Lecture Notes (11th Jan, 2026)

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In this lecture, we learn more about circles and their properties.

§1 Power of a Point

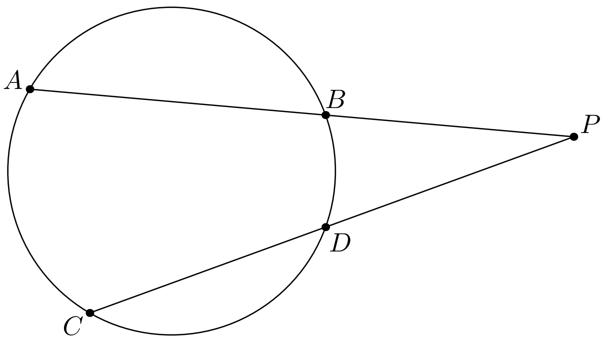
Let's first look at a result that motivates the power of a point.

Theorem 1.1

Suppose ω is a circle and P is a point. Draw two lines ℓ_1 and ℓ_2 passing through P that intersect the circle in points A, B and C, D. Then,

$$\overline{PA} \cdot \overline{PB} = \overline{PC} \cdot \overline{PD}$$

There are two cases to consider, one in which the point P lies inside the circle ω, and another in which it lies outside the circle. Surprisingly, the result holds in both cases. The proof relies on properties of cyclic quadrilaterals to establish pairs of similar triangles, and we therefore omit it here.



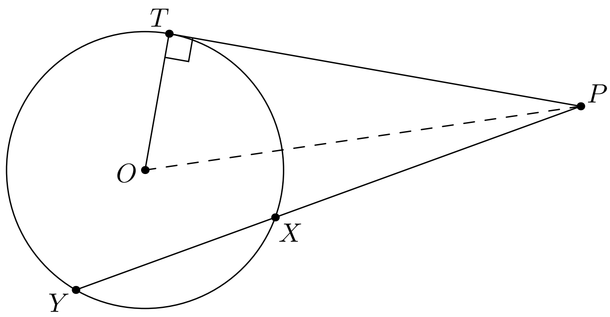
A stronger form of the preceding result is given by the following theorem.

Theorem 1.2

Suppose ω is a circle and P is a point lying outside the circle. Let ℓ be a line that passes through P and cuts the circle in points X and Y. Let T be a point on the circle such that $\overline{PT}$ is tangent to ω. Then

$$\overline{PX} \cdot \overline{PY} = \overline{PT}^2$$

We can give an intuitive explanation for why this result might hold true. If the points X and Y are brought arbitrarily close together so that they effetively coincide, then the secant line becomes a tangent and the product $\overline{PX} \cdot \overline{PY}$ reduces to the square of the length of the tangent drawn.



Upon careful observation, we realise that product of line segments is constant regardless of the choice of the line ℓ. Hence, if define the following quantity for a point P and a circle ω,

$$\text{Pow}_\omega\left(P\right) = \overline{PX} \cdot \overline{PY}$$

where, ℓ is a line that passes through P cutting ω at X and $Y \implies$ the quantity $\text{Pow}_\omega\left(P\right)$ will be a constant for a fixed P and ω. Furthermore, we can rewrite the above expression as

$$\begin{aligned}\text{Pow}_\omega\left(P\right) &= \overline{PX} \cdot \overline{PY} = \overline{PT}^2 \\ &= \overline{OP}^2 - \overline{OT}^2 \\ &= \overline{OP}^2 - r^2\end{aligned}$$

Since the above expression is only dependent on the circle ω and the point P, therefore that motivates us to define the following quantity!

Definition 1.3. Given a circle ω centered at O with radius r and a point P, the **power of point P** is given by

$$\text{Pow}_\omega\left(P\right) = \overline{OP}^2 - r^2$$

With the above definition, we can rewrite the previously stated results as follows.

Theorem 1.4 (Power of a Point Theorem)
Given a circle ω and a point P,

1. the quantity $\text{Pow}_\omega\left(P\right)$ is positive, zero or negative depending on whether P is outside, on or inside ω, respectively.
2. if ℓ is a line through P intersecting ω at two distinct points X and Y, then

$$\overline{PX} \cdot \overline{PY} = \left|\,\text{Pow}_\omega\left(P\right)\right|$$

3. if P is outside ω and $\overline{PA}$ is a tangent to ω at a point A on ω, then

$$\overline{PA}^2 = \text{Pow}_\omega\left(P\right)$$

In fact, the converse of the above result also holds.

Theorem 1.5 (Converse of the Power of a Point Theorem)

Suppose A, B, X, and Y are four distinct points in the plane, and let the lines AB and XY intersect at P. Then the points A, B, X, and Y are concyclic if and only if

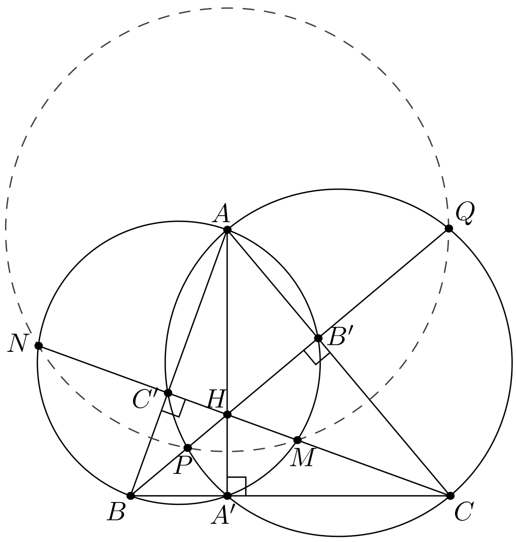
$$\overline{PA} \cdot \overline{PB} = \overline{PX} \cdot \overline{PY}.$$

As it turns out, this is a powerful tool that can be used to prove that a quadrilateral is cyclic. We now look at a few examples illustrating these results.

§1.1 Examples

Problem 1.6 (USA Math Olympiad 1990)

An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.



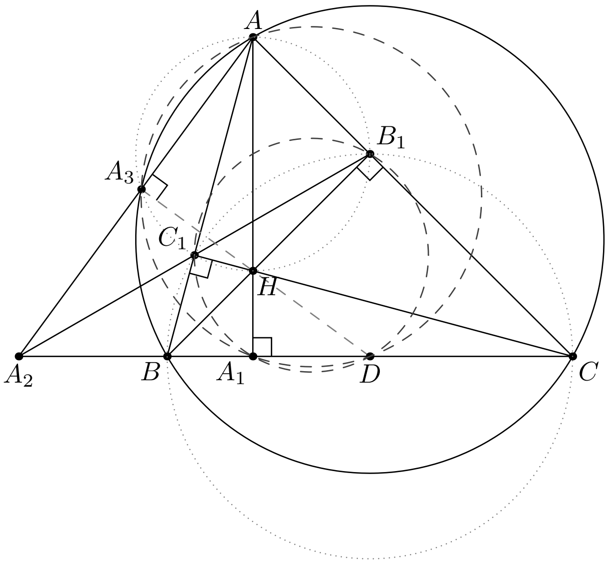
Proof. Let H be the orthocenter of $\triangle ABC$ and A' be the foot of perpendicular dropped from point A onto $\overline{BC}$. By power of a point theorem applied on circles with diameter $\overline{AC}$, $\overline{BC}$ and $\overline{AB}$, we have

$$\begin{aligned} \overline{HP} \cdot \overline{HQ} &= \overline{HC} \cdot \overline{HC'} \\ &= \overline{HB} \cdot \overline{HB'} \\ &= \overline{HM} \cdot \overline{HN} \end{aligned}$$

Therefore, by the converse of power of a point theorem $\implies$ $MNPQ$ is a cyclic quadrilateral. $\square$

Problem 1.7 (USA TSTST 2012)

In scalene triangle ABC, let the feet of the perpendiculars from A to BC, B to CA, C to AB be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides BC, CA, AB. Show that the perpendiculars from D to AA_2, E to BB_2 and F to CC_2 are concurrent.



Proof. We shall show that the perpendiculars from D to $\overline{AA_2}$, E to $\overline{BB_2}$ and F to $\overline{CC_2}$, all pass through the orthocenter H of $\triangle ABC$, which is their concurrency point. Suppose A_3 is the foot of perpendicular from D to $\overline{AA_2}$. Since, $\angle AA_3D = 90°$ and $\angle AA_1D = 90° \implies AA_3A_1D$ is cyclic. However, $B_1C_1A_1D$ is cyclic too because they lie on the nine-point circle of $\triangle ABC$, and BC_1B_1C is cyclic too because $\angle BC_1C = 90°$ and $\angle BB_1C = 90°$. By applying the power of a point theorem on these circles, we have

$$\begin{aligned} \overline{A_2A_3} \cdot \overline{A_2A} &= \overline{A_2A_1} \cdot \overline{A_2D} \\ &= \overline{A_2C_1} \cdot \overline{A_2B_1} \\ &= \overline{A_2B} \cdot \overline{A_2C} \end{aligned}$$

By the converse of power of a point theorem, this implies that $AA_3C_1B_1$ and AA_3BC are cyclic quadrilaterals too. Since $\overline{AH}$ is the diameter of $\odot(AC_1B_1) \implies \angle AA_3H = 90°$. However we know that, $\angle AA_3D = 90° \implies \overline{A_3D}$ passes through H. Similarly, we can show that the others pass through the orthocenter H, thus implying the concurrency. □

§1.2 Exercises

Exercise 1.8. Let $\triangle ABC$ be an acute angled triangle with circumcenter O and orthocenter H. Prove that

$$\overline{OH}^2 = R^2\left(1 - 8\cos A\cos B\cos C\right)$$

Exercise 1.9 (USA Math Olympiad 2023)**.** In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

Exercise 1.10 (USA Math Olympiad 2009)**.** Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

§2 Radical Axis Theorem

So far, we have developed tools that allow us to tackle problems involving a single circle. We now move onto problems involving multiple circles. We begin by introducing a few key definitions.

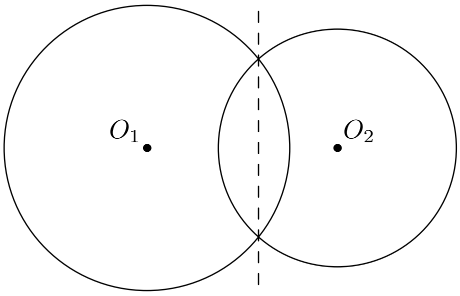
Definition 2.1. Given two circles ω_1 and ω_2 with distinct centers, the **Radical Axis** of the circles is the set of points P such that

$$\operatorname{Pow}_{\omega_1}(P) = \operatorname{Pow}_{\omega_2}(P)$$

The definition of the radical axis may not seem very intuitive. However, it essentially tells us the following.

Corollary 2.2

For two intersecting circles ω_1 and ω_2, their radical axis is the line passing through their points of intersections.



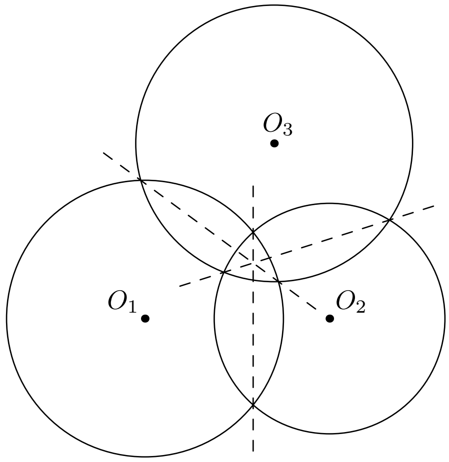
Something even more counterintuitive is that the radical axis is defined for a pair of *non-intersecting* circles as well. A common misconception is to think of it as the perpendicular bisector of the line joining the centers, but this is incorrect. Instead, it is only the locus of points that have equal powers with respect to both circles. So what makes this interesting? The following result is what gives the radical axis its power as a tool in geometry.

Theorem 2.3 (Radical Axis Theorem)

Given three distinct circles ω_1, ω_2 and ω_3, their pairwise radical axes are concurrent. This point of concurrency is known as the **Radical Center** of the three circles.

The proof immediately follows from the definition of radical axis. Infact, the converse of radical axis theorem holds is true and serves as a criterion for proving cyclicity of one of the three circles. Essentially, this is just equivalent to applying the power of a point theorem twice and then concluding via its converse.

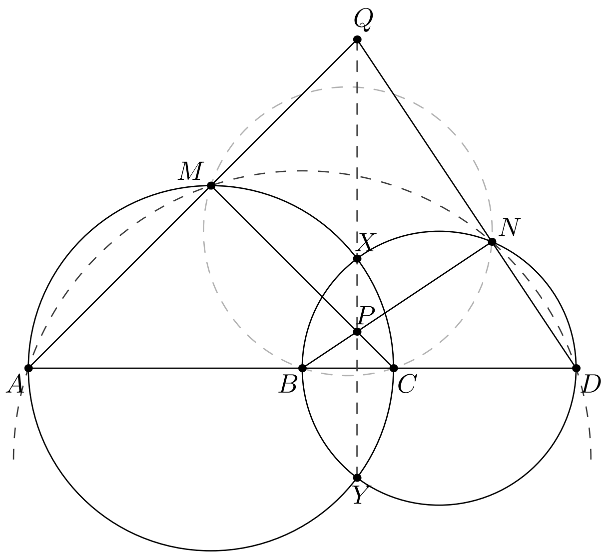
This theorem is particularly useful because it gives us a powerful new tool for proving concurrencies. Let us now explore some examples.



§2.1 Examples

Problem 2.4 (IMO 1995)

Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.



Proof. Suppose ω_1 and ω_2 are the circles with diameters $\overline{AC}$ and $\overline{BD}$. Then $\overline{XY}$ is the radical axis of ω_1 and ω_2. Since P lies on the radical axis of ω_1 and $\omega_2 \implies MBCN$ is cyclic, by the converse of radical axis theorem with P as the radical center. Since,

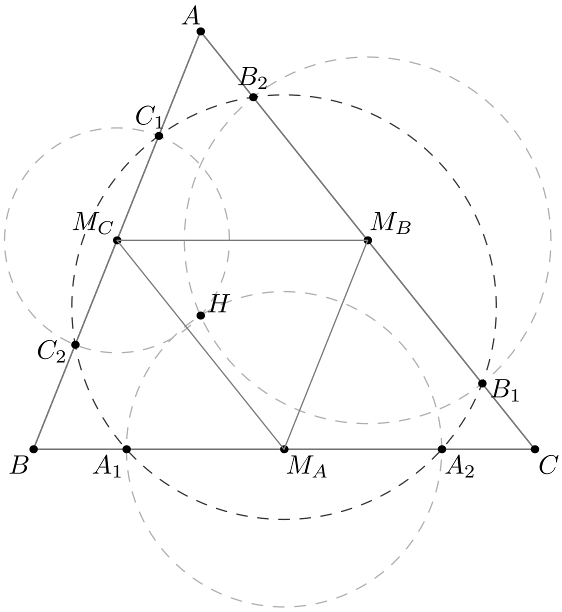
$$\begin{aligned}\angle MAD &= 90^\circ - \angle MCB \\ &= 90^\circ - \angle MNB \\ &= 180^\circ - \angle MND\end{aligned}$$

Therefore, $AMND$ is a cyclic quadrilateral too. Applying the radical axis theorem on circles ω_1, ω_2 and $\odot(AMND)$, we get $\overline{AM}$, $\overline{XY}$ and $\overline{DN}$ are concurrent. $\square$

Problem 2.5 (IMO 2008)

Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.



Proof. We shall show that the quadrilaterals $A_1A_2B_1B_2$, $B_1B_2C_1C_2$ and $C_1C_2A_1A_2$ are all cyclic. Let's prove the first one. From the definition of points A_1 and A_2, we have

$$\begin{aligned}\overline{CA_2} \cdot \overline{CA_1} &= \left(\overline{CM_A} - \overline{M_AH}\right) \cdot \left(\overline{CM_A} + \overline{M_AH}\right) \\ &= \overline{CM_A}^2 - \overline{HM_A}^2\end{aligned}$$

Similarly, $\overline{CB_1} \cdot \overline{CB_2} = \overline{CM_B}^2 - \overline{HM_B}^2$. Since $\overline{CH} \perp \overline{M_AM_B}$, therefore by Pythagoras' Theorem, we have

$$\overline{CM_A}^2 - \overline{HM_A}^2 = \overline{CM_B}^2 - \overline{HM_B}^2 \implies \overline{CA_2} \cdot \overline{CA_1} = \overline{CB_1} \cdot \overline{CB_2}$$

Hence, by the converse of power of a point theorem $A_1A_2B_1B_2$ is cyclic. Similarly, $B_1B_2C_1C_2$ and $C_1C_2A_1A_2$ are cyclic too. Applying the radical axis theorem on their circumcircles we get that the radical center is

$$\overline{BC} \cap \overline{CA} \cap \overline{AB}$$

However, these lines do not have a common point of intersection. This means that all the three circles must be the same. Therefore, we have shown that the points A_1, A_2, B_1, B_2, C_1 and C_2 all lie on the same circle. □

§2.2 Exercises

Exercise 2.6 (USA Junior Math Olympiad 2012)**.** Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).

Exercise 2.7. Let ABC be a triangle and let D and E be points on sides AB and AC, respectively, such that $DE \parallel BC$. Let P be any point interior to triangle ADE, and let F and G be the intersections of DE with the lines BP and CP, respectively. Let Q be the second intersection point of the circumcircles of triangles PDG and PFE. Prove that the points A, P, and Q are collinear.

Exercise 2.8. Let ABC be an acute triangle with incenter I. Points E and F are the midpoints of the shorter arcs $\widehat{AC}$ and $\widehat{AB}$ of the circumcircle $\odot(ABC)$, respectively. Segment EF intersects sides AB and AC at points P and Q, respectively. Point D is defined by the conditions $PD \parallel BI$ and $QD \parallel CI$. Let T be the intersection point of BF and CE. Prove that points T, I, D are collinear.

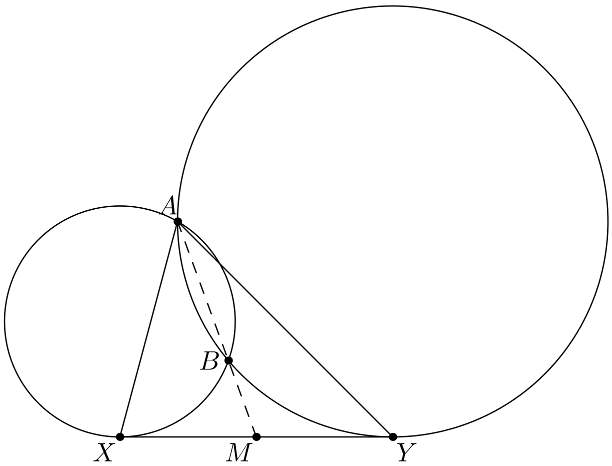
§3 Applications: Common Tangent

There is a very special configuration in the geometry of two intersecting circles. Let look at the following example to motivate it.

§3.1 Medians!

Example 3.1

Given two intersecting circles ω_1 and ω_2, let ℓ be the common tangent of both the circles. Suppose ℓ touches ω_1 and ω_2 at X and Y, then the radical axis of ω_1 and ω_2 bisects $\overline{XY}$.



This result follows immediately from computing the power of M with respect to both the circles. Suppose $AB \cap \overline{XY} = M$, then

$$\overline{XM}^2 = \overline{MB} \cdot \overline{MA} = \overline{MY}^2$$

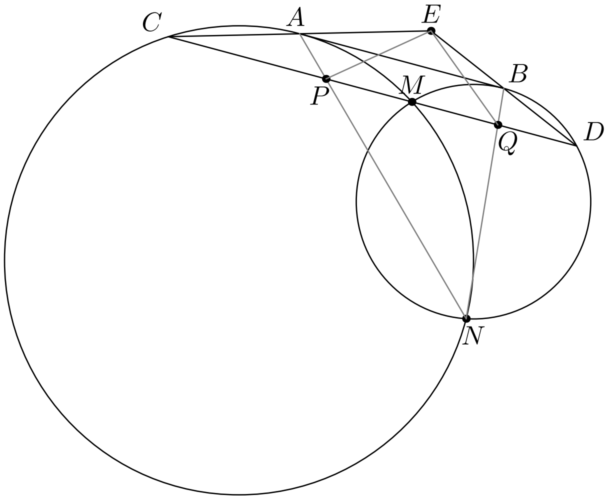
Thus, $\overline{XM} = \overline{MY}$.

Again, the result is easy to prove, but it appears frequently in geometric configurations and one would not want to overlook this elegant apperance of medians in the configuration.

§3.2 Examples

Problem 3.2 (IMO 2000)

Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.



Proof. Since MN bisects $\overline{AB}$ and $\overline{AB} \parallel \overline{PQ} \implies M$ is the midpoint of $\overline{PQ}$. However, $\angle EAB = \angle ECM = \angle BAM$ and similarly, $\angle EBA = \angle MBA$. Therefore by SAS congruence criterion, $\triangle EAB \cong \triangle MAB$. This implies that $EAMB$ is a kite and $\overline{EM}$ is perpendicular to $\overline{AB} \implies \overline{EM} \perp \overline{PQ}$. Again by SAS congruence criterion, $\triangle EMP \cong \triangle EMQ \implies \overline{EP} = \overline{EQ}$. □

§3.3 Exercises

Exercise 3.3 (APMO 1999)**.** Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Exercise 3.4 (IMO Shortlist 2006)**.** Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

§4 Applications: Orthic Axis

There is a very special line related to the orthocenter configuration, that naturally appears as a radical axis of two circles. We shall study the **Orthic Axis** in this section. Let's begin with the some definitions.

§4.1 The Trilinear Polar of H

Proposition 4.1 (Trilinear Polar of a point)

Given a $\triangle ABC$ and a point P inside it. Suppose $\triangle DEF$ is the cevian triangle of P. Let the line EF intersect BC at X and similarly, define Y and Z. Then, the points X, Y and Z are collinear, and this line is called the **Trilinear Polar of P**.

The collinearity can be proven using menelaus' theorem. There is another way to prove this based on a result from projective geometry, called the **Desargues' Theorem**. The trilinear polar is precisely the axis of perspectivity of the cevian triangle and $\triangle ABC$.

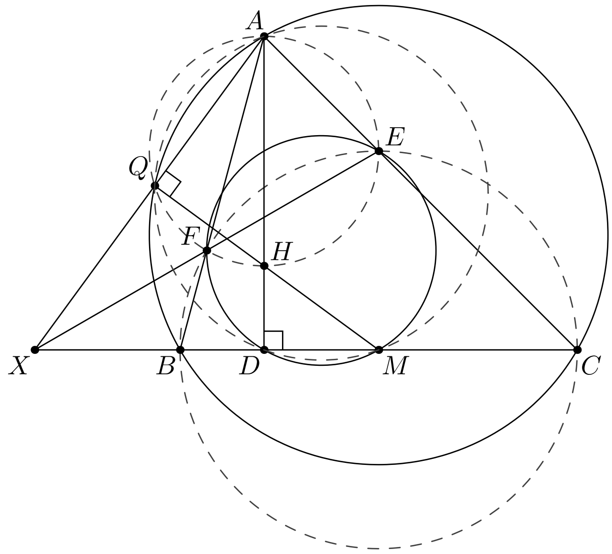
In particular, the trilinear polar of the orthocenter H is the axis of perspectivity of the orthic triangle and $\triangle ABC$. This line is known as the *orthic axis*.

§4.2 Orthic Axis as the Radical Axis

An even more remarkable property of the orthic axis is that it is the radical axis of the nine-point circle and $\odot(ABC)$. We now proceed to prove this fact.

Theorem 4.2

Given a $\triangle ABC$ and its nine-point circle $\odot(N_9)$. Then, the orthic axis of $\triangle ABC$ is the radical axis of $\odot(ABC)$ and $\odot(N_9)$.



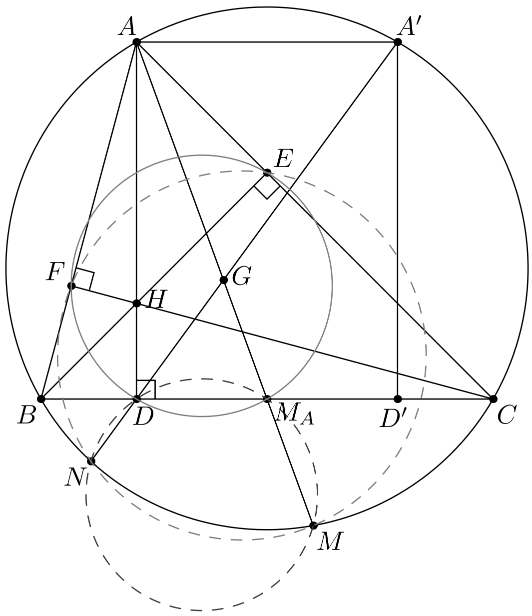
Proof. Let H be the orthocenter of $\triangle ABC$ and $\triangle DEF$ be the orthic triangle of $\triangle ABC$. Let M be the midpoint of $\overline{BC}$ and let the ray $\overrightarrow{MH}$ intersect $\odot(ABC)$ at Q. It's easy to show that Q lies on $\odot(AEF)$ and $\odot(ADM)$. Applying the radical axis theorem on $\odot(AEF)$, $\odot(DEF)$ and $\odot(ADM)$, we get that the lines AQ, EF and BC are concurrent. Applying radical axis theorem on $\odot(ABC)$, $\odot(DEF)$ and $\odot(BFEC)$ implies that the radical axis of $\odot(ABC)$ and $\odot(DEF)$ passes through $EF \cap BC = X$. Similarly, we can show that the radical axis of $\odot(ABC)$ and $\odot(DEF)$ passes through $DF \cap CA$ and $DE \cap AB$, proving the result. $\square$

Let's take a look at an application of this result.

§4.3 Examples

Problem 4.3 (Greece IMO TST 2019)

$\triangle ABC$ is inscribed in a circle (C). Let G the centroid of $\triangle ABC$. We draw the altitudes AD, BE, CF of the given triangle. Rays AG and GD meet (C) at M and N. Prove that points F, E, M, N are concyclic.



Proof. Let M_A be the midpoint of $\overline{BC}$ and let D' be the reflection of D over M_A. Since M_A is the midpoint of $\overline{DD'}$, then $\overline{AM_A}$ is the A-median of $\triangle ADD'$. Since G divides the cevian $\overline{AM_A}$ in the ratio $2:1$, therefore G is the centroid of $\triangle ADD'$ too. Construct a point A' such that $AA'D'D$ is rectangle. Since D' is the reflection of D over the perpendicular bisector of $\overline{BC}$, hence A' is the reflection of A over the perpendicular bisector of $\overline{BC}$ too. Therefore A' lies on $\odot(ABC)$ too. Since G is the centroid of $\triangle ADD'$, therefore $\overline{DG}$ bisects $\overline{AD'}$. Since $AA'D'D$ is a rectangle, therefore $A'D$ bisects $\overline{AD'}$ and hence DG must pass through A'. By converse of reim's theorem, $DNMM_A$ is a cyclic quadrilateral because $\overline{AA'} \parallel \overline{BC}$.

Applying the radical axis theorem on $\odot(ABC)$, $\odot(DNMM_A)$ and $\odot(DEF)$, we get that the lines EF, BC and MN must concur, because $EF \cap BC$ lies on the orthic axis of $\triangle ABC$ which is the radical axis of $\odot(ABC)$ and $\odot(DEF)$. By the converse of radical axis theorem on $\odot(DEF)$ and $\odot(DNMM_A)$, we get that $EFNM$ must be a cyclic quadrilateral. □

§4.4 Exercises

Exercise 4.4 (USA TSTST 2017). Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and that $\angle A \neq 90°$. Let M and N be the midpoints of sides AB and AC, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection of lines AQ and EF. Prove that $PR \perp OH$.

§5 Pratice Problems

Exercise 5.1. Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be the altitudes of a scalene triangle with circumcenter O. Prove that $\odot(AOD)$, $\odot(BOE)$, and $\odot(COF)$ intersect at point X other than O.

Exercise 5.2 (USA Junior Math Olympiad 2024)**.** Let $ABCD$ be a cyclic quadrilateral with $AB = 7$ and $CD = 8$. Point P and Q are selected on segment AB such that $AP = BQ = 3$. Points R and S are selected on segment CD such that $CR = DS = 2$. Prove that $PQRS$ is a cyclic quadrilateral.

Exercise 5.3 (APMO 2020)**.** Let Γ be the circumcircle of $\triangle ABC$. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.

Exercise 5.4 (IMO Shortlist 2011)**.** Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

Exercise 5.5 (USA Math Olympiad 1998)**.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

Exercise 5.6 (USA Math Olympiad 1997)**.** Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

Exercise 5.7 (USA TSTST 2011)**.** Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.

Exercise 5.8 (IMO 2009)**.** Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Exercise 5.9 (USA TSTST 2016)**.** Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.