IMO Shortlist 2023 G1

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§1 Problem

Problem (IMO Shortlist 2023 G1)
Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle AED = 90^\circ$. Suppose that the midpoint of CD is the circumcenter of triangle ABE. Let O be the circumcenter of triangle ACD.

Prove that line AO passes through the midpoint of segment BE.

§2 Solution 1 (Using Reflection & Isogonality)

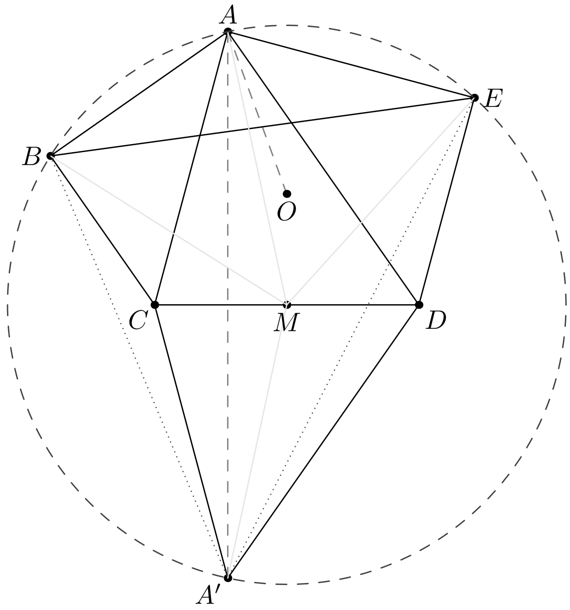
Proof. Reflect A across $\overline{CD}$ to A', and let M be the midpoint of $\overline{CD}$.

Claim 2.1. The quadrilateral $ABA'E$ is cyclic, with M as the center of its circumcircle.

Proof. Since M is the circumcenter of $\triangle ABE$, we have

$$\overline{MB} = \overline{MA} = \overline{ME} = \overline{MA'}$$

so A' also lies on the circumcircle of $\triangle ABE$ whose center is M. $\square$



Claim 2.2. Line $\overline{BC} \parallel \overline{AD}$ and $\overline{AC} \parallel \overline{DE}$.

Proof. Suppose K is the midpoint of $\overline{AB}$ then $\overline{KM} \perp \overline{AB} \implies \overline{KM} \parallel \overline{BC}$. Since K and M are midpoints that would imply $\overline{BC} \parallel \overline{AD}$. Similarly, $\overline{AC} \parallel \overline{DE}$. $\square$

Claim 2.3. Segment $\overline{AA'}$ is the A-symmedian in $\triangle ABE$.

Proof. Since $\angle BCA = \angle EDA$, it follows that $\triangle BCA \sim \triangle EDA$. By angle chasing, we obtain $\angle BCA' = \angle EDA'$ and

$$\frac{\overline{BC}}{\overline{CA'}} = \frac{\overline{BC}}{\overline{CA}} = \frac{\overline{DE}}{\overline{AD}} = \frac{\overline{ED}}{\overline{DA'}}$$

Thus $\triangle BCA' \sim \triangle EDA'$. Hence,

$$\frac{\overline{BA'}}{\overline{EA'}} = \frac{\overline{A'C}}{\overline{A'D}} = \frac{\overline{AC}}{\overline{AD}} = \frac{\overline{AB}}{\overline{AE}}$$

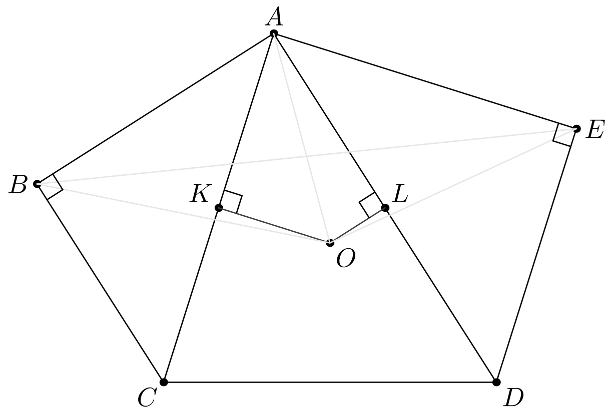
and therefore $\overline{AA'}$ is the A-symmedian in $\triangle ABE$. $\square$

To finish the problem, we observe that lines isogonal with respect to $\angle CAD$ are also isogonal with respect to $\angle BAE$ because

$$\angle BAC = \angle DAE$$

Therefore, taking the isogonal of $\overline{AA'}$ which is the A-symmedian in $\triangle ABE$ and also the A-altitude in $\triangle ACD$, we obtain $\overline{AO}$ in $\triangle ACD$. This corresponds to the A-median in $\triangle ABE$ which proves that $\overline{AO}$ bisects $\overline{BE}$ $\square$

§3 Solution 2 (Using Areas)



Proof. Let K, L and M be the midpoints of sides $\overline{AC}$, $\overline{AD}$ and $\overline{CD}$.

Claim 3.1. Lines $\overline{BC} \parallel \overline{AD}$ and $\overline{AC} \parallel \overline{DE}$.

Proof. Same as Solution 1. □

Claim 3.2. Lines $\overline{AB} \parallel \overline{OL}$ and $\overline{EA} \parallel \overline{OK}$.

Proof. Since $\overline{BC} \parallel \overline{AD}$ and $\angle CBA = 90^\circ \implies \angle BAD = 90^\circ$. Since $\overline{OL} \perp \overline{AD}$, therefore $\overline{AB} \parallel \overline{OL}$. Similarly, $\overline{EA} \parallel \overline{OK}$. □

Claim 3.3. The areas of $\triangle AOB$ and $\triangle AOE$ are equal.

Proof. Since $\overline{AB} \parallel \overline{OL}$, we have

$$\text{Area}\,(\triangle AOB) = \text{Area}\,(\triangle ABL)$$

Because $\overline{AD} \parallel \overline{BC}$, it follows that

$$\text{Area}\,(\triangle ABL) = \text{Area}\,(\triangle ACL) = \frac{1}{2}\,\text{Area}\,(\triangle ACD)$$

where the last equality holds because L is the midpoint of $\overline{CD}$. By a similar argument,

$$\text{Area}\,(\triangle AOE) = \text{Area}\,(\triangle ADK) = \frac{1}{2}\,\text{Area}\,(\triangle ACD)$$

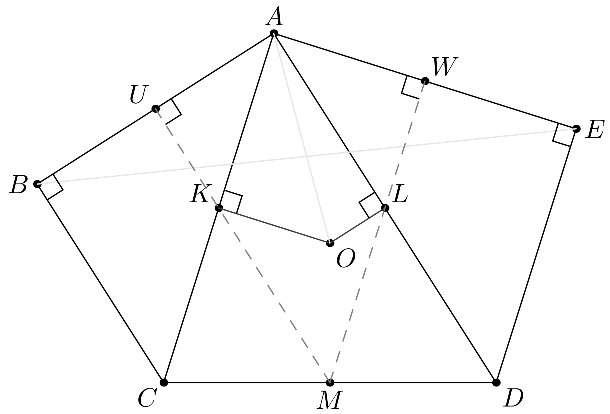
Thus the areas of $\triangle AOB$ and $\triangle AOE$ are equal. □

To finish the proof, if $\overline{AO} \cap \overline{BE} = X$, then

$$\text{Area}\,(\triangle AOB) = \text{Area}\,(\triangle AOE) \implies \overline{BX} = \overline{XE}$$

Thus X is the midpoint of $\overline{BE}$. □

§4 Solution 3 (Using Ratio Lemma)



Proof. Let U, K, M, L and W be the midpoints of $\overline{AB}$, $\overline{AC}$, $\overline{CD}$, $\overline{AD}$ and $\overline{AE}$. Define P as the intersection of $\overline{AO}$ and $\overline{BE}$.

Claim 4.1. Lines $\overline{BC} \parallel \overline{AD}$ and $\overline{AC} \parallel \overline{DE}$.

Proof. Since M is the circumcenter of $\triangle ABE$, we have $\overline{UM} \perp \overline{AB}$. As $\angle CBA = 90°$, it follows that $\overline{UM} \parallel \overline{BC}$. By midpoint theorem, $\overline{UM}$ also passes through K. Thus U, K and M are collinear, with $\overline{BC} \parallel \overline{UK}$ and $\overline{KM} \parallel \overline{AC}$. This implies $\overline{BC} \parallel \overline{AD}$. By a similar argument, $\overline{AC} \parallel \overline{DE}$. □

Claim 4.2. Lines $\overline{AB} \parallel \overline{OL}$ and $\overline{EA} \parallel \overline{OK}$.

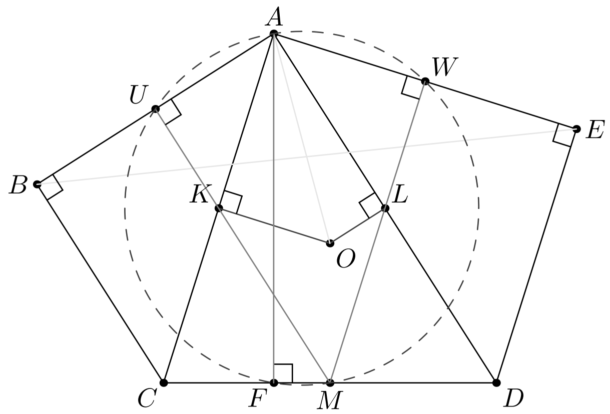
Proof. Same as Solution 2. □

Finally we apply **Ratio Lemma** to $\triangle ABE$

$$
\begin{aligned}
\left(\frac{\overline{BP}}{\overline{PE}}\right) &= \left(\frac{\overline{AB}}{\overline{AE}}\right) \cdot \left(\frac{\sin \angle BAO}{\sin \angle EAO}\right) \\
&= \left(\frac{\overline{AB}}{\overline{AE}}\right) \cdot \left(\frac{\sin \angle AOL}{\sin \angle AOK}\right) \\
&= \left(\frac{\overline{AB}}{\overline{AE}}\right) \cdot \left(\frac{\sin \angle ACD}{\sin \angle ADC}\right) \\
&= \left(\frac{\overline{AB}}{\overline{AE}}\right) \cdot \left(\frac{\overline{AD}}{\overline{AC}}\right) = \frac{\left(\frac{\overline{AB}}{\overline{AC}}\right)}{\left(\frac{\overline{AE}}{\overline{AD}}\right)} \\
&= \left(\frac{\sin \angle ACB}{\sin \angle ADE}\right) = 1
\end{aligned}
$$

Thus, we conclude that $\overline{AO} \cap \overline{BE} = P$ is the midpoint of $\overline{BE}$. □

§5 Solution 4 (Using Harmonic Bundles)



The idea of this proof is similar to Solution 1. The main difference is that we use **Harmonic Cross Ratios** to show that $\overline{AF}$ is the A-symmedian of $\triangle ABE$.

Proof. Continuing the notations from Solution 3. Let F be the foot of perpendicular from A to $\overline{CD}$. By angle chasing, it follows that F lies on $\odot(AUMW)$.

> **Claim 5.1.** Line $\overline{AF}$ is the A-symmedian of $\triangle ABE$.

Proof. Since K and L are the midpoints of $\overline{AC}$ and $\overline{AD}$, we get $\overline{KL} \parallel \overline{CD}$. Let ∞_{KL} be the point at infinity on line $\overline{KL}$. Since $AKML$ is a parallelogram, we have

$$-1 = \left(K, L; \overline{AM} \cap \overline{KL}, \infty_{KL}\right) \stackrel{M}{=} (U, W; A, F)$$

This implies that $\overline{AF}$ is the A-symmedian in $\triangle AUW$. Since $\overline{UW} \parallel \overline{BE}$, there exists a homothety at A sending $\triangle AUW$ to $\triangle ABE$. Thus, $\overline{AF}$ is also the A-symmedian of $\triangle ABE$. □

Rest of the proof is similar to Solution 1. □