$

However, $\odot(A, AI)$ is tangent to $\odot(BIC)$ at I because M (midpoint of arc BC not containing A) is the center of $\odot(BIC)$ and $AM = AI + IM$ which is due these points being collinear. Hence, any point $P \in \odot(BIC)$ is farther away from A than $I \implies AP \geq AI$ where equality holds if and only if P and I coincide. □

§2.5 Exercises

Exercise 2.13. In $\triangle ABC$, let I be the incenter of $\triangle ABC$. Show that,

1. $\overline{AI} = \frac{2bc}{a+b+c}\cos\left(A/2\right)$
2. $\overline{BI} = \frac{2ca}{a+b+c}\cos\left(B/2\right)$
3. $\overline{CI} = \frac{2ab}{a+b+c}\cos\left(C/2\right)$

Exercise 2.14. In the cyclic quadrilateral $ABCD$, let I_1 and I_2 denote the incenters of $\triangle ABC$ and DBC, respectively. Prove that I_1I_2BC is cyclic.

Exercise 2.15. Let ABC be an acute triangle inscribed in circle ω. Let X be the midpoint of the arc BC not containing A and define Y, Z similarly. Show that the orthocenter of $\triangle XYZ$ is the incenter I of $\triangle ABC$

Exercise 2.16. In $\triangle ABC$, let I be the incenter of $\triangle ABC$ and $\triangle I_AI_BI_C$ be the excentral triangle. Show that I is the orthocenter of $\triangle I_AI_BI_C$.