

# On Feuerbach's Theorem

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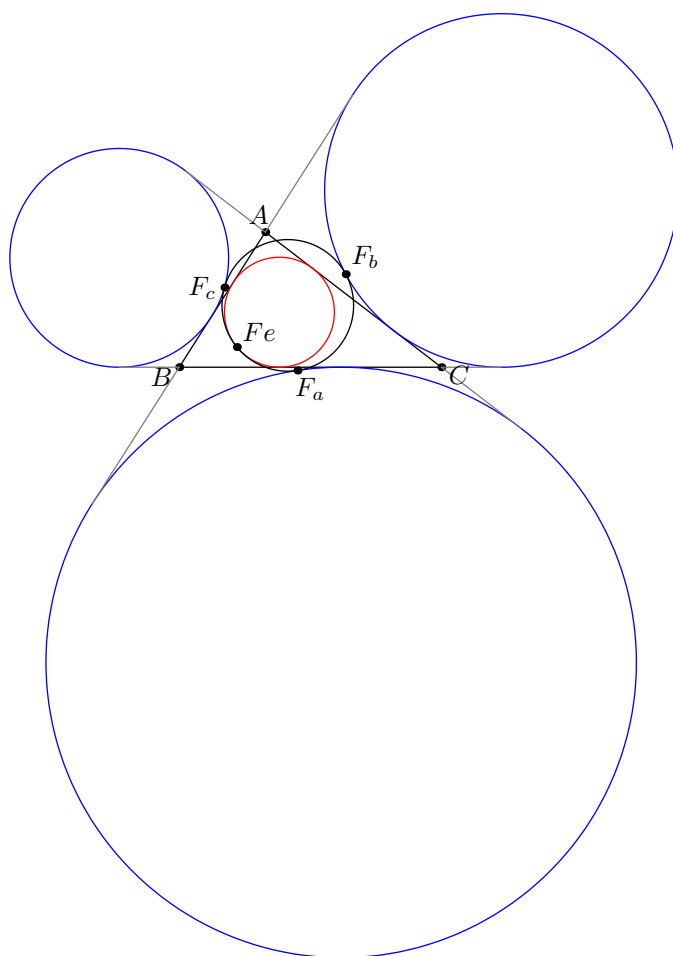
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This article explores a synthetic proof of a celebrated result in euclidean geometry: **Feuerbach's Theorem**

## §1 Statement

### Theorem 1.1 (Feuerbach's Theorem)

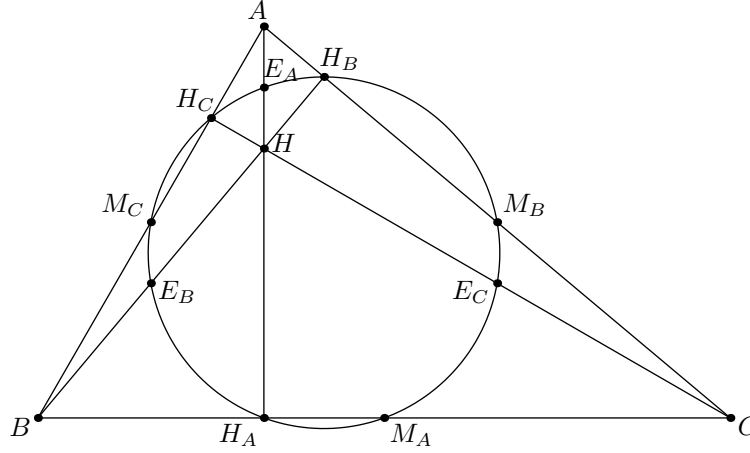
The **Nine-Point Circle** of a triangle is *internally tangent* to the **Incircle** and *externally tangent* to the three **Excircles**. The points of tangencies are the interior and three exterior **Feuerbach Points**.



## §2 Definitions

We need to state some definitions before we begin diving in this configuration.

**Definition 2.1.** The **Nine-Point Circle** of  $\triangle ABC$  is a circle that passes through the midpoints of  $\overline{BC}$ ,  $\overline{CA}$  and  $\overline{AB}$ ; midpoints of  $\overline{AH}$ ,  $\overline{BH}$  and  $\overline{CH}$ , and through the feet of perpendiculars  $H_A$ ,  $H_B$  and  $H_C$  in  $\triangle ABC$ .



The existence of the nine-point circle is a popular result in classical geometry. It follows from the fact that we can magnify the circumcircle  $\odot(ABC)$  by a factor  $\frac{1}{2}$  so that it maps to the nine-point circle.

**Definition 2.2.** The **Incircle** of  $\triangle ABC$  is a circle tangent to all the three sides of the triangle internally.

**Definition 2.3.** The **Excircle** of  $\triangle ABC$  is a circle tangent to one of the sides internally and the other two sides externally.

Some popular triangle names that we shall use.

**Definition 2.4.** The **contact triangle** of  $\triangle ABC$  is the triangle joining the incircle touch-points with the triangle sides.

**Definition 2.5.** The **medial triangle** of  $\triangle ABC$  is the triangle joining the midpoints of the sides of  $\triangle ABC$ .

Finally, we will need the definition of an isogonal conjugate in a triangle.

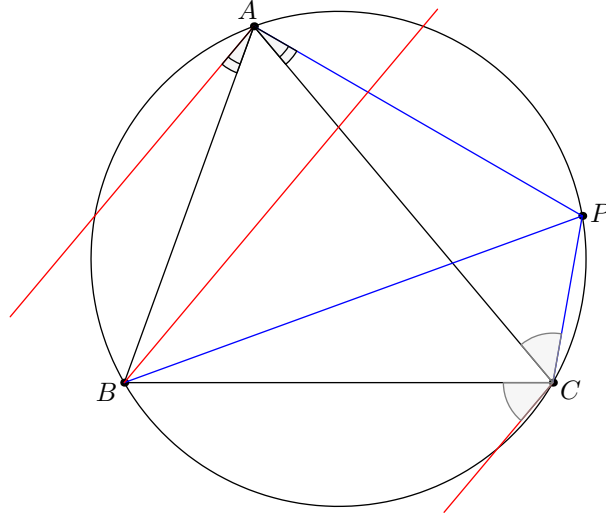
**Definition 2.6.** In a  $\triangle ABC$ , where  $AD$ ,  $BE$  and  $CF$  are  $A$ -cevia,  $B$ -cevia and  $C$ -cevians. Let their concurrency point be  $X$ . Let  $D'$ ,  $E'$  and  $F'$  be points on  $\overline{BC}$ ,  $\overline{CA}$  and  $\overline{AB}$  such that  $\angle BAD = \angle D'AC$ ,  $\angle CBE = \angle E'BA$  and  $\angle ACF = \angle F'CB$ , then the cevians  $AD'$ ,  $BE'$  and  $CF'$  are concurrent too. The point of concurrency is the **Isogonal Conjugate** of point  $X$  with respect to  $\triangle ABC$ .

The proof of the above concurrency follows from **Ceva's Theorem**.

### §3 Important Lemmas

#### Lemma 3.1

The isogonal conjugate of a point  $P$  with respect to  $\triangle ABC$  is the point at infinity along the isogonal of line  $\overline{AP}$  if and only if  $P$  lies on  $\odot(ABC)$ .



**Figure 1:** Proof without words.

#### Lemma 3.2

Let  $I$  be the incenter and  $O$  be the circumcenter of  $\triangle ABC$ . Let  $M$  be the midpoint of  $\overline{BC}$ . Suppose  $\odot(I)$  touches  $\overline{BC}$  at  $D$  and  $M'$  is the reflection of  $M$  over  $\overline{AI}$ . Then  $\overline{DM'} \perp \overline{OI}$ .

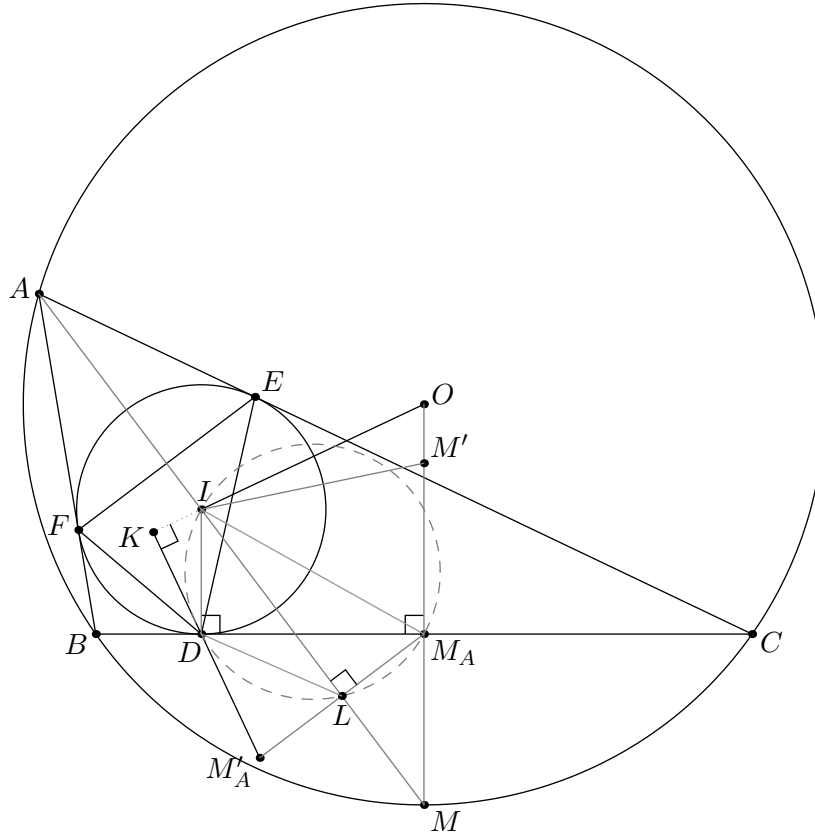
*Proof.* The proof for this result could be broken down into several claims immediately following due to clever constructions. Suppose  $L$  is the foot of perpendicular from  $M_A$  to  $\overline{AI}$ . Let  $M$  be the point of intersection of  $\overline{AI}$  with  $\odot(ABC)$  and  $M'$  be the reflection of  $M$  over  $\overline{BC}$ . Let  $\overline{DM'_A} \cap \overline{OI}$  at  $K$ . We would like to show that  $KDM_AO$  is cyclic, which should imply that  $\angle OKD = 90^\circ$ .

**Claim 3.3.** Quadrilateral  $IDLM_A$  is cyclic.

*Proof.* Since  $\angle IDM_A = 90^\circ$  and  $\angle ILM_A = 90^\circ$ , therefore the concyclicity is implied.  $\square$

**Claim 3.4.**  $\triangle DLM_A \sim \triangle IM_A M$

*Proof.* Since  $\angle DM_AL = \angle M_A M L$  and  $\angle LDM_A = \angle LIM_A$ , thus due to AA similarity criterion we have  $\triangle DLM_A \sim \triangle IM_A M$ .  $\square$



**Claim 3.5.**  $\triangle DM'_A M_A \sim \triangle IM'M$

*Proof.* We already have  $\angle DM'_A M_A = \angle IMM'$  due to the previous claim. Since,

$$\left( \frac{\overline{DM'_A}}{\overline{IM}} \right) = \left( \frac{\overline{LM'_A}}{\overline{M_A M}} \right) = \left( \frac{\overline{M'_A M_A}}{\overline{M' M}} \right)$$

Therefore, due to SAS similarity criterion we have  $\triangle DM'_A M_A \sim \triangle IM'M$ .  $\square$

**Claim 3.6.**  $\triangle IM'M \sim \triangle OIM$

*Proof.* This claim follows due to SAS similarity criterion because,

$$\overline{IM}^2 = \overline{BM}^2 = 2R \cdot \overline{MM_A} = R \cdot \overline{M'M} = \overline{OM} \cdot \overline{M'M}$$

$\square$

Combining all the claims, we finally have  $\angle M'_A D M_A = \angle I O M \implies K D M_A O$  is cyclic. Hence  $\angle OKD = 90^\circ$  which proves the result.  $\square$

## §4 Proof of Feuerbach's Theorem

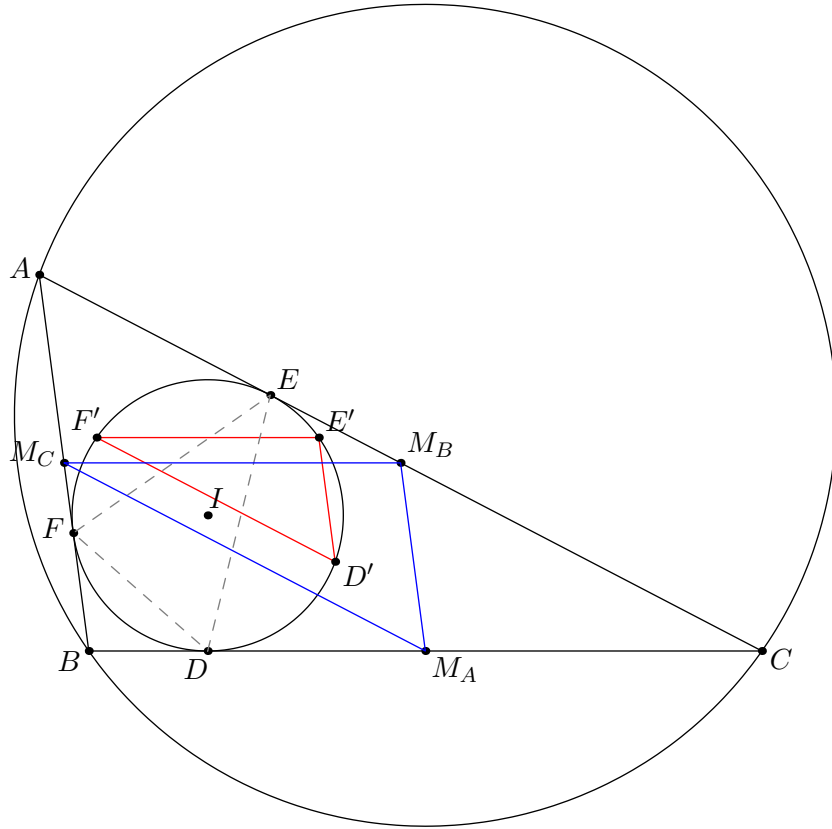
We are now ready to prove the Feuerbach's Theorem. We will proceed with the proof in two steps. The first step establishes that the incircle is tangent to the nine-point circle. In the second step, we show that one of the excircles is tangent to the nine-point circle too, thus proving the statement.

### §4.1 Step 1: Interior Feuerbach Point

To prove two circles are tangent, a common strategy is to find two triangles on each of the circles and show that they are homothetic from a point that lies on one of the circles. In this way, we can ensure that the circles are tangent at the point of homothety.

So we need to pick two triangles, one on the nine-point circle and the other on the incircle. We'll show that the following triangles are homothetic.

**Claim 4.1.** Suppose  $\triangle M_A M_B M_C$  is the medial triangle and  $\triangle DEF$  is the contact triangle of  $\triangle ABC$ . Reflect  $D$  over  $\overline{AI}$  to  $D'$  and similarly define  $E'$  and  $F'$ . We claim that  $\triangle M_A M_B M_C$  and  $\triangle D'E'F'$  are homothetic.



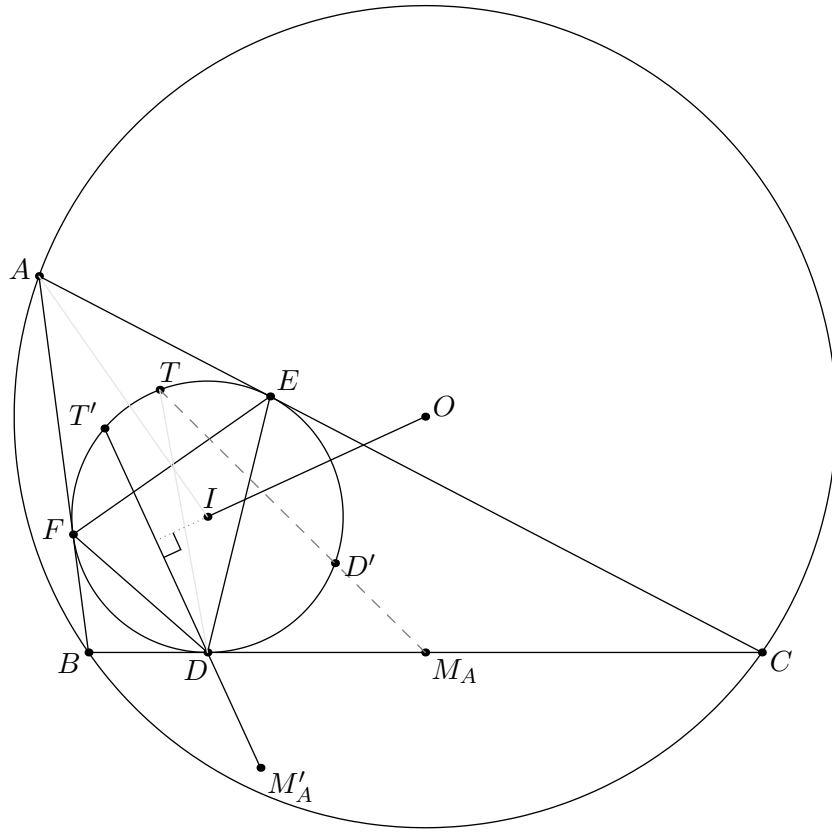
*Proof.* To prove that  $\triangle M_A M_B M_C$  and  $\triangle D'E'F'$  are homothetic, we show that their sides are pairwise parallel. Since  $\overline{EF}$  is perpendicular to  $\overline{AI} \implies \overline{DD'} \parallel \overline{EF}$ . As a result,

$$\angle BFD' = \angle FED' = \angle EFD = 90^\circ - \frac{1}{2}\angle C$$

However,

$$\begin{aligned} \angle FD'E' &= \angle FDE + \angle ED'E' = \angle FDE + 180^\circ - \angle BIF - \angle FIE \\ &= (90^\circ - \frac{1}{2}\angle A) + 180^\circ - (90^\circ - \frac{1}{2}\angle B) - (180^\circ - \angle A) \\ &= \frac{1}{2}(\angle A + \angle B) = 90^\circ - \frac{1}{2}\angle C \end{aligned}$$

Therefore,  $\angle BFD' = \angle FD'E' \implies \overline{D'E'} \parallel \overline{AB} \parallel \overline{M_A M_B}$ . Similarly, we will have  $\overline{E'F'} \parallel \overline{M_B M_C}$  and  $\overline{F'D'} \parallel \overline{M_C M_A}$ . Thus, we conclude that  $\triangle M_A M_B M_C$  and  $\triangle D'E'F'$  are homothetic.  $\square$



So far we have shown that the two triangles have their sides pairwise parallel. Due to **Desargues' Theorem**, we have that these triangles are perspective about a point, which is precisely their homothetic center. In our case, this point will be the tangency point of the nine-point circle and the incircle. This is due to the fact that existence of such homothetic center implies that the circumcircles of the two triangles  $\triangle D'E'F'$  and  $\triangle M_AM_BM_C$  are mapped under the transformation. Thus implying the desired tangency. So it is only left to find out this homothetic center.

**Claim 4.2.** Let  $T$  be the isogonal conjugate with respect to  $\triangle DEF$  of the point at infinity of the line perpendicular to  $\overline{OI}$ . Then  $T$  is the homothetic center of  $\triangle D'E'F'$  and  $\triangle M_AM_BM_C$ .

*Proof.* We will prove that  $\overline{D'M_A}$  passes through  $T$ . Similarly, we should be able to show that  $\overline{E'M_B}$  and  $\overline{F'M_C}$  pass through  $T$  as well.

Let the point at infinity along the line perpendicular to  $\overline{OI}$  be  $\infty_{\perp \overline{OI}}$ . Suppose  $D\infty_{\perp \overline{OI}} \cap \odot(I)$  again at  $T'$ . Then lines  $\overline{DT}$  and  $\overline{DT'}$  are isogonal with respect to  $\angle FDE$ , and  $T'$  is the reflection of  $T$  over  $\overline{AI}$ . Since  $\overline{DT'}$  is perpendicular  $\overline{OI} \implies \overline{DT'}$  passes through  $M'_A$  due to 3.2, where  $M'_A$  is the reflection of  $M_A$  over  $\overline{AI}$ .

Therefore, reflecting the line  $\overline{T'DM'_A}$  over  $\overline{AI}$  implies that the points  $T$ ,  $D'$  and  $M_A$  are collinear. Similarly, we can show that  $\overline{E'M_B}$  and  $\overline{F'M_C}$  pass through  $T$ , thus implying that  $T$  is indeed the homothetic center of  $\triangle D'E'F'$  and  $\triangle M_AM_BM_C$ .  $\square$

The point  $T$  is called the **Feuerbach Point**, which is the point of tangency of the nine-point circle and the incircle. It is also the isogonal conjugate with respect to  $\triangle DEF$  of the point at infinity along the line perpendicular to  $\overline{OI}$ , denoted as  $\infty_{\perp \overline{OI}}$ .

## §4.2 Step 2: Exterior Feuerbach Point

We will show that the  $A$ -excircle is tangent to the nine-point circle. The proof is similar to the previous step. We first need to identify the homothetic triangles to imply the tangency.

**Claim 4.3.** Suppose that the  $A$ -excircle is tangent to the sides  $\overline{BC}$ ,  $CA$  and  $AB$  at points  $T$ ,  $U$  and  $V$ . Reflect  $T$  over  $\overline{AI}$  to  $T'$ . Reflect  $T'$  over  $\overline{UI_A}$  and  $\overline{VI_A}$  to  $V'$  and  $U'$  respectively. Then  $\triangle M_A M_B M_C$  and  $\triangle T' U' V'$  are homothetic.

*Proof.* Since  $\overline{UI_A} \perp \overline{T'V'}$  and  $\overline{VI_A} \perp \overline{T'U'}$   $\implies \overline{T'U'} \parallel \overline{AB} \parallel \overline{M_A M_B}$  and  $\overline{T'V'} \parallel \overline{CA} \parallel \overline{M_A M_C}$ . Since points  $T$ ,  $U$  and  $V$  are pairwise reflections of each other over the lines  $\overline{AI}$ ,  $\overline{BI_A}$  and  $\overline{CI_A}$

$$\overline{TU'} = \overline{UV} = \overline{TV'}$$

Hence,  $\overline{TI_A} \perp \overline{U'V'} \implies \overline{U'V'} \parallel \overline{BC} \parallel \overline{M_B M_C}$ . This implies that  $\triangle M_A M_B M_C$  and  $\triangle T' U' V'$  are homothetic.  $\square$

The next step is similar as well. If we reflect  $M_A$  over  $\overline{AI}$  to  $M'_A$ , then  $\overline{TM'_A} \perp \overline{OI_A}$ , where  $T$  is the point where  $A$ -excircle touches  $\overline{BC}$ .

**Claim 4.4.** Suppose that point  $M_A$  is reflected over  $\overline{AI}$  to  $M'_A$ . Then  $\overline{TM'_A} \perp \overline{OI_A}$ .

*Proof.* The idea is similar to showing  $\overline{DM'_A} \perp \overline{OI}$ . Let  $R$  be the intersection of  $\overline{TM'_A}$  and  $\overline{OI_A}$ . Let  $M$  be the intersection of  $\overline{AI}$  with  $\odot(ABC)$  and  $M'$  be the reflection of  $M$  over  $\overline{BC}$ . Then  $\triangle M'_A M_A T \sim \triangle M' M I_A \sim \triangle I_A M O$ . These similar triangles imply that  $OM_A R T$  is cyclic  $\implies \overline{TM'_A} \perp \overline{OI_A}$ .  $\square$

Finally, we finish the implication by identifying that the exterior feuerbach point  $F_a$  is the isogonal conjugate of point at infinity of the line perpendicular to  $\overline{OI_A}$  with respect to  $\triangle TUV$ . Reflecting the collinearity over  $\overline{AI}$  implies that  $\overline{M_A T'}$  passes through  $F_a$ . Similarly, we can show that  $\overline{U' M_B}$  and  $\overline{V' M_C}$  passes through  $F_a$ , implying that it is indeed the center of homothety for  $\triangle M_A M_B M_C$  and  $\triangle T' U' V'$ .