The Steiner Line of Feuerbach Point

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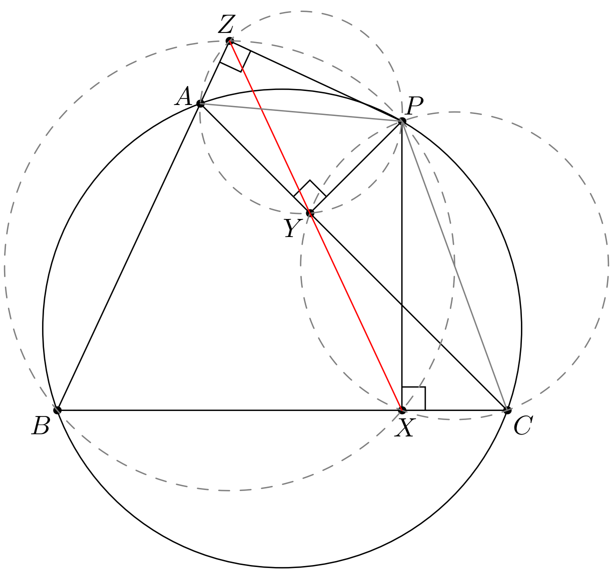
In this article, we shall prove that the **Steiner Line** of the **Feuerbach Point** is with respect to the contact triangle is the $\overline{OI}$ line. Here's a pre-requisite article where we discuss the properties of the feuerbach point stated here.

§1 Simson-Wallace Line

The existence of the **Simson-Wallace Line** is a result closely related to the existence of the **Steiner Line**. So, let's look at that first.

Theorem 1.1 (Simson-Wallace Line)

Given a $\triangle ABC$ and a point P, let X, Y and Z be the foot of perpendicular from P to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$. Points X, Y and Z are collinear if and only if P lies on the circumcircle $\odot(ABC)$.



Proof. Suppose P lies on the circumcircle $\odot(ABC)$. Due to the foot of perpendiculars draw onto the sides, we have that quadrilaterals $PYXC$, $PZAY$ and $PZBX$ are cyclic. Hence

$$\begin{aligned}\angle PYX &= 180^\circ - \angle PCX \\ &= 180^\circ - \angle PCB\end{aligned}$$

$$\begin{aligned}
&= \angle BAP \\
&= 180^\circ - \angle PAZ \\
&= 180^\circ - \angle PYZ
\end{aligned}$$

Therefore points X, Y and Z are collinear. For the other direction, we suppose that the point P does not lie on the circumcircle and points X, Y and Z are collinear. We would still have that $PYXC$, $PZAY$ and $PZBX$ are cyclic. Then

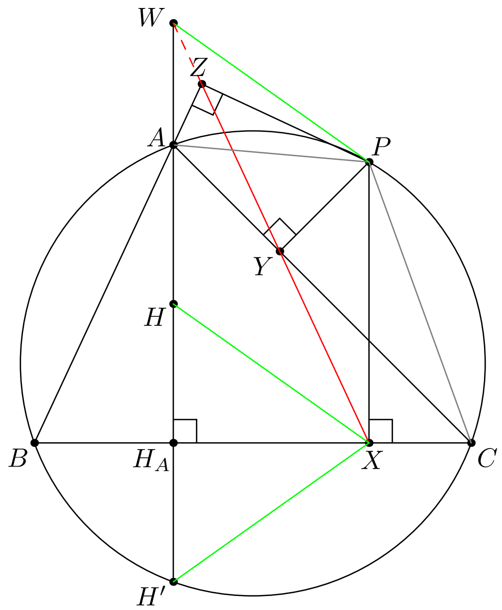
$$\begin{aligned}
\angle APC &= \angle APY + \angle YPC \\
&= \angle AZY + \angle BXY \\
&= 180^\circ - \angle ABC
\end{aligned}$$

which implies that $PABC$ is cyclic, completing the proof. □

Something very remarkable about this line is the following result.

Theorem 1.2 (Simson's Theorem)

Suppose H is the orthocenter of $\triangle ABC$, then the **Simson-Wallace** line of P bisects the segment $\overline{PH}$.



Proof. Suppose H_A is the foot of perpendicular from A onto $\overline{BC}$. If H is reflected over $\overline{BC}$ to H', then it's well known that H' lies on $\odot(ABC)$. Therefore, $\triangle XH'H$ is isosceles. Suppose point W is chosen on line AH such that $HXPW$ forms a parallelogram. Since,

$$\overline{PW} = \overline{HX} = \overline{H'X}$$

Since $\overline{WH'} \parallel \overline{PX} \implies PXH'W$ is an isosceles trapezium. However we can show that W lies on the line $\overline{XYZ}$ because,

$$\angle WXP = \angle WH'P = \angle AH'P = \angle ACP = \angle YXP$$

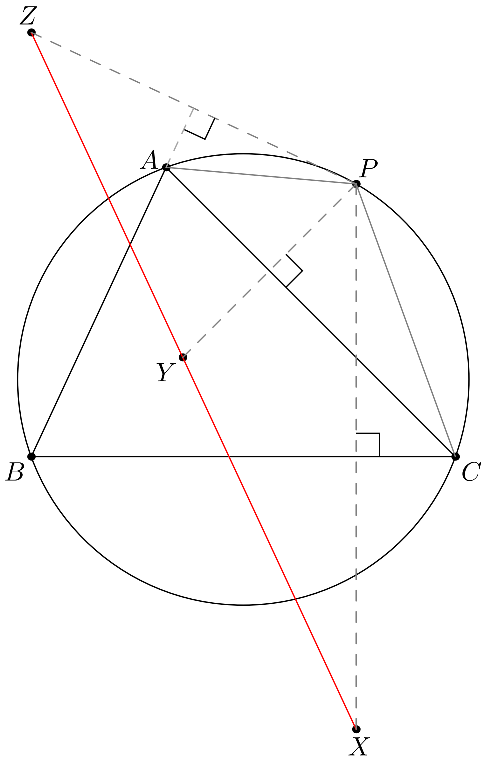
Since the diagonals of parallelogram bisect each other $\implies \overline{XYZ}$ bisects $\overline{PH}$. □

Now we are in the position to discuss the main results of this article.

§2 Steiner Line

Theorem 2.1 (Steiner Line)

Given a $\triangle ABC$ and a point P on its circumcircle $\odot(ABC)$. Let X, Y and Z be the reflections of P over $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$. Then points X, Y and Z are collinear and the line $\overline{XYZ}$ passes through H, the orthocenter of $\triangle ABC$.



Proof. Perform a homothetic transformation at point P with scaling factor 2, and combining the results from 1.1 and 1.2 we get the $\overline{XYZ}$ are collinear and this line passes through the orthocenter of $\triangle ABC$. □

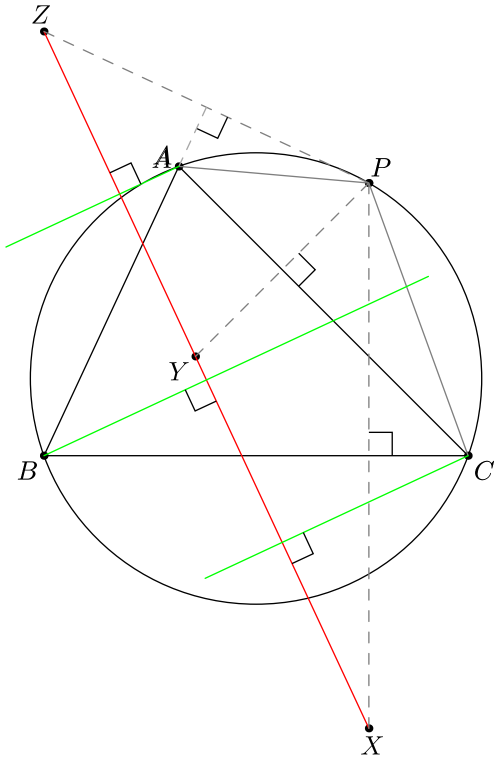
That was easy to prove! Formally we define,

Definition 2.2. Suppose ℓ is the line that passes through the reflections of point P over the sides of $\triangle ABC$, where P lies on $\odot(ABC)$. Then ℓ is said to tbe the **Steiner Line** of point P with respect to $\triangle ABC$, and P is said to be the **Anti-Steiner Point** of ℓ with respect to $\triangle ABC$.

§3 Isogonal Conjugation

Lemma 3.1

Given a $\triangle ABC$, suppose P is a point on $\odot(ABC)$. Let ℓ be the steiner line of P. The isogonals of lines $\overline{AP}$, $\overline{BP}$ and $\overline{CP}$ with respect to $\angle A$, $\angle B$ and $\angle C$ are all perpendicular to ℓ.



Proof. This result follows from a simple angle chase. Suppose ℓ_A is the isogonal line of $\overline{AP}$ with respect to $\angle A$,

$$\angle\left(\ell_A, \overline{AB}\right) = \angle\left(\overline{AC}, \overline{AP}\right) = \angle\left(\overline{ZYX}, \overline{ZP}\right)$$

which implies that $\ell_A \perp \overline{XYZ}$, and hence the result follows. □

§4 Steiner Line of Feuerbach Point

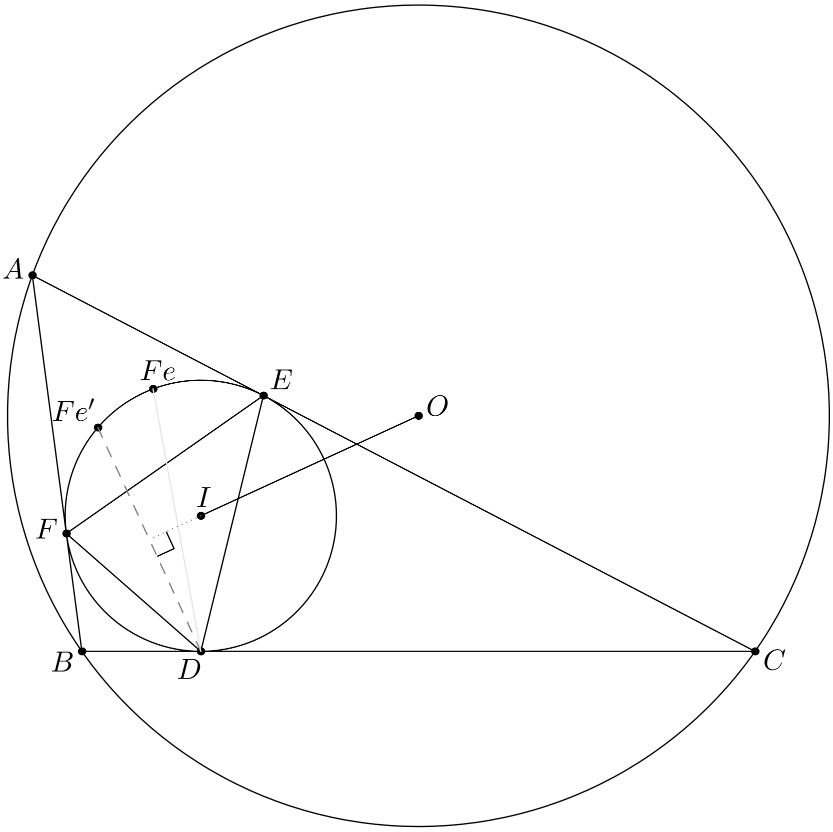
Recall that the **Feuerbach Point** is the point of tangency of the **Incircle** and the **Nine-Point Circle** of the triangle. It is also the isogonal conjugate of the point at infinity along the line perpendicular to $\overline{OI}$, with respect to the contact triangle. We claim that the steiner line of Fe with respect to the contact triangle is the $\overline{OI}$ line.

Lemma 4.1

Given $\triangle ABC$ and its incenter I and circumcenter O. Suppose $\triangle DEF$ is the contact triangle of $\triangle ABC$, then $\overline{OI}$ is the **Euler Line** of $\triangle DEF$.

Proof. Suppose $\triangle I_A I_B I_C$ is the **excentral triangle** of $\triangle ABC$. Since I and O are the orthocenter and the nine-point center of $\triangle I_A I_B I_C \implies \overline{OI}$ is the euler line of $\triangle I_A I_B I_C$. Since $\triangle DEF$ is homothetic to $\triangle I_A I_B I_C \implies$ euler line of $\triangle DEF$ is parallel to euler line of $\triangle I_A I_B I_C$. However, the euler line of $\triangle DEF$ passes through $I \implies \overline{OI}$ is also the euler line of $\triangle DEF$. □

Since it's well known that the isogonal of line $\overline{DFe}$ with respect to $\angle FDE$ is perpendicular to $\overline{OI} \implies$ steiner line of Fe is parallel to $\overline{OI}$. However, the steiner line of $\overline{Fe}$ must also pass through the orthocenter of $\triangle DEF$. Additionally, we have also shown



that $\overline{OI}$ is the euler line of $\triangle DEF \implies \overline{OI}$ passes through the orthocenter of $\triangle DEF$. Therefore, the steiner line of Fe with respect to $\triangle DEF$ is the $\overline{OI}$ line. In other words, the anti-steiner point of $\overline{OI}$ with respect to $\triangle DEF$ is Fe, the feuerbach point of $\triangle ABC$.