# Linearity of Power of a Point

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In this article, we will learn about an advanced technique that exploits the linearity of the power of a point function.

## §1 Background

Let's reiterate over the definition of the power of a point function.

> **Definition 1.1.** For a given circle $\omega$ centered at $O$ and radius $r$, and a point $P$, the **power of a point** is defined as
>
> $$\operatorname{Pow}_\omega(P) = \overline{OP}^2 - r^2$$

The big claim here is that the difference of power of a point computed against two circles is *linear*. In mathematics, when we talk about **linear functions**, we mean functions that satisfy the following conditions

$$f(x+y) = f(x) + f(y)$$
$$f(\alpha x) = \alpha f(x)$$

In general, these conditions can be combined and written as

$$f(\alpha x + (1-\alpha)x) = \alpha f(x) + (1-\alpha) f(x)$$

for some real number $\alpha$.

## §2 Proving Linearity

**Theorem 2.1** (Linearity of Power of a Point)
Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(P) = \operatorname{Pow}_{\omega_1}(P) - \operatorname{Pow}_{\omega_2}(P)$$

for two fixed circles $\omega_1$ and $\omega_2$. Then $f$ is linear.

*Proof.* Suppose the centers of the circles $\omega_1$ and $\omega_2$ are $O_1$ and $O_2$, and their radii are $r_1$ and $r_2$ respectively. Pick two points $A$ and $B$ and choose point $C$ on $\overline{AB}$ such that

$$\frac{\overline{AC}}{\overline{CB}} = \frac{k}{1-k}$$

We would like to show that

$$f(C) = kf(A) + (1-k)f(B)$$

Simplifying the left hand side expression

$$\begin{aligned} f(C) &= \operatorname{Pow}_{\omega_1}(C) - \operatorname{Pow}_{\omega_2}(C) \\ &= \left(\overline{O_1C}^2 - r_1^2\right) - \left(\overline{O_2C}^2 - r_2^2\right) \end{aligned}$$

Applying Stewart's Theorem on $\triangle ABO_1$ and $\triangle ABO_2$,

$$\begin{aligned} &= \left(\overline{O_1C}^2 - r_1^2\right) - \left(\overline{O_2C}^2 - r_2^2\right) \\ &= \Bigg( -k(1-k)\overline{AB}^2 + \left((1-k)\overline{O_1B}^2 + k\overline{O_1A}^2\right) \Bigg) \\ &\quad - \Bigg( -k(1-k)\overline{AB}^2 + \left((1-k)\overline{O_2B}^2 + k\overline{O_2A}^2\right) \Bigg) + r_2^2 - r_1^2 \\ &= k\left(\overline{O_1A}^2 - \overline{O_2A}^2\right) + (1-k)\left(\overline{O_1B}^2 - \overline{O_2B}^2\right) + r_2^2 - r_1^2 \\ &= k\left(\overline{O_1A}^2 - r_1^2\right) - k\left(\overline{O_2A}^2 - r_2^2\right) \\ &\quad + (1-k)\left(\overline{O_1B}^2 - r_1^2\right) - (1-k)\left(\overline{O_2B}^2 - r_2^2\right) \\ &= kf(A) + (1-k)f(B) \end{aligned}$$

thus proving the linearity. □

## §3 Applications: Radical Axis

Linearity of power of a point turns out to be a good criteria to determine if a point lies on the radical axis of two circles. Since the radical axis is the locus of points having equal power, it essentially means that it's the locus of zeros of $f$.

**Problem 3.1** (USA 2020)

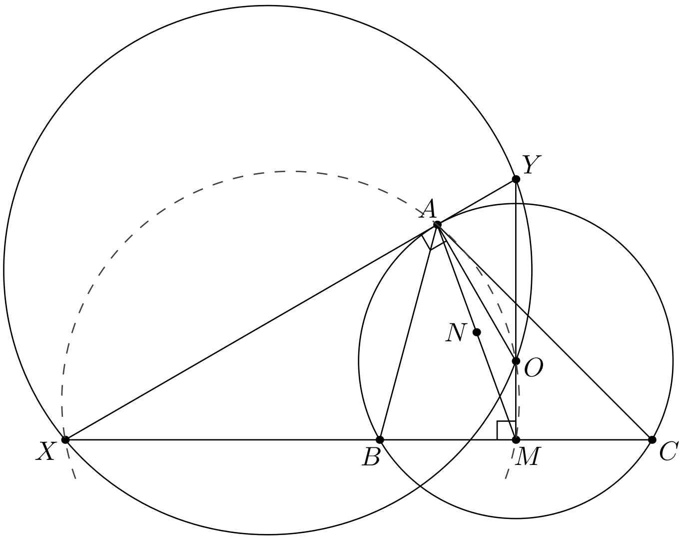
Let $O$ and $\Gamma$ denote the circumcenter and circumcircle, respectively, of scalene $\triangle ABC$. Furthermore, let $M$ be the midpoint of side $\overline{BC}$. The tangent to $\Gamma$ at $A$ intersects $BC$ and $OM$ at points $X$ and $Y$, respectively. If the circumcircle of $\triangle OXY$ intersects $\Gamma$ at two distinct points $P$ and $Q$, prove that $PQ$ bisects $\overline{AM}$.

*Proof.* Suppose $N$ is the midpoint of $\overline{AM}$. We would like to show that $N$ lies on the radical axis of the circles $\odot(ABC)$ and $\odot(OXY)$. Suppose $f$ is defined as

$$f(P) = \operatorname{Pow}_{\odot(ABC)}(P) - \operatorname{Pow}_{\odot(OXY)}(P)$$

Then we would like to show that

$$f(N) = \frac{1}{2}\left(f(A) + f(M)\right) = 0$$




Keeping in mind that the power of a point function is negative for points inside the circle,

$$\begin{aligned} f(A) + f(M) &= (0 - (-\overline{AX} \cdot \overline{AY})) + (-\overline{BM} \cdot \overline{MC} - \overline{OM} \cdot \overline{MY}) \\ &= \overline{AX} \cdot \overline{AY} - \tfrac{1}{4}\overline{BC}^2 - \overline{OM} \cdot \overline{MY} \\ &= \overline{AX} \cdot \overline{AY} + \overline{OM}^2 - \overline{OB}^2 - \overline{OM} \cdot \overline{MY} \\ &= \overline{AX} \cdot \overline{AY} - \overline{OA}^2 - \overline{OM} \cdot \overline{OY} \\ &= \overline{AX} \cdot \overline{AY} + \overline{AY}^2 - \overline{OY}^2 - \overline{OM} \cdot \overline{OY} \\ &= \overline{AY} \cdot \overline{XY} - \overline{OY} \cdot \overline{MY} \end{aligned}$$

Since $\angle OAX = \angle OMX = 90^\circ \implies AOMX$ is a cyclic quadrilateral. Therefore,

$$\overline{AY} \cdot \overline{XY} = \overline{YO} \cdot \overline{YM} \implies f(A) + f(M) = 0$$

hence $f(N) = 0$, implying that the midpoint of $\overline{AM}$ lies on the radical axis $\overline{PQ}$. □

## §3.1 Exercises

**Exercise 3.2** (Taiwan TST 2016)**.** Let $O$ be the circumcenter of triangle $ABC$, and $\omega$ be the circumcircle of triangle $BOC$. Line $AO$ intersects with circle $\omega$ again at the point $G$. Let $M$ be the midpoint of side $BC$, and the perpendicular bisector of $BC$ meets circle $\omega$ at the points $O$ and $N$. Prove that the midpoint of the segment $AN$ lies on the radical axis of the circumcircle of triangle $OMG$, and the circle whose diameter is $AO$.

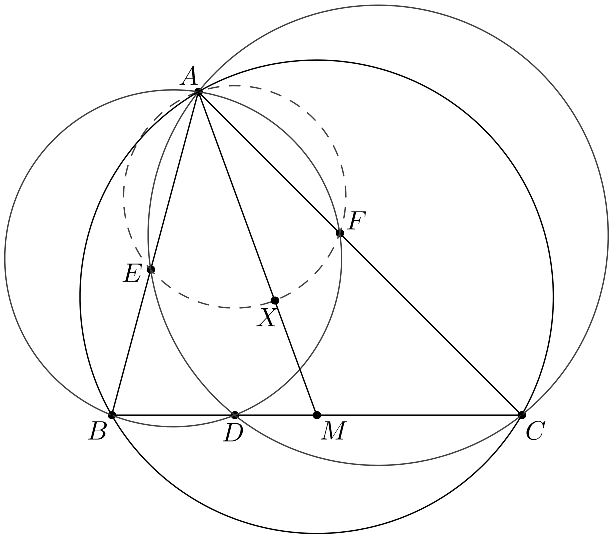
**Exercise 3.3.** Given $\triangle ABC$ inscribed in $\odot(O)$. Let $M$ be the midpoint of $\overline{BC}$, $H$ be the projection of $A$ onto $\overline{BC}$. Suppose $\overline{OH}$ meets $\overline{AM}$ at $P$. Prove that $P$ lies on the radical axis of $\odot(BOC)$ and the nine-point circle of $\triangle ABC$.

**Exercise 3.4** (USA TSTST 2016)**.** Let $ABC$ be a triangle with incenter $I$, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at $D$, $E$, $F$, respectively. Let $K$ be the foot of the altitude from $D$ to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points $C_1$ and $C_2$, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points $B_1$ and $B_2$. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint $M$ of $\overline{DK}$.

# §4 Applications: Identifying Fixed Points & Locus

Often we would like to show that a circle passes through a fixed point. There are several ways to prove such results. One of the ways is to pick a fixed point on the secant through the desired fixed point and show that the power of point is constant.

**Problem 4.1** (ELMO Shortlist 2013)
In $\triangle ABC$, a point $D$ lies on line $BC$. The circumcircle of $ABD$ meets $AC$ at $F$ (other than $A$), and the circumcircle of $ADC$ meets $AB$ at $E$ (other than $A$). Prove that as $D$ varies, the circumcircle of $AEF$ always passes through a fixed point other than $A$, and that this point lies on the median from $A$ to $BC$.




*Proof.* If we can show that the power of $M$ with respect to $\odot(AEF)$ is constant, then that would mean $\odot(AEF)$ passes through a fixed point on $\overline{AM}$. Suppose $\overline{AM} \cap \odot(AEF) = X$. Define $f$ as

$$f(P) = \text{Pow}_{\odot(ABC)}(P) - \text{Pow}_{\odot(AEF)}(P)$$

Using the fact that $f$ is linear, we get

$$\begin{aligned} f(M) &= \tfrac{1}{2}\left(f(B) + f(C)\right) \\ &= \tfrac{1}{2}\left(\left(0 - \overline{BE}\cdot\overline{AB}\right) + \left(0 - \overline{CF}\cdot\overline{AC}\right)\right) \\ &= -\tfrac{1}{2}\left(\overline{BE}\cdot\overline{AB} + \overline{CF}\cdot\overline{AC}\right) \\ &= -\tfrac{1}{2}\left(\overline{BD}\cdot\overline{BC} + \overline{CD}\cdot\overline{BC}\right) \\ &= -\tfrac{1}{2}\overline{BC}^2 \end{aligned}$$

Since,

$$\begin{aligned} f(M) &= \text{Pow}_{\odot(ABC)}(M) - \text{Pow}_{\odot(AEF)}(M) \\ &= -\overline{BM}\cdot\overline{MC} - \text{Pow}_{\odot(AEF)}(M) \\ &= -\tfrac{1}{2}\overline{BC}^2 \end{aligned}$$

Therefore,

$$\text{Pow}_{\odot(AEF)} = \frac{1}{4}\overline{BC}^2$$

which is a constant. This shows that the product $\overline{AM} \cdot \overline{MX}$ is a constant and thus $\odot(AEF)$ passes through the fixed point $X$. □

**Problem 4.2** (USA Math Olympiad 2015)

Quadrilateral $APBQ$ is inscribed in circle $\omega$ with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let $X$ be a variable point on segment $\overline{PQ}$. Line $AX$ meets $\omega$ again at $S$ (other than $A$). Point $T$ lies on arc $AQB$ of $\omega$ such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let $M$ denote the midpoint of chord $\overline{ST}$. As $X$ varies on segment $\overline{PQ}$, show that $M$ moves along a circle.

*Proof.* □

## §4.1 Exercises

**Exercise 4.3** (USA TST 2012)**.** In acute triangle $ABC$, $\angle A < \angle B$ and $\angle A < \angle C$. Let $P$ be a variable point on side $BC$. Points $D$ and $E$ lie on sides $AB$ and $AC$, respectively, such that $BP = PD$ and $CP = PE$. Prove that as $P$ moves along side $BC$, the circumcircle of triangle $ADE$ passes through a fixed point other than $A$.

**Exercise 4.4** (USA TST 2021)**.** Points $A$, $V_1$, $V_2$, $B$, $U_2$, $U_1$ lie fixed on a circle $\Gamma$, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$. Let $X$ be a variable point on the arc $V_1V_2$ of $\Gamma$ not containing $A$ or $B$. Line $XA$ meets line $U_1V_1$ at $C$, while line $XB$ meets line $U_2V_2$ at $D$. Let $O$ and $\rho$ denote the circumcenter and circumradius of $\triangle XCD$, respectively. Prove there exists a fixed point $K$ and a real number $c$, independent of $X$, for which $OK^2 - \rho^2 = c$ always holds regardless of the choice of $X$.

**Exercise 4.5** (IMO 2019)**.** Let $I$ be the incentre of acute triangle $ABC$ with $AB \neq AC$. The incircle $\omega$ of $ABC$ is tangent to sides $BC, CA$, and $AB$ at $D, E$, and $F$, respectively. The line through $D$ perpendicular to $EF$ meets $\omega$ at $R$. Line $AR$ meets $\omega$ again at $P$. The circumcircles of triangle $PCE$ and $PBF$ meet again at $Q$. Prove that lines $DI$ and $PQ$ meet on the line through $A$ perpendicular to $AI$.

# §5 Applications: Point Circles

**Problem 5.1** (USAMO 2013)

In triangle $ABC$, points $P$, $Q$, $R$ lie on sides $BC$, $CA$, $AB$ respectively. Let $\omega_A$, $\omega_B$, $\omega_C$ denote the circumcircles of triangles $AQR$, $BRP$, $CPQ$, respectively. Given the fact that segment $AP$ intersects $\omega_A$, $\omega_B$, $\omega_C$ again at $X$, $Y$, $Z$, respectively, prove that $YX/XZ = BP/PC$.

*Proof.* □

### §5.1 Exercises

**Exercise 5.2** (IMO Shortlist 2015)**.** Let $ABC$ be an acute triangle and let $M$ be the midpoint of $AC$. A circle $\omega$ passing through $B$ and $M$ meets the sides $AB$ and $BC$ at points $P$ and $Q$ respectively. Let $T$ be the point such that $BPTQ$ is a parallelogram. Suppose that $T$ lies on the circumcircle of $ABC$. Determine all possible values of $\frac{BT}{BM}$.

## §6 Applications: Feuerbach's Theorem

Linearity of power of a point can be used to prove the following claim, which turns out to be a major step in proving the **Feuerbach's Theorem**.

**Problem 6.1**

Let $I$ be the incenter and $O$ be the circumcenter of $\triangle ABC$. Let $M$ be the midpoint of $\overline{BC}$. Suppose $\odot(I)$ touches $\overline{BC}$ at $D$ and $M'$ is the reflection of $M$ over $\overline{AI}$. Then $\overline{DM'} \perp \overline{OI}$.

*Proof.* □