# Mass Point Geometry

BMC Int II Spring 2020

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# 1    Introduction

Mass point geometry is a method to solve geometry problems involving triangles and asking for ratios of lengths or areas of the triangles. It uses a basic concept from physics but can have some powerful applications.

Archimedes' principle of lever tells us that the center of mass of two points of mass $m_1$ and $m_2$ is where

$$m_1 \cdot d_1 = m_2 \cdot d_2,$$

where $d_1, d_2$ refer to the distance from the center of mass to the points. Note that the center of mass is the balancing point if the two points were placed on a lever. (This is also where Archimedes famously quoted "Give me a place to stand on, and I will move the Earth") A second important property that we will use is that

> The center of mass of a system is not changed by replacing several point masses with their total mass positioned at their center of mass.

What this means is that we can find the center of mass of several points by successively considering line segments only.
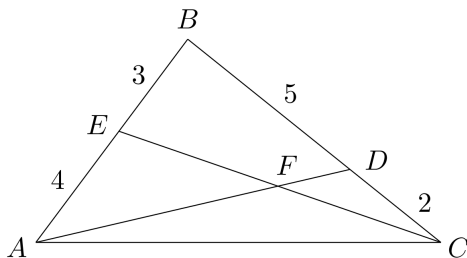
# 2    Intersecting Cevians

## 2.1    Example

**Definition 2.1.** *A **cevian** is a line segment from a vertex of a triangle to the opposite side.*

The first example of using mass points will be in the first case of two intersecting cevians as in figure 1.

**Example 2.2.** *In triangle $ABC$, cevian $\overline{AD}$ and $\overline{CE}$ divide side lengths $\overline{BC}$ and $\overline{AB}$ as shown. Let $F$ be their intersection. What is $\dfrac{CF}{EF} + \dfrac{AF}{DF}$?*

The general strategy will be to pretend to place weights at the vertices $A, B, C$ such that their center of mass is $F$. Before we begin, we set up some notation.

**Definition 2.3.** *A **mass point** is a point $P$ along with a mass $m$. We write $mP$ to denote that there is a mass of $m$ at point $P$.*

**Definition 2.4.** *We can add two mass points $mP$ and $nQ$. Their sum is the point $R$ between $P$ and $Q$ that is their center of mass. Namely $PR \cdot m = RQ \cdot n$. It has mass $m + n$. So $mP + nQ = (m + n)R$.*

**Remark.** *The addition we defined is both associative and commutative. So $(xP + yQ) + zR = xP + (yQ + zR)$.*

Now we want to set up the masses so that $E$ is the center of mass of $A$ and $B$ and $D$ is the center of mass of $B$ and $C$. So we place a mass of $4B$ and hence we get $3A$ and $10C$. Note that we could have just used $1B$ and have fractional masses. Therefore, the weight at $E$ is $3A + 4B = 7E$ and the weight at $D$ is $4B + 10C = 14D$. Finally, the weight at $F$ is $3A + 14D = 17F$. To double check our work and make sure that $F$ is the center of mass, we see that

$$17F = 3A + 4B + 10C = (3A + 4B) + 10C = 3A + (4B + 10C).$$

This means that $F$ is the center of mass of both the line segment $\overline{CE}$ and $\overline{AD}$. So, we use the lever principle one last time to get

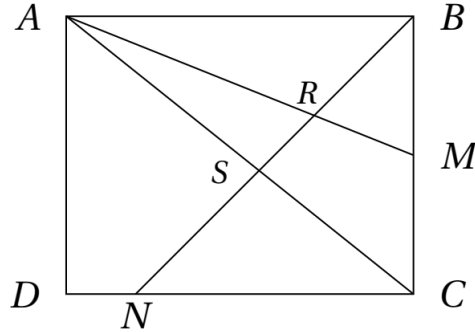$$\frac{CF}{EF} + \frac{AF}{DF} = \frac{7}{10} + \frac{14}{3} = \frac{161}{30}.$$

## 2.2  Problems

**Exercise 2.5.** *In triangle $ABC$, point $E$ is on $\overline{AC}$ so that $CE = 3AE$ and $F$ is on $\overline{AB}$ so that $BF = 3AF$. If $\overline{BE}$ and $\overline{CF}$ intersect at $O$ and the line $AO$ intersects $\overline{BC}$ at $D$, calculate $\frac{OB}{OE} + \frac{OD}{OA}$.*

**Exercise 2.6.** *Let $AD$ be the median bisecting the side $BC$ in triangle $ABC$. Let $O$ be the midpoint of $\overline{AD}$. Suppose that $BO$ intersects side $\overline{AC}$ at $E$. What is $\dfrac{AE}{CE}$?*

**Exercise 2.7.** *Let $\overline{BE}$ and $\overline{CF}$ be medians of triangle $ABC$ and let $X$ be the midpoint of $\overline{BE}$. Suppose that $D$ is on $\overline{BC}$ so that $\overline{AD}$ intersects $\overline{BE}$ at $X$. Let $Z$ be the intersection of $\overline{CF}$ and $\overline{AD}$. Find $\dfrac{CZ}{ZF}$. (Hint: You will have to use two different mass systems. One assigning masses so that the center of mass is $X$ and then another so that the center of mass is $Z$.)*

**Exercise 2.8.** *(Mathcounts) In rectangle $ABCD$, point $M$ is the midpoint of side $BC$, and point $N$ lies on $CD$ such that $DN : NC = 1 : 4$. Line segment $BN$ intersects $AM$ and $AC$ at points $R$ and $S$ respectively. If $NS : SR : RB = x : y : z$, where $x, y, z$ are positive integers, what is the minimum possible value of $x + y + z$?*



**Exercise 2.9.** *(Purple Comet) Triangle $ABC$ has sides with $AB = 39, BC = 57, CA = 70$. Median $\overline{AD}$ is divided into three congruent segments by points $E$ and $F$ and lines $BE$ and $BF$ intersect side $AC$ at points $G$ and $H$ respectively. What is $GH$?*

**Exercise 2.10.** *Points $D, E, F$ are on sides $\overline{BC}, \overline{CA}, \overline{AB}$ respectively so that*

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 2.$$

*Let $P$ be the intersection of $\overline{BE}$ and $\overline{AD}$, $Q$ the intersection of $\overline{BE}$ and $\overline{CF}$, and $R$ the intersection of $\overline{AD}, \overline{CF}$. Find the ratio of the areas $\dfrac{[PQR]}{[ABC]}$.*
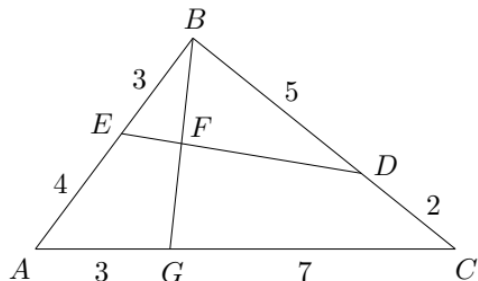
# 3 Transversals

## 3.1 Example

**Definition 3.1.** *A **transversal** is a line that intersects two other distinct lines.*

When dealing with a transversal of a triangle, the point that is on its own side of the transversal will have a split mass.

**Example 3.2.** *Transversal $DE$ joins points $D$ and $E$ on the sides of triangle $ABC$ so that $AE : EB = 4 : 3$ and $CD : DB = 2 : 5$. Cevian $BG$ divides $AC$ in a ratio of $AG : GC = 3 : 7$. Find $\frac{EF}{FD}$.*

To solve this, we will split up the mass at $B$ so that part of it will balance out the mass at $A$ to have a center of mass at $E$, and the other part of the mass at $B$ will balance at $C$ at $D$. We first assign a mass of $21A$ so we have $28B$ and $9C$. Then the contribution at $B$ from $C$

will be $\frac{2}{5}(9C) = \frac{18}{5}B$. Thus, the total mass at $G$ is $21A + 9C = 30G$, the total mass at $E$ is $28B + 21A = 49E$, and at $D$ $9C + 18/5B = 12\frac{3}{5}D$ and at $B$ is $\frac{18}{5}B + 28B = 31\frac{3}{5}B$. To verify that $F$ is indeed the center of mass of the whole system, we see that

$$61\frac{3}{5}F = 21A + 31\frac{3}{5}B + 9C = (21A + 9C) + 31\frac{3}{5}B = 30G + 31\frac{3}{5}B$$

$$= (21A + 28B) + (3\frac{3}{5}B + 9C) = 49E + 12\frac{3}{5}D$$

Thus, we have $\frac{EF}{FD} = \frac{12\frac{3}{5}}{49} = \frac{63}{245} = \frac{9}{35}$.

## 3.2 Problems

**Exercise 3.3.** *In triangle $ABC$, let $E$ be on $AB$ so that $AE : EB = 1 : 3$ and $D$ on $BC$ so that $BD : DC = 2 : 5$ and $F$ on $ED$ so that $EF : FD = 3 : 4$. Let ray $BF$ intersect $AC$ at $G$. Find $BF : FG$.*

**Exercise 3.4.** *In triangle $ABC$, $D$ is the midpoint of $BC$. Point $E$ lies on $AC$ so that $AE : EC = 11 : 9$ and point $F$ lies on cevian $AD$ so that $AF : FD = 2 : 3$. Ray $EF$ intersects $AB$ at $G$. What is $GF : FE$?*

**Exercise 3.5.** *(ARML 1992 I8) In triangle $ABC$, points $D$ and $E$ are on $AB$ and $AC$ respectively. Point $T$ is chosen on $BC$ so that $BT : CT = 2 : 3$ and $AT$ intersects $DE$ at $F$. If $AD = 1, DB = 3, AE = 2$ and $EC = 4$. What is $AF : FT$?*

**Exercise 3.6.** *(AIME I 2009 4) In parallelogram $ABCD$, point $M$ is on $AB$ so that $\frac{AM}{AB} = \frac{17}{1000}$ and point $N$ is on $AD$ so that $\frac{AN}{AD} = \frac{17}{2009}$. Let $P$ be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$.*

Problems were taken from the UCLA Math Circle and Tom Rike's excellent handout for the San Jose Math Circle. Additional problems can be found on the AoPS Wiki here: `https://artofproblemsolving.com/wiki/index.php/Mass_points`