IMO Shortlist 2007 N2

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18 January 2026

https://artofproblemsolving.com/community/c6h214712

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§1 Problem

Problem (IMO Shortlist 2007 N2)
Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

§2 Solution 1 (Using $\nu_p(b)$)

Proof. Since $b > 1$, therefore there must exist a prime p such that $p \mid b$. Choose any such prime p.

Claim 2.1. For any prime p that divides b, we will always have $n \mid \nu_p(b)$

Proof. Using the division algorithm, we can write

$$\nu_p(b) = qn + r, \quad \text{where, } 0 \leq r < n$$

If $r = 0$, we are done. Hence, assume $r > 0$ here onwards. Choose an integer ℓ such that $\ell n > qn + r$. If we set $k = p^{\ell n}$, then

$$p^{\ell n} \mid b - a_k^n \implies \nu_p(b - a_k^n) \geq \ell n$$

If $\nu_p(b) \neq \nu_p(a_k^n)$, then we must have $\min(\nu_p(b), \nu_p(a_k^n)) \geq \ell n$. However,

$$\nu_p(b) = qn + r < \ell n \implies \min(\nu_p(b), \nu_p(a_k^n)) < \ell n$$

this implies that, we must have $\nu_p(b) = \nu_p(a_k^n)$. However, this would mean that

$$\nu_p(a_k^n) = n\nu_p(a_k) = \nu_p(b) = qn + r \implies n \mid qn + r$$

which forces $n \mid r$. Given the bounds on r, this is impossible contradicting our assumption $r > 0$. Hence for any prime p that divides b, we will always have $n \mid \nu_p(b)$. $\square$

From the claim, we have that for any prime p, $\nu_p(b)$ is a multiple of n. Therefore, b must be of the form A^n. $\square$

§3 Solution 2 (Using Construction for k)

Proof. Choose $k = b^2$. This implies,

$$\begin{aligned} b^2 \mid b - a_k^n &\iff b - a_k^n = qb^2 \\ &\iff a_k^n = b\,(1 - qb) \end{aligned}$$

Since $\gcd(b, 1 - qb) = 1$, therefore they do not share any common prime factors. Any prime p that divides b must divide a_k^n. If

$$\nu_p(a_k) = \ell \implies \nu_p(a_k^n) = \ell n$$

Consequently, $\nu_p\,(b(1 - qb)) = \nu_p\,(b) = \ell n \implies b$ is of the form A^n. $\square$