

Lemmas in AoPS Geometry

aka: how you can tell who's done too much olympiad geo

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This document is a compilation of a bunch of silly nicknames and terms that have become popular on the Internet among modern teenagers.

Important disclaimer (how to use this file)

Before writing this handout, even I (Evan) did not know several of the names here. If you are a beginner or intermediate level geometry student, **I do not recommend memorizing the names or configurations in this file**. The reason is:

- If it wasn't already in EGMO at least implicitly, then it doesn't come up often enough to be worth learning for beginners.
- When it *does* come up, intermediate students can often figure the relevant facts on-the-spot anyway. Thus, the only real value is documenting the "slang" name for a fact that they could derive on their own.

Instead, this handout is meant as an encyclopedic reference that you can refer to when you hear some bizarre name on the Internet that does not appear in the textbook EGMO.

Contents

1	Basic configurations	3
1.1	Reim's theorem	3
1.2	The Incenter/Excenter Lemma / Fact 5 / Trillium theorem / chicken feet theorem (EGMO Lemma 1.18)	4
1.3	Iran lemma (EGMO Lemma 1.44)	5
1.4	Shooting lemma / Death Star lemma (EGMO Lemma 4.33)	5
1.5	Salmon theorem	6
1.6	The first isogonality lemma	7
2	Length-based lemmas	8
2.1	Ratio lemma	8
2.2	Prism lemma	8
2.3	EFFT (EGMO Theorem 7.16 and 7.25)	9
3	Special named points inside a triangle	9
3.1	HM/Humpty and Dumpty point	9
3.2	Queue points	10
3.3	Sharky-Devil point	11
4	'Muricaaaaaaa, by i3435	12

§1 Basic configurations

§1.1 Reim's theorem

Reim's theorem is a nickname given for a certain easy angle chasing statement. To fully state it, we give the following somewhat silly definition.

Definition 1.1. We say two lines ℓ_1, ℓ_2 are *antiparallel* with respect to m_1 and m_2 , if the four points $\ell_i \cap m_j$ are the vertices of a cyclic quadrilateral.

Theorem 1.2 (Reim's theorem)

Fix two lines m_1 and m_2 . Prove that if ℓ_1 is antiparallel to ℓ_2 and ℓ_2 is antiparallel to ℓ_3 , then lines ℓ_1 and ℓ_3 are either parallel or coincide.

Proof. Because $\angle(m_1, \ell_1) = -\angle(m_2, \ell_2) = \angle(m_1, \ell_3)$. □

See Figure 1; I've highlighted the angles in green. You'll notice this is literally a two-step angle chase, which means that for practical purposes, **whenever you see the name "Reim's theorem", it's just a shorthand for a certain two-step anglechase.**

So, if you ever come up to me with the excuse "I didn't solve the problem because I didn't know Reim's theorem", I will be sassy in my reply.

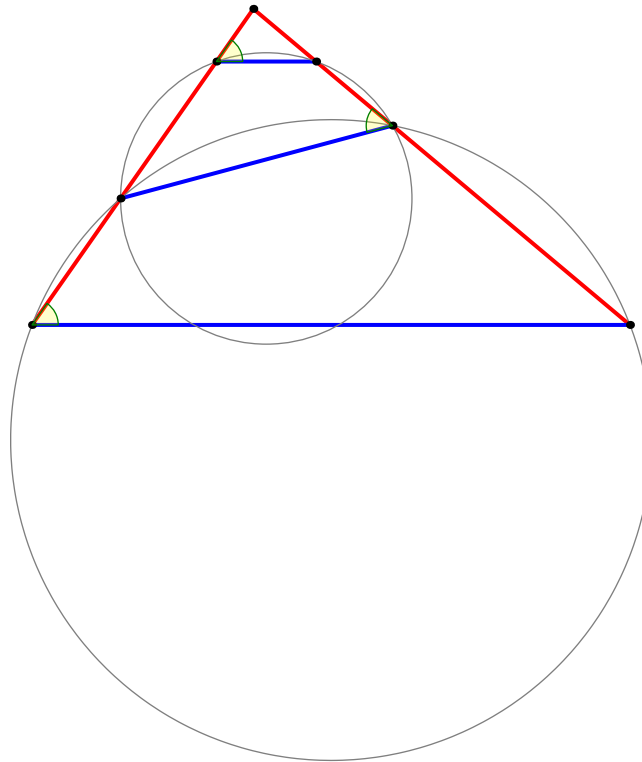


Figure 1: The three angles of Reim's theorem.

§1.2 The Incenter/Excenter Lemma / Fact 5 / Trillium theorem / chicken feet theorem (EGMO Lemma 1.18)

The following theorem from EGMO has several slang names. The most popular one, “Fact 5”, originated from the filename at <https://web.evanchen.cc/handouts/Fact5/Fact5.pdf> (despite the name itself not appearing in the handout at all).

Lemma 1.3 (EGMO Lemma 1.18)

Let ABC be a triangle with incenter I , A -excenter I_A . The circumcenter of cyclic quadrilateral IBI_AC (that is, the midpoint of the diameter $\overline{II_A}$) coincides with the arc midpoint L of minor arc BC .

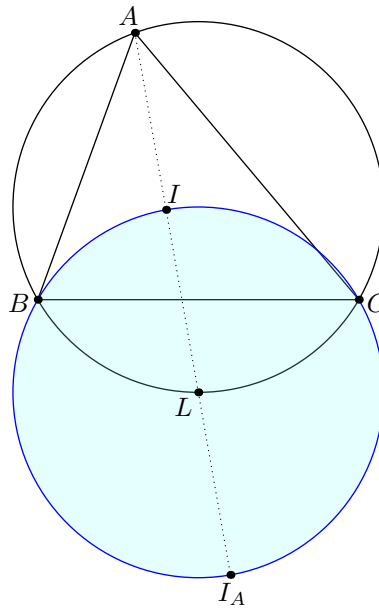


Figure 2: EGMO Lemma 1.18, also known as “Fact 5”, and many other regional nicknames.

Proof. This is just angle chasing; see Figure 2. Let $A = \angle BAC$, $B = \angle CBA$, $C = \angle ACB$, and note that A, I, L are collinear (as L is on the angle bisector). We are going to show that $LB = LI$, the other cases being similar.

First, notice that

$$\angle LBI = \angle LBC + \angle CBI = \angle LAC + \angle CBI = \angle IAC + \angle CBI = \frac{1}{2}A + \frac{1}{2}B.$$

However,

$$\angle BIL = \angle BAI + \angle ABI = \frac{1}{2}A + \frac{1}{2}B.$$

Hence, $\triangle BIL$ is isosceles. So $LB = LI$. The rest of the proof proceeds along these lines. \square

You can remember this lemma as saying that the **circumcenter of BIC is the arc midpoint**.

§1.3 Iran lemma (EGMO Lemma 1.44)

The following lemma in EGMO earned the nickname “Iran lemma” or “Iran incenter lemma” because of its prominence in Problem 11.19 of EGMO (which was from an Iran TST, and from the last chapter of EGMO). This lemma also featured as USAJMO 2014 problem 6.

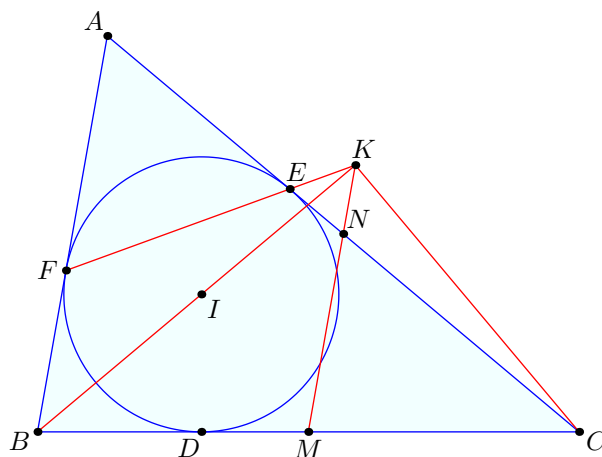


Figure 3: The Iran incenter lemma.

Lemma 1.4 (EGMO Lemma 1.44)

The incircle of $\triangle ABC$ is tangent to \overline{BC} , \overline{CA} , \overline{AB} at D , E , F , respectively. Let M and N be the midpoints of \overline{BC} and \overline{AC} , respectively. Ray BI meets line EF at K . Then $\overline{BK} \perp \overline{CK}$ and K lies on MN .

Proof. See Figure 3. Left as exercise because it’s a problem in EGMO. Hint: focus on the circle with diameter \overline{IC} , which passes through D , E , and K . \square

A good way to remember this lemma is that:

The A -touch chord, B -bisector, and C -midline concur.

§1.4 Shooting lemma / Death Star lemma (EGMO Lemma 4.33)

Lemma 4.33 of EGMO concerns the picture in which you have two internally tangent circles and a segment of the large circle tangent to the smaller one. In Brazil, it is called the **Death Star lemma** because the two circles look like the Death Star spaceship from Star Wars. Here it is:

Lemma 1.5 (EGMO Lemma 4.33)

Let \overline{AB} be a chord of a circle Ω . Let ω be a circle tangent to chord \overline{AB} at K and internally tangent to Ω at T . Then ray TK passes through the midpoint M of the arc \widehat{AB} not containing T . Moreover,

$$MA^2 = MB^2 = MK \cdot MT.$$

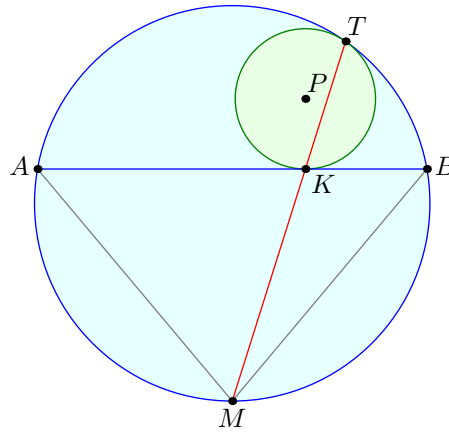


Figure 4: The shooting lemma / death star lemma.

Proof. See Figure 4. The fact that TK passes through the arc midpoint follows by taking the homothety at T mapping ω to Ω : it should map the “south pole” of ω to that of Ω , ergo it maps M to K .

Meanwhile, $MK \cdot MT = MA^2$ follows from $\triangle MKA \sim \triangle MAT$. \square

In the United States, the name **shooting lemma** refers to various parts of this lemma (I never figured out which).

§1.5 Salmon theorem

The name “Salmon theorem” refers to a certain easy spiral similarity. It seems to have first appeared in the official solutions for USAMO 2013 provided by the MAA. Here Salmon is a surname, not a kind of fish.

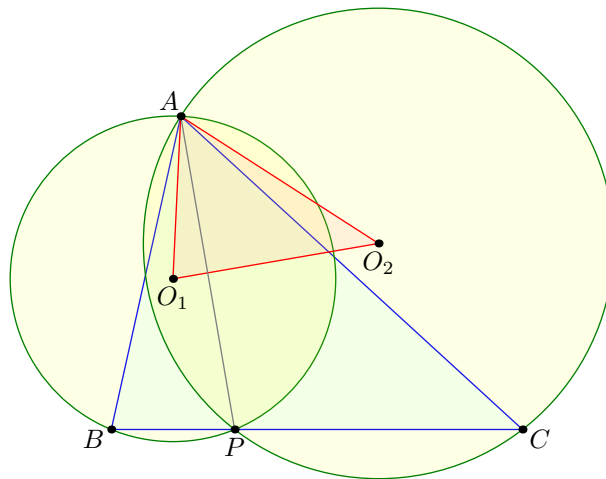


Figure 5: Salmon's theorem.

Theorem 1.6 (Salmon theorem)

Let ABC be a triangle and P a point on line BC . Then we have the spiral similarity

$$\triangle AO_1B \stackrel{+}{\sim} \triangle AO_2C, \quad \triangle ABC \stackrel{+}{\sim} \triangle AO_1O_2.$$

Proof. See Figure 5. Assume without loss of generality that $\angle APB \leq 90^\circ$. Then

$$\angle AO_1B = 2\angle APB$$

but

$$\angle AO_2C = 2(180 - \angle APC) = 2\angle ABP.$$

Hence $\angle AO_1B = \angle AO_2C$. Moreover, both triangles are isosceles, establishing the first similarity.

The second part follows from spiral similarities coming in pairs. \square

Confusingly, there are other results named “Salmon theorem”. See, e.g., [this result](#).

§1.6 The first isogonality lemma

The following lemma appears in *Geometry Revisited*, Section 1.9, Exercise 3. It comes up once in a blue moon, and has earned the name “isogonality lemma”.

Lemma 1.7 (The first isogonality lemma)

Let ABC be a triangle and let P be a point inside it satisfying $\angle ABP = \angle PCA$. Let Q be the reflection of P across the midpoint of \overline{BC} . Then $\angle BAP = \angle CAQ$.

Proof. Most straightforward would be to use barycentric coordinates, but you can see a synthetic proof at <https://aops.com/community/c6h518987>. \square

The condition $\angle ABP = \angle PCA$ is actually usually more naturally interpreted as saying the points $B, C, \overline{BP} \cap \overline{AC}$ and $\overline{CP} \cap \overline{AB}$ are cyclic. Similarly, the parallelogram $BPCQ$ is noteworthy.

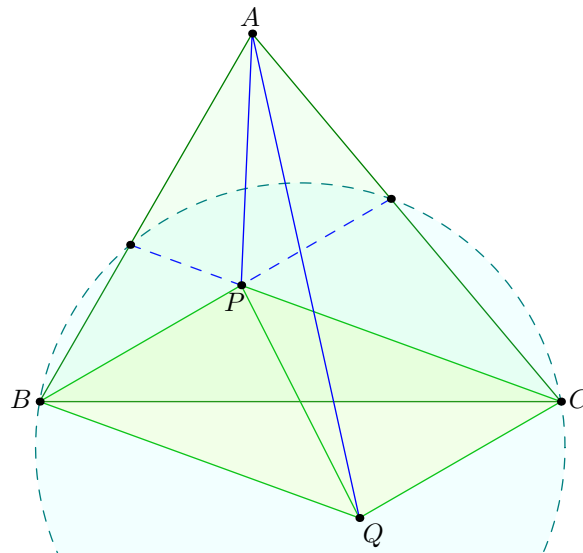


Figure 6: The first isogonality lemma.

Remark 1.8. The *second* isogonality lemma refers to the following statement: Suppose that \overline{AP} and \overline{AQ} are isogonal with respect to $\angle A$. Let $X = \overline{PB} \cap \overline{QC}$ and $Y = \overline{PC} \cap \overline{QB}$. Then \overline{AX} and \overline{AY} are isogonal with respect to $\angle A$.

§2 Length-based lemmas

§2.1 Ratio lemma

In the United States, the following fact is sometimes called the ratio lemma. Like Reim’s theorem, this name is “unnecessary” in that it is only capturing a single natural step.

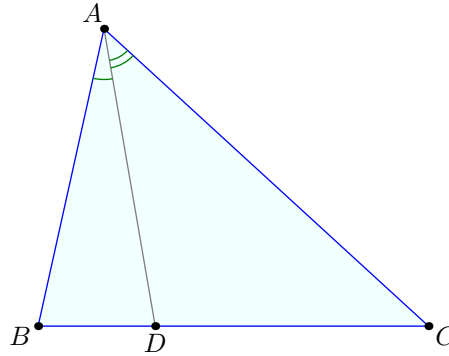


Figure 7: The simple picture for the Ratio Lemma.

Lemma 2.1 (Ratio lemma)

Let ABC be a triangle and D a point in the interior of segment BC . Then

$$\frac{BD}{DC} = \frac{AB \sin \angle BAD}{AC \sin \angle CAD}.$$

Proof. See Figure 7. By the law of sines on $\triangle ABD$ and $\triangle ACD$, we have

$$\frac{BD}{\sin \angle BAD} = \frac{AB}{\sin \angle ADB}, \quad \frac{CD}{\sin \angle CAD} = \frac{AC}{\sin \angle ADC}.$$

Since $\angle ADB + \angle ADC = 180^\circ$, canceling those sines finishes. \square

The lemma is most prominently featured in the handout posted by mira74 at <https://aops.com/community/p19166714>

§2.2 Prism lemma

The following simple lemma from projective geometry is useful for transferring cross ratios. The name is because the picture looks like a pyramid (see Figure 8). I’m told that the origin story came from me mentioning during the test review of TSTST 2019/9 that “this motif comes up often, so we should have a name for it”, and one student suggested “prism lemma”, which stuck.

Lemma 2.2 (Prism lemma)

Two lines ℓ_1 and ℓ_2 meet at a point D . Let A_1, B_1, C_1 be distinct points on ℓ_1 and A_2, B_2, C_2 be distinct points on ℓ_2 , different from D . Then A_1A_2, B_1B_2, C_1C_2 are concurrent if and only if $(A_1B_1; C_1D) = (A_2B_2; C_2D)$.

Proof. This is a restatement of the fact that projection through a point preserves cross ratios. \square

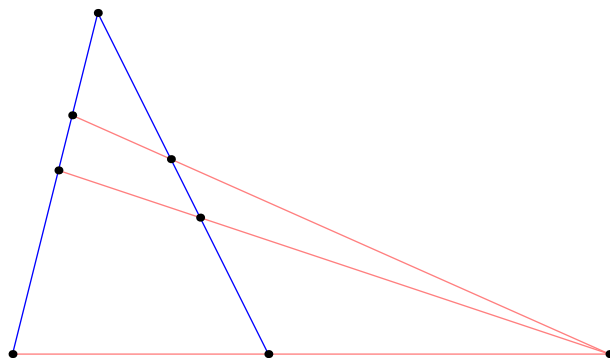


Figure 8: The prism lemma.

§2.3 EFFT (EGMO Theorem 7.16 and 7.25)

In barycentric coordinates, there are two theorems in EGMO that let one decide whether two displacement vectors are perpendicular. Teenager Evan, in 2012, thought it was a good idea to call these “EFFT” and “strong EFFT” respectively, in the handout he posted on Art of Problem Solving. Later efforts to get rid of those names from the slang repository have been largely unsuccessful, and Evan has given up.

Theorem 2.3 (EGMO Lemma 7.16)

Let $\overrightarrow{MN} = (x_1, y_1, z_1)$ and $\overrightarrow{PQ} = (x_2, y_2, z_2)$ be displacement vectors. Then $\overline{MN} \perp \overline{PQ}$ if and only if

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2).$$

Theorem 2.4 (EGMO Lemma 7.25)

Suppose M , N , P , and Q are points with

$$\begin{aligned}\overrightarrow{MN} &= x_1\overrightarrow{AO} + y_1\overrightarrow{BO} + z_1\overrightarrow{CO} \\ \overrightarrow{PQ} &= x_2\overrightarrow{AO} + y_2\overrightarrow{BO} + z_2\overrightarrow{CO}\end{aligned}$$

such that either $x_1 + y_1 + z_1 = 0$ or $x_2 + y_2 + z_2 = 0$. In that case, lines MN and PQ are perpendicular if and only if

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2).$$

§3 Special named points inside a triangle

Here we only give the definition of the points and leave the (many) properties of the handouts to provided external links.

§3.1 HM/Humpty and Dumpty point

As shown in [Figure 9](#), let ABC be a triangle with orthic triangle DEF , and orthocenter H . Let M be the midpoint of \overline{BC} .

Definition 3.1. The foot of the altitude from H to \overline{AM} , denoted Q in Figure 9, is called either the **A-Humpty point** or the **A-HM point**.

Some basic properties of the A -Humpty point are that lines QH , BC , EF are concurrent, and that it lies on the circle (BHC) .

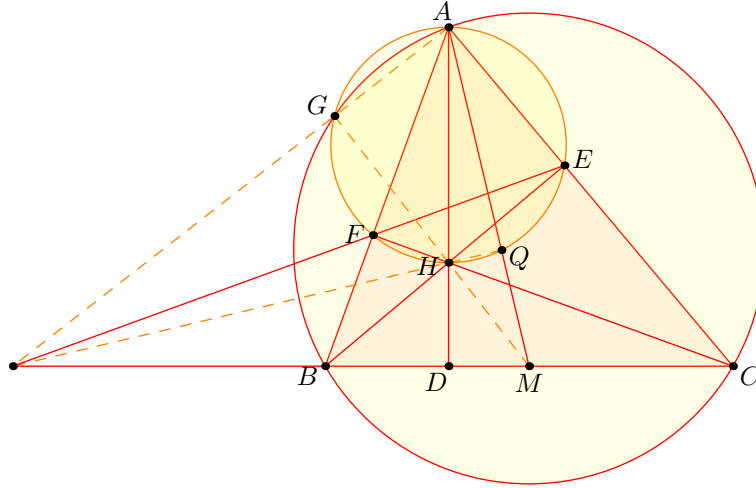


Figure 9: The Humpty point (denoted Q) and the A -queue point (denoted G).

Less commonly seen is:

Definition 3.2. The **A-Dumpty point** is the isogonal conjugate of the A -Humpty point. It turns out to coincide with the midpoint of the A -symmedian chord.

References for these points include:

- <https://math.stackexchange.com/a/3894697/229197>
- [On Two Special Points in a Triangle](#), which seems to be the origin of the name Humpty and Dumpty.
- [A Special Point on the Median](#) which seems to be the origin of the name HM.

§3.2 Queue points

In the Humpty configuration, the point G in Figure 9 also plays an important role. It is named as follows:

Definition 3.3. The **A-queue point** is the second intersection of (AEF) and (ABC) , denoted G in Figure 9. In other words, it is the Miquel point of the cyclic quadrilateral $BFEC$.

The most important properties of the A -queue point are:

Theorem 3.4 (Important properties of the A -queue point)

In the notation of Figure 9:

1. Line AG is concurrent with lines EF , HQ , BC defined before.
2. Point G lies on line HM .

The name seems to have come from the blog post by [math_pi_rate](https://aops.com/community/c701535h1754217) at <https://aops.com/community/c701535h1754217>¹, which documents several additional properties of the line.

§3.3 Sharky-Devil point

As shown in Figure 10, let ABC be a triangle with intouch triangle DEF .

Definition 3.5. The **A-Sharkydevil point**, denoted K below, is defined as the Miquel point K of $BFEC$; i.e. the intersection of the (AEF) and (ABC) .

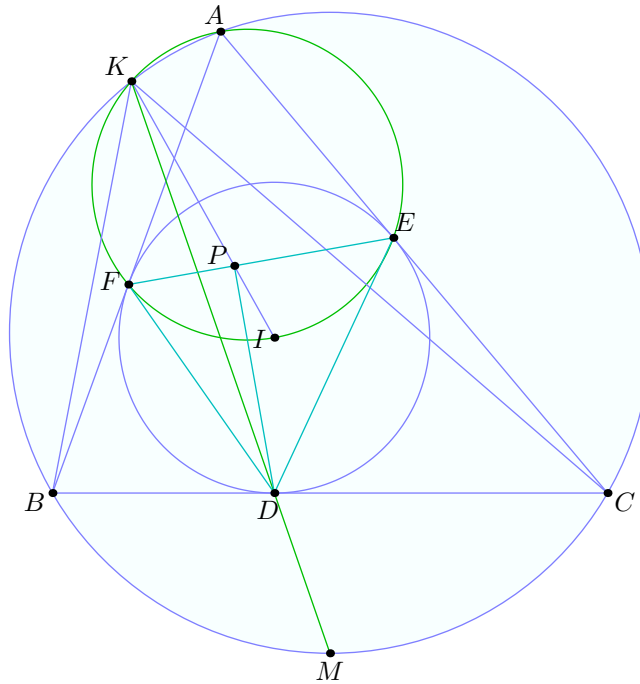


Figure 10: The A-Sharkydevil point.

The name seems coined by [aops29](https://aops.com/community/c946900h1911664) in a blog post, <https://aops.com/community/c946900h1911664>.

The most important properties are:

Theorem 3.6 (Important properties of the Sharkydevil point)

In the notation of Figure 10:

1. Line \overline{KD} bisects $\angle BKC$ and hence passes through the midpoint M of \widehat{BC} .
2. Line \overline{KD} passes through the exsimilicenter of the incircle and circumcircle.
3. Let P denote the foot from D to \overline{EF} . Then K is the inverse of P with respect to the incircle.

¹The concurrence point $\overline{AG} \cap \overline{EF} \cap \overline{HQ} \cap \overline{BC}$ is named the A -ex point in that blog post, but that name seems to not have caught on.

§4 'Muricaaaaaaa, by i3435

I tried to keep this handout brief and to-the-point, since I intended it as a lookup of common idiosyncratic names.

If you want a more complete reference, there is a 2021 handout by i3435 at

<https://aops.com/community/p20459825>

which is *much* more comprehensive, including proofs of the facts here, several properties of the named points not recorded here, and many more configurations. It also features a long set of problems and solutions.