Power of a Point

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In this article, we will learn about the power of a point theorem and it's applications in geometry.

§1 Power of a Point

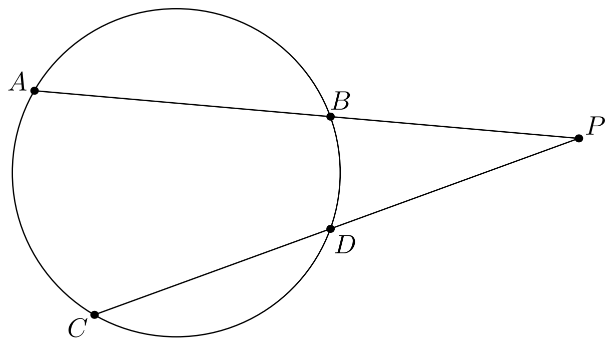
The power of a point theorem is a useful result that helps us attack a whole new set of problems that are out of the realms of angle chasing. We want to primarily deal with circles and be able to convert information about angles to lengths. Now, let's state the result.

Theorem 1.1

Suppose ω is a circle and P is a point. Draw two lines ℓ_1 and ℓ_2 passing through P that intersect the circle in points A, B and C, D. Then,

$$\overline{PA} \cdot \overline{PB} = \overline{PC} \cdot \overline{PD}$$

There are two cases of this result, one when the point P lies inside the circle and when it lies outside the circle ω. Surprisingly, the result holds true in both the cases. Proving this is just uses properties of the cyclic quadrilaterals to establish similar triangles, so we will omit the proof here.



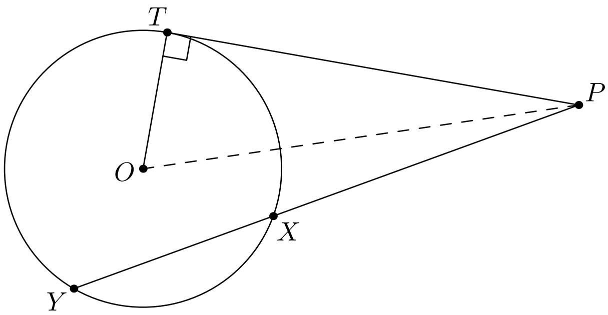
A powerful form of the above stated result is the following theorem

Theorem 1.2

Suppose ω is a circle and P is a point lying outside the circle. Let ℓ be a line that passes through P and cuts the circle in points X and Y. Let T be a point on the circle such that $\overline{PT}$ is tangent to ω. Then

$$\overline{PX} \cdot \overline{PY} = \overline{PT}^2$$

We can intuitively imagine why this might be true. If we bring the points X and Y so close that they basically coincide, then the line is a tangent and we would have square of the length of tangent drawn.



If we look at the result carefully, we realise that regardless of what the line is, we will always have that the product of line segments the line makes will stay equal. Hence, if define the following quantity for a point P and a circle ω,

$$\operatorname{Pow}_\omega(P) = \overline{PX} \cdot \overline{PY}$$

where, ℓ is a line that passes through P cutting ω at X and Y. Then the quantity $\operatorname{Pow}_\omega(P)$ will stay constant for a fixed P and ω. To be even more formal, the function $\operatorname{Pow}_\omega(P)$ should depend only on P and ω. But looks like the way we have define it right now also uses a line ℓ that passes through P and its intersections with X and Y with a circle ω. We can rewrite the above expression as

$$\begin{aligned}\operatorname{Pow}_\omega(P) &= \overline{PX} \cdot \overline{PY} \\ &= \overline{PT}^2 \\ &= \overline{OP}^2 - \overline{OT}^2 \\ &= \overline{OP}^2 - r^2\end{aligned}$$

The above expression is only dependent on the circle ω and the point P. Therefore, we go ahead and define this quantity

Definition 1.3. Given a circle ω centered at O with radius r and a point P, the power of point P is given by

$$\operatorname{Pow}_\omega(P) = \overline{OP}^2 - r^2$$

With the above definition, we can rewrite the previously stated results as

Theorem 1.4 (Power of a Point Theorem)

Given a circle ω and a point P,

1. the quantity $\mathrm{Pow}_\omega(P)$ is positive, zero or negative depending on whether P is outside, on or inside ω, respectively.
2. if ℓ is a line through P intersecting ω at two distinct points X and Y, then
$$\overline{PX} \cdot \overline{PY} = |\,\mathrm{Pow}_\omega(P)\,|$$
3. if P is outside ω and $\overline{PA}$ is a tangent to ω at a point A on ω, then
$$\overline{PA}^2 = \mathrm{Pow}_\omega(P)$$

Infact the converse of the above result holds true as well.

Theorem 1.5 (Converse of Power of a Point Theorem)

Suppose A, B, X and Y are four distinct points in the plane. Let the lines AB and XY intersect at P. Then points A, B, X and Y are concyclic if and only if

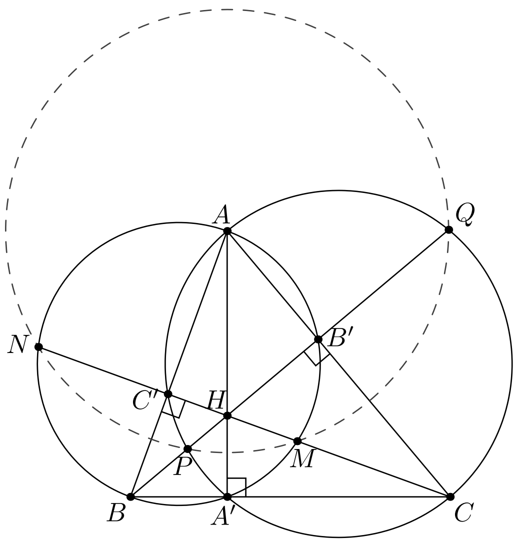
$$\overline{PA} \cdot \overline{PB} = \overline{PX} \cdot \overline{PY}$$

As it turns out, this is a very powerful result in our toolbox that we can use to prove that a quadrilateral is cyclic. Let's look at a few examples of these results.

§1.1 Examples

Problem 1.6 (USA Math Olympiad 1990)

An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.



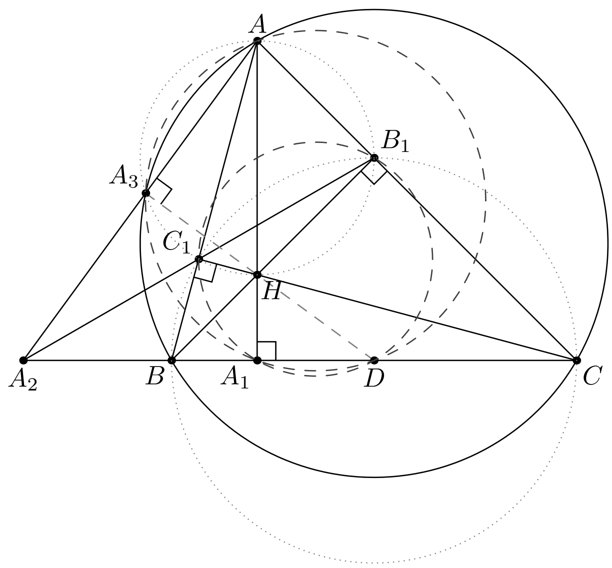
Proof. Let H be the orthocenter of $\triangle ABC$ and A' be the foot of perpendicular dropped from point A onto $\overline{BC}$. By power of a point theorem applied on circles with diameter $\overline{AC}$, $\overline{BC}$ and $\overline{AB}$, we have

$$\begin{aligned}\overline{HP} \cdot \overline{HQ} &= \overline{HC} \cdot \overline{HC'} \\ &= \overline{HB} \cdot \overline{HB'} \\ &= \overline{HM} \cdot \overline{HN}\end{aligned}$$

Therefore, by the converse of power of a point theorem $\implies$ $MNPQ$ is a cyclic quadrilateral. □

Problem 1.7 (USA TSTST 2012)

In scalene triangle ABC, let the feet of the perpendiculars from A to BC, B to CA, C to AB be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides BC, CA, AB. Show that the perpendiculars from D to AA_2, E to BB_2 and F to CC_2 are concurrent.



Proof. We shall show that the perpendiculars from D to $\overline{AA_2}$, E to $\overline{BB_2}$ and F to $\overline{CC_2}$, all pass through the orthocenter H of $\triangle ABC$, which is their concurrency point. Suppose A_3 is the foot of perpendicular from D to $\overline{AA_2}$. Since, $\angle AA_3D = 90^\circ$ and $\angle AA_1D = 90^\circ \implies AA_3A_1D$ is cyclic. However, $B_1C_1A_1D$ is cyclic too because they lie on the nine-point circle of $\triangle ABC$, and BC_1B_1C is cyclic too because $\angle BC_1C = 90^\circ$ and $\angle BB_1C = 90^\circ$. By applying the power of a point theorem on these circles, we have

$$\begin{aligned}\overline{A_2A_3} \cdot \overline{A_2A} &= \overline{A_2A_1} \cdot \overline{A_2D} \\ &= \overline{A_2C_1} \cdot \overline{A_2B_1} \\ &= \overline{A_2B} \cdot \overline{A_2C}\end{aligned}$$

By the converse of power of a point theorem, this implies that $AA_3C_1B_1$ and AA_3BC are cyclic quadrilaterals too. Since $\overline{AH}$ is the diameter of $\odot(AC_1B_1) \implies \angle AA_3H = 90^\circ$. However we know that, $\angle AA_3D = 90^\circ \implies \overline{A_3D}$ passes through H. Similarly, we can show that the others pass through the orthocenter H, thus implying the concurrency. □

§1.2 Exercises

Exercise 1.8. Let $\triangle ABC$ be an acute angled triangle with circumcenter O and orthocenter H. Prove that

$$\overline{OH}^2 = R^2 \left(1 - 8 \cos A \cos B \cos C\right)$$

Exercise 1.9 (IMO Shortlist 2011)**.** Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

Exercise 1.10 (USA Math Olympiad 1998)**.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

Exercise 1.11 (USA Math Olympiad 2009)**.** Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

§2 Radical Axis Theorem

So far, we have developed tools that allow us to tackle problems involving a single circle. We now move onto problems involving multiple circles. We begin by introducing a few key definitions.

Definition 2.1. Given two circles ω_1 and ω_2 with distinct centers, the **Radical Axis** of the circles is the set of points P such that

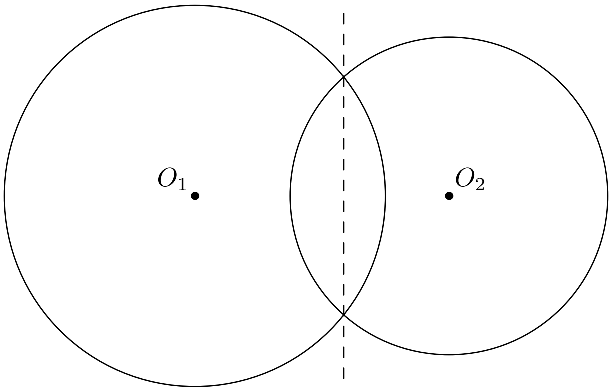
$$\operatorname{Pow}_{\omega_1}(P) = \operatorname{Pow}_{\omega_2}(P)$$

Looking at the definition of the radical axis doesnt seem that intuitive. However, it basically tells us this

Corollary 2.2

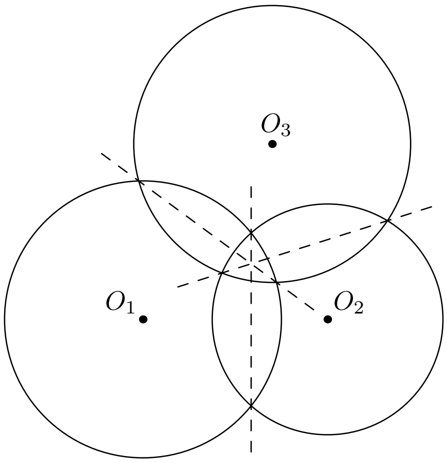
For two intersecting circles ω_1 and ω_2, their radical axis is the line passing through their points of intersections.

Something even more counter-intuitive is that the radical axis is defined even for a pair of *non-intersecting* circles too. A common misconception is to imagine the radical axis as the perpendicular bisector of the line joining the centers, but that is not true. It is only the locus of points in the space that have equal powers from both the circles. So what is interesting about this? The following result is what makes radical axis a useful tool in geometry.



> **Theorem 2.3** (Radical Axis Theorem)
>
> Given three distinct circles ω_1, ω_2 and ω_3, their pairwise radical axes are concurrent. This point of concurrency is known as the **Radical Center** of the three circles.

The proof immediately follows from the definition of radical axis. Infact, the converse of radical axis theorem holds is true and serves as a criterion for proving cyclicity of one of the three circles. Essentially, this is just equivalent to applying the power of a point theorem twice and then concluding via its converse.



This theorem is particularly useful because it gives us a powerful new tool for proving concurrencies. Let us now explore some examples.

§2.1 Examples

> **Problem 2.4** (IMO 1995)
>
> Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

Proof. □

Problem 2.5 (IMO 2008)

Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

Proof. □

§2.2 Exercises

Exercise 2.6 (USA Math Olympiad 1997)**.** Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

Exercise 2.7. Let ABC be a triangle and let D and E be points on sides AB and AC, respectively, such that $DE \parallel BC$. Let P be any point interior to triangle ADE, and let F and G be the intersections of DE with the lines BP and CP, respectively. Let Q be the second intersection point of the circumcircles of triangles PDG and PFE. Prove that the points A, P, and Q are collinear.

Exercise 2.8. Let ABC be an acute triangle with incenter I. Points E and F are the midpoints of the shorter arcs $\widehat{AC}$ and $\widehat{AB}$ of the circumcircle $\odot(ABC)$, respectively. Segment EF intersects sides AB and AC at points P and Q, respectively. Point D is defined by the conditions $PD \parallel BI$ and $QD \parallel CI$. Let T be the intersection point of BF and CE. Prove that points T, I, D are collinear.

§3 Common Tangents

§3.1 Examples

Problem 3.1 (IMO Shortlist 2000)

Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

Proof. □

Problem 3.2 (APMO 1999)

Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Proof. □

§3.2 Exercises

Exercise 3.3. Circles ω_1 and ω_2 intersect at points X and Y. Their two common tangents meet at a point P. A line ℓ through P intersects ω_1 at points A and C and intersects ω_2 at points B and D, where the points A, B, C, D lie on ℓ in this order. Prove that the tangent to ω_1 at C and the tangent to ω_2 at B intersect at a point lying on the line XY.

Exercise 3.4. Let ω_1 and ω_2 be two circles. Line ℓ_1 is tangent to ω_1 at A and to ω_2 at B, such that the two circles lie on the same side of ℓ_1. Line ℓ_2 is tangent to ω_1 at C and to ω_2 at D, such that the two circles lie on different sides of ℓ_2.

Prove that the intersection point of AC and BD lies on the line joining the centers of ω_1 and ω_2.

§4 Practice Problems

Exercise 4.1. Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be the altitudes of a scalene triangle with circumcenter O. Prove that $\odot(AOD)$, $\odot(BOE)$, and $\odot(COF)$ intersect at point X other than O.

Exercise 4.2 (USA Junior Math Olympiad 2024)**.** Let $ABCD$ be a cyclic quadrilateral with $AB = 7$ and $CD = 8$. Point P and Q are selected on segment AB such that $AP = BQ = 3$. Points R and S are selected on segment CD such that $CR = DS = 2$. Prove that $PQRS$ is a cyclic quadrilateral.

Exercise 4.3 (USA Junior Math Olympiad 2012)**.** Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).

Exercise 4.4 (USA Math Olympiad 2023)**.** In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

Exercise 4.5 (IMO 2009)**.** Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.