IZhO 2026 P2

Mmukul Khedekar

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§1 Problem

Problem (IZhO 2026 P2)

We consider a positive integer n for which there exist positive integers a and b for which $\lfloor a\sqrt{10} \rfloor = n = \lfloor b\sqrt{11} \rfloor$. Prove there exists a positive integer c for which $n = \lfloor c(11\sqrt{10} - 10\sqrt{11}) \rfloor$

§2 Solution (Using Rationalization)

Proof. We claim that $\boxed{c = a + b}$ works. Consider the equation $n = \lfloor a\sqrt{10} \rfloor$. This could be written as

$$\begin{aligned} n &\le a\sqrt{10} < n+1 \\ n &\le a\sqrt{10}\,(11-10) < n+1 \\ \frac{n}{\sqrt{11}+\sqrt{10}} &\le a\sqrt{10}\left(\sqrt{11}-\sqrt{10}\right) < \frac{n+1}{\sqrt{11}+\sqrt{10}} \\ \frac{n\sqrt{11}}{\sqrt{11}+\sqrt{10}} &\le 11a\sqrt{10} - 10a\sqrt{11} < \frac{(n+1)\sqrt{11}}{\sqrt{11}+\sqrt{10}} \end{aligned}$$

Similarly, we can write $n = \lfloor b\sqrt{11} \rfloor$ as

$$\begin{aligned} n &\le b\sqrt{11} < n+1 \\ n &\le b\sqrt{11}\,(11-10) < n+1 \\ \frac{n}{\sqrt{11}+\sqrt{10}} &\le b\sqrt{11}\left(\sqrt{11}-\sqrt{10}\right) < \frac{n+1}{\sqrt{11}+\sqrt{10}} \end{aligned}$$

$$\frac{n\sqrt{10}}{\sqrt{11}+\sqrt{10}} \leq 11b\sqrt{10} - 10b\sqrt{11} < \frac{(n+1)\sqrt{10}}{\sqrt{11}+\sqrt{10}}$$

Adding these two inequalities we get,

$$\begin{aligned} n &\leq 11a\sqrt{10} - 10a\sqrt{11} + 11b\sqrt{10} - 10b\sqrt{11} < n+1 \\ n &\leq (a+b)\left(11\sqrt{10} - 10\sqrt{11}\right) < n+1 \end{aligned}$$

this implies that $\left\lfloor (a+b)\left(11\sqrt{10} - 10\sqrt{11}\right) \right\rfloor = n$ as desired. □

§3 Generalization 1

Theorem

For any positive integers n, a and b and positive real numbers x and y, if $n = \lfloor ax \rfloor = \lfloor by \rfloor$, then there exists a positive integer c such that

$$n = \left\lfloor c \cdot \frac{xy}{x+y} \right\rfloor$$

Proof. For the equations

$$n = \lfloor ax \rfloor \quad \text{and} \quad n = \lfloor by \rfloor,$$

we have

$$ax - 1 < n \leq ax \quad \text{and} \quad by - 1 < n \leq by.$$

Dividing on both the sides by x and y gives us

$$a - \frac{1}{x} < \frac{n}{x} \leq a, \qquad b - \frac{1}{y} < \frac{n}{y} \leq b.$$

Adding these two inequalities, we obtain

$$(a+b) - \left(\frac{1}{x} + \frac{1}{y}\right) < n\left(\frac{1}{x} + \frac{1}{y}\right) \leq a+b.$$

Multiplying throughout by $\dfrac{xy}{x+y}$ gives

$$(a+b)\frac{xy}{x+y} - 1 < n \leq (a+b)\frac{xy}{x+y}.$$

Therefore,

$$n = \left\lfloor (a+b)\frac{xy}{x+y} \right\rfloor.$$

For which, $c = a+b$ proves the result. □