Geometry of Conics

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This is a manually typed, English translated version of this remarkable text on the geometry of conics. I have invested time in carefully understanding the original Chinese text and adding diagrams for better understanding. This translation is intended for personal study and educational purposes. Any translation or typographical errors are my own.

Contents

0 Notations

These are some of the notations and conventions that we will follow throughout this book.

1. If X, Y are two points (not lying on the line at infinity $\mathcal{L}_\infty$), then
   - XY denotes the line through X and Y.
   - $\overline{XY}$ denotes the line segment joining X and Y (which does not intersect $\mathcal{L}_\infty$).
2. If K and L are two lines, then
   - $K \cap L$ denotes their intersection point.
   - $\angle(K, L)$ denotes the *unoriented* angle between K and L.
   - $\measuredangle(K, L)$ denotes the *oriented* angle between K and L.
3. We use $\triangle XYZ$ to denote a triangle with vertices X, Y and Z.
   - $\odot(XYZ)$ denotes the circumcircle of $\triangle XYZ$.
   - $\odot(XY)$ denotes the circle with $\overline{XY}$ as its diameter.
   - $\odot(X)$ denotes the circle centered at point X.
4. Given a circle Γ and a point X on the circle, we use $XP \cap \Gamma$ or $\Gamma \cap XP$ to denote the intersection of XP with Γ other than at X (if XP is tangent to Γ, this point is still X).
5. Given three lines a, b and c, we use $\triangle abc$ to denote the triangle formed by these lines, namely $\triangle\ (a \cap b)\ (b \cap c)\ (c \cap a)$.
6. Unless specified otherwise, $\triangle ABC$ is taken as the reference triangle. The points I, G, O and H will denote the incenter, centroid, circumcenter, and orthocenter of $\triangle ABC$. Also, I_X $(X = A, B, C)$ will denote the three excenters opposite to the corresponding vertex X.
7. We say that $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are *directly similar*, written as $\triangle X_1Y_1Z_1 \overset{+}{\sim} \triangle X_2Y_2Z_2$, and for these two triangles the following holds:
$$\measuredangle Y_1X_1Z_1 = \measuredangle Y_2X_2Z_2,\ \measuredangle Z_1Y_1X_1 = \measuredangle Z_2Y_2X_2,\ \measuredangle X_1Z_1Y_1 = \measuredangle X_2Z_2Y_2.$$
8. We say that $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are *oppositely similar*, written as $\triangle X_1Y_1Z_1 \overset{-}{\sim} \triangle X_2Y_2Z_2$, and for these two triangles the following holds:
$$\measuredangle Y_1X_1Z_1 = -\measuredangle Y_2X_2Z_2,\ \measuredangle Z_1Y_1X_1 = -\measuredangle Z_2Y_2X_2,\ \measuredangle X_1Z_1Y_1 = -\measuredangle X_2Z_2Y_2.$$
9. Given a $\triangle ABC$ and a point P, we define
   - the **Cevian Triangle of P** with respect to $\triangle ABC$ is
   $$\triangle\,(AP \cap BC)\,(BP \cap CA)\,(CP \cap AB)\,.$$
   - the **Anti-Cevian Triangle of P** with respect to $\triangle ABC$, is a triangle for which the cevian triangle of P is $\triangle ABC$.

- the **Pedal Triangle of P** with respect to $\triangle ABC$ is

$$\triangle\left(P\infty_{\perp BC}\cap BC\right)\left(P\infty_{\perp CA}\cap CA\right)\left(P\infty_{\perp AB}\cap AB\right).$$

- the **Anti-Pedal Triangle of P** with respect to $\triangle ABC$ is

$$\triangle\left(A\infty_{\perp AP}\right)\left(B\infty_{\perp BP}\right)\left(C\infty_{\perp CP}\right),$$

  whose pedal triangle is $\triangle ABC$.

- the **Circum-Cevian Triangle of P** with respect to $\triangle ABC$ is

$$\triangle\left(AP\cap\odot(ABC)\right)\left(BP\cap\odot(ABC)\right)\left(CP\cap\odot(ABC)\right).$$

10. Given a $\triangle ABC$ and a line ℓ, we define

- the **Cevian Triangle of ℓ** with respect to $\triangle ABC$,

$$\triangle\left(A\left(BC\cap\ell\right)\right)\left(B\left(CA\cap\ell\right)\right)\left(C\left(AB\cap\ell\right)\right).$$

- the **Anti-Cevian Triangle of ℓ** with respect to $\triangle ABC$, is a triangle whose cevian triangle is $\triangle ABC$.

1 Fundamentals of Conic Sections

§1.1 Definitions and Basic Properties

We first introduce one way to define conic sections from high school geometry:

Definition 1.1.1. Let $\odot(O)$ be a circle in space. Draw a line through O such that ℓ is perpendicular to the plane of $\odot(O)$. Take any point V on ℓ such that $V \neq O$. When a moving point $M \in \odot(O)$ moves along the circle, the surface formed by the lines VM is called a **Circular Conic Surface**. The circle $\odot(O)$ is called its **Directrix**, and V is called its **Vertex**.

Definition 1.1.2. A curve $\mathcal{C}$ on a plane E is called a **Conic**, if there exists a circular conical surface S with vertex V such that $\mathcal{C} = S \cap E$.

Later, when the spatial context is not mentioned, we will omit "on plane E". Because the equation of a conic is quadratic, conics are also called *quadratic curves*.

Definition 1.1.3. Let $\mathcal{C}$ be a conic and $\mathcal{L}_\infty$ be the line at infinity. Then:

1. $\mathcal{C}$ is called an **Ellipse**, if $\mid \mathcal{C} \cap \mathcal{L}_\infty \mid = 0$.
2. $\mathcal{C}$ is called an **Parabola**, if $\mid \mathcal{C} \cap \mathcal{L}_\infty \mid = 1$.
3. $\mathcal{C}$ is called an **Hyperbola**, if $\mid \mathcal{C} \cap \mathcal{L}_\infty \mid = 2$.

Hereafter, $\mathcal{L}_\infty$ will be used to denote the **Line at Infinity**. For a line $\ell \neq \mathcal{L}_\infty$, the intersection point

$$\infty_\ell := \ell \cap \mathcal{L}_\infty$$

is called the **Point at Infinity on ℓ**. The following are the more commonly seen equivalent definitions of conic sections:

Proposition 1.1.4

Let $\mathcal{C}$ be a conic. Then

1. $\mathcal{C}$ is an ellipse *if and only if* there exists two points F_1, F_2 and a positive real number α, where $\alpha > \frac{\overline{F_1F_2}}{2}$, such that
$$\mathcal{C} = \{\, P \mid \overline{F_1P} + \overline{F_2P} = 2\alpha\}.$$
2. $\mathcal{C}$ is a parabola *if and only if* there exists a point F and a line ℓ such that
$$\mathcal{C} = \{\, P \mid \overline{FP} = d\,(\ell, P)\}.$$
3. $\mathcal{C}$ is a hyperbola *if and only if* there exists two points F_1, F_2 and a positive real number α, where $\alpha < \frac{\overline{F_1F_2}}{2}$, such that
$$\mathcal{C} = \{\, P \mid |\overline{F_1P} - \overline{F_2P}| = 2\alpha\}.$$

Moreover, the points F_1, F_2 are called the focii of $\mathcal{C}$, and the line ℓ is called the directrix of $\mathcal{C}$.

Proof. // to be added □

Before discussing the next property, we first state the following: *all conic sections are smooth.* We also introduce some more notations.

1. $T(\mathcal{C})$ denotes the set of all tangents to $\mathcal{C}$.
2. $T_P(\mathcal{C})$ denotes the tangent to $\mathcal{C}$ at the point P, where $P \in \mathcal{C}$.
3. $T_\ell(\mathcal{C})$ denotes the point of tangency of the line ℓ with $\mathcal{C}$, where $\ell \in T(\mathcal{C})$.

Thus, it is clear that

$$T(\mathcal{C}) = \{T_P(\mathcal{C}) \mid P \in \mathcal{C}\}, \; \mathcal{C} = \{ T_\ell(\mathcal{C}) \mid \ell \in T(\mathcal{C})\}.$$

§1.2 Cross Ratio

§1.3 Cross Ratio on Conic Sections

§1.4 Some Theorems from Projective Geometry

§1.5 Exercises

Problem 1. Let $\triangle S_1S_2S_3$ be an equilateral triangle, and P be any point. Let Q_1 be the intersection of the perpendicular bisector of PS_1 with S_2S_3, and define Q_2, Q_3 similarly. Prove that Q_1, Q_2 and Q_3 are collinear.

Problem 2. Let $\triangle S_1S_2S_3$ be an equilateral triangle, and P be any point. Prove that the Euler lines of triangles $\triangle PS_2S_3$, $\triangle PS_3S_1$, and $\triangle PS_1S_2$ concur.

Problem 3. Let I be the incenter of $\triangle ABC$, D be the foot of the perpendicular from I to BC, and M be the midpoint of $\overline{BC}$. Prove that IM bisects AD.