## 2.5 Math Olympiad Trigonometry 501: Beyond Euclidean Trigonometry (Egregia Introductio ad Monstruosam Trigonometriam)

We now summon geometric monsters from non-Euclidean spaces in order to do algebraic operations on them. In general, a geometry in which the assumption of flatness of space is disregarded would be a non-Euclidean geometry. This means that spaces studies here are curved, either positively or negatively.

We start by studying the simplest case of a positively-curved geometry called elliptic geometry, and we initially assume that the positively-curved space has a uniform curvature. The most accessible such geometry would be the spherical geometry whose trigonometric calculations are on the way.

Great Circular Arcs on a Sphere

**Problem 407.** The shortest path connecting two points on a sphere is always part of a **great circular arc** whose center is the same as sphere's center. On a given sphere, imagine the lines of latitude as you would see them on a globe. Each line of latitude, except for the equator, is a **small circle** of the sphere. The equator, which splits the sphere into two equal-sized pieces, is a **great circle** of the sphere (see Figure 2.1).

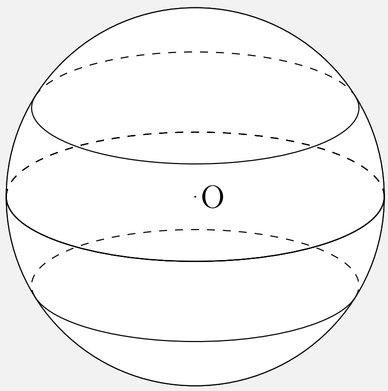


Figure 2.1: Two small circles and a great circle on a sphere.

Any two points on the surface of the sphere divide the great circle joining them into two parts. These two parts will be equal to each other if the two points are **antipodal** (diametrically opposite to one another). Otherwise, one of the two parts will be smaller than the other. Show that:

(a) If $A$ and $B$ are antipodal points, meaning they are the two ends of a diameter of the sphere, there are infinitely many great cirles passing through them.

(b) If $A$ and $B$ are not antipodal, there is exactly one **great circle** passing through them. The smaller arc connecting $A$ to $B$ is associated with a **central angle** connecting the center of the sphere to $A$ & $B$.

### Length of Shortest Arc Between Two Points on Sphere

**Problem 408.** On a sphere of radius $R$, there are two points $A$ and $B$ (Figure 2.2). Prove that the **central angle** associated with the shortest arc connecting $A$ and $B$ is $\alpha$, then the length of the arc is $\alpha R$.

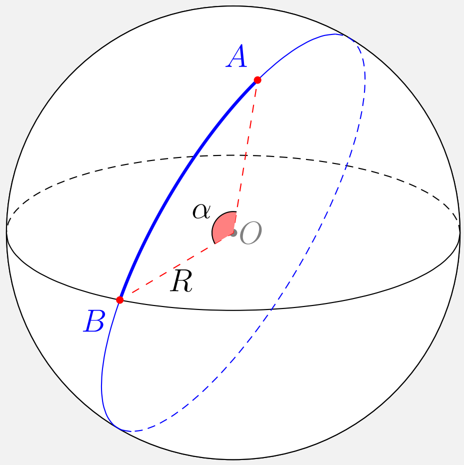


Figure 2.2: The great circle joining $A$ to $B$

### Main Elements of Euler's Spherical Triangle

**Problem 409.** Let $A, B, C$ be points on a unit sphere such that no two of them are antipodals. We draw the three great circles joining the three vertices $A, B, C$ to envision **Euler's Spherical Triangle**. Each angle $A, B, C$ and the corresponding side $a, b, c$ facing it is a **main element** of spherical triangle $ABC$ (Figure 2.3). Prove that the value of each of the main elements of Euler's spherical triangle lies between 0 and $\pi$.

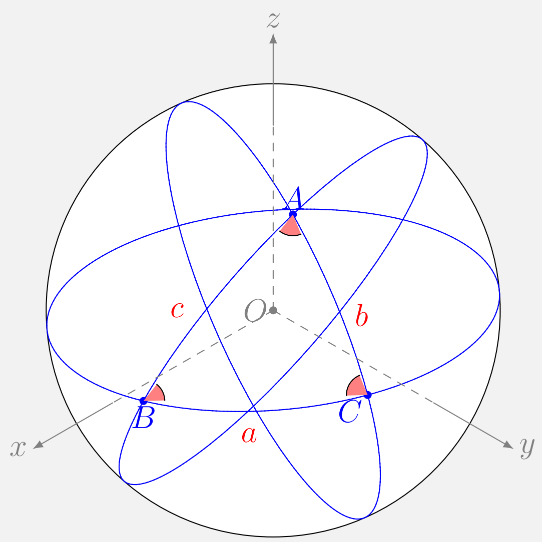


Figure 2.3: Angles $A, B, C$ and sides $a, b, c$ of a spherical triangle.

Trihedral Corner

**Problem 410.** Each spherical triangle $ABC$ corresponds to a **trihedral corner** whose vertex is at the center of the sphere and whose edges are the radii of the sphere connecting the center to the points $A, B, C$ on the surface of sphere. Moreover, each trihedral corner with its vertex at the center of the sphere corresponds to a spherical triangle formed by the edges of the trihedral corner. Figure 2.4 demonstrates a **trirectangle** (a spherical triangle with three right angles) and its associated trihedral corner. Prove that:

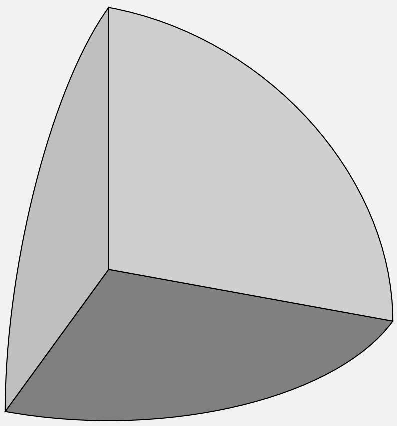

Figure 2.4: A Trihedral corner depicting an eighth of a sphere.

(a) The angles $A, B, C$ of the spherical triangle are equal to the corresponding three dihedral angles of the trihedral corner (angles between the three planes that make the trihedral corner), and (b) The sides $a, b, c$ of the spherical triangle are equal to the three angles formed at the vertex of the trihedral corner.

**Definition.** For any arc $AB$ of any great circular arc on the sphere, if the diameter of the sphere that is perpendicular to the plane of the great circle intersects the surface of the sphere at points $C_1$ and $C_2$, we call $C_1$ and $C_2$ the **poles** of the arc $AB$.

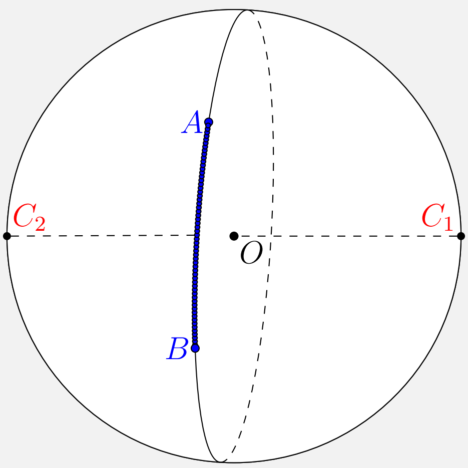


Figure 2.5: The poles of the arc $AB$.

## Distances and Great Circles

**Problem 411.** Let $\ell$ be a great circular arc on the unit sphere, and $P$ a point not on $\ell$. Prove that

1. If $P$ is a pole of $\ell$, then for any point $Q$ on $\ell$, $PQ$ is a quadrant.
2. If for $Q_1$ and $Q_2$ on $\ell$ we have $PQ_1 = PQ_2 = \pi/2$, then $P$ is a pole of $\ell$.
3. If $P$ is a pole of $\ell$ and $Q_1$ and $Q_2$ are on $\ell$, then the distance between $Q_1$ and $Q_2$ equals the spherical angle between $Q_1P$ and $PQ_2$: $Q_1Q_2 = \angle Q_1PQ_2$.

## Polar Triangles

**Problem 412.** For any triangle $ABC$, if the poles of the sides of $ABC$ are vertices of triangle $A'B'C'$, then we call $A'B'C'$ the **polar triangle** of $ABC$. The assumption is that vertex $A'$ is a pole of $BC$, vertex $B'$ is a pole of $CA$, and $C'$ is a pole of $AB$. Prove that:

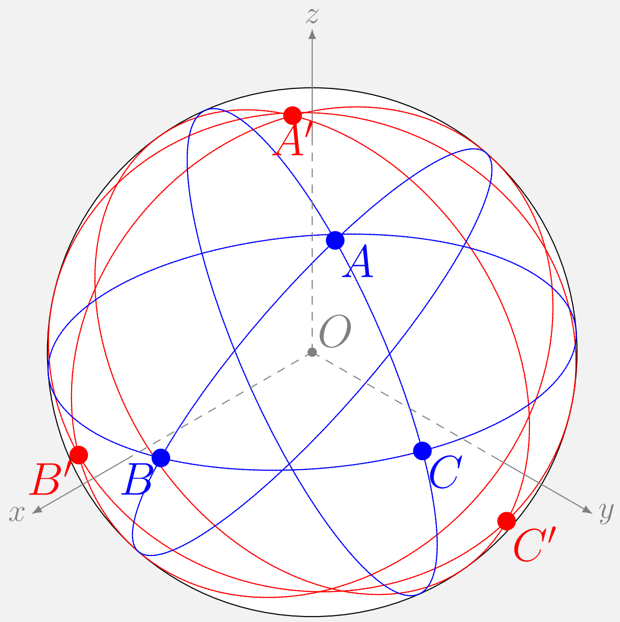


Figure 2.6: Polar triangles $ABC$ and $A'B'C'$ on a sphere.

(a) If $A'B'C'$ is the polar triangle of $ABC$, then $ABC$ is also the polar triangle of $A'B'C'$, so that vertex $A$ is the pole of side $a'$ of $A'B'C'$, vertex $B$ is the pole of side $b'$, and vertex $C$ is the pole of side $c'$.

(b) If $\triangle ABC$ and $\triangle A'B'C'$ are polar triangles of one another, then the sum of each angle and its associated side in the polar triangle equals $\pi$:

$$A + a' = \pi, \quad B + b' = \pi, \quad C + c' = \pi,$$
$$A' + a = \pi, \quad B' + b = \pi, \quad C' + c = \pi.$$

## Classification of Spherical Triangles

**Definition.** Concerning the angles of spherical triangles,

- A spherical triangle could be **acute**, **right**, or **obtuse**, like plane triangles.
- Spherical triangles may have two or three right or obtuse angles, and each angle can be close to $\pi$. We can thus see that the sum of angles of a spherical triangle cannot exceed $3\pi$.

Regarding the sides,

- A spherical triangle may be **scalene**, **isosceles**, or **equilateral**.
- A spherical triangle that has one or more of its **sides** equal to a **quadrant** ($\pi/2$) is called a **quadrantal triangle**.
- A triangle in which one of the vertices is a pole of the opposing side is called a **semilunar** triangle, or a **semilune**.

**Problem 413.** Prove the following statements:

1. It is known in Euclidean geometry that for plane triangles, being equilateral (having equal sides) is equivalent to being equiangular (having equal angles). Prove the same statement for spherical triangles.
2. Furthermore, if a plane triangle is isosceles (has two equal sides), then the angles facing those sides are equal to each other. Show that the same things happens for spherical triangles.
3. Prove the Pythagorean Theorem for Spherical Triangles: $a, b, c$ are side-lengths of a spherical triangle $ABC$ with right angle at $A$ on a unit sphere if and only if
$$\cos a = \cos b \cos c.$$
**Remark**. Remember that for a spherical triangle on a unit sphere, all main elements are smaller than $\pi$; and note that in the Spherical Pythagorean Formula the cosine function is applied to the side-lengths $a, b, c$ rather than angles $A, B, C$. Show that we could, however, find sines of non-right angles:
$$\sin B = \frac{\sin b}{\sin a}, \cos B = \frac{\cos b \sin c}{\sin a}, \quad \text{and} \quad \sin C = \frac{\sin c}{\sin a}, \cos C = \frac{\cos c \sin b}{\sin a}.$$
4. If a spherical triangle has three right angles, all its sides are quadrants.
5. If a spherical triangle has two right angles, the sides facing those angles are quadrants and the third angle is measured by its opposite side.
6. If any two parts, a part being a side or an angle, of a spherical triangle measure $\pi/2$ radians, the triangle is a semilune. Also, the angle at the pole has the same measure as the opposing side. All of the other sides and angles measure $\pi/2$ radians.

## Lunes on a Sphere

**Problem 414.** Two great circles passing through antipodal points on a sphere divide the sphere into four parts like orange slices, each being a **lune** on the sphere (Figure 2.7). Prove that the area of a lune with angle $\alpha$ is $2\alpha$.

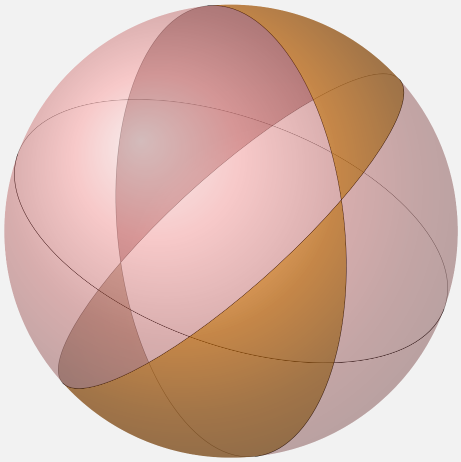

Figure 2.7: Lunes on a sphere divide it into four parts.

## Spherical Triangle Inequalities

**Problem 415.** Prove that for any spherical triangle $ABC$ on a unit sphere with sides $a, b, c$ and angles $A, B, C$,

1. The triangle inequality holds: each side is smaller than the sum of the other two sides:
$$a < b + c, \quad b < c + a, \quad c < a + b.$$
2. Each side is larger than the difference of the other two sides.
3. The sum of the sides of the triangle is positive and smaller than $2\pi$:
$$0 < a + b + c < 2\pi.$$
4. The sum of the angles of the triangle is larger than $\pi$ and smaller than $3\pi$:
$$\pi < A + B + C < 3\pi.$$
5. The larger side of the triangle faces the larger angle of the spherical triangle.
6. The following inequalities hold true for triangle's angles:
$$A + B - C < \pi, \quad A - B + C < \pi, \quad -A + B + C < \pi.$$

### Congruent Spherical Triangles & Gauss–Bonnet Theorem

**Problem 416.** Remember from the Euclidean geometry that two plane triangles are **congruent** (having equal sides and angles) if (a) all three sides are equal (**SSS**), (b) two sides and the angle between them are equal (**SAS**), or (c) two angles and the side joining them are equal (**ASA**). However, two **incongruent** plane triangles may have all three angles equal to one another, simply because we can scale all sides of a plane triangle equally to get a triangle with larger/smaller sides but the same angles. Prove that for spherical triangles,

1. The three Euclidean criteria for congruent plane triangles (**SSS**, **SAS**, **ASA**) also hold true for spherical triangles.
2. Two spherical triangles with equal angles (**AAA**) are congruent.
3. Two congruent spherical triangles have equal areas.
4. On a unit sphere, sum of triangle's angles equals $\pi$ plus the area of triangle:

$$A + B + C = \pi + (\text{area of } \triangle ABC).$$

### Inverse Spherical Triangles

**Problem 417.** Two spherical triangles $ABC$ and $A'B'C'$ have all their corresponding main elements equal to one another, that is,

$$A = A', \quad B = B', \quad C = C', \qquad a = a', \quad b = b', \quad c = c'.$$

Prove that either the two triangles are **directly equal**, meaning one can be moved in space so that its vertices matches the vertices of the other triangle, or they are **inversely equal**, or simply **inverse** of each other, which means that they are reflections of each other with respect to a plane that passes through sphere's center.

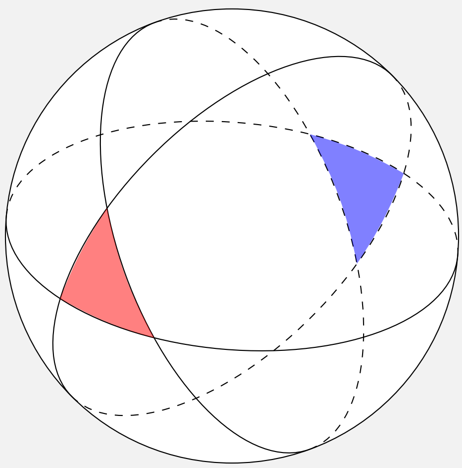

Figure 2.8: Inverse spherical triangles.

### Spherical Law of Cosines for Sides

**Problem 418.** The cosine of one side of a spherical triangle is equal to the product of the cosine of the other two sides plus the product of sines of these two sides times the cosine of the angle between them:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

**Problem 419.** Figure 2.9 shows a spherical triangle $ABC$ with sides $a, b, c$ on a sphere centered at $O$. The tangent at $A$ to the arc $AB$ meets $OB$ at $D$ and the tangent at $A$ to the arc $AC$ meets $OC$ at $E$.

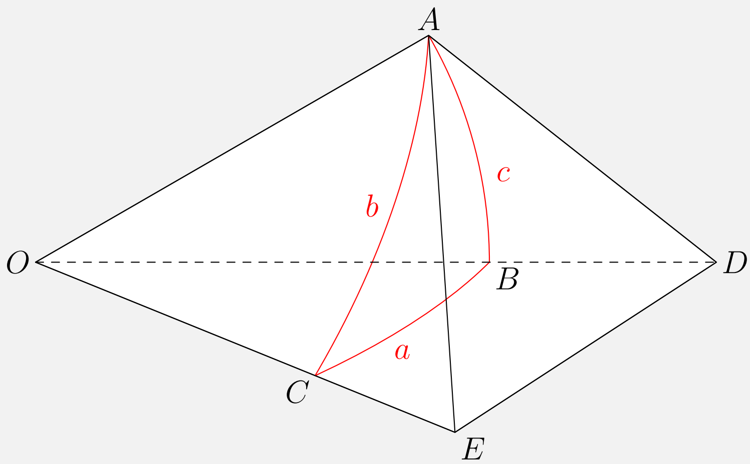


Figure 2.9: Spherical triangle $ABC$ with tangents to arcs at vertex $A$.

1. Using the Law of Cosines in the plane, show that

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

2. Using the Pythagorean identity $\sin^2 A + \cos^2 A = 1$, prove that

$$\sin A = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin b \sin c}.$$

### Spherical Law of Sines

Consider the ratio between the sine of an angle and the sine of its opposite side in a spherical triangle. Prove that this ratio is the same for all three angles:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

## Five–Piece Spherical Trigonometric Identities

In a spherical triangle $ABC$ with side-lengths $a, b, c$, prove the five-piece identities:

**Side's Sine Times Adjacent Angle's Cosine 420.** For any side $x$ and angle $Y$ adjacent to it, the product $\sin x \cos Y$ equals $\cos y \sin z$ **minus** $\sin y \cos z \cos X$:

$$\text{Side } a : \begin{cases} \sin a \cos B = \cos b \sin c - \sin b \cos c \cos A, \\ \sin a \cos C = \cos c \sin b - \sin c \cos b \cos A, \end{cases}$$
$$\text{Side } b : \begin{cases} \sin b \cos C = \cos c \sin a - \sin c \cos a \cos B, \\ \sin b \cos A = \cos a \sin c - \sin a \cos c \cos B, \end{cases}$$
$$\text{Side } c : \begin{cases} \sin c \cos A = \cos a \sin b - \sin a \cos b \cos C, \\ \sin c \cos B = \cos b \sin a - \sin b \cos a \cos C. \end{cases}$$

**Angle's Sine Times Adjacent Side's Cosine 421.** For any angle $X$ and side $y$ adjacent to it, the product $\sin X \cos y$ equals $\cos Y \sin Z$ **plus** $\sin Y \cos Z \cos x$:

$$\text{Angle } A : \begin{cases} \sin A \cos b = \cos B \sin C + \sin B \cos C \cos a, \\ \sin A \cos c = \cos C \sin B + \sin C \cos B \cos a, \end{cases}$$
$$\text{Angle } B : \begin{cases} \sin B \cos c = \cos C \sin A + \sin C \cos A \cos b, \\ \sin B \cos a = \cos A \sin C + \sin A \cos C \cos b, \end{cases}$$
$$\text{Angle } C : \begin{cases} \sin C \cos a = \cos A \sin B + \sin A \cos B \cos c, \\ \sin C \cos b = \cos B \sin A + \sin B \cos A \cos c. \end{cases}$$

## Spherical Law of Cosines for Angles

**Problem 422.** The cosine of an angle of a spherical triangle equals the product of sines of the other two angles and the cosine of the side between them **minus** the product of cosines of the other two angles:

$$\begin{aligned} \cos A &= \sin B \sin C \cos a - \cos B \cos C, \\ \cos B &= \sin C \sin A \cos b - \cos C \cos A, \\ \cos C &= \sin A \sin B \cos c - \cos A \cos B. \end{aligned}$$

**Problem 423.** Imply that

$$\begin{aligned} \cos a &= \frac{\cos A + \cos B \cos C}{\sin B \sin C}, \\ \sin^2 \frac{a}{2} &= -\frac{\cos A + \cos(B + C)}{2 \sin B \sin C}. \end{aligned}$$

## Half–Angle and Half–Side Spherical Formulas

**Problem 424.** For every spherical triangle $ABC$ with angles $A, B, C$ and $a, b, c$, let $p$ be the semiperimeter: $p = (a+b+c)/2$. Prove the following half–angle trigonometric formulas:

1. $\sin^2 \dfrac{A}{2} = \dfrac{\sin\left(\frac{a+b-c}{2}\right)\sin\left(\frac{a-b+c}{2}\right)}{\sin b \sin c}$,
2. $\sin^2 \dfrac{A}{2} = \dfrac{\sin(p-b)\sin(p-c)}{\sin b \sin c}$,
3. $\cos^2 \dfrac{A}{2} = \dfrac{\sin p \sin(p-a)}{\sin b \sin c}$,
4. $\tan^2 \dfrac{A}{2} = \dfrac{\sin(p-b)\sin(p-c)}{\sin p \sin(p-a)}$,
5. $\sin A = \dfrac{2\sqrt{\sin p \sin(p-a)\sin(p-b)\sin(p-c)}}{\sin b \sin c}$.

**Problem 425.** For every spherical triangle $ABC$ with angles $A, B, C$ and $a, b, c$, define $P = (a+b+c)/2$. Prove the following half–side trigonometric identities:

1. $\sin^2 \dfrac{a}{2} = -\dfrac{\cos P \cos(P-A)}{\sin B \sin C}$,
2. $\cos^2 \dfrac{a}{2} = \dfrac{\cos(P-B)\cos(P-C)}{\sin B \sin C}$,
3. $\tan^2 \dfrac{a}{2} = -\dfrac{\cos P \cos(P-A)}{\cos(P-B)\cos(P-c)}$,
4. $\sin a = \dfrac{2\sqrt{-\cos P \cos(P-A)\cos(P-B)\cos(P-C)}}{\sin B \sin C}$.

## Spherical Law of Haversines

**Definition.** Some hundreds of years ago when spherical trigonometry was a hot-topic for mathematicians, there was another periodic function besides cosine and sine, called **versine**, short for **versed sine**, defined by $\operatorname{versin}(\theta) = 1 - \cos\theta$.

**Definition.** Since $1 - \cos\theta = 2\sin^2\theta$, it makes sense to define a **halved versed sine**, or shortly, **haversine**, by $\operatorname{hav}(\theta) = \sin^2\left(\frac{\theta}{2}\right)$.

**Problem 426.** Prove the **Haversine Formula** in a spherical triangle $ABC$ with side-lengths $a, b, c$:

$$\operatorname{hav}(c) = \operatorname{hav}(a-b) + \sin(a)\sin(b)\ \operatorname{hav}(C).$$

### Swimming the Depths of the Algebraic Ocean

Kaywañan is an Algebra Competition, and you may say its motto is "Let No One Ignorant of Algebra Enter." So far, we have been vigorously forging algebraic equations and definitions that are deeply rooted within their applications. It is both an intention and a purpose of Kaywañan to be defined as the collection of most important algebraic equations and identities that one may encounter in dealing with in ordinary, Euclidean flatland geometry, as well as non-Euclidean monsters and witches that might appear in hyperbolic geometry.

We have not even started to discuss the geometry of hyperbola and its associated hyperboloid. The most advanced formula, I would say, in Kaywañan so far is that of Problem 2.5, the Spherical Law of Haversines, which has absolutely fascinating applications in astronomy. The haversine formula for spherical triangles is just an example of myriads of unknown equations in Spherical Geometry, the most special type of Elliptic Geometry. You can only imagine how many more of such algebraic equations may be found, written down, and added to Kaywañan if we consider other elliptic structures than the Sphere, such as the Ellipsoid.

There are other types of non-Euclidean geometries that some might say are even more surprising than the fact that angles of a spherical triangle add up to more than $\pi$. If the Euclidean plane is not infinite in all directions, and in the special case when the plane is limited to a $1 \times 1$ Square of the Euclidean Plane, you can imagine that the corresponding points on opposite sides of the square are actually the same point, as if you map the surface of a doughnut to the $1 \times 1$ Euclidean Square. If we start walking from a certain point in any of the two directions of the Euclidean Square on the surface of a doughnut, we would reach the same point. The same would happen if we map the sphere to the Euclidean plane, but the surface of the doughnut and that of a sphere are clearly distinct to us. If we had no idea of the third dimension, as if we were ants walking around in two-dimensional Euclidean plane, we would never even be able to know whether the surface on which we walk is a doughnut or a sphere.

Here, at the depths of the Algebraic Ocean of Kaywañan, Titan, Moon of Saturn, where we can see the positive curvature of the surface of the core of Titan, nobody doubts the spherical shape of Planet of Algebra, Kaywan. There are rumours that in earlier Eons, Kaywañans (those who live around Saturn) believed that Titan is the most special place in the Sölar System, and after one of them dreamed of Maurits Cornelis Escher's "Angels and Devils" painting, they started to preach a certain belief in a Hyperbolic Geometry of Titan to emphasize their uniqueness among moons of Saturn and other celestial objects.

It is now, however, a ridiculous claim to believe in a hyperbolic geometry for the actually spherical surface of Titan, maybe as ridiculous as believing in a Flat Earth once you have traveled to the moon and seen the biosphere of the Earth from afar. Now that we can swim the depths of Kaywañan and measure the spherical arcs close to the core of Titan, hyperbolic beliefs are but a joke. We may study mythology of such treacheries in the advanced levels of Napirañan Geometry Contest, but here in Kaywañan we stick to the algebra.

### Maclaurin Series of Versine & Haversine

**Problem 427.** Prove that for any complex number $z$,

$$\operatorname{versin}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{(2k)!} \quad \text{and} \quad \operatorname{hav}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{2(2k)!}.$$

Then show that the limit of both $\operatorname{versin}(\theta)/\theta$ and $\operatorname{hav}(\theta)/\theta$ when $\theta \to 0$ is 0.

### The Sine Formulae (from "Spherical Astronomy")

The following method of notation is quoted from "Textbook on Spherical Astronomy" by W. M. Smart and R. M. Green, given in the first chapter as one of the main formulas (formulae **A** and **D**) in Spherical Trigonometry.

**Definition** (Laterangular Function of a Spherical Triangle)**.** For any spherical triangle $ABC$, the **Laterangular Function of** $A$, denoted $X(a, A)$, is defined by

$$(X(a, A))^2 \cdot \sin^2 a \sin^2 b \sin^2 c = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c.$$

**Problem 428.** In a spherical triangle $ABC$ with side-lengths $a, b, c$,

1. Prove the **Astronomical Sine** formula:

$$\sin^2 b \cdot \sin^2 c \cdot \sin^2 A = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c.$$

2. Prove that

$$(X(a, A))^2 = \left(\frac{\sin A}{\sin a}\right)^2,$$

   and imply that the Laterangular Function must be symmetric, so that $X(a, A) = X(b, B) = X(c, C)$.

3. If all main elements of triangle $ABC$ are smaller than $\pi$, then the **Spherical Law of Sines** holds true:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}}{\sin a \sin b \sin c}.$$

**Definition** (Four–Piece Spherical Trigonometric Terminology)**.** In the spherical triangle $ABC$ consider the four consecutive parts $B, a, C, b$. The angle $C$ is contained by the two sides $a$ and $b$ and is called the **inner angle**. The side $a$ is flanked by the two angles $B$ and $C$ and is called the **inner side**.

**Four-Part Identity 429.** Prove that cos(inner side) · cos(inner angle) equals

$$\sin(\text{inner side}) \cdot \cot(\text{other side}) - \sin(\text{inner angle}) \cdot \cot(\text{other angle}).$$

### Delambre's and Napier's Analogies

**Delambre's Aanlogies 430.** In a spherical triangle $ABC$ with sides $a, b, c$,

$$\sin\frac{c}{2}\sin\frac{A-B}{2}=\cos\frac{C}{2}\sin\frac{a-b}{2} \quad \text{and} \quad \sin\frac{c}{2}\cos\frac{A-B}{2}=\sin\frac{C}{2}\sin\frac{a+b}{2},$$
$$\cos\frac{c}{2}\sin\frac{A+B}{2}=\cos\frac{C}{2}\cos\frac{a-b}{2} \quad \text{and} \quad \cos\frac{c}{2}\cos\frac{A+B}{2}=\sin\frac{C}{2}\cos\frac{a+b}{2}.$$

Taking Delambre's equations, which are also called **Gauss's Spherical Equations**, in pairs, we obtain Napier's Analogies:

### Napier's Analogies

**Napier's Aanlogies 431.** In a spherical triangle $ABC$ with sides $a, b, c$,

$$\tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2} \quad \text{and} \quad \tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2},$$
$$\tan\frac{A+B}{2}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2} \quad \text{and} \quad \tan\frac{A-B}{2}=\frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2}.$$

### Napier's Rules

**Napier's Rules for Right Spherical Triangles 432.** If one main element among $a, b, c, A, B, C$ is $\pi/2$, there would be five remaining unknown parts. John Napier suggested Five–Piece Mnemonics shown in Figures 2.10 and 2.11 to prove:

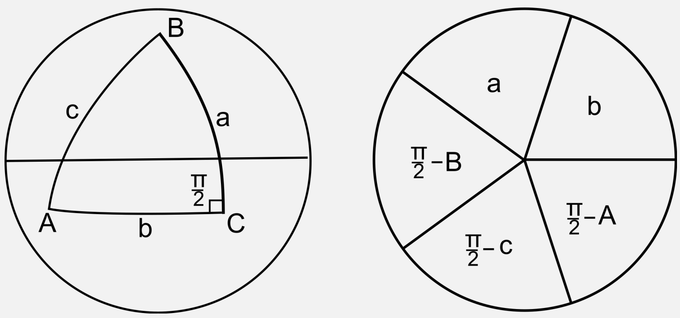


Figure 2.10: Napier's Mnemonics (When One Angle is Right) [Wikipedia]

1. The sine of any middle part equals the product of tangents of adjacent parts.
2. The sine of any middle part equals the product of cosines of opposite parts.

## Napier's Ten Right Spherical Triangle Commandments

**Napier's Ten Commandments 433.** For any spherical triangle $ABC$ with a right angle at $A$, Napier's Ten Rules are the ten equations derived from various spherical trigonometric identities studied in previous problems:

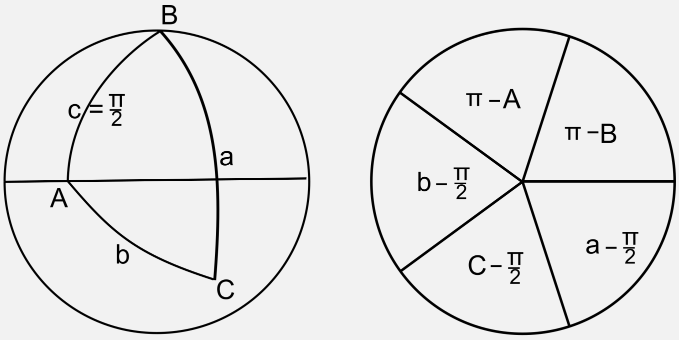


Figure 2.11: Napier's Mnemonics (When One Side is a Quadrant) [Wikipedia]

(I) $\cos a = \cos b \cos c$, A.K.A. *Spherical Pythagorean Theorem*,

(II) $\sin b = \sin a \sin B$, derived from *Spherical Law of Sines*,

(III) $\sin c = \sin a \sin C$, also from *Spherical Law of Sines*,

(IV) $\cos B = \cos b \sin C$, by *Spherical Law of Cosines for Angle* $B$,

(V) $\cos C = \cos c \sin B$, also by *Spherical Law of Cosines*, but for Angle $C$,

(VI) $\cos a = \cot B \cot C$, by *Spherical Law of Cosines for Angle* $A$,

(VII) $\cos B = \cot a \tan c$, by *Napier's Four–Part Identity* for $\cos c \cos B$,

(VIII) $\cos C = \tan b \cot a$, by *Napier's Four–Part Identity* for $\cos b \cos C$,

(IX) $\sin b = \tan c \cot C$, by *Napier's Four–Part Identity* for $\cos b \cos A$,

(X) $\sin c = \cos c \sin B$, by *Napier's Four–Part Identity* for $\cos c \cos A$.

**Problem 434.** Prove that Napier's Ten Commandments can be reduced to the following cosine formulae by changing $b$ and $c$ to $\frac{\pi}{2} - b$ and $\frac{\pi}{2} - c$, respectively:

a) $\cos a = \sin\left(\frac{\pi}{2} - b\right) \sin\left(\frac{\pi}{2} - c\right)$,

b) $\cos a = \cot B \cot C$,

c) $\cos\left(\frac{\pi}{2} - b\right) = \sin a \sin B$,

d) $\cos\left(\frac{\pi}{2} - b\right) = \cot\left(\frac{\pi}{2} - c\right) \cot C$,

e) $\cos\left(\frac{\pi}{2} - c\right) = \sin a \sin C$,

f) $\cos\left(\frac{\pi}{2} - c\right) = \cot\left(\frac{\pi}{2} - b\right) \cot B$,

g) $\cos B = \sin\left(\frac{\pi}{2} - b\right) \sin C$,

h) $\cos B = \cot a \cot\left(\frac{\pi}{2} - c\right)$,

i) $\cos C = \sin\left(\frac{\pi}{2} - c\right) \sin B$,

j) $\cos C = \cot\left(\frac{\pi}{2} - b\right) \cot a$.

Napier's Ten Commandments and their ten cosine forms formulae inspired Napier to make his **Napier Rules for Right Spherical Triangles**.

## The Global Half–Side Identities

**Definition** (Spherical Triangle Half–Side Identities)**.** For any spherical triangle $ABC$, the **Haveside Function of** $A$, denoted $H(a, A)$, is defined by

$$H(a, A) \cdot \cos(P - A) = \tan\left(\frac{a}{2}\right).$$

where $P = (A + B + C)/2$.

**Problem 435.** In a spherical triangle $ABC$ with side-lengths $a, b, c$,

1. Prove the **Astronomical Half–Side Tangent** formula:

$$\tan\left(\frac{a}{2}\right) = \sqrt{\frac{-\cos P}{\cos(P - A)\cos(P - B)\cos(P - C)}}\cos(P - A),$$

2. Prove that

$$(H(a, A))^2 = \frac{-\cos P}{\cos(P - A)\cos(P - B)\cos(P - C)},$$

   and imply that the Laterangular Function must be symmetric, so that $H(a, A) = H(b, B) = H(c, C)$.

3. Prove **Mollweide's** formula in Euclidean plane:

$$\left(\tan\frac{A}{2}\right)^2 = \frac{(a + b - c)(a - b + c)}{(a + b + c)(-a + b + c)}.$$

**Spherical Half–Side Tangent–Cotangent Formulae 436.** In the spherical triangle $ABC$ with sides $a, b, c$ that subtend angles $A, B, C$, prove that

$$\tan\left(\frac{a - b}{2}\right)\cot\left(\frac{a + b}{2}\right) = \tan\left(\frac{A - B}{2}\right)\cot\left(\frac{A + B}{2}\right).$$

## L'Huilier's & Cagnoli's Theorems

**L'Huilier's Theorem 437.** Let a spherical triangle have sides of length $a, b$, and $c$, and semiperimeter $p$. Then the spherical excess $E = (A + B + C) - \pi$ is given by

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{p}{2}\right)\tan\left(\frac{p - a}{2}\right)\tan\left(\frac{p - b}{2}\right)\tan\left(\frac{p - c}{2}\right)}.$$

**Cagnoli's Theorem 438.** $\sin\left(\dfrac{E}{2}\right) = \dfrac{\sqrt{\sin p \sin(p - a)\sin(p - b)\sin(p - c)}}{2\cos\left(\frac{a}{2}\right)\cos\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)}.$

Exercises on Spherical Excess $E$

**Problem 439.** Prove that the area of any spherical triangle $ABC$ equals the Spherical Excess $E = A + B + C - \pi$ times the square of sphere's radius.

**Problem 440.** In a spherical triangle if $A = B = 2C$, show that

$$8 \sin\left(a + \frac{c}{2}\right) \sin^2\left(\frac{c}{2}\right) \cos\left(\frac{c}{2}\right) = \sin^3 a.$$

**Problem 441.** If $A + B + C = 2\pi$, show that

$$\cos^2\left(\frac{a}{2}\right) + \cos^2\left(\frac{b}{2}\right) + \cos^2\left(\frac{c}{2}\right) = 1.$$

**Problem 442.** In spherical triangle $ABC$, if $C$ is a right angle, prove

$$\frac{\sin^2 c}{\cos c} \cos E = \frac{\sin^2 a}{\cos a} + \frac{\sin^2 b}{\cos b}.$$

**Problem 443.** Show that

$$\sin\left(\frac{E}{2}\right) = \sin\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) \sec\left(\frac{c}{2}\right),$$
$$\cos\left(\frac{E}{2}\right) = \cos\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) \sec\left(\frac{c}{2}\right).$$

**Problem 444.** Prove that

$$\sin^2\left(\frac{C}{2} - \frac{E}{4}\right) = \frac{\cos\left(\frac{p}{2}\right) \sin\left(\frac{p-a}{2}\right) \sin\left(\frac{p-b}{2}\right) \sin\left(\frac{p-c}{2}\right)}{\sin\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) \cos\left(\frac{c}{2}\right)},$$
$$\cos^2\left(\frac{C}{2} - \frac{E}{4}\right) = \frac{\sin\left(\frac{p}{2}\right) \cos\left(\frac{p-a}{2}\right) \cos\left(\frac{p-b}{2}\right) \cos\left(\frac{p-c}{2}\right)}{\sin\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) \cos\left(\frac{c}{2}\right)}.$$

**Problem 445.** If $p = (a + b + c)/2$ is the semiperimeter, show that

$$\sin p = \frac{\sqrt{\sin\left(\frac{E}{2}\right) \sin\left(A - \frac{E}{2}\right) \sin\left(B - \frac{E}{2}\right) \sin\left(C - \frac{E}{2}\right)}}{2 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right)}.$$

## Radii of the Spherical Triangle

**Inradius of the Spherical Triangle 446.** In a spherical triangle $ABC$ with sides $a, b, c$, and semiperimeter $p = (a+b+c)/2$, the inradius $r$ of the incircle of triangle may be calculated from

$$\tan r = \tan\left(\frac{A}{2}\right) \cdot \sin(p-a) = \sqrt{\frac{\sin(p-a)\sin(p-b)\sin(p-c)}{\sin p}}.$$

**Circumradius of the Spherical Triangle 447.** In a spherical triangle $ABC$ with sides $a, b, c$, and angles $A, B, C$, define $P = (A+B+C)/2$, the circumradius $R$ of the circumscribed circle of $ABC$ can be calculated from

$$\cot R = \cot\left(\frac{A}{2}\right) \cdot \cos(P-A) = \sqrt{\frac{\cos(P-A)\cos(P-B)\cos(P-C)}{-\cos P}}.$$

## Medians of Spherical Triangles

**Problem 448.** In a spherical triangle $ABC$ with sides $a, b, c$ the length of the median $m_C$ drawn from $C$ (length of $CF$ in Figure 2.12) satisfies

$$\cos m_C = \cos b \cdot \cos\left(\frac{c}{2}\right) + \sin b \cdot \sin\left(\frac{c}{2}\right) \cdot \cos A.$$

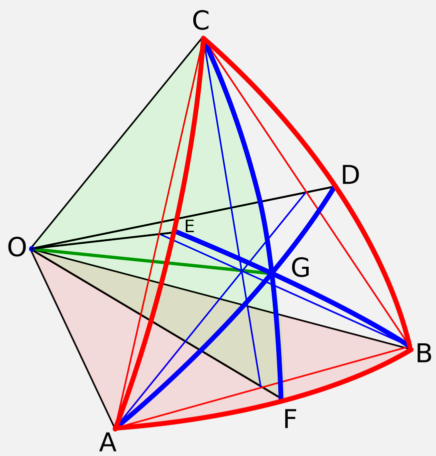


Figure 2.12: Spherical Medians [Wikipedia]

**Problem 449.** Let $O$ be the center of the sphere. If $G$ is the centroid of spherical triangle $ABC$ and $A'B'C'$ is the polar triangle of $ABC$, show that

$$\overrightarrow{OG} = \frac{1}{2E} \cdot \left(\overrightarrow{OC'} \cdot |\overline{AB}| + \overrightarrow{OA'} \cdot |\overline{BC}| + \overrightarrow{OB'} \cdot |\overline{CA}|\right).$$