Maciej Dziuba - Radical axis

Problem 1. Points A, B, C, D lie on the line l in this order. Circles ω_C , ω_B with diameters respectively AC, BD intersect at points X, Y. Point P lies on the line XY, but not on l. Denote by M, N the intersection of respectively CP with ω_C and BP with ω_B. Prove that the lines AM, DN, XY are corcurrent.

Problem 2. Circles ω_1, ω_2 intersect at points X, Y. Two common tangents to these circles intersect at the point P. Line l passing through P and intersect ω_1 at points A, C and ω_2 at points B, D and A, B, C, D lie on l in this order. Prove that tangent to ω_1 in C and tangent to ω_2 in B intersect on the line XY.

Problem 3. Let H be the orthocenter of an acute triangle ABC. The circle with center in the midpoint of the segment BC and passing through H intersects the side BC at A_1, A_2. Similarly, define the points B_1, B_2, C_1, C_2. Prove that the points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic.

Problem 4. Let l_1 and l_2 be the parallel lines and points P, Q lie between them. For point A, which lies on l_1, let A_1 be the intersection of AP and l_2. Now, let A_2 be the intersection of A_1Q and l_1. For point $B \in l_1$ similarly define B_1 and B_2. Prove that the second point of intersection $\bigcirc(PBA_2)$ and $\bigcirc(PAB_2)$ lies on the line PQ.

Problem 5. Points D, E lie on the sides respectively AB, AC of $\triangle ABC$, such that $DE \parallel BC$. Point P lies inside $\triangle ADE$. Segments BP, CP intersect segment DE at points F, G. The circumcircles of the triangles DPG, FPE intersect at points P and Q. Prove that points A, P, Q are collinear.

Problem 6. Let ABC be a acute triangle with incenter I. Points E, F are midpoints of shorter arcs $\widehat{AC}, \widehat{AB}$ respectively (on $\bigcirc(ABC)$). Segment EF intersects the sides AB, AC at points P, Q respectively. Point D satises the conditions $PD \parallel BI, QD \parallel CI$. Let T denote the intersection of BF and CE. Prove that T, I, D are collinear.

Problem 7. Cyclic quadrilateral $ABCD$ has no parallel sides. Prove that the locus of points P, which satisfies condition $\sphericalangle DAP + \sphericalangle CBP = \sphericalangle CPD$ is a circle.

Problem 8. Circles o_1, o_2 intersect each other at two points. Circles ω_1, ω_2 are externally tangent to o_1 at A_1, A_2, internally tangent to o_2 at B_1, B_2 and intersect each other at points C, D. Prove that the lines A_1B_1, A_2B_2 and CD are concurrent.

Problem 9. Let the incircle and $A-$excircle be tangent to side BC at point D, E respectively. Circle ω_B is the reflection of circle inscribed in $\triangle ABE$ in the midpoint of segment AB. Circle ω_C is defined similarly. Prove that D lies on radical axis of ω_B and ω_C.

Problem 10. Let ω_1, ω_2 be the two circles. Line l_1 is tangent to ω_1 in A and to ω_2 in B, such that these circles lie on the same side of l_1. Line l_2 is tangent to ω_1 in C and to ω_2 in D, such that these circles lie on the different sides of l_2. Prove that intersection point of AC and BD lies on the line passing through the centers of ω_1 and ω_2.

Problem 11. Points A, B lie outside the circle ω. Line l passing through point A intersects ω at P, Q and BP, BQ intersect ω at S, R. Prove that all lines RS, for every line l, passing through a fixed point.

Problem 12. Point S lies inside the circle ω. Circles ω_1, ω_2 are externally tangent to each other at S and internally tangent to ω. External common tangents to ω_1 and ω_2 intersect at point P. Prove that all points P lie on one line.

Problem 13. Circle ω is tangent to two parallel lines l_1, l_2. Circle ω_1 is tangent to l_1 at point A and externally tangent to ω at B. Circle ω_2 is tangent to l_2 at point C, externally tangent to ω at D and to ω_1 at E. Lines AD and BC intersect in S. Prove that $SB = SD = SE$.

Problem 14. Consider four lines (No three concurrent and no two parallel). We have four ways to choose three of them, which bound the triangle. Prove that orthocenters of these four triangles are collinear. Line passing through these orthocenters is called Aubert's line.

Problem 15. Let P be the intersection of diagonals AC and BD in cyclic quadrilateral $ABCD$. Point X lies inside the quadrilateral and satisfies the condition $\sphericalangle XAB + \sphericalangle XCB = \sphericalangle XBC + \sphericalangle XDC = 90^\circ$. Prove that line PX passing through the center of $\bigcirc(ABCD)$. (If there exist the circle insribed in ABCD, then P is its center and we have the very useful lemma)

Problem 16. Let ABC be the triangle with $BAC = 90^\circ$. Point D is foot of altitude from A. Point X lies on the segment AD. Point K is on the segment BX and $CA = CK$. Point L is on the segment CX and $BA = BL$. Let P be the intersection of BL and CK. Prove that $PK = PL$.

Problem 17. Let ABC be a triangle with orthocenter H, circumcenter O and K, M, N are the midpoints of sides BC, CA, AB respectively. The line tangent to $\bigcirc(ABC)$ at A intersects MN at P. Let E, F be the feet of altitudes from B, C in $\triangle ABC$. Let T denote the intersection of the ray $\overrightarrow{KH}$ and $\bigcirc(ABC)$. Lines AT and EF intersect at point Q. Prove that $OH \perp PQ$.

Problem 18. Let O, H denote respectively the circumecenter and orthocenter of $\triangle ABC$. Lines BH, CH intersect $\bigcirc(ABC)$ at points E, F respectively. Segment EO intersects side AC at P and segment FO intersects side AB at Q. Tangents to $\bigcirc(ABC)$ in B, C intersect at point T. Prove that $TH \perp PQ$.

Problem 19. Point D lies on the side BC of $\triangle ABC$. Circle ω_1 is tangent to (ABC), segment AD and the segment BD at P. Circle ω_2 is tangent to (ABC), segment AD and the segment CD at Q. Let M be the midpoint of arc $\widehat{BC}$ in $\bigcirc(ABC)$ and N is the midpoint of the segment PQ. Let I denote the incenter of $\triangle ABC$ and K is the midpoint of the segment DI. Prove that M, N, K are collinear.

Problem 20. Let E, F be the tangency point of the circle ω, inscribed in $\triangle ABC$, and the sides respectively AC, AB. Point G is intersection of BE and CF. Let P, Q be the points, such that the quadrilaterals $BCEP$ and $CBFQ$ are parallelograms. Prove that $GP = GQ$.

Problem 21. Let ABC be a triangle and let I and O denote its incenter and circumcenter respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC. The circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line OI.

Problem 22. Let I and H be the incenter and orthocenter of $\triangle ABC$. Let P be the point on BC, which satisfies $PI \perp AI$, and let M be the midpoint of AP. Prove that H lies on the radical axis of circle with diameter IM and the inscribed circle.