Plane Geometry

A Perspective Field looks like a constellation of stars at heaven. All these points are like stars. Each star moves in its own way. Some stars adjourn together in a flow. Just like in astronomy we just watch and sometimes we understand why.

Perspective Fields

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Preface

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And last but certainly not least, we'd like to thank Li4 for writing this book in the first place <3.

Conventions

We adopt some general conventions throughout this project. It's best not to read these until you are confused later on; this section should serve as a reference for strange notation.

Numbering is done with respect to the original book, and any additional remarks are not numbered and done by the translators.

We have attempted to minimize references that break the ordering of chapters, though sometimes the ordering of the original book supersedes this desire. For instances, references to $\mathcal{L}_\infty$ and points at infinity in Chapters 0 and 1.

Most of these are non-normative for Olympiad Contests as of writing.

We typically take the default reference triangle $\triangle ABC$ and have $a = \overline{BC}, b = \overline{CA}, c = \overline{AB}$ be the three sides respectively. We let $\alpha = \measuredangle BAC, \beta = \measuredangle CBA, \gamma = \measuredangle ACB$.

Regarding triangle style config geo specifically:

- I represents the incenter, and I^a, I^b, I^c or I_A, I_B, I_C respectively represent the three excenters (Example 0.3.9). We refer to **extraversion** as the fact that properties true for I typically remain true for I^a, I^b, I^c as well.
- $\omega, \omega^a, \omega^b, \omega^c$ respectively represent the incircle and the three excircles, r, r^a, r^b, r^c respectively represent their radii;
- $\triangle DEF, \triangle D^aE^aF^a, \triangle D^bE^bF^b, \triangle D^cE^cF^c$ respectively represent the intouch triangle and the A, B, C-extouch triangles;
- $\triangle M_aM_bM_c$ represents the medial triangle;
- Ω represents the circumcircle, R represents the circumradius, A^*, B^*, C^* represent the antipodes of A, B, C;
- $\mathcal{E}$ can be the Euler line.
- ϵ represents the nine-point circle;
- $\triangle H_aH_bH_c$ represents the orthic triangle;
- $\triangle N_aN_bN_c$ represents the circumcevian triangle of I, aka the triangle formed by arc midpoints, and ${N_a}^*$, ${N_b}^*$, and ${N_c}^*$ respectively represent the antipodes of N_a, N_b, and N_c.
- $\mathcal{L}_\infty$ is used for the line at infinity.
- $\triangle(AB)(CD)(EF)$ refers to the triangle with vertices at the pairwise intersections of the three lines.

- $\triangle PQ$ means the circumconic of ABC through P, Q.
- $(PQ)^*$ means the circumconic of ABC through P^*, Q^*.
- K^a, K^b, K^c refer to the vertices of the tangential triangle, and $G^aG^bG^c$ refer to the vertices of the anticomplementary triangle.

Along with the original book, we use the following notation for transformations.

- $\mathfrak{h}_{O,k}$ represents a homothety with center O and ratio k.
- $\mathfrak{h}_{\vec{v}}$ represents a translation of $\vec{v}$.
- $\mathfrak{s}_O$ represents a reflection about O.
- $\mathfrak{r}_{O,\theta}$ represents a rotation about O of θ.
- $\mathfrak{I}_\Omega$ represents an inversion about a circle Ω.
- $\mathfrak{I}_{O,k}$ represents an inversion about O with power k.
- $\mathfrak{S}_{O,k,\theta} = \mathfrak{r}_{O,\theta} \circ \mathfrak{h}_{O,k}$ represents a spiral similarity with ratio k and rotation θ.
- $P^{\complement} = \mathfrak{h}_{G,-\frac{1}{2}}(P)$ represents the complement of a point P wrt. $\triangle ABC$. $P^{\complement,\triangle ABC}$ will be used if the reference triangle is unclear.
- $P^{\mho} = \mathfrak{h}_{G,-2}(P)$ represents the anticomplement of a point P wrt. $\triangle ABC$. $P^{\mho,\triangle ABC}$ will be used if the reference triangle is unclear.
- $\mathcal{S}_P$ is the Steiner line of P on the circumcircle.
- $\mathfrak{t}(P)$ is the trilinear polar of P wrt. $\triangle ABC$.
- P' can denote the image under of P under some transformation, or under isotomic conjugation.
- P^* can be used to refer the image of P under isogonal conjugation or the antipode with respect to a circle.
- For some known A, B by context, and C on AB, we define $C^\vee$ as the harmonic conjugate of C with respect to $\{A, B\}$. Often this is just the midpoint.
- $\mathfrak{I}$ as a superscript represents the image of a point under inversion.

Regarding barycentrics and point operations, where A and B are arbitrary points in a triangle, (note that many of these terms are newly coined in this book; Chapter 12 may be a better reference if unfamiliar with vocab.)

- $A \times B$ refers to the (termwise) barycentric product of the two points. Similarly, $A \div B$ refers to the barycentric quotient.
- $\mathcal{A} \cap \mathcal{B}$ refers to the common points on both curves.
- The crosspoint of A and B is $A \pitchfork B$
- the cevapoint is $A \star B$.
- The cross conjugate is $A \Psi B$
- The ceva conjugate is A/B.
- φ_G and φ_K are isogonal and isotomic conjugation respectively, φ^P is the isoconjugation with pole P.
- $P^{\mathbb{D}}$ is the difference point of P.
- P^{s} is the square transformation, and P^{r} is the square root transformation.

Regarding angles,

- We use $\angle$ for undirected angles, taken with config issues.
- We use $\measuredangle ABC = \measuredangle(AB, BC)$ for directed angles, taken $\pmod{180^\circ}$. $\measuredangle(AB, CD)$ refers to the angle between AB, CD.
- We use $\measuredangle AB$ for line arguments $\pmod{180^\circ}$. We use $\perp AB = \measuredangle AB + 90^\circ$. We use $\overrightarrow{AB}$ when things are $\pmod{360^\circ}$
- We use directed points on a circle where $(A+B)_\Omega = \measuredangle AB$ and $(A-C)_\Omega = \measuredangle AXC$ for $X \in \Omega$.
- We use degrees for the most part but use radians sometimes.

Regarding conics, we define (where Γ is an arbitrary conic)

- $\mathbf{T}\Gamma$ as the set of all tangents of Γ; In the degenerate case where Γ is a point P, $\mathbf{T}P$ is the set of lines through P.
- $\mathbf{T}_P\Gamma$ as the tangent line to Γ at P, where $P \in \Gamma$;
- $\mathbf{T}_\ell\Gamma$ as the point of tangency of ℓ and Γ, where $\ell \in \mathbf{T}\Gamma$;
- $\mathfrak{p}_\Gamma(A)$ as the polar of a point A wrt. Γ (for higher degree curve $\mathcal{K}$, $\mathfrak{p}^n_{\mathcal{K}}(P)$ will represent the nth polar of P wrt. $\mathcal{K}$. For clarification, the 1st polar of a point wrt. a degree n curve is degree $n-1$.)
- $\mathfrak{c}(P)$ is the circumconic of $\triangle ABC$ with P as perspector.

- $\mathcal{H}$ generally refers to a hyperbola.
- $\mathcal{H}_{Fe}, \mathcal{H}_K, \mathcal{H}_J$ is the Feuerbach, Kiepert, Jerabek hyperbola.
- $\mathbf{Per}^{\Gamma}_{P}(X)$ and $\mathbf{Li}^{\Gamma}_{P,Q}(X)$ refer to the P-permutation line and P, Q-Li conjugate of X wrt. Γ. If Γ is omitted, assume it to be the circumcircle.

For complete quadrilaterals, we define

- $(AC)(BD)$ refers to the quadrilateral consisting of the four lines AB, BC, CD, DA
- $\triangle_X$ refers to the component triangle not containing vertex X
- $\mathcal{Q}$ as the complete quadrilateral (BC, CE, EF, FB)
- The Miquel point as $M_{\mathcal{Q}}$
- The Newton line as $\tau_{\mathcal{Q}}$
- The Steiner line as $\mathcal{S}_{\mathcal{Q}}$.
- $A_{ij} = \ell_i \cap \ell_j$

Here's a short but important exposition on notation and definitions of projective geometry used in this book; for more detail and rigor check the appendix at the end. (for experts, only recommended to read after Part I).

- $\mathbb{RP}^1$ is the real projective line, which is roughly represented with $[x : y]$ in homogeneous coordinates up to scaling (excluding $[0 : 0]$). and $\mathbb{RP}^2$ is the real projective plane (three variable variant). These can roughly be thought of as $\{\mathbb{R}\} \cup \{P_\infty\}$ and $\{\mathbb{R}^2\} \cup \{\ell_\infty\}$. Analogously, $\mathbb{CP}^1$ and $\mathbb{CP}^2$ are the same over complex numbers.
- When we refer to the point at infinity in $\mathbb{P}^1$, we will typically use $[x : 0]$ for some $x \neq 0$. Similarly, when we refer to the line at infinity in $\mathbb{P}^2$, we will typically use $[x : y : 0]$ for some $(x, y) \neq (0, 0)$.
- $\mathbb{CP}^1$ is called the complex projective line, and has the structure of having one point at infinity, and inversion is done over it. **Important: Note that we do not call this a plane.**
- $\mathbb{CP}^2$ is called the complex projective plane, with coordinates $[x : y : z]$. **Despite similarity to the term "complex plane", these are two very different objects!** The "complex plane" typically refers to $\mathbb{C}^1$, which is technically a one-dimensional space (it's only two-dimensional if you split the complex numbers into real and imaginary parts), while this is two-dimensional (four, if you split into imaginary and real parts). We apologize for the confusion.

- When we don't care about the underlying scalar field, we will use $\mathbb{P}^n$.
- The term "projective map" is used for an automorphism between $\mathbb{P}^1 \to \mathbb{P}^1$ which preserves cross-ratio.
- A homography is an automorphism between $\mathbb{P}^n \to \mathbb{P}^n$ which preserves cross-ratio and collinearity/incidence.
- A collineation is some transformation from $\mathbb{P}^n \to \mathbb{P}^n$ that preserves collinearity of points, but not necessarily cross-ratio. In $\mathbb{RP}^2$ (but not $\mathbb{CP}^2$), these transformations also preserve the cross-ratio and are thus homographies.
- In this book, "projective transformation" is used for an automorphism between $\mathbb{P}^2 \to \mathbb{P}^2$ which preserves cross-ratio and collinearity/concurrence. However, we will only use "projective transformation" when the distinction between homography and collineation does not matter, such as when working in the real projective plane.
- The circle points are the two complex points in $\mathbb{CP}^2$ of $[1 : i : 0], [1 : -i : 0]$ that every circle passes through. These will be denoted as I and J usually, unless this is confusing in context (incenters), in which case we will use ∞_i and ∞_{-i}.
- A Möbius transformation / fractional linear transformation is a homography/projective map specific to $\mathbb{CP}^1$; these are the combination of [inversions + reflections] and orientation-preserving spiral similarities.
- I and J are the circle points $(1\colon i\colon 0), (i\colon 1\colon 0)$ in $\mathbb{CP}^2$.
- The word **pencil** will be used for sets of objects that are similar to $\mathbb{P}^1$, and the word **net** will be used for sets of objects that are similar to $\mathbb{P}^2$.

Some amounts of group theory knowledge will be assumed for this book.

- Hom will represent homomorphisms, which is a map between two sets of objects that preserves some sense of structure (most commonly, the structure is cross-ratio)
- Aut will represent automorphisms, which is a homomorphism from a set of objects to itself.

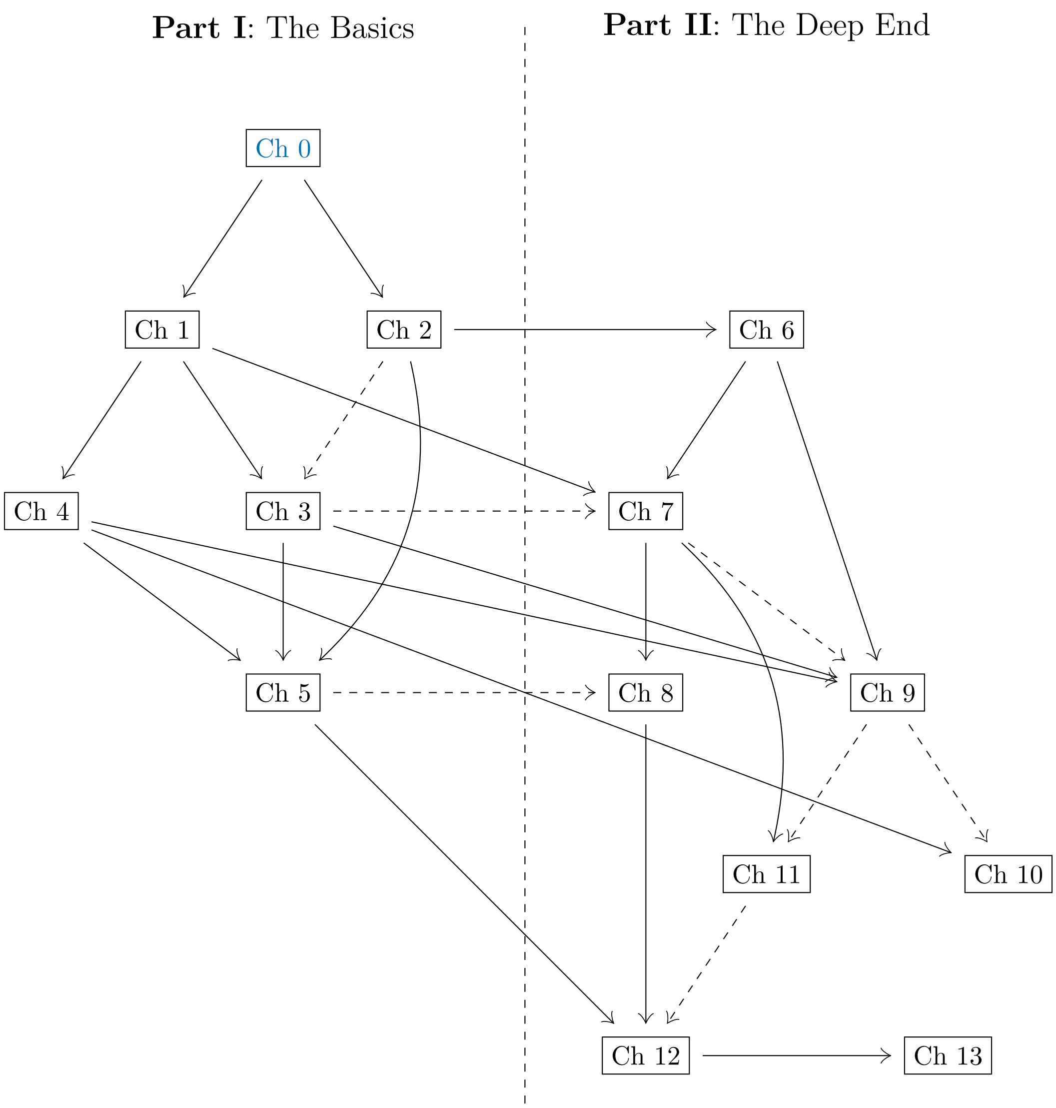
Miscellaneous notation:

- $\Gamma_r^{A,B} = \{P \mid \overline{PA} = r \cdot \overline{PB}\}$ as the r-Apollonius circle.
- $\perp$ as "perpendicular"
- $\mathbf{T}P$ for some point P is the pencil of lines through P.

Online Encyclopedias of points, lines, and figures mentioned in this text are as follows:

- Kimberling's Encyclopedia of Triangle Centers here.
- Bernard Gilbert's Catalogue of Triangle Cubics here.
- Chris van Tienhoven's Encyclopedia of Quadri-figures here.

Definitions of vocabulary and more results related to isoconjugations and isocubics can all be found in Special Isocubics in the Triangle Plane.



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Part I

The Basics

Chapter 0

Lengths and Angles

Two points define a **length** between them, and two lines define an **angle** between them. Here, we assume that lengths and angles are preserved under rotations, translations, and reflections.

0.1 Directed Angles and Line Arguments

♠: In this section, assume that all lines are not the line at infinity $\mathcal{L}_\infty$.

With different geometric diagrams of the same problem, points that lie inside an angle in one diagram can lie outside it in another. These are popularly called "config issues." For example:

Example 0.1.1. If $\angle BAC = 20°, \angle BAD = 40°$, what is angle $\angle CAD$?

Some will answer $20°$, while others will say $60°$, because we don't know whether point C lies inside $\angle BAD$. A common way to resolve this ambiguity is through directed angles.

Definition 0.1.2 (Directed Angles)**.** For two rays / directed lines $\vec{\ell_1}, \vec{\ell_2}$ (where ℓ_1, ℓ_2 represent, respectively, the undirected lines), we define the **directed angle** $\measuredangle(\ell_1, \ell_2)$ between $\vec{\ell_1}, \vec{\ell_2}$ as

(i) If ℓ_1, ℓ_2 aren't parallel, or $\ell_1 \cap \ell_2 \notin \mathcal{L}_\infty$, [1] then it is the amount needed to rotate ℓ_1 counterclockwise to ℓ_2.

(ii) If they are parallel, then it is $0°$ if they face the same direction and else $180°$

For two vectors $\vec{v_1}, \vec{v_2}$, we define the angle $\measuredangle(v_1, v_2)$ between $\vec{v_1}, \vec{v_2}$ by the directed angle formed by taking the vectors' corresponding directed lines; we undirect the lines by taking all angles modulo $180° = \pi$. In other words, $\measuredangle(\ell_1, \ell_2) \equiv \measuredangle(\vec{\ell_1}, \vec{\ell_2}) \pmod{\pi}$.

[1] Here, we use $\ell_1 \cap \ell_2$ as the intersection point of ℓ_1 and ℓ_2.

Definition 0.1.3 (Parallelism and Perpendicularity)**.** Let ℓ_1, ℓ_2 be two lines.

(i) If $\measuredangle(\ell_1, \ell_2) = 0°$, then we denote ℓ_1, ℓ_2 as **parallel**, and we represent this as $\ell_1 \parallel \ell_2$.

(ii) If $\measuredangle(\ell_1, \ell_2) = 90°$, then we denote ℓ_1, ℓ_2 as **perpendicular**, and we represent this as $\ell_1 \perp \ell_2$.

Proposition 0.1.4. Given any three directed lines $\vec{\ell_1}, \vec{\ell_2}, \vec{\ell_3}$, we have that

$$\measuredangle(\vec{\ell_1}, \vec{\ell_2}) + \measuredangle(\vec{\ell_2}, \vec{\ell_3}) = \measuredangle(\vec{\ell_1}, \vec{\ell_3}).$$

Proof. Check the original book for a more in depth proof, but the idea is case work based on which lines are parallel. □

Definition 0.1.5. Given any three points A, B, O, define

$$\measuredangle AOB = \measuredangle(\overrightarrow{OA}, \overrightarrow{OB})$$

Now, let us revisit our beginning example with directed angles.

Example 0.1.6. If $\measuredangle BAC = 20°, \measuredangle BAD = 40°$, what is angle $\measuredangle CAD$?

We can answer this now as

$$\measuredangle CAD = \measuredangle BAD - \measuredangle BAC = 20°.$$

Now, let us reformulate all our previous angle facts for undirected angles in terms of directed angles.

Proposition 0.1.7. Let P be a point and let A, B, C be points on a line such that $A \neq P$. Then $\measuredangle PAB = \measuredangle PAC$.

Proof. Follows since $AB \parallel AC$. □

Proposition 0.1.8. If three points A, B, C satisfy $\overline{CA} = \overline{AB}$, then we have

$$\measuredangle CBA = \measuredangle ACB \pmod{360°}$$

and A lies on the perpendicular bisector of BC.

Proof. See original proof for more details but this is effectively just isosceles triangle and considering angle bisector. □

(This proposition has a converse, namely A lying on the perpendicular bisector on BC / angle condition with nonzero angle imply the other two.)

Proposition 0.1.9. Given a fixed $\triangle ABC$, we consider the three perpendicular bisectors ℓ_A, ℓ_B, ℓ_C of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. These lines then concur at a point called the **circumcenter** O of $\triangle ABC$. O is also the center of the unique circle through A, B, and C, called the **circumcircle**.

Proof. Note that no three of the perpendicular bisectors which are parallel. Let O be the intersection of ℓ_B and ℓ_C, then $\overline{OC} = \overline{OA}$ and $\overline{OA} = \overline{OB}$, so $\overline{OB} = \overline{OC}$ by the transitive property. As such, by Proposition 0.1.8 to get that $O \in \ell_A$, and thus the three perpendicular bisectors of a triangle concur.

Since we have that $\overline{OA} = \overline{OB} = \overline{OC}$, this implies that there is a unique circle centered at O through A, B, and C. Conversely, the center of a circle through A, B, and C lies on ℓ_A, ℓ_B, and ℓ_C, and thus must be O. □

Proposition 0.1.10. Let O be the circumcenter of $\triangle ABC$. Then $\measuredangle BAC + \measuredangle CBO = 90^\circ$.

Proof. By Proposition 0.1.8, we have that

$$\begin{aligned} 2 \cdot \measuredangle BAC &= \measuredangle BAC + \measuredangle BAO + \measuredangle OAC = \measuredangle BAC + \measuredangle OBA + \measuredangle ACO \\ &= \measuredangle OBC + \measuredangle BCO + 180^\circ = 2 \cdot \measuredangle OBC + 180^\circ \pmod{360^\circ} \end{aligned}$$

Dividing by two finishes. □

Example 0.1.11. Let O be the circumcenter of $\triangle ABC$. Let $\ell = A\infty_{\perp BC}$ be the line going through A perpendicular to line BC. Then we have

$$\measuredangle(AB, \ell) + \measuredangle(AC, AO) = \measuredangle BAO + \measuredangle(AC, \ell) = (90^\circ - \measuredangle ACB) + (90^\circ + \measuredangle ACB) = 0^\circ,$$

so we have that $\measuredangle(AB, \ell) = \measuredangle(AO, AC)$.
In general, call any pair of lines (ℓ_1, ℓ_2) **isogonal lines** in $\angle A$ if $\measuredangle(AB, \ell_1) = \measuredangle(\ell_2, AC)$.

Proposition 0.1.12. Given any four points A, B, P, and Q, there is a circle passing through all four points A, B, P, Q if and only if

$$\measuredangle APB = \measuredangle AQB.$$

We call such a quadrilateral **cyclic**. [2] We call the points A, B, P, and Q as being **concyclic**.

Proof. Assume no three of $\{A, B, P, Q\}$ lie on one line. We seek to prove that $\triangle PAB$ and $\triangle QAB$ have the same circumcenters, denote these respectively as O_P and O_Q. Note that both of these points lie on

[2] We will see later on in 6.A that this is equivalent to A, B, P, and Q being conconic with two fixed points at infinity.

the perpendicular bisector of $\overline{AB}$. Thus our condition is equivalent to proving $\measuredangle BAO_P = \measuredangle BAO_Q$. By Proposition 0.1.10, we get

$$\measuredangle APB + \measuredangle BAO_P = 90^\circ = \measuredangle AQB + \measuredangle BAO_Q,$$

so our statement is equivalent to $\measuredangle APB = \measuredangle AQB$. $\square$

The above proposition tells us that given two points A, B and a fixed angle θ, then the locus of points P such that $\measuredangle APB = \theta$ lie on a circle. In other words, $\{P \mid \measuredangle APB = \theta) = \Gamma \cap \{A, B\}$ for some circle Γ through A and B.

When $\theta = 90^\circ$, we get by Proposition 0.1.10 that $\measuredangle BAO = 90^\circ - \measuredangle APB = 0^\circ$, so O lies on AB. We call this circle (AB), where AB is a **diameter** of said circle.

Often when angle-chasing in a cyclic quadrilateral configuration, we use the concept of antiparallel lines.

Definition 0.1.13. We say that a pair of lines are **antiparallel** (L_1, L_2) with respect to (ℓ_1, ℓ_2) if

$$\measuredangle(L_1, \ell_1) + \measuredangle(L_2, \ell_2) = 0^\circ.$$

This is equivalent to $L_1 \cap \ell_1, L_1 \cap \ell_2, L_2 \cap \ell_1, L_2 \cap \ell_2$ lying on a circle.

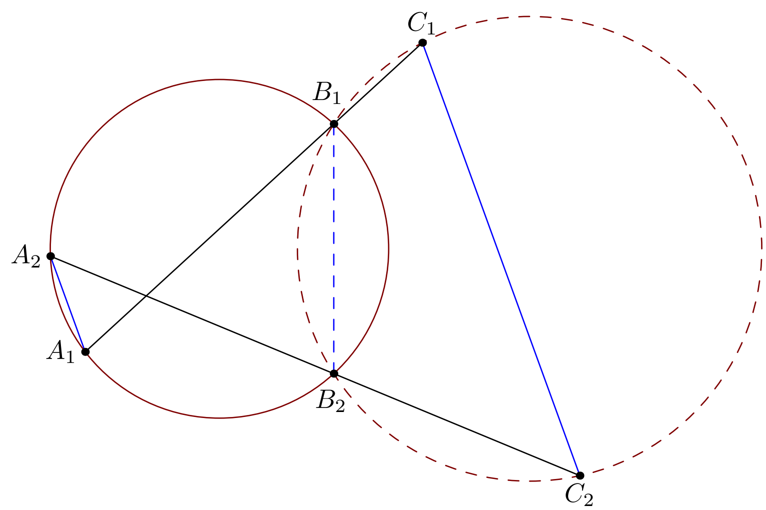
We can check that this concept is symmetric in (L_1, L_2), and also forms a **equivalence relation** $(L_1, L_2) \succ (\ell_1, \ell_2)$ on pairs of lines.

(i) $(\ell_1, \ell_1) \succ (\ell_1, \ell_2)$ follows immediately.

(ii) $L \succ K \iff K \succ L$ follows by symmetry.

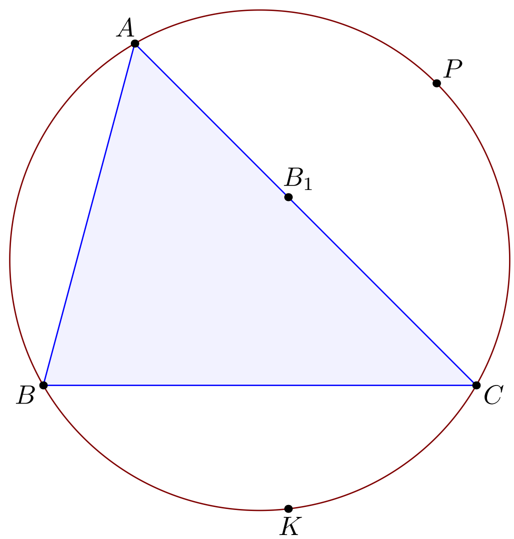
(iii) $A \succ B, B \succ C \implies A \succ C$ follows by angle chasing.

Example 0.1.14 (Reim's Theorem)**.** Let A_1, A_2, B_1, and B_2 be concyclic, let C_1 be a point on the line A_1B_1. Prove that $(B_1B_2C_1)$, A_2B_2, and the line through C_1 parallel to A_1A_2 are concurrent.



Proof. Define C_2 on A_2B_2 such that $C_1C_2 \parallel A_1A_2$. Then since (A_1A_2, B_1B_2) and (A_1B_1, A_2B_2) are antiparallel, it follows that (C_1C_2, B_1B_2) and (B_1C_1, B_2C_2) are antiparallel, so B_1, B_2, C_1, C_2 are concyclic. $\square$

Example 0.1.15. Let Γ be the circumcircle of an acute-angled triangle $\triangle ABC$. Draw the angle bisector of $\angle ABC$, and let this line intersect segment AC at point B_1, let this line intersect arc AC at point P. Draw the perpendicular line to $\overline{BC}$ from B_1 and let this line intersect arc BC at point K. Draw the perpendicular line to AK through B, let this intersect AC at point L. Prove that K, L, and P collinear.



Proof. Instead of proving that K, L, P collinear, we instead prove that $\measuredangle AKL = \measuredangle AKP$. Furthermore, we also know that

$$\measuredangle AKP = \measuredangle ABP = \measuredangle PBC = \measuredangle B_1BC = 90^\circ - \measuredangle KB_1B$$

and also $\measuredangle AKL = 90^\circ - \measuredangle KLB$. Thus we just need to prove $\measuredangle KLB = \measuredangle KB_1B$, or just B, K, L, B_1 concyclic.

Proving B, K, L, B_1 concyclic is equivalent to proving $\measuredangle KB_1L = \measuredangle KBL$, so we just calculate

$$\measuredangle KB_1L = 90^\circ - \measuredangle ACB = 90^\circ - \measuredangle AKB = \measuredangle KBL,$$

and we are done. □

Theorem 0.1.16 (Triangle Miquel (known in Chinese as "three circle theorem"))**.** Given a fixed triangle $\triangle ABC$, let E, F be points on segments CA, AB. Let D be an arbitrary point. Then D lies on segment BC iff $(EAF), (FBD), (DCE)$ are concurrent. (Furthermore, given a triangle $\triangle DEF$ with vertices on the three sides of ABC, denote this point of concurrency as the **Miquel point**.)

Proof. Let P be the intersection point of (FBD) and (DCE) that's not D. We want to prove that P being located on (EAF) is equivalent to point D lying on BC. We proceed with directed angles, we want to show

$$\measuredangle EAF = \measuredangle EPF \iff \measuredangle BDC = 0^\circ.$$

Expand out and get

$$\measuredangle BDC = \measuredangle BDP + \measuredangle PDC = \measuredangle BFP + \measuredangle PEC = \measuredangle FAE + \measuredangle EFP,$$

thus,

$$\measuredangle EAF = \measuredangle EPF \iff \measuredangle BDC = \measuredangle FAE + \measuredangle EFP = 0^\circ.$$

□

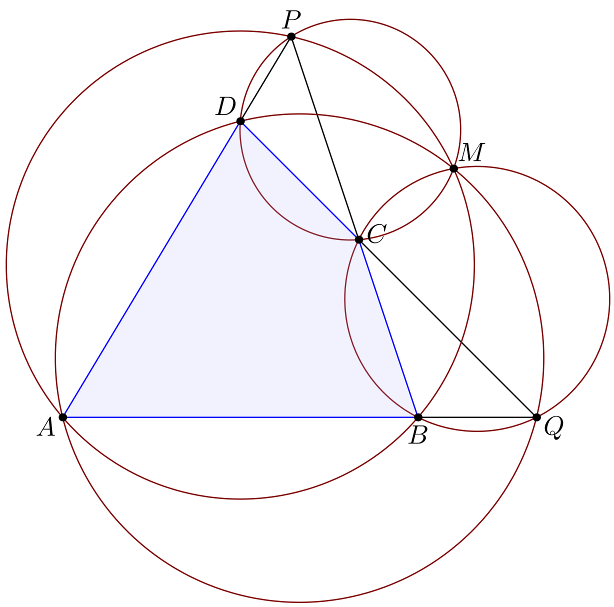
Example 0.1.17. Let $\triangle M_aM_bM_c$ be the **medial triangle** of $\triangle ABC$ (M_a, M_b, M_c are the midpoints of their respective sides). Prove that the Miquel point of $\triangle M_aM_bM_c$ is the circumcenter of $\triangle ABC$.

Note: typically it's hard to characterize the Miquel point of DEF wrt. ABC. It's very hard to characterize the angle $\measuredangle BAM$ given $\triangle DEF$ and $\triangle ABC$.

Proof. It's equivalent to show that O lies on (AM_AM_B) which finishes by symmetry. Then note that $\measuredangle AM_AO = \measuredangle AM_BO = 90^\circ$ which finishes. □

Now consider the above theorem when D, E, F are collinear and lie on AB, BC, CA. Then we get that $(AEF), (DEC), (BDF)$ have a Miquel point. However, since B, D, C are also collinear and lie on AF, FE, AE, it follows that (ABC) also goes through this point. We get the following.

Theorem 0.1.18 (Miquel's Theorem)**.** Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines such that no 3 are concurrent. Then the circumcircles of the four triangles formed by the intersections of these four lines are concurrent at a point called the **Miquel point** with respect to the four lines / that quadrilateral.



The Miquel point is the most important point of a quadrilateral, and we will further address this in Chapter 4.

Definition 0.1.19. Given two triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$,

(i) We call $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ to be **directly similar** if all the corresponding directed angles are equal, and **inversely similar** if the corresponding directed angles have opposite sign.

Denote this as

$$\triangle A_1B_1C_1 \overset{+}{\sim} \triangle A_2B_2C_2, \triangle A_1B_1C_1 \overset{-}{\sim} \triangle A_2B_2C_2$$

respectively.

(ii) We call $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ **directly congruent** if they are directly similar and corresponding side lengths are equal, and **inversely congruent** if they are inversely similar and have equal side lengths.

Denote this as

$$\triangle A_1B_1C_1 \overset{+}{\cong} \triangle A_2B_2C_2, \triangle A_1B_1C_1 \overset{-}{\cong} \triangle A_2B_2C_2$$

respectively.

(iii) Finally, we can just call $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ to be **similar** or **congruent** if we don't care about directed angles, denoted as $\sim$ and $\cong$ respectively.

Proposition 0.1.20. If $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are similar, we have

$$\frac{\overline{B_1C_1}}{\overline{B_2C_2}} = \frac{\overline{C_1A_1}}{\overline{C_2A_2}} = \frac{\overline{A_1B_1}}{\overline{A_2B_2}}.$$

To prove this we will first prove a sub-lemma:

Lemma 0.1.21. If A_1, B_1, C_1 and A_2, B_2, C_2 lie on two different lines ℓ_1 and ℓ_2 respectively, and $A_1A_2 \parallel B_1B_2 \parallel C_1C_2$, then

$$\frac{A_1B_1}{B_1C_1} = \frac{A_2B_2}{B_2C_2}.$$

Note that in this lemma we can exchange some of these parallelisms with concurrencies: for example if $A_1 = A_2$ and $B_1B_2 \parallel C_1C_2$ then the lemma stll holds.

Proof. Since

$$\frac{A_1B_1}{B_1C_1} = \frac{A_2B_2}{B_2C_2} \iff \frac{A_1C_1}{C_1B_1} = \frac{A_2B_2}{B_2C_2}$$

we can "swap the sequence" of A, B, C.

By the fact that $A_1A_2 \parallel B_1B_2$, we get

$$[\triangle A_1B_1B_2] = [\triangle A_2B_1B_2],$$

(brackets represent the (signed) area of the triangle). Similarly, we also have that

$$[\triangle C_1B_1B_2] = [\triangle C_2B_1B_2],$$

and thus

$$\frac{A_1B_1}{B_1C_1} = \frac{[\triangle A_1B_1B_2]}{[\triangle B_1C_1B_2]} = \frac{[\triangle A_2B_1B_2]}{[\triangle B_1C_2B_2]} = \frac{A_2B_2}{B_2C_2}$$

□

Proof of (0.1.20). Suppose $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ were inversely similar. Then just reflect A_2 over B_2C_2 to get $\triangle A_1B_1C_1$ and $\triangle A_2'B_2C_2$ are directly similar.

Thus we can assume WLOG that $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are directly similar.

Let us first look at the translation sending A_2 to A_1. Let this translation send $\triangle A_2B_2C_2$ to $\triangle A_1B_2'C_2'$. (Now points A_2, A_1 are the same so we call them both A). Now take a rotation of $AB_2'C_2'$ around A such that $\overline{AB_2'}$ becomes $\overline{AB_1}$. Since they are directly similar we have that $\measuredangle B_1AC_1 = \measuredangle B_2AC_2$, so this rotation also sends $\overline{A_1C_2'}$ to $\overline{A_1C_1}$. Thus we now have A, B_1, B_2' collinear and $A, C_1.C_2'$ collinear. Also note that

$$\measuredangle(B_1C_1, B_2'C_2') = \measuredangle C_1B_1A - \measuredangle C_2'B_2'A = 0^\circ$$

so we know that $B_1C_1 \parallel B_2'C_2'$. Now we can use our previous lemma! We get

$$\frac{C_1A}{C_2'A} = \frac{B_1A}{B_2'A} = \frac{AB_1}{AB_2'}$$

Doing this for each other vertex finishes.

□

0.1.1 Further Analysis Into Line Arguments

When we calculate angles, we're actually just calculating the difference between the "arguments" of two lines (arc-tangent of slopes). So sometimes when we have an "inaccessible" angle in a problem, we can instead just look at the two lines that form the angle. If we know the arguments of these lines, then we can just subtract them to get the "inaccessible" angle.

(Note that this also implies usage of line arguments is equivalent to drawing a bunch of parallel lines in your problem.)

Here's some examples of what I mean:

(i) Instead of $\measuredangle(\ell_1, \ell_2)$, we can express this in line arguments as $\measuredangle\ell_2 - \measuredangle\ell_1$.

(ii) Instead of $\measuredangle(\ell_1, \ell_2) + \measuredangle(\ell_2, \ell_3) = \measuredangle(\ell_1, \ell_3)$, this becomes $(\measuredangle\ell_2 - \measuredangle\ell_1) + (\measuredangle\ell_3 - \measuredangle\ell_2) = \measuredangle\ell_3 - \measuredangle\ell_1$.

(iii) Two lines are parallel iff they have the same line arguments, or $\ell_1 \parallel \ell_2 \iff \measuredangle\ell_1 = \measuredangle\ell_2$.

(iv) Two lines (L_1, L_2) are antiparallel with respect to (ℓ_1, ℓ_2) iff.

$$(\measuredangle\ell_1 + \measuredangle\ell_2) - (\measuredangle L_1 + \measuredangle L_2) = \measuredangle(L_1, \ell_1) + \measuredangle(L_2, \ell_2) = 0.$$

or equivalently $\measuredangle L_1 + \measuredangle L_2 = \measuredangle\ell_1 + \measuredangle\ell_2$. (This also will give a condition for finding cyclic quadrilaterals, which we will address later.)

This might feel non-well-defined at first, as it is arbitrary what line we set as the "zero" when setting line arguments. This is actually not a problem, since every expression we get in line arguments is homogeneous in addition

Why is this useful? This lets us move away from the world of geometry and enter into the world of addition and subtraction (modulo $180°$, note that $\measuredangle\ell = \measuredangle\ell + 180°$). Previously when we were only looking at directed angles, we were really only considering the differences between line arguments, which makes angle chasing a lot more artificially difficult and harder to see.

We use the shorthand $\perp \ell = \measuredangle\ell + 90°$ as well. We also use $\overrightarrow{\ell}$ when we want to do things mod $360°$.

We can now expand our definition for directed angles, by defining

$$\begin{aligned}\measuredangle(L_1 + \cdots + L_n, \ell_1 + \cdots + \ell_n) &:= (\ell_1 + \cdots + \ell_n) - (L_1 + \cdots + L_n)\\ &= \measuredangle(L_1, \ell_1) + \cdots + \measuredangle(L_n, \ell_n)\end{aligned}$$

and noting that swapping ℓ_i, ℓ_j and swapping L_i, L_j does not change this directed angle.

Now, the most powerful use of line arguments is in cyclic quadrilaterals. We know that if A, B, P, Q are cyclic, then $\measuredangle APB = \measuredangle AQB$. Symbolically, this is just

$$\measuredangle PB - \measuredangle PA = \measuredangle QB - \measuredangle QA.$$

(This can also be shown from antiparallelism).

Now, by repeating this in fact we actually also have

$$\measuredangle AP + \measuredangle BQ = \measuredangle AQ + \measuredangle BP$$

$$\measuredangle AP + \measuredangle BQ = \measuredangle AB + \measuredangle PQ$$

by further exchanging.

By merging this all into one, we have that

$$\measuredangle AB + \measuredangle PQ = \measuredangle AP + \measuredangle BQ = \measuredangle AQ + \measuredangle BP$$

This is the power of line arguments. We have three equalities from just a simple cyclic quadrilateral! In fact, proving any of these equalities is equivalent to proving A, B, P, Q cyclic.

We use the notation $(A + B + P + Q)_\Gamma$ to represent how every pair has the same sum. In general, $(A + C)_\Gamma = \measuredangle AC$ and $(A - C)_\Gamma = A_\Gamma - C_\Gamma$. These will be used somewhat irregularly but the above guarantees that all the operations are well defined.

Let's give another example. Let's convert the proof of Miquel's theorem to a proof with line arguments. We want to prove that D lies on BC iff $(EAF), (FBD), (DCE)$ concyclic, where E, F are known to lie on

CA and AB respectively.

Let $P \neq D$ be the second intersection of (FBD) and (DCE). Then we have

$$\measuredangle PE = \measuredangle PD + \measuredangle CE - \measuredangle CD = \measuredangle PD + \measuredangle AE - \measuredangle CD,$$

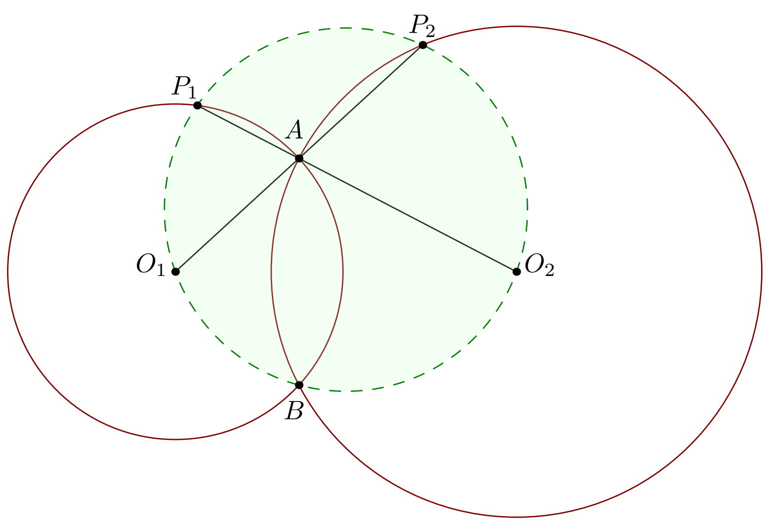
$$\measuredangle PF = \measuredangle PD + \measuredangle BF - \measuredangle BD = \measuredangle PD + \measuredangle AF - \measuredangle BD.$$

And D lying on BC is equivalent to the argument condition of $BD = CD$. So this lets us just subtract the two above equations and get that

$$\measuredangle PE - \measuredangle PF = \measuredangle PD - \measuredangle PD + \measuredangle AE - \measuredangle AF - \measuredangle CD + \measuredangle BD = \measuredangle AE - \measuredangle AF$$

so A, P, E, F are cyclic, we are done. The converse is similar.

Example 0.1.22. Let O_1, O_2 respectively be the two centers of circles c_1 and c_2, and let c_1 and c_2 intersect at points A and B. Additionally, let O_2 be outside c_1 and similarly for O_1. Let line O_1A intersect c_2 at point $P_2 \neq A$, let line O_2A intersect c_1 at point $P_1 \neq A$. Prove that O_1, O_2, P_1, P_2, B are concyclic.



Solution. We use line arguments. Note that since $\triangle O_1AP_1$ and $\triangle O_2AP_2$ are isosceles,

$$\measuredangle O_1P_1 + \measuredangle O_2P_2 = (2\measuredangle AP_1 - \measuredangle O_1A) + (2\measuredangle AP_2 - \measuredangle O_2A) = \measuredangle O_1P_2 + \measuredangle O_2P_1$$

so O_1, O_2, P_1, P_2 are cyclic.

Now, by Proposition 0.1.10, we get that

$$\begin{aligned}\measuredangle BP_1 - \measuredangle BP_2 &= (90^\circ - (\measuredangle AB + \measuredangle AP_1 - \measuredangle AO_1)) - (90^\circ - (\measuredangle AB + \measuredangle AP_2 - \measuredangle AO_2)) \\ &= (2\measuredangle AP_2 - \measuredangle AO_1) - \measuredangle O_1P_2 = \measuredangle O_1P_1 - \measuredangle O_1P_2\end{aligned}$$

so O_1, B, P_1, P_2 is cyclic. □

A nice example of a usage of line arguments to angle-chase effectively is in the proof of Steiner's Orthocenter Theorem.

The rest of this section's contents will be a rigorous definition of line arguments, feel free to skip it. We will look at the addition group

$$\mathcal{A} = \bigoplus_{\ell} \mathbb{Z}\ell := \left\{ \sum_{i=1}^{k} n_i \ell_i \,\middle|\, k \in \mathbb{N} \cup \{0\}, n_i \in \mathbb{Z}, \ell_i \text{ is a line} \right\}$$

with its addition defined as the natural

$$\sum_{i=1}^{k} n_i \ell_i + \sum_{i=1}^{k} m_i \ell_i = \sum_{i=1}^{k} (n_i + m_i) \ell_i.$$

Given a line ℓ_0, we define the homomorphism ϵ

$$\begin{aligned} \epsilon : \qquad \mathcal{A} &\longrightarrow \mathbb{Z} \\ \sum_{i=1}^{k} n_i \ell_i &\longmapsto \sum_{i=1}^{k} n_i \end{aligned}$$

and this homomorphism has a kernel $\mathcal{K} = \ker \epsilon = \{\sum n_i \ell_i \mid \sum n_i = 0\}$ spanned by $\ell - \ell_0$, as such

$$\sum n_i \ell_i = \sum n_i (\ell_i - \ell_0) \in \mathcal{K} = \bigoplus_{\ell \neq \ell_0} \mathbb{Z}(\ell - \ell_0)$$

And now we can define directed angles as taking a mapping mod 180.

$$\begin{aligned} \measuredangle : \qquad \mathcal{K} &\longrightarrow \mathbb{R}^\circ \big/ 180^\circ \\ \sum_{i=1}^{k} n_i (\ell_i - \ell_0) &\mapsto \sum_{i=1}^{k} n_i \cdot \measuredangle(\ell_0, \ell_i). \end{aligned}$$

By Proposition 0.1.4 we know that

$$(\ell_2 - \ell_1) = (\ell_2 - \ell_0) - (\ell_1 - \ell_0) \overset{\measuredangle}{\mapsto} \measuredangle(\ell_0, \ell_2) - \measuredangle(\ell_0, \ell_1) = \measuredangle(\ell_1, \ell_2),$$

or in other words, "addition" is preserved. We can then do our math completely in $\mathcal{K}$ and then convert to actual angles to finish.

As such, $\measuredangle$ is a homomorphism as well. This allows further abuse notation such as the following.

$$\begin{aligned}\measuredangle(\ell_1 + \cdots + \ell_k, L_1 + \cdots + L_k) &= \measuredangle((L_1 + \cdots + L_k) - (\ell_1 + \cdots + \ell_k)) \\ &= \measuredangle(\ell_1, L_1) + \cdots + \measuredangle(\ell_k + L_k).\end{aligned}$$

As seen previously, we can swap any two lines in this and it won't change the directed angles, so we can write the following "equality":

$$L_1 + \cdots + L_k - (\ell_1 + \cdots + \ell_k) = \measuredangle(\ell_1 + \cdots + \ell_k, L_1 + \cdots + L_k)$$

and if $\measuredangle(\ell_1 + \cdots + \ell_k, L_1 + \cdots + L_k) = 0$, we can define this equivalence relation:

$$L_1 + \cdots + L_k = \ell_1 + \cdots + \ell_k.$$

Note that all of this stuff holds for directed lines (i.e. rays) if we work in $\mathbb{R}^\circ/360^\circ$ instead of $\mathbb{R}^\circ/180^\circ$.

Practice Problems

Problem 1 (Orthocenter)**.** Let D be the foot from A to $\overline{BC}$ and defne E and F analogously.

(i) Prove that B, C, E, and F are concyclic. Then similarly, C, A, F, D and A, B, D, E are concyclic.

(ii) Let H be the Miquel Point of $\triangle ABC$ with respect to $\triangle DEF$. Prove that A, H, D are collinear. Similarly, B, H, E and C, H, F are collinear.

Hence the three altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ share a point H. We call H the **orthocenter** of $\triangle ABC$, and $\triangle DEF$ the **orthic triangle** of $\triangle ABC$. Note that A, B, C are also the orthocenters of $\triangle BHC$, $\triangle CHA$, $\triangle AHB$ respectively, so we can also define the four points (A, B, C, H) to be an **orthocentric system**.

(iii) Prove that $\angle BHC = 180^\circ - \angle BAC$. Thus if H_A is the reflection of H about $\overline{BC}$, then H_A lies on (ABC).

Problem 2. Let A, B, C, D be four concyclic points. Define A', C' as the foots of the altitudes from A, C onto $\overline{BD}$, respectively. Similarly, let B', D' be the foots of the altitudes from B and D onto $\overline{AC}$, respectively. Prove that the points A', B', C', D' are concyclic.

Problem 3 (Miquel's Pentagon Theorem, Five Circle Theorem in Chinese)**.** Let $ABCDE$ be a convex pentagon, with $A' = \overline{BC} \cap \overline{DE}$, and B', C', D', E' defined analogously. Let A'' be the intersection of (EAC') and (ABD') other than A, and define B'', C'', D'', E'' analogously. Prove that A'', B'', C'', D'', E'' are concyclic.

Problem 4. Let E and F be two points on the side $\overline{BC}$ of the convex quadrilateral $ABCD$ such that B, E, F, C are on line $\overline{BC}$ in that order. Given that $\angle BAE = \angle FDC$ and $\angle EAF = \angle EDF$, prove that $\angle FAC = \angle BDE$.

Problem 5 (APMO 2010/1)**.** Let ABC be a triangle with $\angle BAC \neq 90^\circ$. Let O be the circumcenter of the triangle ABC and Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be the diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

Problem 6 (ISL 2007 G2)**.** Let M be the midpoint of side BC in an isosceles triangle $\triangle ABC$ with $AC = AB$. Let X be a point on minor arc MA of (AMB). Let T be a point inside of $\angle BMA$ such that $\angle TMX = 90^\circ$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on the choice of X.

Problem 7. Let $\triangle ABC$ be an acute triangle where $AB < AC$. Let D, E respectively be the midpoints of segments $\overline{CA}, \overline{AB}$, and let P be the second intersection of (ADE) and (BCD), similarly let Q be the second intersection of (ADE) and (BCE). Prove $\overline{AP} = \overline{AQ}$.

Problem 8. Let ABC be a triangle with circumcenter O and circumcircle ω. Let D be a point on segment $\overline{BC}$ such that $\angle BAD = \angle OAC$. Let $E = \overline{AD} \cap \omega$. Let M, N, and P be the midpoints of $\overline{BE}$, $\overline{OD}$, $\overline{AC}$ respectively. Prove that M, N, P are collinear.

0.2 Trig and Vector Bashing

Definition 0.2.1. Given an angle $\theta \in (0^\circ, 90^\circ)$, consider $\triangle ABC$ with $\angle ABC = 90^\circ$ and $\angle BAC = \theta \pmod{360^\circ}$. We define

$$\sin\theta = \frac{\overline{BC}}{\overline{CA}} \quad \text{and} \quad \cos\theta = \frac{\overline{AB}}{\overline{CA}}.$$

Note that this definition is independent of the choice of $\triangle ABC$ (by Proposition 0.1.20). We extend this to any angle $\theta \pmod{360^\circ}$ by the following relations:

$$\cos 0^\circ = \sin(90^\circ) = 1,$$
$$\sin\theta = \sin(180^\circ - \theta) = -\sin(-\theta),$$
$$\cos\theta = -\cos 180^\circ - \theta = \cos(-\theta).$$

In addition, define

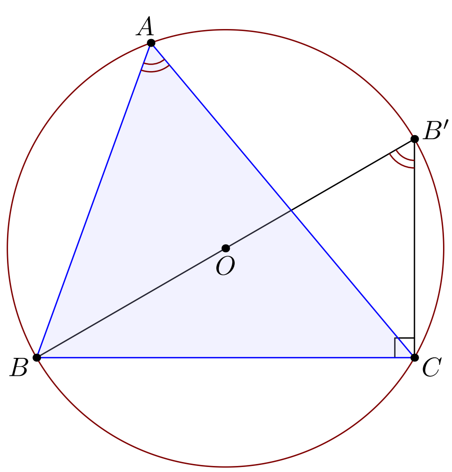
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \in \mathbb{R} \cup \{\infty\}.$$

Under this definition, $|\sin\theta|$ is well-defined for an undirected angle, and $\tan\theta$ is well defined for directed angles $\pmod{180^\circ}$. We can also check that

$$\sin\theta = \cos(90^\circ - \theta).$$

Proposition 0.2.2 (Law of Sines)**.** If the circumradius of ABC is R, then

$$\frac{BC}{|\sin\angle BAC|} = \frac{CA}{|\sin\angle CBA|} = \frac{AB}{|\sin\angle ACB|} = 2R.$$



Proof. Let B^* be the antipode of B on (ABC) - then $BB^* = 2R$ and

$$2R \cdot |\sin\angle BAC| = 2R \cdot |\sin\angle BB^*C| = BC.$$

□

Proposition 0.2.3 (Law of Cosines)**.** For a triangle $\triangle ABC$, let $a = \overline{BC}$ and analogously define b, c. Then

$$\cos \measuredangle BAC = \frac{b^2 + c^2 - a^2}{2bc}.$$

Proof. Let D be the foot from A onto BC. Let $\alpha = \measuredangle BAC \pmod{360^\circ}$ and define β and γ analogously. Then

$$a = |BD + DC| = c\cos\beta + b\cos\gamma \implies a^2 = ca\cos\beta + ab\cos\gamma.$$

Similarly, we have that

$$b^2 = ab\cos\gamma + bc\cos\alpha, \quad c^2 = bc\cos\alpha + ca\cos\beta.$$

Thus, combining these three equations gives the result. □

Corollary 0.2.4 (Pythagorean Theorem)**.** If $\triangle ABC$ satisfies $\angle BAD = 90^\circ$, then

$$BC^2 = CA^2 + AB^2.$$

We also have the identity $\sin^2\theta + \cos^2\theta = 1$.

In addition, we have

Corollary 0.2.5 (Triangle Inequality)**.** For any $\triangle ABC$,

$$BC < CA + AB.$$

Proof. Let $a = BC$ and analogously for b, c. Note that

$$\cos^2\alpha = 1 - \sin^2\alpha < 1$$

so

$$-1 \le \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \implies -2bc < b^2 + c^2 - a^2 \implies a^2 < (b+c)^2.$$

Taking the square root gives that $BC = a < b + c = CA + AB$. □

Since $\cos : (0^\circ, 180^\circ) \to \mathbb{R}$ is injective (see Problem 2 of this section), we can combine this with the Law of Cosines to obtain the converse of Proposition 0.1.20 - namely that if

$$\frac{B_1C_1}{B_2C_2} = \frac{C_1A_1}{C_2A_2} = \frac{A_1B_1}{A_2B_2}$$

then $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$.

By using Law of Sines, we also get the familiar AA and SAS similarity theorems.

Proposition 0.2.6. Let P be a point on circle Γ with center O. Then, a line ℓ passing through P is tangent to Γ if and only if $\ell \perp OP$.

Proof. For any point $Q \in \ell$, by Pythagoras, we have

$$OQ^2 = OP^2 + PQ^2 \geq OP^2 \implies OQ \geq OP,$$

where equality occurs if and only if $P = Q$. Since OP is the circle's radius, ℓ is tangent to Γ. $\square$

Using this allows us to relate the angle of the tangent line with an angle of a chord in the circle:

Proposition 0.2.7. Given $\triangle ABC$, a line L passing through A is tangent to (ABC) if and only if

$$\measuredangle(L, CA) = \measuredangle ABC.$$

Proof. Let O be the circumcenter of $\triangle ABC$. Then by Proposition 0.2.6, L is tangent to (ABC) if and only if $L \perp OA$. From Proposition 0.1.10,

$$\measuredangle OAC = \measuredangle ABC + 90^\circ$$

. Therefore, $L \perp OA$ if and only if $\measuredangle(L, CA) = \measuredangle ABC$. $\square$

Remark. This can also be expressed as the limiting case of antiparallelism: If L is a line through A on (ABC), then L is tangent if and only if $(L, BC) = (CA, AB)$. Using line arguments, this writes as

$$L + BC = \text{“}AA\text{”} + BC = AB + AC.$$

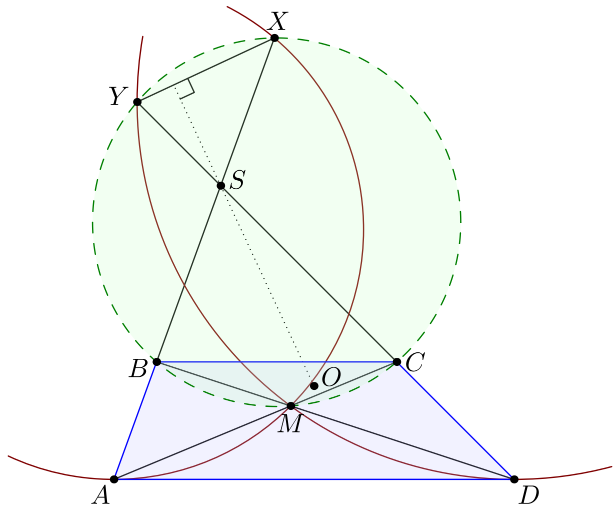
Let's see some applications.

Example 0.2.8 (Argentina 2011/3)**.** Let $ABCD$ be a trapezoid with bases $\overline{BC} \parallel \overline{AD}$, where $AD > BC$, and non-parallel legs $\overline{AB}$ and $\overline{CD}$. Let $M = \overline{AC} \cap \overline{BD}$. Let Γ_1 be a circle that passes through M and is tangent to $\overline{AD}$ at point A; let Γ_2 be a circle that passes through M and is tangent to $\overline{AD}$ at point D. Let $S = \overline{AB} \cap \overline{CD}$, $X \neq A$ be $\Gamma_1 \cap \overline{AS}$, $Y \neq D$ be $\Gamma_2 \cap \overline{DS}$, and O be the circumcenter of triangle ASD. Show that $SO \perp XY$.

Proof. From $\overline{BC} \parallel \overline{AD}$,

$$\measuredangle MXB = \measuredangle MXA = \measuredangle MAD = \measuredangle MCB$$

so X lies on (BCM).



Similarly, Y lies on (BCM). Now (BC, XY) are antiparallel about (SA, SD). By Example 0.1.11, $(SO, S\infty_{\perp AD})$ are antiparallel about (SA, SD). Hence (BC, XY) are antiparallel about $(SO, S\infty_{\perp AD})$. By using $\overline{BC} \parallel \overline{AD}$ again,

$$\measuredangle(SO, XY) = \measuredangle(BC, S\infty_{\perp AD}) = 90°.$$

□

Definition 0.2.9. For two vectors $\vec{v_1}$ and $\vec{v_2}$, we define the **dot product** as

$$\vec{v_1} \cdot \vec{v_2} = |\vec{v_1}| \cdot |\vec{v_2}| \cdot \cos \measuredangle(\vec{v_1}, \vec{v_2})$$

We also define the **outer product** as

$$\vec{v_1} \times \vec{v_2} = |\vec{v_1}| \cdot |\vec{v_2}| \cdot \sin \measuredangle(\vec{v_1}, \vec{v_2}).$$

This is equivalent to the magnitude of the cross product of two vectors in $\mathbb{R}^2$. We can also show that the following hold:

$$\vec{v_1} \cdot (a\vec{v_2}) = (a\vec{v_1}) \cdot \vec{v_2} = a(\vec{v_1} \cdot \vec{v_2}),$$
$$\vec{v_1} \times (a\vec{v_2}) = (a\vec{v_1}) \times \vec{v_2} = a(\vec{v_1} \times \vec{v_2}),$$

as well as the fact that $\vec{v_1} \cdot \vec{v_2} = \vec{v_2} \cdot \vec{v_1}, \vec{v_1} \times \vec{v_2} = -\vec{v_2} \times \vec{v_1}$. Finally, by definition, the dot product of perpendicular vectors and the cross product of parallel vectors are both 0.

Theorem 0.2.10. The dot and cross products are distributive, meaning for any vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, we have

$$\vec{v_1} \cdot (\vec{v_2} + \vec{v_3}) = \vec{v_1} \cdot \vec{v_2} + \vec{v_1} \cdot \vec{v_3}$$
$$\vec{v_1} \times (\vec{v_2} + \vec{v_3}) = \vec{v_1} \times \vec{v_2} + \vec{v_1} \times \vec{v_3}$$

Proposition 0.2.11 (Perpendicularity Criterion)**.** For four points A, B, C, D, $AB \perp CD$ holds iff $AC^2 - AD^2 = BC^2 - BD^2$.

Proof. Note that

$$\begin{aligned}(AC^2 - AD^2) - (BC^2 - BD^2) &= (AC + AD) \cdot (AC - AD) - (BC + BD) \cdot BC - BD) \\ &= -(AC + AD - BC - BD) \cdot -2AB \cdot CD = -2AB \cdot CD\end{aligned}$$

which implies the result. □

Proposition 0.2.12 (Sine Area Formula)**.** In any triangle $\triangle ABC$, the directed area is

$$[ABC] = \frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC} = \frac{1}{2}AB \cdot AC \cdot \sin\measuredangle BAC.$$

Proof. The core idea is that it just reduces to the right angle case, which is immediate.

Formally, if D is the foot from B to AC, then

$$\overrightarrow{AB} \times \overrightarrow{AC} = \overrightarrow{AB} \times \overrightarrow{AD} + \overrightarrow{AB} \times \overrightarrow{DC} = \overrightarrow{DA} \times \overrightarrow{DB} + \overrightarrow{DB} \times \overrightarrow{DC}$$

which reduces to the cases of $\triangle DAB, \triangle DBC$. Both of these cases can be easily checked. □

Notably, the directed area of a triangle is positive if it is oriented counterclockwise and else negative.

Corollary 0.2.13 (Heron's Formula)**.** For any $\triangle ABC$, let $a = \overline{BC}, b = \overline{CA}, c = \overline{AB}$. Let $s = \frac{1}{2}(a + b + c)$ be the semiperimeter. Then

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof. By Sine Area Formula, Law of Cosines, Pythagorean Theorem, we get that

$$\begin{aligned}|[\triangle ABC]| &= \frac{bc}{2}\cdot|\sin\measuredangle BAC|\,| = \frac{bc}{2}\sqrt{1-\cos^2\measuredangle BAC} = \frac{bc}{2}\cdot\sqrt{1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2}\\ &= \frac{1}{4}\cdot\sqrt{(2bc)^2-(b^2+c^2-a^2)^2} = \frac{1}{4}\cdot\sqrt{((b+c)^2-a^2)(a^2-(b-c)^2)}\\ &= \sqrt{s(s-a)(s-b)(s-c)}.\end{aligned}$$

□

Practice Problems

Problem 1 (Sum and Difference Formulas)**.** Show that for any two angles θ_1, θ_2, we have that

(i) $\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$

(ii) $\sin(\theta_1-\theta_2) = \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2$

(iii) $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

(iv) $\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$.

Problem 2. Use Problem 1 to show that $\cos : [0°, 90°] \to [0, 1]$ is strictly decreasing and hence bijective over $[0°, 180°]$.

Problem 3 (Sum and Product Formulas)**.** Show that for any two angles θ_1, θ_2, we have that

(i) $\sin\theta_1 \pm \sin\theta_2 = 2\sin(\frac{\theta_1}{2} \pm \frac{\theta_2}{2})\cos(\frac{\theta_1}{2} \mp \frac{\theta_2}{2})$.

(ii) $\cos\theta_1 + \cos\theta_2 = 2\cos(\frac{\theta_1}{2} + \frac{\theta_2}{2})\cos(\frac{\theta_1}{2} - \frac{\theta_2}{2})$.

(iii) $\cos\theta_1 - \cos\theta_2 = -2\sin(\frac{\theta_1}{2} + \frac{\theta_2}{2})\sin(\frac{\theta_1}{2} - \frac{\theta_2}{2})$

(iv) $\sin\theta_1\cdot\sin\theta_2 = \frac{1}{2}(\cos(\theta_1-\theta_2) - \cos(\theta_1+\theta_2))$

(v) $\sin\theta_1\cdot\cos\theta_2 = \frac{1}{2}(\sin(\theta_1+\theta_2) + \sin(\theta_1-\theta_2))$

(vi) $\cos\theta_1\cdot\cos\theta_2 = \frac{1}{2}(\cos(\theta_1-\theta_2) + \cos(\theta_1+\theta_2))$

Problem 4. Let H be the orthocenter of $\triangle ABC$. Show that $\triangle ABC, \triangle HBC, \triangle HAB, \triangle HAC$ have the same circumradius.

Problem 5 (Austria Federal 2019/2/5)**.** Let ABC be an acute-angled triangle. Let D and E be the feet of the altitudes on the sides $\overline{BC}$ and $\overline{AC}$, respectively. Points F and G are located on the lines AD and BE so that $\frac{AF}{FD} = \frac{BG}{GE}$. The line passing through C and F intersects $\overline{BE}$ at point H and the line passing through C and G intersects $\overline{AD}$ at point I. Prove that points F, G, H and I lie on a circle.

0.3 Length Bashing

Calculating lengths is a powerful tool (compared to calculating angles). Many questions boil down to calculating lengths, so let's see what things we can cook.

Theorem 0.3.1 (Menelaus)**.** Given any $\triangle ABC$ with points D, E, F on lines BC, CA, AB, respectively, we have that points D, E, F are collinear if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1.$$

(here, the ratio $\frac{AB}{BC}$ is negative if B lies outside AC)

Proof. Let $D' = EF \cap BC$ and let X be on BC such that $AX \parallel EF$.

Then we get that

$$\frac{CE}{EA} \cdot \frac{AF}{FB} = -\frac{CD'}{D'X} = \frac{D'X}{D'B} = -\frac{CD'}{BD'}$$

As such, $D = D'$ only holds iff $\frac{BD}{DC} = \frac{BD'}{D'C}$, which is equivalent to their product being -1. $\square$

Theorem 0.3.2 (Ceva)**.** Given any $\triangle ABC$ with points D, E, F on lines BC, CA, AB, respectively, we have that $\overline{AD}, \overline{BE}, \overline{CF}$, concur if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Proof. Let $P = BE \cap CF$ and $D' = AP \cap BC$. Then by Menelaus,

$$\frac{CE}{EA} \cdot \frac{AF}{FB} = \left(-\frac{CB}{BD'} \cdot \frac{D'P}{PA}\right) \cdot \left(-\frac{AP}{PD'} \cdot \frac{D'C}{CB}\right) = \frac{D'C}{BD'}.$$

We have that $D = D'$ iff $\frac{BD}{DC} = \frac{BD'}{D'C}$, which implies the result. $\square$

Example 0.3.3. Let $\triangle M_a M_b M_c$ be the medial triangle of $\triangle ABC$. Then

$$\frac{BM_a}{M_aC} \cdot \frac{CM_b}{M_bA} \cdot \frac{AM_b}{M_bB} = 1 \cdot 1 \cdot 1 = 1,$$

so we get that AM_a, BM_b, CM_c concur. We call this concurrence point the **centroid** of $\triangle ABC$.

Proposition 0.3.4 (Ratio Lemma)**.** Pick a point D on side BC of $\triangle ABC$, then

$$\frac{BD}{DC} = \frac{AB \cdot \sin\measuredangle BAD}{CA \cdot \sin\measuredangle DAC}.$$

Proof. By the Sine Area Formula, we get

$$\frac{BD}{DC} = \frac{[\triangle ABD]}{[\triangle ADC]} = \frac{\frac{1}{2} \cdot DA \cdot AB \cdot \sin \measuredangle BAD}{\frac{1}{2} \cdot CA \cdot AD \cdot \sin \measuredangle DAC} = \frac{AB \cdot \sin \measuredangle BAD}{CA \cdot \sin \measuredangle DAC}.$$

□

Note that the ratio

$$\frac{\sin \measuredangle BAD}{\sin \measuredangle DAC}$$

is the same regardless of the sign of the directed length $\overrightarrow{AD}$. So we can write the above ratio as

$$\frac{\sin \measuredangle(\overrightarrow{AB}, AD)}{\sin \measuredangle(AD, \overrightarrow{AC})}.$$

With this property, we can rewrite Menelaus's theorem in terms of angles.

Theorem 0.3.5 (Trig Menelaus)**.** Given a $\triangle ABC$ and lines d, e, f on BC, CA, AB respectively, and points $D = BC \cap d$,$E = BC \cap e$,$F = AB \cap F$, D, E, F are collinear iff

(i) It holds that
$$\frac{\sin \measuredangle(\overrightarrow{AB}, d)}{\sin \measuredangle(d, \overrightarrow{AC})} \cdot \frac{\sin \measuredangle(\overrightarrow{BC}, e)}{\sin \measuredangle(e, \overrightarrow{BA})} \cdot \frac{\sin \measuredangle(\overrightarrow{CA}, f)}{\sin \measuredangle(f, \overrightarrow{CB})} = -1.$$

(ii) For some fixed point P not on AB, BC, CA, we have that

$$\frac{\sin \measuredangle BPD}{\sin \measuredangle DPC} \cdot \frac{\sin \measuredangle CPE}{\sin \measuredangle EPA} \cdot \frac{\sin \measuredangle APF}{\sin \measuredangle FPB} = -1.$$

Proof. We reduce this to normal Trig Menelaus. Using Ratio Lemma, we get

$$\prod_{\text{cyc}} \frac{BD}{DC} = \prod_{\text{cyc}} \frac{\overline{AB} \cdot \sin \measuredangle(\overrightarrow{AB}, d)}{\overline{CA} \cdot \sin \measuredangle(d, \overrightarrow{AC})} = \prod_{\text{cyc}} \frac{\sin \measuredangle(\overline{AB}, d}{\sin \measuredangle(d, \overline{AC})}.$$

Similarly, we have that

$$\prod_{\text{cyc}} \frac{BD}{DC} = \prod_{\text{cyc}} \frac{\overline{PB} \cdot \sin \measuredangle BPD}{\overline{CP} \cdot \sin \measuredangle DPC} = \prod_{\text{cyc}} \frac{\sin \measuredangle BPD}{\sin \measuredangle DPC}.$$

□

Remark. You may have noticed that the contribution of the sign of AB, BA is irrelevant due to cancelling out.

So this really could also be written as

$$\prod \frac{\sin \measuredangle(AB,d)}{\sin \measuredangle(d,CA)} \cdot \frac{\sin \measuredangle(BC,e)}{\sin \measuredangle(e,AB)} \cdot \frac{\sin \measuredangle(CA,f)}{\sin \measuredangle(f,BC)} = -1.$$

Example 0.3.6. Given $\triangle ABC$, let the tangents to (ABC) at A, B, C intersect BC, CA, AB at X, Y, Z, respectively. Prove that X, Y, Z are collinear.

Solution. We investigate the quantity

$$\frac{\sin \measuredangle(\overrightarrow{AB}, AX)}{\sin \measuredangle(AX, \overrightarrow{AC}}$$

which finishes. Note that this quantity is always negative because AX lies outside AB, AC.

As such, using tangent, angle chasing, we get that

$$\frac{\sin \measuredangle(\overrightarrow{AB}, AX)}{\sin \measuredangle(AX, \overrightarrow{AC})} = -\left|\frac{\sin \measuredangle BAX}{\sin \measuredangle XAC}\right| = -\left|\frac{\sin \measuredangle BCA}{\sin \measuredangle ABC}\right|$$

so the cyclic product of the above cancels out to -1. ☐

Ceva also has a trig version.

Theorem 0.3.7 (Trig Ceva)**.** Given a $\triangle ABC$ and lines d, e, f through A, B, C respectively, then d, e, f are concurrent iff

(i) It holds that

$$\frac{\sin \measuredangle(\overrightarrow{AB}, d)}{\sin \measuredangle(d, \overrightarrow{AC})} \cdot \frac{\sin \measuredangle(\overrightarrow{BC}, e)}{\sin \measuredangle(e, \overrightarrow{BA})} \cdot \frac{\sin \measuredangle(\overrightarrow{CA}, f)}{\sin \measuredangle(f, \overrightarrow{CB})} = 1.$$

(ii) For some fixed point P not on AB, BC, CA, and points $D = BC \cap d$,$E = BC \cap e$,$F = AB \cap F$,

$$\frac{\sin \measuredangle BPD}{\sin \measuredangle DPC} \cdot \frac{\sin \measuredangle CPE}{\sin \measuredangle EPA} \cdot \frac{\sin \measuredangle APF}{\sin \measuredangle FPB} = 1.$$

Proof. It's the same argument as for Trig Menelaus, except you do it for Ceva. Apply ratio lemma again. ☐

Remark. Note that in the above proof of Menelaus we can see that we really only needed P, Q, R to satisfy

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1.$$

This lets us see a certain duality in Ceva and Menelaus's theorems, (which we will formalize later, in chapter 3), but for now we can notice that D, E, F collinear and AD, BE, CF concurrent are both conditions on

$$\frac{\sin \measuredangle BPD}{\sin \measuredangle DPC} \cdot \frac{\sin \measuredangle CPE}{\sin \measuredangle EPA} \cdot \frac{\sin \measuredangle APF}{\sin \measuredangle FPB} = \pm 1$$

just with a change of sign.

Example 0.3.8 (Ceva for Circles)**.** Let $AFBDCE$ be a convex hexagon inscribed in a circle. Then $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ concur if and only if

$$AF \cdot BD \cdot CE = EA \cdot FB \cdot DC.$$

Solution. By Trig Ceva, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur if and only if

$$\frac{\sin \measuredangle(\overrightarrow{AB}, \overline{AD})}{\sin \measuredangle(\overline{AD}, \overrightarrow{AC})} \cdot \frac{\sin \measuredangle(\overrightarrow{BC}, \overline{BE})}{\sin \measuredangle(\overline{BE}, \overrightarrow{BA})} \cdot \frac{\sin \measuredangle(\overrightarrow{CA}, \overline{CF})}{\sin \measuredangle(\overline{CF}, \overrightarrow{CB})} = 1.$$

Because $AFBDCE$ is convex, we know that each of these sines is positive, so by the Law of Sines we have

$$\frac{\sin \measuredangle(\overrightarrow{AB}, \overline{AD})}{\sin \measuredangle(\overline{AD}, \overrightarrow{AC})} = \frac{\sin \measuredangle BAD}{\sin \measuredangle BAC} = \frac{BD}{DC}.$$

Then $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1,$$

which is equivalent to the theorem statement. This theorem is relevant when we discuss perfect six-point sets in Section 8.4. □

Example 0.3.9 (Incenters and Excenters)**.** Let ℓ_A^+, ℓ_A^- be the interior and exterior angle bisectors of $\angle BAC$, i.e. the lines that satisfy

$$\measuredangle(\overrightarrow{AB}, \overrightarrow{\ell_A^+}) = \measuredangle(\overrightarrow{\ell_A^-}, \overrightarrow{AC}), \quad \measuredangle(\overrightarrow{AB}, \overrightarrow{\ell_A^+}) = \measuredangle(\overrightarrow{\ell_A^-}, \overrightarrow{AC}) + 180^\circ \pmod{360^\circ}.$$

Then these lines satisfy

$$\frac{\sin \measuredangle(\overrightarrow{AB}, \ell^+ -_A)}{\sin \measuredangle(\ell_A^+, \overrightarrow{AC})} = 1, \quad \frac{\sin \measuredangle(\overrightarrow{AB}, \ell_A^-)}{\sin \measuredangle(\ell_A^-, \overrightarrow{AC})} = -1.$$

If we define ℓ_B^+, ℓ_B^-, ℓ_C^+, ℓ_C^- in a similar fashion for the angles $\angle CBA$ and $\angle ACB$, respectively, then we have that

$$\frac{\sin \measuredangle(\overrightarrow{AB}, \ell_A^+)}{\sin \measuredangle(\ell_A^+, \overrightarrow{AC})} \cdot \frac{\sin \measuredangle(\overrightarrow{BC}, \ell_B^+)}{\sin \measuredangle(\ell_B^+, \overrightarrow{BA})} \cdot \frac{\sin \measuredangle(\overrightarrow{CA}, \ell_C^+)}{\sin \measuredangle(\ell_C^+, \overrightarrow{CB})} = 1 \cdot 1 \cdot 1 = 1,$$

so ℓ_A^+, ℓ_B^+, and ℓ_C^+ concur at a point I, which we call the **incenter** of $\triangle ABC$. Similarly, we obtain that the triples of lines (ℓ_A^+, ℓ_B^-, ℓ_C^-), (ℓ_A^-, ℓ_B^+, ℓ_C^-), (ℓ_A^-, ℓ_B^-, ℓ_C^+) concur at points I_A, I_B, I_C respectively, which we call the ***A*-excenter**, ***B*-excenter**, and ***C*-excenter** respectively.

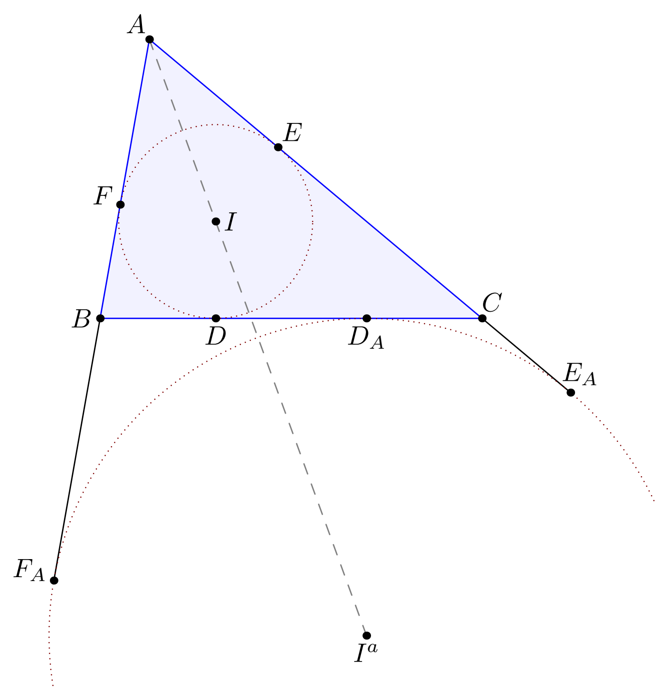
If we define D, E, and F to be the feet of the altitudes from I to $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively, then

$AEIF$ is cyclic, and hence

$$\measuredangle IFE = \measuredangle IAE = \measuredangle FAI = \measuredangle FEI$$
$$\measuredangle AFE = \measuredangle AIE = 90^\circ - \measuredangle EAI = 90^\circ - \measuredangle IAF = \measuredangle FIA = \measuredangle FEA,$$

so $IE = IF$ and $AE = AF$. Similarly, we obtain $ID = IE = IF$, $BF = BD$, and $CD = CE$.

If we define $\omega = (DEF)$, then $\overline{ID} \perp \overline{BC}$, and by Proposition 0.2.6, ω is tangent to $\overline{BC}$ at D. Similarly, ω is tangent to $\overline{CA}$, $\overline{AB}$ at E, F respectively. We define ω to be the **incircle** of $\triangle ABC$, and we let $\triangle DEF$ be the **contact triangle** or **intouch triangle** of $\triangle ABC$.



Since D, E, F lie on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively, we must have

$$\begin{cases} AE = AF, \\ BF = BD, \\ CD = CE, \end{cases} \qquad \begin{cases} BD + DC = a := BC, \\ CE + EA = b := CA, \\ AF + FB = c := AB, \end{cases}$$

so we obtain by solving

$$AE = AF = \frac{b+c-a}{2}, \quad BF = BD = \frac{c+a-b}{2}, \quad CD = CE = \frac{a+b-c}{2}.$$

Similarly, if D_A, E_A, F_A are the feet of the A-excenter to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively, then $\omega_A = (D_A E_A F_A)$

can also be found to be tangent to all the sides of $\triangle ABC$, and we call ω_A the **A-excircle** of $\triangle ABC$. We also call $\triangle D_AE_AF_A$ the **A-extouch triangle** of $\triangle ABC$. We can find that, similarly to the incircle,

$$AE_A = AF_A = \frac{a+b+c}{2}, \quad BF_A = BD_A = \frac{a+b-c}{2}, \quad CD_A = CE_A = \frac{c+a-b}{2}.$$

Note that $BD = D_AC$ and $BD_A = DC$. Thus D and D_A are reflections about the midpoint M_A of $\overline{BC}$.

In a similar fashion, we can define the **B-excircle** and the **C-excircle** as $(D_BE_BF_B)$ and $(D_CE_CF_C)$, respectively. Note that D_B and D_C have the same reflective properties about M_B and M_C, respectively.

Having introduced the incircle and excircles, let's look at a related example:

Example 0.3.10 (19 Czech-Slovak MO P4)**.** Let $\triangle ABC$ be an acute-angled triangle, let P lie on BC satisfying $AB = BP$, and B is between P and C. Let Q be on line BC such that $AC = CQ$, and C is between Q and B. Suppose the A-excircle (J) touches AB and AC at D and E. Let DP intersect EQ at F. Prove that $AF \perp FJ$.

Proof. We need to show that F is on the circle with diameter $\overline{AJ}$, or $(ADJE)$. Let D' be the point where the A-excircle touches $\overline{BC}$. Then $BD' = BD$ and $CD' = CE$, so $\triangle ABD' \stackrel{-}{\sim} \triangle PBD$, $\triangle ACD' \stackrel{-}{\sim} \triangle QCE$. Hence

$$\begin{aligned}\measuredangle DFE &= \measuredangle PDB + \measuredangle DAE + \measuredangle CEQ \\ &= \measuredangle BD'A + \measuredangle DAE + \measuredangle AD'C = \measuredangle DAE,\end{aligned}$$

so $F \in (ADE)$, which is what we wanted. □

Theorem 0.3.11 (Stewart's Theorem)**.** Given three collinear points A, B, C and another point P not on that line, we have

$$PA^2 \cdot \overrightarrow{BC} + PB^2 \cdot \overrightarrow{CA} + PC^2 \cdot \overrightarrow{AB} + \overrightarrow{BC} \cdot \overrightarrow{CA} \cdot \overrightarrow{AB} = 0.$$

Proof. WLOG assume $\measuredangle BAC = 180^\circ \pmod{360^\circ}$. By the Law of Cosines, we have

$$\frac{PA^2 + CA^2 - PC^2}{2 \cdot PA \cdot CA} + \frac{PA^2 + AB^2 - PB^2}{2 \cdot PA \cdot AB} = \cos \measuredangle PAC + \cos \measuredangle PAB = 0,$$

so we have

$$(PA^2 - CA \cdot AB)(AB - CA) + PB^2 \cdot CA - PC^2 + AB = 0.$$

We obtain the directed lengths version by using our assumption that $\measuredangle BAC = 0^\circ \pmod{360^\circ}$, which gives us

$$PA^2 \cdot \overrightarrow{BC} + PB^2 \cdot \overrightarrow{CA} + PC^2 \cdot \overrightarrow{AB} + \overrightarrow{BC} \cdot \overrightarrow{CA} \cdot \overrightarrow{AB} = 180.$$

□

Example 0.3.12 (Apollonius's Theorem)**.** Let $\overline{AM}$ be the A-median of $\triangle ABC$. Then we have

$$AM^2 = \frac{AB^2}{2} + \frac{AC^2}{2} - \frac{BC^2}{4}.$$

Solution. This is a simple corollary of Stewart's Theorem. □

The same method can be used to calculate the length of the angle bisector - namely, if $\overline{AD}$ is the angle bisector with D on $\overline{BC}$, then

$$AD^2 = AB \cdot AC - BD \cdot DC.$$

Theorem 0.3.13 (Subtended Angle Theorem)**.** Pick four distinct points A, B, C, P. Then A, B, C are collinear iff.

$$\frac{\sin \measuredangle BPC}{PA} + \frac{\sin \measuredangle CPA}{PB} + \frac{\sin \measuredangle APB}{PC} = 0.$$

Proof. By the Sine Area Formula, we have

$$[PBC] + [PCA] + [PAB] = \sum_{\text{cyc}} \frac{1}{2} \cdot PB \cdot PC \cdot \sin \measuredangle BPC.$$

If A, B, C are collinear then

$$[ABC] = [PBC] + [PCA] + [PAB] = 0,$$

which is equivalent to

$$\frac{\sin \measuredangle BPC}{PA} + \frac{\sin \measuredangle CPA}{PB} + \frac{\sin \measuredangle APB}{PC} = 0.$$

□

Practice Problems

Problem 1 (Albania MO 2012/5)**.** Let ABC be a scalene triangle. Let P be the foot of the altitude from C to $\overline{AB}$. Let H be the orthocenter and O be the circumcenter of triangle ABC. Let $D = \overline{OC} \cap \overline{AB}$. Let E be the midpoint of $\overline{CD}$. Prove that $\overline{EP}$ bisects $\overline{OH}$.

Problem 2 (Incenter-Excenter Lemma)**.** Let I, I_A be the incenter and A-excenter of $\triangle ABC$. Let $N_A = \overline{AI} \cap (ABC)(\neq A)$. Prove that N_A is the circumcenter of the circle $(BICI_A)$.

Remark. This theorem is also known at "Fact 5" in America or "Chicken Feet Theorem" in Chinese speaking places. The name "Fact 5" is from an old handout detailing certain geometry lemmas. The name "Chicken

Feet Theorem" can be found by observing that the figure consisting of the four segments $\overline{N_AB}$, $\overline{N_AI}$, $\overline{N_AC}$, $\overline{N_AI_A}$ looks like a chicken foot.

Problem 3. Let $ABCD$ be a cyclic quadrilateral. Let I_D and I_A be the incenters of $\triangle ABC$ and $\triangle BCD$, respectively, let J_B be the A-excenter of $\triangle CDA$, and let J_C be the D-excenter of $\triangle DAB$. Prove that I_A, J_B, J_C, and I_D are collinear.

Problem 4. Let the contact triangle of $\triangle ABC$ be $\triangle DEF$. Prove that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur. This concurrency point is called the **Gergonne point** of $\triangle ABC$.

Problem 5. Let the A-excircle, B-excircle, and C-excircle touch $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively (this triangle is called the **extouch triangle** of $\triangle ABC$). Prove that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur. The concurrency point is called the **Nagel Point** of $\triangle ABC$.

Problem 6 (Kariya's Theorem)**.** Let ABC be a triangle with incenter I and intouch triangle DEF. Select X, Y, Z on rays $\overrightarrow{ID}$, $\overrightarrow{IE}$, $\overrightarrow{IF}$ such that $IX = IY = IZ$. Show that lines $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ concur.

Problem 7. Let $\triangle DEF$ be the intouch triangle of $\triangle ABC$. Select points X, Y, Z on the incircle of $\triangle ABC$. Prove that $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ concur if and only if $\overline{DX}$, $\overline{EY}$, $\overline{FZ}$ concur or $\overline{DX} \cap \overline{EF}, \overline{EY} \cap \overline{FD}, \overline{FZ} \cap \overline{DE}$ are collinear.

Remark. This is a special case of Desargues's Theorem.

Problem 8. Let (I) be the incircle of triangle ABC. Let $\triangle DEF$ be the medial triangle of $\triangle ABC$. Draw the tangents to (I) through the points D, E, and F that are *not* the sides of the triangle ABC. These three tangents respectively intersect $\overline{EF}$, $\overline{FD}$, and $\overline{DE}$ at X, Y, Z respectively. Prove that X, Y, and Z are collinear.

Problem 9. In triangle $\triangle ABC$, let $\overline{AD}$ be the internal angle bisector of $\angle BAC$, such that $\angle ADC = 60°$. Construct point E on $\overline{AD}$ such that $\overline{DE} = \overline{DB}$. Let the ray $\overrightarrow{CE}$ intersect $\overline{AB}$ at F. Prove that

$$AF \times AB + CD \times CB = AC^2.$$

Problem 10 (Taiwan MO 2021/4)**.** Let I be the incenter of triangle ABC and let D the foot of altitude from I to BC. Suppose the reflection D' of D with respect to I satisfies $\overline{AD'} = \overline{ID'}$. Let Γ be the circle centered at D' that passes through A and I, and let X, $Y \neq A$ be the intersection of Γ and AB, AC, respectively. Suppose Z is a point on Γ so that AZ is perpendicular to BC.

Prove that lines AD, $D'Z$, XY are concurrent.

0.4 Power of a Point

Definition 0.4.1. Given a fixed circle Γ and a point P, we can define a function denoted as the **power** of point P defined as such:

$$\mathbf{Pow}_\Gamma(P) := OP^2 - R^2$$

where O and R represent the center and radius of Γ.

Proposition 0.4.2. Construct a line going through P that intersects Γ at points A, B. Then

$$\overrightarrow{PA} \cdot \overrightarrow{PB} = \mathbf{Pow}_\Gamma(P).$$

Proof. Let O be the center of Γ, and let M be the midpoint of $\overline{AB}$. Then we know that $OM \perp AB$. So by (0.2.11), we get

$$\begin{aligned}\overrightarrow{PA} \cdot \overrightarrow{PB} &= (\overrightarrow{PM} + \overrightarrow{MA}) \cdot (\overrightarrow{PM} + \overrightarrow{MB}) = PM^2 - MA^2 \\ &= PO^2 - OA^2 = \mathbf{Pow}_\Gamma(P)\end{aligned}$$

□

Corollary 0.4.3. Draw two lines ℓ_1, ℓ_2 through a point P, and then pick two points A_i, B_i on ℓ_i such that

$$\overrightarrow{PA_1} \cdot \overrightarrow{PB_1} = \overrightarrow{PA_2} \cdot \overrightarrow{PB_2}.$$

Then A_1, B_1, A_2, B_2 are concyclic. (The converse of this also holds).

Proof. Let $\Gamma = (A_1B_1A_2)$ and let $B_2' = \Gamma \cap \ell_2$. Now

$$\overrightarrow{PA_2} \cdot \overrightarrow{PB_2} = \overrightarrow{PA_1} \cdot \overrightarrow{PB_1} = \mathbf{Pow}_\Gamma(P) = \overrightarrow{PA_2} \cdot \overrightarrow{PB_2'}$$

so $B_2 = B_2'$ as desired. □

Example 0.4.4 (Azerbaijan TST 2017/2/1)**.** Let ABC be an acute-angled triangle. Points E and F are chosen on the sides AC and AB, respectively, such that

$$BC^2 = BA \cdot BF + CE \cdot CA.$$

Prove that for all such E and F, the circumcircle of the triangle AEF passes through a fixed point different from A.

Solution. Let M be the midpoint of $\overline{BC}$ - we compute $\mathbf{Pow}_{(AEF)}(M)$. If it's fixed, then we are done, because then $AM \cap (AEF)$ is the desired fixed point. We compute

$$\begin{aligned}\textbf{Pow}_{(AEF)}(M) &= OM^2 - r^2 = \frac{1}{2}\left(OB^2 - r^2 + OC^2 - r^2\right) - \frac{1}{4}BC^2 \\ &= \frac{1}{2}\left(\textbf{Pow}_{(AEF)}(B) + \textbf{Pow}_{AEF}(C)\right) - \frac{1}{4}BC^2 \\ &= \frac{1}{2}\left(BA \cdot BF + CE \cdot CA\right) - \frac{1}{4}BC^2 \\ &= \frac{1}{4}BC^2\end{aligned}$$

which is fixed, and we are done. □

Proposition 0.4.5. Suppose a line through P is tangent to a circle Γ at T. Then

$$\textbf{Pow}_\Gamma(P) = \overline{PT}^2.$$

This can be thought of as the limiting case of Proposition 0.4.2.

Proof. By Proposition 0.2.6, $\overline{OT} \perp \overline{PT}$. Hence by the Pythagorean Theorem,

$$PT^2 = OT^2 - OP^2 = \textbf{Pow}_\Gamma(P)$$ □

Corollary 0.4.6. Two lines ℓ_1 and ℓ_2 are drawn through a point P. Let A, $B \in \ell_1$ and $T \in \ell_2$ Then (ABT) is tangent to ℓ_2 at T if

$$PA \cdot PB = PT^2.$$

Example 0.4.7 (CWMO 2019/5)**.** Let O, H be the circumcenter and orthocenter of acute $\triangle ABC$ respectively. The line passing through H and parallel to AB intersects line AC at M, and the line passing through H and parallel to AC intersects line AB at N. L is the reflection of the point H in MN. Line OL and AH intersect at K. Prove that K, M, L, N are concyclic.

Solution. We first begin by trying to prove that $OM = ON$, or that $\textbf{Pow}_{(ABC)}(M) = \textbf{Pow}_{(ABC)}(N)$. Note that

$$\measuredangle MCH = 90 - \measuredangle BAC = \measuredangle HBN, \quad \measuredangle HMC = \measuredangle BAC = \measuredangle BNH.$$

so it follows that $\triangle CMH \stackrel{-}{\sim} \triangle BNH$ Now

$$\begin{aligned}\textbf{Pow}_{(ABC)}(M) &= MA \cdot MC = HN \cdot MC \\ &= HM \cdot NB = NA \cdot NB = \textbf{Pow}_{(ABC)}(N)\end{aligned}$$

as desired. Now $ALMN$ is an isosceles trapezoid and hence cyclic. Hence $\triangle OAM \cong \triangle OLN$. We conclude that (AO, AH) is antiparallel with respect to (AM, AN), so

$$\measuredangle NLK = \measuredangle OAM = \measuredangle NLK$$

and $AKLN$ is cyclic - combined with $ALMN$ cyclic gives the desired. $\square$

Definition 0.4.8. For any two circles Γ_1, Γ_2, we define the **radical axis** of the two circles as

$$\{P \mid \mathbf{Pow}_{\Gamma_1}(P) = \mathbf{Pow}_{\Gamma_2}(P).\}$$

Proposition 0.4.9. The radical axis of Γ_1 and Γ_2 is a straight line perpendicular to the line through the centers of Γ_1 and Γ_2.

Proof. Let O_1 and r_1 be the center and radius of Γ_1, respectively, and analogously define O_2 and r_2. Note that

$$\mathbf{Pow}_{\Gamma_1}(P) = \mathbf{Pow}_{\Gamma_2}(P) \iff O_1P^2 - r_1^2 = O_2P^2 - r_2^2 \implies O_1P^2 - O_2P^2 = r_1^2 - r_2^2$$

so by the Perpendicularity Criterion the locus of P is a straight line perpendicular to $\overline{O_1O_2}$. $\square$

From this proof, we see that any line perpendicular to O_1O_2 is the set of points defined by

$$\{P \mid \mathbf{Pow}_{\Gamma_1}(P) - \mathbf{Pow}_{\Gamma_2}(P) = k\}$$

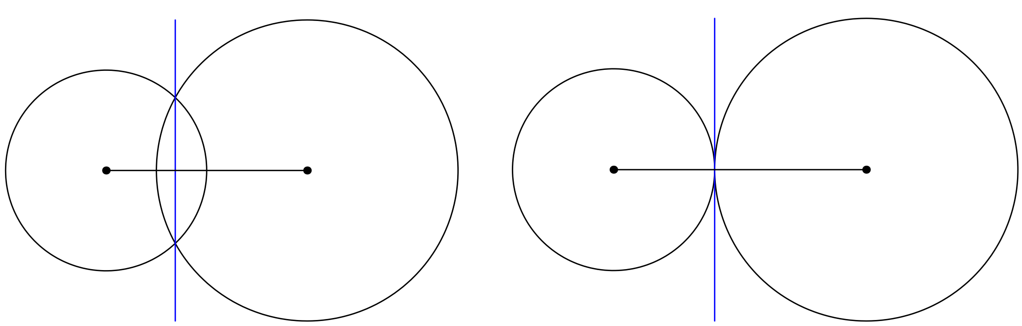
for some $k \in \mathbb{R}$.

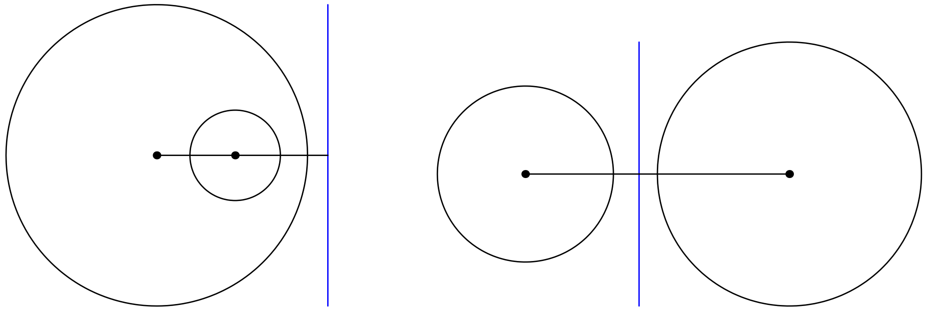
Proposition 0.4.10. The radical axis of two circles Γ_1 and Γ_2 intersecting at A and B is the line $\overline{AB}$.

Proof. Note that

$$\mathbf{Pow}_{\Gamma_1}(A) = \mathbf{Pow}_{\Gamma_2}(A) = \mathbf{Pow}_{\Gamma_1}(B) = \mathbf{Pow}_{\Gamma_2}(B) = 0. \qquad \square$$

so both A, B lie on the radical axis.





Theorem 0.4.11 (Radical Axis Theorem)**.** For any three circles ω_1, ω_2, ω_3, let ℓ_1, ℓ_2, and ℓ_3 be the three radical axis of the pairs of circles (ω_2, ω_3), (ω_3, ω_1), and (ω_1, ω_2) respectively. Then ℓ_1, ℓ_2, and ℓ_3 concur, are all parallel, or are all the same line.

Proof. If $\ell_1 \cap \ell_2 = P \notin \mathcal{L}_\infty$, then

$$\mathbf{Pow}_{\omega_1}(P) = \mathbf{Pow}_{\omega_2}(P) = \mathbf{Pow}_{\omega_3}(P)$$

by the definition of radical axis. The edge cases (all parallel or all the same line) can easily be checked. □

Definition 0.4.12. For any three circles ω_1, ω_2, ω_3, let ℓ_1, ℓ_2, and ℓ_3 be the three radical axis of the pairs of circles (ω_2, ω_3), (ω_3, ω_1), and (ω_1, ω_2) respectively. Then:

- If $\ell_1 = \ell_2 = \ell_3$, then ω_1, ω_2, and ω_3 are said to be **coaxial**.
- Otherwise, ℓ_1, ℓ_2, ℓ_3 concur at the **radical center** of ω_1, ω_2, and ω_3, or they are all parallel.

Example 0.4.13. Let $\triangle DEF$ be the orthic triangle of $\triangle ABC$. Then $BCEF$, $CAFD$, and $ABDE$ are all cyclic, and their radical axis is the intersection of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, which is the orthocenter H.

Example 0.4.14 (ISL 2009 G3)**.** Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms.

Prove that $GR = GS$.

Solution. Let ω_a be the A-excircle, and let it be tangent to $\overline{BC}$, $\overline{CA}$ at X' and Y' respectively. The main idea in this problem is to consider the limiting case of the Radical Axis Theorem, where one of the circles is just a point - i.e. a circle with radius 0. We call this a **point circle**, and this is a very useful technique.

Consider two point circles at R, S. Thus, we have

$$\mathbf{Pow}_{\omega_a}(B) = BX'^2 = CY^2 = BR^2 = \mathbf{Pow}_R(B)$$

$$\mathbf{Pow}_{\omega_a}(Y) = YY'^2 = BC^2 = RY^2 = \mathbf{Pow}_R(Y)$$

where $\mathbf{Pow}_R(P)$ denotes the power of a point with respect to a circle with radius 0 at R. Then $\overline{BY}$ is the radical axis of the point circle at R and ω_a. Similarly, $\overline{CZ}$ is the radical axis of the point circle at S and ω_a. Hence G is the radical center of the point circles at R and S and ω_a. It follows that $GR = GS$ as desired. $\square$

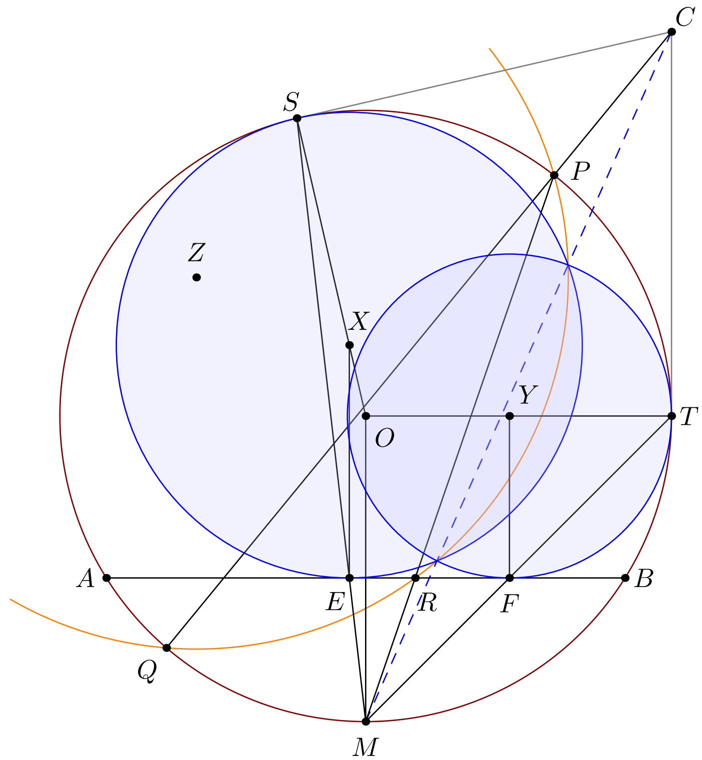
Corollary 0.4.15. Let $\triangle ABC$ be a triangle. Suppose D_1, $D_2 \in \overline{BC}$, E_1, $E_2 \in \overline{CA}$, and F_1, $F_2 \in \overline{AB}$ such that $D_1D_2E_1E_2$, $E_1E_2F_1F_2$, and $F_1F_2D_1D_2$ are all cyclic. Then the hexagon $D_1D_2E_1E_2F_1F_2$ is cyclic.

Proof. Assume otherwise. Let $\Gamma_1 = (D_1D_2E_1E_2)$ and analogously define Γ_2 and Γ_3. Then the radical axis of Γ_1 and Γ_2 is $\overline{CA}$, and similarly the radical axis of Γ_2 and Γ_3 is $\overline{AB}$ and the radical axis of Γ_3 and Γ_1 is $\overline{BC}$. If these circles do not coincide, then by Radical Axis Theorem their radical axis should either be all-parallel, coincide, or concur, but none of these three things are true, contradiction. $\square$

Proposition 0.4.16. If (O_1), (O_2), and (O_3) are three coaxial circles, then O_1, O_2, O_3 are collinear. This line is perpendicular to the common radical axis of the three circles.

Proof. $\overline{O_1O_2}$, $\overline{O_2O_3}$, and $\overline{O_3O_1}$ are all perpendicular to the common radical axis ℓ, which is enough. $\square$

Example 0.4.17. Let $\overline{AB}$ be a chord of (O) and let M be the midpoint of the minor arc AB. Let C be a point outside (O), and suppose that the two tangents from C touch O at S and T, respectively. Let $\overline{MS}$ and $\overline{MT}$ intersect $\overline{AB}$ at E, F respectively. The line perpendicular to $\overline{AB}$ through E and F intersect $\overline{OS}$, $\overline{OT}$ at X, Y respectively. A line through C intersects (O) at P and Q. Let R be the intersection of $\overline{MP}$ and $\overline{AB}$. Finally, let Z be the center of (PQR). Prove that X, Y, and Z are collinear.



Solution. Since $\overline{EX} \perp \overline{AB} \perp MO$ and $\overline{FY} \perp \overline{AB} \perp MO$,

$$\triangle ESX \overset{+}{\sim} \triangle MSO, \quad \triangle FTY \overset{+}{\sim} MTO.$$

Hence $XE = XS$ and $YF = YT$. Let ω_X and ω_Y be the circles centered at X and Y passing through E and F respectively. Then ω_X and ω_Y are tangent to (O) at S and T and tangent to $\overline{AB}$ at E and F, respectively. Since $\measuredangle MSA = \measuredangle MBA = \measuredangle EAM$, MA is tangent to (AES). Hence

$$MA^2 = ME \cdot MS$$

and similarly $MA^2 = MF \cdot MT = MP \cdot MR$ so

$$\mathbf{Pow}_{\omega_X}(M) = \mathbf{Pow}_{\omega_Y}(M) = \mathbf{Pow}_{(PQR)}(M).$$

Also,

$$\mathbf{Pow}_{\omega_X}(C) = CS^2, \quad \mathbf{Pow}_{\omega_Y}(C) = CT^2, \quad \mathbf{Pow}_{(PQR)}(C) = CP \cdot CQ.$$

Hence $\overline{CM}$ is the (common) radical axis of ω_X, ω_Y, and (PQR). By Proposition 0.4.16, we are done. $\square$

For problems requiring proofs of concyclicity, we often use the following theorem:

Theorem 0.4.18 (Ptolemy's Inequality)**.** For any four points A, B, C, D,

$$BC \cdot AD + CA \cdot BD + AB \cdot CD \geq 2\max\{BC \cdot AD, CA \cdot BD, AB \cdot CD\}$$

with equality if and only if A, B, C, D are collinear or concyclic. More commonly, Ptolemy's inequality is written for convex quadrilaterals $ABCD$, in which

$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD.$$

There are many great proofs of this theorem, and we will present one that is completely based on the properties outlined in this chapter. Before we prove this theorem, we need the following lemma, which is the basis of Section 1.2:

Lemma 0.4.19 (Spiral Similarity)**.** If $\triangle ABC \stackrel{+}{\sim} \triangle AB'C'$, then $\triangle ABB' \stackrel{+}{\sim} \triangle ACC'$.

Proof. $\triangle ABC \stackrel{+}{\sim} \triangle AB'C'$ means

$$\measuredangle BAC = \measuredangle B'AC' \pmod{360^\circ}, \quad \frac{AB}{AC} = \frac{AB'}{AC'}$$

which is equivalent to

$$\measuredangle BAB' = \measuredangle CAC' \pmod{360^\circ}, \quad \frac{AB}{AB'} = \frac{AC}{AC'}$$

which implies $\triangle ABB' \stackrel{+}{\sim} \triangle ACC'$. □

Proof of 0.4.18. WLOG assume that $BC \cdot AD$ is maximal among the four points and that A, B, C, D don't lie on a line. Let E be a point such that $\triangle DEB \stackrel{+}{\sim} \triangle DCA$, so by Lemma 0.4.19, $\triangle DCE \stackrel{+}{\sim} \triangle DAB$. Now

$$CA \cdot BD = BE \cdot AD, \quad AB \cdot CD = EC \cdot AD.$$

Thus

$$CA \cdot BD + AB \cdot CD + BC \cdot AD - 2BC \cdot AD = (BE + EC - BC)AD \geq 0$$

by the Triangle Inequality. Equality holds when E lies on $\overline{BC}$, which can be checked to be equivalent to A, B, C, D are concyclic. □

Theorem 0.4.20 (Casey's Theorem)**.** Given four non-intersecting circles $\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_D$, define d_{IJ}^{+} to be the length of the external common tangents of Γ_I and Γ_J, and define d_{IJ}^{-} to be the length of the internal common tangents of Γ_I and Γ_J. There exists a circle Ω tangent to the four circles $\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_D$ iff.

$$d_{BC}d_{AD} \pm d_{CA}d_{BD} \pm d_{AB}d_{CD} = 0,$$

where

$$d_{IJ} = \begin{cases} d_{IJ}^{+}, & \text{if } \Gamma_I,\, \Gamma_J \text{ are both tangent on the same side of } \Omega, \\ d_{IJ}^{-}, & \text{if } \Gamma_I,\, \Gamma_J \text{ are tangent on opposite sides of } \Omega. \end{cases}$$

(This weird definition lets us avoid writing sixteen forms of this theorem. Here, $\pm$ refers to any of the 4 possible equalities holding).

Proof. We first prove the "if" side of the "iff". (The proof of the other side will be left until Cross Ratios under Inversion and Polarity, after we learn to use the power of circle inversion.)

Suppose Ω is tangent to $\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_D$, with O, R respectively being the center of Ω and its radius. Define O_I, r_I similarly for Γ_I.

We first prove that

$$d_{IJ} = \frac{IJ}{R}\sqrt{(R \pm r_I)(R \pm r_J)},$$

where the $R \pm r_n$ is positive if Γ_n is externally tangent, and negative if it's internally tangent. By the Law of Cosines we get

$$\frac{(R \pm r_I)^2 + (R \pm r_J)^2 - O_I O_J^2}{2(R \pm r_I)(R \pm r_J)} = \cos(\measuredangle IOJ) = \frac{R^2 + R^2 - IJ^2}{2R^2},$$

and thus

$$\begin{aligned} d_{IJ}^2 &= O_I O_J^2 - (r_I \pm r_J)^2 \\ &= (R \pm r_I)^2 + (R \pm r_J)^2 - 2(R \pm r_I)(R \pm r_J)\left(1 - \frac{IJ^2}{2R^2}\right) - (r_I \pm r_J)^2 \\ &= \frac{IJ^2}{R^2}(R \pm r_I)(R \pm r_J). \end{aligned}$$

Taking the square root of both sides, we get our desired expression, and we proved this small lemma.

Back to the proof of Casey's; we proceed by Ptolemy's theorem to get that

$$d_{BC}d_{AD} \pm d_{CA}d_{BD} \pm d_{AB}d_{CD} = \frac{\sqrt{(R \pm r_A)(R \pm r_B)(R \pm r_C)(R \pm r_D)}}{R^2} \cdot (BC \cdot AD \pm CA \cdot BD \pm AB \cdot CD)$$

and then we one-shot by our previous lemma, to get this whole nasty expression is just 0.

$\square$

Corollary 0.4.21 (Three Chords Theorem)**.** For any four concyclic points A, B, C, D, we have

$$DA \cdot \sin \measuredangle BDC + DB \cdot \sin \measuredangle CDA + DC \cdot \sin \measuredangle ADB = 0.$$

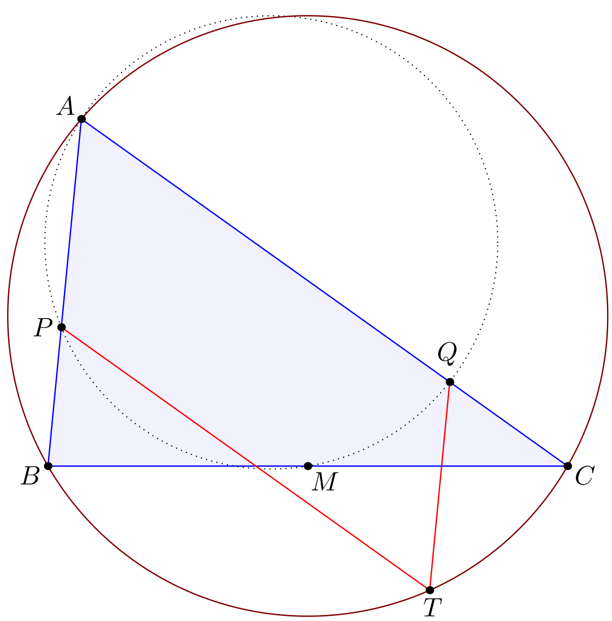
Proof. Similarly to the proof of Ptolemy's Inequality, take a point E such that $\triangle DEB \overset{+}{\sim} \triangle DCA$ - thus B, E, C are collinear. Now

$$\begin{aligned}
&DA \cdot \sin \measuredangle BDC + DB \cdot \sin \measuredangle CDA + DC \cdot \sin \measuredangle ADB \\
&= \frac{DB \cdot DC}{DE} \sin \measuredangle BDC + DB \cdot \sin \measuredangle EDB + DC \cdot \sin \measuredangle CDE \\
&= \frac{2}{DE} \left([DBC] + [DEB] + [DCE] \right) = 0
\end{aligned}$$

since B, E, and C are collinear. □

Remark. If you know what inversion is, this is also the inverted version of Theorem 0.3.13 from earlier.

Example 0.4.22 (ISL 2015 G4)**.** Let ABC be an acute triangle and let M be the midpoint of $\overline{BC}$. A circle ω passing through A and M meets the sides $\overline{AB}$ and $\overline{AC}$ at points P and Q respectively. Let T be the point such that $APTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{AT}{AM}$.



Solution. By the Three Chords Theorem, $APMQ$ cyclic means

$$AP \cdot \sin \measuredangle BAM + AQ \cdot \sin \measuredangle MAC = AM \cdot \sin \measuredangle BAC$$

and $ABTC$ cyclic means

$$AB \cdot \sin \measuredangle TAP + AC \cdot \sin \measuredangle QAT = AT \cdot \sin \measuredangle QAP.$$

By Ratio Lemma, we have

$$\sin \measuredangle BAM : \sin \measuredangle MAC : \measuredangle BAC :: AC : AB : 2AM$$
$$\sin \measuredangle TAP : \sin \measuredangle QAT : \measuredangle QAP :: AQ : AP : AT.$$

Hence the two length equations can be written as

$$AP \cdot AC + AQ \cdot AB = AM \cdot 2AM$$
$$AB \cdot AQ + AC \cdot AP = AT \cdot AT.$$

This gives $AT^2 = 2AM^2$, or equivalently $\frac{AT}{AM} = \sqrt{2}$. □

Example 0.4.23 (APMO 2017/2)**.** Let ABC be a triangle with $AB < AC$. Let D be the intersection point of the internal bisector of angle BAC and the circumcircle of ABC. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of triangle ADZ.

Solution. By the Three Chords Theorem, it suffices to show that

$$AM \cdot \sin \angle DAZ + AZ \cdot \sin \angle MAD = AD \cdot \sin MAZ.$$

Define $a = BC$ and analogously define b and c, and define $\alpha = \angle BAC$ and analogously define β and γ. Then Then the left-hand side of the above equation is

$$\frac{c}{2} + \frac{b}{2\sin\frac{1}{2}\alpha} \cdot \sin\frac{1}{2}\alpha = \frac{b+c}{2}$$

and the right-hand side (from the Law of Sines) is

$$\left(\frac{b}{\sin\beta} \cdot \sin\left(\beta + \frac{1}{2}\alpha\right)\right) \cdot \sin\left(90^\circ + \frac{1}{2}\alpha\right)$$
$$= \frac{b}{\sin\beta} \cdot \frac{1}{2}(\cos(90^\circ - \beta) - \cos(90^\circ + \alpha + \beta)) = \frac{b+c}{2}$$

where the last equality is because

$$\frac{b}{\sin\beta}\cdot\cos(90^\circ+\alpha+\beta)=\frac{c}{\sin\gamma}\cdot(-\sin\gamma)=-c.$$

$\square$

Practice Problems

Problem 1 (Euler's Theorem)**.** Let O and I be the circumcenter and incenter of $\triangle ABC$ with circumradius R and inradius r.

(i) Let F be the foot of I to $\overline{AB}$, N_a be the intersection of $\overline{AI}$ and (ABC), and let $N*_a$ be the antipode of N_a on (ABC). Prove that $\triangle AFI \overset{+}{\sim} \triangle N_a^*BN_a$.

(ii) By the Incenter-Excenter Lemma, show that $OI^2=R^2-2Rr$.

Problem 2 (Taiwan IMOC 2021)**.** Let the A-midline (line connecting midpoints of sides AB and AC) of $\triangle ABC$ intersect (ABC) at two points P, Q. Let the tangent to (ABC) at A intersect BC at T. Prove that $\measuredangle BTQ=\measuredangle PTA$.

Problem 3. Let triangle ABC have centroid G. Let the medians AG, BG, CG intersect (ABC) at points A', B', C'. Prove that

$$\frac{AG}{GA'}+\frac{BG}{GB'}+\frac{CG}{GC'}=3.$$

Problem 4 (Taiwan IMOC 2021)**.** Let $\overline{BE}$ and $\overline{CF}$ be the altitudes of $\triangle ABC$, and let D be the antipode of A on (ABC). The lines $\overline{DE}$ and $\overline{DF}$ intersect(ABC) again at Y and Z, respectively. Prove that YZ, EF, and BC concur.

Problem 5 (Iran TST 2011/1)**.** In acute triangle ABC, $\angle B > \angle C$. Let M be the midpoint of BC. Let D and E be the feet of the altitude from C and B, respectively. Let K and L be the midpoints of $\overline{ME}$ and $\overline{MD}$ respectively. If $\overline{KL}$ intersects the line through A parallel to $\overline{BC}$ in T, prove that $TA=TM$.

Problem 6. Let circles O_1 and O_2 intersect at the two points X and Y. Draw a line ℓ_1 through the center of O_1 that intersects O_2 at two points P, Q, Draw another line ℓ_2 through the center of O_2 that intersects O_1 at two points R, S. Prove that if P, Q, R, S are concyclic, then their circumcenter lies on XY.

Problem 7. Let ABC be a triangle with incenter I and circumcenter O for which $BC < AB < AC$. Let Y and X be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $CX = BC = YB$. Prove that $\overline{XY} \perp \overline{IO}$.

Problem 8 (2019 Estonia TST/2)**.** In an acute-angled triangle ABC, the altitudes intersect at point H, and point K is the foot of the altitude drawn from the vertex A. Circle c passing through points A and K intersects sides AB and AC at points M and N, respectively. The line passing through point A and parallel to line BC intersects the circumcircles of triangles AHM and AHN for the second time, respectively, at points X and Y. Prove that $|XY| = |BC|$.

Problem 9 (2017 ISL G4)**.** In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

Problem 10 (Canada 2016/5)**.** Let $\triangle ABC$ be an acute-angled triangle with altitudes AD and BE meeting at H. Let M be the midpoint of segment AB, and suppose that the circumcircles of $\triangle DEM$ and $\triangle ABH$ meet at points P and Q with P on the same side of CH as A. Prove that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.

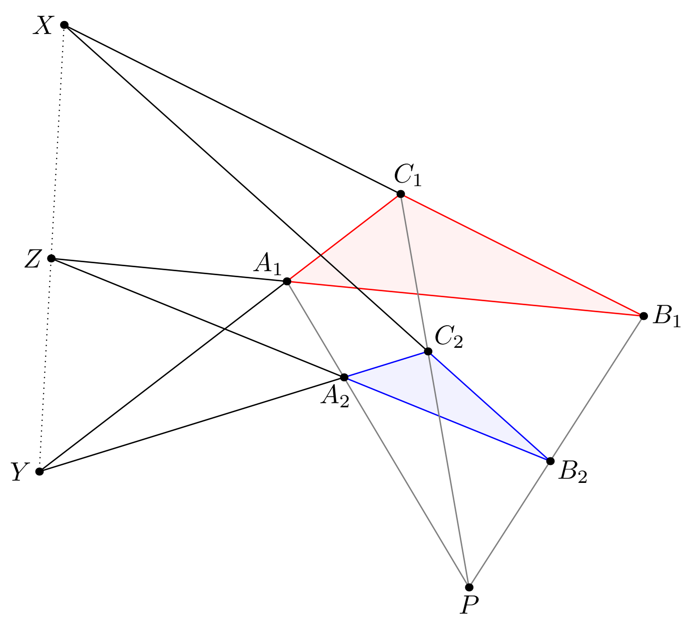
Problem 11 (Taiwan 2015/2J/I1-2)**.** Let the incircle of $\triangle ABC$ be ω, and suppose ω meets $\overline{BC}$ at D. Let $\overline{AD} \cap \omega = L(\neq D)$, and let the A-excenter of ABC be I_A. Let M be the midpoint of $\overline{BC}$ and let N be the midpoint of $\overline{I_A M}$. Prove that $BCNL$ is cyclic.

Chapter 1

Geometric Transformations

1.1 Homothety

Theorem 1.1.1 (Desargues's Theorem)**.** Given two triangles $\triangle A_1B_1C_1, \triangle A_2B_2C_2$, we have that $B_1C_1 \cap B_2C_2, C_1A_1 \cap C_2A_2, A_1B_1 \cap A_2B_2$ are collinear iff. A_1A_2, B_1B_2, C_1C_2 are concurrent.



Note that this is a symmetric relation between triangles.

Proof. Let $X = \overline{B_1C_1} \cap \overline{B_2C_2}$, $Y = \overline{C_1A_1} \cap \overline{C_2A_2}$, and $Z = \overline{A_1B_1} \cap \overline{A_2B_2}$.

First suppose that A_1A_2, B_1B_2, C_1C_2 concur at some point P.

Then by Menelaus, we have that

$$\frac{B_1X}{XC_1}\cdot\frac{C_1C_2}{C_2P}\cdot\frac{PB_2}{B_2B_1}=-1,$$
$$\frac{PC_2}{C_2C_1}\cdot\frac{C_1Y}{YA_1}\cdot\frac{A_1A_2}{A_2P}=-1,$$
$$\frac{B_1B_2}{B_2P}\cdot\frac{PA_2}{A_2A_1}\cdot\frac{A_1Z}{ZB_1}=-1$$

Multiplying all of these three together gives that

$$\frac{B_1X}{XC_1}\cdot\frac{C_1Y}{YA_1}\cdot\frac{A_1Z}{ZB_1}=-1$$

so X, Y, Z lie on a line.

Now suppose that X, Y, Z lie on a line. Consider the triangles $\triangle B_1B_2Z, \triangle C_1C_2Y$. We have that B_1C_1, B_2C_3, YZ concur at X. As such, $A_2 = B_2Z \cap C_2Y, A_1 = ZB_1 \cap YC_1, B_1B_2 \cap C_1C_2$ lie on a line which finishes. □

If this is the case, we say that $\triangle_1 = \triangle A_1B_1C_1$ and $\triangle_2 = \triangle A_2B_2C_2$ are **perspective**. The line through $B_1C_1 \cap B_2C_2, C_1A_1 \cap C_2A_2, A_1B_1 \cap A_2B_2$ is their **perspectrix** and the point that A_1A_2, B_1B_2, C_1C_2 concur at is the **perspector**.

A specific case of this is when $B_1C_1 \parallel B_2C_2, C_1A_1 \parallel C_2A_2, A_1B_1 \parallel A_2B_2$, then $B_1C_1 \cap B_2C_2, C_1A_1 \cap c_2A_2, A_1B_1 \cap A_2B_2$ all lie on the line at infinity $\mathcal{L}_\infty$. We can mimic the proof of Desargues's Theorem (or take a projective transformation, defined in Chapter 2); this means that the two triangles are perspective, and we call them **homothetic**. The perspective center is called the **homothetic center**. If O is the homothetic center, then by the parallel sides we have that

$$\frac{OA_2}{OA_1}=\frac{OB_2}{OB_1}=\frac{OC_2}{OC_1}=k. \qquad (\spadesuit)$$

We call k the **homothetic ratio** of the homothety. If we are given $\triangle_1, O, k$ for finite O, then we can use ($\spadesuit$) to construct $\triangle_2$. Removing $\triangle_1$, we get the following definition.

Definition 1.1.2. Given a point $O \notin \mathcal{L}_\infty$ and $k \neq 0$, we define the **homothety** $\mathfrak{h}_{O,k}$ as a mapping on points $P \mapsto Q$ for $P \notin \mathcal{L}_\infty$ such that $Q \in OP$ and

$$\frac{OQ}{OP}=k.$$

The line at infinity is fixed under this operation.

We also define for a vector $\vec{v}$ the translation $P \mapsto P + \vec{v}$ with the mapping $\mathfrak{h}_{\vec{v}}$. We consider this a homothety with ratio 1 taken at the point $\infty_{\vec{v}}$ at infinity.

Note that a homothetic transformation $\mathfrak{h}$ maps lines to lines. Furthermore, we have that

$$\overrightarrow{\mathfrak{h}(A)\mathfrak{h}(B)} = k \cdot \overrightarrow{AB}$$

so homotheties also preserve circles (consider the center and radius).

If $k = -1$, then $\mathfrak{h}_{O,-1}(P)$ is the reflection of P about O. In that case we use the notation $\mathfrak{s}_O = \mathfrak{h}_{O,-1}$.

Example 1.1.3 (Euler line)**.** Let $\triangle M_aM_bM_c$ be the medial triangle of $\triangle ABC$. We know that

$$M_aM_b \parallel AB, \quad M_bM_c \parallel BC, \quad M_cM_a \parallel CA.$$

Hence, $\triangle ABC$ is homothetic to $\triangle M_aM_bM_c$. The center of homothety of the two triangles is $G = AM_a \cap BM_B \cap CM_c$, and the ratio of the homothety is

$$\frac{GM_a}{GA} = -\frac{1}{2}$$

As such, we know that the circumcenter O of $\triangle ABC$ is the orthocenter of $M_aM_bM_c$ (since $M_aO \perp BC \parallel M_bM_c$). Thus, if H is the orthocenter of $\triangle ABC$, then O, G, H lie on a line and

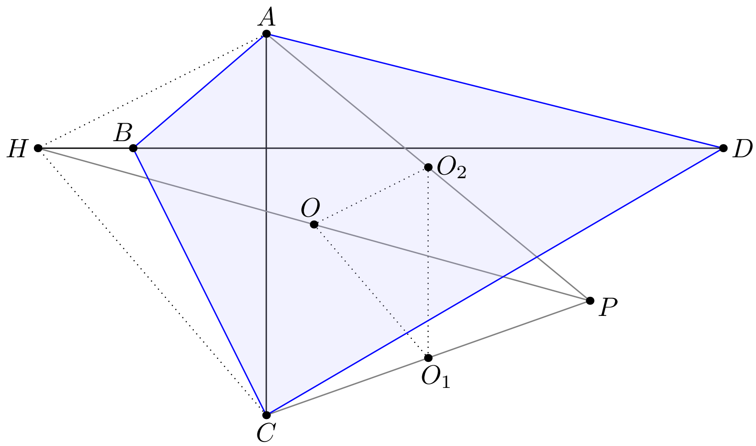
$$\frac{GO}{GH} = -\frac{1}{2}.$$

This line is the **Euler line** of $\triangle ABC$.

⚠: If $\triangle ABC$ is an equilateral triangle, then O, G, H (and more generally most triangle centers) are the same, so no Euler line exists.

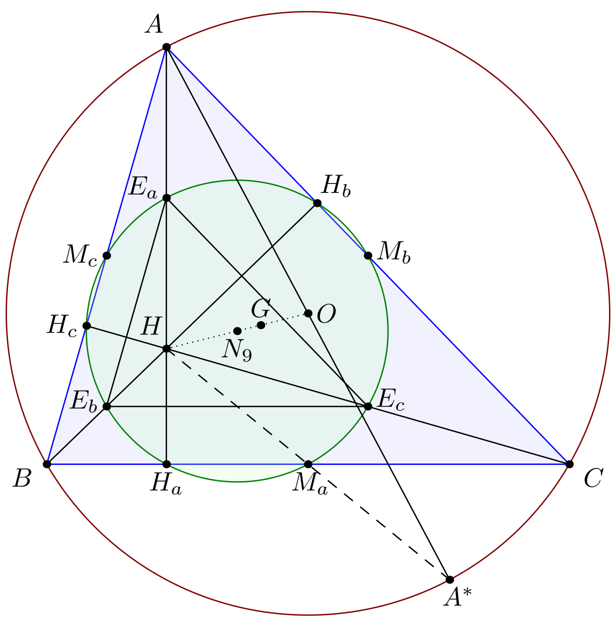
Example 1.1.4 (2021 2J I1-G)**.** Let $ABCD$ be a convex quadrilateral with all distinct side lengths and $AC \perp BD$. Let O_1, O_2 be the circumcenters of $\triangle ABD, \triangle CBD$, respectively. Prove that $\overline{AO_2}, \overline{CO_1}$, and the Euler lines of $\triangle ABC$ and $\triangle ADC$ concur.

Solution. By symmetry, we only need to prove that AO_2, CO_1, and the Euler line of $\triangle ABC$ are concurrent. Let O, H be the circumcenter and orthocenter of $\triangle ABC$. Then we only need to show $\triangle CAH$ and $\triangle O_1O_2O$ are homothetic, so we can just prove that $AH \cap O_2O, HC \cap OO_2, CA \cap O_1O_2$ are collinear. Now note that since $AH \perp BC \perp O_2O, HC \perp AB \perp OO_1, CA \perp BD \perp O_1O_2$, we are done.



□

Example 1.1.5 (Nine-Point Circle)**.** Let H be the orthocenter of $\triangle ABC$, $\triangle M_aM_bM_c, \triangle H_aH_bH_c$ be the medial and orthic triangles of $\triangle ABC$, respectively. Let E_a, E_b, E_c be the midpoints of $\overline{AH}, \overline{BH}, \overline{CH}$, respectively. Then $M_a, M_b, M_c, H_a, H_b, H_c, E_a, E_b, E_c$ all lie on a common circle ϵ, called the **nine-point circle** of $\triangle ABC$.



The full nine-point circle configuration. We will revisit this in Chapters 5 and 6.

Solution. We show that the nine-point circle is the image of (ABC) under $\mathfrak{h}_{H,\frac{1}{2}}$.

Then we get that $\mathfrak{h}_{H,\frac{1}{2}}(A) = E_a$ and so forth. Now, by Orthocenter we get that $\mathfrak{h}_{H,2,(}(H_a) = H_A$ and so forth. It remains to show that $\mathfrak{h}_{H,2,(}(M_a) \in (ABC)$. Let A^* be the A antipode on (ABC).

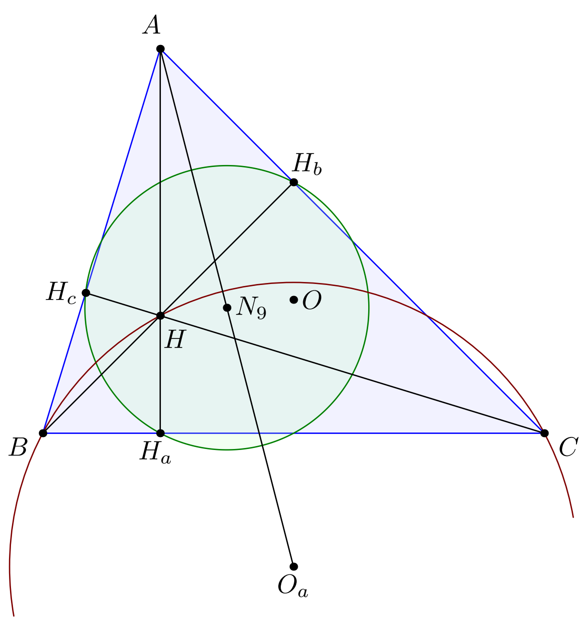
Then we have that $BH \perp CA \perp A^*C, HC \perp BA^*$ so $BHCA^*$ is a parallelogram. This implies the result. Thus, the **nine-point center** N_9 is the image of $\mathfrak{h}_{H,\frac{1}{2}}(O)$, where O is the circumcenter. As such, N_9 is the midpoint of OH, and is thus on the Euler line. In summary,

$$(N) = (M_a M_b M_c) = \mathfrak{h}_{G,-\frac{1}{2}}(ABC).$$

Given $\triangle ABC$, we define the **complement** of a point P as $P^{\complement} = \mathfrak{h}_{G,-\frac{1}{2}}$. If the reference triangle needs specification, $P^{\complement,\triangle ABC}$ will be used. Then the complement of (ABC) is ε and $A^{\complement} = M_a, H^{\complement} = O$.

Likewise, we define the **anti-complement** of P as $P^{\mathbb{C}}$ is the inverse of that operation. We define the **anticomplementary triangle** as the image $(\triangle ABC)^{\mathbb{C}}$. □

Example 1.1.6 (Hong Kong TST 2018/6)**.** Let the orthocenter and circumcenter of $\triangle ABC$ be H and O, respectively, and let the circumcenters of $\triangle HBC, \triangle HCA, \triangle HAB$ be O_a, O_b, O_c, respectively. Prove that $\overline{AO_a}, \overline{BO_b}, \overline{CO_c}$, and $\overline{OH}$ concur.



Solution. Let $\triangle H_a H_b H_c$ be the orthic triangle with respect to $\triangle ABC$.

Notably, $\triangle HBC$, $\triangle HCA$, and $\triangle HAB$ all share the same orthic triangle. As such, $\triangle ABC$, $\triangle HBC$, $\triangle HCA$, and $\triangle HAB$ share a nine point circle and thus the same nine point center N_9.

Since AO_a, BO_b, CO_c all have N_9 as a midpoint, N_9 lies on all such lines. □

Proposition 1.1.7. The composition of two homotheties is itself a homothety whose ratio is the product of the ratios of the earlier two homotheties.

Proof. The original proof does a load of casework involving vectors so we present an alternative proof. Consider points in $\mathbb{R}^2$.

Note that a homothety is effectively a mapping from P to $k(P-O)+O$, which simplifies as $P \mapsto kP+C$ for some constant C.

Then the image of two mappings with ratios K_1, k_2 is of the form $P \mapsto k_1k_2P + C$. □

Proposition 1.1.8. For any two circles, there exist two homotheties $\mathfrak{h}_+, \mathfrak{h}_-$ of positive and negative ratio with the same magnitude respectively, that map Γ_1 to Γ_2. We call $\mathfrak{h}_+$ the **exsimilicenter** and $\mathfrak{h}_-$ the **insimilicenter**.

Proof. If the two circles are concentric, then taking the two homotheties with negative signs at their center suffices. If the two circles have the same radius, taking a translation and reflection suffice.

Else, let the circles have centers $O_1 \neq O_2$ and radii $r_1 \neq r_2$, then define O_+, O_- such that $\frac{O_+O_2}{O_+O_1} = \frac{r_2}{r_1}, \frac{O_1O_2}{O_1O_1} = -\frac{r_2}{r_1}$. Then define $\mathfrak{h}_+ = \mathfrak{h}_{h,O_+,r_2/r_1}$ and $\mathfrak{h}_- = \mathfrak{h}_{h,O_-,-r_2/r_1}$. □

Example 1.1.9. The homotheties that map the nine-point circle ε to the circumcircle are $\mathfrak{h}_{H,2}$ and $\mathfrak{h}_{G,-2}$ respectively.

Proposition 1.1.10. The intersection of the external tangents (if they exist) of two circles Γ_1 and Γ_2 is the exsimilicenter of Γ_1 and Γ_2. Similarly, the intersection of the internal tangents of Γ_1 and Γ_2 is the insimilicenter of the two circles.

Proof. We will only prove one statement — the other is analogous. Suppose the two external tangents to the two circles are K_+ and L_+ that meet at O_+. Let $P_{+,1}$ and $P_{+,2}$ be the intersections of K_+ with Γ_1 and Γ_2, respectively, and define $Q_{+,1}$ and $Q_{+,2}$ for L_+ similarly. Finally, let O_1 and O_2 be the centers of Γ_1 and Γ_2, respectively. Note that $\overline{O_1O_2} = \ell$ is the internal angle bisector of $\measuredangle(K_+, L_+)$, so $P_{+,1}$, $Q_{+,1} \in (O_+O_1)$ and $P_{+,2}$, $Q_{+,2} \in (O_+O_2)$. Hence we have

$$\measuredangle O_+P_{1,+}Q_{1,+} = \measuredangle O_+O_1Q_{1,+} = 90^\circ = \measuredangle(L_+, \ell) = O_+O_2Q_{2,+}, = \measuredangle O_+P_{2,+}Q_{2,+}$$

and similarly $\measuredangle P_{1,+}Q_{1,+}O_+ = \measuredangle P_{2,+}Q_{2,+}O_+$ so $\triangle O_+P_{1,+}Q_{1,+}$ and $\triangle O_+P_{1,+}Q_{1,+}$ are homothetic about O_+ by a homothety $\mathfrak{h} = \mathfrak{h}_{O_+,k}$. Then it is easily checked that $\mathfrak{h}(O_1) = O_2$, and since

$$OP_{2,+} = k \cdot OP_{1,+},$$

$\mathfrak{h}(\Gamma_1) = \Gamma_2$, as desired. □

This allows us to prove the following common lemma:

Example 1.1.11. Let I be the incenter of $\triangle ABC$, D be the A-excircle touchpoint on $\overline{BC}$, and M be the midpoint of $\overline{BC}$. Prove that $\overline{AD} \parallel \overline{IM}$.

Solution. Note that CA, AB are the common tangents between ω and the A-excircle. Thus we get that A is the exsimilicenter of ω and ω_a. Let us look at the homothety $\mathfrak{h}$ from A which sends ω_a to ω. Then we know that $\mathfrak{h}(BC)$ is tangent to the incircle (and obviously it's parallel to BC. Thus $\mathfrak{h}(BC)$ is tangent to ω and parallel to BC, so it must be parallel to the antipode of D, which we will call D^*. Further, we get that $D^* = \mathfrak{h}(D)$, so A, D^*, D lie on a line. Since I and M are respectively the midpoints of $\overline{D^*D}$ and $\overline{DD'}$, we have $IM \parallel D^*D' = AD'$. □

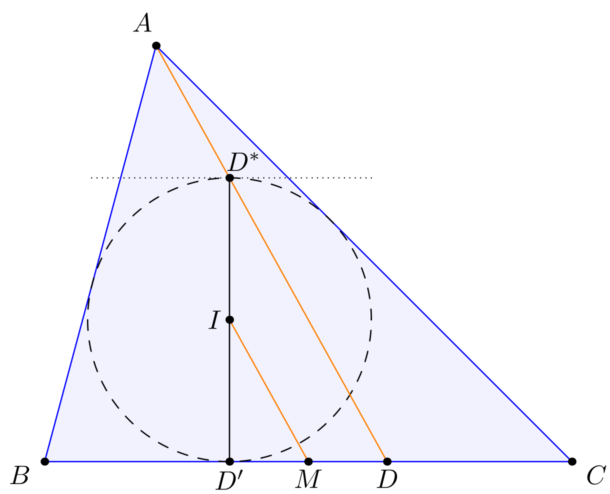


Figure 1.1: Also, $AD' \parallel I_aM$.

Theorem 1.1.12 (Monge's Theorem)**.** For any three circles $\Gamma_1, \Gamma_2, \Gamma_3$, let $O_{ij,+}, O_{ij,-}$ be the respective exsimilicenter and insimilicenters of Γ_i, Γ_j. Then when there are an odd number of plus signs, we get that $O_{12,\pm}, O_{23,\pm}, O_{31,\pm}$ are collinear.

Proof. We assume different radii for simplicity. Let O_i, r_i be the center and radii of Γ_i. Then we have by Proposition 1.1.8 that

$$\frac{O_iO_{ij,\pm}}{O_{ij,\pm}O_j} = \mp\frac{r_i}{r_j}.$$

When there are an odd number of plus signs, we get that

$$\prod_{\text{cyc}} \frac{O_1O_{12,\pm}}{O_{12,\pm O_2}} = \prod_{\text{cyc}} \mp\frac{r_1}{r_2} = -1.$$

so by Menelaus the result follows. □

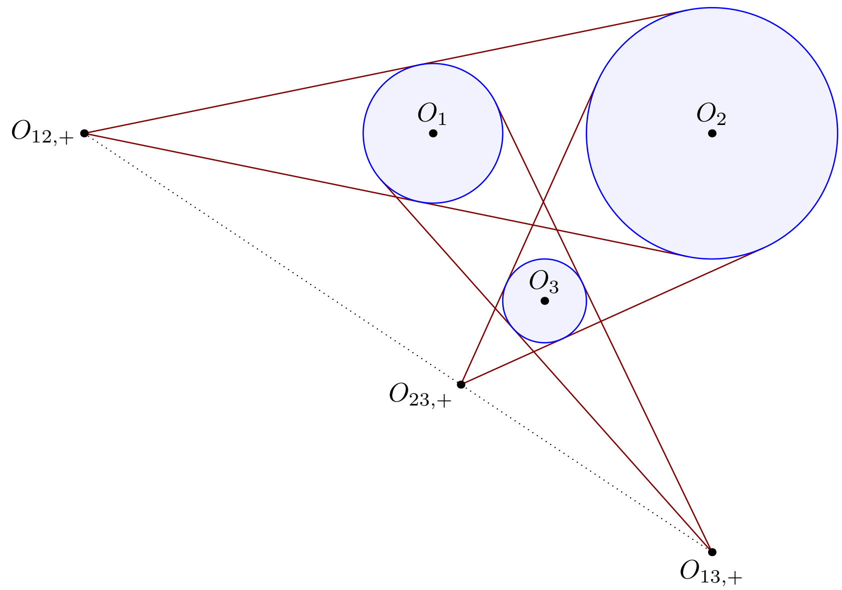


Figure 1.2: An alternate proof of this theorem is that "compositions of homotheties are still homotheties".

Remark. Similarly, by Ceva we have that when the number of plus signs is even, the lines $O_1O_{23,\pm}$, $O_2O_{31,\pm}$, and $O_3O_{12,\pm}$ are concurrent. (This is another example of the duality we referred to earlier.)

However this is usually useless.

Actually, Monge's theorem is better stated as a special case of the following theorem:

Theorem 1.1.13 (Common Perspectrix Implies Collinear Perspectors)**.** Let $\triangle_1 = \triangle A_1B_1C_1$, $\triangle_2 = \triangle A_2B_2C_2$, and $\triangle_3 = \triangle A_3B_3C_3$ be three triangles such that they are pairwise perspective. Let P_{ij}, ℓ_{ij} be the perspector and perspectrix of $\triangle_i$ and $\triangle_j$ respectively. Then

(i) If $\ell_{23} = \ell_{31} = \ell_{12}$, we have P_{23}, P_{31}, P_{12} collinear.

(ii) If $P_{23} = P_{31} = P_{23}$, we have $\ell_{23}, \ell_{31}, \ell_{12}$ are concurrent.

Proof. For (i), we first consider the two triangles $\triangle_b = \triangle B_1B_2B_3$, $\triangle_c = \triangle C_1C_2C_3$. Suppose $\ell_{23} = \ell_{31} = \ell_{12}$. Then we have that $\overline{B_1C_1}, \overline{B_2C_2}, \overline{B_3C_3}$ are concurrent, and thus these two triangles are perspective, so by Desargues's Theorem $P_{23} = \overline{B_2B_3} \cap \overline{C_2C_3}$, $P_{31} = \overline{B_3B_1} \cap \overline{C_3C_1}$, $P_{12} = \overline{B_1B_2} \cap \overline{C_1C_2}$ are collinear.

For (ii), consider the two triangles $\triangle_b = \triangle(C_1A_1)(C_2A_2)(C_3A_3)$ and $\triangle_c = \triangle(A_1B_1)(A_2B_2)(A_3B_3)$. Suppose $P_{23} = P_{31} = P_{12}$. Then we have that $\overline{A_1A_2A_3}$ is the perspectrix $\triangle_b$ and $\triangle_c$, Let $b_i = \overline{C_iA_i}$ and $c_i = \overline{A_iB_i}$. Then $l_{23} = \overline{(b_2 \cap b_3)(c_2 \cap c_3)}, l_{31}, l_{12}$ concur by Desargues's Theorem.

(Note the similarity of the proof of (ii) to the proof of (i).) $\square$

Remark. If you set the line in part (i) as the line at infinity, then we get that $\triangle_1, \triangle_2, \triangle_3$ are all homothetic. Then the theorem implies that all of the centers of homothety are collinear! Further, if you let $\mathfrak{h}_1$ be the homothety sending $\triangle_1 \mapsto \triangle_2$, and let $\mathfrak{h}_2$ be the homothety sending $\triangle_2 \mapsto \triangle_3$, then the composition $\mathfrak{h}_1 \circ \mathfrak{h}_2$ is a homothety sending $\triangle_1 \mapsto \triangle_3$, and all three homothety centers are collinear. This leads to another alternate proof of Monge's Theorem by considering the three triangles' circumcircles.

Practice Problems

Problem 1. Let $ABCD$ be a quadrilateral, and M, N be the midpoints of sides $\overline{AB}, \overline{BC}$, respectively. Let P, Q be two points on $\overline{CD}, \overline{DA}$, respectively, such that $\overline{PQ} \| \overline{MN}$. Prove that $\overline{PN}, \overline{QM}$, and $\overline{BD}$ either concur or are parallel.

Problem 2. For any point P in $\triangle ABC$, let $\triangle DEF$ be the **cevian triangle** [1] of P wrt $\triangle ABC$.

Let $X = EF \cap BC, Y = FD \cap CA, Z = DE \cap AB$. Show that X, Y, Z are collinear. This line is called the **trilinear polar** of P, and is notated as $\mathfrak{t}(P)$.

Problem 3 (Crosspoint definition)**.** Take $\triangle ABC$. Let D, E, F be chosen on AB, BC, CA such that $\triangle ABC, \triangle DEF$ are perspective with center P. Let X, Y, Z be chosen on DE, EF, FD such that $\triangle ABC$ and $\triangle XYZ$ are perspective with center Q. Show that $\triangle XYZ$ and $\triangle DEF$ are perspective with center R. We call R the **crosspoint** of P and Q, which is denote with a $P \pitchfork Q$.

In fact, the crosspoint is symmetric: $P \pitchfork Q = Q \pitchfork P$.

Problem 4. Let $\triangle M_a M_b M_c$ be the medial triangle of $\triangle ABC$. Construct a line ℓ which intersects $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F respectively. Let D' be the reflection of D across M_a, and define E' and F' respectively. Prove that

(i) D^*, E^*, F^* lie on a line ℓ^*.

(ii) The midpoints of AD, BE, CF lie on $\tau_\ell = (\ell^*)^{\complement}$.

We call τ_ℓ as the **Newton line** of the complete quadrilateral $\triangle ABC \cup \ell$. (We will further discuss this line in Section 4.1).

Problem 5. Let H be the orthocenter of $\triangle ABC$, and P an arbitrary point on the circumcircle of $\triangle ABC$. Let E be the foot from B to $\overline{CA}$. Pick points Q, R such that $PAQB$ and $PARC$ are both parallelograms. [2] Let $\overline{AQ}$ and $\overline{HR}$ intersect at X. Prove that $EX \parallel AP$.

[1] Where $D = AP \cap BC, E = BP \cap AC, F = CP \cap AB$.
[2] Q is sometimes called the "parallelogram point" of $\triangle PAB$ with respect to P, and similarly for R.

Problem 6 (Brazil 2012/2)**.** $\triangle ABC$ is a non-isosceles triangle. Let $\triangle T_AT_BT_C$ be the intouch triangle. Let I_A, I_B, I_C be the excenters. Let X_A be the midpoint of I_BI_C and define X_B, X_C similarly. Show that X_AT_A,X_BT_B,X_CT_C, OI concur at a point, where O is the circumcenter and I is the incenter.

(This problem is in the book with X_AT_A replaced with I_AT_A, but it remains true).

Problem 7. Prove line $\overline{IO}$ is the Euler line of the intouch triangle.

Problem 8 (Taiwan TST 2015/1J/I2-2)**.** Given a triangle $\triangle ABC$, let H be the orthocenter and G the centroid. Show that (HG), the circumcircle, and the nine-point circle are coaxial.

Problem 9. Let I be the incenter of $\triangle ABC$. Let D be an arbitrary point on $\overline{BC}$, and let ω_B and ω_C respectively be the incircles of $\triangle ABD$ and $\triangle ACD$. Let ω_B and ω_C touch $\overline{BC}$ at E and F. Let P be the intersection of $\overline{AD}$ and the line connecting the centers of ω_B and ω_C. Let X be the intersection of BI and CP, and let Y be the intersection of CI and BP. Prove that EX and FY intersect on the incircle of $\triangle ABC$.

1.2 Spiral Similarity

Given a fixed point O and a fixed angle θ, we can define a **rotation** as the mapping $\mathfrak{r}_{O,\theta}$ which maps a point P to the point Q such that

$$\overline{OP} = \overline{OQ}, \quad \measuredangle POQ = \theta \pmod{360^\circ}$$

We then have that

$$\mathfrak{r}_{O,\theta_1} \circ \mathfrak{r}_{O,\theta_2} = \mathfrak{r}_{O,\theta_2} \circ \mathfrak{r}_{O,\theta_1} = \mathfrak{r}_{O,\theta_1+\theta_2}$$

In fact, the composition of two rotations is itself a rotation. We also note that a rotation preserves angles and lengths, ie

$$\overline{\mathfrak{r}(A)\mathfrak{r}(B)} = \overline{AB}, \quad \measuredangle \mathfrak{r}(A)\mathfrak{r}(B)\mathfrak{r}(C) = \measuredangle ABC \pmod{360^\circ}$$

As such, for a triangle $\triangle ABC$ we have that $\mathfrak{r}(\triangle ABC) = \triangle \mathfrak{r}(A)\mathfrak{r}(B)\mathfrak{r}(C) \overset{+}{\cong} \triangle ABC$.

$\mathfrak{r}_{O,180^\circ} = \mathfrak{s}_O = \mathfrak{h}_{O,-1}$ is a reflection about O. Now, we merge a rotation with a composition to get a **spiral similarity** $\mathfrak{S}_{O,k,\theta} = \mathfrak{r}_{O,\theta} \circ \mathfrak{h}_{O,k} = \mathfrak{h}_{O,k} \circ \mathfrak{r}_{O,\theta}$. Then we have that

(i) $\mathfrak{S}_{O,k_1,\theta_1} \circ \mathfrak{S}_{O,k_2,\theta_2} = \mathfrak{S}_{O,k_1k_2,\theta_1+\theta_2}$

(ii) $\mathfrak{S}^{-1}_{O,k_1,\theta} = \mathfrak{S}_{O,k_1^{-1},-\theta}$

(iii) $\mathfrak{S}_{O,-k,\theta} = \mathfrak{S}_{O,k,\theta+180^\circ}$

Notably, we can assume that spiral similarities have positive ratios. For completeness, translations can also be considered spiral similarities with center at infinity with $k = 1, \theta = 0$.

Earlier we defined the Miquel point for four lines $\ell_1, \ell_2, \ell_3, \ell_4$ as $\bigcap(\ell_i \ell_{i+1} \ell_{i+2})$, let us now expand on this definition.

Definition 1.2.1. For four points A_1, A_2, B_1, B_2 satisfying $A_1 \neq A_2$, $B_1 \neq B_2$, $A_1 \neq B_1$, $A_2 \neq B_2$ (notably we allow $A_1 = B_2$ or $A_2 = B_1$), we define the **Miquel point** of the quadrilateral $A_1B_1B_2A_2$ to be the Miquel point of the four lines $(A_1B_1, B_1B_2, B_2A_2, A_2A_1)$.

Here are some degenerate cases.

- If any three of these points are collinear (WLOG here A_1, B_1, B_2), then the circumcircle of the degenerate triangle $\triangle(\overline{A_1B_1})(\overline{B_1B_2})(\overline{B_2A_2})$ is the circle passing through B_1, B_2 that is tangent to $\overline{B_2A_2}$.
- If any two lines are parallel (WLOG $A_1B_1 \parallel B_2A_2$), then we take the circumcircle of the triangle formed by $\overline{A_1B_1}$, $\overline{B_1B_2}$, and $\overline{B_2A_2}$) to be $\overline{B_1B_2}$.
- If all four points lie on a line, then M is also on the line such that

$$\frac{MB_1}{MA_1} = \frac{MB_2}{MA_2}.$$

It is easy to show that such a point M still exists in these degenerate cases.

(This book often writes the quadrilateral $A_1B_1B_2A_2$ as $(A_1B_2)(B_1A_2)$ to make the definitions more symmetrical, which will be used in the case of complete quadrilaterals. Here, $\overline{A_1B_2}, \overline{B_2A_1}$ are the $\binom{4}{2} - 2$ lines not in the aforementioned four lines). Then the major property of the Miquel point is the following:

Proposition 1.2.2. If O is the center of the spiral similarity that maps A_1A_2 to B_1B_2, then M is the Miquel point of a complete quadrilateral $(A_1B_2)(B_1A_2)$.

Proof. We will only prove the result in the non-degenerate case outlined above. If $\mathfrak{S}_{O,k,\theta}$ is the transformation, then take k as positive. We then have that

$$k = \frac{\overline{OB_1}}{\overline{OA_1}} = \frac{\overline{OB_2}}{\overline{OA_2}}, \quad \theta = \measuredangle A_1OB_1 = \measuredangle A_2OB_2 \pmod{360^\circ},$$

and that $\triangle OA_1B_1 \overset{+}{\sim} OA_2B_2$. By Spiral Similarity, we also have that $\triangle OA_1A_2 \overset{+}{\sim} \triangle OB_1B_2$. As such, if

$C = A_1B_1 \cap B_2A_2, D = B_1B_2 \cap A_2A_1$, then

$$\begin{aligned}\measuredangle A_1CA_2 &= \measuredangle A_1B_1O + \measuredangle B_1OB_2 + \measuredangle OB_2A_2 \\ &= (\measuredangle A_1B_1O - \measuredangle A_2B_2O) + \measuredangle A_1OA_2 = \measuredangle A_1OA_2, \\ \measuredangle A_1DB_1 &= \measuredangle A_1A_2O + \measuredangle A_2OB_2 + \measuredangle OB_2B_1 \\ &= (\measuredangle A_1A_2O - \measuredangle B_1B_2O) + \measuredangle A_1OB_1 = \measuredangle A_1OB_1.\end{aligned}$$

As such, O lies on (A_1A_2C) and (A_1B_1D). Similarly, O lies on (B_1B_2C) and (A_2B_2D) which finishes. $\square$

Proposition 1.2.3. For any four points A_1, A_2, B_1, B_2 that satisfy $A_1 \neq A_2$, $B_1 \neq B_2$, there exists a unique spiral similarity $\mathfrak{S}$ such that $\mathfrak{S}(A_1) = B_1, \mathfrak{S}(A_2) = B_2$.

Proof. If $A_1 \neq B_1$, $A_2 \neq B_2$, then let O be the Miquel point of $(A_1B_2)(B_1A_2)$. To check all (possibly degenerate) cases.

If there are no degeneracies and $O \neq \infty$, then set $C = \overline{A_1B_1} \cap \overline{B_2A_2}$. We have

$$\begin{cases}\measuredangle OA_1B_1 = \measuredangle OA_1C = \measuredangle OA_2C = \measuredangle OA_2B_2, \\ \measuredangle OB_1A_1 = \measuredangle OB_1C = \measuredangle OB_2C = \measuredangle OB_2A_2,\end{cases} \qquad \triangle OA_1B_1 \overset{+}{\sim} \triangle OA_2B_2$$

so if we set

$$k = \frac{OB_2}{OA_2}, \quad \theta = \measuredangle A_1OB_1 = \measuredangle A_2OB_2 \pmod{360^\circ}$$

then the spiral similarity $\mathfrak{S} = \mathfrak{S}_{O,k,\theta}$ works.

There's some other casework for degenerate cases.

The most notable degenerate case is when $A_1A_2B_1B_2$ is a parallelogram, in which case the spiral similarity becomes a translation. $\square$

Proposition 1.2.4. The composition of two spiral similarities is another spiral similarity. Further, the angle rotated in the composition of two spiral similarities is the sum of the angles rotated in the two individual spiral similarities, and the scale factor in the composition is the product of the scale factors in the two individual spiral similarities.

Proof. We can place our real plane into $\mathbb{C}$ by sending $(a, b) \mapsto a + bi$, where a spiral similarity consists of the mapping $z \mapsto ke^{i\theta} \cdot z + C$ for some complex C. Then applying this twice makes the result immediate. $\square$

Practice Problems

Problem 1. Let circle (P) and circle (Q) intersect in two points. Draw a pair of perpendicular lines through one of these intersection points and let one of these lines intersect $\overline{PQ}$, (P), and (Q) at A, B, and C respectively. The other line intersects $\overline{PQ}$, (P), and (Q) at D, E, and F respectively. Prove that $AB : AC = DE : DF$.

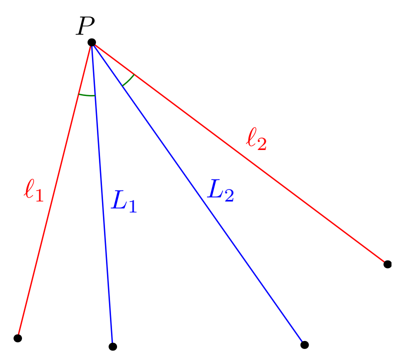
Problem 2. For an arbitrary triangle $\triangle ABC$, let D_1 and D_2 be points on $\overline{BC}$, let E_1 and E_2 be arbitrary points on CA, let F_1 and F_2 be arbitrary points on AB. Let M_1 and M_2 respectively be the Miquel points of $\triangle D_1E_1F_1$ and $\triangle D_2E_2F_2$ wrt. $\triangle ABC$. Prove that $\triangle D_1E_1F_1 \overset{+}{\sim} \triangle D_2E_2F_2$ if and only if $M_1 = M_2$.

Problem 3 (ISL 2014 G4)**.** Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

1.3 Isogonal Conjugation

Isogonal conjugation is one of the most important concepts in plane geometry, and is a commonly used tool in olympiad problems.

Definition 1.3.1. Given two lines ℓ_1, ℓ_2 that intersect at P, call two lines L_1, L_2 passing through P **isogonal lines** with respect to $\angle(\ell_1, \ell_2)$ if and only if L_1, L_2 are antiparallel to ℓ_1, ℓ_2.



Proposition 1.3.2 (Isogonal Ratios)**.** Given a fixed triangle $\triangle ABC$, if D, D^* are on line BC, then AD, AD^* being isogonal lines with respect to $\angle BAC$ implies that

$$\frac{BD}{DC} \cdot \frac{BD^*}{D^*C} = \left(\frac{\overline{AB}}{\overline{CA}}\right)^2$$

Proof. By Proposition 0.3.4, we have that

$$\frac{BC/DC}{\overline{AB}/\overline{CA}} \cdot \frac{BD^*/D^*C}{\overline{AB}/\overline{CA}} = \frac{\sin\measuredangle(\overrightarrow{AB}, AD)}{\sin\measuredangle(AD, \overrightarrow{AC})} \cdot \frac{\sin\measuredangle(\overrightarrow{AB}, AD^*)}{\sin\measuredangle(AD^*, \overrightarrow{AC})} = 1.$$

□

We can further define an important object, the **complete n-gon**, written as

$$\mathcal{N} = (\ell_1, \ell_2, \ldots, \ell_n)$$

representing the n-gon formed by the $\binom{n}{2}$ intersections of ℓ_i, ℓ_j, such that no three lines are concurrent.

Definition 1.3.3. Given a complete n-gon $\mathcal{N} = (\ell_1, \ell_2, \ldots, \ell_n)$, we define $A_{ij} = \ell_i \cap \ell_j$. If for some point P, there exists a point P^* such that for all i, j, $A_{ij}P, A_{ij}P^*$, are isogonal lines with respect to $\angle(\ell_i, \ell_j)$, then we say that P and P^* are **isogonal conjugates** with respect to $\mathcal{N}$. Notably, $(P^*)^* = P$.

Example 1.3.4. The orthocenter H and circumcenter O are isogonal conjugates wrt $\triangle ABC$, and I, I^a, I^b, I^c are fixed points under isogonal conjugation.

Proposition 1.3.5. For a triangle $\triangle ABC$ and point P, a isogonal conjugate P^* of P always exists, potentially at infinity.

Proof. One way to do this is through Ceva:

$$\prod \frac{\sin\measuredangle(\overrightarrow{AB}, A\infty_{AB+AC-AP})}{\sin\measuredangle(A\infty_{AB+AC-AP}, \overrightarrow{AC})} = \prod \frac{\sin\measuredangle(AP, \overrightarrow{AC})}{\sin\measuredangle(\overrightarrow{AB}, AP)} = 1$$

Here's a "synthetic" approach to showing existence. First suppose that $P \notin (ABC), \mathcal{L}_\infty$. Define $\triangle P_aP_bP_c$ as the **circumcevian triangle** of P ($P_a = AP \cap (ABC)$ and so forth).

Let D, E, F be the feet from P to P_bP_c, P_cP_a, P_aP_b, respectively. Then

$$\measuredangle EDF = \measuredangle EDP + \measuredangle PDF = \measuredangle P_aP_cC + \measuredangle BP_bP_c = \measuredangle BAC,$$

so similarly $\measuredangle FED = \measuredangle CBA, \measuredangle DFE = \measuredangle ACB$. As such, $\triangle ABC \sim \triangle DEF$. Let P^* be the point such that $\triangle ABC \cup P^* \overset{+}{\sim} \triangle DEF \cup P$. Then it follows that

$$\measuredangle(AP + AP^*, AB + AC) = \measuredangle P_aAB + \measuredangle PDF = \measuredangle P_aP_bB + \measuredangle PP_bF = 0^\circ$$

so P^* is the desired point.

Now suppose that P is on (ABC) or the line at infinity. Let ℓ_A, ℓ_B, ℓ_C be the isogonal conjugates of AP, BP, CP wrt $\angle A, \angle B < \angle C$ respectively. Then

$$\begin{aligned}\measuredangle(\ell_B, \ell_C) &= \measuredangle(\ell_B, BC) + \measuredangle(BC, \ell_C) = \measuredangle ABP + \measuredangle PCA \\ &= \begin{cases} 0^\circ, & P \in (ABC) \\ \measuredangle BAC, & P \in \mathcal{L}_\infty \end{cases}\end{aligned}$$

As such, if $P \in (ABC)$ then $\ell_A \parallel \ell_B \parallel \ell_C$ converge at infinity. Else, if $P \in \mathcal{L}_\infty$ then we get that $\ell_B \cap \ell_C \in (ABC)$, so $\ell_A \cap \ell_B \cap \ell_C$ converge on (ABC). □

Proposition 1.3.6. Given a triangle $\triangle ABC$, points P, Q are isogonal conjugates if and only if

$$\measuredangle CPA + \measuredangle CQA = \measuredangle CBA, \quad \measuredangle APB + \measuredangle AQB = \measuredangle ACB.$$

Proof. First suppose they are isogonal conjugates. Then

$$\begin{aligned}\measuredangle CPA + \measuredangle CQA &= \measuredangle PCB + \measuredangle CBA + \measuredangle BAP + \measuredangle CQA \\ &= \measuredangle ACQ + \measuredangle CBA + \measuredangle QAC + \measuredangle CQA = \measuredangle CBA\end{aligned}$$

The other angle condition follows by symmetry.

For the other direction, note that the forward direction gives $\measuredangle CQA = \measuredangle CBA - \measuredangle CPA = \measuredangle CP^*A$, so $Q \in (CAP^*)$. As such, by symmetry

$$Q \in (CAP^*) \cap (ABP^*) = \{A, P^*\}$$

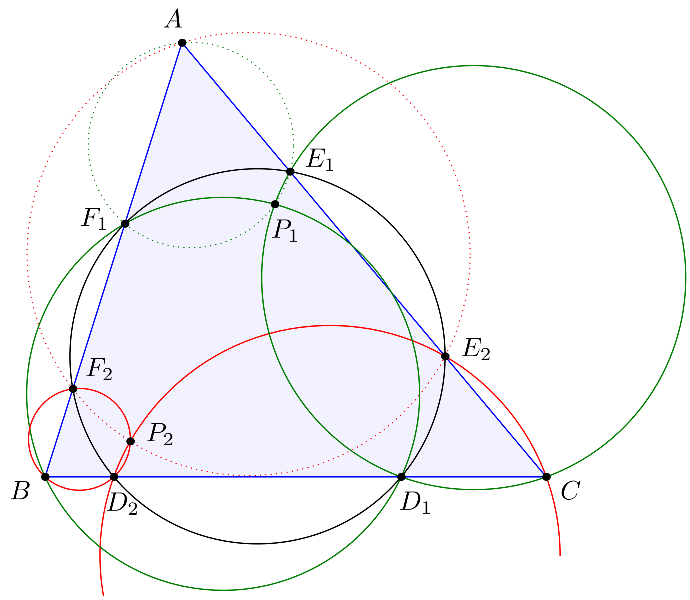
so $Q = P^*$. □

Example 1.3.7 (Generalized Six Point Circle)**.** Suppose the circle Γ intersects the sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D_1 and D_2, E_1 and E_2, and F_1 and F_2 respectively. Let $P_1 = (F_1BD_1) \cap (D_1CE_1) \neq D_1$, and let $P_2 = (F_2BD_2) \cap (D_2CE_2) \neq D_2$. Prove that $\measuredangle BAP_1 = \measuredangle P_2AC$.

Proof. By Triangle Miquel, $P_1 \in (E_1AF_1)$ and $P_2 \in (E_2FA_2)$. Now

$$\begin{aligned}\measuredangle BP_1C + \measuredangle BP_2C &= \measuredangle BP_1D_1 + \measuredangle D_1P_1C + \measuredangle BP_2D_2 + \measuredangle D_2P_2C \\ &= \measuredangle AF_1D_1 + \measuredangle D_1E_1A + \measuredangle F_1F_2D_2 + \measuredangle D_2E_2E_1 \\ &= \measuredangle BAC + \measuredangle E_1D_1F_1 + \measuredangle F_1D_1D_2 + \measuredangle D_2D_1E_1 \\ &= \measuredangle BAC.\end{aligned}$$

We can get analogous equations for all three vertices of the triangle, so by Proposition 1.3.6 P_1 and P_2 are isogonal conjugates in $\triangle ABC$ - in particular $\measuredangle BAP_1 = \measuredangle P_2AC$.



$\square$

Since $\triangle AE_1F_1 \cup P_1 \overset{-}{\sim} \triangle AF_2E_2 \cup P_2$, we also have

$$\begin{aligned}&\measuredangle(\overline{P_1D_1}, \overline{BC}) = \measuredangle(\overline{P_1E_1}, \overline{CA}) = \measuredangle(\overline{P_1F_1}, \overline{AB}) \\ &= -\measuredangle(\overline{P_2D_2}, \overline{BC}) = -\measuredangle(\overline{P_2E_2}, \overline{CA}) = -\measuredangle(\overline{P_2F_2}, \overline{AB})\end{aligned}$$

which generalizes to n-gons as follows:

Proposition 1.3.8 (Generalized Pedal Circles)**.** Given a complete-n gon $\mathcal{N} = (\ell_1, \ell_2, \ldots, \ell_n)$, a point P, and an angle $\alpha \neq 0°$, take a point P_i on ℓ_i such that $\measuredangle(\ell_i, \overline{PP_i}) = \alpha$. Then P has an isogonal conjugate with respect to $\mathcal{N}$ if and only if P_1, P_2, $\ldots$, P_n are concyclic.

In this case, let P^* be this isogonal conjugate. Define P_i^* as the point on ℓ_i so that $\measuredangle(\ell_i, P^*P_i^*) = -\alpha$. Then $P_i^* \in (P_1P_2 \dots P_n) = \Gamma_\alpha$. for all i. (This is a generalization of the six-point circle theorem.) Additionally, the center of Γ_α, or O_α, is on the perpendicular bisector of $\overline{PP^*}$ and satisfies $\angle O_\alpha PP^* = 90° - \alpha$.

Proof. Define $A_{ij} = \ell_i \cap \ell_j$. For the forward direction, let P^* be the isogonal conjugate of P with respect to $\mathcal{N}$, and define a P_i^* for every ℓ_i such that $\measuredangle(\ell_i, P^*P_i^*) = -\alpha$. Then from $\triangle A_{ij}P_iP_j \cup P \overset{-}{\sim} \triangle A_{ij}P_j^*P_i^* \cup P^*$ we know that P_i, P_j, P_i^*, P_j^* are concyclic. Since ℓ_i, ℓ_j, ℓ_k are not concurrent, we can use Corollary 0.4.15 to get that $P_i, P_j, P_k, P_i^*, P_j^*, P_k^*$ are concyclic, and thus $P_1, P_2, \dots, P_n, P_1^*, P_2^*, \dots P_n^*$ are concyclic.

When $\alpha = 90°$, we have that the circumcenter (let's call it $O_{90°}$) lies on all of the perpendicular bisectors of $\overline{P_iP_i^*}$. Given $\alpha \neq 90°$, we can consider the spiral similarity with center P, $\mathfrak{S}_{P,(\sin\alpha)^{-1},\alpha-90°}$, and we get that $\Gamma_a = \mathfrak{S}(\Gamma_{90°})$, therefore $O_\alpha = \mathfrak{S}(O_{90°})$ lies on the perpendicular bisector of $\overline{PP^*}$ and satisfies

$$\measuredangle O_\alpha PP^* = \measuredangle O_\alpha PO_{90°} = 90° - \alpha,$$

which proves this half.

For the backwards direction, let P_i^* be the second intersection of $(P_1P_2P_3 \dots P_n)$ with ℓ_i. For any $\triangle A_{jk}A_{ki}A_{ij}$, take lines

$$\measuredangle(\ell_i, L_i^*) = \measuredangle(\ell_j, L_j^*) = \measuredangle(\ell_k, L_k^*) = -\alpha$$

such that L_i^*, ℓ_i, and $(P_1P_2 \dots P_n)$ occur.

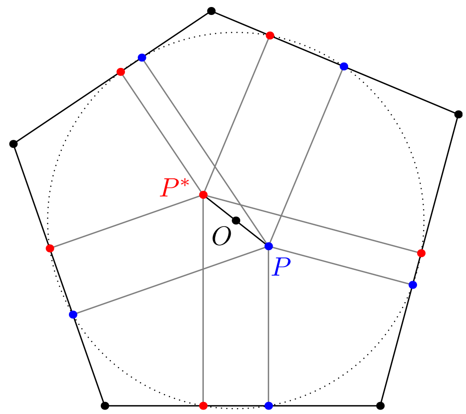
Then by Example 1.3.7, L_i^*, L_j^*, L_k^* concur at the isogonal conjugation P^* of P with respect to $\triangle A_{jk}A_{ki}A_{ij}$. Thus $L_1^*, L_2^*, \cdots, L_n^*$ are concurrent, suppose at point P^*, thus P^* is the isogonal conjugate of P in $\mathcal{N}$.

□

So we can say that it's similar to the situation for a triangle, in which we can say that P is the Miquel point of $(P_1, \dots, P_n)$ in $\mathcal{N} = (\ell_1, \dots, \ell_n)$.

When $\alpha = 90°$, we can get rid of the directed angles.

Proposition 1.3.9. Given a complete n-gon $\mathcal{N}(\ell_1, \ell_2, \dots, \ell_n)$ and a point P, let P_i be the foot from P to ℓ_i. Define P^* as the isogonal conjugate of P and define P_i^* similarly. Then $P_1, P_2, \dots, P_n, P_1^*, P_2^*, \dots, P_n^*$ are concyclic, with circumcenter at the midpoint of $\overline{PP^*}$.



Definition. We call this circle the **pedal circle** of P. Additionally, if $\mathcal{N} = \triangle ABC$, we call the triangle $\triangle P_aP_bP_c$ made by the feet of P to BC, CA, AB the **pedal triangle** of P. These concepts lead to a nice way to prove two points are isogonal conjugates without resorting to cevian angles.

Definition. We also define the **antipedal triangle** of P as the triangle for which $\triangle ABC$ is the P-pedal triangle, so just the triangle formed by the intersections of the perpendiculars to AP, BP, CP at A, B, C.

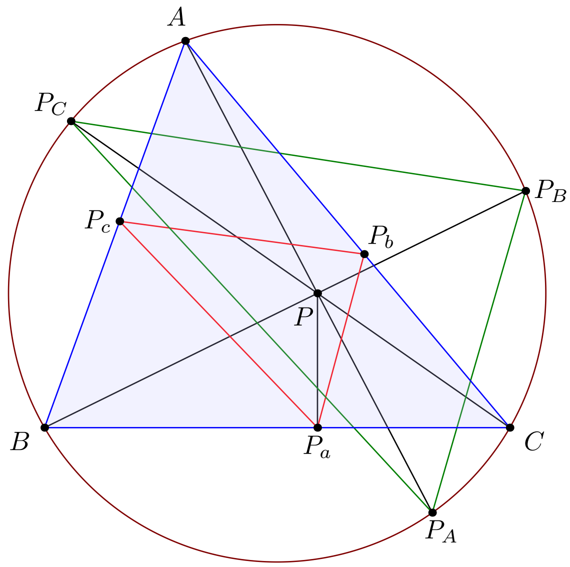
Example 1.3.10. I is its own isogonal conjugate.

We prove this by noting the pedal triangle of I is simply the intouch triangle, and its pedal circle is just the incircle. However, the incircle is tangent to the three sides of the triangle, so $I = I^*$.

Example 1.3.11. Prove O and H are isogonal conjugates.

Simply note that the pedal triangles of O and H are the medial and orthic triangles respectively, and these both lie on the nine-point circle.

Proposition 1.3.12. Let $\triangle P_aP_bP_c$ be the pedal triangle of an arbitrary point P inside $\triangle ABC$, and let $\triangle P_AP_BP_C$ be the circumcevian triangle of P. Then $\triangle P_aP_bP_c \stackrel{+}{\sim} \triangle P_AP_BP_C$.



Proof. The proof is a simple angle-chase, using all the cyclic quads that pedal triangles give us.

$$\begin{aligned}\measuredangle P_bP_aP_c &= \measuredangle P_bP_aP + \measuredangle PP_aP_c = \measuredangle ACP + \measuredangle PBA \\ &= \measuredangle AP_AP_C + \measuredangle P_BP_AA = \measuredangle P_BP_AP_C,\end{aligned}$$

and similarly for the other three angles. □

The above result immediately proves Simson Line as well.

Proposition 1.3.13. The area of the pedal triangle $\triangle P_aP_bP_c$ is given by

$$[\triangle P_aP_bP_c] = \frac{\mathbf{Pow}_{(ABC)}(P)}{4R^2} \cdot [\triangle ABC],$$

where R is the circumradius of $\triangle ABC$.

Proof. By the Sine Area Formula, we have that

$$\begin{aligned}[\triangle P_aP_bP_c] &= \frac{1}{2} \cdot P_cP_a \cdot P_aP_b \cdot \sin\measuredangle P_bP_aP_c \\ &= \frac{1}{2} \cdot (BP \cdot |\sin\measuredangle CBA|) \cdot (CP \cdot |\sin\measuredangle ACB|) \cdot \sin\measuredangle P_BP_AP_C,\end{aligned}$$

where $\triangle P_AP_BP_C$ is the circumcevian triangle of P. By the Law of Sines, we have that

$$\begin{aligned}CP \cdot \sin\measuredangle P_BP_AP_C &= CP \cdot \sin\measuredangle P_BCP \\ &= PP_B \cdot \sin\measuredangle BP_BC = PP_B \cdot \sin\measuredangle BAC,\end{aligned}$$

and therefore

$$\begin{aligned}[\triangle P_aP_bP_c] &= \frac{1}{2}\cdot(BP\cdot PP_B)\cdot\sin\measuredangle BAC\cdot\sin\measuredangle CBA\cdot\sin\measuredangle ACB\\ &= \frac{1}{2}\cdot\mathbf{Pow}_{(ABC)}(P)\cdot\frac{[\triangle ABC]}{2R^2}.\end{aligned}$$

□

By Proposition 1.3.9 where we set $\mathcal{N}$ to be a complete quadrilateral $\mathcal{Q}$, we get that:

Proposition 1.3.14. Given any quadrilateral $(AC)(BD)$, and any point P, P has an isogonal conjugate in this quadrilateral $\mathcal{Q}$ if and only if (PA, PC) is antiparallel to (PB, PD).

Proof. Let P's feet onto AB, BC, CD, DA be W, X, Y, Z respectively. Note that P has an isogonal conjugate in $\mathcal{Q}$ iff. W, X, Y, Z are concyclic.

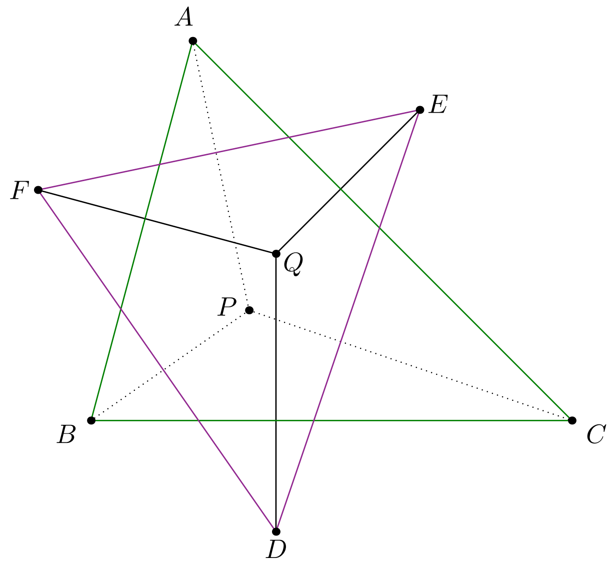
$$\begin{aligned}\measuredangle WXY+\measuredangle YZW=0^\circ &\iff \measuredangle WXP+\measuredangle PXY+\measuredangle YZP+\measuredangle PZW=0^\circ\\ &\iff \measuredangle ABP+\measuredangle PCD+\measuredangle CDP+\measuredangle PAB=0^\circ\\ &\iff \measuredangle APB+\measuredangle CPD=0^\circ.\end{aligned}$$

□

Note that this way to find all points with an isogonal conjugate gives a mysterious locus of points with isogonal conjugates. We will fully hunt down and defeat this locus in Perfection of Isogonal Conjugation, but for now don't worry too much about it.

For deeper investigation, we will explore another way to characterize isogonal conjugates.

Definition 1.3.15. Given two triangles $\triangle ABC$ and $\triangle DEF$, we say they are **orthologic** if the perpendicular $A\infty_{EF}$ from A to EF, $B\infty_{FD}$, and $C\infty_{DE}$ concur at a point P. We call P the **center of orthology**.



Suppose such a P exists. By the Perpendicularity Criterion, we have that

$$AE^2 - AF^2 = PE^2 - PF^2$$
$$BF^2 - BD^2 = PF^2 - PD^2$$
$$CD^2 - CE^2 = PD^2 - PE^2.$$

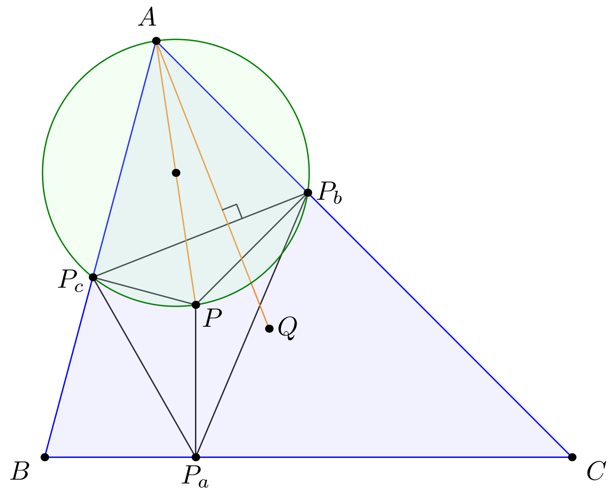
Adding these three equations, we get

$$AE^2 - AF^2 + BF^2 - BD^2 + CD^2 - CE^2 = 0.$$

However now note that the roles of $\triangle ABC$ and $\triangle DEF$ are symmetric!

This gives us that orthology is symmetric!!

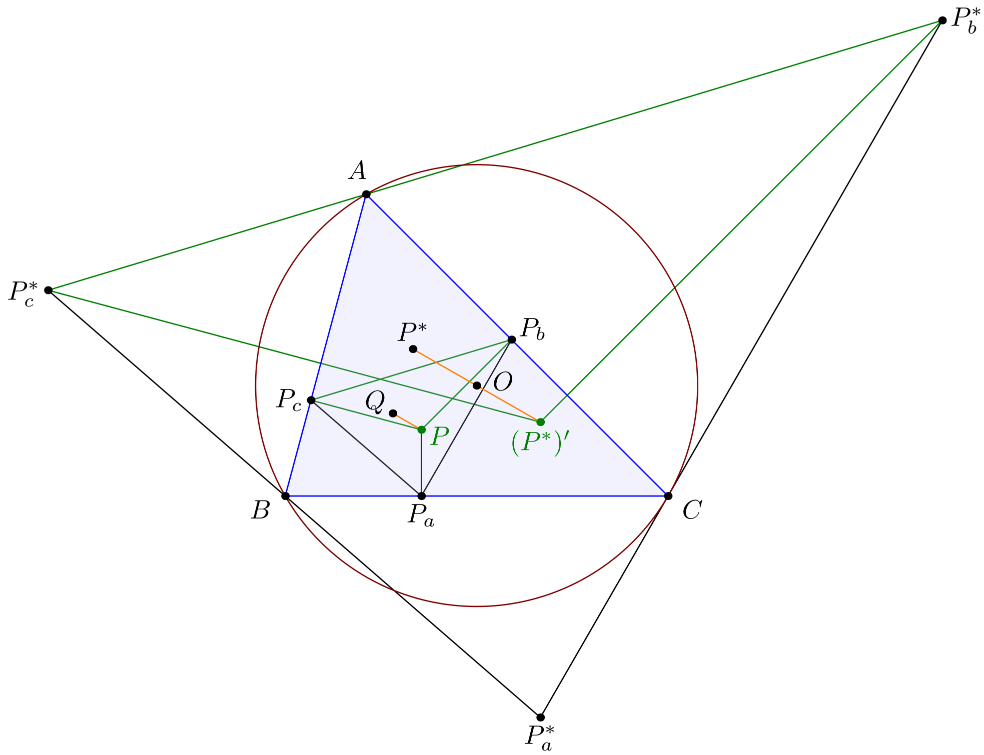
Obviously, the pedal triangle $\triangle P_aP_bP_c$ of any point P not on $\mathcal{L}_\infty$ is orthologic to $\triangle ABC$ (the perpendiculars concur at P). Thus, by symmetry of orthology, the three lines $A\infty_{\perp P_bP_c}$, $B\infty_{\perp P_cP_a}$, $C\infty_{\perp P_cP_a}$ are concurrent. Let this concurrency point be Q. Since P is the antipode of A on (AP_bP_c), we have that the two lines $AQ = A\infty_{\perp P_bP_c}$ and AP are isogonal in $\angle A$. Similarly, BQ, CQ and BP, CP are pairwise isogonal. Thus Q is the isogonal conjugate of P!



By this orthogonality characterization, we can derive some more properties.

Proposition 1.3.16. Given $\triangle ABC$, let O be the circumcenter of $\triangle ABC$, and let (P, P^*) be a pair of isogonal conjugates in $\triangle ABC$. Let $(P^*)'$ be the reflection of P^* in O. Let $\triangle P_aP_bP_c$ be the pedal triangle of P, and let $\triangle P_a^*P_b^*P_c^*$ be the antipedal triangle of P^*. Then we have:

- $(P^*, (P^*)')$ are isogonal in $\triangle P_a^*P_b^*P_c^*$,
- $\triangle P_aP_bP_c \cup P \overset{+}{\sim} \triangle P_a^*P_b^*P_c^* \cup (P^*)'$,
- Let Q be the isogonal conjugate of P in $\triangle P_aP_bP_c$, then $PQ \parallel OP^*$.



Proof. We prove these sequentially.

- Note that the pedal circle of P^* with respect to $\triangle P_a^*P_b^*P_c^*$ is just (ABC), so the isogonal conjugate of P^* wrt. $\triangle P_a^*P_b^*P_c^*$ is just the reflection of P^* over O.

- By definition, we can use two perpendicularities to get that $\triangle PP_bP_c \overset{+}{\sim} \triangle(P^*)'P_b^*P_c^*$, are homothetic, and similarly we get the two other similarities

$$\triangle PP_cP_a \overset{+}{\sim} \triangle(P^*)'P_c^*P_a^*, \quad \triangle PP_aP_b \overset{+}{\sim} \triangle(P^*)'P_a^*P_b^*,$$

  so we get that $\triangle P_aP_bP_c \cup P \overset{+}{\sim} \triangle P_a^*P_b^*P_c^* \cup (P^*)'$.

- Since

$$\triangle P_aP_bP_c \cup P \cup Q \overset{+}{\sim} \triangle P_a^*P_b^*P_c^* \cup (P^*)' \cup P^*,$$

  are homothetic, we get that $PQ \parallel (P^*)'P^* = OP^*$. $\square$

Practice Problems

Problem 1. In $\triangle ABC$, let the reflections of $\overline{BC}$ with respect to $\overline{AB}$ and $\overline{AC}$ meet at K. Prove that $\overline{AK}$ passes through the circumcenter of $\triangle ABC$.

Problem 2. Let M be the midpoint of $\overline{BC}$ in $\triangle ABC$, and let I_1 and I_2 be the incenters of $\triangle ABM$ and $\triangle ACM$, respectively, Prove that (AI_1I_2) passes through the midpoint of the arc $\widehat{BAC}$.

Problem 3. $\triangle ABC$ is an equilateral triangle with circumcircle Γ and let P be a point in the plane not on Γ. Let $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ meet Γ at A_1, B_1, and C_1, respectively. Prove that there are only two points P such that $\triangle A_1B_1C_1$ is a (positive?) triangle, and they are collinear with the circumcenter of $\triangle ABC$.

Problem 4. Let (P, P^*) be a pair of isogonal conjugates with respect to $\triangle ABC$. $\overline{AP}$ and $\overline{AP^*}$ intersect (ABC) at U and B respectively, and $\overline{AP} \cap \overline{BC} = T$. Prove that

$$\frac{PT}{TU} = \frac{AP^*}{P^*V}.$$

Problem 5. Let I be the incenter of $\triangle ABC$, let M be the midpoint of $\overline{AI}$, and let $\overline{BI}$ and $\overline{CI}$ intersect (ABC) at E and F respectively. Take points X, Y on $\overline{AE}$, $\overline{AF}$ respectively so that $\measuredangle XBC = \measuredangle ABM$ and $\measuredangle YCB = \measuredangle ACM$. Prove that I, X, and M are collinear.

Problem 6 (USA TST 2010/7)**.** Let (P, Q) be a pair of isogonal conjugates with respect to $\triangle ABC$, and let D be a point on $\overline{BC}$. Prove that $\measuredangle APB + \measuredangle DPC = 180°$ if and only if $\measuredangle AQC + \measuredangle DQB = 180°$.

Problem 7 (ISL 2007 G2)**.** The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and points P and Q are on opposite sides of line CD. Prove that $\angle BQP = \angle DAQ$.

Problem 8. Let N_9 be the center of the nine-point circle of $\triangle ABC$. Let B' and C' be the reflections of B and C about $\overline{CA}$ and $\overline{AB}$. Prove that $\overline{B'C'}$ is perpendicular to the reflection of $\overline{AN}$ about the angle bisector of $\angle BAC$.

Problem 9 (APMO 2010/4)**.** Let ABC be an acute angled triangle with $AB > BC$ and $AC > BC$. Denote by O and H the circumcenter and orthocenter, respectively, of $\triangle ABC$. Suppose that $(AHC) \cap \overline{AB} = M \neq A$ and $(AHC) \cap \overline{AC} = N \neq A$. Prove that the circumcenter of $\triangle MNH$ lies on $\overline{OH}$.

Problem 10. Given $\triangle ABC$ and a line ℓ, let D, E, and F be the feet of the altitudes from A, B, and C to ℓ respectively. Prove that the altitude from D to $\overline{BC}$, the altitude from E to $\overline{CA}$, and the altitude from F to $\overline{AB}$ concur. This point is called the **orthopole** of ℓ with respect to $\triangle ABC$.

Problem 11 (Taiwan APMOC 2015/5)**.** In $\triangle ABC$, points L, M, and N lie on the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively so that $\triangle ANM$, $\triangle BLN$, and $\triangle CML$ are all acute. Let H_A, H_B, and H_C be the orthocenters of these three triangles (respectively). Suppose $\overline{AH_A}$, $\overline{BH_B}$, and $\overline{CH_C}$ are concurrent. Prove that $\overline{LH_A}$, $\overline{MH_B}$, and $\overline{NH_C}$ are also concurrent.

Problem 12 (Taiwan TST 2021/3J/6)**.** Let $ABCD$ be a rhombus with center O. Select point P lying on segment $\overline{AB}$. Let I, J, and L be the incenters of $\triangle PCD$, $\triangle PAD$, and $\triangle PBC$ respectively. Let H and K be the orthocenters of $\triangle PLB$ and $\triangle PJA$, respectively. Prove that $\overline{OI} \perp \overline{HK}$.

Problem 13 (2022 IRN×TWN P3, modified)**.** Scalene $\triangle ABC$ has intouch triangle $\triangle DEF$ and incenter I. Let Y and Z be the midpoints of $\overline{DF}$ and $\overline{DE}$, respectively. Let J be the isogonal conjugate of I in $\triangle AZY$. Prove that $\overline{IJ} \perp \overline{AS}$.

1.4 Simson and Steiner lines

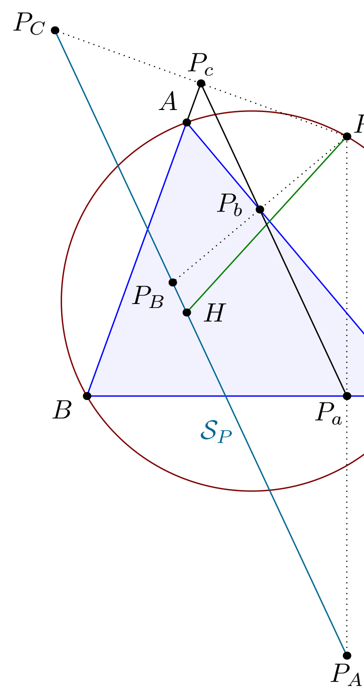
In the previous section, we investigated isogonal conjugates and pedal circles. We know that the isogonal conjugate of the circumcircle is the line at infinity, but it's slightly hard to define pedal triangles and pedal circles for points on the line at infinity. (We will do this in a later chapter!) So we first will instead just look at the pedal circles of points on the circumcircle.

Theorem 1.4.1 (Simson Line)**.** The pedal triangle of any point P on the circumcircle (ABC) is actually just a line.

Proof. Let P_a, P_b, P_c be the feet from P to BC, CA, AB. By Miquel's theorem, we get that P_a lies on P_bP_c if and only if $(BAC), (CP_bP_a), (P_aP_cB)$ concur. The circles concur at P, so we are done.

(Another immediate proof is by usage of Proposition 1.3.13, to get that the area of the pedal triangle is zero.) □

We call the line formed by $\overline{P_aP_bP_c}$ the **Simson line** of P with respect to $\triangle ABC$. Additionally, the line $\overline{P_AP_BP_C}$ formed by the reflections of P over BC, CA, AB (which is just $\mathfrak{h}_{P,2}(\overline{P_aP_bP_c})$) is known as the **Steiner line** of P.



Theorem 1.4.2 (Steiner's Orthocenter Theorem)**.** The Steiner line $\mathcal{S}_P$ of any point P on (ABC) will always pass through the orthocenter H.

Proof. Let Ω be the circumcircle.

Let P_B, P_C respectively be the reflection of P across CA and AB. Let H_B, H_C respectively be the reflections of H across CA and AB. By Problem 1 H_B and H_C lie on (ABC).

$$\begin{aligned}\measuredangle HP_B - \measuredangle HP_C &= (2\measuredangle CA - \measuredangle H_BP) - (2\measuredangle AB - \measuredangle H_CP) \\ &= (2C - H_B - 2B + H)C)_\Omega \\ &= 2C - \perp (C + A - B) - 2B + \perp (A + B - C) = 0^\circ\end{aligned}$$

So H lies on $\overline{P_BP_C} = \mathcal{S}_P$. $\square$

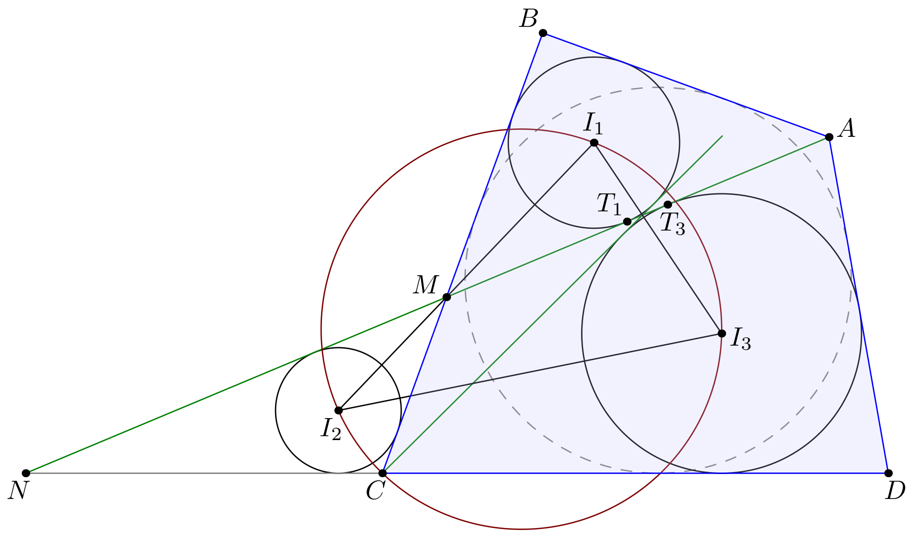
Since the Simson line gets sent to the Steiner line under $\mathfrak{h}_{P,2}$, we get a simple corollary:

Theorem 1.4.3. Let H be the orthocenter of $\triangle ABC$. The Simson line of any point P will bisect $\overline{HP}$.

Example 1.4.4 (2009 ISL G8)**.** Let quadrilateral $ABCD$ have an incircle. Define g to be a line passing through A, which intersects the segments $\overline{BC}$ at M and $\overline{CD}$ at N. Let I_1, I_2, I_3 be the incenters of $\triangle ABM, \triangle MNC, \triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on line g.

Solution. Instead of proving g passes through the orthocenter of $\triangle I_1I_2I_3$, we can instead show that g is the Steiner line for some point P, or that the 3 reflections of g over I_1I_2, I_2I_3, I_3I_1 concur at a point $P \in (I_1I_2I_3)$.

Note that the reflection of g over I_2I_3 is line CD, the reflection of g over I_1I_2 is BC, so we only have to prove that the reflection of g over I_3I_1 (let it be g') passes through C. (Then from converse Simson we get that $C \in (I_1I_2I_3)$.



Since g is one of the common internal tangents to $(I_3), (I_1)$, g' is the other internal tangent. Let T_3, T_1 be g's touchpoints with $(I_3), (I_1)$ respectively, then we just have to prove the length of the tangent from C to (I_3) intersection with (I_1) is just T_1T_3. We get a length of

$$(T_1M + MC) - (T_3N - CN) = MC + CN - T_1T_3 - NM.$$

We can re-express T_1T_3 too, as

$$\begin{aligned} 2 \cdot T_3T_1 &= 2 \cdot AT_1 - 2 \cdot AT_3 = (MA + AB - BM) - (DA + AN - ND) \\ &= (AB - DA) - NM + ND - BM \\ &= (BC - CD) - NM + ND - BM \text{ [since } (AC)(BD) \text{ is tangential quadrilateral]} \\ &= CM + NC - NM. \end{aligned}$$

So the two lengths are equal and we're done. □

Proposition 1.4.5 (Simson-Antipode Isogonality)**.** Given a fixed point P on (ABC), let $\mathcal{S}_P$ be either the Simson or Steiner line of P. Then the isogonal conjugate of the point at infinity along $\mathcal{S}_P$ is the antipode of P in (ABC).

Proof. Since $\infty_{\mathcal{S}_P}$ lies on the line at infinity, the isogonal conjugate of it will lie on (ABC). So we only need to prove that $A\infty_{\mathcal{S}_P} \parallel P_bP_c$, and from this we only need to prove that (P_bP_c, AP^*) is antiparallel with (CA, AB).

We consider the triangle $\triangle AP_bP_c$. We know that $AP, A\infty_{\perp P_bP_c}$ are isogonal in $\angle P_bAP_c$, and combined with $P_bP_c \perp A\infty_{\perp P_bP_c}$ and $AP^* \perp AP$, we get that (P_bP_c, AP^*) is antiparallel with $(P_bA, AP_c) = (CA, AB)$, so we are done. $\square$

Typically we can also directly angle-chase with the line argument of the Simson/Steiner line. Since $\measuredangle(\mathcal{S}_P) = \measuredangle(HP_A)$, where P_A is the reflection of P over $\overline{BC}$, we get

$$\begin{aligned}\measuredangle\mathcal{S}_P + \measuredangle AP &= 2\measuredangle BC - \measuredangle H_aP + \measuredangle AP \\ &= 2\measuredangle BC - \measuredangle H_aA + (\measuredangle AB + \measuredangle AC - \measuredangle BC) \\ &= \measuredangle AB + \measuredangle AC + 90^\circ\end{aligned}$$

which implies that $(\mathcal{S}_P, AP^*)$ is antiparallel with (AB, AC). This can also be written as

$$(A + B + C - P)_{(ABC)} =\perp \mathcal{S}_P$$

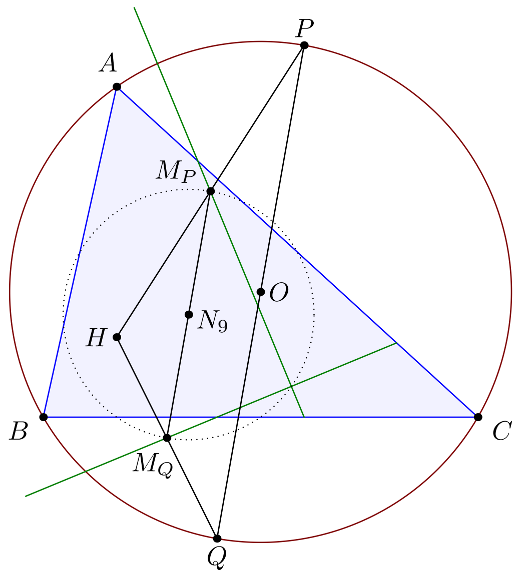
If we have two points $P, Q \in (ABC)$, then $A\infty_{\mathcal{S}_P}, AP^*$ and $A\infty_{\mathcal{S}_Q}, AQ^*$ are pairs of isogonal lines in $\angle BAC$. Therefore from $AP \perp AP^*, AQ \perp AQ^*$, we can get:

Corollary 1.4.6. In $\triangle ABC$ and two points P, Q on (ABC), the angle between their Steiner/Simson lines $\mathcal{S}_P, \mathcal{S}_Q$ is just

$$\measuredangle(\mathcal{S}_P, \mathcal{S}_Q) = \measuredangle QAP.$$

This can also be rephrased as for a fixed Q and a moving P on (ABC), $\measuredangle(\mathcal{S}_P) + \measuredangle AP$ is constant.

Example 1.4.7. Let P, Q be two antipodal points on the circumcircle of $\triangle ABC$. Show that the Simson lines of P, Q intersect at a point on the nine-point circle of $\triangle ABC$.



Solution. Let H be the orthocenter of $\triangle ABC$, and let M_P, M_Q be the midpoints of $\overline{HP}$ and $\overline{HQ}$. Note that there is a homothety $\mathfrak{h}_{H,\frac{1}{2}}$ which sends (ABC) to the nine-point circle, so we have that M_P, M_Q lie on the nine-point circle. From Theorem 1.4.3 we know that $M_P \in \mathcal{S}_P, M_Q \in \mathcal{S}_Q$.

Since PQ passes through the circumcenter of $\triangle ABC$, we get that $\overline{M_P M_Q}$ is the diameter of the nine-point circle. So from Corollary 1.4.6 we get that

$$\measuredangle(\mathcal{S}_P, \mathcal{S}_Q) = 90^\circ,$$

and therefore $\mathcal{S}_P, \mathcal{S}_Q$ intersect on the nine-point circle. (We can also use Fontene's second theorem, introduced in Chapter 8.) $\square$

Practice Problems

Problem 1. Let H be the orthocenter of $\triangle ABC$, let M be the midpoint of $\overline{BC}$, and let N be the midpoint of $\widehat{BAC}$ on (ABC). Let D be a point on $\overline{CA}$ so that $\overline{MD}$ bisects $\overline{HN}$. Prove that $CD = DA + AB$.

Problem 2 (IMO 2007/2)**.** Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the angle bisector of $\angle DAB$.

Problem 3. Given $\triangle ABC$, let P_1, P_2, $P_3 \in (ABC)$. Prove that the Simson lines of P_1, P_2, and P_3 concur if and only if

$$\measuredangle ABP_1 + \measuredangle BCP_2 + \measuredangle CAP_3 = 0°.$$

Problem 4. Let H be the orthocenter of $\triangle ABC$ and let $P \in (ABC)$. Suppose $\overline{HP} \cap \overline{CA} = Q$ and $\overline{HP} \cap \overline{AB} = R$. Prove that $APQR$ is cyclic.

Problem 5 (Taiwan TST 2020/3J/5)**.** Let O and H be the circumcenter and the orthocenter, respectively, of an acute triangle ABC. Points D and E are chosen from sides AB and AC, respectively, such that A, D, O, E are concyclic. Let P be a point on the circumcircle of triangle ABC. The line passing P and parallel to OD intersects AB at point X, while the line passing P and parallel to OE intersects AC at Y. Suppose that the perpendicular bisector of $\overline{HP}$ does not coincide with XY, but intersect XY at Q, and that points A, Q lies on the different sides of DE. Prove that $\angle EQD = \angle BAC$.

1.5 Isotomic Conjugation and Trilinear Polars

We now define the evil twin of isogonal conjugation — isotomic conjugation.

Definition 1.5.1. Given two points P_1, P_2, we can say that two points A_1, A_2 on $\overline{P_1P_2}$ are **isotomic points** if the midpoint of $\overline{A_1A_2}$ is also the midpoint of $\overline{P_1P_2}$.

And now we will extend this definition to complete n-gons, which will be represented as $(P_1, P_2, \ldots, P_n)$, meaning the polygon formed by these n points and the $\binom{n}{2}$ lines connecting them, where no three points are collinear.

Definition 1.5.2. Given any complete n-gon $(P_1, P_2, \ldots, P_n)$, define $L_{ij} = \overline{P_iP_j}$. Given any line ℓ, if there exists a line ℓ_*, such that for distinct i, j, $L_{ij} \cap \ell$ and $L_{ij} \cap \ell_*$ are isotomic points in $\overline{P_iP_j}$, then we call ℓ and ℓ_* **isotomic transversals** in $(P_1, P_2, \ldots, P_n)$.

Example 1.5.3 (M_aM_b is Self-Isotomic)**.** For an arbitrary triangle $\triangle ABC$, the sides of the medial triangle $\triangle M_aM_bM_c$ are isotomic transversals with themselves in $\triangle ABC$. Also, the line at infinity $\mathcal{L}_\infty$ is its own isotomic transversal.

From Problem 4 in Spiral Similarity, we know that:

Proposition 1.5.4. Any line ℓ that is not the three sides of $\triangle ABC$ has an isotomic transversal ℓ_*.

Note that this problem also tells us that ℓ_* is the complement of the Newton line of $\triangle ABC \cup \ell$. Back in Proposition 1.3.14 we got a condition for when a point has an isogonal conjugate in a complete quadrilateral, and now we will consider when a line has an isotomic transversal in a complete quadrangle (a set of four points and six lines).

(Note the "duality" showing up again!)

Proposition 1.5.5. Given a complete quadrangle $\Pi = (A, B, C, D)$ with no points at infinity, a line $\ell \neq \mathcal{L}_\infty$, let $P_{XY} = XY \cap \ell$. Then ℓ has an isotomic transversal if and only if the midpoint of $\overline{P_{CA}P_{BD}}$ is also the midpoint of $\overline{P_{AB}P_{CD}}$.

Proof. Let P_{XY*} be the isotomic conjugate with respect to $\overline{XY}$ and let M_{XY} be the midpoint of $\overline{XY}$. Then

$$\begin{aligned} \overrightarrow{P_{CA*}P_{AB*}} &= 2 \cdot \overrightarrow{M_{CA}M_{AB}} - \overrightarrow{P_{CA}P_{AB}} \\ \overrightarrow{P_{BD*}P_{CD*}} &= 2 \cdot \overrightarrow{M_{BD}M_{CD}} - \overrightarrow{P_{BD}P_{CD}}. \end{aligned}$$

Note that $(M_{CA}M_{BD})(M_{AB}M_{CD})$ is a parallelogram so $\overrightarrow{M_{CA}M_{AB}} = -\overrightarrow{M_{BD}M_{CD}}$. These two equations hence add up to

$$\overrightarrow{P_{CA*}P_{AB*}} + \overrightarrow{P_{BD*}P_{CD*}} = -(\overrightarrow{P_{CA}P_{AB}} + \overrightarrow{P_{BD}P_{CD}}) \qquad (♋)$$

If the midpoints of $\overline{P_{CA}P_{BD}}$ and $\overline{P_{AB}P_{CD}}$ coincide, then the RHS of our above equation is 0, so $\overline{P_{CA*}P_{AB*}} \parallel \overline{P_{BD*}P_{CD*}}$ (or the lines coincide). Note that P_{BC*}, P_{CA*}, and P_{AB*} are collinear, and P_{BC*}, P_{BD*}, P_{CD*} are collinear - since P_{BC*} lies on both lines, in fact these lines coincide on a line. Since P_{CA*} lies on line $\overline{P_{AD*}P_{DC*}}$, P_{CA*} also lies on this line, so this line is the isotomic transversal of ℓ.

Conversely, suppose ℓ has an isotomic transversal ℓ_*, or that P_{XY*} are collinear for all choices of X, $Y \in \{A, B, C, D\}$, then (♋) tells us that

$$\overrightarrow{P_{CA}P_{AB}} + \overrightarrow{P_{BD}P_{CD}} = 0 \quad \text{or} \quad \ell \parallel \ell_*.$$

In the former case, the conclusion is clearly valid. In the other case, we get that $\overrightarrow{M_{AB}M_{CD}}$ is parallel to

$\overrightarrow{M_{AC}M_{BD}}$ and so forth so either $AD \parallel BC$ or $\frac{A+D}{2} + \frac{B+C}{2}$ and thus II is a parallelogram which implies the result. $\square$

The real aim of this section is actually to expand the idea of isotomic transversals from lines to points. We do this with the concept of trilinear poles (see Exercise 2 in section 1.1).

Formally, we have:

Definition 1.5.6 (Trilinear Polarity)**.** In triangle $\triangle ABC$,

- For a point $P \neq A, B, C$, let $\triangle P_a P_b P_c$ be the **cevian triangle** of P (As a reminder, this is the triangle with vertices $AP \cap BC, BP \cap AC, CP \cap AB$). (This is also denoted with $\triangle_P$ sometimes). We define the **trilinear polar** of P wrt. $\triangle ABC$ as the perspectrix of $\triangle ABC$ and $\triangle P_a P_b P_c$, notated as $\mathfrak{t}(P)$.
- For a line $\ell \neq BC, CA, AB$, let $\triangle \ell_a \ell_b \ell_c$ be the **cevian triangle** of ℓ (defined as the triangle formed by the lines $\ell_a = \overline{A(BC \cap \ell)}, \ell_b = \overline{B(CA \cap \ell)}, \ell_c = \overline{C(AB \cap \ell)}$). Then we define the **trilinear pole** of ℓ as the perspector of $\triangle \ell_a \ell_b \ell_c$ and $\triangle ABC$, also notated as $\mathfrak{t}(P)$.

These both exist because of Desargues's Theorem.

We can then use more Desargues's Theorem to show that this operation "$\mathfrak{t}$" is an involution (i.e. $\mathfrak{t}(\mathfrak{t}(P)) = P$), since $\mathfrak{t}(\mathfrak{t}(P)) = P$ and $\mathfrak{t}(\mathfrak{t}(\ell)) = \ell$.

Example 1.5.7. The cevian triangle of $\mathcal{L}_\infty$ with respect to $\triangle ABC$ is

$$\triangle(A\infty_{BC})(B\infty_{CA})(C\infty_{AB}),$$

the anticomplementary triangle of $\triangle ABC$, $\triangle A^{\mathfrak{I}} B^{\mathfrak{I}} C^{\mathfrak{I}}$. Thus the trilinear pole of $\mathcal{L}_\infty$ is the centroid G.

Example 1.5.8. Let H be the orthocenter of $\triangle ABC$. Then the trilinear polar of H wrt. $\triangle ABC$ is the radical axis of the circumcircle (ABC) and the nine-point circle ϵ (this is known as the **orthic axis** in English).

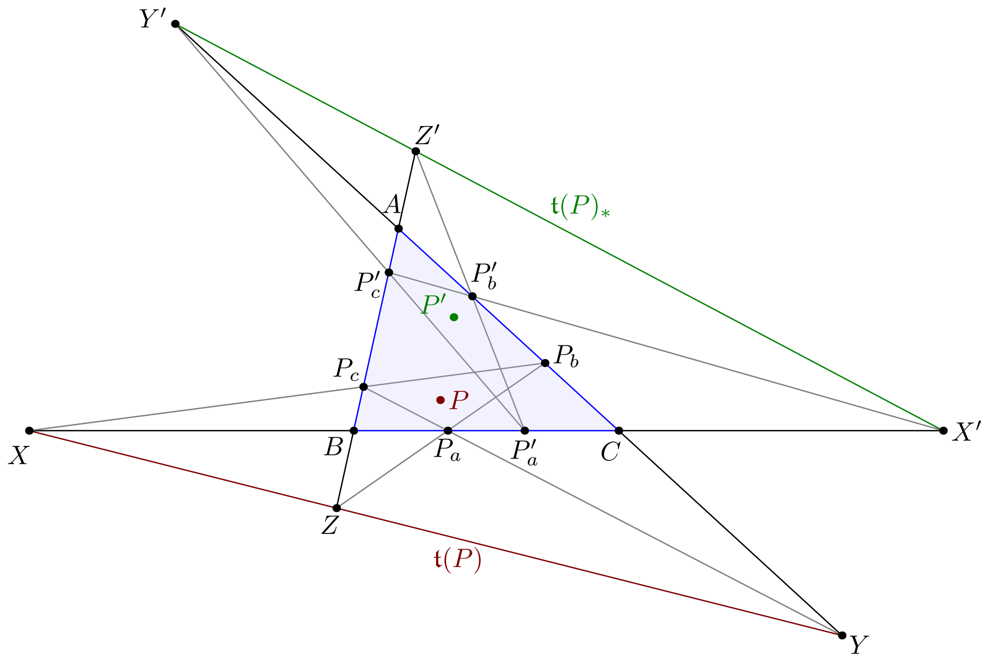
Definition 1.5.9. We define the **isotomic conjugate** of a point P as

$$P' = \mathfrak{t}(\mathfrak{t}(P)_*)$$

(the trilinear pole of the isotomic transversal of P's trilinear polar).

This might sound like a very arbitrary definition, but we can actually characterize P' much easier.

Proposition 1.5.10. Given a triangle $\triangle ABC$ and $P \neq A, B, C$, let P' be the isotomic conjugate of P. Then we have that $\overline{AP} \cap \overline{BC}$, $\overline{AP'} \cap \overline{BC}$ are isotomic points in segment $\overline{BC}$. Similarly, $\overline{BP} \cap \overline{CA}$, $\overline{BP'} \cap \overline{CA}$ are isotomic in $\overline{CA}$, and $\overline{CP} \cap \overline{BC}$, $\overline{CP'} \cap \overline{BC}$ are isotomic in segment $\overline{AB}$.



Proof. Let ℓ, ℓ' respectively be the trilinear polars of P, P' wrt. $\triangle ABC$, and let $\triangle P_aP_bP_c$ and $\triangle P'_aP'_bP'_c$ be the cevian triangles of P and P'. Then we have that $\overline{BC}, \ell, \overline{P_bP_c}$ are concurrent by Desargues's Theorem, let this concurrency point b/e Q_a. Similarly define Q'_a as the concurrency point of $\overline{BC}, \ell', \overline{P'_bP'_c}$. By Menelaus and Ceva (or harmonics, as we wil later show in Proposition 2.2.8), we get

$$\frac{BP_a}{P_aC} = \frac{BP_c}{P_cA} \cdot \frac{AP_b}{P_bC} = -\frac{BQ_a}{Q_aC} = -\frac{CQ'_a}{Q'_aB} = \frac{CP'_b}{P'_bA} \cdot \frac{AP'_c}{P'_cB} = \frac{CP'_a}{P'_aB},$$

and thus $P_a = AP \cap BC, P'_a = AP' \cap BC$ are isotomic in $\overline{BC}$. $\square$

For the rest of the section we will use this definition of isotomic conjugation.

Example 1.5.11. From Example 1.5.3 and Example 1.5.7, we can get that for $\triangle ABC$, the vertices of its anticomplementary triangle $\triangle A^{\complement}B^{\complement}C^{\complement}$ are self-isotomic in $\triangle ABC$. Also G is self-isotomic.

Another example of isotomic conjugates are the intouch and extouch points. From Incenters and Excenters we have that

$$BD = D_aC, CE = E_bA, AF = F_cB,$$

which gets us that the Gergonne point $Ge = AD \cap BE \cap CF$ and the Nagel point $Na = AD_a \cap BE_b \cap CF_c$ are isotomic conjugates.

(Note the similarity between the roles of the anticomplementary triangle + centroid in isotomic conjugation, and the excentral triangle + incenter in isogonal conjugation. This is not a coincidence!)

Remark. We know that isogonal conjugation sends the line at infinity to the circumcircle (ABC). Isotomic conjugation sends the line at infinity to an ellipse passing through A, B, C, called the **Steiner circumellipse**. We will explore this further in Section 7.3 and Section 7.4.

We present another application of isotomic conjugation.

Proposition 1.5.12. Given an arbitrary triangle $\triangle ABC$ and a point $P \neq A, B, C$, let $\triangle P_aP_bP_c$ be the cevian triangle of P. Let Q_a, Q_b, Q_c be the second intersections of $(P_aP_bP_c)$ with sides BC, CA, AB. Then AQ_a, BQ_b, CQ_c concur at a point Q, known as the **cyclocevian conjugate** of P.

Proof. This is direct by an application of Carnot's Theorem, but we will present an elementary proof by Ceva. First,

$$\frac{BP_a}{P_aC} \cdot \frac{CP_b}{P_bA} \cdot \frac{AP_c}{P_cB} = 1.$$

By Power of a Point,

$$AP_c \cdot AQ_c = AP_b \cdot AQ_b, \quad BP_a \cdot BQ_a = BP_c \cdot BQ_c, \quad CP_b \cdot CQ_b = CP_a \cdot AQ_a,$$

so

$$\begin{aligned} \frac{BQ_a}{Q_aC} \cdot \frac{CQ_b}{Q_bA} \cdot \frac{AQ_c}{Q_cB} &= \left(\frac{BP_a}{P_aC} \cdot \frac{CP_b}{P_bA} \cdot \frac{AP_c}{P_cB}\right) \cdot \left(\frac{BQ_a}{Q_aC} \cdot \frac{CQ_b}{Q_bA} \cdot \frac{AQ_c}{Q_cB}\right) \\ &= \frac{BP_a \cdot BQ_a}{P_aC \cdot Q_aC} \cdot \frac{CP_b \cdot CQ_b}{P_bA \cdot Q_bA} \cdot \frac{AP_c \cdot AQ_c}{P_cB \cdot Q_cB} = 1. \end{aligned}$$ □

Finally, we combine isogonal conjugation, isotomic conjugation, and complementing into a monstrous way to characterize cyclocevian conjugates.

Theorem 1.5.13 (Grinberg)**.** Let P and Q be cyclocevian conjugates. Then

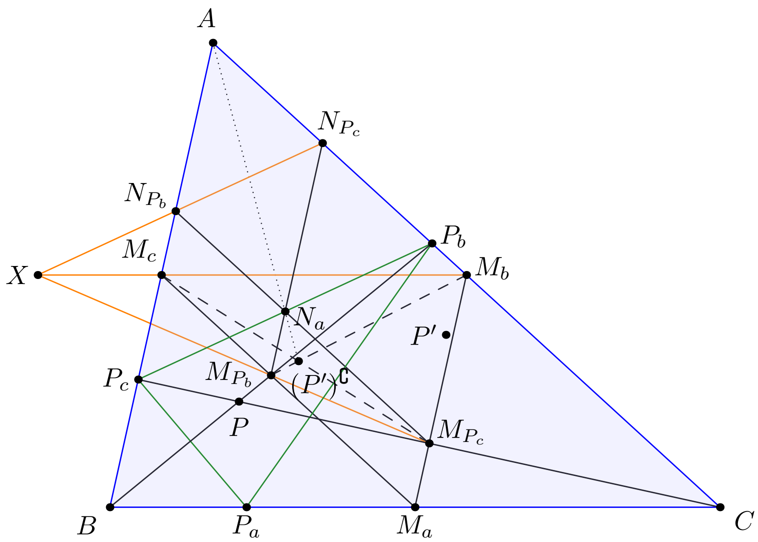
$$(P')^{\complement}, \quad (Q')^{\complement}$$

are isogonal conjugates. In other words, if we let φ^G, φ^K represent the maps $[P \to P']$, $[P \to P^*]$ (isotomic and isogonal conjugation) respectively. (This notation will make more sense in chapter 7.), then the map sending P to its cyclocevian conjugate is just

$$\varphi^G \circ (P)^{\complement^{-1}} \circ \varphi^K \circ (P)^{\complement} \circ \varphi^G.$$

For the proof of this ridiculous theorem, we first introduce a small lemma.

Lemma 1.5.14. For points P and $(P')^{\mathsf{C}}$, their crosspoint $P \pitchfork (P')^{\mathsf{C}}$ is the centroid of the cevian triangle $\triangle P_aP_bP_c$ of P. Denote this point as G_P, then we also have that the medial triangle of $\triangle P_aP_bP_c$ and $\triangle ABC$ are perspective, with perspector $(P')^{\mathsf{C}}$.



Proof. We wish to prove that $A(P')^{\mathsf{C}}$ bisects $\overline{P_bP_c}$. Let $\triangle M_aM_bM_c$ be the medial triangle of $\triangle ABC$, and let M_{P_b} and M_{P_c} be the midpoints of $\overline{BP_b}$ and $\overline{CP_c}$ respectively. Then from $M_bM_{P_b} \parallel BP'$ we get that $M_bM_{P_b} = (BP')^{\mathsf{C}}$, so $(P')^{\mathsf{C}} \in M_bM_{P_b}$. By the same logic we get that $(P')^{\mathsf{C}} \in M_cM_{P_c}$. Let $\triangle N_aN_{P_b}N_{P_c}$ be the medial triangle of AP_bP_c. Then $N_a = M_{P_b}N_{P_c} \cap M_{P_c}N_{P_b}$. By Desargues's Theorem, $A, N_a, (P')^{\mathsf{C}}$ are collinear iff. $M_{P_b}M_{P_c}, M_bM_c, N_{P_c}N_{P_b}$ are concurrent. By Menelaus we convert this into the length condition

$$\frac{M_bN_{P_c}}{N_{P_c}A} \cdot \frac{AN_{P_b}}{N_{P_b}M_c} = \frac{M_bM_{P_c}}{M_{P_c}M_a} \cdot \frac{M_aM_{P_b}}{M_{P_b}M_c}.$$

However the LHS and RHS are both equal to

$$\frac{CP_b}{P_bA} \cdot \frac{AP_c}{P_cB},$$

so we are done. □

Proof of (1.5.13). Let $\triangle P_aP_bP_c$, $\triangle Q_aQ_bQ_c$ respectively be the cevian triangles of P and Q. Let M_P and M_Q be the midpoints of $\overline{P_bP_c}$ and $\overline{Q_bQ_c}$. From $\triangle AP_bP_c \stackrel{-}{\sim} \triangle AQ_cQ_b$, we can get that $\triangle AP_bP_c \cup M_P \stackrel{-}{\sim} \triangle AQ_cQ_b \cup M_Q$. Therefore from Lemma 1.5.14, we get that

$$\measuredangle A(P')^{\mathsf{C}} + \measuredangle A(Q')^{\mathsf{C}} = \measuredangle AM_P + \measuredangle AM_Q = \measuredangle AB + \measuredangle AC,$$

so $\overline{A(P')^{\mathsf{C}}}$ and $\overline{A(Q')^{\mathsf{C}}}$ are isogonal in $\angle BAC$. By symmetry we get that $(P')^{\mathsf{C}}$ and $(P')^{\mathsf{C}}$ are isogonal conjugates in $\triangle ABC$. $\square$

Remark. A faster proof with some conic theory is noting that $(P')^{\mathsf{C}}$ (called the **isotomcomplement** of P) is the center of the inellipse (see chapter 6) tangent to $\triangle ABC$ at the vertices $P_aP_bP_c$ of the cevian triangle of P. (This can be proven by taking an affine transformation sending the inellipse to a circle). Let $\triangle DEF$ be the medial triangle of $\triangle P_aP_bP_c$. Then, we have that $A, D, (P')^{\mathsf{C}}$ collinear (let this line be ℓ) by pole/polar properties, but for two cyclocevian conjugates P, Q with corresponding lines ℓ_P, ℓ_Q, ℓ_P and ℓ_Q are isogonal since P_bP_c and Q_bQ_c are antiparallel.

Example 1.5.15. In $\triangle ABC$, we know that H and G are cyclocevian conjugates (see: nine-point circle). Then by Grinberg's theorem, we have that

$$\begin{aligned}((((H')^{\mathsf{C}})^*)^{\reflectbox{C}})' &= G\\ (((H')^{\mathsf{C}})^*)^{\reflectbox{C}} &= G' = G\\ ((H')^{\mathsf{C}})^* &= G^{\mathsf{C}} = G\\ (H')^{\mathsf{C}} &= G^* = K.\end{aligned}$$

So we get the isotomic conjugate of the orthocenter is the symmedian point of the anticomplementary triangle, X_{69}. (We will further elaborate on this point in chapter 5).

Here's a result combining the Steiner line and isotomic transversals:

Proposition 1.5.16. Let O, H be the circumcenter and orthocenter of triangle $\triangle ABC$. Let ℓ be a line through O, and let P be the anti-Steiner point of ℓ in the medial triangle $\triangle M_aM_bM_c$ (note O is the orthocenter of $\triangle M_aM_bM_c$). Then ℓ's isotomic transversal ℓ_* is the perpendicular to HP at P.

Proof. On the nine-point circle $(M_aM_bM_c)$, choose a point Q such that PQ is perpendicular to HP. Since the nine-point center N is the midpoint of $\overline{OH}$, we know PQ is perpendicular to OQ. In other words, P's antipode P^* on $(M_aM_bM_c)$ lies on OQ. Let D be the intersection of ℓ and BC, and let D_* be the isotomic point of D on $\overline{BC}$. We just need to prove $\measuredangle OQD_* = 90^\circ$. Since

$$\measuredangle OD_*M_a = \measuredangle M_aDO = \ell - M_bM_c = M_a - P + 90^\circ = M_a - P^* = \measuredangle OQM_a,$$

we know O, D_*, M_a, Q are concyclic, therefore $\measuredangle OQD_* = \measuredangle OM_aD_* = 90^\circ$. $\square$

Practice Problems

Problem 1. Let the orthocenter of $\triangle ABC$ be H. Let P be a point on (ABC) and let $\mathcal{S}$ be the Simson line of P with respect to $\triangle ABC$. Prove $\mathcal{S}$ has an isotomic transversal with respect to the complete quadrilateral $\triangle ABC \cup H$.

Problem 2. Let (P, P') be a pair of isotomic conjugates in $\triangle ABC$. Let Q be the isotomic conjugate of the anticomplement of P, $Q = (P^{\reflectbox{$\complement$}})'$. Prove that P, P', Q are collinear.

1.6 Morley's Equilateral Triangle

For this section, we will consider the triangle $\triangle ABC$ with its labels labeled anticlockwise. We will notate the angle trisectors of $\angle BAC$ as ℓ_A^{Bi}, with $i \in \{-1, 0, 1\}$ such that when $i = 0$, the angle trisectors are internal, and

$$\measuredangle(\ell_A^{Bi}, AC) = 2 \cdot \measuredangle(AB, \ell_A^{Bi}), \measuredangle(\ell_A^{Bi}, \ell_A^{Bj}) = (j - i) \cdot 60^\circ.$$

Then ℓ_A^{Ci} is defined as being an isogonal line to ℓ_A^{Bi} in $\angle BAC$. We can similarly $\ell_B^{Ci}, \ell_B^{Ai}, \ell_C^{Ai}, \ell_C^{Bi}$ where the second and forth are isogonal to the first and third respectively. Finally, define

$$a_{ij} = \ell_B^{Ci} \cap \ell_C^{Bj},$$

and b_{ij}, c_{ij} similarly.

(All of this notation is for avoiding the config-issues in this theorem. It turns out that only 18 of the 27 triangles you can get are actually equilateral!)

Theorem 1.6.1 (Morley's Trisector Theorem)**.** For any $(i, j, k) \in \{-1, 0, 1\}^3$ such that $i + j + k \not\equiv 2 \pmod 3$, $\triangle a_{jk} b_{ki} c_{ij}$ is a equilateral triangle. Additionally, $(b_{ki}c_{ij}, BC)$ and $(a_{jk}B, a_{jk}C)$ are antiparallel and so forth cyclically.

Proof. We construct $\triangle ABC$ from the equilateral triangle $\triangle a_{jk} b_{ki} c_{ij}$.

Let $\alpha = \measuredangle(\ell_A^{B0}, \ell_A^{C0})$ and define β, γ similarly.

We then have that

$$\theta \coloneqq \alpha + \beta + \gamma = (i + j + k)^\circ + \sum_{\text{cyc}} \measuredangle(\ell_A^{B0}, \ell_A^{C0}) = (1 + i + j + k)^\circ = \pm 60^\circ.$$

and thus $\theta = \pm 60^\circ \pmod{360^\circ}$. Then we let $\triangle abc$ be an equilateral triangle so $\measuredangle bac = \measuredangle cab = \measuredangle acb = \theta$. We then define A_0 such that

$$\measuredangle cA_0b = \alpha, \quad \measuredangle bcA_0 = \theta + \beta, \quad \measuredangle A_0bc = \theta + \gamma,$$

which is possible because $\alpha + (\theta + \beta) + \theta + \gamma = 3\theta = 180^\circ$. Define B_0, C_0 similarly.

Let b', c' be the reflections of b, c over C_0a, B_0a respectively. Then we get that $\overline{ab'} = \overline{ab} = \overline{ac} = \overline{ac'}$ and that $\triangle ab'c'$ is isosceles. Notice that $2\measuredangle C_0ab, 2\measuredangle caB_0$ are well defined $\pmod{360^\circ}$, so then we let

$$\measuredangle c'a'b' = 360^\circ - 2\measuredangle C_0ab - \measuredangle bac - 2\measuredangle caB_0 = -2(\theta + \beta) - \theta - 2(\theta + \gamma) = 2\alpha - \theta \pmod{360^\circ}$$

so since $3\theta/2 = 90^\circ \pmod{180^\circ}$, we have that

$$\measuredangle b'c'a = 90^\circ - \frac{1}{2}(2\alpha - \theta) = (-\theta + \alpha) = -\measuredangle B_0ca = \measuredangle B_0c'a \pmod{180^\circ}$$

so $B_0 \in b'c'$. Similarly, we get C_0 lies on $b'c'$. As such, we get that

$$\measuredangle C_0B_0a = \measuredangle c'B_0a = \measuredangle aB_0c = \beta, \quad \measuredangle B_0C_0a = \gamma.$$

And thus, we get that $\triangle A_0B_0C_0 \cup \triangle abc \overset{+}{\sim} \triangle ABC \cup a_{jk}b_{ki}c_{ij}$, so $\triangle a_{jk}b_{ki}c_{ij} \cong \triangle abc$ is equilateral.

Now we angle chase, and we get that

$$\begin{aligned}
\measuredangle b_{ki}c_{ij} &= \measuredangle a_{jk}C + \measuredangle b_{ki}a_{jk}C + \measuredangle c_{ij}b_{ki}a_{jk} \\
&= \measuredangle a_{jk}C - (\theta + \beta) + \theta \\
&= \measuredangle a_{jk}C + \measuredangle CBa_{jk} = \measuredangle a_{jk}B + \measuredangle a_{jk}C - \measuredangle BC
\end{aligned}$$

as desired. $\square$

We can prove a similar proposition.

Proposition 1.6.2 (Morley Variant)**.** Using the same notation as above, for $(i, j) \in \{-1, 0, 1\}^2$,

$$\triangle a_{ij}a_{(i+1),(j-1)}a_{(i-1)(j+1)}$$

is equilateral and B, C lies on its circumcircle. Furthermore, $(a_{(i+1)(j-1)}a_{(i-1)(j+1)}, BC)$ and $(a_{ij}B, a_{ij}C)$ are antiparallel.

Proof. For conciseness we use $i^{\pm 1} = i \pm 1, j^{\pm 1} = j \pm 1$.

Then we have that

$$\measuredangle Ba_{ij}C = \ell_C^{Bj} - \ell_B^{Ci} = (\ell_C^{B0} - \ell_B^{C0}) + (i+j)\cdot 60^\circ$$

so this only depends on $i+j$ which is fixed for $(i,j),(i^+,j^-),(i^-,j^+)$ which implies that B,C lie on the circumcircle.

Next, note that

$$\measuredangle a_{i^+}a_{ij}a_{i^-j^+} = \measuredangle a_{i^+j^-}Ca_{i^-j^+} = \measuredangle(\ell_C^{Bi^+},\ell_C^{Bi^-}) = 60^\circ,$$

and the same holds for the other two angles, giving an equilateral triangle.

Finally, we have that

$$\measuredangle a_{i^+j^-}a_{i^-j^+} = a_{i^+j^-}B + a_{i^-,j^+}C - BC = a_{ij}B + a_{ij}C - BC.$$

□

The 18 triangles specified in Morley's Trisector Theorem and in Morley Variant give us the 27 **Morley triangles** (though in certain contexts only the first 18 count, and most normal people only know 1 Morley triangle). In fact, we only need to find three Morley triangles due to the following proposition.

Proposition 1.6.3. For $(i,j,k) \in \{-1,0,1\}^3$ such that $3 \mid 1+i+j+k$, the six points

$$a_{(j+1),(k-1)}, a_{(j-1),(k+1)}, b_{(k+1),(i-1)}, b_{(k-1),(i+1)}, c_{(i+1),(j-1)}, c_{(i-1),(j+1)}$$

are collinear.

Proof. Reuse the $i^\pm, j^\pm, k^\pm$ from above. By Morley triangles, we have that

$$\triangle a_{j^-k^+}b_{k^+i^-}c_{i^-j^-}, \quad \triangle a_{jk}a_{j^+k^-}a_{j^-k^+}$$

are both equilateral triangles and

$$b_{k^+i^-}c_{i^-j^-} + BC = a_{j^-k^+}B + a_{j^-k^+}C, \quad a_{j^+k^-}a_{j^-k^+} + BC = a_{jk}B + a_{jk}C$$

so we get that

$$\begin{aligned}
\measuredangle a_{j^+k^-}a_{j^-k^+}b_{k^+i^-} &= \measuredangle c_{i^-j^-}b_{k^+i^-}a_{j^-k^+} + (a_{jk}B + a_{jk}C) - (a_{j^-k_+}B + a_{j^-k^+}C) \\
&= (1 + i^- + j^- + k^+)\cdot 60^\circ + \measuredangle a_{jk}Ba_{j^-k^+} + \measuredangle a_{jk}Ca_{j^-k^+} \\
&= -60^\circ + (j^- - j)\cdot 60^\circ + (k - k^+)\cdot 60^\circ = 0^\circ
\end{aligned}$$

which gives that $a_{j^+k^-}, a_{j^-k^+}, b_{k^+i^-}$ are collinear. Similar angle chases gives us all 6 points. $\square$

We now call $\triangle a_{ii}b_{ii}c_{ii}$ for $i = 0, 1, -1$ the **first**, **second**, and **third Morley triangle** in that order.

We will revisit this theorem when we talk about cardioids later in Morley's Beautiful Cardioids.

Chapter 2

The Cross Ratio

In the previous chapter we have defined lengths and angles, and now through some simple calculations we will discover that it is easier to work with something called the **cross-ratio**. For this section, we will work in the **real projective plane**, but for this chapter it'll be good enough to imagine it as the "real plane + one line at infinity".

The concept of points/lines "at infinity" might be familiar to you if you have done three-point perspective drawing before. If you have not, first think of a flat grid of lines in 2d space. Then think about "moving the camera" upwards, in three-dimensional space. Then, all of the previously parallel lines will appear to concur at some points at infininity, which all lie on a common horizon. For Part I of this book, we will call this camera movement a **projective transformation**, and we call this horizon the **line at infinity**, $\mathcal{L}_\infty$. (We will completely rigorously define projective space later on, and change the terminology to "homography".)

The cross ratio is our main tool to work with projective transformations: as we will prove later, it is invariant under them.

2.1 Introduction to Cross-Ratio

Definition 2.1.1. For four collinear points P_1, P_2, P_3, P_4,

$$(P_\bullet) \coloneqq (P_1, P_2; P_3, P_4) \coloneqq \frac{P_1P_3/P_3P_2}{P_1P_4/P_4P_2}$$

is called the **cross ratio** of the bundle of points P_1, P_2, P_3, P_4.

(Note: a much more beautiful way to think of this is "if you take a projective transformation sending the line $\overline{P_1P_2P_3P_4}$ to the real number line such that we send P_1 to 0, P_2 to 1, P_4 to infinity, where does P_3 go?")

This definition is actually a bit problematic, because we have not dealt with the case $P_i \in \mathcal{L}_\infty$, but for any two points $A, B \notin \mathcal{L}_\infty$, we can define

$$\frac{A\infty_{AB}}{\infty_{AB}B} = -1.$$

This definition does not fully handle the case where P_1, P_2, P_3, P_4 are all on the line of infinity, but we will address this later - right now we are just building basic concepts.

First, let's look at the fundamental result for cross ratios:

Theorem 2.1.2. Given four concurrent lines ℓ_1, ℓ_2, ℓ_3, ℓ_4, if a line $L \neq \ell_1, \ell_2, \ell_3, \ell_4$ not at infinity intersects ℓ_i at P_i respectively, then $(P_\bullet)$ is fixed (ie. does not depend on the choice of L). If $\bigcap \ell_i \notin \mathcal{L}_\infty$, then

$$(P_\bullet) = \frac{\sin\measuredangle(\ell_1, \ell_3)/\sin\measuredangle(\ell_3, \ell_2)}{\sin\measuredangle(\ell_1, \ell_4)/\sin\measuredangle(\ell_4, \ell_2)}.$$

Proof. Let $L' \not\ni \bigcap \ell_i$ be another line which intersects ℓ_i at P_i' respectively. If $\bigcap \ell_i \in \mathcal{L}_\infty$, then we clearly have

$$\frac{P_1P_3}{P_3P_2} = \frac{P_1'P_3'}{P_3'P_2'}, \frac{P_1P_4}{P_4P_2} = \frac{P_1'P_4'}{P_4'P_2'} \implies (P_\bullet) = (P_\bullet').$$

If $A := \bigcap \ell_i \notin \mathcal{L}_\infty$, and without loss of generality assume $P_1, P_2, P_3 \notin \mathcal{L}_\infty$, then by 0.3.4 we have

$$\frac{\sin\measuredangle P_1AP_3}{\sin\measuredangle P_3AP_2} = \frac{P_1P_3/P_3P_2}{\overline{AP_1}/\overline{AP_2}}, \frac{\sin\measuredangle P_1AP_4}{\sin\measuredangle P_4AP_2} = \frac{P_1P_4/P_4P_2}{\overline{AP_1}/\overline{AP_2}}$$

(note that the above still holds if $P_4 \in \mathcal{L}_\infty$). Combining the two equations we get

$$(P_\bullet) = \frac{\sin\measuredangle P_1AP_3/\sin\measuredangle P_3AP_2}{\sin\measuredangle P_1AP_4/\sin\measuredangle P_4AP_2}.$$

which does not depend on the choice of L. $\square$

Since the cross ratio $(P_\bullet)$ is the same regardless what line L we choose, $(P_\bullet)$ is really an inherent property of the four lines $\ell_1, \ell_2, \ell_3, \ell_4$.

Definition 2.1.3. For four concurrent lines ℓ_1, ℓ_2, ℓ_3, ℓ_4,

$$(\ell_\bullet) := (L \cap \ell_\bullet)$$

(where $L \not\ni \bigcap \ell_i$ is a line not at infinity) is called the cross ratio of the bundle of lines ℓ_1, ℓ_2, ℓ_3, ℓ_4.

And the cross ratio of points on the line of infinity:

Definition 2.1.4. For four collinear points $P_1,\ P_2,\ P_3,\ P_4 \in \mathcal{L}_\infty$,

$$(P_\bullet) := \frac{\sin\measuredangle P_1AP_3/\sin\measuredangle P_3AP_2}{\sin\measuredangle P_1AP_4/\sin\measuredangle P_4AP_2},$$

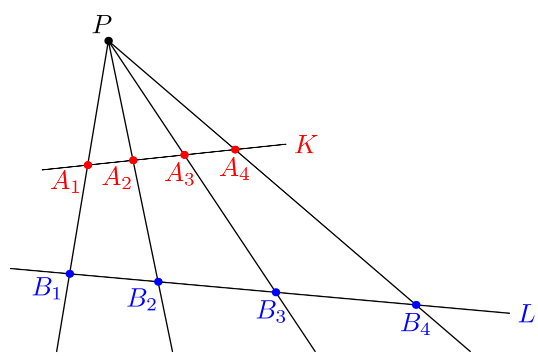
where $A \notin \mathcal{L}_\infty$ is a point.

From this definition, Theorem 2.1.2 still holds when $L = \mathcal{L}_\infty$.

Example 2.1.5. Let a pencil of lines $\ell_1, \ell_2, \ell_3, \ell_4$ intersect line K at A_1, A_2, A_3, A_4 and intersect line L at B_1, B_2, B_3, B_4. Suppose

$$A_1A_2 = A_2A_3 = A_3A_4 = 1, B_1B_2 = 2, B_2B_3 = 3.$$

Calculate B_3B_4.



Solution. Let $B_3B_4 = x$. From the cross ratio property $(A_\bullet) = (B_\bullet)$, we have

$$\frac{2/(-1)}{3/(-2)} = \frac{5/(-3)}{(5+x)/(-3-x)}.$$

So we just need to solve the linear equation $5(3+x) = 4(5+x)$ and we obtain $x = 5$. ☐

See how this calculation is very easy, but without cross ratios you might need to bash with Menelaus.

From the definition it is easy to see that cross ratios are preserved under any spiral similarity or homothety ϕ, as ratios and angles are preserved.

With all these results about equal cross-ratios, here's some important facts about their actual numeric values.

Proposition 2.1.6. For four collinear points P_1, P_2, P_3, P_4, if $(P_\bullet) = \lambda$, then

(i) $(P_2, P_1; P_3, P_4) = (P_1, P_2; P_3, P_4) = \lambda^{-1}$

(ii) $(P_1, P_3; P_2, P_4) = 1 - \lambda$.

The proof is easy:

$$\lambda^{-1} = \left(\frac{P_1P_3/P_3P_2}{P_1P_4/P_4P_2}\right)^{-1} = \begin{cases} \dfrac{P_2P_3/P_3P_1}{P_2P_4/P_4P_1} = (P_2, P_1; P_3, P_4), \\ \dfrac{P_1P_4/P_4P_2}{P_1P_3/P_3P_2} = (P_1, P_2; P_4, P_3), \end{cases}$$
$$\begin{aligned} \lambda + (P_1, P_3; P_2, P_4) &= \frac{P_1P_3/P_3P_2}{P_1P_4/P_4P_2} + \frac{P_1P_2/P_2P_3}{P_1P_4/P_4P_3} \\ &= \frac{P_3P_1 \cdot P_4P_2 + P_1P_2 \cdot P_4P_3}{P_2P_3 \cdot P_1P_4} = 1, \end{aligned}$$

where the last equality is the special case of Ptolemy's Inequality where the four points lie on a straight line. (The directed version is equivalent to

$$a(b-c) + b(c-a) + c(a-b) = 0.)$$

Of course, this is also true for cross ratios of lines instead of points. Actually, from this property we can represent all $(P_{\sigma(\bullet)})$ in terms of $(P_\bullet)$ where $\sigma \in S_4$ is any permutation of $\{1, 2, 3, 4\}$. In other words, we have $(P_{\sigma(\bullet)}) = \rho(\sigma)(P_\bullet)$, where $\rho : S_4 \to \mathrm{Aut}(\mathbb{P}^1) \subset \mathbb{R}(\lambda)$ is defined as

$$(1\ 2), (3\ 4) \mapsto [\lambda \mapsto \lambda^{-1}], (2\ 3) \mapsto [\lambda \mapsto 1 - \lambda].$$

Next up is the most important property of cross ratios, which is the lifeforce of all projective geometers:

Proposition 2.1.7 (Uniqueness of Cross-Ratio for Collinear Points)**.**

(i) For collinear points P_1, P_2, P_3, P_4, $P_\heartsuit$, we have that $P_\heartsuit = P_4$ if and only if $(P_1, P_2; P_3, P_4) = (P_1, P_2; P_3, P_\heartsuit)$,

(ii) For concurrent lines ℓ_1, ℓ_2, ℓ_3, ℓ_4, $\ell_\heartsuit$, we have that $\ell_\heartsuit = \ell_k$ if and only if $(\ell_1, \ell_2; \ell_3, \ell_4) = (\ell_1, \ell_2; \ell_3, \ell_\heartsuit)$. This also applies to cyclic permutations of the indexes.

Note that (ii) follows from (i) and (i) can be proved from the definitions. We will prove the result for $P_\heartsuit = P_4$ - this is o:

$$(P_\bullet) = (P_{\bullet'}) \iff \frac{P_1P_3/P_3P_2}{P_1P_4/P_4P_2} = \frac{P_1P_3/P_3P_2}{P_1P_\heartsuit/P_\heartsuit P_2} \iff \frac{P_1P_4}{P_4P_2} = \frac{P_1P_\heartsuit}{P_\heartsuit P_2} \iff P_4 = P_\heartsuit.$$

Now let's define some convenient notation:

Definition 2.1.8.

(i) For any five points P_1, P_2, P_3, P_4, A,

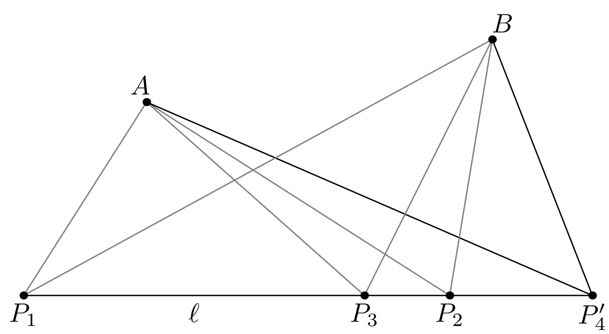
$$A(P_\bullet) \coloneqq A(P_1, P_2; P_3, P_4) \coloneqq (\overline{AP_1}, \overline{AP_2}; \overline{AP_3}, \overline{AP_4}).$$

(ii) For any five lines ℓ_1, ℓ_2, ℓ_3, ℓ_4, L,

$$L(\ell_\bullet) \coloneqq L(\ell_1, \ell_2; \ell_3, \ell_4) \coloneqq (L \cap \ell_1, L \cap \ell_2; L \cap \ell_3, L \cap \ell_4).$$

This allows to express the following fact simply, which is a useful tool in proving collinearity.

Proposition 2.1.9. Given collinear points P_1, P_2, $P_3 \in \ell$ and points A, B not on ℓ, then for any point P_4 not on $\overline{AB}$, we have $A(P_\bullet) = B(P_\bullet)$ if and only if $P_4 \in \ell$.



Proof. The "if" direction is easy. For the other direction, let $P'_\bullet = P_\bullet$ for $i = 1, 2, 3$ and $P'_4 = \overline{AP_4} \cap \ell$, then

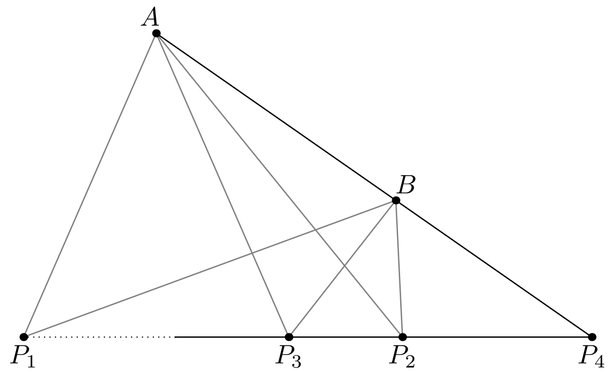
$$B(P'_\bullet) = (P'_\bullet) = A(P'_\bullet) = A(P_\bullet) = B(P_\bullet),$$

hence $P'_4 \in \overline{BP_4}$, which implies $P'_4 = \overline{AP_4} \cap \overline{BP_4} = P_4$, so $P_4 \in \ell$. $\square$

Proposition 2.1.10. For points P_1, P_2, P_3 and points $A, B \notin \bigcup \overline{P_iP_j}$,

$$A(P_1, P_2; P_3, B) = B(P_1, P_2; P_3, A)$$

if and only P_1, P_2, P_3 are collinear.



Proof. Once again, the "if" direction is easy. For the other direction, let $P_4 = \overline{AB} \cap \overline{P_2P_3}$, then P_2, P_3, P_4 are collinear and

$$A(P_1, P_2; P_3, P_4) = A(P_1, P_2; P_3, B) = B(P_1, P_2; P_3, A) = B(P_1, P_2; P_3, P_4),$$

so $P_1 \in \overline{P_2P_3}$. $\square$

Of course, similar results hold when we swap points with lines. The proofs are similar and we will omit them.

Proposition 2.1.11. Given lines ℓ_1, ℓ_2, ℓ_3 concurrent at P and lines K, L not passing through P, then for any line ℓ_4 not passing through $K \cap L$, we have $K(\ell_\bullet) = L(\ell_\bullet)$ if and only if $P \in \ell_4$.

Remark. The converse of the degenerate case of this, when ℓ_4 passes through $K \cap L$ is also very useful - we get that for three points K_1, K_2, K_3 and L_1, L_2, L_3, that if

$$(K_1, K_2; K_3, K \cap L) = (L_1, L_2; L_3, K \cap L),$$

then K_iL_i are concurrent. This is often referred to as the "prism lemma".

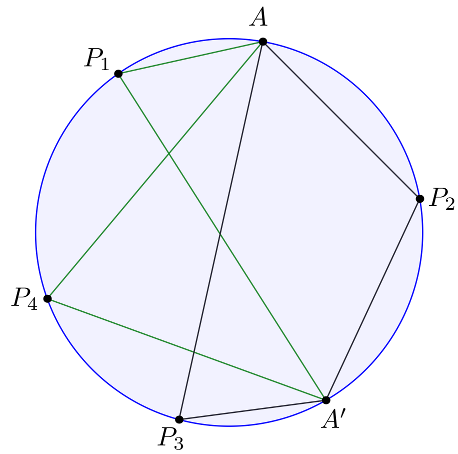
Proposition 2.1.12. For points ℓ_1, ℓ_2, ℓ_3 and lines K, L not passing through $\ell_i \cap \ell_j$ for any $i \neq j$,

$$K(\ell_1, \ell_2; \ell_3, L) = L(\ell_1, \ell_2; \ell_3, K)$$

if and only if ℓ_1, ℓ_2, ℓ_3 are concurrent.

Next, we are going to define cross ratio on a circle.

Proposition 2.1.13 (Cross Ratio on a Circle)**.** For points P_1, P_2, P_3, P_4 lying on circle Ω, if we choose a point $A \in \Omega$ that is different from all the P_i's, then $A(P_\bullet)$ does not depend on the choice of A on the circle.



Proof. Let $A' \in \Omega$ be another point different from all the P_i's, then from circle properties we get $\measuredangle P_i A P_j = \measuredangle P_i A' P_j$. So from Theorem 2.1.2 we have $A(P_\bullet) = A'(P_\bullet)$, so $A(P_\bullet)$ is fixed. □

Remark. For the result above, if we allow $A = P_i$ and define $\overline{AA}$ as the tangent $\mathbf{T}_A\Omega$ to Ω at A, then the statement remains true. We will adopt this notation for the conic chapters as well.

This allows us to define the cross ratio of four points on a circle:

Definition 2.1.14. For points P_1, P_2, P_3, P_4 lying on circle Ω,

$$(P_\bullet) \coloneqq (P_1, P_2; P_3, P_4) \coloneqq A(P_\bullet),$$

where $A \in \Omega$.

Remark. If we view the (projective) plane as $\mathbb{CP}^1$, the complex projective line, rather than $\mathbb{RP}^2$, then the formula

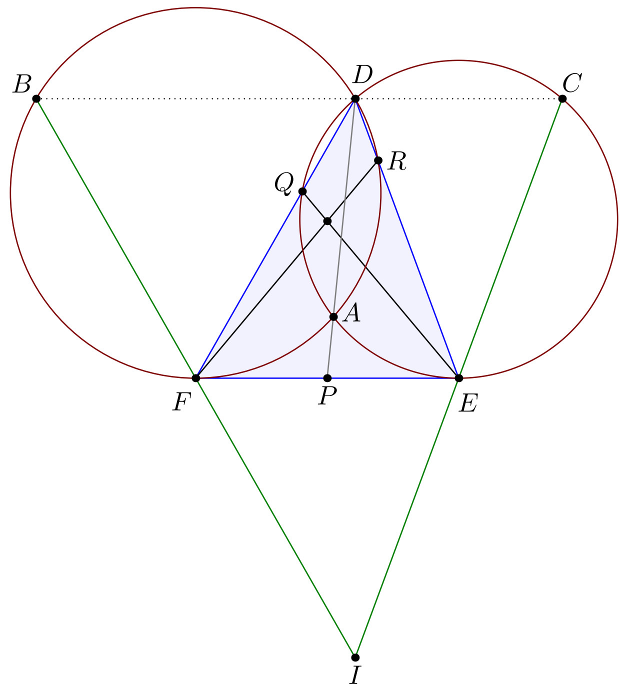
$$(z_1, z_2; z_3, z_4) \coloneqq \frac{(z_3 - z_1)/(z_2 - z_3)}{(z_4 - z_1)/(z_2 - z_4)}$$

works for both concyclic and collinear points z_1, z_2, z_3, z_4. In particular, the cross ratio is real if and only if they are either concyclic or collinear. (We will further discuss this in Section 6.A.)

Example 2.1.15. For two parallel lines BC and EF, $\overline{BF}$ and $\overline{CE}$ intersect at I, and point D lies inside segment BC. Suppose that circles $\mathcal{K} = (CDE), \mathcal{L} = (BDF)$ are tangent to $\overline{EF}$ at E and F respectively. Let $\mathcal{K}$ and $\mathcal{L}$ intersect again at $A \neq D$. $\overline{DF}$ and $\mathcal{K}$ intersect again at $Q \neq D$, $\overline{DE}$ and $\mathcal{L}$ intersect again at $R \neq D$. Let $\overline{EQ}$ and $\overline{FR}$ intersect at M. Prove that I, A, M are collinear.

Solution. By Proposition 2.1.10, it suffices to show that

$$E(I, A; M, F) = F(I, A; M, E).$$



Note that

$$E(I, A; M, F) = (C, A; Q, E) = D(C, A; Q, E) \overset{EF}{=} (\infty_{EF}, \overline{AD} \cap \overline{EF}; F, E).$$

If we let $P = \overline{AD} \cap \overline{EF}$, then P lies on the radical axis of K and L, so

$$PE^2 = \mathbf{Pow}_K(P) = \mathbf{Pow}_L(P) = PF^2,$$

so P is the midpoint of $\overline{EF}$. This gives us

$$E(I, A; M, F) = (\infty_{EF}, P; F, E) = -1.$$

Similarly, we have $F(I, A; M, E) = -1$, so I, A, M are collinear. $\square$

We have previously defined cross-ratio for four points on a circle. Analogously to how we defined cross-ratio for four collinear points and then defined it for four concurrent lines, we can also define cross-ratio for four tangents to a common circle.

Let Γ be a circle, and define

(i) $\mathbf{T}\Gamma$ as the set of all tangents of Γ;

(ii) $\mathbf{T}_P\Gamma$ as the tangent line to Γ at P, where $P \in \Gamma$;

(iii) $\mathbf{T}_\ell\Gamma$ as the point of tangency of ℓ and Γ, where $\ell \in \mathbf{T}\Gamma$;

(iv) In the degenerate case where Γ is a point P, $\mathbf{T}P$ is the set of lines through P.

To define cross ratio on $\mathbf{T}\Gamma$, we will need:

Proposition 2.1.16. Given lines $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathbf{T}\Gamma$, then for any line $L \in \mathbf{T}\Gamma$ different from the ℓ_i's, $L(\ell_\bullet)$ is fixed (does not depend on the choice of L).

Proof. Let O be the center of Γ, then from $O(L \cap \ell_i) \perp (\mathbf{T}_L\Gamma)(\mathbf{T}_{\ell_i}\Gamma)$ we get

$$L(\ell_\bullet) = \mathbf{T}_L\Gamma(\mathbf{T}_{\ell_\bullet}\Gamma) = (\mathbf{T}_{\ell_\bullet}\Gamma),$$

which does not depend on the choice of L, hence proved. □

So we can define:

Definition 2.1.17. For lines $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathbf{T}\Gamma$,

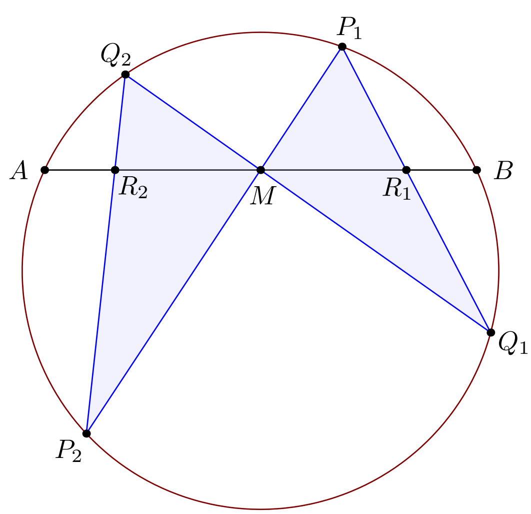
$$(\ell_\bullet) \coloneqq (\ell_1, \ell_2, \ell_3, \ell_4) \coloneqq L(\ell_\bullet),$$

where L is a tangent to Γ.

Remark. Similarly to the point case, if we allow $L = \ell_i$ and define $L \cap L$ as the tangency point $\mathbf{T}_L\Gamma$, then the statement remains true. We will adopt this notation for the conic chapters as well.

For an application of cross ratio on a circle, let's prove the butterfly theorem:

Theorem 2.1.18 (Butterfly Theorem)**.** Let $\overline{AB}$ be a chord of circle Γ and M be the midpoint of $\overline{AB}$. Construct chords $\overline{P_1P_2}, \overline{Q_1Q_2}$ through M where $P_1, P_2, Q_1, Q_2 \in \Gamma$. Let $R_1 = \overline{P_1Q_1} \cap \overline{AB}, R_2 = \overline{P_2Q_2} \cap \overline{AB}$. Then M is the midpoint of $\overline{R_1R_2}$.



Proof. We have

$$(A, B; M, R_1) \overset{P_1}{=} (A, B; P_2, Q_1) \overset{Q_2}{=} (A, B; R_2, M)$$

so we get

$$\frac{AM/MB}{AR_1/R_1B} = \frac{AR_2/R_2B}{AM/MB} \implies \frac{QR_1}{R_1B} = \frac{BR_2}{R_2A} \implies \frac{AB}{R_1B} = \frac{BA}{R_2A},$$

so $\overrightarrow{R_1B} = \overrightarrow{AR_2}$. Since M is the midpoint of $\overline{AB}$, we get that it is also the midpoint of $\overline{R_1R_2}$. $\square$

The butterfly theorem has nice generalizations which have to do with involutions (see Section 7.2). Here is an example problem using the butterfly theorem:

Example 2.1.19. Let I, O be the incenter and circumcenter of $\triangle ABC$ respectively. The line through I perpendicular to $\overline{OI}$ meets the external angle bisector of $\angle BAC$ and $\overline{BC}$ at P and Q respectively. Prove that $IP = 2QI$.

Solution. Let the line through I perpendicular to $\overline{OI}$ intersect (ABC) at U and V, then I is the midpoint of chord $\overline{UV}$. Let N_b and N_c be the second intersections of $\overline{BI}$ and $\overline{CI}$ with (ABC), then by the butterfly theorem, $R = \overline{N_bN_c} \cap \overline{UV}$ is the reflection of Q across I.

By the Incenter-Excenter Lemma, N_b, N_c both lie on the perpendicular bisector of $\overline{AI}$, so $\overline{N_bN_c}$ is the perpendicular bisector of $\overline{AI}$. Noting that the external angle bisector of $\angle BAC$ is the image of the perpedicular bisector of $\overline{AI}$ under homothety at I with factor 2, so P is the image of R under homothety at I with factor 2. Hence $IP = 2IR = 2QI$. $\square$

Practice Problems

Problem 1. Let $\triangle ABC$ be an isosceles triangle with apex A. points P, Q satisfy

$$\measuredangle ABP = \measuredangle ACQ \quad \text{and} \quad \measuredangle PCA = \measuredangle QBC.$$

Prove that A, P, Q are collinear.

Problem 2. Let P be a point inside $\triangle ABC$, and $\overline{BP}$, $\overline{CP}$ meet $\overline{CA}$, $\overline{AB}$ at E, F respectively. Let $\overline{EF}$ intersect (ABC) at B', C', and $\overline{B'P}$, $\overline{C'P}$ meet $\overline{BC}$ at C'', B'' respectively. Prove that $\overline{B'B''}, \overline{C'C''}, (ABC)$ concur.

Problem 3. Let $\triangle ABC$ be acute with altitudes $\overline{AH_1}, \overline{BH_2}$ and angle bisectors $\overline{AL_1}, \overline{BL_2}$. If O and I are the circumcenter and incenter of $\triangle ABC$ respectively. Prove that O lies on $\overline{L_1L_2}$ if and only if I lies on $\overline{H_1H_2}$.

Problem 4 (Cross Ratio Equality)**.** Let P_1, P_2, P_3, P_4 be collinear points and Q_1, Q_2, Q_3, Q_4 be collinear points (on different lines). Let $R_i = \overline{P_iQ_{i+1}} \cap \overline{P_{i+1}Q_i}$. Prove that R_1, R_2, R_3 are collinear if and only if $(P_\bullet) = (Q_\bullet)$.

Problem 5. On line ℓ lies points P_1, P_2, P_3, P_4, P_4'. Prove that it is possible to construct, using ruler only, points Q and R on ℓ such that

$$(P_1, P_2; P_3, Q) = (P_1, P_2; P_3, P_4) + (P_1, P_2; P_3, P_4')$$
$$(P_1, P_2; P_3, R) = (P_1, P_2; P_3, P_4) \cdot (P_1, P_2; P_3, P_4').$$

Problem 6. Prove the following generalization of Problem 3: Let $(P, P^*), (Q, Q^*)$ be pairs of isogonal conjugate points in $\triangle ABC$, and $\triangle P_aP_bP_c$, $\triangle Q_aQ_bQ_c$ be the Ceva triangles of P, Q with respect to $\triangle ABC$. Prove that P^* lies on $\overline{Q_bQ_c}$ if and only if Q^* lies on $\overline{P_bP_c}$.

Problem 7 (Taiwan 2014/2J/I2-1)**.** Let I and O be the incenter and circumcenter of $\triangle ABC$ respectively. Let L be the tangent to the incircle parallel to $\overline{BC}$. L meets $\overline{IO}$ at X, and Y is on L such that $\overline{YI} \perp \overline{IO}$. Prove that A, X, O, Y are concyclic.

Problem 8 (ISL 1996 G3)**.** Let O and H be the circumcenter and orthocenter of acute $\triangle ABC$ respectively. F is the foot of the altitude $\overline{CH}$ of $\triangle ABC$. The line through F perpendicular to $\overline{OF}$ intersects $\overline{AC}$ at P. Prove that $\angle FHP = \angle BAC$.

Problem 9 (Taiwan 2021/2J/P3)**.** Let O and H be the circumcenter and orthocenter of scalene $\triangle ABC$ respectively. P is a point inside $\triangle AHO$ such that $\angle AHP = \angle POA$. M is the midpoint of $\overline{OP}$. Let $\overline{BM}$ and $\overline{CM}$ intersect (ABC) again at X, Y respectively. Prove that $\overline{XY}$ passes through the circumcenter of $\triangle APO$.

2.2 Harmonic Bundles

Definition 2.2.1. We say four points P_1, P_2, Q_1, Q_2 are **harmonic** iff.

$$(P_1, P_2; Q_1, Q_2) = -1,$$

and we call these four points a **harmonic bundle**. (note that the lengths in the definition are signed). Similarly, we define four lines K_1, K_2, L_1, L_2 to be **harmonic** iff.

$$(K_1, K_2; L_1, L_2) = -1,$$

and we call these four lines a **harmonic pencil**.

Notably, by Proposition 2.1.6, harmonic bundles/pencils are fixed under swapping P_1, P_2, swapping Q_1, Q_2, and swapping P, Q. As such, these four points can be split into an unordered set $\{P_1, P_2\}, \{Q_1, Q_2\}$.

Remark. If you're wondering about the term "harmonic" in harmonic bundles, note that

$$\begin{aligned}(P_1, P_2; Q_1, Q_2) = -1 &\iff \frac{\overrightarrow{P_1Q_1}}{\overrightarrow{Q_1P_2}} + \frac{\overrightarrow{P_1Q_2}}{\overrightarrow{Q_2P_2}} = 0 \\ &\iff \frac{\overrightarrow{P_1P_2}}{\overrightarrow{Q_1P_2}} + \frac{\overrightarrow{P_1P_2}}{\overrightarrow{Q_2P_2}} = 2 \\ &\iff \frac{1}{\overrightarrow{P_2Q_1}} + \frac{1}{\overrightarrow{P_2Q_2}} = \frac{2}{\overrightarrow{P_2P_1}},\end{aligned}$$

so $\overrightarrow{P_2P_1}$ is the harmonic mean of $\overrightarrow{P_2Q_1}$ and $\overrightarrow{P_2Q_2}$!

Define the **harmonic conjugate** of a point P on line $\overline{AB}$ as the point $Q \in \overline{AB}$ such that $(A, B; P, Q) = -1$. Note that by the previous swapping property, the harmonic conjugate of A in $\overline{PQ}$ is B. We will notate harmonic conjugation of P in $\overline{AB}$ as $P^\vee_{AB}$.

Example 2.2.2. For some idea on why we care about this, here are some easy and common examples of harmonic bundles.

- If M is the midpoint of $\overline{AB}$, then A, B, M, ∞_{AB} make a harmonic bundle since

$$(A, B; M, \infty) = \frac{\overrightarrow{AM}/\overrightarrow{MB}}{\overrightarrow{A\infty}/\overrightarrow{\infty B}} = \frac{1}{-1} = -1.$$

  and thus the midpoint of $\overline{AB}$ is $\infty^\vee_{AB}$.

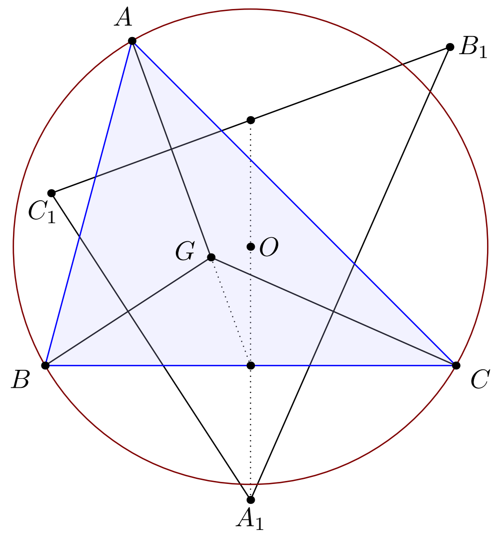
- If in $\triangle ABC$, $\angle BAC$'s internal and external bisectors intersect $\overline{BC}$ at D and D', then

$$(B, C; D, D') = \frac{BD/DC}{BD'/D'C} = \frac{AB/AC}{-AB/AC} = -1.$$

- In $\triangle ABC$, the circumcenter O, the nine-point center N, the centroid G, and the orthocenter H make a harmonic bundle.

$$(H, G; N, O) = \frac{HN/NG}{HO/OG} = \frac{3}{-3} = -1.$$

Example 2.2.3. Let G and O be the centroid and circumcenter of $\triangle ABC$. Let the perpendicular bisectors of GA, GB, GC pairwise intersect at A_1, B_1, C_1. Prove that O is the centroid of $A_1B_1C_1$.



Solution. By symmetry, we only need to prove that A_1O is the median of B_1C_1. So we just need to prove that

$$(OA_1, O\infty_{B_1C_1}; OB_1, OC_1) = (OA_1 \cap B_1C_1, \infty_{B_1C_1}; B_1, C_1) = -1.$$

Then note that A_1, B_1, C_1 are the circumcenters of $\triangle GBC, \triangle GCA, \triangle GAB$, so we have that $OA_1 \perp BC, OB_1 \perp CA, OC_1 \perp AB$. So from $B_1C_1 \perp AG$, we have

$$\begin{aligned}(OA_1, O\infty_{B_1C_1}; OB_1, OC_1) &= (\infty_{OA_1}, \infty_{B_1C_1}; \infty_{OB_1}, \infty_{OC_1}) \\ &\stackrel{\mathfrak{t}}{=} (\infty_{BC}, \infty_{AG}; \infty_{CA}, \infty_{AB}) \\ &= (A\infty_{BC}, AG; AC, AB) = -1,\end{aligned}$$

since $\overline{AG}$ bisects $\overline{BC}$. Here, $\mathfrak{t}$ is a 90° rotation on $\mathcal{L}_\infty$. (This is a common trick.). $\square$

Here's another way to get harmonic bundles that extends on the internal/external bisector harmonic bundle.

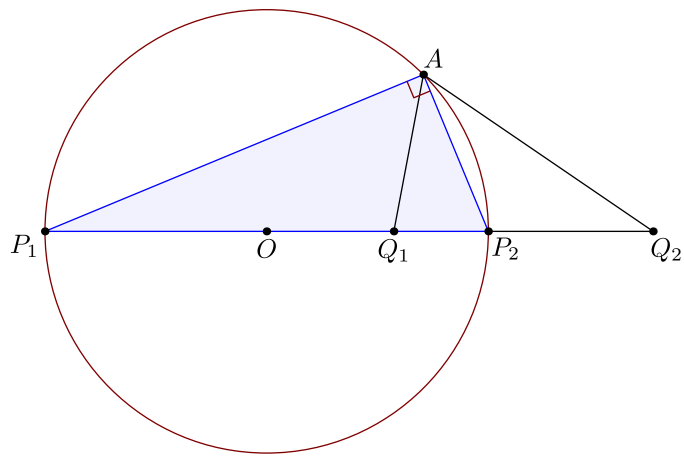
Proposition 2.2.4 (Harmonic Bisectors of Right Angles)**.** Given four distinct points P_1, P_2, Q_1, Q_2 on a line and a point A not on that line, then any two of the following four conditions implies the rest of the four conditions.

(i) $(P_1, P_2; Q_1, Q_2) = -1$.

(ii) $\measuredangle P_1AP_2 = 90^\circ$

(iii) AP_1 bisects $\measuredangle Q_1AQ_2$ internally,

(iv) AP_2 bisects $\measuredangle Q_1AQ_2$ externally.



Proof. By the second bullet point of Example 2.2.2 and angle chasing we already have most of these. Here's the proof of the remaining few by phantom points.

- (i) and one of $\{(\text{iii}), (\text{iv})\}$ implies the other follows by converse principle. If we let AP_2' be the angle bisector of $\angle Q_1AQ_2$ that AP_1 isn't, then

$$(P_1, P_2'; Q_1, Q_2) = -1 = (P_1, P_2; Q_1; Q_2)$$

so $AP_2 = AP_2'$.

- (i), (ii) implies (iii), (iv): take Q_2' so AP_1 bisects $\angle Q_1AQ_2$. By (ii) we have that AP_2 bisects $\measuredangle Q_1AQ_2$, so it follows that

$$(P_1, P_2; Q_1, Q_2') = -1 = (P_1, P_2; Q_1, Q_2),$$

and thus $Q_2' = Q_2$. $\square$

Now we characterize a bunch of length-conditions as harmonics.

Proposition 2.2.5. Given four collinear points P_1, P_2, Q_1, Q_2, and let M be the midpoint of $\overline{P_1P_2}$. Then the following are equivalent:

- P_1, P_2, Q_1, Q_2 are harmonic;
- $MQ_1 \cdot MQ_2 = MP_1^2 = MP_2^2$;
- $P_1P_2 \cdot Q_1Q_2 = 2 \cdot P_1Q_1 \cdot P_2Q_2$;
- $P_1P_2 \cdot Q_2Q_1 = 2 \cdot P_1Q_2 \cdot P_2Q_1$;
- $Q_1M \cdot Q_1Q_2 = Q_1P_1 \cdot Q_1P_2$;

- $Q_2M \cdot Q_2Q_1 = Q_2P_1 \cdot Q_2P_2$.

Proof. Clearly, you can just use a bunch of algebra to "explode" every statement by setting $P_1 = -1, P_2 = 1, Q_1 = s, Q_2 = t$. However let's do the length-bash sensibly so that we learn some length characterizations along the way.

First we prove that (i) $\iff$ (iii), (iv). From Proposition 2.1.6, we have

$$1 - (P_1, P_2; Q_1, Q_2)^{-1} = 1 - (P_1, P_2; Q_2, Q_1) = (P_1, Q_2; P_2, Q_1) = \frac{P_1P_2/P_2Q_2}{P_1Q_1/Q_1Q_2}$$

$$1 - (P_1, P_2; Q_1, Q_2) = (P_1, Q_1; P_2, Q_2) = \frac{P_1P_2/P_2Q_1}{P_1Q_2/Q_2P_1}$$

Thus $(P_1, P_2; Q_1, Q_2) = -1$ iff

$$\frac{P_1P_2/P_2Q_2}{P_1Q_1/Q_1Q_2} = 2 \iff P_1P_2 \cdot Q_1Q_2 = 2 \cdot P_1Q_1 \cdot P_2Q_2,$$

that is (iii), is also equivalent to

$$\frac{P_1P_2/P_2Q_1}{P_1Q_2/Q_2Q_1} = 2 \iff P_1P_2 \cdot Q_2Q_1 = 2 \cdot P_1Q_2 \cdot P_2Q_1,$$

so we get (iv) as well.

We now prove (i) $\implies$ (ii), (v), (vi), geometrically: from the fact that $P_1Q_1/Q_1P_2, P_1Q_2/Q_2P_2$ have opposite signs, we get that there has to be one of Q_1, Q_2 that lies on segment P_1P_2. So we can just assume WLOG that $Q_1 \in \overline{P_1P_2}$. Draw a perpendicular to P_1P_2 through Q_1 and let this intersect the circle $\omega = (P_1P_2)$ at A, B. From Proposition 2.2.4, we have that AP_1, AP_2 bisect $\angle Q_1AQ_2$, therefore we have

$$\measuredangle P_2AQ_2 = \measuredangle Q_1AP_2 = 90^\circ - \measuredangle AP_2Q = \measuredangle P_2P_1A,$$

so AQ_2 is tangent to ω at A. Thus we also have by symmetry that MA is tangent to (AQ_1Q_2), and that AQ_2 is tangent to (MAQ_1). As such, we have that

$$\measuredangle(MA + Q_1Q_2, AQ_1 + AQ_2) = 90^\circ + 90^\circ = 0^\circ$$
$$\measuredangle(AQ_2 + Q_1Q_2, MA + AQ_1) = 90^\circ + 90^\circ = 0^\circ$$

Thus we have that

$$MQ_1 \cdot MQ_2 = MA^2 = MP_1{}^2 = MP_2{}^2, Q_2M \cdot Q_2Q_1 = Q_2A^2 = Q_2P_1 \cdot Q_2P_2.$$

Finally, our construction of B isn't completely useless: since $\measuredangle MAQ_2 = \measuredangle MBQ_2 = 90^\circ$, M, Q_2, A, B are concyclic. Thus we have $Q_1M \cdot Q_1Q_2 = Q_1A \cdot Q_1B = Q_1P_1 \cdot Q_1P_2$. □

Example 2.2.6 (Finland 2019/3)**.** Cyclic quadrilateral $ABCD$ has AB as its diameter. Let AC and BD intersect at E, and let AD and BC intersect at F. Let EF intersect the circumcircle of $ABCD$ at G. Extend EF to intersect AB at H. Suppose that $FG = GH$. Prove that $GE = EH$ also.

Solution. We consider the reflection of G across AB, call this point G'. Then we know that G' lies on the circle with diameter AB. Thus we have

$$HE \cdot HF = -HA \cdot HB = -HG \cdot HG' = HG^2 = HG'^2,$$

so E, F, G, G' are harmonic. Thus

$$FG = GH \implies 3FG = FG' \implies 3GE = EG' \implies GE = EH.$$

□

Example 2.2.7 (Bulgaria 2016/5)**.** Isosceles triangle $\triangle ABC$ satisfies $AC = BC$, and let D be on ray AC such that C is between A, D and $AC > CD$. Let the angle bisector of $\angle BCD$ intersect BD at N. Let the midpoint of BD be M. Draw the tangent to circle (AMD) at M, and let it intersect BC at P. Prove A, P, M, N concyclic.

Solution. This suspicious tangency, and the appearance of perpendicular and angle bisectors, motivates us to find harmonic bundles. We consider the reflection of D across AB, point D'. Since B, C, D' are collinear, we have A, B, D, D' are concyclic. Thus

$$\measuredangle AD'P = \measuredangle ADM = \measuredangle AMP,$$

and A, D', M, P are concyclic. Thus we only need to prove that A, D', M, N are concyclic. Let E be the intersection of AD' and BD, then E lies on the perpendicular bisector of AB, so it also lies on the external angle bisector of $\angle BCD$. So B, D, N, E is a harmonic bundle! So by the above length conditions, we have

$$EN \cdot EM = EB \cdot ED = EA \cdot ED',$$

so we have that A, D', M, N are concyclic by PoP. □

Proposition 2.2.8 (Quadrilateral Brokard)**.** Given a complete quadrilateral $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$ and one diagonal $\overline{A_{ij}A_{kl}}$, the two other diagonals $\overline{A_{ik}A_{lj}}, \overline{A_{il}A_{jk}}$ intersect the other diagonal in a harmonic bundle with its endpoints. ($A_{ij} = \ell_i \cap \ell_j$, this notation will be used more in Chapter 4.)

Proof. Let $P = \overline{A_{ij}A_{kl}} \cap \overline{A_{ik}A_{lj}}$, and $Q = \overline{A_{ij}A_{kl}} \cap \overline{A_{il}A_{jk}}$. Because P, A_{ik}, A_{lj} collinear, Menelaus gives

$$\frac{\overrightarrow{A_{ij}P}}{\overrightarrow{PA_{kl}}} \cdot \frac{\overrightarrow{A_{kl}A_{lj}}}{\overrightarrow{A_{lj}A_{il}}} \cdot \frac{\overrightarrow{A_{il}A_{ik}}}{\overrightarrow{A_{ik}A_{ij}}} = -1$$

From $A_{il}Q, A_{kl}A_{lj}, A_{kl}A_{ik}$ concurrent at A_{jk}, we can use Ceva to get

$$\frac{A_{ij}Q}{QA_{kl}} \cdot \frac{A_{kl}A_{lj}}{A_{lj}A_{il}} \frac{A_{il}A_{ik}}{A_{ik}A_{ij}} = 1.$$

Dividing these two equations we can get

$$(A_{ij}, A_{kl}; P, Q) = \frac{A_{ij}P/PA_{kl}}{A_{ij}Q/QA_{kl}} = -1,$$

so A_{ij}, P, A_{kl}, Q are harmonic.

□

Example 2.2.9. Let AH_a be the height from A in $\triangle ABC$. Let P be a point on AH_a. Let $E = BP \cap CA$, let $F = CP \cap AB$, let $D = EF \cap AP$. Draw a line ℓ through D that intersects PE, AF at X, Y. Prove that AH_a is the angle bisector of $\angle XH_aY$.

Solution. Since $AH_a \perp BC$, by Proposition 2.2.4 we can see that AH_a is the angle bisector of $\angle XH_aY$ if and only if

$$(X, Y; D, \ell \cap BC) = -1.$$

By projecting, we see $(X, Y; D, \ell \cap BC) \stackrel{B}{=} (E, F; D, EF \cap BC)$,

However then just use the previous lemma on the complete quadrilateral (CA, AB, BE, CF) to get that

$$(E, F; D, EF \cap BC) = (E, F; EF \cap AP, EF \cap BC) = -1.$$

□

Definition 2.2.10. We now define harmonic quadrilaterals. A cyclic quadrilateral A, B, C, D is harmonic iff.

$$(A, B; C, D) = -1.$$

Proposition 2.2.11. A quadrilateral inscribed in Γ, $(P_1P_2)(Q_1Q_2)$ is a harmonic iff the intersection of the tangents $\mathbf{T}_{P_1}\Gamma$, and $\mathbf{T}_{P_2}\Gamma$, Q_1, and Q_2 are collinear.

Proof. Let $A = P_1P_2 \cap Q_1Q_2$, $B_1 = \mathbf{T}_{P_1}\Gamma \cap Q_1Q_2$, $B_2 = \mathbf{T}_{P_2}\Gamma \cap Q_1Q_2$. Then,

$$(Q_1, Q_2; A, B_1) = P_1(Q_1, Q_2; P_2, P_1) = (Q_1, Q_2; P_2P_1),$$
$$(Q_1, Q_2; A, B_2) = P_2(Q_1, Q_2; P_2, P_1) = (Q_1, Q_2; P_1P_2).$$

Therefore, the intersection of the tangents $\mathbf{T}_{P_1}\Gamma$, and $\mathbf{T}_{P_2}\Gamma$, Q_1, and Q_2 are collinear is equivalent to $B_1 = B_2$ that is equivalent to

$$(Q_1, Q_2; P_2, P_1) = (Q_1, Q_2; P_1, P_2) = (Q_1, Q_2; P_2, P_1)^{-1}$$

which implies that the cross ratio is indeed -1, because $(Q_2, Q_1; P_2, P_1) \neq 1$ as $P_2 \neq P_1$, and $Q_2 \neq Q_1$. This implies that $(P_1P_2)(Q_1Q_2)$ is harmonic. □

Example 2.2.12 (Bosnia Grade 9 2018/5)**.** Let H be the orthocenter of $\triangle ABC$, and let the foot from H to the internal angle bisector of $\angle BAC$ be D. Let the foot from H to the external angle bisector of $\angle BAC$ be E. Let the midpoint of BC be M. Prove M, D, E are collinear.

Proof. We consider the circle (AH). Let the feet of the altitudes from B to CA and C to AB be H_b and H_c. Then we have that M is the circumcenter of (BCH_bH_c). We also have

$$\measuredangle MH_bA = \measuredangle MH_bC = \measuredangle ACB = \measuredangle H_bHA,$$

and thus MH_b is tangent to (AH). Similarly we have MH_c tangent to (AH). Since AD and AE are the angle bisectors of $\angle BAC$, we have

$$-1 = (AB, AC : AD, AE) = (H_c, H_b; D, E),$$

so by the above characterization we know that DE goes through $H_bM \cap H_cM = M$. □

Example 2.2.13 (2013 APMO P5)**.** Let $ABCD$ be a quadrilateral inscribed in circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

Proof. Because PB and PD are tangent to ω, $ABCD$ is harmonic. We also get that $ADCE$ is harmonic because of the way it is constructed (AE intersects the tangents at C and D.) By the definitions of harmonic quadrilaterals we get that $(A, C; B, D) = -1$. Because E lies on ω proving that $(EA, EC; ER, ED) = -1$ solves as it means we can project point B to point R through point E, or that they are collinear.

$$E(A, C; R, D) = (Q, C; R, DE \cap CQ) = D(Q, C; R, E) = (D, C; A, E) = -1$$

Thus we are done. $\square$

Another commonly used idea is:

Proposition 2.2.14. Let $P_1P_2Q_1Q_2$ be a harmonic quadrilateral, and let M be the midpoint of $\overline{Q_1Q_2}$. Then we have that P_1M, P_1P_2 are isogonal lines in $\measuredangle Q_1P_1Q_2$. (We call P_1P_2 the P_1-**symmedian** of $\triangle Q_1P_1Q_2$.)

Proof. Let P_1' be the reflection of P_1 across the perpendicular bisector of $\overline{Q_1Q_2}$. Then using the fact that $P_1Q_1Q_2P_1'$ is a isosceles trapezoid, we get

$$-1 = (P_1, P_2; Q_1, Q_2) = (P_1'P_1, P_1'P_2; P_1'Q_1, P_1'Q_2) = (\infty_{Q_1Q_2}, P_1'P_2 \cap Q_1Q_2; Q_1, Q_2),$$

so $P_1'P_2$ passes through the midpoint of $\overline{Q_1Q_2}$, point M. Since

$$\begin{aligned}\measuredangle(P_1M + P_1P_2, P_1Q_1 + P_1Q_2) &= \measuredangle MP_1Q_1 + \measuredangle P_2P_1'Q_2 \\ &= \measuredangle MP_1Q_1 + \measuredangle MP_1'Q_2 = 0^\circ,\end{aligned}$$

thus P_1M, P_1P_2 are isogonal lines in $\measuredangle Q_1P_1Q_2$. $\square$

Example 2.2.15. Let $ABCD$ be a harmonic quadrilateral. Then the midpoints of AC and BD are isogonal conjugates in $ACBD$.

Example 2.2.16. Let Ω be the circumcircle of $\triangle ABC$ and let T_A, T_B, and T_C be the tangents to Ω at A, B, and C, and let D, E, F be $T_B \cap T_C, T_C \cap T_A, T_A \cap T_B$ respectively. Let X be the other intersection of AD and Ω. Then $(AX)(BC)$ is a harmonic quadrilateral. Hence $\measuredangle AD = \measuredangle AX$ is the isogonal line of the median $\measuredangle AM_a$ with respect to $\angle BAC$. Similarly, BE, CF are the isogonals of BM_b, CM_c with respect to $\angle CBA, \angle ACB$. Hence AD, BE, CF is the isogonal conjugate of the centroid $G = AM_a \cap BM_b \cap CMa_c$ of $\triangle ABC$. We call this point K the **symmedian point** of $\triangle ABC$.

Example 2.2.17. In acute triangle $\triangle ABC$, let M be the midpoint of $\overline{BC}$. Let P be a point inside $\triangle ABC$ that satisfies $\measuredangle BAM = \measuredangle PAC$. Let the circumcenters of $\triangle ABC, \triangle ABP, \triangle APC$ be O, O_1, O_2 respectively. Prove that AO bisects $\overline{O_1O_2}$.

Solution. Let D be the intersection point of AP and (ABC). Then $ABDC$ is harmonic. Since O_1O_2, O_2O, and OO_1 are perpendicular to $AP = AD$, CA, and AB, we can get

$$\begin{aligned}(OA, O\infty_{O_1O_2}, OO_1, OO_2) &= (\infty_{AO}, \infty_{O_1O_2}; \infty_{OO_1}, \infty_{O_2O}) \\ &= (\infty_{\perp AO}, \infty_{AD}; \infty_{AB}, \infty_{AC}) \\ &= (AA, AD, AB, AC) = -1,\end{aligned}$$

where the second equality is due to rotation on $\mathcal{L}_\infty$ by $90°$. Thus OA bisects O_1O_2. □

We now briefly define the inverse operation of cevian triangles.

Definition (Anticevian Triangle)**.** For a point P, there exists a triangle $\triangle^P := \triangle P^a P^b P^c$ such that the cevian triangle of P with respect to $\triangle^P$ is $\triangle ABC$. This is the **anticevian triangle** of P wrt. $\triangle ABC$. This is defined similarly for lines.

Proof. Let $P_aP_bP_c$ be the cevian triangle of P (so $P_a = AP \cap BC$). Define P^A such that $(P^A, P; A, P_a) = -1$ and so forth cyclically. Then we have that

$$(P^B, P^C; BC \cap P^BP^C, CA \cap P^BP^C) \overset{A}{=} (P^B, P; B, P_B) = -1$$

Similarly, $(P^C, P^B; BC \cap P^BP^C, BA \cap P^BP^C) = -1$ which implies that A lies on P_bP_c, which finishes by symmetry.

The proof for lines is similar. □

Practice Problems

Problem 1 (09 Costa Rica Final round P6)**.** Let ΔABC with incircle Γ, let D, E and F the tangency points of Γ with sides BC, AC and AB, respectively, and let P the intersection point of AD with Γ. Prove that, BC, EF and the straight line tangent to Γ for P concur at a point A'. Now, define B' and C' in an analogous way than A'. Prove that A', B', C' are collinear.

Problem 2. Let $\triangle ABC$ with incircle ω tangent to CA, and AB at E, F respectively. Let $P = BC \cap EF$. Draw the line parallel to BC that is tangent to ω, and let this intersect CA and AB at Y, Z. Prove that the tangent from P to ω that is not BC bisects segment $\overline{YZ}$.

Problem 3. Let $\triangle ABC$ with circumcircle Γ, and K be the symmedian point. Prove that for any point X on Γ, the trilinear polar of X, $\mathfrak{t}(X)$ pass through K.

(This is proven in Chapter 5).

Problem 4. Let Ω be the A-excircle of $\triangle ABC$. Let this excircle touch $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F. Pick two points P, Q on Ω such that EP and FQ are both parallel to the line connecting D and the midpoint of segment $\overline{EF}$. Let X be the intersection point of BP and CQ. Prove that AM is the internal angle bisector of $\angle XAD$.

Problem 5 (2018 Taiwan P6)**.** Let P be a point inside $\triangle ABC$ that satisfies $\angle BAC + \angle BPC = 180°$ and $\frac{AB}{AC} = \frac{PB}{PC}$. Prove that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Problem 6 (Taiwan TST 1 Day 3 P2)**.** Given a triangle $\triangle ABC$, A', B', C' are the midpoints of $\overline{BC}, \overline{AC}, \overline{AB}$, respectively. B^*, C^* lie in $\overline{AC}, \overline{AB}$, respectively, such that $\overline{BB^*}, \overline{CC^*}$ are the altitudes of the triangle ABC. Let $B^\#, C^\#$ be the midpoints of $\overline{BB^*}, \overline{CC^*}$, respectively. $\overline{B'B^\#}$ and $\overline{C'C^\#}$ meet at K, and $\overline{AK}$ and $\overline{BC}$ meet at L. Prove that $\angle BAL = \angle CAA'$

Problem 7 (2019 Taiwan P4)**.** Let Γ be a circle, and choose A outside of Γ, and draw the two tangents to Γ from A. Let these tangents touch Γ at B and C. Choose D on Γ such that $BC = BD$. Connect AD, let it intersect Γ again at E. Prove $DE = 2 \cdot CE$.

Problem 8. Let a $\triangle ABC$ with incircle ω tangent to sides BC, AC, AB at points D, E, F, let X be any point on ω with XB, XC intersecting ω again at Y, Z. Prove that DX, EY, FZ concur.

Problem 9 (APMOC 2011/1)**.** Let the incircle of $\triangle ABC$ touch BC, CA, AB at P, Q, R. Let O, I be the circumcenter and incenter respectively, and suppose the orthocenter H of $\triangle ABC$ lies on QR. Let N be the A-extouch point on BC. Prove

- $PH \perp QR$.
- I, O, N are collinear.

Problem 10. Let $\triangle ABC$ be rectangle at A with circumcircle Γ. Then line tangent to Γ through A intersects BC at D, and let E be the reflection of A with respect to BC, and X is the projection of A to BC. The midpoint of AX is Y and the second intersection of BY with Γ is Z. Prove that BD is tangent to the circumcircle of ADZ.

Problem 11 (2015 Day 2 P4)**.** Let a $\triangle ABC$ with incircle ω and circumcircle Γ. Let D be the tangency point of BC and ω, and let M be the midpoint of ID and A' be the antipode of A in Γ. Let X be the other intersection of $A'M$ with Γ. Prove that the circumcircle of $\triangle AXD$ is tangent to BC

Problem 12 (Buffed 2014 ISL G6)**.** Let acute triangle $\triangle ABC$ have points E and F on CA and AB. Let M be the midpoint of EF. Let the perpendicular bisector of EF intersect line BC at K. Let the perpendicular bisector of MK intersect CA, AB at S, T. Suppose $KSAT$ is concyclic. Prove $\angle KEF = \angle KFE = \angle A$.

2.3 A Few Projective Theorems

The first theorem is Desargues's Theorem, which we have already looked at:

Theorem 2.3.1 (Desargues)**.** Given two triangles $\triangle A_1B_1C_1, \triangle A_2B_2C_2$, $P = B_1C_1 \cap B_2C_2, Q = C_1A_1 \cap C_2A_2, R = A_1B_1 \cap A_2B_2$ lie on a line iff A_1A_2, B_1B_2, C_1C_2 are concurrent.

Proof. Let $X = B_1B_2 \cap C_1C_2, Z_1 = A_1A_2 \cap B_1C_1, Z_2 = A_1A_2 \cap B_2C_2$ Notice that $X \in A_1A_2 \iff X \in Z_1Z_2$ holds if and only if

$$(P, C_1; B_1, Z_1) \overset{X}{=} (P, C_2; B_2, Z_2).$$

By Proposition 2.1.10, P, Q, R collinear if and only if

$$A_1(P, Q; R, A_2) = A_2(P, Q; R, A_1);$$

(Note that $A_i \notin QR \cup RP \cup PQ$). Thus,

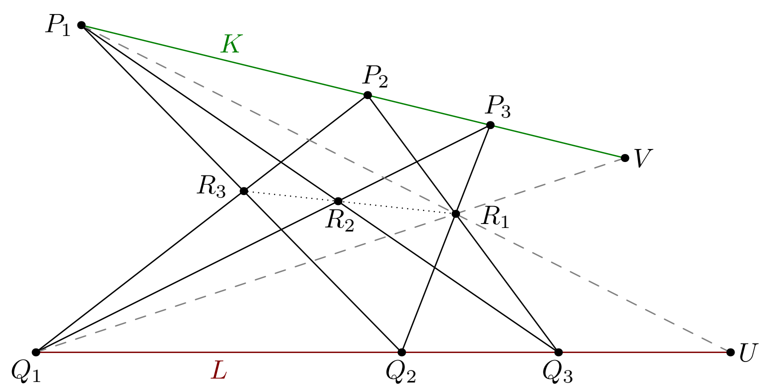
$$A_1(P, Q; R, A_2) = (P, C_1; B_1, Z_1), A_2(P, Q; R, A_2) = (P, C_2; B_2, Z_2).$$

Thus the 2 conditions are equivalent. □

Theorem 2.3.2 (Pappus)**.** Let P_1, P_2, P_3, and Q_1, Q_2, Q_3 lie on lines K and L respectively. Then

$$P_2Q_3 \cap P_3Q_2, P_3Q_1 \cap P_1Q_3, P_1Q_2 \cap P_2Q_1$$

are collinear.



Proof. Let $R_i = P_{i+1}Q_{i-1} \cap P_{i-1}Q_{i+1}, U = P_1R_1 \cap K, V = Q_1R_1 \cap L$, thus

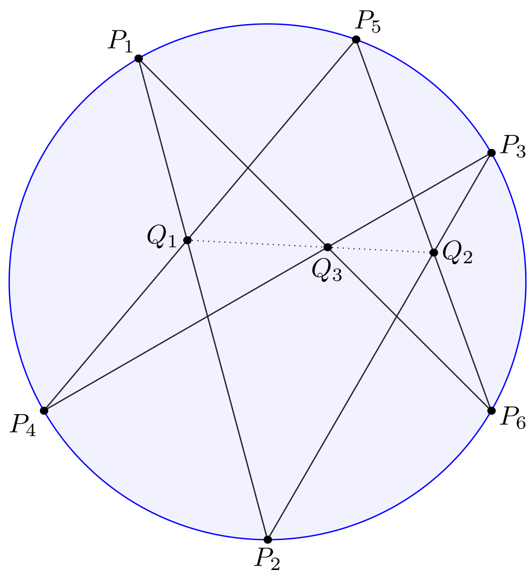
$$\begin{aligned} P_1(R_1, R_2; R_3, Q_1) &= (U, Q_3; Q_2, Q_1) \stackrel{R_1}{=} (P_1, P_2; P_3, V) \\ &= (V, P_3; P_2, P_1) = Q_1(R_1, R_2; R_3, P_1). \end{aligned}$$

Thus by Proposition 2.1.10 R_1, R_2, R_3 are collinear. □

Theorem 2.3.3 (Pascal's Theorem)**.** Let the points $P_1, P_2, P_3, P_4, P_5, P_6$ lie on a circle. Then

$$P_1P_2 \cap P_4P_5, P_2P_3 \cap P_5P_6, P_3P_4 \cap P_6P_1$$

are collinear.



Proof. Let $Q_1 = P_1P_2 \cap P_4P_5, Q_2 = P_2P_3 \cap P_5P_6, Q_3 = P_3P_4 \cap P_6P_1$. Thus

$$\begin{aligned} Q_1(P_1, P_3; P_4, Q_3) &= (P_1Q_1 \cap P_3P_4, P_3; P_4, Q_3) \\ &= P_1(P_2, P_3; P_4, P_6) = P_5(P_2, P_3; P_4, P_6) \\ &= (P_2, P_3; P_4Q_1 \cap P_2P_3, Q_2) \\ &= Q_1(P_1, P_3; P_4, Q_2) \end{aligned}$$

Thus Q_1, Q_2, Q_3 are collinear. □

Example 2.3.4 (Belarus TST 2019/6/P1)**.** Two circles Ω, ω are tangent at A. Let $\overline{BC}$ be a chord of Ω that is tangent to ω at L. Let AB, BC intersect ω at M, N respectively. Reflect M, N about AL to obtain M_1, N_1, and reflect M, N about BC to obtain M_2, N_2. Let $K = M_1M_2 \cap N_1N_2$. Prove that $AK \perp BC$.

Proof. Let l be the line through A tangent to both circles. Then

$$\measuredangle BAL = \measuredangle(BA, l) + \measuredangle(l, AL) = \measuredangle BCA + \measuredangle ALB = \measuredangle LAC$$

That is, AL is bisects $\angle BAC$. Thus, M_1, N_1 lie on CA, AB respectively and L is the midpoint of arc $\widehat{MN}$ (not passing through A). Notice that M, N, M_1, N_1 are concyclic, call this circle Γ. The center of this circle is the intersection of the perpendicular bisectors $\overline{MN}$ and $\overline{MM_1}$ and $\overline{AL}$, or L. Because

$$\overline{LM_2} = \overline{LM} = \overline{LN} = \overline{LN_2}$$

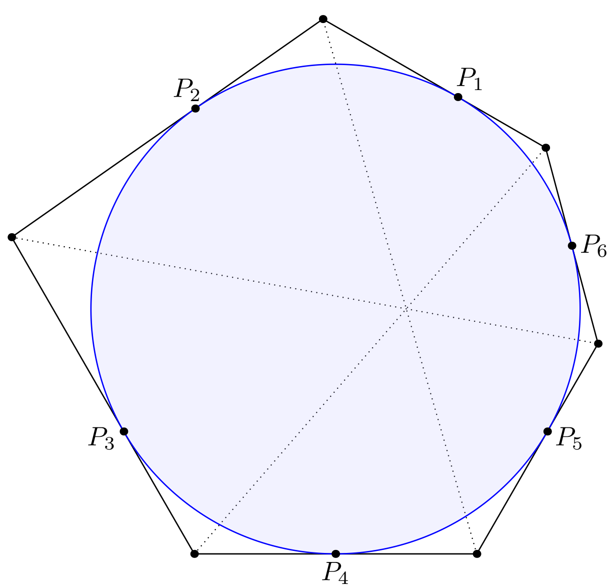
We find that M_2, N_2 also lie on Γ. Now we have that the 6 points $MN_1N_2NM_1M_2$ are concyclic on Γ. By Pascal's Theorem, we get that $A, K, \infty_{\perp BC} = N_2N \cap M_2M$ are collinear. Thus, $AK \perp BC$. $\square$

Note the "duality" between the following theorem and Pascal's:

Theorem 2.3.5 (Brianchon's Theorem)**.** Let $l_1, l_2, l_3, l_4, l_5, l_6$ are all tangent to a common circle. Then

$$(l_1 \cap l_2)(l_4 \cap l_5), (l_2 \cap l_3)(l_5 \cap l_6), (l_3 \cap l_4)(l_6 \cap l_1)$$

are concurrent.



Proof. We present a proof similar to Pascal's Theorem. Let $L_1 = (l_1 \cap l_2)(l_4 \cap l_5), L_2 = (l_2 \cap l_3)(l_5 \cap l_6), L_3 =$

$(l_3 \cap l_4)(l_6 \cap l_1)$. Then

$$\begin{aligned} L_1(l_1, l_3; l_4, L_3) &= ((l_1 \cap L_1)(l_3 \cap l_4), l_3; l_4, L_3) \\ &= l_1(l_2, l_3; l_4, l_6) = l_5(l_2, l_3; l_4, l_6) \\ &= (l_2, l_3; (l_4 \cap L_1)(l_2 \cap l_3), L_2) \\ &= L_1(l_1, l_3; l_4, L_2) \end{aligned}$$

Consequently, L_1, L_2, L_3 are concurrent. □

Remark. The two theorems above, Pascal's Theorem and Brianchon's Theorem, actually allow for 2 points (or lines) to coincide. In this case, the line joining the 2 points (or the intersection of the 2 lines) becomes a tangent (point) to the line, and vice versa.

Example 2.3.6 (APMO 2016/3)**.** Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P. Let N be the intersection of lines PR and AC and let M be the intersection of line AB and the line through R parallel to AC. Prove that MN is tangent to ω

Proof. Let U be the point at infinity along the line AC. Let L be the tangent to ω from M which isn't AB. Let L intersect AC at N'. Note that PU is tangent to ω because $PU \parallel AC$. Consider the hexagon $UEN'MFP$ with incircle ω. By Brianchon's Theorem, $UM, EF, N'P$ concur. Since $R = UM \cap EF$ it follows that N', R, P are collinear. Thus $N = N'$ and MN is tangent to ω □

Practice Problems

Problem 1. Let ABC be a triangle with circumcenter O, and let P, Q be points on CA and AB respectively such that P, O, Q collinear. Let M, N be the midpoints of BP, CQ respectively. Prove that $\angle BAC = \angle MON$.

Problem 2. Let ω be the incircle of $\triangle ABC$ and let D, E, F be the intouch points. Let K be the second intersection of AD with ω. Let the line tangent to ω at K intersect FD, DE at Y, Z respectively. Prove that AD, BZ and CY concur.

Problem 3. Let ABC be the triangle with incircle ω and l be a tangent to ω. Let A', B', C' be points other than the intouch points lying on BC, CA, AB respectively. The tangent from A' (other than BC) intersects l at A^*, B^* and C^* are defined similarly. Prove that AA^*, BB^*, CC^* are concurrent.

Chapter 3

Inversion and Polarity

3.1 Basic Inversion

In previous sections we looked at transformations that preserved similarities. Now let's look at a transformation that doesn't even preserve shapes.

Remark. Like what we did implicitly in the Spiral Similarity section, we will also work in $\mathbb{CP}^1$, the **complex projective line** or the **inversive plane**. Imagine this as the complex plane, represented by $z := a + bi$, and exactly one point at infinity P_∞. (This is called a line because it is defined by exactly one complex number, so it ends up being dimension 1 over the complex numbers.)

Definition 3.1.1. Given a point O and a constant $k \neq 0$ (potentially negative), we can define an **inversion** $\mathfrak{J}_{O,k}$ at O (the **center of inversion**) with **power** k as such:

- If P isn't O or P_∞, then $P^{\mathfrak{J}} = \mathfrak{J}_{O,k}(P)$ is the point on OP such that such that
$$\overrightarrow{OP} \cdot \overrightarrow{OP^{\mathfrak{J}}} = k.$$
- O swaps with P_∞, under this transformation.

Notably, $\mathfrak{J}_{O,k} = \mathfrak{s}_O \circ \mathfrak{J}_{O,-k}$. When $k < 0$, this is called a **negative inversion**.

It is more natural to phrase inversion with circles. For a circle Γ with center O and radius r, we can define the inversion $\mathfrak{J}_\Gamma$ as the inversion centered at O with power $k = r^2$. (This is also why r is called the power of the inversion, as for $P \in \Gamma$, $\mathbf{Pow}_\Gamma(P) = r^2$.)

Note that $\mathfrak{J}_\Gamma$ is an **involution**, that is $\mathfrak{J}_\Gamma(\mathfrak{J}_\Gamma(P)) = P$. Furthermore, Γ is fixed under $\mathfrak{J}_\Gamma$.

In this section, by default the notation $P^{\mathfrak{J}}$ will refer to the image of P under the inversion $\mathfrak{J}$ which is known by context. We define the image of a line ℓ under inversion as $\ell^{\mathfrak{J}} = \{P^{\mathfrak{J}} \mid P \in \ell\}$.

Note for any two points P, Q, that aren't O or P_∞, we have that

$$\overrightarrow{OP} \cdot \overrightarrow{OP^{\mathfrak{J}}} = k = \overrightarrow{OQ} \cdot \overrightarrow{OQ^{\mathfrak{J}}},$$

so $P, Q, P^{\mathfrak{J}}, Q^{\mathfrak{J}}$ are concyclic by power of a point.

Proposition 3.1.2. Given any two points $P, Q \neq O, \infty$, we have that

$$\measuredangle OP^{\mathfrak{J}}Q^{\mathfrak{J}} = \measuredangle PQO, \quad P^{\mathfrak{J}}Q^{\mathfrak{J}} = \frac{|k|}{OP \cdot OQ} \cdot PQ.$$

In line arguments, we can rephrase the first formula as $\measuredangle P^{\mathfrak{J}}Q^{\mathfrak{J}} + \measuredangle PQ = \measuredangle OP + \measuredangle OQ$.

(The formula for $P^{\mathfrak{J}}Q^{\mathfrak{J}}$ is commonly known as the **inversion distance formula**.)

Proof. Since $P, Q, P^{\mathfrak{J}}, Q^{\mathfrak{J}}$ are concyclic, we get that

$$\measuredangle OP^{\mathfrak{J}}Q^{\mathfrak{J}} = \measuredangle PP^{\mathfrak{J}}Q^{\mathfrak{J}} = \measuredangle PQQ^{\mathfrak{J}} = \measuredangle PQO.$$

This gives us the similarity $\triangle OPQ \overset{-}{\sim} \triangle OQ^{\mathfrak{J}}P^{\mathfrak{J}}$, therefore

$$P^{\mathfrak{J}}Q^{\mathfrak{J}} = \frac{OQ^{\mathfrak{J}}}{OP} \cdot PQ = \frac{k}{OP \cdot OQ} \cdot PQ. \qquad \square$$

Note that $\mathfrak{J}_{O,k_1} = \mathfrak{h}_{O,\frac{k_1}{k_2}} \circ \mathfrak{J}_{O,k_2}$, so if we don't overlay the pre-inversion and post-inversion diagrams, it doesn't matter what power we choose, as the inverted diagram will be the same up to scaling. In these cases, we will say "invert about P" which means an inversion with arbitrary power.

We will now characterize what inversion does to lines and circles.

Proposition 3.1.3. Given an inversion centered at O, we have that:

(i) If ℓ is a line passing through O, then $\ell^{\mathfrak{J}} = \ell$;

(ii) If ℓ is a line not passing through O, then $\ell^{\mathfrak{J}}$ is a circle passing through O;

(iii) If Ω is a circle passing through O, then $\Omega^{\mathfrak{J}}$ is a straight line not passing through O;

(iv) If Ω is a circle not passing through O, then $\Omega^{\mathfrak{J}}$ is a circle not passing through O.

Proof. Part (i) is obvious, since $O, P, P^{\mathfrak{I}}$ are collinear. For (ii), given any two points P, Q on ℓ, we have

$$\measuredangle OQ^{\mathfrak{I}}P^{\mathfrak{I}} = \measuredangle QPO = \measuredangle(\ell, OP).$$

Varying Q on ℓ or $Q^{\mathfrak{I}}$ on $(OP^{\mathfrak{I}}Q^{\mathfrak{I}})$ gives that $\ell^{\mathfrak{I}} = (OP^{\mathfrak{I}}Q^{\mathfrak{I}})$.. Part (iii) follows by the above logic in reverse.

Finally, for part (iv), note that for any three points $P, Q, R \in \Omega$, we have

$$\measuredangle Q^{\mathfrak{I}}P^{\mathfrak{I}}R^{\mathfrak{I}} = \measuredangle Q^{\mathfrak{I}}P^{\mathfrak{I}}O + \measuredangle OP^{\mathfrak{I}}R^{\mathfrak{I}} = \measuredangle OQP + \measuredangle PRO = \measuredangle QOR + \measuredangle RPQ$$

is fixed as P varies, so $\Omega^{\mathfrak{I}} \subseteq (P^{\mathfrak{I}}Q^{\mathfrak{I}}R^{\mathfrak{I}})$.

Since the image of O is P_∞, and Ω does not pass through the point at infinity, its image will not pass through O, either. $\square$

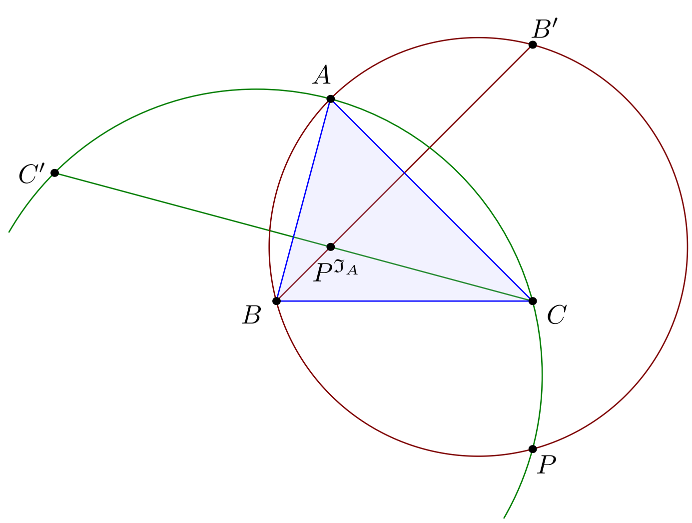
Remark. Note that **inversion does not preserve circle centers.** This is a very common mistake.

Given $\triangle ABC$, consider the involution $\mathfrak{I}_A$ resulting from first applying an inversion $\mathfrak{I}_A$ with radius $\sqrt{AB \cdot AC}$ followed by a reflection about the angle bisector of $\angle BAC$. This swaps B and C, AB and AC, and BC and (ABC). We will use the term $\sqrt{bc}$ **inversion** with respect to $\triangle ABC$ as this involution. We will also use $\mathfrak{I}_A$ to just refer to the inversion part.

(The original book uses "Reflect-Invert about A", but this is more clear and more widely used.)

To get a handle as to why this is useful, let's look at some examples:

Example 3.1.4 (Croatia MO 2010/7)**.** Given $\triangle ABC$, let B' be the reflection of B about $\overline{AC}$ and let C' be the reflection of C about $\overline{AB}$. Let $(ABB') \cap (ACC') = P \neq A$. Prove that $\overline{AP}$ passes through the circumcenter O of $\triangle ABC$.



Solution. We $\sqrt{bc}$-invert the problem. Let us first find where everything goes. Note that $AB' = AB, AC' = AC$ and that AB', AC' are isogonal, so we have the following pairings:

$$\overline{AB} \leftrightarrow \overline{AC}, \quad B' \leftrightarrow C', \quad (ABB') \leftrightarrow \overline{CC'},$$
$$(ACC') \leftrightarrow \overline{BB'}, \quad P \leftrightarrow P^{\mathfrak{I}_A} := \overline{BB'} \cap \overline{CC'}.$$

As such, showing that $O \in \overline{AP'}$ is thus equivalent to showing that the orthocenter H lies on $AP^{\mathfrak{I}_A}$. However, $P^{\mathfrak{I}_A} = BB' \cap CC'$ is just the orthocenter. ☐

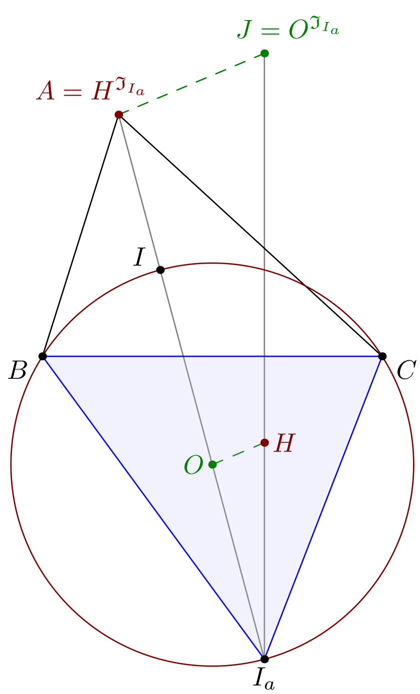
Proposition 3.1.5. Given $\triangle ABC$, if (P, Q) are isogonal conjugates with respect to $\triangle ABC$, then $\mathfrak{I}_A(P)$ and $\mathfrak{I}_A(Q)$ are also isogonal conjugates with respect to $\triangle ABC$ after a $\sqrt{bc}$ inversion.

Proof. Using the fact that $\triangle ABP \overset{+}{\sim} \triangle A\mathfrak{I}_A(P)C, \triangle ABQ \overset{+}{\sim} \triangle A\mathfrak{I}_A(Q)C$, we get that

$$\measuredangle(C\mathfrak{I}_A(P) + C\mathfrak{I}_A(Q), CA + CB) = \measuredangle BPA + \measuredangle BQA + \measuredangle ACB = 0^\circ,$$

where the last equality follows from Proposition 1.3.6. As such, $C\mathfrak{I}_A(P)$ and $C\mathfrak{I}_A(Q)$ are isogonal. Since $B\mathfrak{I}_A(P)$ and $B\mathfrak{I}_A(Q)$ are also isogonal by symmetry, the result follows. ☐

Example 3.1.6. In $\triangle ABC$, let I^a be the A-excenter, let J be the reflection of I^a about $\overline{BC}$, let H be the orthocenter of $\triangle BI^aC$, and let O be the circumcenter of $\triangle BI^aC$. Prove that $\overline{AJ} \parallel \overline{OH}$.



Solution. Since

$$\measuredangle BA + \measuredangle BJ = (2\measuredangle BI^a - \measuredangle BC) + (2\measuredangle BC - \measuredangle BI^a) = \measuredangle BI^a + \measuredangle BC$$

$$\measuredangle CA + \measuredangle CJ = (2\measuredangle CI^a - \measuredangle BC) + (2\measuredangle BC - \measuredangle CI^a) = \measuredangle CI^a + \measuredangle BC$$

we have that (A, J) are isogonal conjugates in $\triangle BI^aC$. Now, the $\sqrt{bc}$ inversion at I^a wrt. $\triangle BI^aC$ sends O to J. However since (O, H) are isogonal conjugates in $\triangle BI^aC$, we get that this also maps H to A.

Thus, by $\measuredangle I^aAJ = -\measuredangle HOI^a = \measuredangle I^aOH$, we get $AJ \parallel OH$. □

As a matter of fact, inversion also preserves "angles" in a sense, or is a **conformal** mapping:

Definition 3.1.7. Let γ_1 and γ_2 be two smooth curves (in our case, this will be circles or lines for the foreseeable future). Let P be an intersection of γ_1 and γ_2. Then the **angle** between γ_1 and γ_2 at P is the angle between the tangent line $\mathbf{T}_P(\gamma_1)$ and $\mathbf{T}_P(\gamma_2)$, and its value is $\measuredangle(\mathbf{T}_P(\gamma_1), \mathbf{T}_P(\gamma_2)) = \measuredangle_P(\gamma_1, \gamma_2)$.

If this angle is $90°$, then the two curves are said to be **orthogonal** which is a symmetric relation.

Proposition 3.1.8 (Inversion is a Anticonformal Map)**.** Let γ_1, γ_2 be two smooth curves with an intersection point at P. Then the angle between γ_1 and γ_2 at P is equal and opposite to the angle between $\gamma_1^{\mathfrak{I}}$ and $\gamma_2^{\mathfrak{I}}$ at $P^{\mathfrak{I}}$.

Proof. Let ℓ_1, ℓ_2 be the tangents at P to γ_1, γ_2 respectively. Because $\gamma_i^{\mathfrak{I}}$ and $\ell_i^{\mathfrak{I}}$ are tangent at $P^{\mathfrak{I}}$, we can just consider $\ell_1^{\mathfrak{I}}, \ell_2^{\mathfrak{I}}$. By Proposition 3.1.3, we get that $\ell_i^{\mathfrak{I}}$ are both circles through P and the center of inversion O.

As such, both $\ell_1^{\mathfrak{I}}$ and $\ell_2^{\mathfrak{I}}$ are symmetric about the perpendicular bisector of $\overline{OP^{\mathfrak{I}}}$, so

$$\measuredangle_{P^{\mathfrak{I}}}(\ell_2^{\mathfrak{I}}, \ell_1^{\mathfrak{I}}) = \measuredangle_O(\ell_1^{\mathfrak{I}}, \ell_2^{\mathfrak{I}}).$$

Let L_i be the tangent at O to $\ell_i^{\mathfrak{I}}$. By Proposition 3.1.3, $L_i^{\mathfrak{I}} = L_i$. Since $O \in \ell_i^{\mathfrak{I}} \cap L_i$ it follows that $P_\infty \in \ell_i \cap L_i$, so $\ell_i \parallel L_i$. As such,

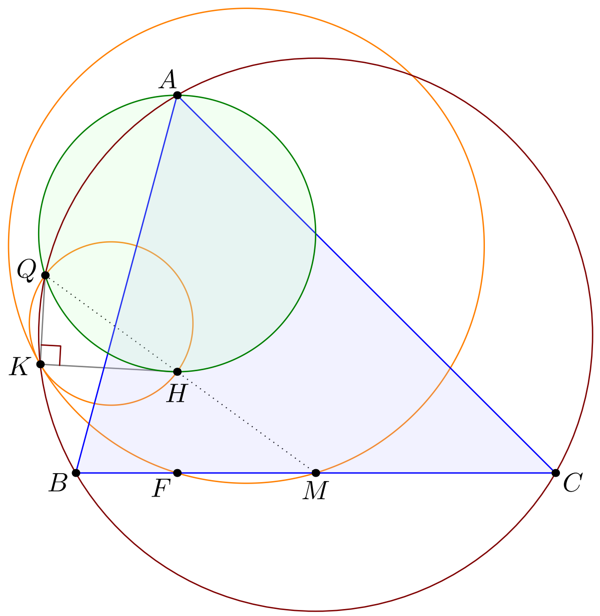
$$\measuredangle_O(\ell_1^{\mathfrak{I}}, \ell_2^{\mathfrak{I}} = \measuredangle(L_1, L_2) = \measuredangle(\ell_1, \ell_2) = \measuredangle_P(\gamma_1, \gamma_2)$$

□

Example 3.1.9 (Buffed IMO 2015/3)**.** Let $\triangle ABC$ be an acute triangle with $AB < AC$, and let Γ be the circumcircle and let H be the orthocenter as usual. Let F be the foot from A. Let M be the midpoint of $\overline{BC}$, and let Q be a point on Γ that satisfies $\angle HQA = 90°$. Let K be the point on Γ that satisfies $\angle HKQ = 90°$. It is given that the points A, Q, K, B, C lie on Γ in that order.

Prove that the circumcircle of $\triangle KQH$ is tangent to the circumcircle of $\triangle FKM$.

(Note: Point Q is known as the **Queue** point in English.)



Proof. Consider the negative inversion $\mathfrak{I}$ with center H and radius $\sqrt{HA \cdot HF}$. This sends the vertices of $\triangle ABC$ to their feet on the opposite side, so $\Gamma^{\mathfrak{I}} = \epsilon$ is the nine-point circle of $\triangle ABC$. Hence $Q^{\mathfrak{I}}$ lies on ε with $\measuredangle Q^{\mathfrak{I}}FH = 90°$, so $Q^{\mathfrak{I}}$ is the midpoint M of $\overline{BC}$.

Note that $K^{\mathfrak{I}} \in \epsilon$ satisfies $\measuredangle K^{\mathfrak{I}}MH = 90°$, and (KQH) being tangent to (FKM) is equivalent to $\overline{K^{\mathfrak{I}}M}$ being tangent to $(AK^{\mathfrak{I}}Q)$. Let M_A and M_Q be the midpoints of $\overline{HA}$ and $\overline{HQ}$, respectively, so that both midpoints lie on ϵ. Since $\overline{MM_A}$ is a diameter of ε, we have

$$\measuredangle M_AM_Q = \measuredangle AQ = \measuredangle \perp QM = \measuredangle MK^{\mathfrak{I}} \quad \Longrightarrow \quad (M_Q - K^{\mathfrak{I}})_\Gamma = (M - M_A)_\Gamma,$$

so $\overline{K^{\mathfrak{I}}M_Q}$ is also a diameter of ε.

Because $\overline{K^{\mathfrak{I}}M_A} \perp \overline{M_AM_Q} \parallel \overline{AQ}$, M_A lies on the median $\overline{AQ}$, so $K^{\mathfrak{I}}$ also lies on this median. Therefore

$$\measuredangle QK^{\mathfrak{I}}M = 90° - \measuredangle MQK^{\mathfrak{I}} = \measuredangle K^{\mathfrak{I}}QA = \measuredangle QAK^{\mathfrak{I}},$$

which implies the conclusion. $\square$

Note that if Γ_1, Γ_2 intersect in two points P, Q, then

$$\measuredangle_P(\Gamma_1, \Gamma_2) = -\measuredangle_Q(\Gamma_1, \Gamma_2).$$

So if two circles are orthogonal at P, then they are also orthogonal at Q. In this case, we call Γ_1 **orthogonal** to Γ_2. Further, since Inversion is a Anticonformal Map tells us that orthogonality is preserved under inversion.

Proposition 3.1.10. Let Γ_1, Γ_2 be two circles with O_1, O_2 as centers and r_1, r_2 as radii. Then the following are equivalent:

- Γ_1 and Γ_2 are orthogonal;
- ${O_1O_2}^2 = r_1^2 + r_2^2$;
- $\mathbf{Pow}_{\Gamma_2}(O_1) = r_1^2$.

Proof. Pythagorean Theorem on $\triangle O_1PO_2$ where P is one of their intersection points. □

Corollary 3.1.11 (Orthogonal Circles Are Preserved Under Inversion)**.** If we define $\mathfrak{J} = \mathfrak{J}_\Gamma$, then for any point P and circle $\Omega \ni P$, Ω is orthogonal to Γ if and only if $P^{\mathfrak{J}} \in \Omega$.

Proof. Let Γ, Ω have centers O_1, O_2 and radii r_1, r_2. By power of a point, we have that $P^{\mathfrak{J}} \in \Omega$ iff $r_1^2 = OP \cdot OP^{\mathfrak{J}} = O_1O_2^2 - r_2^2$ so the result follow by the above. □

This gives us a key property of orthogonal circles: inverting one around the other maps it to itself.

Let us go back to the concept of "negative inversion". In this case, we don't have actual real points for calculating angles however, by Proposition 3.1.10 we can still define orthogonality. Consider a circle $\mathcal{E}$ centered at O with $\sqrt{k}$ as its "radius". Set $\mathfrak{J} = \mathfrak{J}_{\mathcal{E}}$.

- When $k > 0$, $\mathcal{E}$ is a normal, real circle.
- When $k = 0$, $\mathcal{E}$ is the point circle we encountered in Section 0.4.
- When $k < 0$, $\mathcal{E}$ is a circle with imaginary radius.

Remark. To rigorously define circles with imaginary radius, we need to go back to the complex projective plane $\mathbb{CP}^2$, as the only way to give $\mathbb{CP}^1$ a two-dimensional structure is by representing it as $(a, b), a, b \in \mathbb{R}$. We will rigorously interpret this in later chapters, don't worry about it right now. What's important to know now is that inversion about a circle with imaginary radius is equivalent to a combination of inversion about a circle with the magnitude of that radius and reflection about some axis.

Definition 3.1.12. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be circles centered at O_1, O_2 with radius $\sqrt{k_1}$, $\sqrt{k_2}$ respectively. We say that Γ_1, Γ_2 are **orthogonal** if

$$O_1O_2^2 = k_1 + k_2.$$

In this setting, we can replace Γ in Orthogonal Circles Are Preserved Under Inversion with any circle $\mathcal{E}$ as follows:

Proposition 3.1.13. If $\mathfrak{J} = \mathfrak{J}_{\mathcal{E}}$, then for any point P and a circle $\Omega \ni P$, Ω and $\mathcal{E}$ are orthogonal if and only if $P^{\mathfrak{J}} \in \Omega$.

Recall that the radical axis is defined as the set of points with equal power wrt. two circles.

Definition 3.1.14. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be circles centered at O_1, O_2 with radius $\sqrt{k_1}$, $\sqrt{k_2}$ respectively. We define the **radical axis** of $\mathcal{E}_1$, $\mathcal{E}_2$ to be

$$\{P \mid \mathbf{Pow}_{\mathcal{E}_1}(P) = \mathbf{Pow}_{\mathcal{E}_2}(P).\}$$

(Power remains defined for circles with imaginary radii).

By a similar proof as Proposition 0.4.9, the radical axis of $\mathcal{E}_1$ and $\mathcal{E}_2$ is a straight line perpendicular to $\overline{O_1O_2}$. Then we can derive the Radical Axis theorem for any circles. Now, we return to our discussion of orthogonality with the following theorem:

Proposition 3.1.15. Given two circles $\mathcal{E}_1$ and $\mathcal{E}_2$, any circle $\mathcal{E}$ that is orthogonal to $\mathcal{E}_1$ and $\mathcal{E}_2$ has its center O on the radical axis of $\mathcal{E}_1$ and $\mathcal{E}_2$. Conversely, for any point O on the radical axis of $\mathcal{E}_1$ and $\mathcal{E}_2$, there exists exactly one circle $\mathcal{E}$ centered at O and orthogonal to $\mathcal{E}_1$ and $\mathcal{E}_2$.

Proof. Let O_1, O_2 respectively be the centers of $\mathcal{E}_1$ and $\mathcal{E}_2$, and let $\sqrt{k_1}$ and $\sqrt{k_2}$ respectively be the radii of $\mathcal{E}_1$ and $\mathcal{E}_2$.

Let $\mathcal{E}$ be a circle orthogonal to both of these circles, and let $\sqrt{k}$ be the radius of this circle. Then by the orthogonality length condition we get that

$$\mathbf{Pow}_{\mathcal{E}_1}(O) - \mathbf{Pow}_{\mathcal{E}_2}(O) = k - k = 0,$$

so $O \in \ell$.

To prove the converse, let O be an arbitrary point on ℓ, let $k = \mathbf{Pow}_{\mathcal{E}_1}(O) = \mathbf{Pow}_{\mathcal{E}_2}(O)$, then we have the circle with radius $\sqrt{k}$ is orthogonal to both $\mathcal{E}_1$ and $\mathcal{E}_2$ by the same logic as before. $\square$

Proposition 3.1.16. If the centers of two inversions $\mathfrak{J}_1$ and $\mathfrak{J}_2$ do not coincide, then their composition $\mathfrak{J}_2 \circ \mathfrak{J}_1$ is an inversion $\mathfrak{J}$ coupled with a reflection about some line $\mathfrak{s}$. In fact, if we set $\mathfrak{J}_1 = \mathfrak{J}_{\mathcal{E}_1} = \mathfrak{J}_{O_1,k_1}$ and $\mathfrak{J}_2 = \mathfrak{J}_{\mathcal{E}_2} = \mathfrak{J}_{O_2,k_2}$, then $\mathfrak{J}$ has center $O = O_2^{\mathfrak{J}_1}$ and $\mathfrak{s}$ is the reflection about the radical axis ℓ of $\mathcal{E}_1$ and $\mathcal{E}_2$.

Proof. We show that $\mathfrak{J} \coloneqq \mathfrak{s} \circ \mathfrak{J}_2 \circ \mathfrak{J}_1$ is an inversion with center O.

For a point P, let $P_1 = P^{\mathfrak{J}_1}, P_2 = P_1^{\mathfrak{J}_2}, P_3 = \mathfrak{s}(P_2)$. By Proposition 3.1.13, we have that (PP_1P_2) is orthogonal with $\mathcal{E}_1\mathcal{E}_2$, so its center lies on ℓ. As such, it follows that P_3 also lies on (PP_1P_2), and that

$P_2P_3 \perp \ell \perp O_1O_2$. Furthermore, P, P_1, O, O_2 are cyclic (because $P_1 = \mathfrak{J}_1(P), O = \mathfrak{J}_1(O_2)$). As such, we get that

$$\measuredangle P_1PO = \measuredangle P_1O_2O = \measuredangle P_1P_2P_3 = \measuredangle P_1PP_3$$

so O, P, P_3 are collinear (This is just Reim's Theorem).

We now find $OP \cdot OP_2'$. Let $O_3 = \mathfrak{s}(O_2)$, so we then have that

$$\measuredangle O_1PP_3' = \measuredangle P_1PO = \measuredangle P_1O_2O = \measuredangle P_2O_2O_3 = \measuredangle O_2O_3P_3 = \measuredangle O_1O_3P_3$$

so O_1, O_3, P, P_3 are concyclic. As such, $OP \cdot OP_3 = OO_1 \cdot OO_3$ is fixed, which finishes. □

Proposition 3.1.17. Given an inversion $\mathfrak{J}$ centered at O, for any circle Ω, the circle $\Omega^{\mathfrak{J}}$ has center $\mathfrak{J}(\mathfrak{J}_\Omega(O))$.

Proof. Let O_Ω be the center of Ω, let $O_{\Omega^{\mathfrak{J}}}$ be the center of $\Omega^{\mathfrak{J}}$. Let A, B be the two intersection points of $\overline{OO_\Omega}$ with Ω. Then since $\overline{AB}$ is orthogonal to Ω we get that $\overline{A^{\mathfrak{J}}B^{\mathfrak{J}}}$ is orthogonal to $\Omega^{\mathfrak{J}}$. Therefore $\Omega^{\mathfrak{J}} = (A^{\mathfrak{J}}B^{\mathfrak{J}})$. From

$$\frac{OA^{\mathfrak{J}}}{OB^{\mathfrak{J}}} = \frac{OB}{OA}$$

we can get that $\Omega^{\mathfrak{J}}, \Omega$ are homothetic with scale factor $\frac{OA^{\mathfrak{J}}}{OB}$. Thus

$$O\mathfrak{J}_\Omega(O) \cdot OO_\Omega = \mathbf{Pow}_\Omega(O) = OA \cdot OB = k \cdot \frac{OB}{OA^{\mathfrak{J}}} = k \cdot \frac{OO_\Omega}{OO_{\Omega^{\mathfrak{J}}}},$$

which gets us that $O\mathfrak{J}_\Omega(O) \cdot OO_{\Omega^{\mathfrak{J}}} = k$, and thus $\mathfrak{J}(\mathfrak{J}_\Omega(O)) = O_{\Omega^{\mathfrak{J}}}$. □

Proposition 3.1.18. Given an inversion $\mathfrak{J}$ centered at O with radius k, for any circle Ω not centered at O, $\Omega^{\mathfrak{J}}$ has radius

$$\frac{rk}{\mathbf{Pow}_\Omega(O)}$$

where r is the radius of Ω.

Proof. Let A, B be the intersection points of OO_Ω and Ω. Then the radius of $\Omega^{\mathfrak{J}}$ is

$$\begin{aligned} \frac{1}{2} \cdot A^{\mathfrak{J}}B^{\mathfrak{J}} = \frac{1}{2} \cdot \left|OB^{\mathfrak{J}} - OA^{\mathfrak{J}}\right| &= \frac{1}{2} \cdot \left|\frac{k}{OB} - \frac{k}{OA}\right| \\ = \frac{|k|}{2} \cdot \left|\frac{OA - OB}{OA \cdot OB}\right| &= \left|\frac{rk}{\mathbf{Pow}_\Omega(O)}\right|. \end{aligned}$$

□

Practice Problems

Problem 1. In $\triangle ABC$ with $\measuredangle A = 90°$, let D be a point on side $\overline{AB}$. Two circles are tangent to $\overline{BC}$ at B and C and intersect at D and E. Prove that $\measuredangle CBA = \measuredangle DEA$.

Problem 2 (Taiwan TST 2015/3J/I3-2)**.** In a scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of triangle AEF at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter ST is orthogonal to the nine-point circle of triangle BIC.

Problem 3 (Taiwan TST 2019/1J/M6)**.** Given $\triangle ABC$, denote its incenter and orthocenter by I and H, respectively. If there is a point K with

$$AH + AK = BH + BK = CH + CK,$$

show that H, I, and K are collinear.

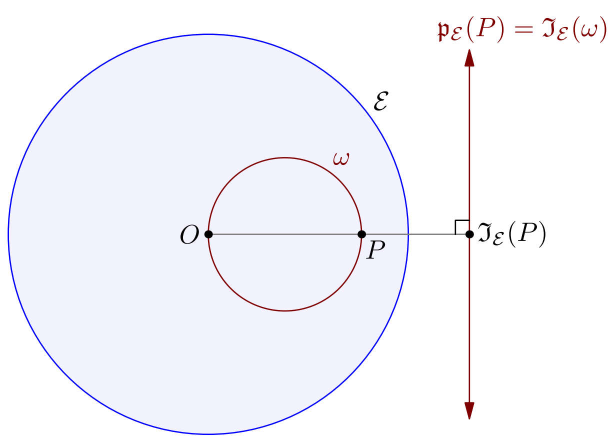
3.2 Basic Polarity

Polarity is a way to exchange the roles of points and lines. This is the explanation for "duality" we saw in previous sections.

In this section, we are back to working in $\mathbb{CP}^2$ (So imagine a line at infinity again.).

Definition 3.2.1. Take a circle $\mathcal{E}$ and let O be its center.

- For an arbitrary point $P \neq O, P \notin \mathcal{L}_\infty$, we define the **polar** $\mathfrak{p}_\mathcal{E}(P)$ of P wrt. $\mathcal{E}$ as the image of the circle (OP) under $\mathfrak{J}_\mathcal{E}$. (The polar of O is the line at infinity $\mathcal{L}_\infty$.)
- For P on the line at infinity, the polar of P is the line $O\infty_{\perp P}$.



From this definition we can see that $\mathfrak{p}_{\mathcal{E}}(P)$ always passes through $\mathfrak{I}_{\mathcal{E}}(P)$ and is always perpendicular to $\overline{OP}$ (every line is perpendicular to the line at infinity, we will elaborate on this later).

When P lies outside $\mathcal{E}$, also note that the polar of P is the line through the touch-points of the two tangent lines to $\mathcal{E}$ from P. (The tangency points lie on both $\mathcal{E}$ and (OP) and are fixed under inversion).

Let M be the midpoint of P and $P^{\mathfrak{I}}$. for $P \notin \mathcal{L}$. We then have that

$$\begin{aligned}\mathbf{Pow}_{\mathcal{E}}(M) &= OM^2 - OP \cdot OP^{\mathfrak{I}} \\ &= (OP^2 + 2 \cdot OP \cdot PM + PM^2) - OP \cdot OP^{\mathfrak{I}} = PM^2 = \mathbf{Pow}_P(M),\end{aligned}$$

and thus M lies on the radical axis of $\mathcal{E}$ and the point circle P.

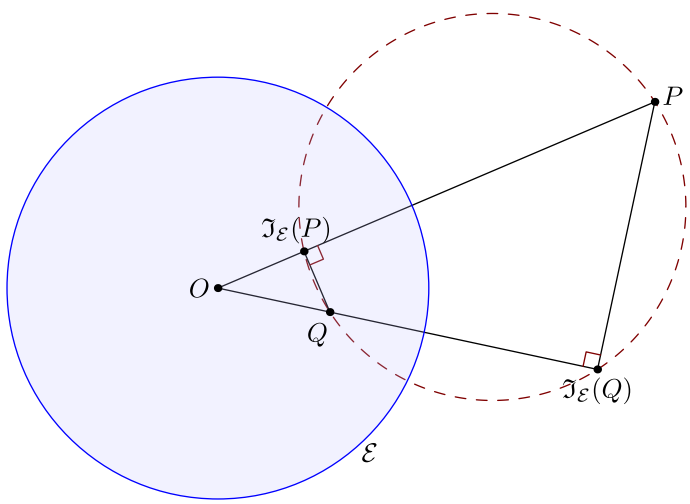
Proposition 3.2.2. For a point $P \notin \mathcal{L}_\infty$ and a circle $\mathcal{E}$, the polar of P across $\mathcal{E}$ is the image of the radical axis of $P, \mathcal{E}$ under a homothety of scale factor 2 from P.

We can redefine polarity in a nicer way: if the radius of $\mathcal{E}$ is $\sqrt{k}$, then

$$\mathfrak{p}_{\mathcal{E}}(P) = \{Q \mid \overrightarrow{OP} \cdot \overrightarrow{OQ} = k\} \cup \{\infty_{\perp OP}\}.$$

From this, we get the most important principle of polarity.

Proposition 3.2.3 (La Hire's Theorem)**.** Given two points P and Q, and a circle $\mathcal{E}$, we have that $P \in \mathfrak{p}_{\mathcal{E}}(Q) \iff Q \in \mathfrak{p}_{\mathcal{E}}(P)$.



Proof 1. We know P, Q and their inversion points are concyclic. Hence

$$P \in \mathfrak{p}_{\mathcal{E}}(Q) \iff \measuredangle P\mathfrak{I}_{\mathcal{E}}(Q)Q = 90^\circ \iff \measuredangle P\mathfrak{I}_{\mathcal{E}}(P)Q = 90^\circ \iff Q \in \mathfrak{p}_{\mathcal{E}}(P). \qquad \square$$

Proof 2. Let P, Q not be O or $\mathcal{L}_\infty$ which can be checked manually. Then we have that

$$P \in \mathfrak{p}_{\mathcal{E}}(Q) \iff \overrightarrow{OP} \cdot \overrightarrow{OQ} = k \iff Q \in \mathfrak{p}_{\mathcal{E}}(P).$$

If $P = O$ or $P \in \mathcal{L}_\infty$, then Q lies on $\mathcal{L}_\infty$ or $\perp OP$ respectively, and the result follows by definitions. $\square$

This is very useful.

Definition 3.2.4. Let $\mathcal{E}$ be a circle, and let P and Q be two points such that P lies on the polar of Q (by La Hire's, we also have vice-versa). Then we call P, Q **conjugates** in $\mathcal{E}$.

Now, suppose that three points P, P_1, P_2 are collinear, then the intersection Q of $\mathfrak{p}_{\mathcal{E}}(P_1)$ and $\mathfrak{p}_{\mathcal{E}}(P_1)$, satisfies that $\mathfrak{p}_{\mathcal{E}}(Q) = P_1P_2$. Since $P \in P_1P_2$, it follows that $Q \in \mathfrak{p}_{\mathcal{E}}(P)$. Varying P, this tells us that for a line K and a point $P \in K$, we have that $\mathfrak{p}_{\mathcal{E}}(P)$ passes through a fixed point regardless of P.

Definition 3.2.5. Let $\mathcal{E}$ be a circle and K be a line. For a varying point $P \in K$, define the fixed point that $\mathfrak{p}_{\mathcal{E}}(P)$ passes through to be the **pole** of line K wrt. $\mathcal{E}$.

From this definition, we get the following relations:

$$\mathfrak{p}_{\mathcal{E}}(P) \cap \mathfrak{p}_{\mathcal{E}}(Q) = \mathfrak{p}_{\mathcal{E}}(PQ), \quad \mathfrak{p}_{\mathcal{E}}(K)\mathfrak{p}_{\mathcal{E}}(L) = \mathfrak{p}_{\mathcal{E}}(K \cap L).$$

Additionally, we get that the function $\mathfrak{p}_{\mathcal{E}}$ is an involution (we have $\mathfrak{p}_{\mathcal{E}}(\mathfrak{p}_{\mathcal{E}}(K)) = K$.) By Proposition 3.2.3 we can get that:

Proposition 3.2.6 (Dual of La Hire's)**.** Let $\mathcal{E}$ be a circle and let K, L be two lines in the plane. Then

$$\mathfrak{p}_{\mathcal{E}}(K) \in L \iff \mathfrak{p}_{\mathcal{E}}(L) \in K.$$

Definition 3.2.7. Let $\mathcal{E}$ be a circle, we call two lines K, L **conjugate** wrt. $\mathcal{E}$ if $\mathfrak{p}_{\mathcal{E}}(K) \in L$, and vice versa.

Remark. This is the "duality" we have been alluding to in previous chapters!! Given a configuration of points and lines, we can take the **polar dual** of the entire diagram, which swaps lines to points, and swaps concurrencies into collinearities. We call this kind of transformation **incidence-preserving**. For example, the polar dual of Ceva is Menelaus, and the polar dual of Desargues's Theorem is the converse of Desargues's Theorem. We will further study this idea in chapter 7, and we will give a rigorous basis for why a transformation like this even exists in the appendix.

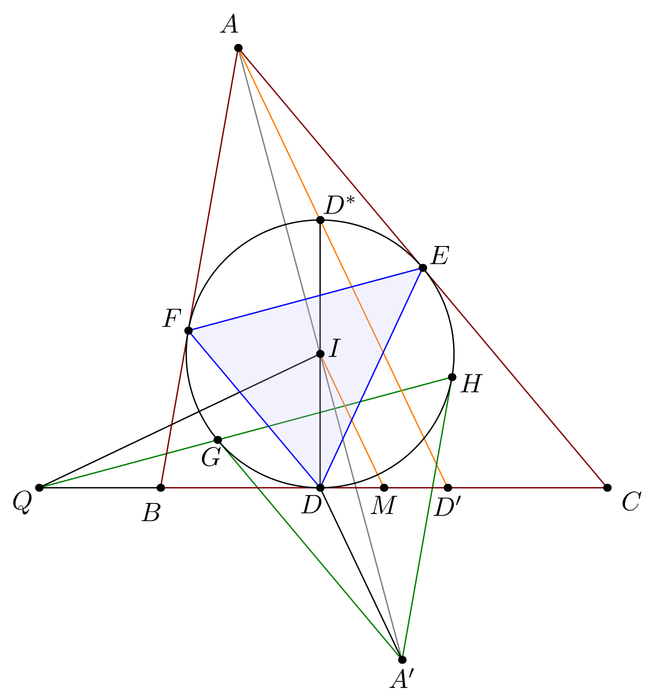
Example 3.2.8. Let non-equilateral triangle $\triangle ABC$ have intouch triangle $\triangle DEF$. Let X be the intersection point of EF and BC, and let P be the second intersection point of AD with the incircle. Prove that XP is tangent to the incircle.

Solution. Note that $X = EF \cap BC = \mathfrak{p}_\omega(A) \cap \mathfrak{p}_\omega(D)$, so

$$P \in AD = \mathfrak{p}_\omega(X) \implies X \in \mathfrak{p}_\omega(P).$$

Since P lies on the incircle, we know $P \in \mathfrak{p}_\omega(P)$ and that $\mathfrak{p}_\omega(P)$ is the tangent to the incircle at P. Thus XP is tangent to the incircle. □

Example 3.2.9 (Taiwan 2014/1J/P3)**.** Let $\triangle ABC$ have incenter I, and let the incircle touch CA, AB at E, F. Let the reflections of E, F across I be G, H. Let Q be the intersection of GH and BC. Let M be the midpoint of BC. Prove $IQ \perp IM$.



Solution. Let the incircle be ω. Let A' be the reflection of A across I. Since $\overline{AE}$ and $\overline{AF}$ are tangent to ω, we know that $\overline{A'G}$ and $\overline{A'H}$ are tangent to ω, so $\overline{GH} = \mathfrak{p}_\omega(A')$. If we let D be the intouch point on BC, then $BC = \mathfrak{p}_\omega(D)$. Thus $Q = \mathfrak{p}_\omega(A') \cap \mathfrak{p}_\omega(D) = \mathfrak{p}_\omega(A'D)$. So we have $IQ \perp A'D$, so all we need to prove to finish the problem is that $A'D \parallel IM$.

If we let D^* be the antipode of D in ω, and let D' be the reflection of D across M (the A-extouch point), then we can get that $A'D \parallel AD^*$ and $IM \parallel D^*D'$ (Since I, M are the midpoints of DD^* and DD' respectively). Therefore we only need to prove that A, D^*, D' are collinear, but this is obvious by homothety at A sending the incircle to the excircle. □

Proposition 3.2.10. Given a circle $\mathcal{E}$, then two points P and Q are **conjugate** if and only if the circle with diameter PQ is orthogonal to $\mathcal{E}$.

Proof. By Proposition 3.1.13, we get that (PQ) is orthogonal to $\mathcal{E}$ iff. $P^{\mathfrak{I}} \in (PQ)$. But this just means that $P^{\mathfrak{I}}Q \perp OP$, so $Q \in \mathfrak{p}_{\mathcal{E}}(P)$. $\square$

Proposition 3.2.11. Given a circle $\mathcal{E}$, and two conjugate points P, Q such that $\overline{PQ}$ intersects $\mathcal{E}$ (at real points) A, B, then

$$(P, Q; A, B) = -1.$$

Proof. If P and Q are conjugates in $\mathcal{E}$, then if we let M be the midpoint of $\overline{AB}$, then from $\measuredangle OMQ = 90^\circ = \measuredangle OP^{\mathfrak{I}}Q$ we know that $O, P^{\mathfrak{I}}, Q, M$ are concyclic. Thus

$$PA \cdot PB = \mathbf{Pow}_{\mathcal{E}}(P) = PP^{\mathfrak{I}} \cdot PO = PQ \cdot PM,$$

so $(P, Q; A, B) = -1$, by fractions. $\square$

Definition 3.2.12. Let $\mathcal{E}$ be a circle such that $\triangle ABC$ has A, B, C pairwise conjugate in $\mathcal{E}$ (which is the same thing as BC, CA, AB pairwise conjugate). Then we call $\triangle ABC$ a **self-conjugate triangle** wrt. $\mathcal{E}$.

Let the center of $\mathcal{E}$ be O and let $\mathcal{E}$ have radius $\sqrt{k}$. Then we have $OA \perp BC, OB \perp CA, OC \perp AB$, so O is the orthocenter of $\triangle ABC$ and

$$\overrightarrow{OA} \cdot \overrightarrow{OD} = \overrightarrow{OB} \cdot \overrightarrow{OE} = \overrightarrow{OC} \cdot \overrightarrow{OF} = k,$$

where $\triangle DEF$ is the orthic triangle of $\triangle ABC$.

Switching our frame of reference, we get the following.

Definition (Polar Circle)**.** Given any triangle $\triangle ABC$, there exists a circle $\mathcal{E}$ centered at H (possibly with imaginary radius) such that $\triangle ABC$ is self-conjugate. This circle is the circle the **polar circle** of $\triangle ABC$.

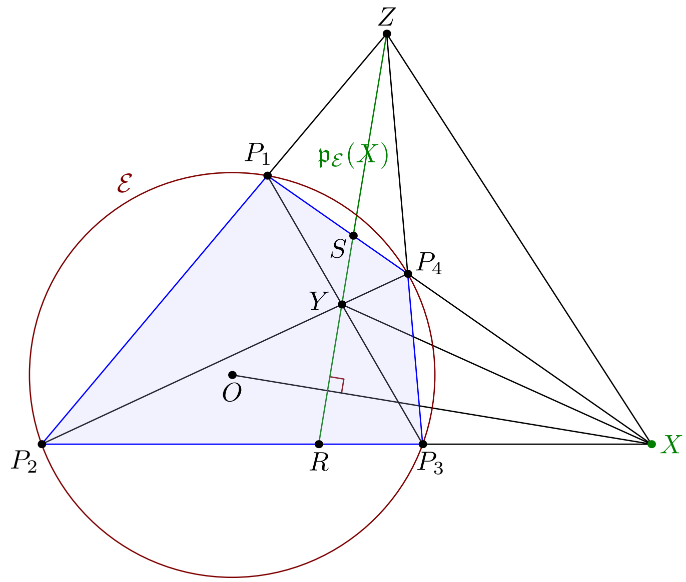
Remark. If H lies inside $\triangle ABC$, then this circle has imaginary radius, as k is negative. You can just think of polarity across this circle as the composition of polarity across the real circle with radius $\sqrt{-k}$ and reflecting about H.

We now look at a new way to find self-conjugate triangles.

Proposition 3.2.13 (Brokard's Theorem)**.** Let $\mathcal{E}$ be a circle and let P_1, P_2, P_3, P_4 be four points on this circle. Set

$$X = P_2P_3 \cap P_1P_4, Y = P_3P_1 \cap P_2P_4, Z = P_1P_2 \cap P_3P_4,$$

then $\triangle XYZ$ is self-conjugate wrt. $\mathcal{E}$.



Proof. Let YZ intersect P_2P_3 and P_1P_4 at R and S. Then by Quadrilateral Brokard we have that

$$(X, R; P_2, P_3) = (X, S; P_1, P_4) = -1.$$

and from Proposition 3.2.11 we know that $R, S \in \mathfrak{p}_{\mathcal{E}}(X)$, so $\mathfrak{p}_{\mathcal{E}}(X) = RS = YZ$. □

A quick corollary of this is that the orthocenter of $\triangle XYZ$ is the center of $\mathcal{E}$, which is very useful.

Example 3.2.14. Let cyclic quadrilateral $ABCD$ be inscribed in Ω. Let the diagonals AC and BD intersect at E, let AB and CD intersect at F, let AD and BC intersect at G, and let M be the midpoint of $\overline{FG}$. Let segment EM intersect Ω at T. Prove that (FGT) is tangent to Ω.

Solution. Note that if AC and EM coincide, we win immediately: By symmetry, WLOG let $T = C$. Then from M being the midpoint of $\overline{FG}$ we can use Quadrilateral Brokard (or Menelaus) to get that $BD \parallel FG$. Thus from

$$\begin{aligned}\measuredangle(\mathbf{T}_C(FGC), \mathbf{T}_C\Omega) &= \measuredangle(\mathbf{T}_C(FGC), CG) + \measuredangle(BC, \mathbf{T}_C\Omega)\\ &= \measuredangle CFG + \measuredangle CDB = 0^\circ\end{aligned}$$

we get that (FGT) is tangent to Ω.

Thus it remains to show that we can pick points A, B, C, D on Ω for fixed E, F, G that satisfy the above statement. By Proposition 3.2.13, $\triangle EFG$ is self-conjugate wrt. Ω.

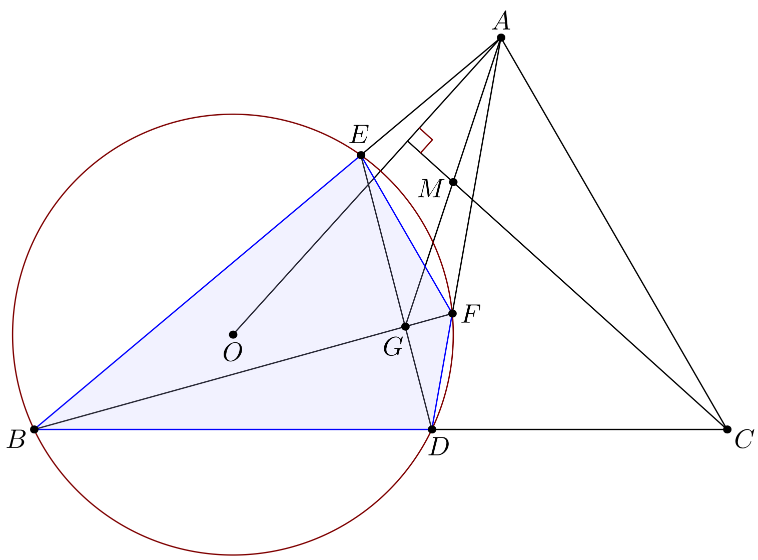
Now, let $A' = T$ and let B', D', C' be the other intersections of FA', GA', FD' with the circle respectively. Now, by Proposition 3.2.13 again, we find that $E' := A'C' \cap B'D', G' := A'D' \cap B'C'$ both lie on $\mathfrak{p}_\omega(F) = GE$. As such,

$$G' = A'D' \cap B'C' = GA' \cap B'C' \cap \mathfrak{p}_\omega(F) = GA' \cap GE = G.$$

and thus E' lies on $\mathfrak{p}_\omega(G) = EF$, and thus $E' = GE \cap EF = E$.

Then (A', B', C', D') is our desired points. $\square$

Example 3.2.15 (2009 China TST Day 1 P1)**.** Let D be a point on segment BC of $\triangle ABC$ such that $\angle CAD = \angle ABC$. The circle with center O through B and D intersects AB, AD again at E and F respectively. Let $G = BF \cap DE$ and let M be the midpoint of AG. Show that $CM \perp AO$.



Solution. Let ω be the circle centered at O with radius BO. Since $\angle CAD = \angle ABC$, we get that CA is tangent to (ABD), so $CA^2 = CB \cdot CD$ and C lies on the radical axis of ω and point circle A. Then, by Proposition 3.2.13 G lies on the polar of A wrt. ω. Now, by Proposition 3.2.3 M thus lies on the radical axis of ω and A as well. As such, CM is the radical axis which finishes. $\square$

Practice Problems

Problem 1 (2014 ISL G3)**.** Let Ω and O be the circumcircle and circumcenter of acute triangle $\triangle ABC$ with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P, Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \| AC$.

3.3 Cross Ratios under Inversion and Polarity

The next theorem shows that the pole-polar transformation preserves cross-ratios, coming in handy often.

Theorem 3.3.1 (Polarity preserves cross-ratios)**.** Let Γ be a circle, and L, l_1, l_2, l_3, l_4 be five lines, and let their respective poles be A, P_1, P_2, P_3, P_4, then

$$A(P_\bullet) = L(l_\bullet)$$

Proof. Let O be the center of Γ, $Q_i = \ell_i \cap L$, $K_i = AP_i$, then $K_i = \mathfrak{p}_\Gamma(Q_i)$, and we want to show that $(Q_\bullet) = (K_\bullet)$. We have two scenarios:

(i) If $O \notin L$, then $OQ_i \perp K_i$, and hence

$$\measuredangle Q_i O Q_j = \measuredangle(OQ_i, OQ_j) = \measuredangle(K_i, K_j).$$

So by the angular definition of the cross-ratio,

$$(Q_\bullet) = O(Q_\bullet) = (K_\bullet).$$

(ii) If $O \in L$, construct an arbitrary line ℓ that does not pass O, and let Q_i' be the projection of Q_i onto ℓ, and let K_i' be the polar of Q_i' with respect to Γ. Assume $R_i = K_i \cap K_i'$, then R_1, R_2, R_3, R_4 are on the same line $O\infty_\ell$. So we have

$$(Q_\bullet) = (Q_\bullet') = (K_\bullet') = (R_\bullet) = (K_\bullet).$$

□

We can then use the above theorem to prove that the inverse also preserves the cross ratio.

Proposition 3.3.2. For any collinear/concyclic points P_1, P_2, P_3, P_4,

$$(P_\bullet^{\mathfrak{J}}) = (P_1^{\mathfrak{J}}, P_2^{\mathfrak{J}}, P_3^{\mathfrak{J}}, P_4^{\mathfrak{J}}) = (P_\bullet)$$

where $\mathfrak{J}$ is taken wrt a circle $\mathcal{E}$.

Note that because P_1, P_2, P_3, P_4 are collinear/concyclic, then $P_1^{\mathfrak{J}}, P_2^{\mathfrak{J}}, P_3^{\mathfrak{J}}, P_4^{\mathfrak{J}}$ is also collinear/concyclic.

Proof. Let the center of inversion be O. We consider four cases:

(i) If P_1, P_2, P_3, P_4 are collinear and $O \in \ell$, then from Theorem 3.3.1 we have

$$(P_\bullet) = (\mathfrak{p}_{\mathcal{E}}(P_\bullet)) = \ell(\mathfrak{p}_{\mathcal{E}}(P_\bullet)) = (P_\bullet^{\mathfrak{J}}).$$

(ii) If P_1, P_2, P_3, P_4 are collinear and $O \notin \ell$, then $P_1^{\mathfrak{J}}, P_2^{\mathfrak{J}}, P_3^{\mathfrak{J}}, P_4^{\mathfrak{J}}$ are concyclic, so

$$(P_\bullet^{\mathfrak{J}}) = (OP_\bullet^{\mathfrak{J}}) = ((OP_\bullet) = (P_\bullet).$$

(iii) If P_1, P_2, P_3, P_4 are concyclic on Γ and $O \in \Gamma$, then $P_1^{\mathfrak{J}}, P_2^{\mathfrak{J}}, P_3^{\mathfrak{J}}, P_4^{\mathfrak{J}}$ are collinear but do not pass through O, so by the above case we get $(P_\bullet) = (P_\bullet^{\mathfrak{J}})$.

(iv) If P_1, P_2, P_3, P_4 are concyclic on Γ and $O \notin \Gamma$, then by the invariance of the cross-ratio under inversion, we can choose the power of inversion to be $\mathbf{Pow}_\Gamma \neq 0$. Then it follows that $P_i^{\mathfrak{J}} \in \Gamma$, and let $Q_{ij} = P_iP_j^{\mathfrak{J}} \cap P_i^{\mathfrak{J}}P_j$. By Proposition 3.2.13, Q_{ij} lies on the polar line $\mathfrak{p}_\Gamma(O)$ of O with respect to Γ. Therefore, since Q_{14}, Q_{24}, Q_{34} are collinear, we have

$$(P_\bullet) = P_4^{\mathfrak{J}}(P_\bullet) = (Q_{14}, Q_{23}; Q_{34}, P_4P_4^{\mathfrak{J}} \cap \mathfrak{p}_\Gamma(O)) = P_4(P_\bullet^{\mathfrak{J}}) = (P_\bullet^{\mathfrak{J}}).$$

$\square$

As an example application of inversion preserving cross-ratio, we will re-prove previously mentioned Casey's Theorem. This time, we will also prove the converse, which we left "for later" back in chapter 0.

Theorem (Casey's Theorem, Repeated)**.** Given four non-intersecting circles $\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_D$, define d_{IJ}^+ to be the length of the external common tangents of Γ_I and Γ_J, and define d_{IJ}^- to be the length of the internal common tangents of Γ_I and Γ_J. There exists a circle Ω tangent to the four circles $\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_D$ iff.

$$d_{BC}d_{AD} \pm d_{CA}d_{BD} \pm d_{AB}d_{CD} = 0,$$

where

$$d_{IJ} = \begin{cases} d_{IJ}^+, & \text{if } \Gamma_I,\ \Gamma_J \text{ are both tangent on the same side of } \Omega, \\ d_{IJ}^-, & \text{if } \Gamma_I,\ \Gamma_J \text{ are tangent on opposite sides of } \Omega. \end{cases}$$

We first prove the following lemma:

Lemma 3.3.3. Let r_I, r_J be the radii of Γ_I, Γ_J, then the ratio

$$\frac{(d_{IJ}^\pm)^2}{r_I r_J}$$

is invariant under inversion about any point X.

Proof. Let O_I and O_J be the centers of circles Γ_I and Γ_J, respectively. Let P_I, Q_I be the intersection points of the line segment O_IO_J and circle Γ_I, and let P_J, Q_J be the intersection points of the line segment O_IO_J and circle Γ_J. Then

$$\begin{aligned}\frac{\left(d_{IJ}^{\pm}\right)^2}{r_Ir_J} &= \frac{\overline{O_IO_J}^2 - (r_I \pm r_J)^2}{r_Ir_J}\\ &= 4\cdot\frac{\left(\overline{O_IO_J} + r_I \pm r_J\right)\left(\overline{O_IO_J} - r_I \mp r_J\right)}{(2r_I)\,(2r_J)}\\ &= 4\cdot\frac{\overline{P_IP_J}\cdot\overline{Q_IQ_J}}{\overline{P_IQ_I}\cdot\overline{P_JQ_J}} \quad \text{or} \quad 4\cdot\frac{\overline{P_IQ_J}\cdot\overline{Q_IP_J}}{\overline{P_IQ_I}\cdot\overline{P_JQ_J}}\\ &= 4\cdot\left|(P_I,Q_J;P_J,Q_I)\right| \quad \text{or} \quad 4\cdot\left|(P_I,P_J;Q_J,Q_I)\right|. \end{aligned} \qquad (\spadesuit)$$

Let circle ω be orthogonal to Γ_I, Γ_J and intersects Γ_I, Γ_J at R_I, S_I, R_J, S_J. Then the center of ω lies on the radical axis L of Γ_I, Γ_J. Let one of the intersections of L and ω be point A, then there exists a inversion centered at A that preserves Γ_I, Γ_J, and sends ω to a line ℓ. Since ω is orthogonal to these two circles, ℓ is also orthogonal, so ℓ is just O_IO_J. Since inversion preserves cross ratios, we have

$$(P_I,Q_J;P_J,Q_I),(P_I,P_J;Q_J,Q_I) = (R_I,S_J;R_J,S_I),(R_I,R_J;S_J,S_I). \qquad (\clubsuit)$$

If Γ_I', Γ_J' are the images of Γ_I, Γ_J under inversion at X (similarly define $r_I', r_J', d_{IJ}^{\pm'}$), we know that O_IO_J will invert to a circle ω' that is orthogonal to Γ_I', Γ_J'. Let ω' intersects Γ_I' at P_I, Q_I, which have images P_I', Q_I' under inversion at X. Similarly define P_J, Q_J, P_J', Q_J'. By ($\spadesuit$), ($\clubsuit$) and Proposition 3.3.2, we have

$$\begin{aligned}\left\{\frac{(d_{IJ}^{+})^2}{r_Ir_J}, \frac{(d_{IJ}^{-})^2}{r_Ir_J}\right\} &= \{4\cdot|(P_I,Q_J;P_J,Q_I)|, 4\cdot|(P_I,P_J;Q_J,Q_I)\}\\ &= \{4\cdot|(P_I',Q_J';P_J',Q_I'), 4\cdot|(P_I',P_J';Q_J',Q_I')|\}\\ &= \left\{\frac{(d_{IJ}^{+'})^2}{r_Ir_J}, \frac{(d_{IJ}^{-'})^2}{r_Ir_J}\right\}\end{aligned}$$

Since $d_{IJ}^{+} > d_{IJ}^{-}, d_{IJ}^{+'} > d_{IJ}^{-'}$, we have that $\left(\frac{(d_{IJ}^{+})^2}{r_Ir_J}, \frac{(d_{IJ}^{-})^2}{r_Ir_J}\right)$ is invariant and does not swap under inversion. $\square$

We now prove the converse of Casey's theorem.

Proof. WLOG let Γ_d have the smallest radii r_D. We now use a technique called **expansion** on AoPS.

We shrink the radii of circles Γ_D and other circles on the same side of the desired ω by r_D, and increase the radii of the circles on the opposite by r_D.

Then in this new configuration, a circle ω' exists if and only if a circle ω existed in the original configuration, but also shrinked / expanded by r_D.

We can also check that this operations preserves both the existence and lengths of d_{IJ}. This also preserves the non-intersection condition for circles with internal tangencies. As such, we now reduce to the case where D is a point circle

Invert about D, which due to the non-intersection condition maps $\Gamma_A, \Gamma_B, \Gamma_C$ to circles $\Gamma'_A, \Gamma'_B, \Gamma'_C$. We now wish show the existence of a line tangent to all three of the inverted images. Let ρ^2 be the power of the inversion. As such, we have that

$$\begin{aligned} 0 &= d_{BC}d_{AD} \pm d_{CA}d_{BD} \pm d_{AB}d_{CD} \\ &= \left(d'_{BC}\sqrt{\frac{r_Br_C}{r'_Br'_C}}\right)d_{AD} \pm \left(d'_{CA}\sqrt{\frac{r_Cr_A}{r'_Cr'_A}}\right)d_{BD} \pm \left(d'_{AB}\sqrt{\frac{r_Ar_B}{r'_Ar'_B}}\right)d_{CD} \\ &= \sqrt{\frac{r_Ar_Br_C}{r'_Ar'_Br'_C}}\left(d'_{BC}d_{AD}\sqrt{\frac{r'_A}{r_A}} \pm d'_{CA}d_{BD}\sqrt{\frac{r'_B}{r_B}} \pm d'_{AB}d_{CD}\sqrt{\frac{r'_C}{r_C}}\right) \end{aligned}$$

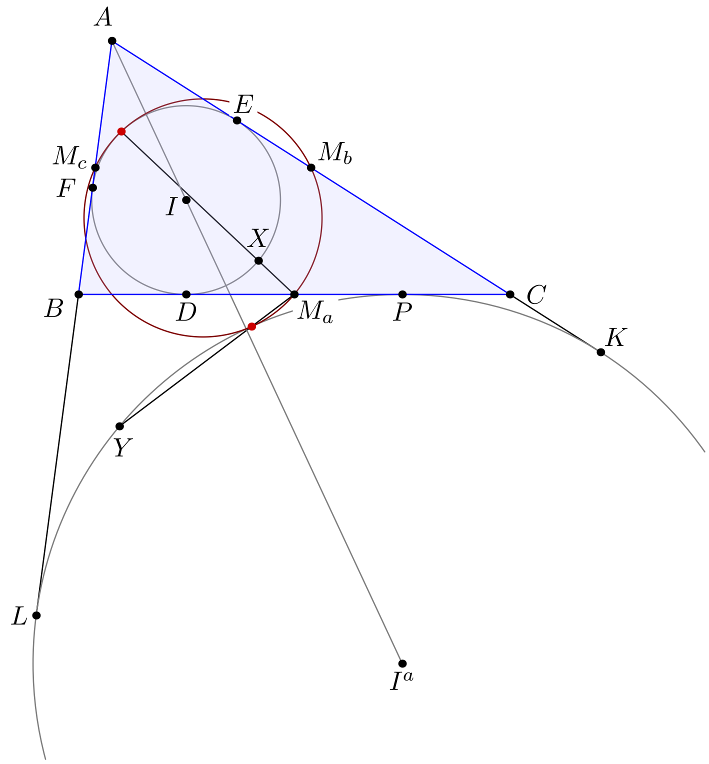
If $r_I = 0$ for some $I \in \{A, B, C\}$, then we set $\frac{r'_I}{r_I} := \frac{\rho^2}{d_{ID}^2}$, which occurs as a limiting case to the above lemma by Proposition 3.1.18, which says $r'_I = \frac{\rho^2 r_I}{d_{ID}^2}$. Substituting this into the above expression, it simplifies as

$$0 = d'_{BC}d_{AD}\sqrt{\frac{r'_A}{r_A}} \pm d'_{CA}d_{BD}\sqrt{\frac{r'_B}{r_B}} \pm d'_{AB}d_{CD}\sqrt{\frac{r'_C}{r_C}} = \rho(d'_{BC} \pm d'_{CA} \pm d'_{AB}).$$

Now, we once again apply expansion, and WLOG let this map Γ'_A into the point circle A'. Let QR be a tangent to Γ'_B, Γ'_C such that $Q \in \Gamma'_B, R \in \Gamma'_C, QR = d'_{BC}$. Then take P on QR such that $\overline{RP} = d'_{CA}, \overline{PQ} = d'_{AB}$ (which holds because one of the lengths is the sum of the two others). Then the locus of points $\mathbf{Pow}_{\Gamma'_B}(S) = d'_{AB}$ is a circle containing A and R with center O'_B. Similarly, the locus of points $\mathbf{Pow}_{\Gamma'_C}(S) = d'_{CA}$ also contains A and R and has center O'_C. Thus, the intersection of points in these loci are $\{R, R'\}$ where R' is the reflection of R about O_BO_C. Thus, $A \in \{R, R'\}$, and R, R' both lie on a common tangent of Γ'_B and Γ'_C, which suffices as the desired line. □

Example 3.3.4 (Feuerbach's Theorem)**.** For $\triangle ABC$, the nine-point circle ϵ and the incircle ω are internally tangent, and the nine-point circle and the three excircles are externally tangent. We call the incircle tangency point the **Feuerbach point** of $\triangle ABC$, and we call the three excircle tangency points the **A,B,C-Feuerbach points**.

⚠: When $\triangle ABC$ is a equilateral triangle, the Feuerbach point is not defined, but we can still use the A,B,C-Feuerbach points.



Solution. Let $\triangle M_aM_bM_c$, $\triangle DEF$ be the medial and intouch triangles of $\triangle ABC$. We use Casey's theorem to prove that there's a circle through M_a, M_b, M_c that's tangent to $\omega = (DEF)$. By Casey we only need to prove

$$M_bM_c \cdot M_aD \pm M_cM_a \cdot M_bD \pm M_cM_a \cdot M_cD = 0.$$

Let $a = BC, b = CA, c = AB$. Then assume WLOG that $a \geq b \geq c$, then

$$M_bM_c \cdot M_aD \pm M_cM_a \cdot M_bE \pm M_cM_a \cdot M_cF = \frac{a}{2} \cdot \frac{b-c}{2} \pm \frac{b}{2} \cdot \frac{a-c}{2} \pm \frac{c}{2} \cdot \frac{a-b}{2} = 0.$$

Thus ϵ and ω are tangent. □

There is a very simple construction of the Feuerbach point that will be very important later on.

Let I, I^a respectively be the incenter and A-excenter, let X be the reflection of D across AI, then Fe is just the second intersection of M_aX and ω. The proof is relatively straightforward.

Proof. Let T be $AI \cap BC$, then TX is clearly tangent to ω. We consider the inversion $\mathfrak{J}$ at M_a with power $\overline{M_aD}^2$, and note that $\mathfrak{J}(\omega) = \omega$. Thus we just need to prove that $\mathfrak{J}(TX) = \epsilon$.

Let D' be the reflection of D in M_a, let H_a be the foot from A to BC. Then

$$(H_a, T; D, D') \overset{\infty_{\perp BC}}{=} (A, T; I, I^a) = (BA, BC, BI, BI^a) = -1.$$

Thus we have $M_aH_a \cdot M_aT = M_aD^2$, which tells us that $\mathfrak{J}(T) = H_a$, thus $\mathfrak{J}(TX)$ is a circle passing through H_a, M_a with the tangent at M_a parallel to TX. Thus we only need to prove that the tangent from M_a to ϵ is also parallel to TX and we're done.

We angle chase

$$\measuredangle(H_aM_a, TX) = 2 \cdot \measuredangle H_aTA = 2 \cdot (\measuredangle H_aAI + 90^\circ) = \measuredangle H_aAI + \measuredangle IAO = \measuredangle H_aAO,$$

where O is the circumcenter of $\triangle ABC$. Let E_a be the second intersection of AH_a and ϵ, which is just the midpoint of A and H. We have $E_aM_a \parallel AO$ by considering $\mathfrak{h}_{H,2}$, so

$$\measuredangle(H_aM_a, TX) = \measuredangle H_aAO = \measuredangle H_aE_aM_a,$$

and TX is tangent to $(H_aM_aE_a) = \epsilon$ as desired. □

For more details on the Feuerbach point view section Feuerbach.

3.4 Apollonian Circles

The Apollonian circle is the locus of points with a constant ratio of distances from two other points.

Proposition 3.4.1. Given two points A, B and a positive real $r \neq 1$, the set

$$\Gamma_r^{A,B} = \{P \mid \overline{PA} = r \cdot \overline{PB}\}$$

is a circle, and A, B are inversive images in this circle. We call $\Gamma_r^{A,B}$ the **r-Apollonius circle**.

Proof. If $r = 1$, then Γ_r is just the perpendicular bisector of segment $\overline{AB}$. So assume $r \neq 1$, then pick two points on AB (call them P_+, P_-) such that

$$\frac{\overrightarrow{AP_+}}{\overrightarrow{P_+B}} = r, \frac{\overrightarrow{AP_-}}{\overrightarrow{P_-B}} = -r,$$

and it's obvious that $(A, B; P_+, P_-) = -1$. We know that $P \in \Gamma_r^{A,B}$ if and only if PP_+ is the angle bisector of $\measuredangle APB$. However by Harmonic Bisectors of Right Angles, this is equivalent to $\measuredangle P_-PP_+$ being a right angle. Thus $\Gamma_r^{A,B}$ must be a circle with $\overline{P_+P_-}$ as diameter.

Since $(A, B; P_+, P_-) = 1$, by properties of inversion and cross ratio Proposition 3.2.11, we get that A and B are inversive images over $\Gamma_r^{A,B}$. □

We then have the following

- $\Gamma_r^{A,B} = \Gamma_{r^{-1}}^{B,A}$;
- $\Gamma_r^{A,B}$ is orthogonal to (AB) by Corollary 3.1.11;
- There exists an Apollonian circle wrt. A, B for any two inverses in the circle (AB) on line $\overline{AB}$.

By some algebraic calculations, we can get the following:

Proposition 3.4.2. Given any two points A, B and a positive real r, the circumcenter O_r of the Apollonian circle $\Gamma_r^{A,B}$ and the radius ρ_r satisfy

$$\frac{\overrightarrow{O_rA}}{\overrightarrow{O_rB}} = r^2, \rho_r = \frac{r}{|r^2-1|} \cdot \overline{AB}.$$

Proof. Choose point P on Apollonian circle $\Gamma_r^{A,B}$, then since A, B are inversive images in $\Gamma_r^{A,B}$ we have $\triangle O_rAP \overset{-}{\sim} \triangle O_rPB$. Therefore since O_r lies outside segment $\overline{AB}$ by the inversion property, we have

$$\frac{\overrightarrow{O_rA}}{\overrightarrow{O_rB}} = \frac{\overrightarrow{O_rA}}{\overrightarrow{O_rP}} \cdot \frac{\overrightarrow{O_rP}}{\overrightarrow{O_rB}} = \frac{\overrightarrow{AP}}{\overrightarrow{PB}} \cdot \frac{\overrightarrow{AP}}{\overrightarrow{PB}} = r^2.$$

Combined with

$$AO_r \cdot AB = \mathbf{Pow}_{\Gamma_r}(A) = O_rA^2 - \rho_r^2, O_rA = \frac{AP}{PB} \cdot O_rP = r \cdot \rho_r,$$

we get

$$AB = \frac{|r^2-1|}{r} \cdot \rho_r,$$

and we are done. □

We can actually extend this as follows.

Proposition 3.4.3. Let P be a point on the r-Apollonius circle $\Gamma_r^{A,B}$ of A, B. Then $\Gamma_r^{A,B}$ is orthogonal to (PAB).

Proposition 3.4.4. Let O be the circumcenter of $\triangle ABC$. For any three positive reals r, s, t, the power of O wrt. the three Apollonian circles $\Gamma_r^{B,C}, \Gamma_s^{C,A}, \Gamma_t^{A,B}$ is equal. Furthermore, $\Gamma_r^{B,C}, \Gamma_s^{C,A}, \Gamma_t^{A,B}$ are coaxial if and only if $rst = 1$.

Proof. Let O_r, O_s, O_t be the three circumcenters of $\Gamma_r^{B,C}, \Gamma_s^{C,A}, \Gamma_t^{A,B}$. Let Ω be the circumcircle of $\triangle ABC$. From Proposition 3.4.3 we get that Ω is orthogonal to $\Gamma_r^{B,C}$, so we have by Proposition 3.1.10 that $\mathbf{Pow}_{\Gamma_r^{B,C}}(O)$ $= R^2$, where R is the circumradius. Applying this symmetrically gives the result.

For the coaxal condition, we know $\Gamma_r^{B,C}, \Gamma_s^{C,A}, \Gamma_t^{A,B}$ are coaxal if and only if O_r, O_s, O_t are collinear. By Menelaus and Proposition 3.4.2, this is the same thing as

$$-1 = \frac{BO_r}{O_rC} \cdot \frac{CO_s}{O_sA} \cdot \frac{AO_t}{O_tB} = (-r^2) \cdot (-s^2) \cdot (-t^2) = -(rst)^2,$$

or $rst = 1$. □

Remark. When A, B, C are collinear, then the condition of $\Gamma_r^{B,C}, \Gamma_s^{C,A}, \Gamma_t^{A,B}$ being coaxial still holds if and only if $rst = 1$. We can't use the above proof because O is on the line at infinity and it's too degenerate, but simple algebra works in this case.

Now we look at Apollonian circles going through a fixed point.

Definition 3.4.5. Given two points A, B and a point P, we define the **P-Apollonian circle** $\Gamma_p^{A,B}$ as the unique Apollonian circle of A, B that goes through P. In this case, r is obviously just $\frac{PA}{PB}$. Further, we obviously have $\Gamma_p^{A,B} = \Gamma_P^{B,A}$.

Given two points A, B, with respect to two points P, Q we know that

$$\frac{PA}{PB} = \frac{QA}{QB} \iff \frac{AP}{AQ} = \frac{BP}{BQ}.$$

As such, $\Gamma_p^{A,B} = \Gamma_Q^{A,B}$ if and only if $\Gamma_A^{P,Q} = \Gamma_B^{P,Q}$. From Proposition 3.4.3, we know that $\Gamma_p^{A,B}$ is orthogonal to (PAB).

Proposition 3.4.6. Given $\triangle ABC$, let $\mathfrak{I}_A$ be a $\sqrt{bc}$-inversion at A. Then

- $\mathfrak{I}_A(\Gamma_A^{B,C})$ is the perpendicular bisector of $\overline{BC}$;
- $\mathfrak{I}_A(\Gamma_B^{C,A})$ is the circle through C with center B;
- $\mathfrak{I}_A(\Gamma_C^{A,B})$ is the circle through B with center C.

Proof. Since $\Gamma_A^{B,C}$ is orthogonal to both BC and (ABC) by Proposition 3.4.3, and $\Gamma_A^{B,C}$ passes through A, the image of $\Gamma_A^{B,C}$ is a line orthogonal to (ABC) and BC, which is just the perpendicular bisector of $\overline{BC}$.

Since $\Gamma_B^{C,A}$ is orthogonal to CA and (ABC) and passes through B, $\mathfrak{I}_A(\Gamma_B^{C,A})$ is orthogonal to $\mathfrak{I}_A(CA) = AB, \mathfrak{I}_A((ABC)) = BC$ and passes through $\mathfrak{I}_A(B) = C$, so it must be the circle centered at B going through C, and same logic for $\Gamma_C^{A,B}$. □

Corollary 3.4.7. In $\triangle ABC$, the three Apollonius circles $\Gamma_A^{B,C}, \Gamma_B^{C,A}, \Gamma_C^{A,B}$ all pass through two common points S_1, S_2. Further, these points are inversive pairs under inversion over (ABC).

Proof. This follows by the Angle-bisector theorem and Proposition 3.4.4. Alternatively, let $\mathfrak{I}_A$ represent $\sqrt{bc}$-inversion at A. Choose T_1, T_2 such that $\triangle T_1BC, \triangle T_2BC$ are equilateral triangles. By Proposition 3.4.6, we can see that $\mathfrak{I}_A(\Gamma_A^{B,C}), \mathfrak{I}_A(\Gamma_B^{C,A}), \mathfrak{I}_A(\Gamma_C^{A,B})$ concur at T_1, T_2. Thus $\Gamma_A^{B,C}, \Gamma_B^{C,A}, \Gamma_C^{A,B}$ concur at $S_1 = \mathfrak{I}_A(T_1), S_2 = \mathfrak{I}_A(T_2)$.

Next, from Proposition 3.4.4, we get that O has equal power R^2 wrt. $\Gamma_A^{B,C}, \Gamma_B^{C,A}, \Gamma_C^{A,B}$. Thus O lies on the radical axis S_1, S_2 of the three circles, and

$$OS_1 \cdot OS_2 = \mathbf{Pow}_{\Gamma_A^{B,C}}(O) = R^2,$$

so S_1, S_2 are inversive pairs under inversion in (ABC). $\square$

Thus we know that $S_1 \neq S_2$ and one of them lies inside (ABC), and one of them lies outside of (ABC). (Note neither can lie on the circumcircle because $T_1, T_2 \notin BC$.) So assume WLOG that S_1 lies inside (ABC).

Definition 3.4.8. We call S_1 the **first isodynamic point** and we call S_2 the **second isodynamic point** of $\triangle ABC$.

Proposition 3.4.9. Given $\triangle ABC$, construct three external equilateral triangles $\triangle A_1BC, \triangle B_1CA, \triangle C_1AB$ on sides BC, CA, AB. Similarly define A_2, B_2, C_2 by constructing internal equilateral triangles. Then AA_1, BB_1, CC_1 concur at a point F_1, and AA_2, BB_2, CC_2 concur at a point F_2. Further, F_1 and F_2 are the isogonal conjugates of S_1 and S_2 respectively.

Proof. Since S_1 is inside of (ABC), under a $\sqrt{bc}$-inversion we get that $T_1 = \mathfrak{I}_A(S_1)$ is on the opposite side of line BC as point A. Formally, note that by Proposition 3.1.2,

$$\frac{\overline{AB} \cdot \overline{AC}}{\overline{A\mathfrak{I}_A(O)} \cdot \overline{AT_1}} \cdot \overline{\mathfrak{I}_A(O)T_1} = \overline{OS_1} < \overline{OA} = \frac{\overline{AB} \cdot \overline{AC}}{\overline{A\mathfrak{I}_A(O)}}.$$

Since $\mathfrak{I}_A(O)$ is the reflection of A over BC, and this implies $\overline{\mathfrak{I}_A(O)T_1} < \overline{AT_1}$, the result follows. Similarly $T_2 = \mathfrak{I}_A(S_2)$ is on the same side of BC as point A. Thus by the proof of Proposition 3.4.1, we get $T_1 = A_2, T_2 = A_2$.

Let $i \in \{1, 2\}$. Then AA_i and AS_i are isogonal, and thus by symmetry AA_i, BB_i, CC_i concur at the isogonal conjugate of S_i. $\square$

From

$$\measuredangle BAB_1 = \measuredangle BAC + 60^\circ = \measuredangle C_1AC \pmod{360^\circ}, \quad \frac{AB}{B_1A} = \frac{AC_1}{CA}$$

we get

$$\triangle ABB_1 \overset{+}{\sim} \triangle AC_1C.$$

Thus

$$\measuredangle BF_1C = \measuredangle(BB_1, C_1C) = \measuredangle(AB, AC_1) = 120^\circ.$$

Similarly we have the other two internal angles $\measuredangle CF_1A = \measuredangle AF_1B = 120^\circ$. In a similar nature, F_2 we have $\measuredangle BF_2C = \measuredangle CF_2A = \measuredangle AF_2B = 60^\circ$.

As such, we can characterize F_1, F_2 in a completely different way.

Definition 3.4.10. Given $\triangle ABC$ labeled counterclockwise, the points F_1, F_2 where F_i satisfies $\measuredangle BF_iC = \measuredangle CF_iA = \measuredangle AF_iB = -i \cdot 60^\circ$ are called the **first Fermat point** and **second Fermat point** respectively.

Remark. When no angle of a triangle exceeds 120°, F_1 minimizes the sum of $\overline{AX} + \overline{BX} + \overline{CX}$ over points X.

Finally, Proposition 3.4.9 lets us recharacterize the isodynamic points.

Proposition 3.4.11. The pedal triangles wrt the two isodynamic points S_i and $\triangle ABC$ are equilateral triangles.

Proof. Fix S_i. Then let $\triangle S_aS_bS_c$ represent the pedal triangle of S_i, then

$$\measuredangle S_bS_aS_c = \measuredangle S_bS_aS + \measuredangle SS_aS_c = \measuredangle S_bCS + \measuredangle SBS_c = \measuredangle F_iCB + \measuredangle CBF_i = \measuredangle(CF_i, BF_i) = i \cdot 60^\circ,$$

and similarly on the other corners of the pedal triangle to get that all angles are equal. $\square$

Remark. Additionally, by Proposition 3.4.6 inverting $\triangle ABC$ around S maps it to an equilateral triangle.

We will revisit Fermat and isodynamic points with the Neuberg cubic in later chapters.

3.5 Apollonius's Circle Problem

Let's now figure out how to construct a circle tangent to three other circles, $\Gamma_1, \Gamma_2, \Gamma_3$.

Suppose Ω is tangent to $\Gamma_1, \Gamma_2, \Gamma_3$. We will find some characterizations of Ω.

Proposition 3.5.1. Let R be the radical center of $\Gamma_1, \Gamma_2, \Gamma_3$, and let T_i be the touchpoint of Ω and Γ_i, let P_i be the pole of line RT_i across Γ_i. Then we have P_1, P_2, P_3 are collinear. Further, let $\mathfrak{J}$ be an inversion centered at R with power $\mathbf{Pow}_{\Gamma_i}(R)$. Then $\overline{P_1P_2P_3}$ is the radical axis of Ω and $\Omega^{\mathfrak{J}}$.

Proof. Since inversion preserves tangency, from $\Gamma_i^{\mathfrak{J}} = \Gamma_i$, we get that $\Omega^{\mathfrak{J}}$ is tangent to $\Gamma_1, \Gamma_2, \Gamma_3$ as well (note this does not necessarily imply $\Omega = \Omega^{\mathfrak{J}}$.) Note that the tangency point of $\Omega^{\mathfrak{J}}$ is just $T_i^{\mathfrak{J}}$. Since P_iT_i is the radical axis of Ω and Γ_i and $P_iT_i^{\mathfrak{J}}$ is the radical axis of $\Omega^{\mathfrak{J}}, \Gamma_i$, P_i is the radical center of $\Omega, \Omega^{\mathfrak{J}}, \Gamma_i$, so P_i lies on the radical axis ℓ of $\Omega, \Omega^{\mathfrak{J}}$.

□

From this characterization we can know that the pole of ℓ wrt. Γ_i (let it be Q_i) must lie on RT_i.

Proposition 3.5.2. Extending the notation from Proposition 3.5.1, let $T_i^{\mathfrak{J}}$ be the touchpoint of $\Omega^{\mathfrak{J}}$ and Γ_i, then T_iT_j and $T_i^{\mathfrak{J}}T_j^{\mathfrak{J}}$ intersect at one of the similicenters of Γ_i and Γ_j, U_{ij}. Further U_{ij} lies on the radical axis ℓ of Ω and $\Omega^{\mathfrak{J}}$.

Proof. Since T_i is one of the similicenters of Ω and Γ_i, T_j is one of the similicenters of Ω and Γ_j, we can use Monge's Theorem to get that T_iT_j passes through one of the similicenters of Γ_i, Γ_j. Similarly, we get that $T_i^{\mathfrak{J}}T_j^{\mathfrak{J}}$ passes through one of the similicenters of Γ_i, Γ_j. So for the following section we will prove that these two similicenters have the same sign, or instead that $O_iO_j, T_iT_j, T_i^{\mathfrak{J}}T_j^{\mathfrak{J}}$ are concurrent. Let O_i be the center of circle Γ_i, and let $O, O^{\mathfrak{J}}$ be the centers of $\Omega, \Omega^{\mathfrak{J}}$. Note that $\triangle OT_iT_j$ and $\triangle O^{\mathfrak{J}}T_i^{\mathfrak{J}}T_j^{\mathfrak{J}}$ are perspective, let their perspector be R. By Desargues's Theorem we have

$$T_iT_j \cap T_i^{\mathfrak{J}}T_j^{\mathfrak{J}}, O_j = T_jO \cap T_j^{\mathfrak{J}}O^{\mathfrak{J}}, O_i = OT_i \cap O^{\mathfrak{J}}T_i^{\mathfrak{J}}$$

are collinear.

Thus $O_iO_j, T_iT_j, T_i^{\mathfrak{J}}T_j^{\mathfrak{J}}$ are concurrent.

Since the radical axis of Ω and $(T_iT_jT_i^{\mathfrak{J}}T_j^{\mathfrak{J}})$ is line T_iT_j, and the radical axis of $\Omega^{\mathfrak{J}}$ and $(T_iT_jT_i^{\mathfrak{J}}T_j^{\mathfrak{J}})$ is $T_i^{\mathfrak{J}}T_j^{\mathfrak{J}}$, we have that U_{ij} is the radical center of $\Omega, \Omega^{\mathfrak{J}}, (T_iT_jT_i^{\mathfrak{J}}T_j^{\mathfrak{J}})$, thus U_{ij} also lies on the radical axis of Ω and $\Omega^{\mathfrak{J}}$. □

By Monge's Theorem, we have that the pairwise similicenters of the same sign $O_{ij,\pm}$ of $\Gamma_1, \Gamma_2, \Gamma_3$ form a complete quadrilateral with sides $(\ell_1, \ell_2, \ell_3, \ell_4)$, where

$$\ell_1 = O_{23,+}O_{31,-}O_{12,-}, \ell_2 = O_{23,-}O_{31,+}O_{12,-},$$

$$\ell_3 = O_{23,-}O_{31,-}O_{12,+}, \ell_4 = O_{23,+}O_{31,+}O_{12,+}.$$

So by Proposition 3.5.2 we have that the radical axis of Ω and $\Omega^{\mathfrak{J}}$ has to be a line in $\{\ell_1, \ell_2, \ell_3, \ell_4\}$.

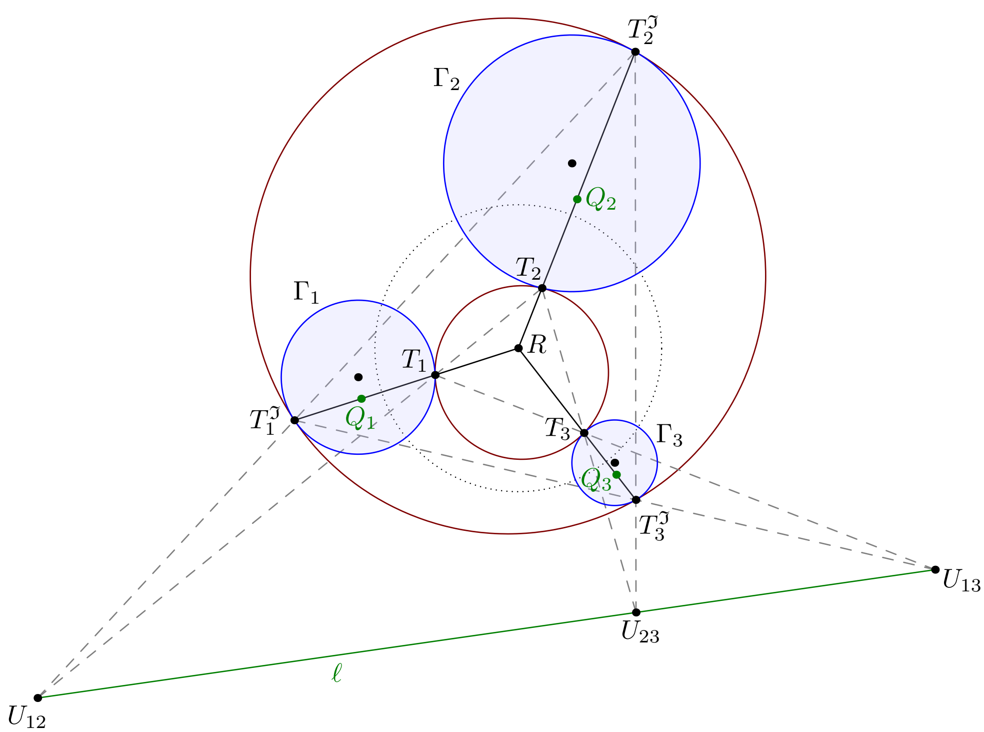
We already know most of the information needed to reconstruct Ω, we just need to construct Ω given the complete quadrilateral formed by the various similicenters. Let R and Q_i be defined as before. If RQ_i and Γ_i do not intersect, then we can't construct Ω, so assume they intersect for some i. These intersection points will be T_i and $T_i^{\mathfrak{I}}$. Now, it remains to distinguish between $T_i, T_i^{\mathfrak{I}}$ such that $(T_1T_2T_3)$, $(T_1^{\mathfrak{I}}T_2^{\mathfrak{I}}T_3^{\mathfrak{I}})$ are all tangent to $\Gamma_1, \Gamma_2, \Gamma_3$. To prove this we just need:

Proposition 3.5.3. Let R be a point on the radical axis of Γ_1, Γ_2, and let U be one of their similicenters. Let ℓ be a line through U, and let Q_i be the pole of ℓ wrt. Γ_i. If T_1 is one of the intersection points of RQ_1 and Γ_1, then UT_1, RQ_2, Γ_2 are concurrent.

Proof. We consider the homothety $\mathfrak{h}$ with center U sending $\mathfrak{h}(\Gamma_1) = \Gamma_2$. We have

$$\mathfrak{h}(Q_1) = \mathfrak{h}(\mathfrak{p}_{\Gamma_1}(\ell)) = \mathfrak{p}_{\mathfrak{h}(\Gamma_1)}(\mathfrak{h}(\ell)) = \mathfrak{p}_{\Gamma_2}(\ell) = Q_2.$$

Let $S = \mathfrak{h}(T_1)$, let T_2 be the second intersection point of ST_1 and Γ_2. Let S', T_2' respectively be the second intersections of SQ_2, T_2Q_2 with Γ_2. By Brokard's theorem Proposition 3.2.13 we have $ST_2 \cap S'T_2'$ is on the polar of $Q_2 = SS' \cap T_2T_2'$, so $U \in S'T_2'$. By Reim's Theorem we also have that since S, T_2, S', T_2' are concyclic, so T_1, T_2, T_1', T_2' concyclic, where $T_1' = \mathfrak{h}^{-1}(S')$. Since T_1T_1' is the radical axis of $(T_1T_2T_1'T_2')$ and Γ_1, and T_2T_2' is the radical axis of $(T_1T_2T_1'T_2')$ and Γ_2, we have $T_1T_1' = T_1Q_1$ and $T_2T_2' = T_2Q_2$ intersect at the radical axis of Γ_1, Γ_2, so R, T_2, Q_2 are collinear. □



We now work on constructing ω given Γ_i. We first construct T_1, T_2, T_3 and their inverses, repeating definitions for clarity: Let O_i be the centers of Γ_i and take ℓ as one of the above four Monge lines. Define $U_{ij} = O_iO_j \cap \ell$. Take R as the radical center of Γ_i, Q_i as the pole of ℓ wrt Γ_i and let $T_1, T_1^{\mathfrak{I}}$ arbitrarily be the intersections of Γ_1 and RQ_1. We can then construct T_2 as $T_1U_{12} \cap Q_2T_2'$, and $T_2^{\mathfrak{J}}$ as the other intersection of T_2Q_2 with Γ_2, such that $U_{12} = T_1T_2 \cap T_1^{\mathfrak{J}}T_2^{\mathfrak{J}}$. We can do the same for $T_3, T_3^{\mathfrak{J}}$ such that $T_1T_3 \cap T_1^{\mathfrak{J}}T_3^{\mathfrak{J}} = U_{31}$.

Notice then that $\triangle T_1T_2T_3$ and $\triangle T_1^{\mathfrak{J}}T_2^{\mathfrak{J}}T_3^{\mathfrak{J}}$ are perspective at R, so by Desargues's Theorem, we have

$$T_2T_3 \cap T_2^{\mathfrak{J}}T_3^{\mathfrak{J}}, T_3T_1 \cap T_3^{\mathfrak{J}}T_1^{\mathfrak{J}} = U_{31}, T_1T_2 \cap T_1^{\mathfrak{J}}T_2^{\mathfrak{J}} = U_{12}$$

are collinear, so we have $U_{23} = T_2T_3 \cap T_2^{\mathfrak{J}}T_3^{\mathfrak{J}}$ passes through U_{23}. Now, we only need to prove that $(T_1T_2T_3), (T_1^{\mathfrak{J}}T_2^{\mathfrak{J}}T_3^{\mathfrak{J}})$ are tangent to $\Gamma_1, \Gamma_2, \Gamma_3$.

Proposition 3.5.4. Let T_1, T_2, T_3 respectively lie on circles $\Gamma_1, \Gamma_2, \Gamma_3$ such that T_iT_j passes through one of the similicenters U_{ij} of Γ_i, Γ_j, and U_{23}, U_{31}, U_{12} are collinear. Then the circumcircle of $\triangle T_1T_2T_3$ is tangent to $\Gamma_1, \Gamma_2, \Gamma_3$.

Proof. Continuing to use the previous notation, let R be the radical center of $\Gamma_1, \Gamma_2, \Gamma_3$, let Q_i be the pole of $\overline{U_{23}U_{31}U_{12}}$ wrt. Γ_i. Let S_{ij} be the second intersection of T_iT_j and Γ_j, and let $\mathfrak{h}_{ij}$ be the homothety centered at U_{ij} sending $\mathfrak{h}_{ij}(\Gamma_i) = \Gamma_j$, and $S_{ij} = \mathfrak{h}_{ij}(T_i)$, which is on Γ_j.

Since $\mathfrak{h}_{ij}(Q_i) = Q_j$, we have $S_{ij}Q_j \parallel Q_iR$, and thus by spiral similarity, $\triangle RT_iT_j \overset{+}{\sim} \triangle Q_jS_{ij}T_j$. Combining $\triangle RT_2T_1 \overset{+}{\sim} \triangle Q_1S_{21}T_1, \triangle RT_3T_1 \overset{+}{\sim} \triangle Q_1S_{31}T_1$, we get $\triangle T_1T_2T_3 \overset{+}{\sim} \triangle T_1S_{21}S_{31}$, thus $\Omega = (T_1T_2T_3)$ is tangent to $\Gamma_1 = (T_1S_{21}S_{31})$, and similarly Ω is tangent to Γ_2, Γ_3. $\square$

Since Ω is tangent to each Γ_i, $O = O_1T_1 \cap O_2T_2 \cap O_3T_3$ is the circumcenter of $\triangle T_1T_2T_3$, and $\triangle O_1O_2O_3$ and $\triangle T_1T_2T_3$ have $U_{12}U_{23}U_{31}$ as a perspectrix.

This can also be proven through Menelaus's Theorem. If O is the perspector of $\triangle O_1O_2O_3$ and $\triangle T_1T_2T_3$ (which exists by Deargues's), then by Menelaus's

$$\frac{O_1T_1}{T_1O} \cdot \frac{OT_2}{T_2O_2} = -\frac{O_1U_{12}}{U_{12}O_2} = \pm\frac{r_1}{r_2}$$

where r_i is the radius of Γ_i. This then implies that $\overline{OT_1} = \overline{OT_2}$, which follows by symmetry.

Remark. Each of the 4 choices of ℓ gives us a different pair of circles, which gives us a total of 8 circles as solutions to being tangent to all of Γ_i.

In the above, we solved the problem when $\Gamma_1, \Gamma_2, \Gamma_3$ are circles. However, Apollonius's problem also exists for the degenerate cases where circles are instead replaced with lines and points. The proof is similar to Proposition 3.5.4 when there's multiple circles, and is left to reader.

Now, here's the constructions.

(i) All three are circles: See above section.

(ii) $\Gamma_1 = L_1$ is a line, the rest are circles: Let R be the intersection of the radical axis of Γ_2, Γ_3 with L_1, let $O_{12,\pm}$ be the intersections of $O_2\infty_{\perp L_1}$ with Γ_2, define $O_{31,\pm}$ similarly. Take a monge line $U_{23}U_{31}U_{12}$. Then let $T_2 = RQ_2 \cap \Gamma_2$ be one intersection, then $T_1 = U_{12}T_2 \cap L_1, T_3 = U_{23}T_2 \cap L_1$.

(iii) Γ_1 is a circle, $\Gamma_2 = L_2, \Gamma_3 = L_3$ are lines: Let $R = L_2 \cap L_3$, define $O_{12,\pm}, O_{31,\pm}$ as the same as in (i). Define $O_{23,\pm}$ as the points along infinity on the angle bisectors between L_2 and L_3. Take a monge line $U_{23}U_{31}U_{12}$, and finish the same as above.

(iv) All $\Gamma_1 = L_1, \Gamma_2 = L_2, \Gamma_3 = L_3$ are lines: Then Ω is just an incircle or excircle of $\triangle L_1L_2L_3$.

(v) $\Gamma_1 = P_1$ is a point, Γ_2, Γ_3 are circles: R is still the radical center, let $O_{31,\pm}$ and $O_{12,\pm}$ all be P_1. Take a Monge line $P_1O_{23,\pm}$ and let $U_{23} = O_{23,\pm}$. Then define T_2 as an intersection of RQ_2 with Γ_2, define T_3 normally, and let $T_1 = P_1$.

(vi) Γ_1 is a circle, $\Gamma_2 = P_2, \Gamma_3 = P_3$ are points: Define R as the radical center, and take the Monge line P_2P_3. Define $T_1 = RQ_1 \cap L_1$, and take $T_2 = P_2, T_3 = P_3$

(vii) $\Gamma_1 = P_1, \Gamma_2 = P_2, \Gamma_3 = P_3$ are points: Then Ω is just the circumcircle of $\triangle P_1P_2P_3$.

(viii) Γ_1 is a circle, $\Gamma_2 = L_2$ is a line, $\Gamma_3 = P_3$ is a point: R is the intersection of the radical axis of P_3, Γ_1 with L_2. Define $O_{12,\pm}$ as in the first case, define $O_{23,\pm}O_{31,\pm}$ as P_3. Take a Monge line $P_3O_{12,\pm}$, $U_{12} = O_{12,\pm}$, and define $T_1 = RQ_1 \cap L_1$, $T_2 = U_{12}T_1 \cap L_2$, $T_3 = P_3$.

(ix) $\Gamma_1 = L_1$ is a line, $\Gamma_2 = P_2, \Gamma_3 = P_3$ are points: Let $U = L_1 \cap P_2P_3$. Take T_1 on L_1 such that $\overline{UT_1}^2 = UP_2 \cdot UP_3$, and $\Omega = (T_1P_2P_3)$

(x) $\Gamma_1 = P_1$ is a point, $\Gamma_2 = L_2, \Gamma_3 = L_3$ are lines: Let ℓ be a line through P_1 parallel to one of the angle bisectors of L_2, L_3. Let U_2, U_3 be the intersections of ℓ with L_2, L_3 respectively. Let P_1' be the reflection of P about the midpoint of U_2U_3. Then Ω is the circle through P_1, P_1' tangent to L_2.

Chapter 4

Complete Quadrilaterals

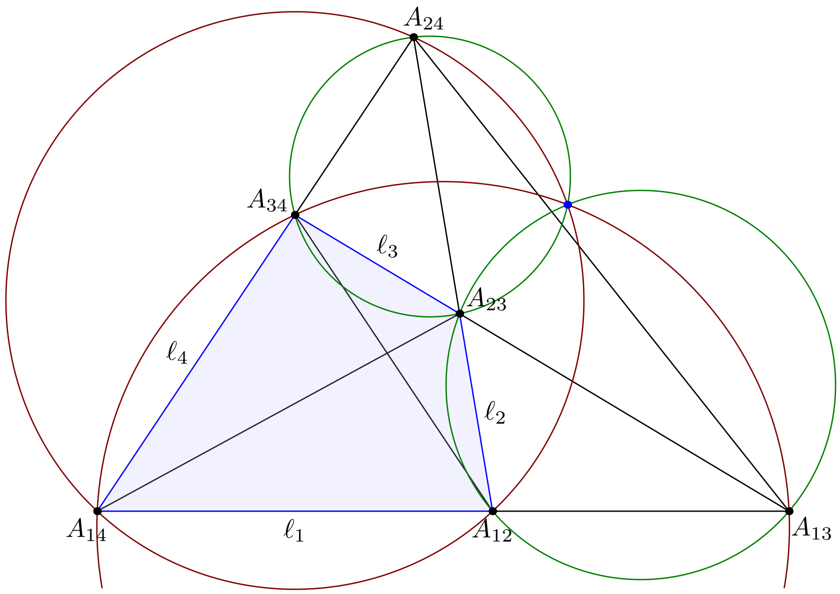
A complete quadrilateral is just four non-concurrent lines and the six points they intersect at. Taking just three non-concurrent lines gave us a triangle, giving us such beautiful theorems as the Euler line, the nine-point circle, Feuerbach's tangency, etc.

What secrets do four lines hold, and how can we apply these back to triangle geometry later on?

4.1 Basic Objects In a Complete Quadrilateral

In this section we will define and characterize some well-known points in a complete quadrilateral (note that many of these can be defined for a normal convex quadrilateral too, but for simplicity we just consider the complete quadrilateral case.). Given a complete quadrilateral $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$, we define

- Six intersection points: $A_{ij} = \ell_i \cap \ell_j$;
- Four triangles: $\triangle_i := \triangle \ell_{i+1}\ell_{i+2}\ell_{i+3}$;
- Three diagonals: $\overline{A_{ij}A_{kl}}$ (or (A_{ij}, A_{kl}));
- Three quadrilaterals: $A_{ij}A_{jk}A_{kl}A_{li}$.
- The diagonal triangle (cevian triangle): δ, formed by the extensions of the three diagonals previously defined.



We also use notation from here below, where say QL-P1 is the Miquel Point.

Proposition 4.1.1 (QL-P1, the Miquel Point)**.** The four circumcircles of $\triangle_1, \triangle_2, \triangle_3, \triangle_4$ concur.

Proof. We have proved this in Theorem 0.1.18, but it is just angle chasing. □

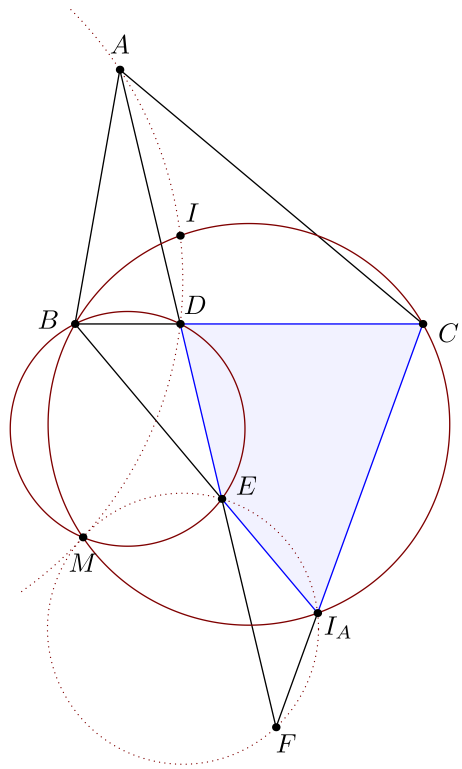
Example 4.1.2 (APMO 2015/1)**.** Let ABC be a triangle, and let D be a point on side BC. A line through D intersects side AB at X and ray AC at Y . The circumcircle of triangle BXD intersects the circumcircle ω of triangle ABC again at point Z distinct from point B. The lines ZD and ZY intersect ω again at V and W respectively. Prove that $AB = VW$.

Solution. (AoPS, TelvCohl) Note that Z is the Miquel point of complete quadrilateral $\{AB, BC, CA, XY\}$, so

$$Z \in (CDY) \implies \angle VZW = 180^\circ - \angle ACB \implies AB = VW.$$ □

Example 4.1.3 (ISL 2020 G6)**.** Let ABC be a triangle with $AB < AC$, incenter I, and A excenter I_A. The incircle meets BC at D. Define $E = AD \cap BI_A$, $F = AD \cap CI_A$. Show that the circumcircle of $\triangle AID$ and $\triangle I_A EF$ are tangent to each other.

Solution. Notice that if (AID) and $(I_A EF)$ are tangent at the Miquel point of $\triangle BI_A C \cap AD$, then angle chasing in the problem becomes almost trivial. So what if we just guess that the tangency point is M?



First we prove that M lies on (AID) (since M already lies on (I_AEF), we don't need to care about that). From $M \in (BI_AC)$, we get

$$\measuredangle DMI = \measuredangle DMB + \measuredangle BMI = \measuredangle DEB + \measuredangle BI_AI = \measuredangle DAI,$$

and thus M lies on (AID). Next, we prove tangency at M by usage of line-arguments.

$$\begin{aligned}\mathbf{T}_M(AID) &= MI + MD - ID = MI_A + \measuredangle(BC, MD) \\ &= MI_A + \measuredangle BEM = MI_A + ME - I_AE = \mathbf{T}_M(I_AEF)\end{aligned}$$ □

Despite that we literally just blindly guessed the Miquel point appearing here, the "motivation" for it is that if the tangency point is not the Miquel point, then we have absolutely no way to chase all of these angles. So whenever you see a tangency condition (aka: angles) in a complete quadrilateral problem, just guess the Miquel point; you have nothing to lose.

Proposition 4.1.4 (QL-L1, the Newton Line)**.** The three midpoints of the diagonals of the segments $\overline{A_{23}A_{14}}$, $\overline{A_{31}A_{24}}$, $\overline{A_{12}A_{34}}$ are collinear on a line τ.

Proof. Let the three midpoints be M_1, M_2, M_3 respectively, and let $\triangle N_1N_2N_3$ be the medial triangle of $\triangle A_{23}A_{31}A_{12}$. Then we get that M_1, M_2, M_3 lie on N_2N_3, N_3N_1, N_1N_2 respectively. Then by Menelaus we get

$$\frac{N_2M_1}{M_1N_3} \cdot \frac{N_3M_2}{M_2N_1} \cdot \frac{N_1M_3}{M_3N_2} = \frac{A_{12}A_{14}}{A_{14}A_{31}} \cdot \frac{A_{23}A_{24}}{A_{24}A_{12}} \cdot \frac{A_{31}A_{34}}{A_{34}A_{34}} = -1,$$

and then we do Menelaus once again on M_1, M_2, M_3 and win. $\square$

Proposition 4.1.5 (QL-Ci3, the Miquel Circle)**.** Let O_i be the circumcenter of $\triangle_i$. Then M, O_1, O_2, O_3, O_4 are concyclic.

Proof. We prove that M lies on $(O_1O_2O_3)$ which finishes by symmetry. We know that the reflections of M across O_2O_3, O_3O_1, O_1O_2 are points A_{14}, A_{24}, A_{34} respectively. These three points all lie on ℓ_4. Thus by the converse of Simson Line, we get that M lies on $(O_1O_2O_3)$. Do this symmetrically and we're done. $\square$

Note that by Steiner's Orthocenter Theorem we can get that the orthocenter H_l' of $\triangle O_iO_jO_k$ lies on ℓ_l.

Proposition 4.1.6 (QL-L2, Orthocenter Line, Steiner Line)**.** Let H_i be the orthocenter of $\triangle_i$. Then H_1, H_2, H_3, H_4 are collinear.

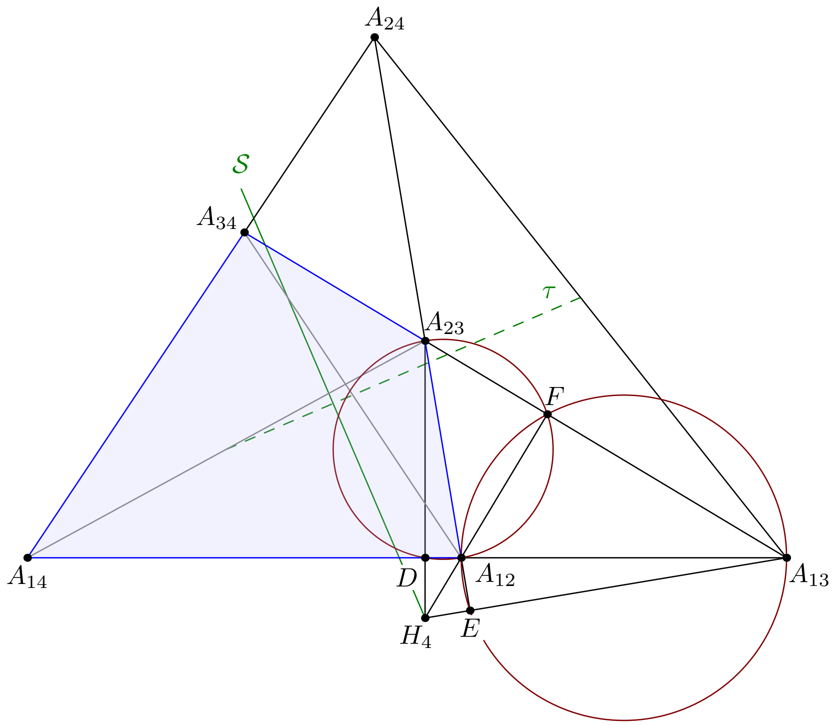
Proof. Let M_i' be the reflection of M across ℓ_i. By Steiner's theorem, we have that H_i, M_j', M_k', M_l' are collinear. Do this cyclically to get that $H_1, H_2, H_3, H_4, M_1', M_2', M_3', M_4'$ are collinear. $\square$

Remark. Take $n \geq 5$, and a complete n-gon $\mathcal{N}(\ell_1, \ldots, \ell_n)$. If M lies on the circumcircle of $\triangle \ell_i\ell_j\ell_k$ for all i, j, k, then the orthocenters of $\triangle \ell_i\ell_j\ell_k$ also lie on a line.

Proposition 4.1.7 (Gauss-Bodenmiller Line)**.** The three circles

$$(A_{23}A_{14}), (A_{31}A_{24}), (A_{12}A_{34})$$

are coaxial, and the Steiner line $\mathcal{S}$ is their common radical axis.



Proof. We prove that H_4, the orthocenter of $\triangle A_{23}A_{31}A_{12}$, has the same power wrt. all of these three circles. Let $\triangle DEF$ be the orthic triangle of $\triangle A_{23}A_{31}A_{12}$, then

$$\begin{cases} \mathbf{Pow}_{(A_{23}A_{14})}(H_4) & = H_4A_{23} \cdot H_4D, \\ \mathbf{Pow}_{(A_{31}A_{24})}(H_4) & = H_4A_{31} \cdot H_4E, \\ \mathbf{Pow}_{(A_{12}A_{34})}(H_4) & = H_4A_{12} \cdot H_4F, \end{cases}$$

Since A_{31}, A_{12}, E, F are concyclic we know that $H_4A_{31} \cdot H_4E = H_4A_{12} \cdot H_4F$, and similarly we have

$$H_4A_{23} \cdot H_4D = H_4A_{31} \cdot H_4E = H_4A_{12} \cdot H_4F.$$

Thus H_4 lies on the radical axis, and by symmetry we have that $H_1, H_2, H_3 \in \mathcal{S}$ also lie on the radical axis. □

Because the radical axis of two circle is perpendicular to the line between their centers, by Proposition 0.4.9, it follows by the above that:

Corollary 4.1.8. The Newton line τ is perpendicular to the Steiner line $\mathcal{S}$.

So in reality we could have also used radical axis properties to prove the Newton line directly, instead of spamming Menelaus.

Example 4.1.9. Let I be the incenter of $\triangle ABC$, and let H be its orthocenter. Let E, E' be the CA-intouch and extouch points. Let F, F' be the AB-intouch and extouch points. Let I' be the reflection of I in EF. Prove $HI' \perp E'F'$.

Solution. Note that I' is the orthocenter of $\triangle AEF$, so HI' is the Steiner line of complete quadrilateral $\triangle ABC \cup EF = (BC, CA, AB, EF)$. So proving $HI' \perp E'F'$ is equivalent to proving $E'F'$ is parallel to the Newton line of (BC, CA, AB, EF).

Let X, Y, Z respectively be the midpoints of $\overline{EF}, \overline{BE}, \overline{CF}$, then

$$ZX = \frac{CE}{2} = \frac{E'A}{2}, XY = \frac{FB}{2} = \frac{AF'}{2},$$

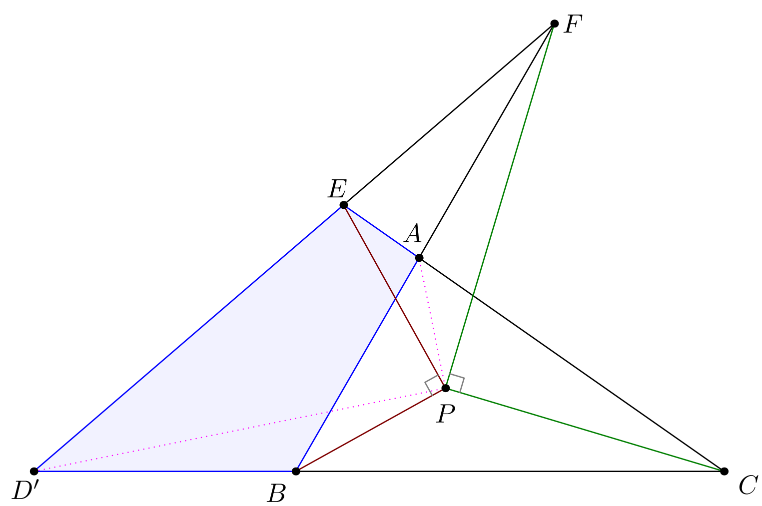
so we have $ZX \parallel E'A$, $XY \parallel AF'$. Thus we get that $\triangle XYZ \overset{+}{\sim} AF'E'$. Thus YZ (which is just the Newton line of $\triangle ABC \cup EF$) is parallel to $E'F'$, we are done. □

Another easy corollary is:

Corollary 4.1.10. Given $\triangle ABC$ and a point P, choose D, E, F on BC, CA, AB such that

$$\measuredangle APD = \measuredangle BPE = \measuredangle CPF = 90^\circ,$$

then D, E, F are collinear. We call this line the **orthotransversal** of P.



Proof. We proceed by phantom points. Let D' be the intersection of EF and BC. Let $\mathcal{Q}$ be the complete quadrilateral $\triangle ABC \cup EF$, then we know that the circles with diameters as the three diagonals of $\mathcal{Q}$, circles $(AD'), (BE), (CF)$ are coaxal. Note that P is one of the intersections of (BE) and (CF), so P also lies on (AD'). Thus $D = D'$. □

We will further expand on orthotransversals in **??**.

Finally, we look at a super-useful theorem.

Theorem 4.1.11. The Miquel point M of a complete quadrilateral $\mathcal{Q}$ is the isogonal conjugate of the point at infinity ∞_τ along the Newton line.

Proof. Denote the antipode of M in $\triangle_1$ as M_1^*, and similarly for M_2^*, M_3^*, M_4^*. From Simson-Antipode Isogonality we have that the isogonal conjugate of M_1^* in $\triangle_1$ is on the line at infinity, and furthermore it's the point at infinity along the Steiner line, $\infty_{\mathcal{S}}$. Thus the isogonal conjugate of M in $\triangle_1$ is just $\infty_{\perp\mathcal{S}} = \infty_\tau$. Similarly the isogonal conjugate of M in every other triangle is also ∞_τ, so we are done. $\square$

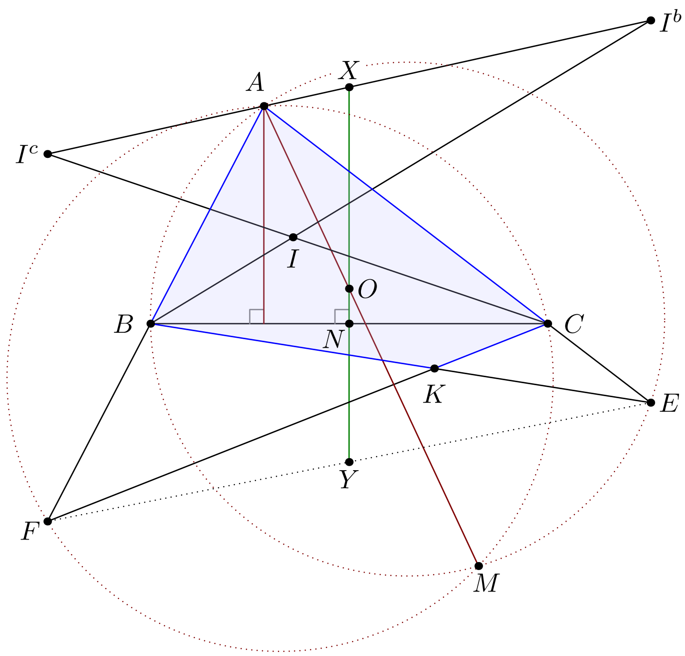
This theorem can be more easily expressed in line arguments as

$$\measuredangle(\tau) = \measuredangle(A_{jk}A_{ki}) + \measuredangle(A_{jk}A_{ij}) - \measuredangle(A_{jk}M).$$

This lets us locate the Miquel point through the Newton line, which we often combine with Problem 4 and Corollary 4.1.8.

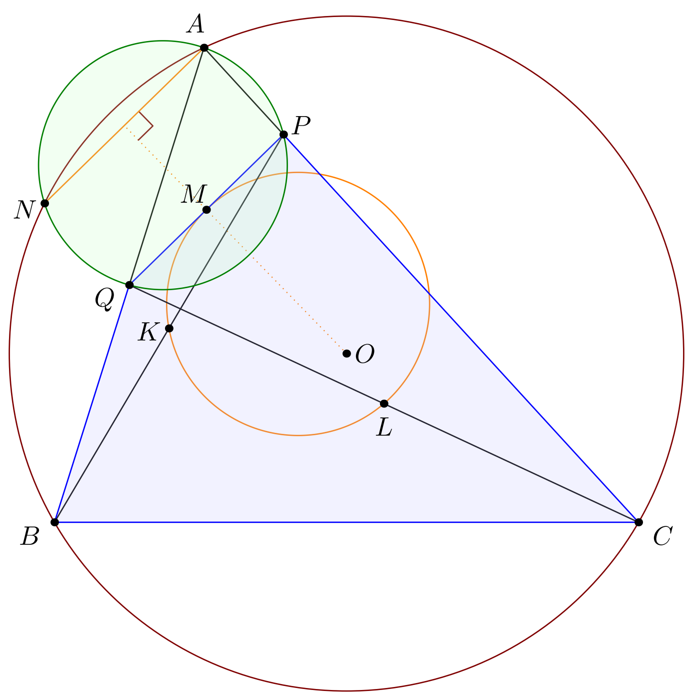
We will show more examples of the utility of this fact.

Example 4.1.12. Let I^b, I^c be the B, C-excenters of $\triangle ABC$. Let the perpendiculars dropped from I^b, I^c to BC intersect CA, AB at E, F. Prove that the circumcenter of $\triangle ABC$ lies on the radical axis of (ABE) and (ACF).



Solution. Let M be the Miquel point of (CA, AB, BE, CF). Then the radical axis of (ABE) and (ACF) is AM. As such, we need to prove that AM goes through the circumcenter of $\triangle ABC$, or that the isogonal line of AM in $\angle BAC$ is the perpendicular to BC. By the previous theorem, we know the isogonal line of AM in $\angle BAC$ is $A\infty_\tau$, and it remains to show that τ is perpendicular to BC. Since τ is the line connecting the midpoints of BC and EF, we just need to prove that the midpoint of EF lies on the perpendicular bisector of BC. However $I^bE, I^cF \perp BC$, so it is equivalent to prove the midpoint of I^bI^c lies on the perpendicular bisector of $\overline{BC}$. But by nine-point circle on $I_aI_bI_c$, we know it lies on $\widehat{BAC}$, so we are done. □

Example 4.1.13 (IMO 2009/2)**.** Let O be the circumcenter of $\triangle ABC$. Let P, Q be two points in segments CA and AB. Let K, L, M respectively be the midpoints of BP, CQ, PQ. Let Γ be a circle through K, L, M. Suppose the line PQ is tangent to Γ. Prove that $OP = OQ$.



Solution. Let N be the Miquel point of $\mathcal{Q} = \triangle ABC \cup PQ$. Since the Newton line of this quadrilateral is KL, we have

$$\measuredangle(AB) + \measuredangle(AC) - \measuredangle(AN) = \measuredangle(KL) = \measuredangle(MK) + \measuredangle(ML) - \measuredangle(PQ) = \measuredangle(AB) + \measuredangle(AC) - \measuredangle(PQ),$$

so $AN \parallel PQ$. Since A, N, P, Q are concyclic, $AN \parallel PQ$ tells us that the perpendicular bisector of PQ is the same as the perpendicular bisector of AN, which passes through O, so $OP = OQ$. □

Practice Problems

Problem 1. For complete quadrilateral $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$, prove $\triangle_i$ and $\triangle O_{i+1}O_{i+2}O_{i+3}$ are spirally similar, with spiral center as the Miquel point of $\mathcal{Q}$.

Problem 2. Prove the Miquel point of a complete quadrilateral lies on the nine-point circle of the diagonal triangle.

Problem 3 (IMO 2011/6)**.** Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Problem 4 (APMO 2014/5)**.** Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.

Problem 5. Let the incenter of $\triangle ABC$ be I and let the circumcenter be O. Let the intouch points on BC, CA, AB be D, E, F. Let FD, DE intersect CA, AB at Y, Z. Let K be the circumcenter of $\triangle DYZ$. Prove that $\measuredangle AIO = \measuredangle KID$.

Problem 6 (2014 Iran TST3/6)**.** The incircle of a non-isosceles triangle ABC with the center I touches the sides BC at D. Let X be a point on arc BC from circumcircle of triangle ABC such that if E, F are feet of perpendicular from X on BI, CI and M is midpoint of EF, then we have that $MB = MC$. prove that $\angle BAD = \angle XAC$

Problem 7. Given acute triangle $\triangle ABC$, let O be the circumcenter. Let Γ be the circumcircle of $\triangle OBC$, and let G be a point on Γ. Let the circumcircles of $\triangle ABG$, $\triangle ACG$ intersect CA, AB respectively again at E, F. Let $K = BE \cap CF$. Prove AK, BC, OG are concurrent.

4.2 Complete Cyclic Quadrilaterals

In this section we will briefly discuss what happens to a complete quadrilateral when four of its six vertices are concyclic. This is a very common configuration to see in problems, for example if $BCEF$ is cyclic, $A = FB \cap CE$, $D = BC \cap EF$ (example: a pedal triangle), we can apply this section.

We let $\mathcal{Q}$ be the complete quadrilateral (BC, CE, EF, FB) and let the Miquel point be $M_{\mathcal{Q}}$, let the Newton line be $\tau_{\mathcal{Q}}$, let the Steiner line be $\mathcal{S}_{\mathcal{Q}}$.

Proposition 4.2.1. The Miquel point lies on AD.

Proof. We have $(FBC), (CDM), (MAF)$ concur at E, so by triangle Miquel on $\triangle BDA$ and C, M, F lies on AD. $\square$

Proposition 4.2.2. We have that $BOEM$ and $COFM$ are concyclic.

Proof. Since A, D, M are collinear, we can just directly chase angles to get

$$\measuredangle BOE = \measuredangle BCE + \measuredangle BFE = \measuredangle BMD + \measuredangle AME = \measuredangle BME,$$

The result follows by symmetry C, O, F, M are concyclic. $\square$

So we can re-define M as the second intersection point of (BOE) and (COF). Let P be the intersection of the diagonals BE and CF of $BCEF$. Now consider inverting about the circumcircle.

Corollary 4.2.3. O, P, M are collinear, and P, M are exchanged under inverting about the circumcircle $(BCEF)$.

Proof. Consider inverting about $(BCEF)$, which maps $X \mapsto X^*$. Then we have $B^* = B, C^* = C, E^* = E, F^* = F$. Thus $(BOE)^* = BE$ and $(COF)^* = CF$. By Proposition 3.1.3 we thus have

$$M^* = ((BOE) \cap (COF) \setminus \{O\})^* = BE \cap CF = P.$$

$\square$

By Proposition 3.2.13 we also have that $\triangle ADP$ is self-polar in (O), so we have $OP \perp AD$. Combining this with the above theorem we get

Proposition 4.2.4. $OM \perp AD$.

We can also prove this with direct PoP.

Proof. Since A lies on the radical axis FB of $(BCEF), (BDF)$, we have

$$\mathbf{Pow}_{(BCEF)}(A) = \mathbf{Pow}_{(BDF)}(A) = AM \cdot AD = AM^2 - MA \cdot MD,$$

and by the same logic we get that $\mathbf{Pow}_{(BCEF)}(D) = DM^2 - MA \cdot MD$. Therefore

$$\begin{aligned} OA^2 - OD^2 &= \mathbf{Pow}_{(BCEF)}(A) - \mathbf{Pow}_{(BCEF)}(D) \\ &= (AM^2 - MA \cdot MD) - (DM^2 - MA \cdot MD) \\ &= AM^2 - DM^2, \end{aligned}$$

so by Proposition 0.2.11 we have $OM \perp AD$. □

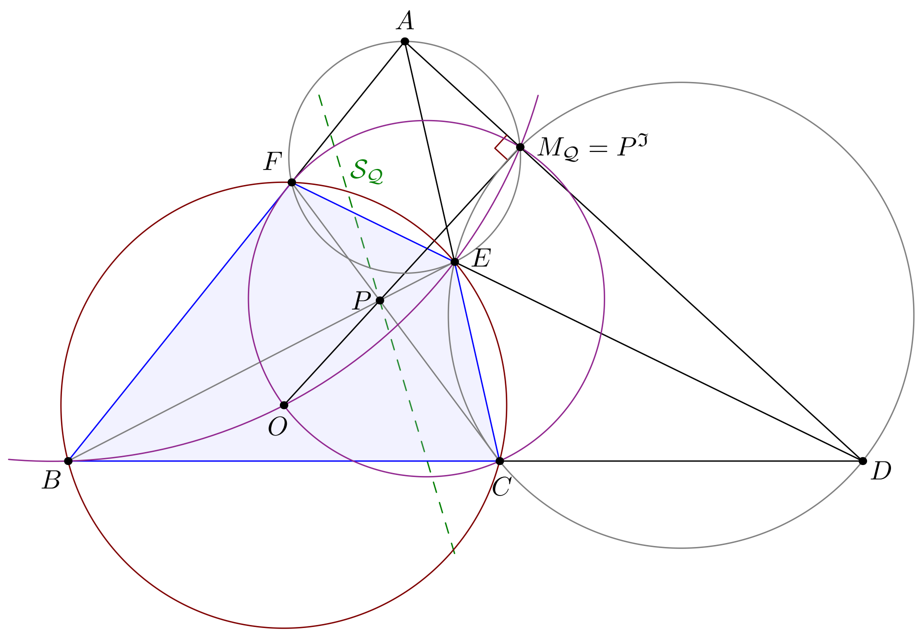
Proposition 4.2.5. *P lies on the Steiner line $\mathcal{S}$.*

Proof. We know that $\mathcal{S}$ is the radical axis of $(AD), (BE), (CF)$, and we have

$$\mathbf{Pow}_{(BE)}(P) = PB \cdot PE = PC \cdot PF = \mathbf{Pow}_{(CF)}(P),$$

so we have $P \in \mathcal{S}$. □

Here is a full diagram about what we did earlier:



Example 4.2.6 (2009 G4)**.** Given cyclic quadrilateral $ABCD$, let diagonals AC and BD intersect at E, and let AD and BC intersect at F. Let G, H respectively be the midpoints of AB and CD. Prove that EF is tangent to (EGH) at E.

Proof. Let M be the midpoint of EF. Then we have that M, G, H is the Newton line of $ABCD$. Let X be the intersection of AB and CD. Then EF is the pole of X across $ABCD$. Therefore, EF is the radical axis of (OX) and $(ABCD)$. Since E, F are inversive pairs in $(ABCD)$, we have that the circle (EF) is orthogonal to $(ABCD)$. Since M is the midpoint of EF, it is the center of (EF), so we have

$$ME^2 = \mathbf{Pow}_{(ABCD)}(M) = \mathbf{Pow}_{(OX)}(M) = MG \cdot MH,$$

so (EGH) is tangent to $ME = EF$. □

4.3 Complete Tangential Quadrilaterals

For conciseness, in the following section we will keep using $A_{ij} = \ell_i \cap \ell_j$. We will also assume that $A_{42}A_{23}A_{31}A_{14}$ is a convex quadrilateral, $A_{41}A_{12}A_{23}A_{34}$ is a concave quadrilateral, and $A_{42}A_{31}A_{12}A_{24}$ is a self-intersecting quadrilateral.

Proposition 4.3.1 (Pitot's Theorem)**.** For a complete quadrilateral $\mathcal{Q} = (l_1, l_2, l_3, l_4)$, the following statements are equivalent.

(i) $\mathcal{Q}$ has an incircle, or is **tangential**;

(ii) $\overline{A_{42}A_{23}} + \overline{A_{31}A_{14}} = \overline{A_{23}A_{31}} + \overline{A_{14}A_{42}}$

(iii) $\overline{A_{41}A_{12}} + \overline{A_{23}A_{34}} = \overline{A_{12}A_{23}} + \overline{A_{34}A_{41}}$

(iv) $\overline{A_{43}A_{31}} + \overline{A_{12}A_{24}} = \overline{A_{31}A_{12}} + \overline{A_{24}A_{43}}$

Proof. If $\mathcal{Q}$ has incircle ω, let T_i be the point of tangency of l_i to ω. For all permutations i, j, k, l of $1, 2, 3, 4$ we get

$$\begin{aligned}\overline{A_{ij}A_{jk}} + \overline{A_{kl}A_{li}} &= \overline{A_{ij}T_j} + \overline{T_jA_{jk}} + \overline{A_{kl}T_l} + \overline{T_lA_{li}} \\ &= \overline{A_{ij}T_i} + \overline{T_kA_{jk}} + \overline{A_{kl}T_k} + \overline{T_iA_{li}} = \overline{A_{jk}A_{kl}} + \overline{A_{li}A_{ij}}\end{aligned}$$

where lengths are directed along each line in a specific way. This implies (ii), (iii), (iv).

Next, assume condition (ii) is satisfied, we need to prove that $\mathcal{Q}$ has an incircle □

Proposition 4.3.2. To check if $\mathcal{Q}$ has an excircle, the following properties are equivalent:

- $\mathcal{Q}$ has an excircle;
- $\overline{A_{42}A_{23}} - \overline{A_{31}A_{14}} = \overline{A_{23}A_{31}} - \overline{A_{14}A_{42}}$
- $\overline{A_{41}A_{12}} - \overline{A_{23}A_{34}} = \overline{A_{12}A_{23}} - \overline{A_{34}A_{41}}$
- $\overline{A_{43}A_{31}} - \overline{A_{12}A_{24}} = \overline{A_{31}A_{12}} - \overline{A_{24}A_{43}}$

Proof. The proof is the exact same as Proposition 4.3.1, so we will omit it. □

Example 4.3.3. Let $ABCD$ be a convex quadrilateral. Let P, Q, R, S respectively be four points on AB, BC, CD, DA. Let X be the intersection point of PR and QS. Suppose $SAPX, PBQX, QCRX, RDSX$ are all tangential quadrilaterals. Prove that $ABCD$ is also tangential.

Solution. Let $\omega_A, \omega_B, \omega_C, \omega_D$ be the incircles of $SAPX, PBQX, QCRX, RDSX$ respectively. Let T_I^{DC} for a tangent DC to ω_I be the touch point. By Proposition 4.3.1, we want to show that $\overline{AB} + \overline{CD} = \overline{BC} + \overline{DA}$.

We have that

$$\begin{aligned}
\overline{AB} + \overline{CD} &= \left(\overline{AT_A^A} + \overline{T_A^A T^{PB}} + \overline{T_B^{PB} B}\right) + \left(\overline{CT_C^{CR}} + \overline{T_C^{CR} T_D^{RD}} + \overline{T_D^{RD} D}\right) \\
&= \left(\overline{AT_A^{SA}} + \overline{T_A^{XS} T_B^{QX}} + \overline{T_B^{BQ} B}\right) + \left(\overline{CT_C^{QC}} + \overline{T_C^{XQ} T_D^{SX}}\overline{T_D^{SX} D}\right), \\
\overline{BC} + \overline{DA} &= \left(\overline{T_B^{BQ} B} + \overline{T_C^{QC} T_B^{BQ}} + \overline{CT_C^{QC}}\right) + \left(\overline{T_D^{SX} D} + \overline{T_A^{SA} T_D^{SX}} + \overline{AT_A^{SA}}\right)
\end{aligned}$$

and thus it remains to show that $\overline{T_A^{XS} T_B^{QX}} + \overline{T_C^{XQ} T_D^{SX}}$ and $\overline{T_C^{QC} T_B^{BQ}} + \overline{T_A^{SA} T_D^{SX}}$ are the same value. To do this, note that

$$\begin{aligned}
\overline{T_A^{XS} T_B^{QX}} + \overline{T_C^{XQ} T_D^{SX}} &= \left(\overline{T_A^{SX} X} + \overline{XT_B^{QX}}\right) + \left(\overline{T_C^{XQ} X} + \overline{XT_D^{SX}}\right) \\
&= \overline{T_A^{PX} X} + \overline{XT_B^{XP}} + \overline{T_C^{RX} X} + \overline{XT_D^{XR}} \\
&= \overline{T_C^{RX} T_B^{XP}} + \overline{T_A^{PX} T_D^{XR}} = \overline{T_C^{QC} T_B^{BQ}} + \overline{T_A^{SA} T_D^{SX}}
\end{aligned}$$

□

For the rest of the section let I be the center of the quadrilateral's incircle.

Proposition 4.3.4. I lies on the Newton line τ of $\mathcal{Q}$.

Proof. In reality, this is Newton's conic theorem Theorem 6.3.13 on circles. However let's give an elementary proof: let T_i be the tangency point of line ℓ_i and ω, let M_2, M_3 be the midpoints of $A_{31}A_{24}, A_{12}A_{34}$, then we have using (signed area)

$$\begin{aligned}
[\triangle IM_2M_3] &= \frac{1}{4}([\triangle IA_{31}A_{12}] + [\triangle IA_{24}A_{12}] + [\triangle IA_{31}A_{34}] + [\triangle IA_{24}A_{34}]]) \\
&= \frac{1}{4}([\triangle IA_{31}T_1] + [\triangle IT_1A_{12}] + [\triangle IA_{24}T_2] + [\triangle IT_2A_{12}] \\
&\quad + [\triangle IA_{31}T_3] + [\triangle IT_3A_{34}] + [\triangle IA_{24}T_4] + [\triangle IT_4A_{34}]) = 0
\end{aligned}$$

with the final equality following because

$$\begin{aligned}
&[\triangle IA_{31}T_1] = [\triangle IT_3A_{31}], \quad [\triangle IT_1A_{12}] = [\triangle IA_{12}T_2] \\
&[\triangle IA_{24}T_2] = [\triangle IT_4A_{24}], \quad [\triangle IT_3A_{34}] = [\triangle IA_{34}T_4].
\end{aligned}$$

So $I \in \tau = M_2M_3$. □

Proposition 4.3.5. Let M be the Miquel point of $\mathcal{Q}$. Then MI is the internal angle bisector of the angles $\angle A_{23}MA_{14}$, $\angle A_{31}MA_{24}$, and $\angle A_{12}MA_{34}$. Further, if we let J to be the reflection of I across M, we have

$(IJ)(A_{23}A_{14}), (IJ)(A_{31}A_{24}), (IJ)(A_{12}A_{34})$ are all cyclic harmonic quadrilaterals.

Proof. Let T_i be the tangency point of ℓ_i with ω. We consider the inversion $\mathfrak{J}$ about the incircle. We have $A^*_{ij} := \mathfrak{J}(A_{ij})$ as the midpoint of $\overline{T_iT_j}$, and

$$M^* := \mathfrak{J}(M) = \bigcap(A^*_{jk}A^*_{ki}A^*_{ij}).$$

Let G be the centroid of (T_1, T_2, T_3, T_4), then G is also the common midpoint of $\overline{A^*_{23}A^*_{14}}, \overline{A^*_{31}A^*_{24}}, \overline{A^*_{12}A^*_{34}}$. We consider the reflection about G, call this transformation $\mathfrak{s}$. Then we have $\mathfrak{s}(A^*_{ij}) = A^*_{kl}$, so

$$\mathfrak{s}(M^*) = \bigcap \mathfrak{s}((A^*_{jk}A^*_{ki}A^*_{ij})) = \bigcap(A^*_{il}A^*_{jl}A^*_{kl}) = I.$$

Therefore $(IM^*)(A_{23}A_{14})$ is a parallelogram, and thus

$$\measuredangle A_{23}MI - \measuredangle IMA_{14} = \measuredangle IA^*_{23}M^* - \measuredangle M^*A^*_{14}I = 0^\circ,$$

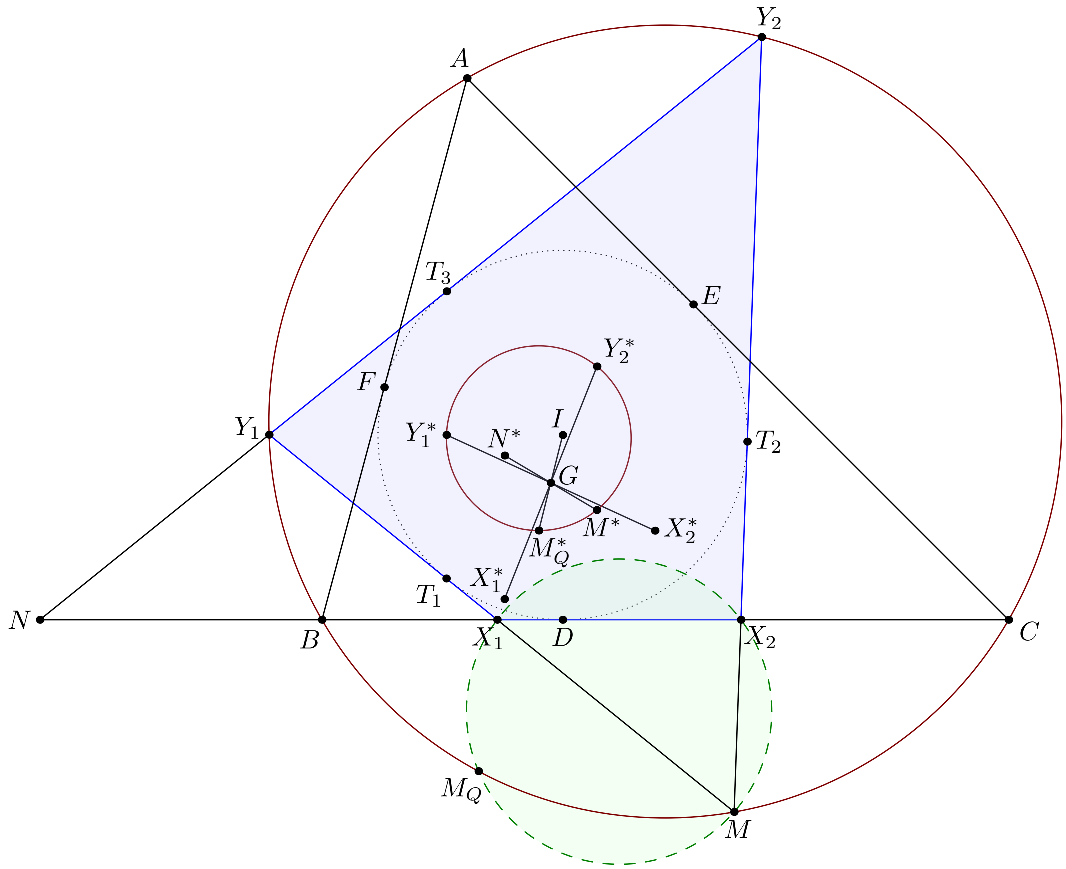
so MI bisects $\angle A_{23}MA_{14}$. Similarly, MI also bisects $\angle A_{31}MA_{24}, \angle A_{12}MA_{34}$. Since J is the reflection of I across M, we know that $J^* := \mathfrak{J}(J)$ is the midpoint of $\overline{IM}$, which is G. From the fact that $J^* = G$ is the common midpoint of $\overline{A^*_{23}A^*_{14}}, \overline{A^*_{31}A^*_{24}}, \overline{A^*_{12}A^*_{34}}$, we get that $(IJ)(A_{23}A_{14}), (IJ)(A_{31}A_{24}), (IJ)(A_{12}A_{34})$ are all harmonic and cyclic. □

Remark. In reality, M^* is the Poncelet point of (T_1, T_2, T_3, T_4), which we will investigate in Feuerbach, so it will lie on the cevian circle of (T_1, T_2, T_3, T_4) and will lie on the pedal circle of T_l wrt. $\triangle T_iT_jT_k$ (which is just the Simson line L_l). So when we invert this question about ω, we can get that M lands on the pedal circle of I wrt. $\triangle(A_{23}A_{14})(A_{31}A_{24})(A_{12}A_{34})$ and $\mathfrak{J}(L_\ell)$.

Example 4.3.6 (Buffed 2014 Taiwan TST 3 P3)**.** Let M be a moving point on the circumcircle of $\triangle ABC$. Draw the two tangents from M to the incircle ω of $\triangle ABC$. Let these two tangents intersect BC at X_1, X_2. Prove that the circumcircle of MX_1X_2 passes through a fixed point.

(Note: In the original problem we want to prove the fixed point is the A-mixtouch point.)

Proof. Let Y_1, Y_2 be the second intersections of MX_1, MX_2 with the circumcircle of (ABC). By Poncelet's closure theorem, we have that Y_1Y_2 is tangent to the incircle ω of $\triangle ABC$. Thus $\mathcal{Q} = \triangle MY_1Y_2 \cup BC$ is a complete tangential quadrilateral with incircle ω. Further we get the second intersection of (MX_1X_2) and (ABC) is the Miquel point of $\mathcal{Q}$. Let N be the intersection of Y_1Y_2 and BC.



We now invert about the incircle. From Proposition 4.3.5 we get that $IM_Q^*, X_1^*Y_2^*, X_2^*Y_1^*, M^*N^*$ have a common midpoint, let this be G. . Let $\mathfrak{s}$ represent reflection across G. Then we have

$$\mathfrak{s}(\Omega^*) = \mathfrak{s}((Y_1^*Y_2^*M^*)) = (X_2^*X_1^*N^*) = (BC)^* = (ID),$$

where Ω and D are the circumcircle and BC-intouch point respectively. This means that G is fixed as M varies, so M_Q^* as the reflection of I over G is also fixed, and thus M_Q is fixed.

If the reader is familar with mixtilinear incircles, then we can prove that M_Q is the A-mixtouch point. Let M_a be the second intersection of AI and Ω, then M_Q is the A-mixtouch point if and only if $\measuredangle IM_QM_A = 90°$, however by inverting around ω we get this condition is equivalent to

$$\angle M_Q^*M_A^*A^* = \angle M_Q^*M_A^*I = 90°,$$

or that $\overline{A^*M_Q^*}$ is the diameter of Ω^*. If $\triangle DEF$ be the intouch triangle of $\triangle ABC$, then $\Omega^* = (N_I)$ is the nine-point circle of $\triangle DEF$! Let H_I be the orthocenter of $\triangle DEF$ and note that $O_I = I$ is the circumcenter. As such A^* is the midpoint of $\overline{EF}$. If N_9' is the center of (ID), then G is the midpoint of $\overline{N_9N_9'}$ so $N_9IN_9'M_Q^*$

is a parallelogram. Taking a $2\times$ homothety at I and using properties of the Nine-Point Circle, it follows that $M_{\mathcal{Q}}^*$ is the midpoint of $\overline{H_I D}$, and thus it is the antipode of A^*. □

4.4 Euler Quadrilaterals

Serving as the last section of chapter 4, let's conclude with looking at a complete quadrilateral configuration with a large amount of internal structure.

Proposition 4.4.1 (Gossard's Theorem)**.** Let the complete quadrilateral $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$ have no angle between two lines equal to $\pm 60°$. Then if for some i, ℓ_i is parallel to $\triangle_i = \triangle \ell_{i+i}\ell_{i+2}\ell_{i+3}$'s Euler line $\mathcal{E}_i$, then the other three sides ℓ_j are parallel to the other three corresponding $\mathcal{E}_j$ as well.

Proof. For this proof, set $i = 4$ for ease of notation, and assume $\mathcal{E}_4$ is parallel to ℓ_4. Let O_i, H_i respectively be the circumcenters and orthocenters of triangles $\triangle_i$. Label the vertices of $\triangle_4 = \triangle A_{23}A_{31}A_{12}$ as A, B, C, let $\mathcal{E}_4$ be the Euler line of $\triangle_4 = \triangle ABC$. Label $A_{14} = D$ as the intersection of $\ell_1 = BC$ and ℓ_4.

Through translating ℓ_4 (which is parallel to $\mathcal{E}_4$), we can assume it passes through A.

We only need to prove that $\mathcal{E}_2$ is parallel to ℓ_2. Let X be the intersection of $\mathcal{E}_1$ and BC. Note that $H_4X \parallel AD \perp BH_2$ and $BX \perp AH_2 = H_2H_4$. Therefore, X is the orthocenter of $\triangle BH_2H_4$, thus $H_2X \perp H_4B \perp CA$. From

$$\measuredangle O_4XB = \measuredangle ADB = \measuredangle O_4O_2B$$

we get that O_2, O_4, B, X are concyclic. Thus we have

$$\measuredangle O_2X = \measuredangle BX + \measuredangle BXO_2 = \measuredangle BC + \measuredangle BO_4O_2 = \measuredangle BC + \measuredangle BCA = \measuredangle CA.$$

Thus H_2X, O_2X are both parallel to $CA = \ell_2$, so $\mathcal{E}_2 = O_2H_2$ is also parallel to ℓ_2. □

Definition 4.4.2. We call a quadrilateral $\mathcal{Q}$ satisfying Proposition 4.4.1 an **Euler quadrilateral**.

Theorem 4.4.3 (Zeeman-Gossard)**.** Let $\mathcal{Q}$ be an Euler quadrilateral, with $\mathcal{E}_i$ being the Euler line of $\triangle_i$. Then the complete quadrilateral $\mathcal{Q}^{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4)$ is homothetic to $\mathcal{Q}$ with a ratio of -1.

Proof. By symmetry, we only need to prove that $\triangle XYZ \cong \triangle \mathcal{E}_1\mathcal{E}_2\mathcal{E}_3$, $\triangle ABC \cong \triangle_4 \cong \triangle \ell_1\ell_2\ell_3$ and that the similarity ratio is -1 (oppositely oriented). Similar to the proof of Proposition 4.4.1, we translate ℓ_4. In the proof of *Proposition* 4.4.1, we find that when ℓ_4 passes through $A = A_{23}$, then $\triangle X_AY_AZ_A = \triangle XYZ$ satisfies

- $X_A = \mathcal{E}_4 \cup \ell_1$,

- A, Y_A, Z_A are collinear,
- $Z_AX_A \parallel CA, X_AY_A \parallel AB$.

Analogously, $\triangle X_BY_BZ_B, \triangle X_CY_CZ_C$ must also satisfy these properties.

Consider the isotomic transversal of $\mathcal{E}_4$ with respect to $\triangle_4$, call it $\mathcal{E}_4^*$. Let U_*, V_*, W_* respectively be the intersection points of $\mathcal{E}_4^*$ with lines BC, CA, AB. Then X_A, X_B, X_C respectively are the reflections of U_*, V_*, W_* across the midpoint of BC. Thus, note that $\triangle BX_AX_B$ is the reflection of $\triangle CU_*V_*$ across point M_A (since if we draw the line between Y_B's reflection across M_A and V_*, this line will be parallel to AB).

Therefore X_A, X_B, X_C lie on a line L_X, which is the reflection of $\mathcal{E}_4^*$ across M_A. Similarly, we get that L_Y, L_Z are the reflections of $\mathcal{E}_4^*$ across M_B, M_C.

Continuing, we prove that for any line ℓ_4, X lies on L_X. When we translate ℓ_4, $\mathcal{E}_2$ also translates. Let X' be the intersection point of $\mathcal{E}_2$ and L_X. Then

$$\frac{X_BX'}{X_BX_C} = \frac{BD}{BC}.$$

Analogously, if we let X'' to be the intersection point of $\mathcal{E}_2$ and L_X, then

$$\frac{X_CX''}{X_CX_B} = \frac{CD}{CB}.$$

These two expressions tell us that $X' = X''$, implying that $X = X' = X''$ lies on L_X. Similarly, Y, Z respectively lie on L_Y, L_Z. Since ZX is parallel to CA, we have that X's ratios on $X_AX_BX_C$ and Z's ratios on $Z_AZ_BZ_C$ are the same. This tells us that X's reflection across M_A coincides with the reflection of Z across M_C (since they have the same ratios on $U_*V_*W_*$). Let this point be R. Then R is also the reflection of Y across M_B. So $\triangle XYZ$ is the medial triangle of $\triangle M_AM_BM_C$ under the homothety $\mathfrak{h}_{R,2}$, so $\triangle ABC$ is congruent with a -1 homothety ratio. $\square$

Through this proof, we know that $\mathcal{Q}^{\mathcal{E}}$ is the image of the reflection of $\mathcal{Q}$'s sides over the point $\mathfrak{h}_{G,-1/2}(R)$, where G is the centroid of $\triangle ABC$. We call this point the **Gossard** point of $\mathcal{Q}$ and we will notate it as Go. We will denote reflection across Go as $(-)^{\mathcal{E}}$. Therefore $X = A^{\mathcal{E}}, Y = B^{\mathcal{E}}, Z = C^{\mathcal{E}}, \mathcal{E}_i = \ell_i^{\mathcal{E}}$. Since $O_4 = \mathfrak{h}_{G,-1/2}(H_4)$, we know that R is the reflection of H_4' across O_4.

Remark. An obvious example of a Euler quadrilateral is the quadrilateral $\triangle ABC \cup \mathcal{E}$ where $\mathcal{E}$ is the Euler line. In this case, the Gossard point Go is called the **Gossard perspector** of $\triangle ABC$, and it is X_{402} in the Encyclopedia of Triangle Centers.

Proposition 4.4.4. Let Go be the Gossard point of Euler quadrilateral $\mathcal{Q}$. Then Go lies on the Newton line τ of $\mathcal{Q}$. Since $\mathcal{Q}, \mathcal{Q}^{\mathcal{E}}$ have the same Gossard point, this also implies that they share the same Newton line.

Proof. Let $R = \mathfrak{h}_{G,-2}(Go)$ be the point R defined in the proof of Theorem 4.4.3. Since the Newton line τ is the image of the isotomic transversal of ℓ_4 wrt. $\triangle_4$ (call it ℓ_4^*) under the homothety $\mathfrak{h}_{G,-1/2}$ (see Problem 4), we only need to prove that $R \in \ell_4^*$.

Let D, E, F respectively be the intersection points of ℓ_4 with BC, CA, AB respectively, let D_*, E_*, F_* be the intersection points of ℓ_4^* with BC, CA, AB, and let U, V, W be the intersection points of $\mathcal{E}_4$ with BC, CA, AB. From the proof of Theorem 4.4.3,

$$\frac{V_*R}{RW_*} = \frac{X_BX}{XX_C} = \frac{X_BX_C}{XX_C} - 1 = \frac{CB}{CD} - 1 = \frac{DB}{CD}.$$

Therefore, from

$$\frac{V_*R}{RW_*} \cdot \frac{W_*F_*}{F_*A} \cdot \frac{AE_*}{E_*V_*} = \frac{DB}{CD} \cdot \frac{WF}{FB} \cdot \frac{CE}{EV} = \frac{DB}{CD} \cdot \frac{UD}{DB} \cdot \frac{CD}{DU} = -1$$

and Menelaus's theorem, we get that $R \in E_*F_* = \ell_4^*$. □

Proposition 4.4.5. Let $O_4^{\mathcal{E}}$ be the circumcenter of $\triangle \mathcal{E}_1\mathcal{E}_2\mathcal{E}_3$. Then the five points $D, X, O_2, O_3, O_4^{\mathcal{E}}$ are concyclic.

Proof. Since

$$\begin{aligned}
\measuredangle XO_4^{\mathcal{E}} + \measuredangle DO_3 &= \measuredangle AO_4 + \measuredangle DO_3 \\
&= \perp (\measuredangle AB + \measuredangle AC - \measuredangle BC) + \perp (\measuredangle DE + \measuredangle DC - \measuredangle EC) \\
&= \measuredangle \ell_3 + \measuredangle \ell_4 = \measuredangle XO_3 + \measuredangle DO_4^{\mathcal{E}},
\end{aligned}$$

we have $D, X, O_3, O_4^{\mathcal{E}}$ are concyclic. Similarily, we have $D, X, O_2, O_4^{\mathcal{E}}$ are concyclic. □

Corollary 4.4.6. $O_4^{\mathcal{E}}$ is the orthocenter of $\triangle O_1O_2O_3$.

Proof. From symmetrically applying the above characteristics, we know that $O_4^{\mathcal{E}}$ is the triangle Miquel point of $\triangle O_1O_2O_3$ wrt. $\triangle XYZ$, so we get

$$\angle O_4^{\mathcal{E}}O_2O_1 = \angle O_4^{\mathcal{E}}ZO_1 = \angle O_1YO_4^{\mathcal{E}} = \angle O_1O_3O_4^{\mathcal{E}}$$

(and two other expressions cyclically). These angle conditions uniquely determine $O_4^{\mathcal{E}}$ to be the orthocenter (rigorously, we know that the isogonal conjugate of $O_4^{\mathcal{E}}$ is the circumcenter of $\triangle O_1O_2O_3$). □

Proposition 4.4.7. Let $\mathcal{S}$ be the Steiner line of Euler quadrilateral $\mathcal{Q}$, then for the complete quadrilateral (non-Euler) $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{S})$, we have that

- The Miquel point of the complete quadrilateral, $M_{\mathcal{S}}^{\mathcal{E}}$, lies on the isotomic transversal $\mathcal{E}_4^*$ in triangle $\triangle_4$ of the Euler line $\mathcal{E}_4$, the Miquel circle of $\mathcal{Q}^{\mathcal{E}}$, and simultaenously is also the perspector of $\triangle XYZ$ and $\triangle O_1^{\mathcal{E}} O_2^{\mathcal{E}} O_3^{\mathcal{E}}$,
- The Newton line $\tau_{\mathcal{S}}^{\mathcal{E}}$ of the complete quadrilateral passes through Go,
- The Steiner line of the complete quadrilateral is just ℓ_4.

Proof. We first prove (iii). In reality, we prove (D, X, H_2, H_3) is an orthocentric system (then by symmetry, the Steiner line of $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{S})$ is $DEF = \ell_4$), which is because $DH_2 \perp BF \parallel XH_3$ and $DH_3 \perp EC \parallel XH_2$. Therefore $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{S})$ □

Remark. Since Go is the intersection of $\mathcal{Q}$'s Newton line and $(\ell_1, \ell_2, \ell_3, \mathcal{S}^{\mathcal{E}})$'s Newton line, through Theorem 6.3.13 we know that there exists a shared inconic of $\mathcal{Q}, \mathcal{S}^{\mathcal{E}}$ with center at Go.

Additionally, $\mathcal{Q}^{\mathcal{E}}$'s Miquel circle's other intersection with $\mathcal{E}_4^*$ lies on GoO_4:

Lemma 4.4.8. Point $O_4^{\mathcal{E}}$ lies on the perpendicular bisector of $\overline{H_4M}$ and

$$\measuredangle MH_4O_4^{\mathcal{E}} = \measuredangle(\ell_4, \mathcal{S}).$$

Proof. Since $O_4^{\mathcal{E}}$ is the orthocenter of $\triangle O_1O_2O_3$ Corollary 4.4.6, and $\triangle O_1O_2O_3$'s spiral center with $\triangle ABC$ is the Miquel point M, we have $\triangle ABC \cup H_4 \cup M \overset{+}{\sim} \triangle O_1O_2O_3 \cup O_4^{\mathcal{E}} \cup M$. Notably, we have $\triangle O_4^{\mathcal{E}} H_4 M \overset{+}{\sim} \triangle O_1AM$ as an isosceles triangle, and

$$\measuredangle MH_4O_4^{\mathcal{E}} = \measuredangle MAO_1 = (\measuredangle \ell_1 + \measuredangle \ell_3 - \measuredangle \ell_4) + 90^\circ - (\ell_2 + \ell_3 - \tau) = \measuredangle(\ell_4, \mathcal{S}).$$

□

Proposition 4.4.9. The center of the Miquel circle O_M of a complete quadrilateral $\mathcal{Q}$ lies on the Steiner line of $\mathcal{Q}^{\mathcal{E}}$, and also lies on the perpendicular bisector of $\overline{O_i^{\mathcal{E}} H_i^{\mathcal{E}}}$.

Proof. Denote the Steiner line of $\mathcal{Q}^{\mathcal{E}}$ as $\mathcal{S}^{\mathcal{E}}$. Consider the reflections of $O_4^{\mathcal{E}}$ across the three sides YZ, ZX, XY, denote these as O_X, O_Y, O_Z. Then $\triangle O_XO_YO_Z$ and $\triangle O_1O_2O_3$ have spiral center at $O_4^{\mathcal{E}}$ (since they are both similar to $\triangle XYZ$, and $O_4^{\mathcal{E}}$ is their common orthocenter!). Therefore, their circumcenter H_4' satisfies, with O_M, $\triangle O_4^{\mathcal{E}} H_4^{\mathcal{E}} O_M \overset{+}{\sim} \triangle O_4^{\mathcal{E}} O_X O_1$. This proves O_M lies on the perpendicular bisector of $\overline{O_i^{\mathcal{E}} H_i^{\mathcal{E}}}$.

Since

$$\measuredangle H_4^{\mathcal{E}} O_4^{\mathcal{E}} O_M = \measuredangle O_X O_4^{\mathcal{E}} O_1 = \measuredangle YO_1O_4^{\mathcal{E}} + 90^\circ = \measuredangle YEO_4^{\mathcal{E}} + 90^\circ,$$

we know that $O_M H_4^{\mathcal{E}} \perp YE$. Note that $\overline{BY}$'s midpoint Go, and BE's midpoint, all lie on τ, so $YE \parallel \tau \perp \mathcal{S}^{\mathcal{E}}$, so $O_M \in \mathcal{S}^{\mathcal{E}}$. □

Define $\odot(Go)$ as the circle with center Go which passes through the Miquel point of $\mathcal{Q}$. The antipode of M on this circle is the Miquel point of $\mathcal{Q}^{\mathcal{E}}$, which is $M^{\mathcal{E}}$.

Proposition 4.4.10. Let $P_i \in \mathcal{E}_{i^*}$ be the anti-Steiner point of $\mathcal{E}_i$ in triangle $\triangle_i$, let Q_i be the foot of $H_i^{\mathcal{E}}$ to $\mathcal{E}_{i^*}$. Then $P_i^{\mathcal{E}}, Q_i^{\mathcal{E}}$ are the two intersections of the circle $\odot(Go)$ with $\mathcal{E}_{i^*}$.

Proof. From Proposition 1.5.16, P_i is the foot of H_i on $\mathcal{E}_{i^*}$. Since $H_i, H_i^{\mathcal{E}}$ are reflections across Go and $P_iQ_i \perp H_i^{\mathcal{E}}Q_i$, we know $Q_i \in \odot(Go)$ and can derive (?) $P_i \in \odot(Go)$. Define $R_i \in \mathcal{E}_{i^*}$ as the reflection of $H_i^{\mathcal{E}}$ across O_i, in other words, the center of homothety between $\triangle_i^{\mathcal{E}}$ and $\triangle_i$'s medial triangle. From Lemma 4.4.8, we know $O_i^{\mathcal{E}}$ lies on the perpendicular bisector of $\overline{H_iM}$, so O_i lies on the perpendicular bisector of $\overline{H_i^{\mathcal{E}}M_i^{\mathcal{E}}}$. Therefore, $M^{\mathcal{E}}$ and Q_i all lie on the circle with diameter $H_i^{\mathcal{E}}R_i$. Specifically, O_i lies on the perpendicular bisector of $\overline{M^{\mathcal{E}}Q_i}$. Note that R_i's reflection over H_i is the reflection of $H_i^{\mathcal{E}}$ over $O_i^{\mathcal{E}}$ (denote this as $L_i^{\mathcal{E}}$), and R_i's reflection over P_i is the Miquel point $M_{Si}^{\mathcal{E}}$ of $\triangle_i^{\mathcal{E}} \cup \mathcal{S}$. Therefore, this Miquel point is the foot of $L_i^{\mathcal{E}}$ onto $\mathcal{E}_{i^*}$. Further, the Miquel point lying on $(\triangle_i^{\mathcal{E}})$ tells us the foot of $H_i^{\mathcal{E}}$ onto $\mathcal{E}_{i^*}$ (which is just Q_i) lies on $(\triangle_i^{\mathcal{E}})$ as well. Especially, $O_i^{\mathcal{E}}$ lies on the perpendicular bisector of $\overline{M^{\mathcal{E}}Q_i}$. Since Go is the midpoint of O_i and $O_i^{\mathcal{E}}$, we have it also lies on the perpendicular bisector of $\overline{M^{\mathcal{E}}Q_i}$, and thus $Q_i \in \odot(Go)$. $\square$

Chapter 5

The Best Of X_n

Triangles are the most important objects you encounter in geometry, implying the centers of these triangles are naturally a key topic for us to address. How exactly do we define a "center" of a triangle? If we want to rigorously define triangle centers, we have to use either trilinear or barycentric coordinates on the plane. These are effectively just giving the ratios of the areas of $\triangle APB, \triangle BPC, \triangle CPA$, or the ratio of distances from P to the three sides. In other words, a "center" must be symmetric in each of these coordinate systems, so i.e. we don't count excenters as triangle centers, since their barycentric coordinates are not symmetric.

Since 1994, the mathematician Clark Kimberling started categorizing an index of these centers - the Encyclopedia of Triangle Centers. He assigned every center a specific ID X_n. Our previously seen incenter, centroid, circumcenter, orthocenter are respectively X_1, X_2, X_3, X_4. If we made an entry for every single symmetric function in these coordinate functions corresponding to a triangle center, this encyclopedia could explode. So commonly we just make an entry for the points people care about.

For the sake of conciseness, we will use conventions a lot here, so review them if necessary. Here, X_n refers to triangle centers from the Encyclopedia of Triangle Centers.

5.1 X_1

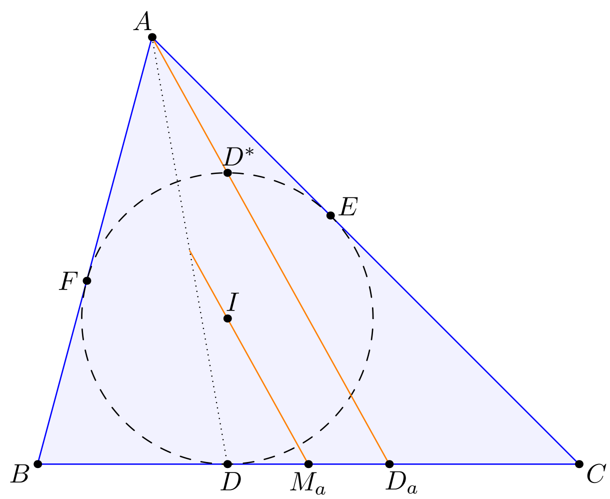
- $X_1 = I$ is the **incenter**, defined as the intersection point of the three internal angle bisectors of $\triangle ABC$. Intersecting any two external bisectors and the other internal bisector gets us three excenters, I^a, I^b, I^c. (Note that these are not triangle centers as they are not symmetric.)

  Also recall that ω is the incircle and $\omega_a, \omega_b, \omega_c$ are the A, B, C-excircles.

Proposition 5.1.1. Let D^* be the antipode of D on ω. Then A, D^*, and D^a are collinear.

Proof. Consider the homothety $\mathfrak{h}$ at A that sends ω to ω^a. Then $\mathfrak{h}^{-1}(D^a)$ is on ω and satisfies $\overrightarrow{\mathfrak{h}^{-1}(D^a)I} \parallel \overrightarrow{I^a D^a} \parallel \overrightarrow{DI}$, so $D^* = \mathfrak{h}^{-1}(D^a)$ which is enough. $\square$

In the same vein, we have that D^a, D^b, D^c's antipodes on the three excircles, $\omega^a, \omega^b, \omega^c$ will lie on AD, AD^c, AD^b respectively. Also note that the midpoint of $\overline{DD_a}$ is the midpoint of $\overline{BC}$ (denoted as M_a). This gives us that:



Corollary 5.1.2. $\overline{IM_a} \parallel \overline{AD_a}$ and $\overline{IM_a}$ bisects $\overline{AD}$.

Proposition 5.1.3. Let $D^\vee = \overline{EF} \cap \overline{BC}$. Then $D^\vee$ is the harmonic conjugate of D wrt B, C.

Proof. Menelaus. Specifically, we have

$$-\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1 = \frac{BD^\vee}{D^\vee C} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB}$$

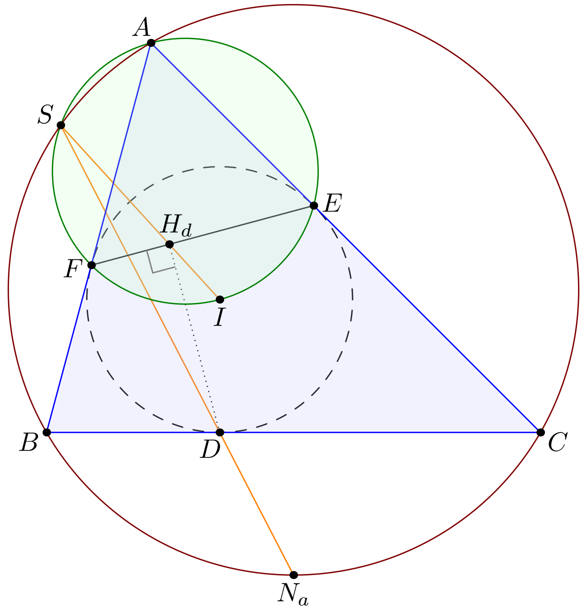
so

$$(B, C : D, D^\vee) = \frac{BD/DC}{BD^\vee/D^\vee C} = -1.$$

Alternatively, using harmonic properties on the cevian triangle of the Gergonne point suffices. $\square$

Proposition 5.1.4. Let the second intersection of the circle $(AI) = (AEIF)$ with Ω be S (in America, we call this the **A-Sharkydevil Point** of $\triangle ABC$). Then D, S, and N_a are collinear.

Proof. Consider the inversion around $(BIC) = (N_a)$. This sends the circle $(AIEF)$ to the circle with diameter $\overline{IX}$, where $X = \overline{AI} \cap \overline{BC}$. Then the inverse of S is $(IX) \cap \overline{BC} = D$, so S, N_a, and D are collinear. $\square$



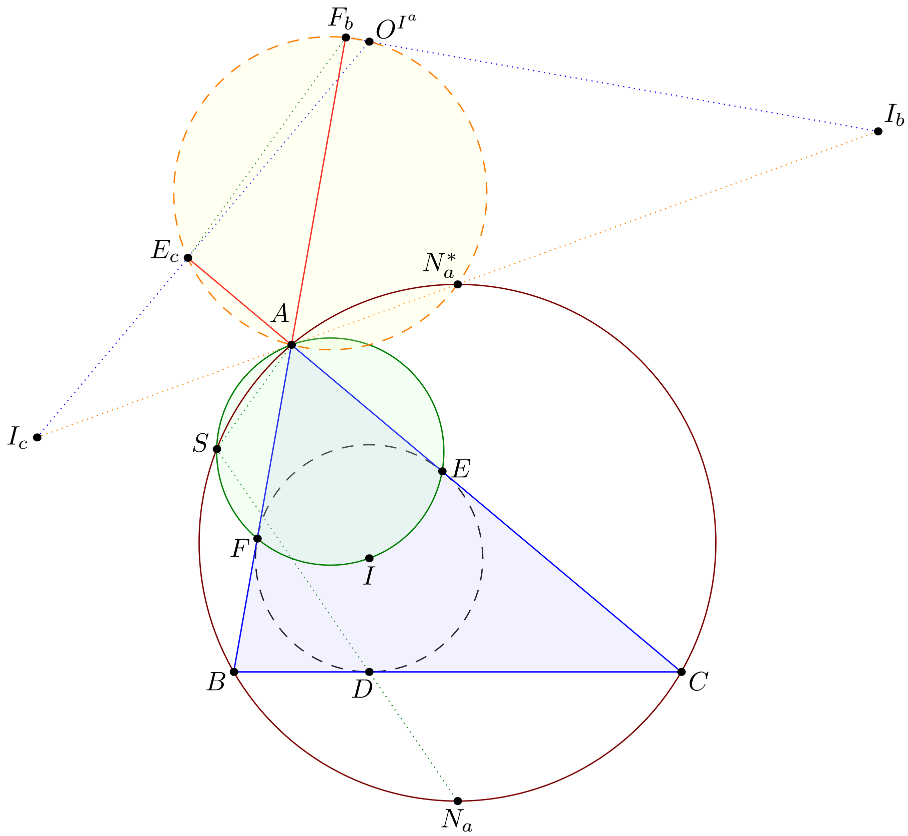
Proposition 5.1.5. Continuing notation from above, we have that IS goes through the foot from D to $\overline{EF}$ (denoted as H_d).

Proof. Let $H'_d = \overline{IS} \cap \overline{EF}$. Since the tangent at I to (AI) is parallel to FE and the tangent at N_a is parallel to BC, by spiral similarity we have that $\triangle SBC \cup N_a \overset{+}{\sim} \triangle SFE \cup I$. By the above claim, we get that $\triangle SBC \cup D \overset{+}{\sim} \triangle SFE \cup H'_d$, so $\triangle SBF \overset{+}{\sim} \triangle SDH'_d$. Thus

$$\measuredangle SN_aA = \measuredangle SBA = \measuredangle SBF = \measuredangle SDH'_D \implies \overline{DH'_d} \parallel \overline{N_aA} \perp \overline{EF},$$

which tells us that $H'_d = H_d$. $\square$

Proposition 5.1.6. Let O^{I^a} be the intersection of $\overline{I^bF^b}$ and $\overline{I^cE^c}$ (which is also the circumcenter of $\triangle II^bI^c$ by angle chasing). Then $A, N_a^*, F_b, E_c, O^{I^a}$ are concyclic.



Proof. Note that $\triangle ABC$ is the orthic triangle of $\triangle I_A I_B I_C$, so (ABC) is the nine-point circle of $\triangle I_A I_B I_C$. As such, it follows that N_a^*, which is the reflection of the midpoint of II_A over O, is the midpoint of $I_B I_C$. Then,

$$\measuredangle AF_bO^{I^a} = \measuredangle AE_bO^{I^a} = \measuredangle AN_a^*O^{I_a} = 90^\circ. \qquad \square$$

Proposition 5.1.7. $\overline{AS} \parallel \overline{F_bE_c}$.

Proof. Using the above proof, we have that

$$\measuredangle F_bE_C + 90^\circ = \measuredangle AF_b + \measuredangle AE_c - \measuredangle AO^{I^a} = 2 \cdot I_BI_C - AO^{I^a} = AO^I,$$

where O^I is the circumcenter of $\triangle I^aI^bI^c$ (which is the reflection of O^{I^a} over I^bI^c).

By Proposition 5.2.4, it then follows that O^I is the reflection of I over O, so AO^I is parallel to A^*I $= AS + 90^\circ$ where A^* is the A antipode. $\square$

Proposition 5.1.8. $\overline{AM_a}$, $\overline{EF}$, and $\overline{ID}$ concur.

Proof 1. Let T be the intersection of $\overline{ID}$ and $\overline{EF}$. Draw a line through T parallel to $\overline{BC}$, denoted as $T\infty_{BC}$. Let this intersect sides $\overline{CA}$ and $\overline{AB}$ at C_T, B_T. Then the feet of I onto $\overline{B_TC_T}, \overline{C_TA}, \overline{AB_T}$ are collinear. Thus by Simson Line, we have that I lies on (AB_TC_T). Further, since $\triangle AB_TC_T \cap I$ is homothetic to $\triangle ABC \cap N_a$ (with A as center of homothety), we have A, the midpoint T of $\overline{B_TC_T}$, M_a are collinear.

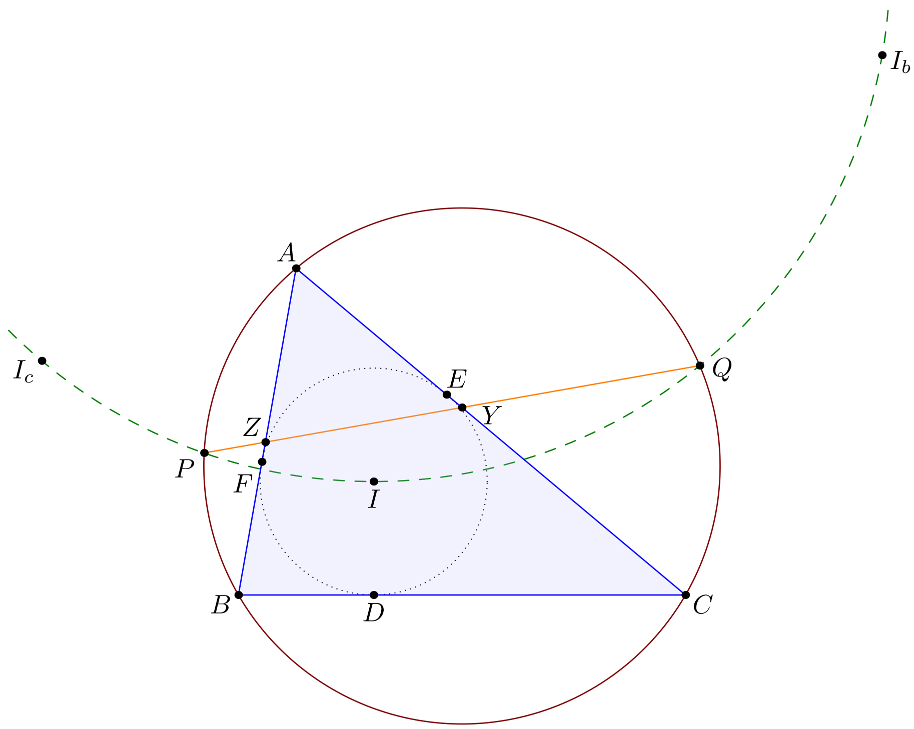
□

Proof 2. Let $T = \overline{ID} \cap \overline{EF}$, and draw a line $A\infty_{\overline{BC}}$ parallel to $\overline{BC}$ through A, and let it intersect $\overline{EF}$ at S. Then S and T are both on the pole $\mathfrak{p}_\omega(A)$ of A about ω. Since $\overline{IT} \perp \overline{AS}$, $\mathfrak{p}_\omega(T) = \overline{AS}$, so

$$A(B, C; T, \infty_{BC}) \overset{\mathfrak{p}_\omega}{=} (F, E; S, T) = -1.$$

Hence $\overline{AT}$ bisects $\overline{BC}$. □

Proposition 5.1.9. Let $\overline{BI}$ and $\overline{CI}$ intersect $\overline{CA}$, $\overline{AB}$ at Y and Z respectively. Let the two intersections of $\overline{YZ}$ and Ω be P and Q. Then I, I^b, I^c, P, and Q are concyclic.



Proof. Y is on the radical axis of $(ACPQ)$ and $(ACII^b)$, so I, I^b, P, and Q are concyclic. Similarly I^c lies on (IPQ). □

Proposition 5.1.10. Let H_a be the foot from A to $\overline{BC}$. Then

$$(H_a, \overline{AI} \cap \overline{BC}; D, D_a) = -1.$$

Proof. From

$$(A, \overline{AI} \cap \overline{BC}; I, I^a) = B(A, C; I, I^a) = -1,$$

we can project $(A, C; I, I^a)$ through the point at infinity $\infty_{\perp BC}$ to get that

$$(H_a, AI \cap BC; D, D_a) = -1.$$

□

Proposition 5.1.11. $\overline{DI^a}$ bisects $\overline{AH_a}$.

Proof.

$$(A, H_a; \infty_{AH_a}, \overline{DI^a} \cap \overline{AH_a}) = D(A, \overline{AI} \cap \overline{BC}; I, I^a) = -1.$$

□

In the rest of this section, let H_A. H_B, and H_C be the orthocenters of $\triangle BIC$, $\triangle CIA$, and $\triangle AIB$ respectively (this notation is not super standard).

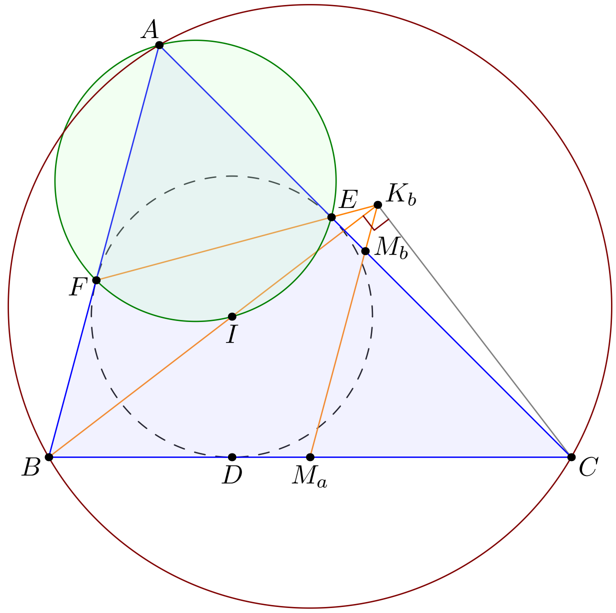
Proposition 5.1.12. Point H_A is the reflection of I^a about M^a.

Proof. We have (by line arguments) that

$$\measuredangle BH_A = \measuredangle CI + 90^\circ = \measuredangle CI^a, \quad \measuredangle CH_A = \measuredangle BI + 90^\circ = \measuredangle BI^a.$$

□

Proposition 5.1.13. [Iran Lemma] Let K_b be the foot from B to $\overline{CI}$, and similarly define K_c as the foot from C to $\overline{BI}$. Then $K_b = \overline{EF} \cap \overline{M_cM_a}$, and $K_c = \overline{EF} \cap \overline{M_bM_a}$ similarly.



Proof. Since M_a is the circumcenter of (BK_bK_cC), we have that

$$\measuredangle M_aK_b = \measuredangle K_bB + \measuredangle K_bC - \measuredangle BC + 90^\circ = 2 \cdot \measuredangle CI - \measuredangle BC = \measuredangle CA,$$

and because of this we have that $K_b \in \overline{M_cM_a}$. Note that since B, I, K_b, F are concyclic, we have

$$\measuredangle IFK_b = \measuredangle BIC + 90^\circ = \measuredangle IAE = \measuredangle IFE,$$

and thus $K_b \in \overline{EF}$. By the same logic we have that $K_c = \overline{EF} \cap \overline{M_aM_b}$ as well. □

Proposition 5.1.14. We have

$$(D, \overline{ID} \cap \overline{EF}; I, H_A) = -1.$$

Proof. Let K_bDK_c be the orthic triangle of $\triangle BIC$. Then we have

$$K_b(D, \overline{ID} \cap \overline{EF}; I, H_A) = (D, D^\vee; B, C) = -1.$$

□

Combining this with the properties of cevian triangles (see Proposition 5.3.2), we get:

Corollary 5.1.15. The cevian triangle of I with respect to $\triangle H_AH_BH_C$ is the intouch triangle $\triangle DEF$.

We can also characterize H_A as follows:

Proposition 5.1.16. The polar $\mathfrak{p}_\omega(H_A)$ is the A-midline $\overline{M_bM_c}$, or the image of H_A about inversion by ω is the foot from I to the A-midline.

Proof. Let $T = \overline{ID} \cap \overline{EF}$ as usual, then $\mathfrak{p}_\omega(T)$ is $A\infty_{\overline{BC}}$, the line parallel to BC through A. Since polarity preserves cross-ratios, we have that

$$(\overline{BC}, A\infty_{\overline{BC}}; \mathcal{L}_\infty, \mathfrak{p}_\omega(H_A)) \stackrel{\mathfrak{p}_\omega}{=} (D, T; I, H_A) = -1,$$

so $\mathfrak{p}_\omega(H_A)$ is the A-midline $\overline{M_bM_c}$. □

5.2 X_2 through X_5

- X_2 is the **centroid** (Example 0.3.3), the intersection of the three medians, typically represented as G;
- X_3 is the **circumcenter** (Proposition 0.1.9). the intersection of the three perpendicular bisectors, typically written as O;
- X_4 is the **orthocenter** (Problem 1), the intersection of the three altitudes, typically written as H;
- X_5 is the **nine-point** center, the center of ϵ, the Nine-Point Circle, typically written as N_9.

The most important properties will be listed below.

Proposition 5.2.1. O and H are isogonal conjugates.

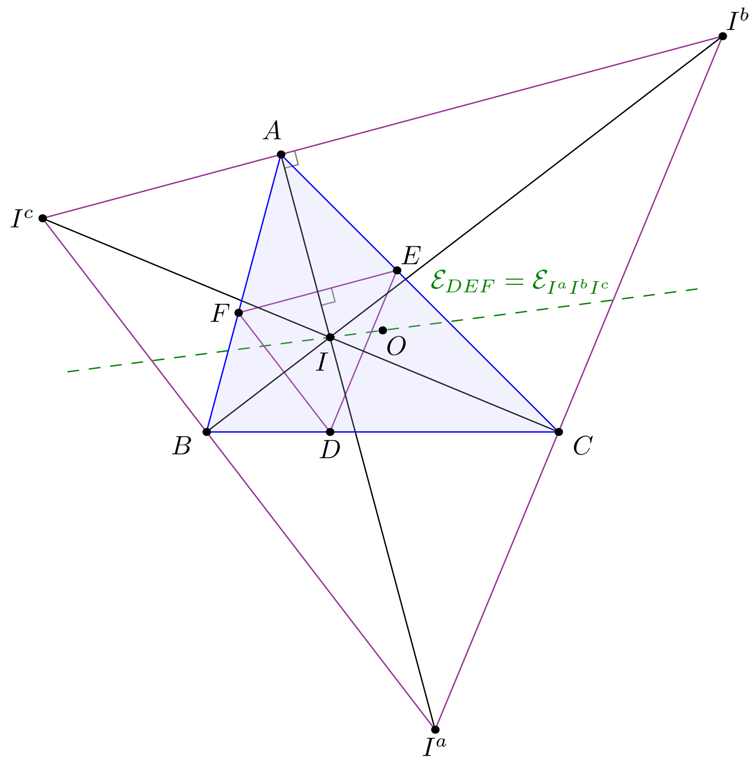
Proposition 5.2.2. G, O, H, N all lie on the Euler line $\mathcal{E}$, and

$$\frac{HG}{GO} = \frac{OG}{GN} = 2, (G, H; O, N) = -1.$$

These two were proven in previous chapters.

The rest of this section will relate H and O to I, the incenter.

Proposition 5.2.3. The Euler line of the intouch triangle $\triangle DEF$ is OI.



Proof. This problem was given as an exercise in a previous chapter, but we will re-present the proof here. Note that the Euler line of the excentral triangle, $\triangle I^aI^bI^c$ is line OI, and furthermore note that $\triangle DEF$ and $\triangle I^aI^bI^c$ are homothetic, so the Euler lines of these two triangles are parallel. However, I clearly lies on the Euler line of $\triangle DEF$, so we are done.

(Note from translators: the center of homothety between the intouch and excentral triangles is X_{57}, the isogonal conjugate of the Mittenpunkt.) □

Proposition 5.2.4. The circumcircle Ω is the nine-point circle of all four of $\triangle I^aI^bI^c$, $\triangle II^bI^c$, $\triangle I^aII^c$, and $\triangle I^aI^bI$.

Proof. Note that I, I^a, I^b, I^c make an orthocentric system and that $\triangle ABC$ is the orthic triangle wrt any of the triangles. □

Proposition 5.2.5. The trilinear polar of I with respect to $\triangle ABC$, line $\mathfrak{t}(I)$, is perpendicular to OI. Further, define Y and Z as $\overline{I^cI^a} \cap \overline{CA}$, $\overline{I^aI^b} \cap \overline{AB}$. Then we have $OI \perp YZ$.

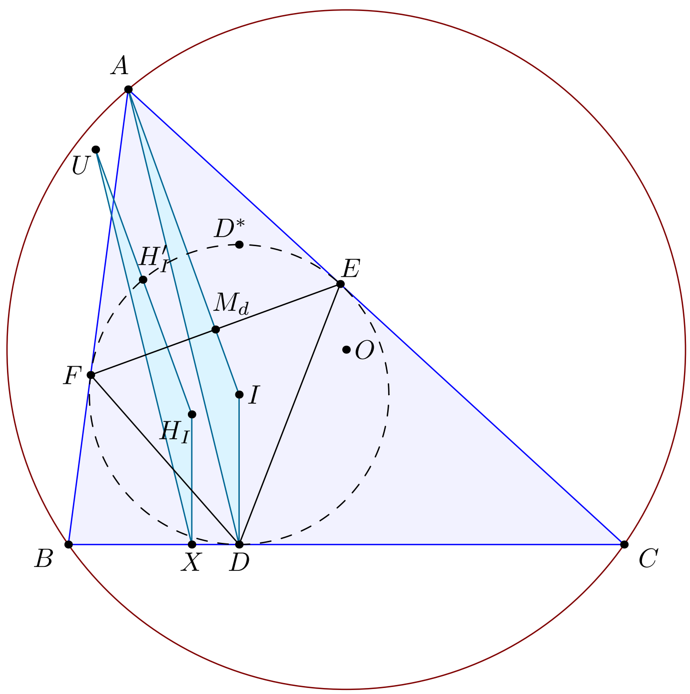
Proof. From

$$YI^c \cdot YI^a = YC \cdot YA, ZI^a \cdot ZI^b = ZA \cdot ZB,$$

we have that YZ is the radical axis of $(ABC), (I^aI^bI^c)$, so it is perpendicular to the line connecting O and the circumcenter of $\triangle I^aI^bI^c$. Further note that O is the nine-point center of $\triangle I^aI^bI^c$, so this line is just the Euler line of the excentral triangle, which is just line OI.

Then it follows by harmonic properties that YZ is also $\mathfrak{t}(I)$ which finishes. □

Proposition 5.2.6. Let U be the reflection of D across EF (where $\triangle DEF$ is the intouch triangle). Then AU, BC, IO are concurrent.



Proof. Let D^* be the antipode of D wrt. ω, and let H_I be the orthocenter of $\triangle DEF$. Let H_I' be the reflection of H_I across EF. Then note that H_I' lies on ω. Let the foot from H_I to $\overline{BC}$ be point X. Then

$$\measuredangle XH_ID = \measuredangle D^*DH_I' = -\measuredangle H_I'DD^*, \measuredangle DXH_I = 90^\circ = -\measuredangle D^*H_I'D,$$

tells us that $\triangle H_IXD \overset{-}{\sim} \triangle DH_I'D*$. Thus if we let M_d be the midpoint of $\overline{EF}$, by

$$\frac{H_IX}{H_IU} = \frac{H_IX}{DH_I'} = \frac{DH_I}{D^*D} = \frac{IM_d}{ID^*} = \frac{ID}{IA},$$

we get that $\triangle IAD$ and $\triangle H_IUX$ are homothetic. Combining Proposition 5.2.3 with this result, we get that $IH_D = IO, AU, DX = BC$ are concurrent. □

Proposition 5.2.7. I, H are isogonal conjugates in the orthic triangle $\triangle H_dH_eH_f$ of the intouch triangle $\triangle DEF$.

Proof. By symmetry, all we need to prove is that

$$\measuredangle(H_dI) + \measuredangle(H_dH) = \measuredangle(H_dH_e) + \measuredangle(H_dH_f) = 2 \cdot \measuredangle EF.$$

By the proof of Proposition 5.1.5, we have that the complete pentagon $\triangle ABC \cup EF \cup DH_d$ has a Miquel point S. Thus by Proposition 4.1.6 we have that the orthocenters of $\triangle AEF$, $\triangle ABC$, and H_d are collinear. Note that the orthocenter of $\triangle AEF$ is the reflection of I across $\overline{EF}$ (let this be I'). Thus

$$\measuredangle(H_dI) + \measuredangle(H_dH) = \measuredangle(H_dI) + \measuredangle(H_dI') = 2 \cdot \measuredangle EF.$$

$\square$

Proposition 5.2.8. The perspector of the triangle formed by $\overline{E^aF^a}, \overline{F^bD^b}, \overline{D^cE^c}$ and $\triangle ABC$ is H. Furthermore, H is the circumcenter of this triangle. (We will denote this triangle as $\triangle'_I$.)

Proof. Note that $\overline{F^bD^b} \perp \overline{BI}, \overline{D^cE^c} \perp \overline{CI}$.

By symmetry, all we need to prove is that the corresponding sides of $\triangle AF^bE^c$ and $\triangle IBC$ are orthologic, or that $I\infty_{\perp F^bE^c}$, $BA^* = B\infty_{\perp E^cA}$, $CA^* = C\infty_{\perp AF^b}$ are concurrent. By Proposition 5.1.7, F^bE^c is perpendicular to IA^*, so the concurrence point of the three lines listed above is just A^*.

Then it follows that the foot from A to BC passes through $F^b\infty_{\perp BI} \cap E^c\infty_{\perp CI} = \overline{F^bD^b} \cap \overline{D^cE^c}$ as desired. $\square$

Proposition 5.2.9. We have the following relation between line arguments:

$$BC + CA + AB = 2 \cdot OI + IN + 90^\circ.$$

Proof. This proof uses results from later sections in this chapter, but is not a circular proof. From Proposition 5.5.2, we have

$$2 \cdot OI = 2 \cdot (AI + BC - DFe) = (BC + CA + AB) + BC - 2 \cdot DFe.$$

We also have

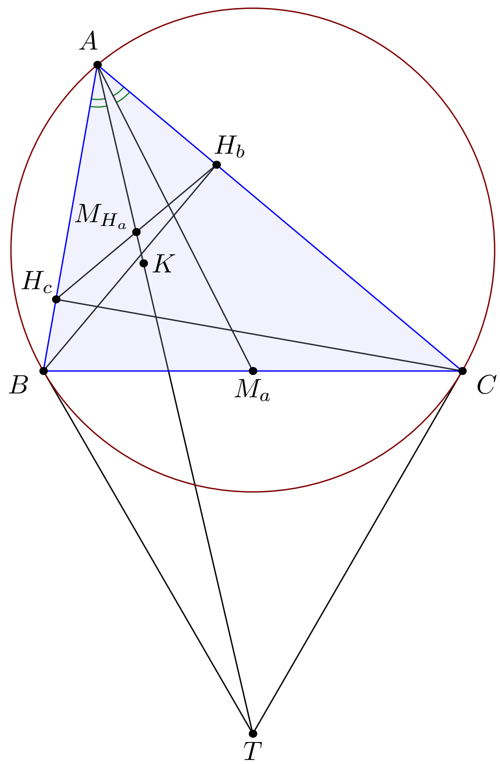
$$IN = IFe = (2Fe - 90^\circ)_\omega = 2DFe - BC - 90^\circ.$$

$\square$

5.3 X_6

- X_6 is the symmedian point Example 2.2.16, the isogonal conjugate of G.

Proposition 5.3.1. AK bisects $\overline{H_bH_c}$.



Proof. Let M_{H_a} be the midpoint of $\overline{H_bH_c}$. Since $\triangle ABC \overset{-}{\sim} \triangle AH_bH_c$, we have

$$\measuredangle H_bAM_{H_a} = -\measuredangle BAM_a = \measuredangle CAK,$$

so A, M_{H_a}, K collinear. □

Proposition 5.3.2. Let the three triangles $\triangle K_aK_bK_c, \triangle K_AK_BK_C, \triangle K^aK^bK^c$ respectively be the K-cevian triangle, circumcevian triangle, and anticevian triangle. Then

$$(A, K_a; K, K_A) = (B, K_b; K, K_B) = (C, K_c; K, K_C) = -2,$$

$$(A, K_A; K_a, K^a) = (B, K_B; K_b, K^b) = (C, K_C; K_c, K^c) = -1.$$

Proof. We have

$$(A, K_a; K, K_A) \overset{B}{=} (A, C; K_B, K_A) = (A, C; K_B, B) \cdot (A, C; B, K_A) = (-1) \cdot 2 = -2,$$

and similarly for the rest. For the harmonic one we can note that $\triangle K^aK^bK^c$ is the tangential triangle of $\triangle ABC$, so we get

$$(A, K_A; K_a, K^a) \stackrel{B}{=} (A, K_A; C, B) = -1,$$

and similarly for the rest. □

Proposition 5.3.3. [Schwatt Line] M_aK bisects segment AH_a.

Proof. Let T_a be the intersection point of the tangents at B, C to (ABC), then A, K, T_a are collinear and M_aT_a is perpendicular to BC. Thus

$$(A, H_a; M_aK \cap AH_a, \infty_{AH_a}) \stackrel{M_a}{=} (A, AK \cap BC; K, T_a) \stackrel{B}{=} (A, C; BK \cap (ABC), B) = -1,$$

so M_aK bisects AH_a. □

Proposition 5.3.4. The trilinear polar $\mathfrak{t}(X)$ of any point X on the circumcircle will always pass through K.

Proof. Let $\triangle X_aX_bX_c$ be the cevian triangle of X with respect to $\triangle ABC$. Construct $X_b^\vee$ and $X_c^\vee$ on lines CA, AB such that

$$(C, A; X_b, X_b^\vee) = (A, B; X_c, X_c^\vee) = -1$$

(so just the harmonic conjugates of X_b and X_c).

Note that $\mathfrak{t}(X)$ is then just $X_b^\vee X_c^\vee$. From

$$(C, A; B, XX_b^\vee \cap \Omega \stackrel{X}{=} (C, A; X_b, X_b^\vee) = -1 = (C, A; B, BK \cap \Omega),$$

we can get $K_B \coloneqq XX_b^\vee \cap BK \in \Omega$, and similarly we get $K_C \coloneqq XX_b^\vee \cap BC \in \Omega$. Now we insta-kill by Pascal on ABK_BXK_CC to get that $K, X_b^\vee, X_c^\vee$ are collinear, so $K \in \mathfrak{t}(X)$. □

Proposition 5.3.5. Given any point P, let $\triangle P_AP_BP_C$ be its circumcevian triangle, and let this triangle have symmedian point K_P. Then K_P, K, P collinear.

Proof. Invert at P with power $\mathbf{Pow}_\Omega(P)$, call this transformation $\mathfrak{J}$. Then we have that

$$A \stackrel{\mathfrak{J}}{\mapsto} P_A, \quad B \stackrel{\mathfrak{J}}{\mapsto} P_B, \quad C \stackrel{\mathfrak{J}}{\mapsto} P_C.$$

Since inversion preserves cross-ratios, we get

$$(P_B, P_C; P_A, \mathfrak{J}(K_A)) = (B, C; A, K_A) = -1.$$

so $\mathfrak{J}(K_A) = P_AK_P \cap \Omega =: K_{P_A}$. Let K_a be the intersection of AK_A and BC, then we get that $\mathfrak{J}(K_a)$ is the second intersection of $(PP_AK_{P_A})$ and (PP_BP_C). By Radical Axis Theorem, we get that $P\mathfrak{J}(K_a), P_AK_{P_A}, P_BP_C$ concurrent, and thus $K_a, P, P_AK_P \cap P_BP_C$ are collinear. By (Proposition 5.3.2), we get

$$(P_A, P_AK_P \cap P_BP_C; PK \cap P_AK_P, K_{P_A}) \overset{P}{=} (A, K_a; K, K_A) = -2 = (P_A, P_AK_P \cap P_BP_C; K_P, K_{P_A}),$$

and thus K, K_P, P are collinear. $\square$

Remark. X_6 is the center of the inscribed ellipse of $\triangle ABC$ with touchpoints at $\triangle H_aH_bH_c$. (This is because it is the complement of the isotomic conjugate of H, as we will see in X_{69}).

5.4 X_7 through X_{10}

- X_7 is the **Gergonne** point (Problem 4), defined as the perspector of the intouch triangle and the reference triangle, commonly denoted as Ge;
- X_8 is the **Nagel** point (Problem 5), defined as the perspector of the extouch triangle and the reference triangle, commonly denoted as Na;
- X_9 is the **Mittenpunkt**, defined as the perspector of the excentral triangle and the medial triangle, commonly denoted as Mt;
- X_{10} is the **Spieker center**, the incenter of the medial triangle, commonly denoted as Sp.

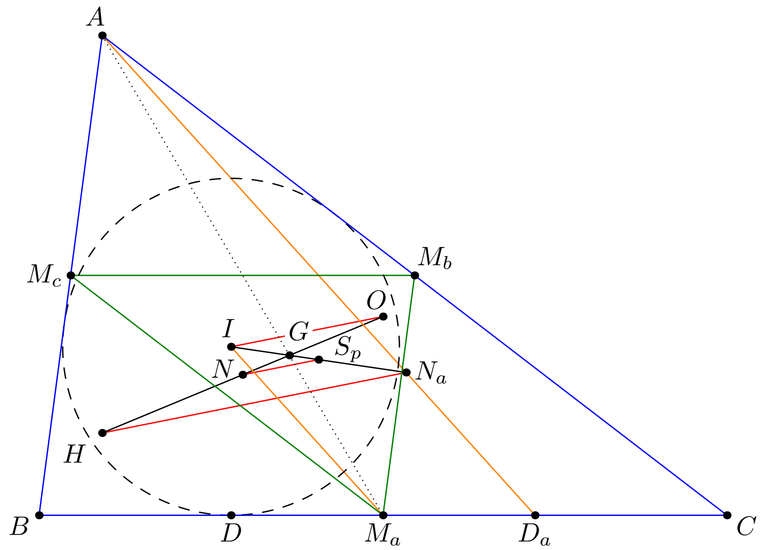
Proposition 5.4.1. The complement of the Gergonne point is the Mittenpunkt.

Proof. By an A-version of Corollary 5.1.2, we get that $\overline{M_aMt} = \overline{M_aI^a} \parallel \overline{AGe}$. Symmetrically we get that $\overline{M_bMt} \parallel \overline{BGe}$, $\overline{M_cMt} \parallel \overline{CGe}$. Thus we get that $\triangle ABC \cap Ge \overset{+}{\sim} \triangle M_aM_bM_c \cap Mt$, so $Mt = Ge^{\complement}$. $\square$

Proposition 5.4.2. The complements of Na, I are I, Sp, respectively. Also

$$(G, Na; I, Sp) = -1$$

on the Nagel line, and $\overline{HNa} \parallel \overline{OI} \parallel \overline{NSp}$.



Proof. From Corollary 5.1.2, we get $\overline{M_aI} \parallel \overline{ANa}$. Similarly we have $\overline{M_bI} \parallel \overline{BNa}$, $\overline{M_cI} \parallel \overline{CNa}$. Thus we get that $\triangle ABC \cap Na \cap I \overset{+}{\sim} \triangle M_aM_bM_c \cap I \cap Sp$, and thus $I = Na^{\complement}$ and $Sp = I^{\complement}$. The harmonic is trivial by the ratios from the definition of the complement. □

Proposition 5.4.3. Mt is the symmedian point of $\triangle I^aI^bI^c$.

Proof. Since $\triangle ABC$ is the orthic triangle of $\triangle I^aI^bI^c$, by Proposition 5.3.1, we have that the symmedian point of the excentral triangle X satisfies

$$X = \overline{I^aM_a} \cap \overline{I^bM_b} \cap \overline{I^cM_c} = Mt.$$

□

Proposition 5.4.4. The line H_AGe bisects segment $\overline{EF}$.

Proof. We use the same notation as in Proposition 5.1.14. Let $\overline{AD} \cap \overline{EF}$ at point U. Let the midpoint of $\overline{EF}$ be M_d. Then we get that

$$(D, \overline{EF} \cap \overline{BC}; B, C) \overset{E}{=} M_d(D, U; A, Ge) = -1 = M_d(D, S; I, H_A),$$

so Ge, M_d, H_A collinear. □

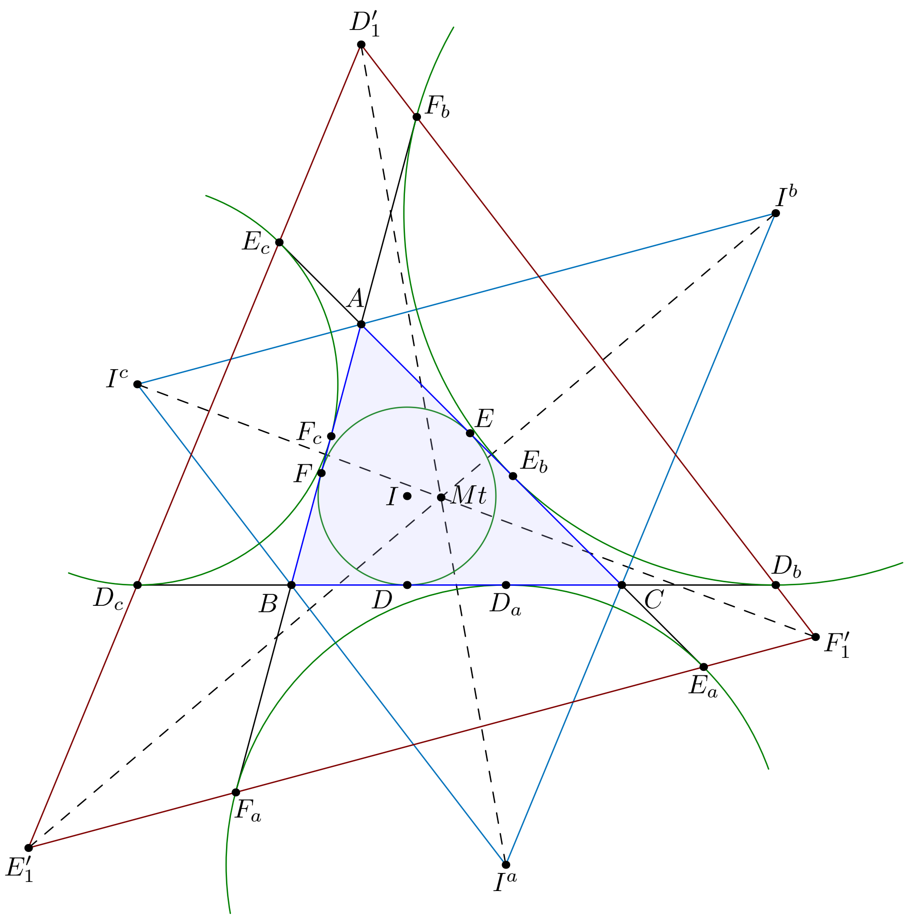
Proposition 5.4.5. Line $\overline{AI}$ is the complement of $\overline{H_ANa}$.

Proof. Let X be the reflection of Na across M_a. Then G is the centroid of AXN_a. Further, we get that I is the midpoint of $\overline{AX}$. Note that since the reflection of H_A across M_a is I^a, we can get that $\overline{H_ANa} \parallel \overline{I^aX} = \overline{AI}$. Then, we can then use the fact that I is the complement of Na to tell us that $\overline{AI} = (\overline{H_ANa})^{\complement}$. □

Proposition 5.4.6. I, K, Mt are collinear.

Proof. By Proposition 5.3.5, we get that $\overline{IK}$ passes through the symmedian point of the arc-midpoint triangle (circumcevian triangle of I). Let this point be K_I. Then take the homothety at I sending $\triangle I^aI^bI^c$ to $\triangle N_aN_bN_c$. Since Mt is the symmedian point of $\triangle I^aI^bI^c$, we get that I, K_I, Mt are collinear, and thus I, K, Mt are collinear. □

Proposition 5.4.7. The triangle made by lines $\overline{E_aF_a}, \overline{F_bD_b}, \overline{D_cE_c}$ is homothetic to $\triangle I^aI^bI^c$ with homothety center Mt. Furthermore, the ratio of homothety is $-\frac{2R+r}{2R}$, where R, r represent the circum- and inradius respectively.



Proof. Let $X = F_bD_b \cap D_cE_c = D_1'$. Then all we need to prove is that X, I^a, M_a are collinear, but this is because the center of homothety between $\triangle XD_bD_c$ and $\triangle I^aBC$ is M_a (note that the midpoint of $\overline{D_bD_c}$ is M_a).

To prove this we first observe that by Proposition 5.2.8, we have $XH_A = I^aM_a \parallel AD$, and thus $AX \parallel DH_A$. Let O^I be the circumcenter of $\triangle I^aI^bI^c$. Then the quadrilateral $I^aO^IN_aN_a^*$ is actually a parallelogram. Thus

$$\begin{aligned}\overrightarrow{XH} &= \overrightarrow{XA} + \overrightarrow{AH} = (\overrightarrow{ID} - \overrightarrow{IH_A}) + 2 \cdot \overrightarrow{OM_a} \\ &= r - 2 \cdot \overrightarrow{N_aM_a} + 2 \cdot \overrightarrow{OM_a} \\ &= r + 2R, \\ \overrightarrow{I^aO^I} &= \overrightarrow{N_aN_a^*} = -2R.\end{aligned}$$

and we get that the ratio of homothety is $-\frac{2R+r}{2R}$, by comparison of circumradii. □

Proposition 5.4.8. [Mittenpunkt Line] H, Mt, Sp are collinear.

Proof. See Proposition 5.6.8. □

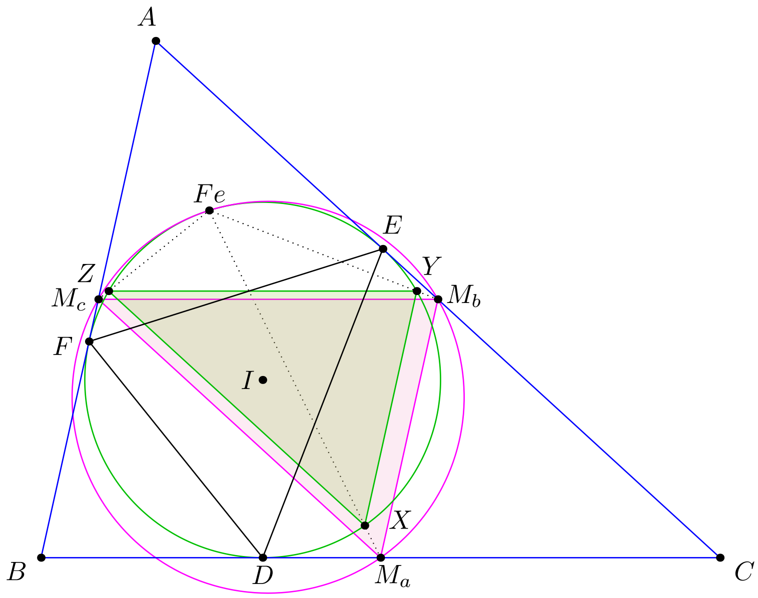
5.5 X_{11}

- X_{11} is the Feuerbach point (see Example 3.3.4, defined as the tangency point of the incircle ω and the nine-point circle ϵ, commonly denoted as Fe.

Proposition 5.5.1. The three points I, N, Fe are collinear.

Proof. Trivial. □

Proposition 5.5.2. Let X, Y, Z respectively be the reflections of D, E, F (intouch) across the perpendicular bisectors of $\overline{EF}, \overline{FD}, \overline{DE}$. (which are just the three angle bisectors of $\triangle ABC$.) Then Fe is the homothety center of $\triangle XYZ$ and $M_aM_bM_c$.



Proof. We have

$$YZ = ((F + D - E) + (D + E - F))_\omega = 2 \cdot D_\omega = \measuredangle M_bM_c.$$

By symmetry, we get that $\triangle XYZ$ and $\triangle M_aM_bM_c$ are homothetic. What is left is the proof by construction of the existence of Fe (See Example 3.3.4). $\square$

Proposition 5.5.3. We have $\triangle AOI \overset{+}{\sim} \triangle FeM_aD$

Proof. Let X be the reflection of D over the perpendicular bisector of $\overline{EF}$. Then we get Fe, X, M_a are collinear by Proposition 5.5.2, and further, by M_aD tangent to ω we get $\triangle FeM_aD \overset{-}{\sim} \triangle DM_aX$. Thus our original problem reduces into proving $\triangle AA^*I \overset{+}{\sim} \triangle DD'X$.

Notice that N_a (the intersection of $\overline{XD}$'s perpendicular bisector with $\overline{DD'}$'s perpendicular bisector) is the circumcenter of $\triangle DD'X$, and from the fact that A^*I intersects N_aD on Ω by Proposition 5.1.4, we know that

$$\measuredangle XD'D = \measuredangle AN_aD = \measuredangle AA^*I = -\measuredangle IA^*A.$$

and also

$$\measuredangle D'DX = \perp AI - \measuredangle BC = \measuredangle HAI = \measuredangle IAO = -\measuredangle A^*AI.$$

Hence $\triangle AA^*I \overset{+}{\sim} \triangle DD'X$, and we are done.

□

Corollary 5.5.4. Fe is the exsimilicenter of ϵ and ω.

Proof. From Proposition 5.5.2 and Proposition 5.5.3, we have

$$\frac{FeX}{FeM_a} = 1 - \frac{M_aX}{M_aFe} = 1 - \left(\frac{\overline{M_aD}}{\overline{M_aFe}}\right)^2$$

□

This just proves that Fe is not the insimilicenter. By a similar proof this can tell us that the three extraversions of the Feuerbach point Fe^a, Fe^b, Fe^c respectively are the insimilicenters of $\omega^a, \omega^b, \omega^c$ and ϵ.

Corollary 5.5.5. Fe is the anti-Steiner point of OI with respect to $\triangle M_aM_bM_c$ and $\triangle DEF$.

Proof. By Proposition 5.5.3, we have that

$$\begin{aligned}(M_a + M_b + M_c - Fe)_\varepsilon &= (M_a + M_b + M_c - H_a)_\varepsilon + \measuredangle H_aM_aFe \\ &= \measuredangle \mathbf{T}_{M_a}\varepsilon + \measuredangle AOI =\perp OI, \\ (D + E + F - Fe)_\omega &= (E + F)_\omega + \measuredangle FeDM_a \\ &=\perp AI + \measuredangle AIO =\perp OI.\end{aligned}$$

Thus we know that Fe is the anti-Steiner point of OI with respect to both of these triangles. Further, by Simson-Antipode Isogonality we have that the Steiner line of Fe with respect to $\triangle DEF$ is parallel to OI. Also note that OI is the Euler line of $\triangle DEF$ (since O, I of $\triangle ABC$ become H, O of $\triangle DEF$).

Alternatively, the result follows by Proposition 8.1.20 and Proposition 8.1.21. □

Proposition 5.5.6. $\mathfrak{J}_{\Omega,}(I)Fe$ is parallel to the Euler Line.

Proof. We length-bash.

$$\frac{O\mathfrak{J}_{\Omega,}(I)}{OI} = \frac{R^2}{\overline{OI}^2} = \frac{R^2}{R^2 - 2Rr} = \frac{\frac{R}{2}}{\frac{R}{2} - r} = \frac{N_9Fe}{N_9I}$$

□

5.6 $X_n, n < 99$

5.6.1 X_{20}

- X_{20} is the **de Longchamps point**, the reflection of H over O. For this section, we will call it L. Obviously, it lies on the Euler line.

Proposition 5.6.1.

$$(O, H; G, L) = (N_9, O; G, H) = -1.$$

Proof. Use length relations between all these points. □

Proposition 5.6.2. Let L_a be $AL \cap BC$. Then OL_a bisects $\overline{AH_a}$.

Proof. Harmonics!

$$(A, H_a; OL_a \cap AH_a, \infty_{AH_a}) \stackrel{O}{=} (AO \cap BC, H_a; L_a, M_a) \stackrel{A}{=} (O, H; L, G) = -1$$

□

Proposition 5.6.3. The three points I, Ge, L are collinear, and have length ratios of

$$\frac{IGe}{GeL} = -\frac{r}{4R + 2r}.$$

Proof. We take the complements of all 3 points. We have $Sp = I^{\mathsf{C}}, Mt = Ge^{\mathsf{C}}, H = L^{\mathsf{C}}$, which are shown to be collinear in Proposition 5.6.8. To get the length ratio,

$$\frac{HMt}{MtBe} = \frac{2R + r}{2R}, \frac{HSp}{SpBe} = 1 \implies \frac{IGe}{GeL} = \frac{SpMt}{MtH} = -\frac{r}{4R + 2r}.$$

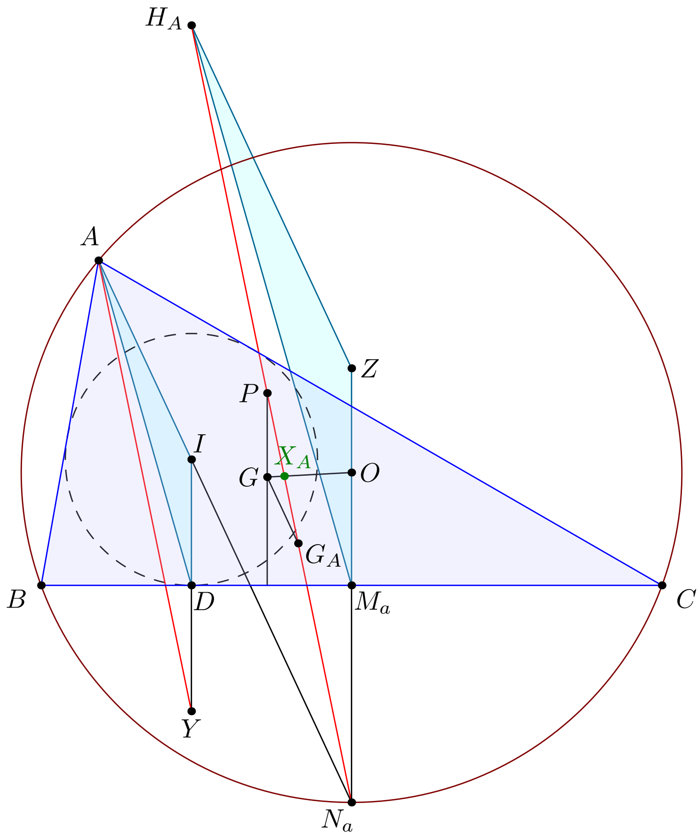
□

5.6.2 X_{21}

Proposition 5.6.4. The Euler lines of triangles $\triangle ABC$, $\triangle IBC$, $\triangle AIC$, $\triangle ABI$ all concur in a common point. Denote this point as $Sc = X_{21}$, the **Schiffler point**. Additionally, we have

$$\frac{GSc}{ScO} = \frac{2r}{3R},$$

where r and R represent the inradius and circumradius as usual.



Proof. Let $\mathcal{E}$, $\mathcal{E}_A$ respectively be the Euler lines of $\triangle ABC$ and $\triangle IBC$. Let $X_A = \mathcal{E} \cap \mathcal{E}_A$, and symmetrically define $\mathcal{E}_B, \mathcal{E}_C, X_B, X_C$. We directly go for

$$\frac{GX_A}{X_AO} = \frac{2r}{3R},$$

since if we prove this we're done by symmetry.

Let Y, Z respectively be the reflections of I, N_a over $\overline{BC}$. Since H_A is the reflection of M_a over I^a, we can apply the A-extraversion of theorem Corollary 5.1.2 to get that

$$\triangle H_A M_a Z \overset{+}{\sim} \triangle I^a M_a N_a \overset{+}{\sim} \triangle ADI.$$

If we define P as the intersection point of the perpendicular line to BC drawn from G and the line $\mathcal{E}_A$, and let G_A be the centroid of $\triangle IBC$, then this tells us that

$$\triangle G_A PG \overset{+}{\sim} \triangle H_A N_a Z \overset{+}{\sim} \triangle AYI$$

by AA similarity. Thus,

$$\frac{GX_A}{X_AO} = \frac{GP}{N_aO} = \frac{G_AG}{AI} \cdot \frac{IY}{N_aO} = \frac{2r}{3R}.$$

(There is also a very elegant proof of existence with Pascal on the Feuerbach hyperbola, which will be shown in later chapters.) $\square$

Proposition 5.6.5. The cevian triangle of Sc, $\triangle Sc_aSc_bSc_c$ is perspective with $\triangle I^aI^bI^c$, with perspector O.

Proof. Let $X = ASc \cap OM_a$. By Menelaus we have

$$\frac{M_aX}{XO} = -\frac{M_aA}{AG} \cdot \frac{GSc}{ScO} = \frac{r}{R}.$$

Thus,

$$(Y = \overline{AI} \cap \overline{BC}, \overline{AI} \cap \overline{OSc_a}; A, N_a) \stackrel{Sc_a}{=} (M_a, O; X, N_a) = \frac{r/R}{M_aN_a/NaO} = \frac{DI}{M_aN_a}.$$

We can get that $\triangle ADY \cup I \stackrel{+}{\sim} \triangle I^aM_aY \cup N_a$. Then by inverting through (IBC), it follows that

$$\frac{DI}{M_aN_a} = \frac{AY}{I^aY} = (A, I^a; Y, \infty_{AI}) \stackrel{(IBC)}{=} (Y, I^a; A, N_a)$$

$\square$

5.6.3 X_{40}

- X_{40} is the **Bevan point**, defined as the reflection of I over O, typically written as Be.

This sounds like a very useless point, but it shows up a lot more often than you expect. For example, we have

Proposition 5.6.6. Be is the circumcenter of the excentral triangle ($\triangle I^aI^bI^c$).

Proof. Since $\overline{N_aO}$ is perpendicular to $\overline{BC}$, by 2x homothety at I we can get that $\overline{I^aBe} \perp \overline{BC}$. Since the excentral triangle is homothetic to the intouch triangle, we get that the circumcenter of the excentral triangle O^I satisfies $\overline{I^aO^I} \parallel \overline{DI} \parallel \overline{I^aBe}$. It follows then by symmetry that $O^I = Be$.

$\square$

Proposition 5.6.7. Be is the midpoint of $\overline{NaL}$.

Proof. Since

$$\frac{GNa}{NaI} \cdot \frac{IBe}{BeO} \cdot \frac{OL}{LG} = \left(\frac{2}{3}\right) \cdot (-2) \cdot \left(\frac{3}{4}\right) = -1,$$

by Menelaus we get Na, Be, L are collinear. By Menelaus again we get that

$$\frac{NaBe}{BeL} = -\frac{NaI}{IG} \cdot \frac{GO}{OL} = -(-3) \cdot \frac{1}{3} = 1,$$

and we're done. $\square$

Proposition 5.6.8. H, Mt, Sp, Be are collinear, and

$$\frac{HMt}{MtBe} = \frac{2R+r}{2R}, \frac{HSp}{SpBe} = 1.$$

Proof. Define $\triangle_I'$ to be the triangle referenced in Proposition 5.4.7. By Proposition 5.2.8, we have that H is the circumcenter of $\triangle_I'$. We also know that Be is the circumcenter of $\triangle I^a I^b I^c$, so by homothety we get that $\overline{HBe}$ passes through their center of homothety, which is just Mt by Proposition 5.4.7. We also have that

$$\frac{HMt}{MtBe} = \frac{2R+r}{2R}.$$

Since

$$(O, \infty_{OI}; I, HSp \cap OI) \stackrel{H}{=} (G, Na; I, Sp) = -1,$$

HSp passes through the reflection of I about O, which is just the Bevan point. Thus H, Mt, Sp, Be collinear. $\square$

5.6.4 X_{54}

- X_{54} is the **Kosnita point**, the isogonal conjugate of the nine-point center N_9. This is typically denoted as Ko or as N_9^*.

Proposition 5.6.9. Let O_{OA} be the circumcenter of $\triangle OBC$. Then A, Ko, O_{OA} are collinear.

Proof. Note that $\triangle ABC \overset{-}{\sim} \triangle AH_bH_c$ and the circumcenter of $\triangle AH_bH_c$ is the midpoint of $\overline{AH_a}$ (so it lies on the nine-point circle ϵ). Thus $\triangle ABC \cup O_{OA} \overset{-}{\sim} \triangle AH_bH_c \cup N$. It follows then that AO_{OA}, AN are isogonal which implies the result. $\square$

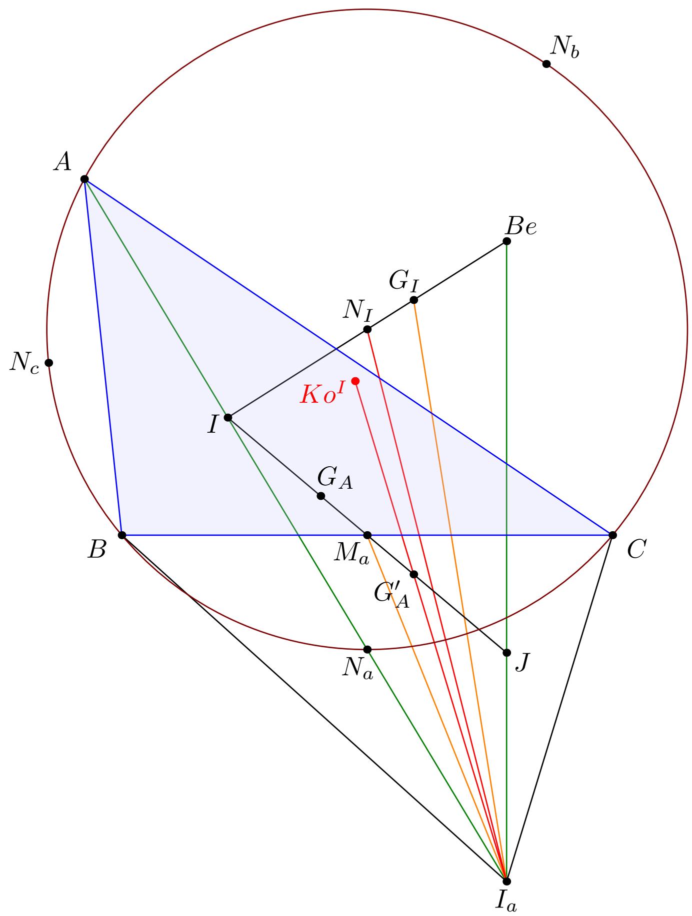
Proposition 5.6.10. Let B', C' respectively be the reflections of B, C across $\overline{CA}$ and $\overline{AB}$. Then $\overline{AKo} \perp \overline{B'C'}$.

Proof. Let X be the reflection of the circumcenter O across point A. Let the tangents to the circumcircle at B and C intersect at point D. Then all we need to prove is $XD \perp B'C'$. Note that

$$\frac{B'C}{CD} = \frac{XO}{OD} = \frac{C'B}{BD}, \measuredangle B'CD = \measuredangle XOD = \measuredangle C'BD,$$

and thus D is the spiral center of similarity sending $\triangle B'XC' \stackrel{+}{\sim} \triangle COB$, so we're done. $\square$

Proposition 5.6.11. The Schiffler point Sc is the Kosnita point of $\triangle N_aN_bN_c$.



Proof. By symmetry, we only need to prove that $\triangle IBC$'s Euler line $\mathcal{E}_A$ goes through $Ko(\triangle N_aN_bN_c)$. Let G_A be the centroid of $\triangle IBC$, and let G'_A be the reflection of I across G_A. Then we have that the condition of $\mathcal{E}_A = N_aG_A$ passing through the Kosnita point of $\triangle N_aN_bN_c$ is equivalent to the condition of $I^aG'_A$ passing through $Ko^I = Ko(\triangle I^aI^bI^c)$. Let J be the reflection of I with respect to M_a, and note that J is thus the orthocenter of $\triangle I^aBC$. Then we have

$$(I, G'_A; M_a, J) = -1 = (Be = O^I, N^I; G^I, I = H^I).$$

Since $(\overline{I^aI}, \overline{I^aBe})$, $(\overline{I^aM_a}, \overline{I^aG^I})$, and $(\overline{I^aJ}, \overline{I^aI})$ all are pairs of isogonal lines in $\angle I^bI^aI^c$, we have that $\overline{I^aG'_A}$ and $\overline{I^aN^I}$ are also isogonal lines. So we have that $Ko^I \in I^aG'_A$. $\square$

Corollary 5.6.12. The Euler line of the orthic triangle $\triangle H_aH_bH_c$ is parallel to OKo.

5.6.5 X_{55}, X_{56}

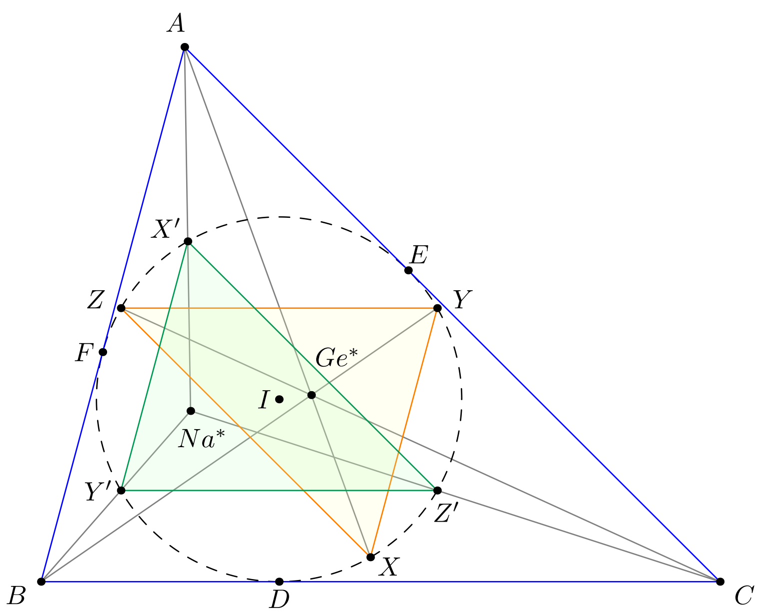
- X_{55} is the insimilicenter of Ω and ω,
- X_{56} is the exsimilicenter of Ω and ω.

Proposition 5.6.13. I, O, X_{55}, X_{56} are collinear, and

$$(I, O; X_{55}, X_{56}) = -1.$$

Proof. See definition. □

Proposition 5.6.14. X_{55} is the isogonal conjugate of Ge, and X_{56} is the isogonal conjugate of Na. (Thus we will denote them as Ge^* and Na^*.)



Proof. Let X, Y, Z respectively be the reflections of D, E, F across AI, BI, CI. Then (suppose $\triangle ABC$ is labeled counterclockwise), we have that

$$\begin{aligned}\overrightarrow{XI} &= 2 \cdot \overrightarrow{AI} - \overrightarrow{DI} = \overrightarrow{AB} + \overrightarrow{AC} - \overrightarrow{ID} + 180^\circ \\ &= \overrightarrow{AO} + 180^\circ = -\overrightarrow{AO},\end{aligned}$$

Similarly we have that $\overrightarrow{YI} = -\overrightarrow{BO}, \overrightarrow{ZI} = -\overrightarrow{CO}$. Thus $\triangle ABC$ and $\triangle XYZ$ are homothetic, with a negative scale factor, which literally just means "insimilicenter." Since (ABC) is the circumcenter and (XYZ) is the incircle, as

$$\measuredangle GeAI = \measuredangle DAI = \measuredangle IAX = \measuredangle IAX_{55}$$

we get that X_{55} is the isogonal conjugate of Ge. We can do a similar argument for X_{56}, except we use the antipodes of X, Y, Z. □

Remark (A suprising connection with mixtilinear incircles)**.** By considering a $\sqrt{bc}$ inversion, the A-mixtilinear incircle is sent to the A-excircle. Let T_A be the point in which the A-mixtilinear incircle touches the circumcircle (ABC); then the image of T_A after a $\sqrt{bc}$ inversion is the A-extouch point. In other words, AT_A, BT_B, and CT_C concur at X_{56}. (A similar concurrency with the mixtilinear excircles is at X_{55}.)

Proposition 5.6.15. G, Fe, Ge^* are collinear, and H, Fe, Na^* are collinear.

Proof. Monge on incircle, circumcircle, and nine-point circle. □

5.6.6 X_{65}

- X_{65} is the orthocenter of the intouch triangle.

Proposition 5.6.16. X_{65} is the isogonal conjugate of the Schiffler point, denoted as Sc^*.

Proof. By Proposition 5.6.5, ASc, OI^a, BC are concurrent. Applying an extraverted version of Proposition 5.2.6 on A, we get $A(D^a)', OI^a, BC$ are concurrent, where $'$ represents reflection over E^aF^a. Thus we just need to prove that $A(D^a)', AX_{65}$ are isogonal conjugates in $\angle BAC$. But this is because $\triangle AEF \cap X_{65} \overset{+}{\sim} \triangle AF^aE^a \cap D'$. Thus we represent X_{65} as Sc^*. □

Proposition 5.6.17. Point Sc^* lies on OI, with

$$\frac{ScI}{IO} = \frac{r}{R}$$

Proof. We have $\triangle DEF \cup I \cup Sc^* \overset{+}{\sim} \triangle N_aN_bN_c \cap O \cap I$, thus I, Sc^*, O concurrent and $\frac{Sc^*I}{IO} = \frac{ID}{ON_a} = \frac{r}{R}$. □

Proposition 5.6.18. O is the midpoint of Ge^* and Sc^*, further, we have,

$$(I, Be; Ge^*, Sc^*) = -1.$$

Proof. $\frac{ScO}{IO} = \frac{r+R}{R} = \frac{IO}{Ge^*O}$ □

Proposition 5.6.19.

$$GeNa^* \cap Ge^*Na = Sc;$$

$$GeNa \cap Ge^*Na^* = Sc^*.$$

Proof. Since

$$\frac{GSc}{ScO} \cdot \frac{OGe^*}{Ge^*I} \cdot \frac{INa}{NaG} = \frac{2r}{3R} \cdot \frac{R}{r} \cdot \left(-\frac{3}{2}\right) = -1,$$

By Menelaus's theorem, Sc, Ge^*, Na are collinear. Similarly, by Proposition 5.6.3, we have

$$\frac{LSc}{ScO} \cdot \frac{ONa^*}{Na^*I} \cdot \frac{IGe}{GeL} = \left(-\frac{12R+6r}{3R}\right) \cdot \left(-\frac{R}{r}\right) \cdot \left(-\frac{r}{4R+2r}\right) = -1,$$

and thus Sc, Na^*, Ge are similarly collinear. By Corollary 7.4.5, a theorem on isoconjugation, we easily have that $Sc^* = GeNa \cap Ge^*Na^*$, but we can also give an elementary proof with previous material; we already know that $OI = Ge^*Na^*$, so by Proposition 5.6.3, Proposition 5.6.7, Proposition 5.6.17 combined, we get that

$$\frac{LSc}{ScO} \cdot \frac{ONa^*}{Na^*I} \cdot \frac{IGe}{GeL} = \left(-\frac{12R+6r}{3R}\right) \cdot (-\frac{R}{r}) \cdot \left(-\frac{r}{4R+2r}\right) = -1$$

and thus we get Sc, Na^*, Ge are collinear by Menelaus. $\square$

Proposition 5.6.20. Line $\overline{SpSc^*}$ is the complement of ISc.

Proof. Since $I^{\complement} = Sp$, we just need to prove that $Sc * Sc^{\complement}$ is parallel to ISc. By Proposition 5.6.4 and Proposition 5.6.17,

$$\frac{Sc * I}{IO} = \frac{r}{R} = \frac{3}{2} \cdot \frac{GSc}{ScO} = \frac{Sc^{\complement}Sc}{ScO},$$

so $Sc * Sc^{\complement} \parallel ISc$. $\square$

5.6.7 X_{69}

- X_{69} is the anticomplement $K^{\reflectbox{$\complement$}}$ of the symmedian point.

Proposition 5.6.21. X_{69} is the isotomic conjugate of H.

Proof. Let the isotomic conjugate of H be H'. Let M be the midpoint of $\overline{AH_A}$. Then by Proposition 5.3.3, we get that $A, H', M^{\reflectbox{$\complement$}}$ are collinear, and we can check that $M^{\reflectbox{$\complement$}}$ lies on BC. Furthermore,

$$\frac{BM^{\reflectbox{$\complement$}}}{M^{\reflectbox{$\complement$}}C} = \frac{M_bM}{MM_c} = \frac{CH_a}{H_aB}$$

tells us that $M^{\text{Ɔ}}$ is the reflection of H_a across the midpoint of $\overline{BC}$, and thus A, X_{69}, H' are collinear. Now note that every step in our proof can be done with a different vertex, so we can apply it cyclically. We then have

$$H' = AX_{69} \cap BX_{69} \cap CX_{69} = X_{69}.$$

□

Proposition 5.6.22. *Ge, Na, X_{69} are collinear.*

Proof. We take the complement. Then this collinearity is equivalent to proving $Mt = Ge^{\mathsf{C}}, I = Na^{\mathsf{C}}, K = (X_{69})^{\mathsf{C}}$ are collinear, which finishes by Proposition 5.4.6. Alternatively, we can prove that

$$B(Na, H'; Ge, C) = C(Na, H'; Ge, B).$$

Let B' be the reflection of B about N_b. Since N_b is the midpoint of $\overline{II^b}$, we have

$$\begin{aligned} B(Na, H'; Ge, C) &= (E', BH' \cap CA; E, C) \\ &= (I^b, B'; I, \infty_{\perp CA}C \cap BI) \\ &= (I, B; I^b, \infty_{\perp CA}A \cap BI), \end{aligned}$$

and similarly we get that

$$C(Na, H'; Ge, B) = (I, C; I^c, \infty_{\perp AB}A \cap CI).$$

Thus we only need to prove that $BC, I^bI^c, \overline{(\infty_{\perp CA}A \cap BI))(\infty_{\perp AB}A \cap CI))}$ are concurrent. This is equivalent to proving that

$$I^bI^c \cap BC, (\infty_{\perp CA}A \cap BI)), (\infty_{\perp AB}A \cap CI))$$

are collinear, but this is exactly the orthotransversal of A with respect to $\triangle BIC$!!

□

5.7 $X_n, n \geq 99$

This will be completed later. (This is what the original book says.)

5.8 Others

5.8.1 X_{19}

Definition 5.8.1. Let ℓ_a be the common internal tangent to ω and ω^a that is not line $\overline{BC}$. Symmetrically, define ℓ_b and ℓ_c. Let ℓ'_a be the common external tangent to ω^b and ω^c that is not line $\overline{BC}$. Again, symmetrically define ℓ'_b and ℓ'_c. We denote $\triangle \ell_a\ell_b\ell_c$ as the **intangents triangle**($\triangle T_aT_bT_c$) and $\triangle \ell'_a\ell'_b\ell'_c$ as the **extangents triangle**($\triangle T'_aT'_bT'_c$).

Obviously, the intangents triangle's three sides $\overline{T_bT_c}, \overline{T_cT_a}, \overline{T_aT_b}$ are the reflections of $\overline{BC}$ over $\overline{II^a}$, etc. Similarly, the extangents triangle's three sides $\overline{T'_bT'_c}, \overline{T'_cT'_a}, \overline{T'_aT'_b}$ are the reflections of $\overline{BC}$ over $\overline{I^bI^c}$, etc.

Proposition 5.8.2. The orthic triangle $\triangle H_aH_bH_c$ is homothetic to both the intangents triangle $\triangle T_aT_bT_c$ and the extangents triangle $\triangle T_aT_bT_c$ (note that the homothety centers are different).

Proof. We proceed with line arguments: we want to prove that $\measuredangle(H_bH_c) = \measuredangle(\ell_a) = \measuredangle(\ell'_a)$. Since

$$\begin{aligned}\measuredangle H_bH_c &= \measuredangle AB + \measuredangle AC - \measuredangle BC,\\ \measuredangle \ell_a &= 2\cdot\measuredangle II^a - \measuredangle BC = \measuredangle AB + \measuredangle AC - \measuredangle BC,\\ \measuredangle \ell'_a &= 2\cdot\measuredangle I^bI^c - \measuredangle BC = \measuredangle AB + \measuredangle AC - \measuredangle BC,\end{aligned}$$

and we get the desired homotheties. $\square$

Now let's look at the two centers of homothety:

- X_{19} is the **Clawson** point, defined as the homothety center between the extangents triangle and the orthic triangle, typically written as Cl;
- X_{33} is the homothety center between the intangents triangle and the orthic triangle.

Proposition 5.8.3. $T_aI \perp BC$, and T_aI bisects $\angle T_bT_aT_c$. Further,$T'_aI^a \perp BC$, and it bisects $\angle T'_bT'_aT'_c$.

Proof. First we characterize T'_a. Let Y', Z' respectively be the intersections of $\overline{T'_cT'_a}, \overline{T'_aT'_b}$ with $\overline{BC}$. Then

$$\overline{Y'D'} = \overline{AF_a} = \overline{E_aA} = \overline{D'Z'},$$

so D' is the midpoint of $\overline{Y'Z'}$. This tells us that I^a lies on the perpendicular bisector of $\overline{Y'Z'}$, and also

$$T'_aY' + T'_aZ' = (2I^cI^a - CA) + (2I^aI^b - AB) = 2BC = 2Y'Z'$$

tells us that T'_a also lies on this perpendicular bisector. Thus we get that $\overline{T'_aI^a}$ is perpendicular to $\overline{BC}$ and that it bisects $\angle T'_bT'_aT'_c = \angle Y'T'_aZ'$.

The proof for T_a is analogous. □

Note that $I^a\infty_{\perp BC}, I^b\infty_{\perp CA}, I^c\infty_{\perp AB}$ concur at Be, and thus we get

Corollary 5.8.4.

$$\triangle H_aH_bH_c \cap H \overset{+}{\sim} \triangle T_aT_bT_c \cap I \overset{+}{\sim} \triangle T'_aT'_bT'_c \cap Be.$$

Notably, this gets us that the Clawson point lies on the Proposition 5.6.8 (H, Mt, Sp, Cl, Be collinear), and that H, I, X_{33} are collinear.

Proposition 5.8.5. The circle centered at Be tangent to the three sides of $\triangle T'_aT'_bT'_c$ exists (call it ω_{Be}), and has radius $2R + r$.

Proof. Let D_{Be} be the tangency point of $T'_bT'_c$ to ω_{Be}, and let Be^a be the reflection of Be over I^bI^c. Then by the A-version of Proposition 5.6.6, we have that $Be^aD \perp BC$, so we can then reflect $BeD_{Be} \perp T'_bT'_c$ over I^bI^c to get that D is the reflection of D_{Be} over I^bI^c. Thus,

$$BeD_{Be} = Be^aD = |\overrightarrow{Be^aI} + \overrightarrow{ID}| = |2\overrightarrow{ON_a} + \overrightarrow{ID}| = 2R + r.$$

Note additionally that this also gets us that $\overrightarrow{OA} \parallel \overrightarrow{BeD_{Be}} \parallel -\overrightarrow{ID_I}$, where D_I is the tangency point of ω and T_bT_c. □

Lemma 5.8.6. Let Y, Z, Y', Z' respectively be the intersection points of T_cT_a, T_aT_b, $T'_cT'_a$, $T'_aT'_b$ with BC. Then the circumcenter of $\triangle AYZ'$ is I^b, and the circumcenter of $\triangle AY'Z$ is I^c.

Proof. This is because $\overline{Z'A}$ and $\overline{AY}$ have I^aI^b and II^b respectively as perpendicular bisectors, so they concur at I^b. (Similarly for $\triangle AY'Z$.) □

Lemma 5.8.7. The intangents triangle $\triangle T_aT_bT_c$, the extangents triangle $\triangle T'_aT'_bT'_c$, the incentral triangle (cevian triangle of I) $\triangle X_aX_bX_c$, and the tangential triangle $\triangle T_AT_BT_c$, are all perspective. Furthermore, their perspector is Ge^*, the isogonal conjugate of Ge.

Proof. We have that

$$\begin{aligned} \triangle T_AT_BT_C \cup O \cup \triangle ABC &\overset{+}{\sim} \triangle T_aT_bT_c \cup I \cup \triangle D_IE_IF_I \\ &\overset{+}{\sim} \triangle T'_aT'_bT'_c \cup Be \cup \triangle D_{Be}E_{Be}F_{Be}. \end{aligned}$$

Additionally,

$$\frac{Ge^*O}{Ge^*Be} = \frac{R}{2R+r} = \frac{\overrightarrow{OA}}{\overrightarrow{BeD_{Be}}}, \frac{Ge^*I}{Ge^*Be} = \frac{-r}{2R+r} = \frac{\overrightarrow{ID_I}}{\overrightarrow{BeD_{Be}}}$$

tells us that $Ge^* \in OI$ as the homothetic center of $\triangle T_A T_B T_c$, $\triangle T_a T_b T_c$, $\triangle T'_a T'_b T'_c$. By Lemma 5.8.6, we have that X_a lies on the radical axis of (AYZ') and $(AY'Z)$, which is just $A\infty_{\perp I^b I^c}$, and thus

$$X_aY \cdot X_aZ' = X_aY' \cdot X_aZ \implies \frac{X_aY}{X_aZ} = \frac{X_aY'}{X_aZ'}$$

combined with the fact that $\triangle T_a Y Z \overset{+}{\sim} \triangle T'_a Y' Z'$ gives us that T_a, T'_a, X_a are collinear, and thus we get that Ge^* is the perspector of $\triangle T_a T_b T_c$, $\triangle T'_a T'_b T'_c$, $\triangle X_a X_b X_c$. $\square$

Since X_{25} is the homothetic center between $\triangle T_A T_B T_C$ and the orthic triangle $\triangle H_a H_b H_c$, by Common Perspectrix Implies Collinear Perspectors we have that

Corollary 5.8.8. Cl, X_{25}, X_{33}, Ge^* collinear.

Part II

The Deep End

Chapter 6

Basic Conic Theory

6.1 Definitions and Basic Properties

We first give a standard high-school definition for a conic section:

Definition 6.1.1. Let $\Omega \in \mathbb{R}^3$ be a circle with center O, and ℓ be the line through O perpendicular to the plane containing Ω. For any point $V \neq O$ on ℓ, the surface $\mathcal{S}$ formed by the union of all straight lines through V and some point $M \in \Omega$ is a **right circular cone**. Ω is the **directrix** of $\mathcal{S}$ and V is the **vertex** of $\mathcal{S}$.

Remark. This is really a double-napped cone, which is much more natural to work with for conic sections. Moreover, whenever a "cone" appears, assume it is a right circular cone.

Definition 6.1.2. On plane E, a curve $\mathcal{C}$ is a **conic** if there exists a cone $\mathcal{S}$ with vertex $V \notin E$ such that $\mathcal{C} = \mathcal{S} \cap E$.

Remark (Technical details)**.** This is the definition of a conic in the Euclidean plane. To define a conic on a projective plane E instead, we first define the cone $\overline{\mathcal{S}}$ in $\mathbb{P}^3 := \mathbb{P}^3_{\mathbb{R}}$ which contains $\mathcal{S}$ as well as the points of infinity on all possible lines VM. Then a conic on E is $\overline{\mathcal{S}} \cap E$.

In what follows, everything happens on the real projective plane $\mathbb{RP}^2 = \mathbb{R}^2 \cup \mathcal{L}_\infty$ and in $\mathbb{R}^3$ (unless the space explicitly is mentioned).

Definition 6.1.3. Let $\mathcal{C}$ be a conic and $\mathcal{L}_\infty$ be the real line at infinity. Then $\mathcal{C}$ is

(i) an **ellipse** if $|\mathcal{C} \cap \mathcal{L}_\infty| = 0$;

(ii) a **parabola** if $|\mathcal{C} \cap \mathcal{L}_\infty| = 1$;

(iii) a **hyperbola** if $|\mathcal{C} \cap \mathcal{L}_\infty| = 2$.

Here are some alternative characterizations for each of these types of conics.

Proposition 6.1.4 (Foci of Conics)**.** Let $\mathcal{C}$ be a conic in a plane. Then

1. $\mathcal{C}$ is an ellipse if and only if there exists points F_1, F_2 in the plane and $a \geq \overline{F_1F_2}/2$ such that
$$\mathcal{C} = \{P \mid \overline{F_1P} + \overline{F_2P} = 2a\};$$
2. $\mathcal{C}$ is a parabola if and only if there exists point F and line L such that
$$\mathcal{C} = \{P \mid \overline{FP} = d(L, P)\};$$
3. $\mathcal{C}$ is a hyperbola if and only if there exists points F_1, F_2 and $a < F_1F_2/2$ such that
$$\mathcal{C} = \{P \mid |\overline{F_1P} - \overline{F_2P}| = 2a\},$$

here F_1, F_2 or F are called the **foci** (plural of **focus**) of $\mathcal{C}$, while L is the **directrix** of $\mathcal{C}$ (if applicable).

Proof. We will only prove the ellipse case, the other cases are similar.

By the definition of a conic, there exists a cone $\mathcal{S}$ and plane E such that $\mathcal{C} = \mathcal{S} \cap E$. Let $\mathcal{B}_1, \mathcal{B}_2$ be spheres tangent to both $\mathcal{S}$ and E, and let $\omega_1 = \mathcal{S} \cap \mathcal{B}_1, \omega_2 = \mathcal{S} \cap \mathcal{B}_2, F_1 = E \cap \mathcal{B}_1, F_2 = E \cap \mathcal{B}_2$. For any point $P \in \mathcal{S}$, let A_1, A_2 be the intersections of PV with ω_1 and ω_2 respectively, then clearly the length $\overline{A_1A_2}$ is fixed. Take $a = \overline{A_1A_2}/2$, then
$$\overline{F_1P} + \overline{F_2P} = \overline{A_1P} + \overline{A_2P} = \overline{A_1A_2} = 2a$$
so $\mathcal{C} \subseteq \{P \mid \overline{F_1P} + \overline{F_2P} = 2a\}$. If $P \in E \setminus \mathcal{C}$ satisfies $\overline{F_1P} + \overline{F_2P} = 2a$, then if ray F_1P intersects $\mathcal{S}$ at $P' \in \mathcal{C}$, then
$$\overline{F_1P} + \overline{F_2P} = 2a = \overline{F_1P'} + \overline{F_2P'} \implies \overline{F_2P} = \overline{F_2P'} \pm \overline{PP'}$$
so F_2, P, P' are collinear with P and P' on the same side of F_2, contradiction, hence $\mathcal{C} = \{P \mid \overline{F_1P} + \overline{F_2P} = 2a\}$.

As such, the foci are in general the touch points of $\mathcal{B}_1, \mathcal{B}_2$ with the plane. □

Remark. A video form of the proof can be found at here.

For now, We will use without proof the fact that all (non-degenerate) conics are smooth (as in - a unique tangent line exists at every point on the conic. a proof can be found later in Definition 6.2.1), and continue using notations for circle tangency-related things for conics:

(i) $\mathbf{T}\mathcal{C}$ is the set of all tangents of $\mathcal{C}$;

(ii) $\mathbf{T}_P\mathcal{C}$ is the tangent to $\mathcal{C}$ at P, where $P \in \mathcal{C}$;

(iii) $\mathbf{T}_\ell\mathcal{C}$ is the point of tangency of $\mathcal{C}$ and ℓ, where $\ell \in \mathbf{T}\mathcal{C}$.

We then have that $\mathbf{T}\mathcal{C} = \{\mathbf{T}_P\mathcal{C} | P \in \mathcal{C}\}, \mathcal{C} = \{\mathbf{T}_\ell\mathcal{C} | \ell \in \mathbf{T}\mathcal{C}\}$.

Proposition 6.1.5. Let $\mathcal{C}$ be a conic and ℓ be a line. Then ℓ intersects $\mathcal{C}$ at exactly one point if and only if $\ell \in \mathbf{T}\mathcal{C}$.

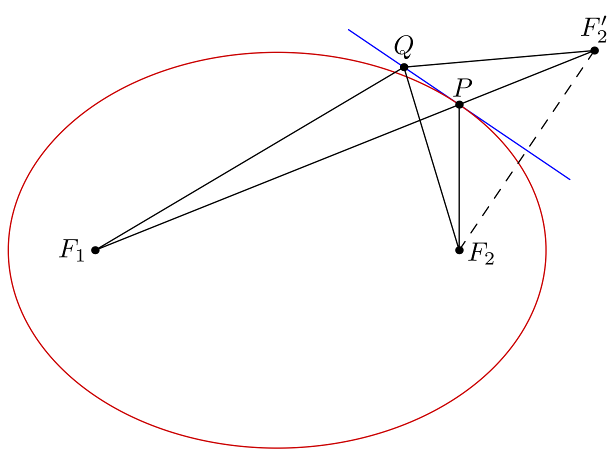
Proof. As before, let $\mathcal{C} = \mathcal{S} \cap E$ where E is a plane and $\mathcal{S}$ is a cone with directrix Ω. Let E_1 be the plane containing ℓ and the vertex of $\mathcal{S}$, E_2 be the plane containing Ω, then by projecting through V, it follows that

$$1 = |\ell \cap \mathcal{C}| = |E \cap E_1 \cap \mathcal{S}| \iff 1 = |(E_1 \cap E_2) \cap \Omega| = |E_1 \cap E_2 \cap \mathcal{S}|.$$

hence we may assume $\mathcal{C}$ is a circle, and the result clearly holds. □

Now we prove the so-called optical property of conics:

Proposition 6.1.6 (Optical Property for Parabolas/Hyperbolas)**.** Let $\mathcal{C}$ be a conic (which is not a parabola) with foci F_1, F_2. Then for any $P \in \mathcal{C}$, the tangent $\mathbf{T}_P\mathcal{C}$ bisects $\angle F_1PF_2$.



Proof. We only prove the case where $\mathcal{C}$ is an ellipse, the hyperbola case is similar. Let T'_P be the external angle bisector of $\angle F_1PF_2$, F'_2 be the reflection of F_2 across T'_P, then F_1, P, F'_2 are collinear. So for any $Q \neq P$ on T'_P,

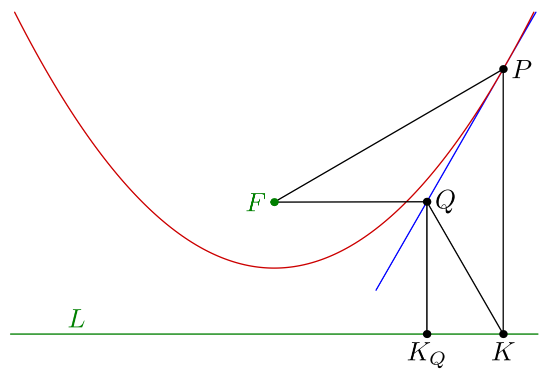
$$\overline{F_1Q} + \overline{F_2Q} = \overline{F_1Q} + \overline{F'_2Q} > \overline{F_1F'_2} = \overline{F_1P} + \overline{F'_2P} = \overline{F_1P} + \overline{F_2P}$$

and so $Q \notin \mathcal{C}$, hence $T'_P = \mathbf{T}_P\mathcal{C}$. □

Remark 6.1.7. There is a nice interpretation of this which is similar to the concept of Lagrange multipliers- if P is the point on line ℓ that minimizes the sum of distances F_1, F_2, then we have the reflection of F_2 across ℓ has F_1, P, F_2 collinear, so lines PF_1 and PF_2 are reflections across line ℓ. Then consider the set of conics with F_1, F_2 as foci - these are the "level curves" of $PF_1 + PF_2$. As we slowly move conics in this set out, the first one in this set that touches ℓ will be tangent to ℓ, however from before we know that the tangency point must be P.

And here's the optical property for a parabola:

Proposition 6.1.8 (Optical Property for Parabolas)**.** Let $\mathcal{P}$ is a parabola with focus F and directrix L. Let P be a point on $\mathcal{P}$. Let K be the projection of P onto L. Then $\mathbf{T}_P\mathcal{P}$ is the internal angle bisector of $\angle FPK$ and also the perpendicular bisector of FK.



Proof. Let T'_P be the internal angle bisector of $\angle FPK$, from $FP = KP$ we know T'_P is the perpendicular bisector of FK. For any point $Q \neq P$ on T'_P, let K_Q be the projection of Q onto L, then we have

$$\overline{K_Q Q} < \overline{KQ} = \overline{FQ}$$

so $Q \notin \mathcal{P}$, hence $T'_P = \mathbf{T}_P\mathcal{P}$. $\square$

Remark. Considering the above properties, we can view the other "focus" of a parabola as the point of infinity on the parabola, so that the optical property still holds.

Below, we consider parabolas as the limiting case for the definitions.

Definition 6.1.9. For any conic $\mathcal{C}$ with foci F_1, F_2, the **center** O of $\mathcal{C}$ is defined as the midpoint of F_1F_2.

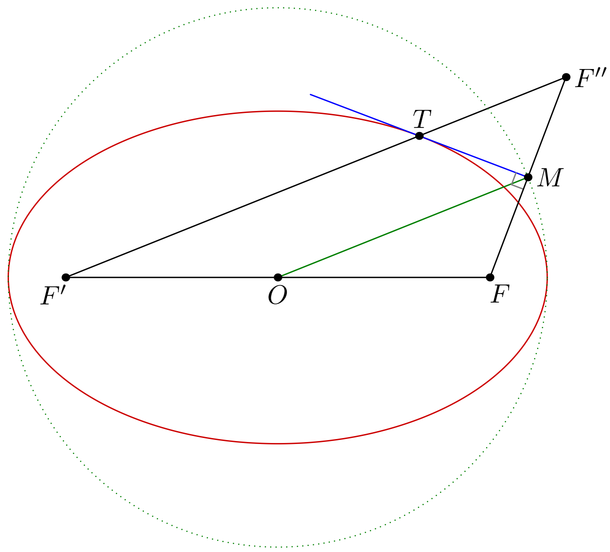
From the definition of foci, it follows that:

Proposition 6.1.10. For any non-parabola conic $\mathcal{C}$ with center O, $\mathcal{C}$ is symmetric about O.

Definition 6.1.11. For a conic $\mathcal{C}$, we define the **major axis** of $\mathcal{C}$ as the line segment whose endpoints are the intersections of $\mathcal{C}$ and F_1F_2, where F_1, F_2 are the foci of $\mathcal{C}$.

Proposition 6.1.12. Let F be a focus of conic $\mathcal{C}$, and Γ be the circle whose diameter of the major axis of $\mathcal{C}$. Then for any $\ell \in \mathbf{T}\mathcal{C}$, the projection of F onto ℓ lies on Γ.

Proof. In the (limiting) case where $\mathcal{C}$ is a parabola, it is not hard to show that this is equivalent to Optical Property for Parabolas: In particular, Γ is the tangent to $\mathcal{C}$ which is parallel to the directrix.



We will prove the result for the ellipse case; the hyperbola case is similar. Let F' be the other focus of $\mathcal{C}$, O be the center of $\mathcal{C}$, and $T = \mathbf{T}_{\mathcal{C}}\ell$. By the optical property, $F'T$ passes through the reflection F'' of F across ℓ, so the projection M of F onto ℓ is the midpoint of FF'', and so

$$\overline{OM} = \frac{1}{2} \cdot \overline{F'F''} = \frac{1}{2}\overline{(FT + F'T)}$$

but $\overline{FT + F'T}$ is the length of the major axis, so $M \in \Gamma$. □

Practice Problems

Problem 1 (Eccentricity)**.** Prove that $\mathcal{C}$ is a conic if and only if $\mathcal{C}$ is a circle or there exists point F, line L and $e > 0$ such that

$$\mathcal{C} = \{P \mid FP = e \cdot d(L, P)\}$$

where

(i) if $e < 1$, then $\mathcal{C}$ is an ellipse;

(ii) if $e = 1$, then $\mathcal{C}$ is a parabola;

(iii) if $e > 1$, then $\mathcal{C}$ is a hyperbola.

The circle case can be seen as the limiting case where $L = \mathcal{L}_\infty$ and $e = 0$, such that $e \cdot d(L, P)$ and F is the center of the circle. Similarly, if $e = \infty$, the conic is just a line.

Problem 2 (Isogonal property of ellipses)**.** Let F_1 and F_2 be the foci of conic $\mathcal{C}$. Let $\ell_1, \ell_2 \in \mathbf{T}\mathcal{C}$ and A be the intersection of ℓ_1 and ℓ_2. Prove that ℓ_1 and ℓ_2 are isogonal in $\angle F_1AF_2$.

Problem 3 (Director circle)**.** Given an ellipse $\mathcal{E}$, prove that the locus of points P such that the two tangents from P to $\mathcal{E}$ are perpendicular is a circle.

6.2 Cross-Ratio on Conics

We have already defined cross ratio for a circle (see Cross Ratio on a Circle), but now we want to extend this definition to conics (plus, conics are really just projected circles after all).

Remark (Technical Details)**.** Here, our conics are considered over $\mathbb{RP}^2$ exposition-wise, but everything here naturally carries over to $\mathbb{CP}^2$. This will remain true for the next few sections.

Definition 6.2.1. Let E_1 and E_2 be planes in the space $\mathbb{R}^3$, and let V be a point not on either plane. Define $\varphi : E_1 \to E_2$ as the map

$$\varphi(Q) = VQ \cap E_2$$

then φ is said to be the **perspective transformation** from planes E_1 to E_2 with center V.

Proposition 6.2.2. Let E_1, E_2 be planes and $\varphi : E_1 \to E_2$ be a perspective transformation, then for any line $\ell \in E_1$, $\varphi(\ell)$ is also a straight line.

Proof. Let V be the center of φ as above, and let G be the plane containing V and ℓ. Then

$$\varphi(\ell) = \{\varphi(Q) \mid Q \in \ell\} = \{PQ \cap E_2 \mid Q \in \ell\} = \{PQ \mid Q \in \ell\} \cap E_2 = G \cap E_2,$$

which is a straight line as it is the intersection of two distinct planes. $\square$

Theorem 6.2.3. Let E_1, E_2 be planes and $\varphi : E_1 \to E_2$ be a perspective transformation with center V, then for four collinear points $P_1, P_2, P_3, P_4 \in E_1$, we have that the cross ratios $(P_\bullet) := (P_1, P_2; P_3, P_4)$ and $(\varphi(P_\bullet)) := (\varphi(P_1), \varphi(P_2); \varphi(P_3), \varphi(P_4))$ are equal.

Proof. Since ℓ and $\varphi(\ell)$ are coplanar, we have

$$(P_\bullet) = V(P_\bullet) = V(\varphi(P_\bullet)) = (\varphi(P_\bullet)).$$

□

Theorem 6.2.4. Let E_1, E_2 be planes and $\varphi : E_1 \to E_2$ be a perspective transformation with center V, then for concurrent lines $\ell_1, \ell_2, \ell_3, \ell_4 \in E_1$, we have $(\ell_\bullet) = (\varphi(\ell_\bullet))$.

Proof. Let E_3 be a plane containing V but not containing the concurrency point of $\ell_\bullet$. Let $R_i = \ell_i \cap E_3$, then

$$(\ell_\bullet) = (R_\bullet) = (\varphi(R_\bullet)) = (\varphi(\ell_\bullet)).$$

□

From the definition of a conic, we also get:

Proposition 6.2.5. Let E be a plane and $\mathcal{C} \subset E$ be a conic. Then there exists a plane E_1 and a circle $\Omega \subset E_1$ such that there is a perspective transformation $\varphi : E \to E_1$ such that $\varphi(\mathcal{C}) = \Omega$.

To prove the next result, we will (for now) accept without proof that given five distinct noncollinear points, there exists a unique conic through them. Similarly, given five distinct nonconcurrent lines, there exists a unique conic tangent to these five lines. In lieu of this, we will use $(P_1P_2P_3P_4P_5)$ to denote the conic through P_1, P_2, P_3, P_4, P_5 and $(\ell_1\ell_2\ell_3\ell_4\ell_5)$ to denote the conic tangent to $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$ when there is no risk of ambiguity.

Theorem 6.2.6 (Fundamental Theorem of Conic Sections)**.** Let $P_1, P_2, P_3, P_4, A, A'$ be points in the plane E, no three collinear. Then $P_1, P_2, P_3, P_4, A, A'$ lie on a common conic or are **conconic** if and only if $A(P_\bullet) = A'(P_\bullet)$.

Proof. The main idea is to take a perspective transformation sending the conic to a circle.

Let $\mathcal{C} = (P_1P_2P_3AA')$. Then there exists plane E_1 and circle $\Omega \in E_1$ and a perspective transformation $\varphi : E \to E_1$ such that $\varphi(\mathcal{C}) = \Omega$.

($\Rightarrow$) Given $P_4 \in \mathcal{C}$, this implies that $\varphi(P_4) \in \Omega$, so

$$A(P_\bullet) = \varphi(A)(\varphi(P_\bullet)) = \varphi(A')(\varphi(P_\bullet)) = A'(P_\bullet).$$

as desired.

($\Leftarrow$) Define $P_i' = P_i$ for $i = 1, 2, 3$, $P_4' = AP_4 \cap \mathcal{C} \setminus \{A\}$, then by ($\Rightarrow$) we have

$$A'(P_\bullet') = A(P_\bullet) = A'(P_\bullet),$$

so $P_4' = A'P_4 \cap AP_4 = P_4$, so $P_1, P_2, P_3, P_4, A, A'$ are conconic.

□

The dual result is:

Theorem 6.2.7. Let $\ell_1, \ell_2, \ell_3, \ell_4, L, L'$ be lines in the plane, no three concurrent. Then $\ell_1, \ell_2, \ell_3, \ell_4, L, L'$ are tangent to a common conic if and only if $L(\ell_\bullet) = L'(\ell_\bullet)$.

The proof is similar. The above theorem guarantees us the existence of cross ratios on a conic.

Definition 6.2.8. Let P_1, P_2, P_3, P_4 be points on a conic $\mathcal{C}$. Then

$$(P_\bullet)_{\mathcal{C}} := (P_1, P_2; P_3, P_4)_{\mathcal{C}} := A(P_\bullet),$$

for any $A \in \mathcal{C}$, is the cross ratio of (P_1, P_2, P_3, P_4) with respect to $\mathcal{C}$.

Definition 6.2.9. Let $\ell_1, \ell_2, \ell_3, \ell_4$ be tangents to a conic $\mathcal{C}$. Then

$$(\ell_\bullet)_{\mathcal{C}} := (\ell_1, \ell_2; \ell_3, \ell_4)_{\mathcal{C}} := L(\ell_\bullet),$$

where $L \in \mathbf{T}\mathcal{C}$, is the cross ratio of $(\ell_1, \ell_2, \ell_3, \ell_4)$ with respect to $\mathcal{C}$.

Let's also bring the concept of harmonic quadrilaterals to conics:

Definition 6.2.10. For points P_1, P_2, P_3, P_4 on a conic $\mathcal{C}$, the quadrilateral $(P_1P_2)(P_3P_4)$ is said to be harmonic on $\mathcal{C}$ if and only if $(P_\bullet)_{\mathcal{C}} = -1$.

Similar to Proposition 2.2.11, we have:

Proposition 6.2.11. For points P_1, P_2, P_3, P_4 on a conic $\mathcal{C}$, let the tangents to $\mathcal{C}$ at P_1 and P_2 met at A. Then $(P_1P_2)(P_3P_4)$ is harmonic if and only if A lies on P_3P_4.

6.2.1 Conics are Degree-2 Curves

Let's now pay our debt of promising to prove something earlier – the fact that five points define a conic, and five tangent lines determine a conic. The proof will be in two parts.

This section freely uses basic linear algebra and calculus and ideas about homogeneous coordinates. Reading the first appendix is recommended if unfamiliar, and skipping this section is possible.

In this section, we will be using the convention of points as vertical 3×1 vectors, and lines as horizontal 1×3 matrices.

(i) Any five points (lines), no four collinear (concurrent), determine a unique degree-2 curve through them (tangent to them);

(ii) Conics are the same as non-degenerate degree-2 curves.

Here, a **degree-2 curve** is simply a curve defined on the projective plane $\mathbb{P}^2$ (with points in this plane represented as vertical vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$) by the homogeneous quadratic implicit function

$$F(x:y:z) := Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 0$$

This rewrites as $v^\intercal M_{\mathcal{C}} v = 0$ for vertical vector $v = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ and

$$M_{\mathcal{C}} = \begin{pmatrix} A & F & E \\ F & B & D \\ E & D & C \end{pmatrix}$$

For this to be solvable over the reals, it must be that the matrix $v^\intercal M_{\mathcal{C}} v$ is not always positive or negative (in linear algebra this is called being positive-definite or negative-definite (so $v^\intercal M_{\mathcal{C}} v$), and that $M_{\mathcal{C}}$ is not the zero matrix. Furthermore, $\mathcal{C}$ is degenerate (a double line) if and only if $\det M_{\mathcal{C}} = 0$.

Part One Let's first show (i) or any five points, no four collinear, determine a degree-2 curve through them. (This still holds if four are collinear, but this gives infinitely many curves).

Hence, WLOG assume P_1, P_2, P_3 are not collinear. Through setting a suitable projective frame, we can also assume

$$P_1 = [1:0:0], \quad P_2 = [0:1:0], \quad P_3 = [0:0:1].$$

Now, plugging this into the original equation of a degree-2 curve, we get $A = B = C = 0$. So all degree-2 curves through P_1, P_2, P_3 are of the form

$$f(x,y,z) = 2Dyz + 2Ezx + 2Fxy = 0.$$

Now if $P_4 = [x_1 : y_1 : z_1], P_5 = [x_2 : y_2 : z_2]$, we want to solve

$$\begin{cases} Dy_1z_1 + Ez_1x_1 + Fx_1y_1 = 0, \\ Dy_2z_2 + Ez_2x_2 + Fx_2y_2 = 0 \end{cases}$$

where D, E, F are not all zero. We know that this equation has three variables and thus must have a nonzero solution, and we want this solution set to be one dimensional (as (D, E, F) and $(\lambda D, \lambda E, \lambda F)$ describe the same line), which implies a unique cubic.

Note that by distincness $(y_1z_1, z_1x_1, x_1y_1), (y_2z_2, z_2x_2, x_2y_2)$ both aren't $(0, 0, 0)$. Define $Q_1 = [y_1z_1 : z_1x_1 : x_1y_1], Q_2 = [y_2z_2 : z_2x_2 : x_2y_2]$. If $Q_1 \neq Q_2$, then $[D : E : F]$ is the coefficients of the unique line Q_1Q_2 which finishes. Else, suppose $Q_1 = Q_2$ which means that

$$y_1z_1 = \lambda y_2z_2, z_1x_1 = \lambda z_2x_2, x_1y_1 = \lambda x_2y_2, \lambda \neq 0.$$

If one of $x_2, y_2, z_2 = 0$, WLOG say $x_2 = 0$, then $y_2z_2 \neq 0$. This tells us then that $x_1 = 0$, however, then P_2, P_3, P_4, P_5 are collinear, contradiction.

As such, we get that

$$\lambda = \frac{y_1}{y_2} \cdot \frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{x_1}{x_2} = \frac{x_1}{x_2} \cdot \frac{y_1}{y_2} \implies \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2} = \pm\sqrt{\lambda}$$

or $P_4 = P_5$, contradiction. This proves the first part of (i).

Next, let's show that five lines determine a conic, however this is effectively the same as the five line case: we can assign lines to points in $\mathbb{P}^2$

$$\ell_i : a_ix + b_iy + c_iz = 0 \longleftrightarrow \ell_i^\vee : [a_i : b_i : c_i].$$

Then ℓ_i having no four concur is the same as no four of $\ell_i^\vee$ being collinear, so there is a unique conic $\mathcal{C}^\vee$ through $\ell_i^\vee$. As such, it remains to find a map $\mathcal{C}^\vee \mapsto \mathcal{C}$ that maps the points on $\mathcal{C}^\vee$ to the tangents of $\mathcal{C}$.

Proposition 6.2.12. Given a conic $\mathcal{C} : Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 0$ and a point $P = [x_0 : y_0 : z_0]$ on it, then the tangent at P is $L : ax + by + cz = 0$ for

$$(a, b, c) = (Ax_0 + Fy_0 + Ez_0, Fx_0 + By_0 + Dz_0, Ex_0 + Dy_0 + Cz_0).$$

Proof. The proof requires some calculus: the tangent to the contour $f(x, y, z) = 0$ at $P = [x_0 : y_0 : z_0]$ is the line

$$\nabla \cdot (x, y, z) = \left(\frac{\partial f}{\partial x}\right)_P \cdot x + \left(\frac{\partial f}{\partial x}\right)_P \cdot y + \left(\frac{\partial f}{\partial z}\right)_P \cdot z = 0.$$

which gives the result by Euler's homogeneous function theorem.

A simpler way to find L (parameterizing (x, y) as $[x : y : 1]$) is to consider the line $\left(\frac{x_0}{z_0} - \frac{b}{c}t, \frac{y_0}{z_0} + \frac{a}{c}t\right), c = -(ax_0 + by_0)$ (for P that isn't the origin or at infinity), and then consider when the quadratic $f(x, y, 1) \coloneqq f(t)$ has a double root (by considering the discriminant). $\square$

As such, it follows that $\mathcal{C}^\vee$ is mapped to $\mathcal{C}$ by the map

$$[x : y : z] \mapsto [Ax + Fy + Ez : Fx + By + Dz : Ex + Dy + Cz] =: [a : b : c]$$

Let's consider when we can explicitly describe $\mathcal{C}$ as $g(a, b, c) = w^\intercal M_{\mathcal{C}^\vee} w = 0$ where w is a vertical vector in (a, b, c). By taking a change of basis, we get:

$$g(a, b, c) = w^\intercal M_{\mathcal{C}^\vee} w \iff f(a, b, c) = v^\intercal M_{\mathcal{C}} M_{\mathcal{C}^\vee} M_{\mathcal{C}} v$$

so $M_{\mathcal{C}^\vee} = M_{\mathcal{C}}^{-1}$, so this holds for nonsingular $M_{\mathcal{C}}$.

Since $(M^{-1})^{-1} = M$ holds for nonsingular matrices M, we get that $(\mathcal{C}^\vee)^\vee = \mathcal{C}$, and the map $\mathcal{C} \mapsto \mathcal{C}^\vee$ is bijective in this case. As such, if $\mathcal{C}^\vee$ is not degenerate, then the lines ℓ_i uniquely correspond to a conic $\mathcal{C}$. In the case of $\mathcal{C}^\vee$ being degenerate, then three lines, WLOG ℓ_1, ℓ_2, ℓ_3 concur at P. Then if $Q = \ell_4 \cap \ell_5$, then $\mathcal{C}$ can be taken as a double cover of the line PQ.

Part Two Finally, we need to show that quadratic curve are equivalent to conics.

By rotating and translating, we can WLOG assume E is the plane $z = 0$ or the points $[x : y : 0]$.

Now, take an arbitrary circular conic surface $\mathcal{S} \colon \mathbb{R}^3$ whose central circle O is the locus of

$$(x_0, y_0, z_0) + r\cos\theta \cdot \vec{e_1} + r\sin\theta \cdot \vec{e_2}$$

where $\vec{e_1}, \vec{e_2}$ are two unit orthogonal vectors. If we let $\vec{e_3} = \vec{e_1} \times \vec{e_2}$, then for some $s \neq 0$, we have the vertex V satisfies

$$V \coloneqq (x_1, y_1, z_1) = (x_0, y_0, z_0) + s \cdot \vec{e_3}.$$

If we let $U = \begin{pmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \end{pmatrix}$, then $\mathcal{S}$ is of points $(x : y : z)$ satisfying

$$a^2 + b^2 = \left(\frac{r}{s} \cdot c\right)^2, \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = U^{-1} \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix}$$

By scaling, we can assume that $s = 1$. Let $\vec{v} = \left(\begin{smallmatrix} x-x_1 \\ y-y_1 \\ z-z_1 \end{smallmatrix}\right)$ and $\vec{e} = \vec{e_3} = \left(\begin{smallmatrix} e_x \\ e_y \\ e_z \end{smallmatrix}\right)$. This means that

$$|\vec{v}|^2 = a^2 + b^2 + c^2 = (r^2 + 1)c^2 = (r^2 + 1)(\vec{e} \cdot \vec{v})^2$$

Shifting, we can also assume $x_1 = y_1 = 0$. Rotating about the z axis, we assume $e_y = 0$. Thus, through solving, the conic $\mathcal{S} \cap \mathcal{E}$ becomes

$$Ax^2 + By^2 + C + 2Dy + 2Ex + 2Fxy = 0$$

where

$$\begin{aligned} &A = 1 - (r^2+1)e_x^2, \quad B = 1, \quad C = z_1^2(1-(r^2+1)e_z^2) \\ &E = (r^2+1)e_x e_z z_1, \quad D = F = 0 \end{aligned} \tag{$\clubsuit$}$$

where $e_x^2 + e_z^2 = 1$ is the only restriction. Similarly, the projective version $\overline{\mathcal{S}} \cap \mathcal{E}$ thus becomes

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy = 0$$

Since any conic $\mathcal{C}$ can be expressed as $\overline{\mathcal{S}} \cap \mathcal{E}$ for some cone $\mathcal{S}$ and plane $\mathcal{E}$, every conic $\mathcal{C}$ is a degree-2 curve.

Let's now show that any degree-2 curve is a conic to finish. Now assume a non-degenerate degree-2 curve $\mathcal{C}$ is of the form

$$f(x,y,z) = Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 0.$$

We now try to find $\mathcal{S}$. Taking a rotation

$$(x,y) \mapsto (x',y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta),$$

we can find an angle θ such that the coefficient of $2x'y'$ which is

$$F' = (A-B)\sin(2\theta) + F\cos(2\theta) = 0$$

so we WLOG $F = 0$. Since $\mathcal{C}$ is nondegenerate, we may WLOG assume $B \neq 0$. Since we now want to solve ($\clubsuit$), by using the existence of foci, we want the foci to be on the x-axis. As such, by scaling, we can assume $B = 1, |A| \leq 1$, and by shifting $y \mapsto y + Dz$, we can assume $D = 0$ as well.

(i) If $A = 0$, through scaling, we can assume $B = 1$, so by taking a shift $y \mapsto y + Dz$.

(ii) If $A \neq 0$ and is the same sign as B, then we can WLOG assume $B = 1$ and $|A| \leq 1$.

(iii) If $A \neq 0$ and is a different sign, we can scale such that $B = 1$ and WLOG B, C are the same sign, so $C = 1$ holds as well.

In all three cases, we can shift $y \mapsto y + Dz$ and scale such that $D = 0$, which canels out the linear case.

The above three cases correspond to parabolas, ellipses, and hyperbolas respectively. Regardless of the case, we end up with the equation

$$Ax^2 + y^2 + Cz^2 + 2Exz = 0$$

and the x axis intersects $\mathcal{C}$ at two (potentially infinite) points X_1, X_2.

Remark (Justification for the intersections)**.** Why does this follow? In the first case, $A = 0$ so this is a linear equation with a definite solution (and the other intersection is at infinity). In the second and third case, we consider the discriminant $AC - E^2 = \det M_{\mathcal{C}} < 0$. For the second case, note that $M_{\mathcal{C}}$ is not positive definite, so by Sylvester's criterion, one of the principal minor determinants $A, A, AC - E^2$ can't be positive, so we are done since $A > 0$. For the third case, since $A < 0$ the discriminant must be negative.

If $A = 1$, then $\mathcal{C}$ is a circle, so it's obviously a conic section, if $A < 1$, then we can shift $x \mapsto x + a$ to make $X_1 = (0, 0)$ (since $X_1 \neq X_2$ we WLOG X_1 is not at infinity), so $C = 0$. It now remains to solve

$$\begin{cases} 1 - (r^2+1)e_x^2 = A, \\ z_1^2(1 - (r^2+1)e_z^2) = C = 0, \\ (r^2+1)e_x e_z z_1 = E, e_x^2 + e_z^2 = 1. \end{cases}$$

Since $z_1 \neq 0$, we obtain

$$\begin{aligned} e_z &= \frac{1}{\sqrt{r^2+1}}, e_x = \frac{r}{\sqrt{r^2+1}} \implies A = 1 - r^2 \implies r = \sqrt{1-A} \\ E &= (r^2+1)e_x e_z z_1 = \sqrt{1-A} z_1 \implies z_1 = \frac{E}{\sqrt{1-A}} \end{aligned}$$

This solves the above equation, so $\mathcal{C}$ is a conic.

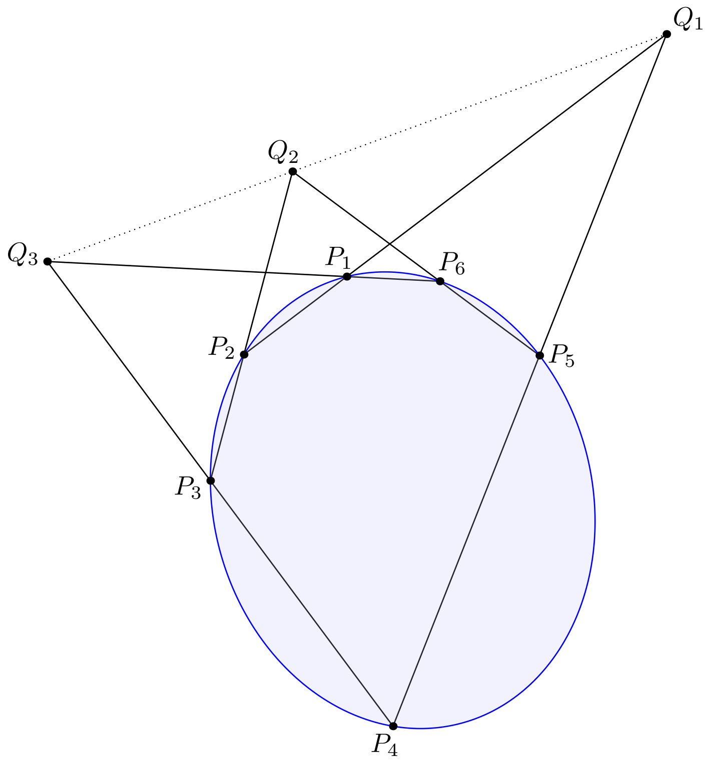
6.3 Various Conic Theorems

Now that we can deal with cross-ratio in conics, we can build on some older theorems. Namely, let's meet Pascal's Theorem and Brianchon's Theorem in their full conic forms.

Theorem 6.3.1 (Pascal's Theorem)**.** Given points $P_1, P_2, P_3, P_4, P_5, P_6$, with no three collinear, and all conconic, we have that

$$P_1P_2 \cap P_4P_5,\ P_2P_3 \cap P_5P_6,\ P_3P_4 \cap P_6P_1$$

are collinear.



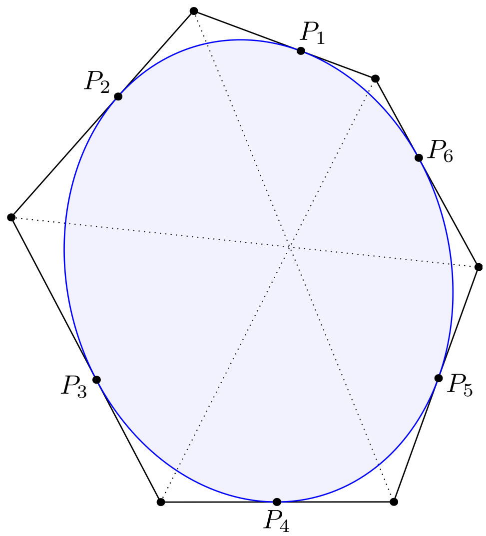
Proof. We can just use the same proof as seen at Pascal's Theorem. □

Once again, the dual of Pascal's is:

Theorem 6.3.2 (Brianchon's)**.** Given lines $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6$ with no three concurrent, and all six tangent to the same conic, then

$$(\ell_1 \cap \ell_2)(\ell_4 \cap \ell_5),\ (\ell_2 \cap \ell_3)(\ell_5 \cap \ell_6),\ (\ell_3 \cap \ell_4)(\ell_6 \cap \ell_1)$$

are concurrent.



The proof follows as the dual of Pascal's Theorem.

Remark. Like in the case of a circle, we can still take limiting cases of letting two points or lines coincide.

Theorem 6.3.3 (Carnot's Theorem)**.** Given $\triangle ABC$, and points (D_1, D_2), (E_1, E_2), and (F_1, F_2) on segments BC, CA, and AB, the following three conditions are equivalent:

(i) $D_1, D_2, E_1, E_2, F_1, F_2$ are conconic;

(ii) $AD_1, AD_2, BE_1, BE_2, CF_1, CF_2$ are tangent to one conic;

(iii) $\dfrac{BD_1}{D_1C} \cdot \dfrac{BD_2}{D_2C} \cdot \dfrac{CE_1}{E_1A} \cdot \dfrac{CE_2}{E_2A} \cdot \dfrac{AF_1}{F_1B} \cdot \dfrac{AF_2}{F_2B} = 1.$

Proof. We will prove (i) $\iff$ (iii), (ii) $\iff$ (iii) which suffices. First assume (iii). Let E_2F_1, F_2D_1, D_2E_1 intersect BC, CA, AB at X, Y, Z respectively. Menelaus gives

$$\frac{BX}{XC} \cdot \frac{CE_2}{E_2A} \cdot \frac{AF_1}{F_1B} = \frac{BD_1}{D_1C} \cdot \frac{CY}{YA} \cdot \frac{AF_2}{F_2B} = \frac{BD_2}{D_2C} \cdot \frac{CE_1}{E_1A} \cdot \frac{AZ}{ZB} = -1.$$

Then by Pascal's, $D_1, D_2, E_1, E_2, F_1, F_2$ is conconic if and only if X, Y, Z are collinear, if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YZ} \cdot \frac{AZ}{ZB} = -1 \iff \frac{BD_1}{D_1C} \cdot \frac{BD_2}{D_2C} \cdot \frac{CE_1}{E_1A} \cdot \frac{CE_2}{E_2A} \cdot \frac{AF_1}{F_1B} \cdot \frac{AF_2}{F_2B} = 1. \qquad \square$$

which gives the first equivalence.

Now, define $A_1 = BE_2 \cap CF_1, A_2 = BE_1 \cap CF_2, B_1 = CF_2 \cap AD_1, B_2 = CF_1 \cap AD_2, C_1 = AD_2 \cap BE_1, C_2 = AD_1 \cap BE_2$.

For part (ii), by Brianchon on $AC_1BA_1CB_1$, we get that an inconic of the hexagon exists if and only if $\triangle ABC$ and $\triangle A_1B_1C_1$ are perspective.

Let the trilinear pole of line $\overline{XYZ}$ be the point P, with cevian triangle $X^\vee Y^\vee Z^\vee$. so $(B, C; X, X^\vee) = -1$. By harmonic bundles in quadrilateral BF_1E_2C, we have that

$$A(B, C; A_1 := BE_2 \cap CF_1, X) = -1$$

. Thus AA_1 passes through $X^\vee$, and also passes through P. As such, it follows cyclically to get that $\triangle A_1B_1C_1$ is perspective with $\triangle ABC$ with perspector at P.

Definition 6.3.4. Given $\triangle ABC$,

(i) We say that a conic $\mathcal{C}$ is a **circumconic** of $\triangle ABC$ if $A, B, C \in \mathcal{C}$;

(ii) We say that a conic $\mathfrak{c}$ is an **inconic** of $\triangle ABC$ if $BC, CA, AB \in \mathbf{T}_{\mathfrak{c}}$.

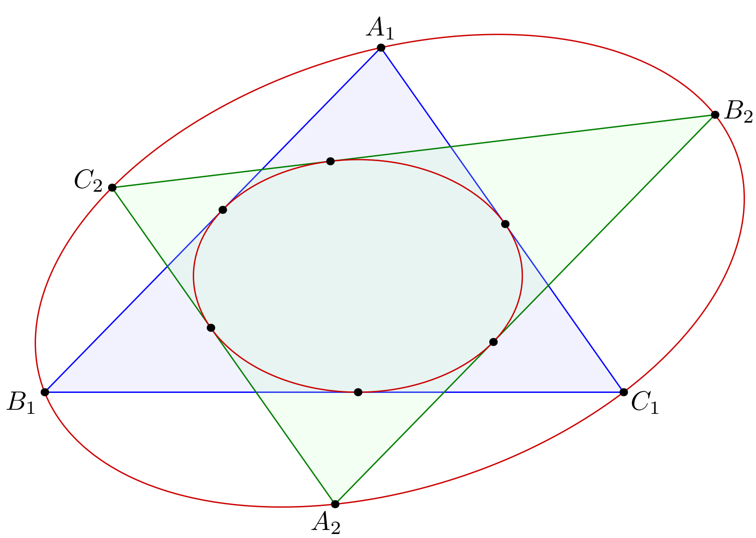
Much like the circumcircle, we have for a conic:

Definition 6.3.5. (i) Given $\triangle ABC$ with circumconic $\mathcal{C}$, and any point P, we define with P and $\triangle ABC$ the **$\mathcal{C}$-cevian triangle** with $\triangle(AP \cap \mathcal{C})(BP \cap \mathcal{C})(CP \cap \mathcal{C})$.

(ii) Given $\triangle ABC$ with inconic $\mathfrak{c}$, and any line ℓ, we define with ℓ and $\triangle ABC$ the **$\mathfrak{c}$-cevian triangle** with $\triangle((BC \cap \ell)\mathfrak{c})((CA \cap \ell)\mathfrak{c})((AB \cap \ell)\mathfrak{c})$ where $(BC \cap \ell)\mathfrak{c})$, where $(BC \cap \ell)\mathfrak{c}$ denotes the tangent from $BC \cap \ell$ to $\mathfrak{c}$ that isn't BC and so forth.

The next theorem is stated as the special case when $n = 3$, but that is when it is most useful.

Theorem 6.3.6 (Poncelet's Porism)**.** Given two triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ on the plane, $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ share a circumconic if and only if they share an inconic.



Proof. Notice that

$$A_1(B_1, C_1; B_2, C_2) = (B_2C_2)(A_1B_1, C_1A_1; A_2B_2, C_2A_2),$$
$$A_2(B_2, C_2; B_1C_1) = (B_1C_1)(A_2B_2, C_2A_2; A_1B_1, C_1A_1),$$

then $\triangle A_1B_1C_1, \triangle A_2B_2C_2$ share a circumconic if and only if

$$\begin{aligned} &A_1(B_1, C_1; B_2, C_2) = A_2(B_2, C_2; B_1, C_1) \\ \iff &(B_2C_2)(A_1B_1, C_1A_1; A_2B_2, C_2A_2) = (B_1C_1)(A_1B_1, C_1A_1; A_2B_2, C_2A_2) \end{aligned}$$

which is true if and only if $\triangle A_1B_1C_1, \triangle A_2B_2C_2$ share an inconic by Theorem 6.2.7. $\square$

Let us recall that a complete quadrangle $q = (P_1, P_2, P_3, P_4)$'s **cevian triangle** is the triangle with vertices

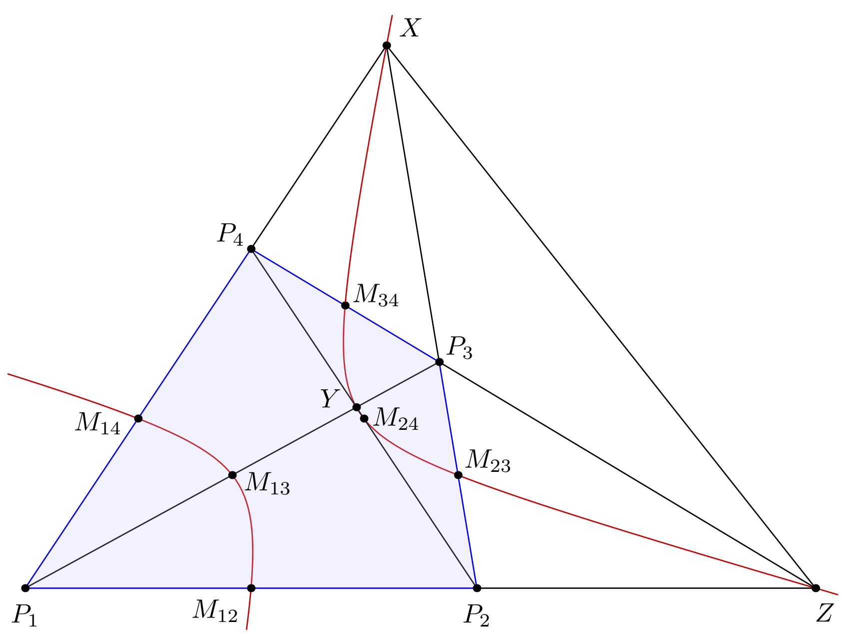
$$P_2P_3 \cap P_1P_4,\ P_3P_1 \cap P_2P_4,\ P_1P_2 \cap P_3P_4$$

so it's actually possible to define P_4 completely from $\triangle P_1P_2P_3$ and the cevian triangle.

Theorem 6.3.7 (Nine-Point Conic)**.** For complete quadrangle $q = (P_1, P_2, P_3, P_4)$, define M_{ij} to be the midpoint of $\overline{P_iP_j}$, and let $X \coloneqq P_2P_3 \cap P_1P_4, Y \coloneqq P_3P_1 \cap P_2P_4, Z \coloneqq P_1P_2 \cap P_3P_4$ be the vertices of the cevian triangle. Then

$$M_{23}, M_{14}, M_{31}, M_{24}, M_{12}, M_{34}, X, Y, Z$$

are conconic.



We can actually extend this:

Theorem 6.3.8 (Extended Nine-Point Conic)**.** Define $q = (P_1, P_2, P_3, P_4)$ and XYZ as above. Take an arbitrarily line ℓ not containing two points of q, and define $Q_{ij} = P_iP_j \cap \ell$. Choose R_{ij} on P_iP_j such that

$$(P_i, P_j; Q_{ij}, R_{ij}) = -1$$

then

$$R_{23}, R_{14}, R_{31}, R_{24}, R_{12}, R_{34}, X, Y, Z$$

lie on one conic.

Proof. By repeated Ceva and Menelaus we have

$$\frac{P_2X}{XP_3} \cdot \frac{P_3Y}{YP_1} \cdot \frac{P_1Z}{ZP_2} \cdot \frac{P_2R_{23}}{R_{23}P_3} \cdot \frac{P_3R_{31}}{R_{31}P_1} \cdot \frac{P_1R_{12}}{R_{12}P_2} = -\frac{P_2Q_{23}}{Q_{23}P_3} \cdot \frac{P_3Q_{31}}{Q_{31}P_1} \cdot \frac{P_1Q_{12}}{Q_{12}P_2} = 1,$$

and then by Carnot's theorem we get that $X, Y, Z, R_{23}, R_{31}, R_{12}$ are conconic on some conic $\mathcal{C}$. Note that for pairwise distinct i, j, k, Q_{jk}, R_{ij}, R_{ik} are collinear by Ceva's. Therefore from

$$R_{31}(Y, Z; R_{23}, R_{14}) \stackrel{\ell}{=} Z(Q_{31}, R_{31}; Q_{12}, Q_{34}) = (Q_{31}, R_{31}; P_1, P_3) = -1$$

$$R_{12}(Y, Z; R_{23}, R_{14}) \stackrel{\ell}{=} Y(R_{12}, Q_{12}; Q_{31}, Q_{24}) = (R_{12}, Q_{12}; P_1, P_2) = -1$$

we know that $R_{14} \in \mathcal{C}$, and analogously we have $R_{24}, R_{34} \in \mathcal{C}$, which finishes. $\square$

Remark. Taking P_1, P_2, P_3, P_4 as an orthocentric system and ℓ as the line at infinity gives the nine-point conic as just the familiar nine-point circle.

This extension is also the natural generalization obtained from taking a homography of the nine-point conic, sending $\mathcal{L}_\infty$ to ℓ.

The dual of this theorem is also useful:

Theorem 6.3.9. For complete quadrilateral $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$ that has cevian triangle with sides x, y, z, choose a point P and define $l_{ij} = (\ell_i \cap \ell_j)P$. Choose k_{ij} in the pencil of lines through $(\ell_i \cap \ell_j)$ such that

$$(\ell_i, \ell_j; l_{ij}, k_{ij}) = -1,$$

then $k_{23}, k_{14}, k_{31}, k_{24}, k_{12}, k_{34}, x, y, z$ are all tangent to a common conic.

Definition 6.3.10. We call the conic in Theorem 6.3.8 the **ℓ-nine-point conic** of the complete quadrangle q wrt. line ℓ. If line ℓ is not specified, then assume it's just $\mathcal{L}_\infty$.

When $\ell = \mathcal{L}_\infty$, then $(R_{23}R_{14})(R_{31}R_{24})(R_{12}R_{34})$ make a hexagon with parallel sides, so the center of the nine-point conic is just the midpoint of $R_{23}R_{14}$, which is just the centroid of complete quadrangle q.

Similarly,

Definition 6.3.11. We call the conic in Theorem 6.3.9 the ***P*-nine-line conic** of the complete quadrilateral $\mathcal{Q}$ wrt. point P.

6.3.1 Newton's Three Theorems

These are three big theorems about complete quadrilaterals and conic sections.

We've already seen the first one in Proposition 4.1.4.

Theorem 6.3.12 (Newton I, Newton line)**.** For complete quadrilateral $\mathcal{Q}$, the midpoints of the three diagonals are collinear.

Theorem 6.3.13 (Newton II, Conic Newton Line)**.** For complete quadrilateral $\mathcal{Q}$, let $\mathcal{C}$ be a conic tangent to the four sides of $\mathcal{Q}$, then the center of $\mathcal{C}$ lies on the Newton line. Conversely, for a point on the Newton line, there is a conic $\mathcal{C}$ centered at it tangent to all four sides of $\mathcal{Q}$.

Proof. Assume WLOG that ℓ_1 isn't parallel to ℓ_4 and take an inconic $\mathcal{C}$. Let $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4), P_{ij} = \ell_i \cap \ell_j, R_2, R_3$ be the midpoints of $\overline{P_{31}P_{24}}, \overline{P_{12}P_{34}}$ respectively, and let O be the center of $\mathcal{C}$. Draw ℓ_1', ℓ_4' as the reflections of ℓ_1, ℓ_4 over O, then $P_{14}' = \ell_1' \cap \ell_4'$ is the reflection of P_{14} over O. Let Q_2, Q_3 respectively be the reflections of P_{14} over R_2, R_3, since $\mathcal{Q}$ and ℓ_1', ℓ_4' are all tangent to $\mathcal{C}$, we have

$$\begin{aligned}\infty_{\ell_1}(Q_2, Q_3; P_{14}', \infty_{\ell_4}) &= (P_{24}, P_{34}; \ell_1' \cap \ell_4, \infty_{\ell_4}) \overset{\mathcal{C}}{=} (P_{12}, P_{31}; \infty_{\ell_1}, \ell_1 \cap \ell_4') \\ &= \infty_{\ell_4}(Q_3, Q_2; \infty_{\ell_1}, P_{14}') = \infty_{\ell_4}(Q_2, Q_3; P_{14}', \infty_{\ell_1}),\end{aligned}$$

and thus by Proposition 2.1.9 we have that Q_2, Q_3, P_{14}' are collinear, and by a homothety with scale factor $\frac{1}{2}$ from P_{14} we get that $O \in R_2R_3$.

Reversing this argument gives the converse. $\square$

Theorem 6.3.14 (Newton III, British Flag Theorem)**.** For complete quadrilateral $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4), P_{ij} = \ell_i \cap \ell_j$, if inconic $\mathcal{C}$ is tangent to the four sides at Q_1, Q_2, Q_3, Q_4, then $Q_1Q_3, Q_2Q_4, P_{12}P_{34}, P_{23}P_{41}$ are concurrent.

Proof. Brianchon on $(\ell_1\ell_1\ell_2\ell_3\ell_3\ell_4)$ and $(\ell_1\ell_2\ell_2\ell_3\ell_4\ell_4)$ gets us that $Q_1Q_3, P_{12}P_{34}, P_{23}P_{41}$ are concurrent and $Q_2Q_4, P_{12}P_{34}, P_{23}P_{41}$ are concurrent. $\square$

Practice Problems

Problem 1. Let $\triangle S_1S_2S_3$ be a equilateral triangle, and let P be an arbitrary point. Prove that the Euler lines of $\triangle PS_2S_3, \triangle PS_3S_1, \triangle PS_1S_2$ concur.

Problem 2. Let I be the incenter of $\triangle ABC$, and let D be the foot from I to BC, and let M be the midpoint of BC. Prove IM bisects AD.

Problem 3 (Schwatt Line)**.** Prove the symmedian point, midpoint of A-altitude, and midpoint of BC are collinear.

Problem 4. Let $ABCD$ be a cyclic quadrilateral and let P be intersection of diagonals AC and BD. Let $M_{AB}, M_{BC}, M_{CD}, M_{DA}$ respectively be the midpoints of arcs AB, BC, CD, DA, let $I_{AB}, I_{BC}, I_{CD}, I_{DA}$ respectively be the incenters of $\triangle PAB, \triangle PBC, \triangle PCD, \triangle PDA$. Prove that $M_{AB}I_{AB}$, $M_{BC}I_{BC}$, $M_{CD}I_{CD}$, and $M_{DA}I_{DA}$ are concurrent.

Problem 5. Let H be the orthocenter of $\triangle ABC$, and let M be the midpoint of AH. Let E, F be the feet from B, C to CA, AB. Choose point R on EM such that $RBC = 90°$, and choose S on FM such that $BCS = 90°$. Prove that A, R, S are collinear.

Problem 6. Let $\triangle DEF$ be the arc-midpoint triangle of $\triangle ABC$, and let $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ be the intersections of $\triangle ABC$ and $\triangle DEF$ (in the sequence counterclockwise $D, Z_1, Z_2, E, X_1, X_2, F, Y_1, Y_2, D$). Let $P_{bc} = EY_1 \cap FZ_2, P_{cb} = FZ_1 \cap EY_2$, and similarly define $P_{ca}, P_{ac}, P_{ab}, P_{ba}$. Prove that $P_{bc}P_{cb}, P_{ca}P_{ac}, P_{ab}P_{ba}$ are concurrent.

Problem 7. Choose six points on the sides of equilateral triangle $\triangle ABC$ (two on each side), where A_1, A_2 are the points on BC, etc, such that $A_1A_2B_1B_2C_1C_2$ is a equilateral hexagon. Prove that A_1B_2, B_1C_2, C_1A_2 are concurrent.

Problem 8. Let I be the incenter of $\triangle ABC$, and let P be the inversive image of I in (ABC). Let line ℓ be a tangent from P to I, and let ℓ intersect (ABC) at X, Y. Prove that $\angle XIY = 120°$.

Problem 9. Suppose there exists eight points $P_1, P_2, \ldots, P_8$ in the plane, and $Q_i = P_{i-2}P_{i-1} \cap P_{i+1}P_{i+2}$. Prove that $P_1, \ldots, P_8$ are conconic if and only if $Q_1, \ldots, Q_8$ are conconic.

6.A A Journey to The Hidden Circle Points

The so-called plane geometry is essentially the real projective plane $\mathbb{RP}^2$ along with complex two circle points I, J. Now, let's talk about what that actually means.

First, let's extend angles in complex numbers.

For a real line in $\mathbb{R}^2$ with slope $t \in \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$, after a rotation by θ, its slope becomes

$$\frac{\sin\theta + t\cos\theta}{\cos\theta - t\sin\theta}.$$

Note that if we let $\Theta = e^{2i\theta}$, then by Euler's theorem $e^{i\theta} = \cos\theta + i\sin\theta$, so we have

$$\frac{\sin\theta}{\cos\theta} = \frac{(e^{i\theta} - e^{-i\theta})/2i}{(e^{i\theta} + e^{-i\theta})/2} = \frac{\Theta - 1}{i(\Theta + 1)},$$

Definition (Complex Rotation)**.** As such, in $\mathbb{C}^2$, we can define a complex slope $t \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, and then after a "rotation" by $\Theta \in \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$, we get a slope of

$$t^\Theta := \frac{(\Theta - 1) + it(\Theta + 1)}{i(\Theta + 1) - t(\Theta - 1)}.$$

As an exercise that this isn't complete nonsense, you can verify $(t^{\Theta_1})^{\Theta_2} = t^{\Theta_1\Theta_2}$.

Let's now try to extract Θ if we know $t_2 \coloneqq t_1^\Theta$. After an arduous process, we obtain that

$$\Theta = \frac{t_1 + i}{t_1 - i}\Big/\frac{t_2 + i}{t_2 - i}$$

Hey, this looks like the form of a cross ratio! Let's extend our definition of a cross ratio to cover this.

Definition (Cross Ratios over $\mathbb{CP}^1$)**.** For $a, b, c, d \in \mathbb{CP}^1$, their cross ratio is defined as

$$(a, b; c, d) \coloneqq \frac{a - c}{a - d}\Big/\frac{b - c}{b - d}.$$

With this our definition, we can say $\Theta = (t_1, t_2; -i, i)$.

Definition (Complex Angles)**.** Now we can finally extend angles in complex numbers.

$$\begin{aligned}
\measuredangle^e(t_1, t_2) &= (t_1, t_2; -i, i) \\
\measuredangle(t_1, t_2) &= \frac{1}{2i}\log\measuredangle^e(t_1, t_2) = \left(\frac{1}{2\pi i}\measuredangle^e(t_1, t_2)\right) \cdot 180^\circ \in \mathbb{C}/\pi\mathbb{Z} = \mathbb{C}^\circ/180^\circ\mathbb{Z}
\end{aligned}$$

and for any two real lines ℓ_1, ℓ_2, we define $\measuredangle(\ell_1, \ell_2) = \measuredangle(\infty_{\ell_1}, \infty_{\ell_2})$

Proof. See section 1.5 of here. $\square$

Remark (Optional)**.** If we define $\mathbb{S}^1$ as $\operatorname{Spec}\mathbb{Z}[r, s]/\langle r^2 + s^2 - 1\rangle$, there is a group action of rotations on slopes as follows: Identify the line at infinity $\mathcal{L}_\infty$ with $\mathbb{RP}^1$ via $[x : y : 0] \mapsto t = y/x$ and identify

$$\mathbb{S}^1(\mathbb{R}) = \{(r, s) \in \mathbb{R}^2 \mid r^2 + s^2 = 1\} \subseteq \mathbb{R}^2$$

with $\{\Theta \in \mathbb{C} \mid \|\Theta\| = 1\} \subset \mathbb{C}^\times$ via $(r, s) \mapsto r + is$. We then have the action

$$\begin{array}{ccc} \mathbb{S}^1(\mathbb{R}) \times \mathbb{RP}^1 & \longrightarrow & \mathbb{RP}^1 \\ (r,s,t) = (e^{2i\theta}, t) & \longmapsto & \frac{\sin\theta + t\cos\theta}{\cos\theta - t\sin\theta} = \frac{s+t(r+1)}{(r+1)-ts} \end{array}$$

The extended action on $\mathbb{CP}^1$ given above is the same function on $\mathbb{C}$, as we can identify

$$\mathbb{S}^1(\mathbb{C}) = \{(r, s) \in \mathbb{C}^2 \mid r^2 + s^2 = 1\} \subseteq \mathbb{C}^2$$

and $\mathbb{C}^\times \ni \Theta$ via $\Theta = r + is$, $(r, s) = \left(\frac{\Theta+\Theta^{-1}}{2}, \frac{\Theta-\Theta^{-1}}{2i}\right)$, yielding

$$\frac{s + t(r + 1)}{(r + 1) - ts} = \frac{(\frac{\Theta-\Theta^{-1}}{2i}) + t(\frac{\Theta+\Theta^{-1}}{2} + 1)}{(\frac{\Theta+\Theta^{-1}}{2} + 1) - t(\frac{\Theta-\Theta^{-1}}{2i})} = t^\Theta.$$

Now, notably, when $t = \pm i$, $t^\Theta = t$ for all $\Theta \in \mathbb{C}^\times$. These points are pretty important so let's give them a name.

Definition (Circle Points)**.** These t correspond to the titular **circle points** $[1 : \pm i : 0] \in \mathbb{CP}^2$, which we call I, J.

(The original text uses $\infty_{\pm i}$ instead to avoid confusion with the incenter.)

These complex points are important because they lie on *every* circle. Let's see how: take a homogenized circle equation

$$(x - x_0z)^2 + (y - y_0z)^2 - kz^2 = 0$$

for some x_0, y_0. When we plug in $z = 0, y = 1, x = 1$, this becomes

$$1^2 + (\pm i)^2 + 0 = 0$$

which is true. So the circumcircle (ABC) of any triangle $\triangle ABC$ is actually just the circumconic $(ABCIJ)$ for A, B, C, I, J! Thus we can actually work with circles purely projectively now, as a family of conics through two fixed points!

In particular, $(t_1, t_2; -i, i) = -1$ if and only if $\measuredangle(t_1, t_2) = \frac{\pi}{2} = 90°$.

Here are some examples of how some plane geometry reduces with $\mathbb{CP}^2$ and the circle points.

Example 6.A.1. Since (ABC) is actually just the circumconic $(ABCIJ)$, for a point P on (ABC),

$$\measuredangle BPC = \frac{1}{2i}\log P(B,C;I,J) = \frac{1}{2i}\log A(B,C;I,J) = \measuredangle BAC,$$

which proves the inscribed angle theorem.

Example 6.A.2. Let X be the intersection of BC with the line at infinity $\mathcal{L}_\infty := IJ$, and let $X^\vee$ be a point on $\mathcal{L}_\infty$ such that $(I,J;X,X^\vee) = -1$, then $AX^\vee$ is the perpendicular from A to BC. Thus we can actually use the circle points to construct the orthocenter H of $\triangle ABC$ purely projectively. (See Example 7.1.25 for more information).

Proposition 6.A.3. *I and J are isogonal conjugates in any triangle $\triangle ABC$.*

Proof. Let ℓ_+, ℓ_- be the internal and external bisectors of $\angle BAC$. Since these are perpendicular, we have

$$(AI, AJ; \ell_+, \ell_-) = -1,$$

As such, it follows that AI, AJ are isogonal so the result follows by symmetry.

Alternatively, note that the two circle points are the two intersections of the line at infinity and the circumcircle, and since the line at infinity and the circumcircle swap under isogonal conjugation, it follows that either I, J are fixed points or are isogonal conjugates. However, they can't be fixed points since the internal and external bisectors are already fixed under isogonal conjugation. □

Remark. If we think of $\triangle ABC$ on the complex line ($\mathbb{C}^1$ which is isomorphic to the Euclidean plane) and define barycentric coordinates, then the barycentric coordinates of I, J are just $[C-B : A-C : B-A], [\overline{C-B} : \overline{A-C} : \overline{B-A}]$. Furthermore, note that the barycentric product of $I \times J$ is just $[a^2 : b^2 : c^2]$, and which implies isoconjugation (this is defined at Section 7.3, Section 7.4).

Remark. If we further generalize the above remark to $\triangle ABC$ as three points in $\mathbb{C}^2$, we then need to have a "complex coordinate system" in A, B, C via the mapping

$$\begin{array}{ccccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & \mathbb{C} \\ (x,y) & \longmapsto & 1 \otimes x + i \otimes y & \longrightarrow & x + iy \end{array}$$

where conjugation $\overline{1 \otimes x + i \otimes y} = 1 \otimes x - i \otimes y$ is defined with the first $\mathbb{C}$.

For a real conic $\mathcal{C}$, let ℓ_{++}, ℓ_{+-} be the two tangents from I to $\mathcal{C}$, and let ℓ_{-+}, ℓ_{--} be the respective two tangents from J to $\mathcal{C}$ which are obtained by taking a barycentric complex conjugation $[x:y:z] \mapsto [\overline{x} : \overline{y} : \overline{z}]$

Proposition 6.A.4 (Projective Definition Of Foci)**.** *The two points $F_+ = \ell_{++} \cap \ell_{-+}, F_- = \ell_{+-} \cap \ell_{--}$ are the two foci of $\mathcal{C}$. In general, the set of intersections of pairs of tangents from I, J to a curve $\mathcal{K}$ are known as the foci of the curve.*

Proof. Since the complex conjugates of $F_\pm$ are themselves, the two points $F_\pm$ are real. For a point $P \in \mathcal{C}$, let F_P be the reflection of F_+ across the tangent $\mathbf{T}_P\mathcal{C}$. We show that F_P lies on PF_2 which implies the result as then $\overline{F_1P} + \overline{F_2P}$ is fixed.

Let X_P, Y_P, Z_P respectively be the intersections of $T \coloneqq \mathbf{T}_P\mathcal{C}$ with $\ell_{++}, \ell_{-+}, FF'_P$. Now consider complex quadrilateral $\infty_T Y_P F_1 I$ where $\infty_T Y_P \cap IF_+ = X_P, \infty_T I \cap Y_P F_+ = J$.

Since $I_\infty F F_P$ is the harmonic conjugate of I_T with respect to IJ, it follows that $Z_P = F_+\infty_{FF_P} \cap T$ so F_P, as the harmonic conjugate of F_+ with respect to IY_P, X_PJ, is equal to $IY_P \cap JX_P$.

Now, by Brianchon with respect to $\mathcal{C}$ on $PX_PIF_-JY_P$, we get that $Y_PI \cap X_PJ$ lies on PF_2 as desired.

□

We can also define a pair of "imaginary" foci $F_i = \ell_{++} \cap \ell_{--}, F_{-i} = \ell_{+-} \cap \ell_{-+}$. Note that we can't distinguish between (F_+, F_-) and (F_i, F_{-i}) purely projectively, so we can at best define the two pairs of foci for a given real conic: the pair of real points and the pair of imaginary ones.

Practice Problems

Problem 1. Prove that $(t^{\Theta_1})^{\Theta_2} = t^{\Theta_1\Theta_2}$.

Problem 2.

6.A.1 Geometric Transformations Revisited

Let's try to define geometric transformations purely projectively with just the circle points I and J and see what remains. We replace I, J with $\mathbf{W} = \{W^+, W^-\}$ to indicate that we only consider I, J projectively. We first redefine the line at infinity as $\mathcal{L} = \overline{W^+W^-}$.

This is useful to solve problems in the context of taking homographies sometimes, as it gives us a way to deal with the standard "non-projective" conditions of angles and perpendicularities under homographies, by taking I, J to suitable points.

As shown in the previous section, we can define angles between two lines ℓ_1, ℓ_2 completely in terms of W^+, W^- with cross-ratio. We introduce the notation $\measuredangle^e_{\mathbf{W}}(\ell_1, \ell_2) = (\mathcal{L}_\infty \cap \ell_1, \mathcal{L}_\infty \cap \ell_2; W^+, W^-), \mathcal{L}_\infty \neq \mathcal{L}$.

We now redefine two lines to be parallel if $\measuredangle^e_{(\mathbf{W}}(\ell_1, \ell_2) = 1$ or $\ell_1 \cap \ell_2 \in \mathcal{L}$. Similarly perpendicular lines happen when $\measuredangle^e_{\mathbf{W}}(\ell_1, \ell_2)$ formed is -1, and two lines through W^+, W^- are perpendicular.

Affine transformations are then projective transformations that preserve the line at infinity $\mathcal{L}$:

$$\mathscr{A}_{\mathbf{W}} = \{\varphi : \mathbb{P}^2 \to \mathbb{P}^2 \mid \varphi(\mathcal{L}_\infty) = \mathcal{L}_\infty\}.$$

Spiral similarities are projective transformations that fix the circle points, and thus fix angles and circles:

$$\mathscr{S}_{\mathbf{W}} = \{\varphi \in \mathscr{A}_{\mathbf{W}} \mid \varphi(W^+) = W^+, \varphi(W^-) = W^-\}.$$

Homotheties are projective transformations in $\mathscr{P}$ that fix all points on $\mathcal{L}$, so we define the group of homotheties as:

$$\mathscr{H}_{\mathbf{W}} = \{\varphi \in \mathscr{P}_{\mathbf{W}} \mid \varphi(P) = P, \forall P \in \mathcal{L}\}.$$

Let's see now that the notion of the center of a homothety is preserved.

For $\varphi \in \mathscr{H}$ and three points P, Q, R, we have

$$\mathcal{L} \cap QR = \varphi(\mathcal{L} \cap QR) = \varphi(\mathcal{L}) \cap \varphi(Q)\varphi(R) = \mathcal{L} \cap \varphi(Q)\varphi(R).$$

So therefore $QR \cap \varphi(Q)\varphi(R) \in \mathcal{L}_\infty$. By the same logic we have $RP \cap \varphi(R)\varphi(P), PQ \cap \varphi(P)\varphi(Q) \in \mathcal{L}$. So by Desargues's theorem, we have $P\varphi(P), Q\varphi(Q), R\varphi(R)$ concur at a point O. Since P, Q, R are arbitrary, we have that $O_\varphi = O$ is the common point of all $P\varphi(P)$, and we call O the center of φ. Similarly, for any $\varphi, \Psi \in \mathscr{H}$, we have $O_\varphi, O_\Psi, O_{\varphi\circ\Psi}$ are collinear, so a variant of Monge's remains true.

If the center of homothety O_φ lies on $\mathcal{L}$, we call φ a translation. The set of translations also forms a group:

$$\mathscr{T}_{\mathbf{W}} = \{\varphi \in \mathscr{H}_{\mathbf{W}} \mid O_\varphi \in \mathcal{L}\}.$$

For two vectors $\overrightarrow{v_1} = \overrightarrow{P_1Q_1}, \overrightarrow{v_2} = \overrightarrow{P_2Q_2}, P_i, Q_i \notin \mathcal{L}, \{I, J\} \notin \overrightarrow{v_i}, P_i \neq Q_i$, we can find a spiral similarity $\varphi \in \mathscr{S}$ such that $\varphi(P_1) = P_2, \varphi(Q_1) = Q_2$ (so our φ is just $\varphi(\overrightarrow{v_1}) = \overrightarrow{v_2}$), and we abuse notation by defining

$$\varphi = \frac{\overrightarrow{v_2}}{\overrightarrow{v_1}}.$$

However this has a subtle problem: we have not proved commutativity. In general,

$$\frac{\overrightarrow{v_2}}{\overrightarrow{v_1}} \circ \frac{\overrightarrow{w_2}}{\overrightarrow{w_1}} \neq \frac{\overrightarrow{w_2}}{\overrightarrow{w_1}} \circ \frac{\overrightarrow{v_2}}{\overrightarrow{v_1}},$$

so some of these φ do not commute. Fortunately, we have the next best thing: there exists a translation $t \in \mathscr{T}_{\mathbf{W}}$ between

$$\frac{\overrightarrow{v_2}}{\overrightarrow{v_1}} \circ \frac{\overrightarrow{w_2}}{\overrightarrow{w_1}} \overset{t}{\mapsto} \frac{\overrightarrow{w_2}}{\overrightarrow{w_1}} \circ \frac{\overrightarrow{v_2}}{\overrightarrow{v_1}}$$

Proposition 6.A.5. The group $\mathscr{T}_{\mathbf{W}}$ is the **commutator subgroup** of $\mathscr{S}_{\mathbf{W}}$, i.e.

$$\mathscr{T}_{\mathbf{W}} = [\mathscr{S}_{\mathbf{W}}, \mathscr{S}_{\mathbf{W}}] := \left\langle [\varphi, \Psi] := \varphi\Psi\varphi^{-1}\Psi^{-1} \mid \varphi, \Psi \in \mathscr{S}_{\mathbf{W}} \right\rangle$$

so the **abelianization** of $\mathscr{S}_{\mathbf{W}}$ is the abelian group $\mathscr{S}_{\mathbf{W}}^{\mathrm{ab}} = \mathscr{S}_{\mathbf{W}}/\mathscr{T}_{\mathbf{W}}$.

So we can define Φ to be the element in $\mathscr{S}^{\mathrm{ab}}$ that corresponds to $\frac{\overrightarrow{v_2}}{\overrightarrow{v_1}}$. Thus we have

$$\frac{\overrightarrow{v_2}}{\overrightarrow{v_1}} \circ \frac{\overrightarrow{w_2}}{\overrightarrow{w_1}} = \frac{\overrightarrow{w_2}}{\overrightarrow{w_1}} \circ \frac{\overrightarrow{v_2}}{\overrightarrow{v_1}} \in \mathscr{S}^{\mathrm{ab}}.$$

Since

$$\frac{\overrightarrow{v_3}}{\overrightarrow{v_1}}(\overrightarrow{v_1}) = \overrightarrow{v_3} = \frac{\overrightarrow{v_3}}{\overrightarrow{v_2}} \circ \frac{\overrightarrow{v_2}}{\overrightarrow{v_1}}(\overrightarrow{v_1}),$$

we have

$$\frac{\overrightarrow{v_3}}{\overrightarrow{v_1}} = \frac{\overrightarrow{v_3}}{\overrightarrow{v_2}} \circ \frac{\overrightarrow{v_2}}{\overrightarrow{v_1}} = \frac{\overrightarrow{v_2}}{\overrightarrow{v_1}} \circ \frac{\overrightarrow{v_3}}{\overrightarrow{v_2}}.$$

so we can notate elements of $\mathscr{S}_{\mathbf{W}}^{\mathrm{ab}}$ similar to directed angles.

To avoid confusion with the "$\cdot$" used in the inner product/dot product, we will use the composition symbol "$\circ$" to represent multiplication in $\mathscr{S}_{\mathbf{W}}^{\mathrm{ab}}$. Further, $\mathscr{S}_{\mathbf{W}}^{\mathrm{ab}}$ will be our familiar complex coordinate system on $\mathbb{C}^2$.

Remark. Consider the exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \mathscr{H}_{\mathbf{W}}/\mathscr{T}_{\mathbf{W}} \arrow[r] \arrow[d, equal] & \mathscr{S}_{\mathbf{W}}/\mathscr{T}_{\mathbf{W}} \arrow[r] \arrow[d, equal] & \mathscr{S}_{\mathbf{W}}/\mathscr{H}_{\mathbf{W}} \arrow[r] \arrow[d, "\sim"] & 1 \\
1 \arrow[r] & \mathscr{H}_{\mathbf{W}}^{\mathrm{ab}} \arrow[r] & \mathscr{S}_{\mathbf{W}}^{\mathrm{ab}} \arrow[r] & \mathbb{S}^1 \arrow[r] & 1 \\
 & & \frac{\vec{v}_2}{\vec{v}_1} \arrow[r, mapsto] & \measuredangle^{\mathrm{e}}_{\mathbf{W}}(v_1, v_2). &
\end{tikzcd}$$

This nicely explains how spiral similarities are the composition of dilations and rotations, and also justifies the notation $\vec{v_2}/\vec{v_1}$ as a natural multiplicative form of directed angles. In other words, if we think of v_i, w_i as the lines defined by extending $\vec{v_i}, \vec{w_i}$, then

$$\prod_{i=1}^{n} \vec{v_i} = \prod_{i=1}^{n} \vec{w_i} \implies \sum_{i=1}^{n} \measuredangle v_i = \sum_{i=1}^{n} \measuredangle w_i.$$

Chapter 7

Projective Space

7.1 Conic Polarity

We've already seen that the idea of polarity for circles is pretty nice for projective geometry. And we know that there is a projective transformation sending conics to circles. Let's try to define something similar to polarity for our general conics.

Definition 7.1.1. Given a conic $\mathcal{C}$ and an arbitrary point P, construct a moving line through P that intersects $\mathcal{C}$ at M_1, M_2. Choose Q such that $(P, Q; M_1, M_2) = -1$. Then the locus of Q is a line. We call this line the **polar** $\mathfrak{p}_{\mathcal{C}}(P)$ of P with respect to $\mathcal{C}$.

We can generalize a bunch of analogous properties from polarity in circles to properties in conics:

Theorem 7.1.2 (La Hire's)**.** Let $\mathcal{C}$ be a conic, and let P, Q be two points in the plane. Then

$$P \in \mathfrak{p}_{\mathcal{C}}(Q) \iff Q \in \mathfrak{p}_{\mathcal{C}}(P).$$

Definition 7.1.3. Let $\mathcal{C}$ be a conic, and suppose two points P and Q satisfy that the polar of P wrt. $\mathcal{C}$ goes through Q. Then we call P, Q **conjugate** in $\mathcal{C}$.

Definition 7.1.4. Let $\mathcal{C}$ be a conic, and let K be a line. Let Q be a moving point on $\mathcal{K}$. Then the fixed point that $\mathfrak{p}_{\mathcal{C}}(Q)$ passes through is the **pole** of K wrt. $\mathcal{C}$ and is denoted as $\mathfrak{p}_{\mathcal{C}}(K)$.

Proposition 7.1.5 (Dual of La Hire's)**.** Let $\mathcal{C}$ be a conic, and let K, L be two lines, then

$$\mathfrak{p}_{\mathcal{C}}(K) \in L \iff \mathfrak{p}_{\mathcal{C}}(L) \in K.$$

Definition 7.1.6. Let $\mathcal{C}$ be a conic, if two lines K, L satisfy $\mathfrak{p}_{\mathcal{C}}(K) \in L$, then we call these two lines K, L conjugate in $\mathcal{C}$.

Definition 7.1.7. Let $\mathcal{C}$ be a conic, if $\triangle ABC$ satisfies that A, B, C are pairwise conjugate wrt. $\mathcal{C}$, we call $\triangle ABC$ a **self-conjugate triangle** wrt. $\mathcal{C}$, and we call $\mathcal{C}$ the **diagonal conic** of $\triangle ABC$.

(It's called a diagonal conic because $\triangle ABC$ is the diagonal triangle of some quadrilateral on $\mathcal{C}$.)

Proposition 7.1.8 (Brokard's Theorem)**.** Let $\mathcal{C}$ be a conic, and let P_1, P_2, P_3, P_4 be four points on $\mathcal{C}$. Let

$$X = P_1P_2 \cap P_3P_4, Y = P_1P_3 \cap P_4P_2, Z = P_1P_4 \cap P_2P_3,$$

then $\triangle XYZ$ is self-conjugate wrt. $\mathcal{C}$.

Theorem 7.1.9 (Polarity preserves cross-ratios)**.** Let $\mathcal{C}$ be a conic, and let the five lines $\ell_1, \ell_2, \ell_3, \ell_4, L$ have P_1, P_2, P_3, P_4, A as poles wrt. $\mathcal{C}$, then

$$A(P_\bullet) = L(\ell_\bullet).$$

The following results are more connected to cross-ratios.

Corollary 7.1.10. Given a conic $\mathcal{C}$, then six points P_1, P_2, P_3, P_4, A, B are conconic if and only if the six lines $\mathfrak{p}_{\mathcal{C}}(P_1, P_2, P_3, P_4, A, B)$ are all tangent to one conic.

Proof. Since

$$A(P_\bullet) = \mathfrak{p}_{\mathcal{C}}(A)(\ell_\bullet), B(P_\bullet) = \mathfrak{p}_{\mathcal{C}}(B)(\ell_\bullet),$$

we know P_1, P_2, P_3, P_4, A, B are conconic if and only if

$$\mathfrak{p}_{\mathcal{C}}(A)(\ell_\bullet) = A(P_\bullet) = B(P_\bullet) = \mathfrak{p}_{\mathcal{C}}(B)(\ell_\bullet),$$

but this is equivalent to the desired result. $\square$

Inspired by this, let's define the following:

Definition 7.1.11. Let $\mathcal{C}$ be a conic, we call two conic sections $\mathcal{C}_1, \mathcal{C}_2$ **conjugate** wrt. $\mathcal{C}$ (notated as $\mathcal{C}_2 = \mathfrak{p}_{\mathcal{C}}(\mathcal{C}_1)$) if

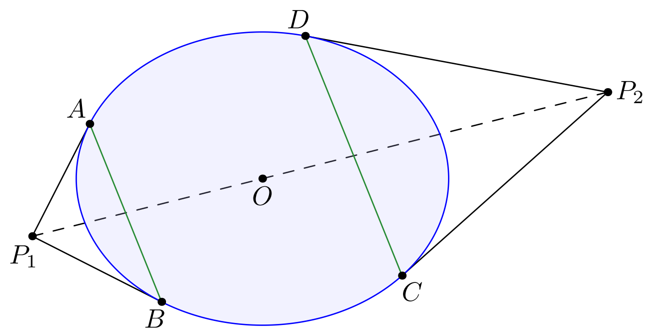
$$\{\mathfrak{p}_{\mathcal{C}}(P) \mid P \in \mathcal{C}_1\} = \mathbf{T}\mathcal{C}_2.$$

Through Corollary 7.1.10, we can ensure that for a conic $\mathcal{C}$ and a arbitrary conic $\mathcal{C}_1$, there exists a $\mathcal{C}_2$ such that $\mathcal{C}_1, \mathcal{C}_2$ are conjugate with respect to $\mathcal{C}$. Further, for two conics $\mathcal{C}_1, \mathcal{C}_2$, there exists another conic $\mathcal{C}$ such that $\mathcal{C}_1, \mathcal{C}_2$ are conjugate in $\mathcal{C}$.

Proposition 7.1.12. Let $\mathcal{C}$ be a conic and let O be the center of this conic. Then $\mathfrak{p}_{\mathcal{C}}(O) = \mathcal{L}_\infty$.

Proof. Draw two lines ℓ_1, ℓ_2 intersecting $\mathcal{C}$ through O. Let these lines intersect $\mathcal{C}$ respectively at (P_1, Q_1) and (P_2, Q_2). Since the midpoints of $\overline{P_1Q_1}$ and $\overline{P_2Q_2}$ are O, it follows that $\infty_{\ell_1}, \infty_{\ell_2} \in \mathfrak{p}_{\mathcal{C}}(O)$, so $\mathfrak{p}_{\mathcal{C}}(O) = \mathcal{L}_\infty$. $\square$

Corollary 7.1.13 (Parallel Chords Theorem)**.** Let $\mathcal{C}$ be a conic, and let O be the center of $\mathcal{C}$. Let P_1, P_2 be two points in the plane, then O, P_1, P_2 are collinear if and only if $\mathfrak{p}_{\mathcal{C}}(P_1)$ is parallel with $\mathfrak{p}_{\mathcal{C}}(P_2)$.

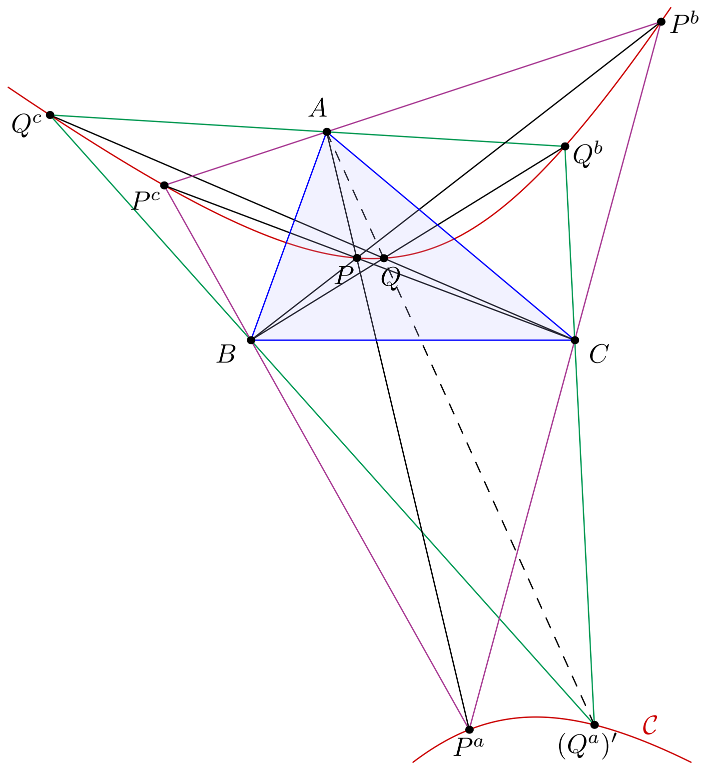


Proof. Note that O, P_1, P_2 are collinear if and only if $\mathfrak{p}_{\mathcal{C}}(O) = \mathcal{L}_\infty, \mathfrak{p}_{\mathcal{C}}(P_1), \mathfrak{p}_{\mathcal{C}}(P_2)$ are concurrent. $\square$

Corollary 7.1.14. Let $\mathcal{C}$ be a conic, and let O be the center of $\mathcal{C}$. If $\overline{AB}$ is a chord of $\mathcal{C}$, then $\overline{O\mathfrak{p}_{\mathcal{C}}(AB)}$ bisects $\overline{AB}$.

Proof. Let $T = AB \cap O\mathfrak{p}_{\mathcal{C}}(AB), U = \infty_{AB}$, then $O\mathfrak{p}_{\mathcal{C}}(AB) = \mathfrak{p}_{\mathcal{C}}(U)$, so $(U, T; A, B) = -1$ and thus T bisects $\overline{AB}$. $\square$

Theorem 7.1.15 (Eight Point Conic)**.** Given $\triangle ABC$, let P, Q be two arbitrary points and let $\triangle P^aP^bP^c$, and $\triangle Q^aQ^bQ^c$ be the anticevian triangles of P and Q wrt. $\triangle ABC$. Then $P, P^a, P^b, P^c, Q, Q^a, Q^b, Q^c$ are all conconic. This is also referred to as the **bianticevian conic** of P and Q.



Proof. Let $\mathcal{C}$ be the conic through P, P^a, P^b, P^c, Q, then by Brokard's, $\mathfrak{p}_{\mathcal{C}}(A), \mathfrak{p}_{\mathcal{C}}(B)$ is BC, so let AQ intersect $\mathcal{C}$ again at $(Q^a)'$, then we have

$$(A, AQ \cap BC; Q, (Q^a)') = -1 = (A, AQ \cap BC; Q, Q^a)$$

and thus $Q^a = (Q^a)'$, so Q^b, Q^c also lie on the conic by symmetry. $\square$

The dual of the eight-point conic is the eight-line conic.

Theorem 7.1.16 (Eight Line Conic)**.** Let K, L be two lines, let $\triangle K^a K^b K^c, \triangle L^a L^b L^c$ respectively be the anticevian triangles of K, L wrt. $\triangle ABC$ (triangle such that perspectrix of it and $\triangle ABC$ is K or L). Then $K, K^a, K^b, K^c, L, L^a, L^b, L^c$ are all tangent to a common conic.

With the definition of polarity, we define a bunch of binary operations on points.

Definition 7.1.17. Given two points P, Q and $\triangle ABC$, we define the **crosspoint** $P \pitchfork Q$ of P and Q as the pole of $\overline{PQ}$ in $(ABCPQ)$.

Some readers may have realize that this definition is different from the definition we gave in Crosspoint definition. However, here the commutativity of this definition is obvious. So let's show that these two definitions are equivalent, and get commutativity as a side effect.

Recall that our earlier definition was that the cevian triangle of $P \pitchfork Q$ in $\triangle P_aP_bP_c$ is perspective with ABC, with perspector Q. So it suffices to prove the following:

Proposition 7.1.18. Let $\triangle P_aP_bP_c$ be the cevian triangle of P wrt. $\triangle ABC$. Then $\overline{P_a(P \pitchfork Q)}, \overline{P_bP_c}, \overline{AQ}$ are concurrent.

Lemma 7.1.19 (Seydewitz-Staudt)**.** Given $\triangle ABC$ and a circumconic $\mathcal{C}$, let $P \coloneqq \mathfrak{p}_{\mathcal{C}}(BC)$, and let ℓ be a line through P that intersects CA, AB at U, V respectively. Then U, V are conjugate in $\mathcal{C}$.

Proof. Let D be the second intersection of BU and $\mathcal{C}$. By Pascal's Theorem on the hexagon $ABBDCC$, it follows that AB, CD, PU concur at V, and thus from Brokard's Theorem, U, V are conjugates in $\mathcal{C}$. $\square$

Proof of (7.1.18). Apply the above lemma wrt. $\triangle APQ$ and $\mathcal{C} = (ABCPQ)$, so $P_a(P \pitchfork Q) \cap AQ$ lies on $\mathfrak{p}_{\mathcal{C}}(P_a) = P_bP_c$. $\square$

Proposition 7.1.20. If $R = P \pitchfork Q$, and $\triangle P_aP_bP_c, \triangle Q_aQ_bQ_c$ are the cevian triangles of P, Q, then both of $P_cP_a \cap Q_aQ_b, P_aP_b \cap Q_cQ_a$ lie on line AR.

Proof. By taking polarity over $\mathcal{C} = (ABCPQ)$, it suffices to prove that tangent at A to $\mathcal{C}, P_bQ_c, P_cQ_b, PQ$ are concurrent.

Note that $P_bP_c = \mathfrak{p}_{\mathcal{C}}(P_a)$ and $Q_bQ_c = \mathfrak{p}_{\mathcal{C}}(Q_a)$ both go through the pole U of BC (i.e. line P_aQ_a) over the conic $(ABCPQ)$. As such, we have

$$(B, C, AU \cap \mathcal{C}, A)_{\mathcal{C}} = -1 = A(B, C; U, P_bQ_c \cap P_cQ_b),$$

so $\overline{A(P_bQ_c \cap P_cQ_b)}$ is tangent to $\mathcal{C}$.

We want to prove P_bQ_c, P_cQ_b, PQ concurrent, so by Desargues's theorem we only need to prove that $U = P_bP_c \cap Q_bQ_c, CP \cap BQ = P_cP \cap QQ_b, BP \cap CQ = PP_b \cap Q_cQ$ are collinear. However, these three points all lie on the polar of $BC \cap PQ$. $\square$

Proposition 7.1.21. Continuing the above notation, let $P_a^\vee, P_b^\vee, P_c^\vee$ respectively be the intersection points of the trilinear polar of P wrt. BC, CA, AB, and similarly define $Q_a^\vee, Q_b^\vee, Q_c^\vee$. Then $P_bQ_c^\vee \cap P_cQ_b^\vee, P_b^\vee Q_c \cap P_c^\vee Q_b$ also lie on AR.

Proof. Let $\triangle T^aT^bT^c$ be the polar of $\triangle ABC$ wrt. $(ABCPQ)$. Then $\mathfrak{p}_{\mathcal{C}}(Q_a^\vee) = Q_aT^a$, and similarly to the proof of Proposition 7.1.20, it suffices to prove that $P_cP_a \cap Q_cT^c, P_aP_b \cap Q_bT^b, \mathfrak{p}_{\mathcal{C}}(AR) = P_bQ_c \cap P_cQ_b$ are concurrent. This follows by Desargues's theorem since $P_bQ_b \cap P_cQ_c = A$, T^b, T^c are collinear. $\square$

Proposition 7.1.22. Let R be the crosspoint $P \pitchfork Q$. Then the perspectrix of the anticevian triangle of R, $\triangle^R$ and the cevian triangle of Q, $\triangle_Q$, is the trilinear polar of P.

Proof. Let X be the intersection point of $P_b^\vee P_c^\vee$ and Q_bQ_c. Then we only need to prove

$$(AB, AC; AR, AX) \stackrel{A}{=} (Q_c, Q_b; AR \cap Q_bQ_c, X) = -1.$$

For the complete quadrilateral $(P_b^\vee P_c^\vee)(Q_bQ_c)$, we have $P_b^\vee Q_c \cap P_c^\vee Q_b \in AR$ by Proposition 7.1.21. As such, this implies that $AR \cap Q_bQ_c, X$ are harmonic conjugates with respect Q_b, Q_c. □

We define the **cross conjugate** of R and Q as $R \Psi Q$, as the trilinear pole of the perspector of the anticevian triangle $\triangle^R$ of R and the cevian triangle $\triangle_Q$ of Q. Note that by the previous proposition we have

$$(R \Psi Q) \pitchfork Q = R.$$

Definition 7.1.23. Let $\triangle P^aP^bP^c, \triangle Q^aQ^bQ^c$ be the anticevian triangles of P and Q. We define the **cevapoint** of P and Q as the pole of PQ in the eight point conic $(PP^aP^bP^cQQ^aQ^bQ^c)$, notated as $P \star Q$.

We further define the **ceva conjugate** of S and Q as the point P such that $S = P \star Q$, notated as S/Q.

From the definition of crosspoints, we discover that the cevapoint $P \star Q$ is also the crosspoint of P and Q wrt. $\triangle P^aP^bP^c$ or $Q^aQ^bQ^c$. As such, the ceva conjugate S/Q is the perspector of the cevian triangle of $\triangle_S$ and the anticevian triangle $\triangle^Q$.

Proposition 7.1.24. Given $\triangle ABC$ and two points P, Q, let $S = P \star Q$. Then we have

$$(AP, AQ; AS, A(PQ \cap BC)) = -1.$$

Proof. Let $\mathcal{D} = (PP^aP^bP^cQQ^aQ^bQ^c)$. Since the polar of $PQ \cap BC$ wrt. $\mathcal{D}$ is $\mathfrak{p}_\mathcal{D}(PQ)\mathfrak{p}_\mathcal{D}(BC) = SA$, we have

$$(AP, AQ; AS, A(PQ \cap BC)) = (P, Q; SA \cap PQ, PQ \cap BC) = -1.$$

□

Example 7.1.25. Since

$$(I, J; \infty_{\perp BC}, \infty_{BC}) = -1$$

holds where I, J are the circle points, we have the cevapoint H' of I, J satisfies $AH' \perp BC$. By symmetry, we get that $H = H'$, so $H = I \star J$.

Furthermore, the bianticevian conic Ω of I, J is a circle of which ABC is self-conjugate about. Furthermore, the center of this circle is the polar of $\mathcal{L}_\infty = IJ$ about Ω, or H. This is called the Polar Circle, and has imaginary radius

$$\sqrt{\overrightarrow{HA} \cdot \overrightarrow{HH_A}} = \sqrt{\overrightarrow{HB} \cdot \overrightarrow{HH_B}} = \sqrt{\overrightarrow{HC} \cdot \overrightarrow{HH_C}}$$

where $\triangle H_A H_B H_C$ is the orthic triangle.

Proposition 7.1.26. Let F be a focus of conic $\mathcal{C}$. Let L be a line through F. Then $F\mathfrak{p}_{\mathcal{C}}(L)$ is perpendicular to L.

Proof. If we use the characterization of a conic's foci from Proposition 6.A.4, this follows since

$$(F\mathfrak{p}_{\mathcal{C}}(L), L; FI, FJ) = -1 \implies F\mathfrak{p}_{\mathcal{C}}(L) \perp L.$$

□

Similar to inversion / polarity about an imaginary circle, we can also define polarity with respect to a purely imaginary conic $\mathcal{X}$, as the composition of polarity across $\mathcal{X}$ and reflection across the center O of $\mathcal{X}$.

Practice Problems

Problem 1. Let O be the center of Γ. Prove that for any conic Ω, $\mathfrak{p}_\Gamma(\Omega)$ is a conic with O as one of its foci.

(The polar of a conic is just the locus of the poles of its tangent lines.)

Problem 2. Given a fixed conic $\mathcal{C}$ and $\triangle ABC$, prove that

$$\mathfrak{p}_{\mathcal{C}}(\triangle ABC) := \triangle \mathfrak{p}_{\mathcal{C}}(BC)\mathfrak{p}_{\mathcal{C}}(CA)\mathfrak{p}_{\mathcal{C}}(AB)$$

is perspective with $\triangle ABC$.

Problem 3. Given $\triangle ABC$ and P as an arbitrary point, let $\triangle P_aP_bP_c$ be the pedal triangle of P wrt. $\triangle ABC$. Choose points D, E, F on PP_a, PP_b, PP_c such that

$$PP_a \cdot PD = PP_b \cdot PE = PP_c \cdot PF =: k.$$

Prove that

- $\triangle ABC$ is perspective with $\triangle P_aP_bP_c$;
- When k varies, the locus of the perspector is a conic passing through A, B, C, P and the orthocenter H,
- When k varies, the envelope of the perspectrix is a parabola tangent to BC, CA, AB.

Problem 4 (Extension of Newton's 2nd Theorem)**.** Let complete quadrilateral $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$ have inconic $\mathcal{C}$. Let $A_{ij} = \ell_i \cap \ell_j$. If line L and $\mathcal{Q}$'s diagonal $A_{ij}A_{kl}$ intersect at P_{ij}, choose Q_{ij} on $A_{ij}A_{kl}$ such that $(A_{ij}, A_{kl}; P_{ij}, Q_{ij}) = -1$. Prove that

- Q_{14}, Q_{24}, Q_{34} are collinear;
- $\mathcal{C}Q_{14}Q_{24}Q_{34} \in L$.

Problem 5. Given triangle $\triangle ABC$, for a point P and a line ℓ, let $\mathcal{C}$ be the nine-point conic of A, B, C, P wrt. ℓ Theorem 6.3.8. Then the trilinear pole of ℓ wrt. $\triangle ABC$, the pole of ℓ wrt. $\mathcal{C}$, and P are collinear.

Problem 6. Given two triangles $\triangle_1 = \triangle A_1B_1C_1, \triangle_2 = \triangle A_2B_2C_2$, $A_1, B_1, C_1, A_2, B_2, C_2$ are conconic if and only if there exists a conic $\mathcal{C}$ such that $\triangle_1, \triangle_2$ are self-polar wrt. $\mathcal{C}$.

Problem 7. Prove in the notation of Proposition 7.1.21 that $P_bQ_c^\vee \cap P_cQ_b^\vee$ lies on $P_a\mathfrak{p}_{(ABCPQ)}(AR)$.

7.2 Involutions

Here, a **pencil** is the set of lines through a point or the set of points through a line. A more precise and general notion of a pencil can be found in Section 7.A, of which this section has some very minor dependencies on for proofs.

Definition 7.2.1. Let X be a pencil. We say an **automorphism** is a bijection on X which preserves the cross ratio. Then $\varphi \in \operatorname{Aut}(X)$ an **involution** if $\varphi^2 = \mathrm{id}_X$ and $\varphi \neq \mathrm{id}_X$. In other words, φ is an automorphism that has order 2 in the group of automorphisms $\operatorname{Aut}(X)$.

Example 7.2.2. Let X be a line, then reflecting about a point P on the line X is an involution on the pencil of the set of points on line X.

Example 7.2.3. Let X be $\mathbf{T}P$, the set of lines through a point $P \notin \mathcal{L}_\infty$, then rotating $90°$ degrees around P is also an involution.

Proposition 7.2.4. Let X be a pencil, if φ is an automorphism such that for some point A, $\varphi(A) \neq A, \varphi(\varphi(A)) = A$, then φ is an involution.

Proof. Note that $\varphi \neq \mathrm{id}_X$ since A is not fixed. Then for any $P \in X$, we have

$$(A, \varphi(A); P, \varphi(P))_X = (\varphi(A), A; \varphi(P), \varphi(\varphi(P)))_X = (A, \varphi(A); \varphi^2(P), \varphi(P))_X$$

so $\varphi^2(P) = P$. □

Remark. In general, every involution on a pencil has two fixed points which uniquely determine it.

Likewise, an involution is determined by two pairs $(A, B), (C, D)$ under this involution or the image of three elements under it.

This will be built on later in Section 7.A.

Here's one of the most famous and important theorems about involutions, commonly referred to as "DIT".

Theorem 7.2.5 (Desargues's Involution Theorem)**.** Given complete quadrangle $\mathfrak{q} = (P_1, P_2, P_3, P_4)$ and a line ℓ not going through the vertices, let $Q_{ij} = P_iP_j \cap \ell$, then

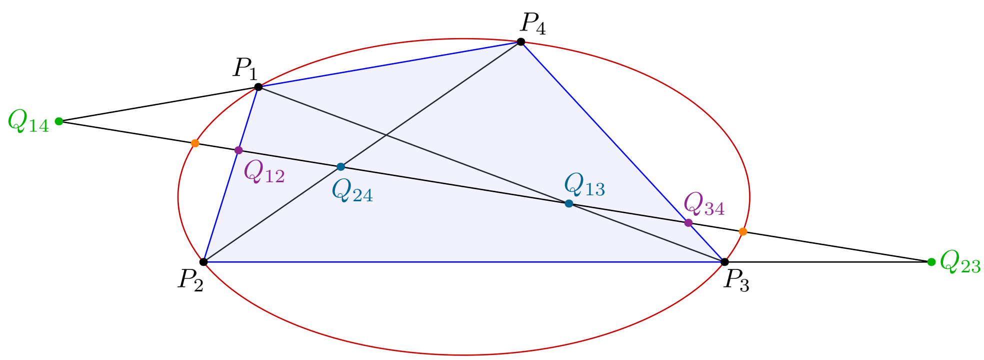
- There exists exactly one involution on ℓ, $\varphi \in \mathrm{Aut}(\ell)$, such that the three pairs

$$(Q_{23}, Q_{14}), (Q_{31}, Q_{24}), (Q_{12}, Q_{34})$$

  swap under the involution;

- If a conic $\mathcal{C}$ intersects ℓ at A, B, then (A, B) are interchanged by φ. Conversely, P_1, P_2, P_3, P_4, A, B are conconic (potentially degenerate) if and only if (A, B) swap under φ.

We will call the involution φ given by (i) as the involution with respect to ℓ and $\mathfrak{q}$.



Proof. Note that Q_{23}, Q_{14}, Q_{12} be pairwise distinct, and Q_{23}, Q_{14}, Q_{34} are pairwise distinct. Define a projective map from $\varphi : \ell \to \ell$ such that $Q_{23} \mapsto Q_{14}, Q_{14} \mapsto Q_{23}, Q_{12} \mapsto Q_{34}$ (from Proposition 7.A.7, we know that φ exists and is unique). Then from Proposition 7.2.4, it remains to show φ has a fixed point to be an involution.

(i) By properties of the cross-ratio we know that

$$(Q_{23}, Q_{14}; Q_{12}, Q_{31}) \stackrel{P_1}{=} (Q_{23}, P_1P_4 \cap P_2P_3; P_2, P_3) = (Q_{23}, Q_{14}; Q_{24}, Q_{34}) \stackrel{P_4}{=} (Q_{14}, Q_{23}; Q_{34}, Q_{24})$$

and therefore $\varphi(Q_{31}) = Q_{24}$. Thus, our projective map φ must swap (Q_{31}, Q_{24}) and thus is an involution.

(ii) This is essentially running the proof in (i) backwards. By cross-ratio properties we know

$$P_1(A, B; P_2, P_3) = P_1(A, B; Q_{12}, Q_{31}), P_4(A, B; P_2, P_3) = P_4(A, B; Q_{24}, Q_{34}).$$

Since P_1, P_2, P_3, P_4, A, B are conconic (possibly degenerate), we have

$$P_1(A, B; Q_{12}, Q_{31}) = P_4(A, B; Q_{24}, Q_{34}) = P_4(B, A; Q_{34}, Q_{24}).$$

If we define $\varphi' \in \operatorname{Aut}(\ell)$ as a projective map satisfting $A \mapsto B, B \mapsto A, Q_{12} \mapsto Q_{34}$, then $Q_{31} \mapsto Q_{24}$ and φ' is a projective map, so by symmetry $\varphi' = \varphi$. Therefore, (A, B) swap under φ.

□

Example 7.2.6 (Harmonic Bundles from Complete Quadrilaterals)**.** Take a complete quadrilateral $\mathcal{Q} = (P_1, P_2, P_3, P_4)$. Let ℓ be the line through $P_3P_1 \cap P_2P_4$ and $P_1P_2 \cap P_3P_4$. Then the DIT involution from ℓ has $P_3P_1 \cap P_2P_4$ and $P_1P_2 \cap P_3P_4$ as fixed points, so we immediately get the involution on ℓ is just harmonic conjugation in these two points, which gives us Proposition 2.2.8, the harmonic bundle property of a quadrilateral.

Example 7.2.7. Let H and Ω be the orthocenter and circumcircle respectively, and let M_a be the midpoint of $\overline{BC}$, let X_a be the second intersection of AM_a with Ω, and let Y_a be the intersection of ray M_aH with Ω. Similarly define X_b, Y_b, X_c, Y_c. Prove that X_aY_a, X_bY_b, X_cY_c concur on the Euler line.

Solution. Let O be the circumcenter and G the centroid. Let A', B', C" be the antipodes of A, B, C in Ω. Note that O lies on AA' and H lies on $A'Y_A$. Let U, V be the two intersections of OH with Ω. By applying DIT with OH with respect to (A, X_A, Y_A, A') and Ω, we get an involution φ exchanging

$$(H, G), (X_AY_A \cap OH, O), (U, V).$$

Since this involution φ only depends on H, G, U, V which are fixed, it follows that $\varphi(O) \in X_AY_A$, so by symmetry the result follows. □

Example 7.2.8 (Pascal's Octagrammum Mysticum)**.** Let $A, A', B, B', C, C', D, D'$ be eight conconic points. Consider the sixteen intersections of the two complete quadrilaterals $(AA')(BB')$ and $(CC')(DD')$. Prove that:

- The eight intersection points with an odd number of apostrophes (such $AB \cap C'D$) are conconic;
- The eight intersection points with an even number of apostrophes (such as $AB \cap CD$)) are also conconic.

Solution. Let's set up some notation. Let I be a subset $1, 2, 3, 4$, and let P_I be $A^{a_1}B^{a_2} \cap C^{a_3}D^{a_4}$, where a_i is an apstrophe if and only if $i \in I$. For example, $P_{134} = A'B \cap C'D', P_{12} = A'B' \cap CD$. We only need to prove that $P, P_{12}, P_{13}, P_{23}, P_{14}, P_{24}$ are conconic, and the rest follows by symmetry. Consider taking a DIT involution with respect to (A, A', B, B') and CD. Since we are given that A, A', B, B', C, D are conconic, we have by DIT that

$$(C, D), (P_1, P_2), (P, P_{12})$$

are pairs under some involution φ. However, if we consider the DIT involution on $(P_{13}, P_{14}, P_{23}, P_{24})$ and line CD, then $(C, D), (P_1, P_2)$ are also pairs under some involution φ'. Since two pairs uniquely determine an involution, it follows that $\varphi \equiv \varphi'$, so by the converse of DIT it follows that $P_{13}, P_{14}, P_{23}, P_{24}, P, P_{12} = \varphi(P)$ are conconic. $\square$

Remark. An alternative proof follows from quartic Cayley-Bacharach **??** on the quartics made out of the union of the two conics, the union of the four lines of quadrilateral $(AB)(CD)$ and $(A'B')(C'D')$.

Note that Desargues's Involution Theorem is completely projective, so let's look at its dual (which we will call DDIT):

Theorem 7.2.9 (Dual of Desargues's Involution Theorem)**.** Given complete quadrilateral $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$ and a point P that doesn't lie on any of the component lines of $\mathcal{Q}$, define points $L_{ij} = (\ell_i \cap \ell_j)P$, then

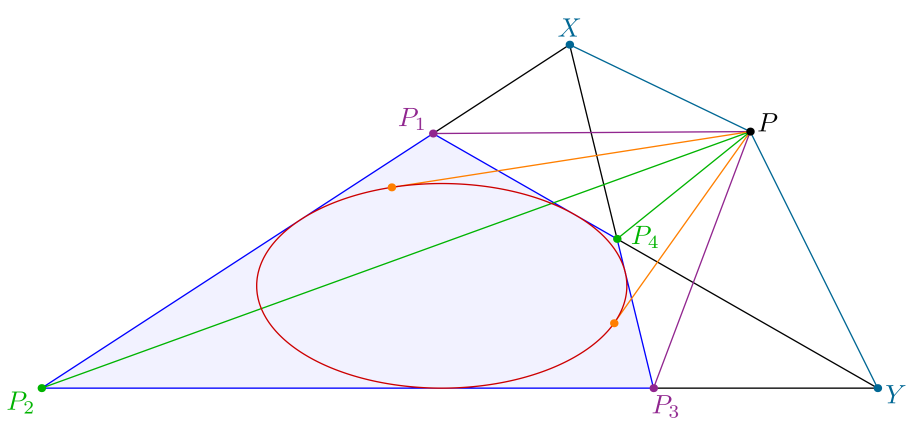
- There exists a unique involution φ in the pencil of lines through P ($\mathbb{T}P$) such that

  $$(L_{23}, L_{14}), (L_{31}, L_{24}), (L_{12}, L_{34})$$

  swap under φ,

- For any conic $\mathcal{C}$ tangent to all of $\ell_1, \ell_2, \ell_3, \ell_4$, the two tangents from P to this conic also swap under φ. Similarly to DIT, the converse also holds; for two lines s, t through point P, $\ell_1, \ell_2, \ell_3, \ell_4, s, t$ are tangent to a common conic (possibly degenerate) if and only if (s, t) swap under φ.

We will call the involution φ given by (i) as the DDIT involution from point P with respect to $\mathcal{Q}$.



Remember the nine-point (or nine-line) conic? Definition 6.3.10. The involution given by (D)DIT is actually just polarity with respect to this conic!

Proposition 7.2.10. Let φ be the involution given by DIT with respect to line ℓ and complete quadrangle $\mathcal{q}$, and let $\mathcal{C}$ be the ℓ-nine point conic of $\mathcal{q}$. Then (A, B) swap under φ if and only if A, B are conjugate points in $\mathcal{C}$.

Proof. Consider the involution $\varphi' = [A \to \mathfrak{p}_{\mathcal{C}}(A) \cap \ell] \in \operatorname{Aut}(\ell)$, and R_{ij} be the point such that $(P_i, P_j; Q_{ij}, R_{ij}) = -1$. Then we know that

$$Q_{23} = R_{12}R_{31} \cap R_{24}R_{34}, Q_{14} = R_{12}R_{24} \cap R_{31}R_{34},$$

so by Brokard, etc, we have that Q_{23}, Q_{14} are conjugate in $\mathcal{C}$, or $\varphi'(Q_{23}) = Q_{14}, \varphi'(Q_{14}) = Q_{23}$. Similarly we have $\varphi'(Q_{31}) = Q_{24}$, and since involutions are determined by two pairs, $\varphi = \varphi'$. Thus (A, B) swap under φ if and only if $\varphi'(A) = B$, equivalent to A, B being conjugate in $\mathcal{C}$. $\square$

Similarly, as a dual:

Proposition 7.2.11. Let φ be the involution given by DDIT, and let $\mathcal{C}$ be the P-nine line conic of complete quadrangle $\mathcal{q}$. Then (s, t) swap under φ if and only if s, t are conjugate lines in $\mathcal{C}$.

This gives us a corollary:

Corollary 7.2.12. The fixed points of φ (DIT) in ℓ are the two intersections of $\mathcal{C}$ and ℓ.

Corollary 7.2.13. The fixed lines of φ (DDIT) in $\mathbf{T}P$ are the two tangents from P to $\mathcal{C}$.

It's pretty useful to know all involutions, so let's characterize them.

Proposition 7.2.14 (Desargues's Assistant Theorem)**.** Let ℓ be a line, and let $\varphi : \ell \mapsto \ell$ be a automorphism, then φ is an involution if and only if it's some inversion centered on ℓ (possibly a negative inversion or just a reflection, also.)

Proof. We have already shown that inversions are involutions earlier, so let's show the other direction. Let $A = \varphi(\infty_\ell)$, so we consider two cases

(i) If $A \neq \infty_\ell$, then take distinct two pairs $(P, P'), (Q, Q')$ with respect to φ. Then we can note that

$$(A, \infty_\ell; P, Q) = (\infty_\ell, A; P', Q') \implies \frac{AP}{AQ} = \frac{Q'A}{P'A} \implies \overrightarrow{AP} \cdot \overrightarrow{AP'} = \overrightarrow{AQ} \cdot \overrightarrow{AQ'}$$

so $\vec{AP} \cdot \vec{AP'}$ is fixed, and φ is inversion about A.

(ii) If $A = \infty_\ell$, then there take a non-fixed point (P, P'), then for any involution (Q, Q'), then

$$(\infty_\ell, P; Q, Q') = (\infty_\ell, P'; Q', Q) \implies \frac{Q'P}{QP} = \frac{QP'}{Q'P'} \implies \vec{PQ} = \vec{Q'P'}$$

, so φ is reflection about the midpoint M of $\overline{PP'}$.

One way to do it is with knowledge of Möbius transformations (as real-line-preserving Möbius transformations are either a rotation of the real line or a inversion and reflection on the real line), but a straight length-bash to prove cross ratios being preserved under inversion is obviously possible, by considering cross ratios with the point at infinity. This is left as an exercise to the reader. $\square$

Remark. An alternative proof exists by working in $\mathbb{CP}^2$, the complex projective plane. Let I, J be the circle points, and let $(Q, Q'), (R, R')$ be pairs of the involution.

Now consider the DIT involution φ' given by the complete quadrangle (P_1, P_2, I, J) and line ℓ, where I, J are the two circle points,

$$P_1 = QI \cap RJ, P_2 = Q'J \cap R'I,$$

such that $(Q, Q'), (R, R')$ are two pairs that swap under involution φ', so $\varphi' = \varphi$. This tells us that for any involutive pair (P, P'), we have P_1, P_2, P, P', I, J lie on one conic, or P_1, P_2, P, P' are concyclic. If we let $A = P_1P_2 \cap \ell$, then φ is just inversion with power $AP_1 \cdot AP_2$.

Proposition 7.2.15 (Involutions on a Conic)**.** Let $\mathcal{C}$ be a conic, and let $\varphi : \mathcal{C} \to \mathcal{C}$ be a projective map from $\mathcal{C}$ to itself, then φ is an involution if and only if there exists a point $A \notin \mathcal{C}$ such that $A, P, \varphi(P)$ are collinear.

Proof. We've shown the converse before as well. Let (P, P') and (Q, Q') be distinct pairs that swap under the involution φ, and define $A = PP' \cap QQ'$, then for a pair of points (R, R') that swap under the involution, we have

$$\begin{aligned} P(R, R'; A, Q) &= (R, R'; P', Q)_\mathcal{C} = (R', R; P, Q')_\mathcal{C} \\ &= (R, R'; Q', P)_\mathcal{C} = Q(R, R'; A, P), \end{aligned}$$

so $A \in RR'$. $\square$

A corollary of this is Proposition 3.3.2:

Corollary 7.2.16. Inversion preserves cross ratio.

Considering the dual of this proposition, we get

Proposition 7.2.17. Let $\mathcal{C}$ be a conic, and let $\varphi : \mathbf{T}\mathcal{C} \to \mathbf{T}\mathcal{C}$ be a transformation on the set of lines tangent to $\mathcal{C}$. Then φ is an involution on this pencil if and only if there exists a unique line $K \notin \mathbf{T}\mathcal{C}$ such that for all $S \in \mathbf{T}\mathcal{C}$, $K, S, \varphi(S)$ are concurrent.

As such, if we want to prove that the three lines formed by six points on a fixed conic are concurrent, this is equivalent to proving that they're three pairs under an involution on the conic. (dually, if we want to prove six lines tangent to a conic that intersect at three points are collinear, we can do the same).

This allows us use projective maps/transformations to send the six points to others on the conic, as we will show in the following example.

Example 7.2.18. Let Ω be the circumcircle of $\triangle ABC$, and let M be a point on arc BC of Ω. Let B', C' respectively lie on CA, AB such that BB', CC' are both parallel with AM. Let MB', MC' respectively intersect Ω again at P, Q. Finally, let S be the intersection of PQ and BC. Prove that AS is the tangent to Ω at A.

Solution. We want to prove that the tangent at A, PQ, BC are concurrent. This is equivalent to proving that there's an involution on Ω that swaps

$$(A, A), (P, Q), (B, C)$$

where P, Q are the second intersections of MB', MC' with Ω. Since the DDIT involution on complete quadrilateral (AB, CA, BB', CC') exchanges

$$(MB, MC), (MP, MQ), (MA, M\infty_{AM}),$$

projecting from M onto Ω finishes. □

DIT also gives us a way to show four points are cylic.

Proposition 7.2.19 (DIT on the Line at Infinity)**.** Four points P_1, P_2, P_3, P_4 are concyclic if and only if the involution φ given by DIT on $\mathcal{L}_\infty$ and complete quadrangle (P_1, P_2, P_3, P_4) has two fixed points ∞_I, ∞_J which are orthogonal to each other.

Proof. Represent ∞_θ as the point at infinity with angle θ. First suppose that $P_1P_2P_3P_4$ is cyclic. Recalling the circle angle notation $(P_1, P_2, P_3, P_4)_\Omega = \measuredangle P_1P_2 + \measuredangle P_3P_4 = \measuredangle P_1P_4 + \measuredangle P_2P_3 = \measuredangle P_1P_4 + \measuredangle P_2P_3$, this means that

$$\varphi(\infty_W) = \infty_{(P_1+P_2+P_3+P_4)_\omega - W}$$

In other words, for a line ℓ whose reflection across the angle bisector of $\measuredangle(P_1P_2, P_3P_4)$ is ℓ', we exchange ∞_ℓ and $\infty_{\ell'}$.

This gives the two fixed points by taking $W = \frac{1}{2}(P_1 + P_2 + P_3 + P_4)$, or the internal and external angle bisectors of $\measuredangle(P_1P_2, P_3P_4)$.

Conversely, suppose the two points $F_1, F_2 \in \mathcal{L}_\infty$ are fixed and orthogonal. Then $2F_1 = 2F_2$, so

$$\varphi(W) = \infty_{2F_1 - W}$$

or ℓ, ℓ' are related by reflection about a line through ∞_{F_1}, so

$$\measuredangle P_3P_4 + \measuredangle P_2P_4 = 2F_1 = \measuredangle P_1P_2 + \measuredangle P_3P_4$$

and thus $(P_1P_2P_3P_4)$ is cyclic. □

Proposition 7.2.20. Given complete quadrangle q and line ℓ, let F_1, F_2 be the fixed points of the DIT involution φ. Then for any circumconic $\mathcal{C}$ of q, F_1, F_2 are conjugate wrt. $\mathcal{C}$.

Proof. If $\mathcal{C}$ and ℓ intersect at points $A \neq B$, then this follows since

$$(A, B; F_1, F_2) = -1$$

by the definition of an involution. If $A = B$ then the second fixed point is ∞_ℓ, and the result follows.

If $\mathcal{C}$ and ℓ don't intersect at any (real) points, consider a homography (defined in section Section 7.A) of φ that sends ℓ to the line at infinity $\mathcal{L}_\infty$. Here, since $\mathcal{C}$ does not intersect ℓ, we have $\varphi(\mathcal{C})$ is an ellipse. Then take an affine transformation sending $\mathcal{C}$ to a circle (that preserves the line at infinity). Therefore, by Proposition 7.2.19, the fixed points are perpendicular on the line at infinity, so they are conjugate in the circle. □

Example 7.2.21. By setting ℓ as the line between $P_3P_1 \cap P_2P_4$ and $P_1P_2 \cap P_3P_4$, we get the duality property (La Hire's) of polars.

Dually, we have

Proposition 7.2.22. Given complete quadrilateral $\mathcal{Q}$ and point P, let f_1, f_2 be the fixed lines of the DDIT involution φ. Then for any inconic $\mathcal{C}$ of $\mathcal{Q}$, f_1, f_2 are conjugate wrt. $\mathcal{C}$.

Proposition 7.2.23. Let P and Q have cevapoint S in $\triangle ABC$. Then P and Q are conjugate in a circumconic $\mathcal{C}$ of $\triangle ABC$ if and only if $\mathcal{C}$ passes through S.

Proof. Let X, Y, Z respectively be the intersections of BC, CA, AB with PQ, and let $X^\vee, Y^\vee, Z^\vee$ be the intersections of AS, BS, CS with PQ. Then there exists an involution $\varphi : PQ \to PQ$ such that

$$(X, X^\vee), (Y, Y^\vee), (Z, Z^\vee)$$

are exchanged. Then from Proposition 7.1.24,

$$(X, X^\vee; P, Q) = (Y, Y^\vee; P, Q) = (Z, Z^\vee; P, Q) = -1,$$

so P, Q are the two fixed points of φ. Therefore from Proposition 7.2.20, for any conic through A, B, C, point S lies on $\mathcal{C}$ iff. P, Q are conjugate in $\mathcal{C}$. $\square$

Practice Problems

Problem 1 (2017 China TST 2 P3)**.** Let $ABCD$ be a quadrilateral, and let ℓ be a line that intersects AB, CD, BC, DA, AC, BD at X, X', Y, Y', Z, Z'. Prove that the circles with diameters XX', YY', ZZ' are coaxal.

Problem 2. For $\triangle ABC$ and three points P, Q, R, let D, E, F respectively be the intersections of QR with BC, CA, AB, let AP and QR intersect at A_1, and let A_2 be the point on QR such that

$$(Q, R; F, A_1) = (R, Q; E, A_2),$$

and similarly define B_2, C_2. Prove that AA_2, BB_2, CC_2 are concurrent.

Problem 3 (Circum-ex config)**.** Let I^a be the the A-excircle of triangle $\triangle ABC$. Let the common tangents from I^a and (ABC) intersect BC at X, Y. Prove that AX, AY are isogonal lines in $\angle A$.

Problem 4. Given $\triangle ABC$, let D be a point on side BC, let I, I_1, and I_2 be the incenters of $\triangle ABC$, $\triangle ABD$, and $\triangle ADC$. Let M and N respectively be the second intersection of (ABC) with (IAI_1) and (IAI_2). Prove that when D moves on BC, MN passes through a fixed point.

Problem 5. Let I^b, I^c be the B and C excenters of triangle $\triangle ABC$. Let ω be the incircle of $\triangle ABC$, and let D be the A-intouch point. Let DI^b, DI^c intersect ω again at X, Y, prove that AD, BX, CY are concurrent.

Problem 6 (Taiwan 2019 Test 1 I3-2)**.** Given convex pentagon $ABCDE$, let A_1 be the intersection point of sides BD and CE, let B_1 be the intersection point of CE and DA, and similarly define C_1, D_1, E_1. Let A_2 be the second intersection of (ABD_1) with (AEC_1), let B_2 be the second intersection of $(BCE_1), (BAD_1)$, and similarly define C_2, D_2, E_2. Prove $AA_2, BB_2, CC_2, DD_2, EE_2$ are concurrent.

Problem 7. Let I^b, I^c be the B, C-excenters of acute triangle $\triangle ABC$, and let D be a point on the circumcircle (ABC) such that $AD \perp BC$. Let DI^b, DI^c intersect (ABC) again at Y, Z, and let $X = YZ \cap BC$. Prove that the triangle formed by lines AX, AO, BC is isosceles.

Problem 8 (2022 EGMO P6)**.** Let O be the circumcenter of cyclic quadrilateral $ABCD$, let X be the intersection of the angle bisector of angle A and angle B, and let Y be the intersection of the angle bisector of B and C, and define Z, W similarly. Let P be the intersection of AC and BD. Suppose X, Y, Z, W, O, P are all distinct. Prove that O, X, Y, Z, W are concyclic if and only if P, X, Y, Z, W are concyclic.

7.3 Isogonal Conjugation

We've already introduced isogonal conjugation (see Section 1.3), but this time we will show their relatoinship to conics.

First, recall this property (see Proposition 1.3.9):

Proposition 7.3.1. For complete n-gon $\mathcal{N}(\ell_1, \ell_2, \dots, \ell_n)$ and a point P, let P_i be the foot from P to ℓ_i. Then P has an isogonal conjugate in the n-gon $\mathcal{N}$ if and only if $P_1, P_2, \dots, P_n$ are concyclic.

Also remember that if P^* is the isogonal conjugate of P wrt. $\mathcal{N}$, then the pedal polygon of P^* and P are concyclic at the midpoint of PP^*, and the feet from from P^* are the other intersections of the pedal circle with the sides.

Here's how we will relate isogonal conjugation to conic sections.

Theorem 7.3.2. For complete n-gon $\mathcal{N}(\ell_1, \ell_2, \dots, \ell_n)$ and two points P, P^*, then (P, P^*) are a pair of isogonal conjugates in $\mathcal{N}$ if and only if there exists an inconic $\mathcal{C}$ with (P, P^*) as foci, internally tangent to $\mathcal{N}$.

Proof. Let $P_1^*, P_2^*, \dots, P_n^*$ respectively be the reflections of P^* across the sides $\ell_1, \ell_2, \dots, \ell_n$. Then P_n^* are all concyclic with circumcenter at P.

- ($\Rightarrow$) Let T_i be the intersection of PP_i^* with ℓ_i. Then $\overline{PT_i} \pm \overline{P^*T_i} = \overline{PP_i^*}$ is constant (with the $\pm$ depending on signed length whether P is outside $\triangle ABC$ or not). This is the exact definition of an ellipse/hyperbola, so now we claim the conic with P, P^* through all T_i is our desired inconic, which is true because of the optical property for conics; since ℓ_i is the angle bisector of $\angle PT_iP^*$, so $\mathcal{C}$ is tangent to all ℓ_i and is thus an inconic of $\mathcal{N}$.
- ($\Leftarrow$) Let $T_i = \mathbf{T}_{\ell_i}\mathcal{C}$ be the touchpoint of the inconic on side ℓ_i with foci P, P^*. Then the reflections of P^* across the sides of $\mathcal{N}$ lie on a circle, so the pedal circle of P has center as the midpoint of $\overline{PP^*}$ by a homothety of $\frac{1}{2}$ at P.

□

Remark. Another correct proof is through the use of Proposition 6.A.4's definition of the foci of a conic and the circle points. Suppose we have an inconic $\mathcal{C}$ of n-gon $\mathcal{N}$, with foci P, P^*. Then by DDIT with P on $IPJP^*$, there exists an involution swapping

$$(A_{ij}P, A_{ij}P^*), (A_{ij}I, A_{ij}J), (\ell_i, \ell_j)$$

for all i, j, where ℓ_i, ℓ_j are the two tangents from P. Thus $\measuredangle A_{ij}P + \measuredangle A_{ij}P^* = \ell_i + \ell_j$.

For the converse, consider DDIT on the inconic $\mathcal{C}$ of $PI, PJ, P^*I, P^*J, \ell_1$. Thus there's an involution swapping

$$(A_{1j}P, A_{1j}P^*), (A_{1j}I, A_{1j}J), (\ell_1, \ell_j)$$

So ℓ_j is tangent to $\mathcal{C}$ as well, for all j.

In the triangle case of this theorem, we have:

Proposition 7.3.3. Let (P, P^*) be a pair of isogonal conjugates in $\triangle ABC$, and let $\mathcal{C}$ be an inconic with P, P^* as foci. Let T be the BC-touchpoint of $\mathcal{C}$. If X is the $AP \cap BC$, then

$$\frac{BT}{TC} \times \frac{BX}{XC} = \left(\frac{BP}{PC}\right)^2.$$

Before proving this, we need a small lemma:

Lemma 7.3.4. Let (P, P^*) be a pair of isogonal conjugates in $\triangle ABC$, and let P_a^* be the reflection of P^* across BC. Then (A, P_a^*) are isogonal conjugates in $\triangle PBC$.

Proof. Angle chase the following:

$$\measuredangle CBA = \measuredangle CBP^* + \measuredangle P^*BA = \measuredangle P_a^*BC + \measuredangle CBP = \measuredangle P_a^*BP,$$
$$\measuredangle ACB = \measuredangle ACP^* + \measuredangle P^*CB = \measuredangle PCB + \measuredangle BCP_a^* = \measuredangle PCP_a^*.$$

□

Proof of 7.3.3. By the optical property, the tangent T of the ellipse with (P, P^*) as foci to BC lies on PP_a^*. Additionally, PX, PT are isogonal in $\angle BPC$ by the above lemma. As such, by Proposition 1.3.2, we have

$$\frac{BT}{TC} \times \frac{BX}{XC} = \left(\frac{BP}{PC}\right)^2.$$

□

Since T is defined symmetrically wrt. P, P^*, we also have the corresponding equation

$$\frac{BT}{TC} \times \frac{BX^*}{X^*C} = \left(\frac{BP^*}{P^*C}\right)^2.$$

Multiplying these two expressions and using Proposition 1.3.2 we get:

Corollary 7.3.5. Let (P, P^*) be a pair of isogonal conjugates in triangle $\triangle ABC$, and let $\mathcal{C}$ be a inconic of $\triangle ABC$ with (P, P^*) as foci. Let T be the BC-touchpoint, then

$$\frac{BT}{TC} = \pm \frac{CA}{AB} \times \frac{BP}{PC} \times \frac{BP^*}{P^*C},$$

where the sign is positive when P, P^* are in $\angle BAC$.

With Proposition 7.1.26, we get

Proposition 7.3.6. Let P be one of the foci of inconic $\mathcal{C}$ of complete n-gon $\mathcal{N} = (\ell_1, \ldots, \ell_n)$, tangent to each side ℓ_i at T_i. Choose a point S_i on every ℓ_i such that $\measuredangle S_i P T_i = 90°$. Then $S_1, \ldots, S_n$ are collinear.

Proof. Follows from Proposition 7.1.26 since $S_i = \mathfrak{p}_{\mathcal{C}}(T_i) \cap P\infty_{\perp PT_i}$ lies on the polar of P wrt. $\mathcal{C}$. $\square$

Proposition 7.3.7. For complete n-gon $\mathcal{N} = (\ell_1, \ldots, \ell_n)$, let $T_1, \ldots, T_n$ be n non-collinear points on lines $\ell_1, \ldots, \ell_n$. Then the following statements are equivalent:

- There exists a conic $\mathcal{C}$ such that $\mathcal{C}$ is tangent to every ℓ_i at T_i.
- There exists a point P such that where if we define $A_{ij} = \ell_i \cap \ell_j$, line PA_{ij} bisects angle $\angle T_i P T_j$.

Then P is also one of the foci of the conic.

Proof. We first show (i) $\implies$ (ii). Let T_i be the tangency point from $\mathcal{C}$ to ℓ_i, and let P, P^* be the two focii of $\mathcal{C}$, and WLOG P is not on the line at infinity. By Theorem 7.3.2 we have that P, P^* are isogonal conjugates. Let P_i^* be the reflection of $P*$ over ℓ_i, so $T_i = PP_i^* \cap \ell_i$ by Proposition 6.1.6. Now note that $A_{ij}P \perp P_i^* P_j^*$ and $\overline{PP_i^*} = \overline{PP_j^*}$, so

$$\measuredangle PT_i + \measuredangle PT_j = \measuredangle PP_i^* + \measuredangle PP_j^* = 2 \cdot \measuredangle P_i^* P_j^* = 2 \cdot \measuredangle PA_{ij},$$

so PA_{ij} bisects $\angle T_i P T_j$.

Now suppose (ii) holds. Suppose there exists a point P such that PA_{ij} bisects $\angle T_i P T_j$. We first consider the case of $n = 3$. Let P^* be the isogonal conjugate of P in $\triangle A_{23} A_{31} A_{12}$, and let the inconic $\mathcal{C}$ with foci

P, P^* be tangent to ℓ_i at T_i'. Then we get that

$$\begin{aligned} 2 \cdot \measuredangle PT_1 &= (2\measuredangle PA_{31} - \measuredangle PT_3) + (2PA_{12} - \measuredangle PT_2) = 2(\measuredangle PA_{31} + \measuredangle PA_{12} - \measuredangle PA_{23}) \\ &= (2\measuredangle PA_{31} - \measuredangle PT_3') + (2\measuredangle PA_{12} - \measuredangle PT_2') = 2\measuredangle PT_1', \end{aligned}$$

by the other direction. This means that either $PT_1 = PT_2$. In the latter case, we get that

$$\measuredangle PT_2 = 2\measuredangle PA_{12} - \measuredangle PT_1 =\perp (2\measuredangle PA_{12} - \measuredangle PT_1') =\perp PT_2'$$

and similarly $\measuredangle PT_3 =\perp PT_3'$. However, by Proposition 7.3.6, this implies that T_1, T_2, T_3 lie on a line, contradiction, so $PT_1 = PT_1'$, and the result follows by symmetry.

Now take any $n > 3$, and consider $\triangle A_{jk}A_{ki}A_{ij}$. From the $n = 3$ case, we have that there exists a conic $\mathcal{P}_{ijk}$ tangent to ℓ_i, ℓ_j, ℓ_k at T_i, T_j, T_k with P as a focii. Note that knowing $P, T_i, T_j, \ell_i, \ell_j$ uniquely determines the conic, so

$$P^* = T_1\infty_{2\ell_1 - T_1P} \cap T_2\infty_{2\ell_2 - T_1P}$$

so all $\mathcal{C}_{12i}$ are the same, call this conic $\mathcal{C}$. Then $\mathcal{C}$ is tangent to ℓ_i at T_i for all i. □

Let's now consider how "isogonal conjugation" works in quadrilaterals. We will elaborate on this in Chapter Perfect Six-Point Sets and The Isoptic Cubic, but currently we only need this small result.

Proposition 7.3.8. The midpoint of any isogonal conjugates in a complete quadrilateral on $\mathcal{Q}$ lies on the Newton line.

Proof. These are foci of an inconic, so the result follows by Newton's second theorem Theorem 6.3.13. □

Through Theorem 4.1.11 we have:

Theorem 7.3.9. The isogonal conjugate of the Miquel point in a complete quadrilateral is the point at infinity along the Newton line.

Proof. Let $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$. Consider the Steiner line of this quadrilateral Proposition 4.1.6. The "circumcenter" of these four reflected points is the point at infinity perpendicular to the Steiner line, which is the point at infinity along the Newton line ∞_τ. □

Remark. A nice proof of this is by considering the unique parabola QL-Co1 tangent to all four sides of the quadrilateral. By the optical property of the parabola, the reflection of the focus of this parabola across all four tangent lines must lie on the directrix.

So the focus and directrix are the Miquel point and Steiner line respectively. Therefore, the parabola opens in the direction perpendicular to the Steiner line. However, a parabola is just an ellipse with one focus at the Miquel point and the other focus at ∞_τ, so they must be isogonal conjugates.

Let's return to triangles. Let φ^K represent isogonal conjugation in triangle $\triangle ABC$. For some object/set of points X, $\varphi^K(X)$ will represent the curve formed by isogonally conjugating all points in X.

Proposition 7.3.10. Let ℓ be a line and let φ^K represent isogonal conjugation in $\triangle ABC$. Then

(i) If ℓ passes through a vertex $V \in \{A, B, C\}$, then $\varphi(\ell)$ is another line through V.

(ii) Otherwise, $\varphi(\ell)$ is a circumconic of $\triangle ABC$. Conversely, the image of a circumconic under φ is a line.

Proof. For (i), let A be a point on ℓ, then for a point P on ℓ, we have

$$\measuredangle(A\varphi(P)) = \measuredangle(AB) + \measuredangle(AC) - \measuredangle(AP) = \measuredangle(AB) + \measuredangle(AC) - \measuredangle(\ell)$$

is a fixed constant, so $\varphi(\ell) = A\infty_{\measuredangle(AB)+\measuredangle(AC)-\measuredangle(\ell)}$, and it's easy to see all points on $\varphi(\ell)$ correspond to lines on ℓ. For (ii), let P_1, P_2, P_3, P_4 be four points on ℓ, then

$$B(\varphi(P_\bullet)) = B(P_\bullet) = C(P_\bullet) = C(\varphi(P_\bullet)),$$

so $\varphi(\ell)$ is either a conic through B, C or a line L not passing through either. However, the latter implies ℓ passes through A, a vertex, contradiction. Therefore $\varphi(\ell)$ is a conic through B, C. Apply cyclically to get it also passes through A.

□

Proposition 7.3.11. Given $\triangle ABC$, let I, I^a, I^b, I^c be the incenter and three excenters. Let

$$\mathscr{F} = \{\mathcal{D} \mid I, I^a, I^b, I^c \in \mathcal{D}\}$$

be the pencil of conics through I, I^a, I^b, I^c. Then two isogonal conjugates (P, P^*) will be conjugate points in any conic $\mathcal{D}$ in this pencil.

Proof. Assume $P \neq P^*$. Consider the DIT involution on complete quadrangle (I, I^a, I^b, I^c) and PP^*. This involution $\psi \in \mathrm{Aut}(PP^*)$ swaps

$$(II^a \cap PP^*, I^bI^c \cap PP^*), (II^b \cap PP^*, I^cI^a \cap PP^*), (II^c \cap PP^*, I^aI^b \cap PP^*)$$

Since this corresponds to harmonic conjugation about P, P^*, Proposition 7.2.20 gives us that they are conjugate points in all circumconics $\mathcal{D}$ of this quadrangle. $\square$

Remark. This leads to an alternate definition of isogonal conjugates as the common point of the polar of P wrt. all conics in this pencil. We will elaborate on this in the following section, with general isoconjugations.

Practice Problems

Problem 1. Denote φ to be isogonal conjugation in $\triangle ABC$. Denote ℓ to be a straight line that does not pass through the points A, B and C. Prove that $A(\ell \cap BC)$ and $\mathbf{T}_A\varphi(\ell)$ are isogonal lines in $\angle BAC$.

Problem 2 (2000 ISL G3)**.** Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

Problem 3. Let (P, P^*) and (Q, Q^*) be two pairs of isogonal conjugate points about $\triangle ABC$ such that $PP^* \parallel QQ^*$. Denote I to be the incenter and I^a, I^b, I^c to be the three excenters of $\triangle ABC$. Denote M and N to be midpoints of $\overline{PP^*}$ and $\overline{QQ^*}$ respectively. Prove that I, I^a, I^b, I^c, M, N are conconic.

Problem 4 (Saragossa's Theorems)**.** For point P and triangle $\triangle ABC$, let $\triangle P_aP_bP_c$ be its cevian triangle and let $\triangle P'_aP'_bP'_c$ be its circumcevian triangle.

- Let $U = P_bP'_c \cap P_cP'_b, V = P_cP'_a \cap P_aP'_c, W = P_aP'_b \cap P_bP'_a$. Prove that AU, BV, CW concur at a point $Sa_1(P)$, and that $Sa_1(P)$ lies on conic $(ABCKP)$, where K is the symmedian point of $\triangle ABC$.
- Prove that P_aU, P_bV, P_cW concur at a point $Sa_2(P)$, and $K, Sa_1(P), Sa_2(P)$ are collinear.
- Prove that P'_aU, P'_bV, P'_cW concur at a point $Sa_3(P)$, and that $Sa_3(P)$ is the crosspoint of $P, Sa_1(P)$ wrt. $\triangle ABC$, and also that $P, Sa_2(P), Sa_3(P)$ are collinear.

This also holds for redefining the circumcevian triangle with any circumconic $\mathcal{C}$, and redefining the symmedian point as the perspector of $\mathcal{C}$ and $\triangle ABC$.

7.4 General Isoconjugations

In this section, we will generalize isogonal conjugation, which will be notated as $\varphi^K(P) = P^*$. We will use the projective properties of isogonal conjugation we developed in Proposition 7.3.11.

Definition 7.4.1. Given $\triangle ABC$, we can say that

$$\varphi : \mathbb{P}^2 \backslash (BC \cap CA \cap AB) \to \mathbb{P}^2 \backslash (BC \cap CA \cap AB)$$

is an **isoconjugation on points** if there exists a pencil $\mathscr{F} = \mathscr{F}_\varphi$ in the set of conics through four points $\{X, X^A, X^B, X^C\}$ such that

- For any conic $\mathcal{D}$ in this pencil (a **diagonal conic** of the isoconjugation), $\triangle ABC$ is self-conjugate in $\mathcal{D}$.
- For any point P and any conic $\mathcal{D}$ in the pencil, $P, \varphi(P)$ are conjugate in $\mathcal{D}$.

Proposition 7.4.2 (Fundamental Theorem of Isoconjugations)**.** Given a fixed triangle $\triangle ABC$, a isoconjugation on points φ is uniquely determined by two distinct diagonal conics in the pencil of this isoconjugation $\mathcal{D}_0, \mathcal{D}_\infty$. (This is equivalent to saying $\mathfrak{p}_{\mathcal{D}_0}(P) \neq \mathfrak{p}_{\mathcal{D}_\infty}(P)$.)

Proof. We consider the transformation $\varphi(P) = \mathfrak{p}_{\mathcal{D}_\infty}(\mathfrak{p}_{\mathcal{D}_0}(P))$, then we have $\varphi(A) = A, \varphi(B) = B, \varphi(C) = C$. Suppose there exists a $P_0 \neq BC \cap CA \cap AB$ such that $\varphi(P_0) = P_0$. However, this transformation can only have three fixed points by Proposition 7.A.10, so thus if this P_0 exists, we have $\varphi = \mathrm{id}$, so for any point P in the plane, we have $\mathfrak{p}_{\mathcal{D}_0}(P) = \mathfrak{p}_{\mathcal{D}_\infty}(P)$, so $\mathcal{D}_0 = \mathcal{D}_\infty$, contradiction. $\square$

Definition 7.4.3. Given $\triangle ABC$, we say

$$\varphi : (\mathbb{P}^2)^\vee \backslash (\mathbf{T}A \cap \mathbf{T}B \cap \mathbf{T}C) \in (\mathbb{P}^2)^\vee \backslash (\mathbf{T}A \cap \mathbf{T}B \cap \mathbf{T}C)$$

is a **isoconjugation on lines** if there exists a pencil of conics $\mathscr{F}$ through four points such that

- For any $\mathcal{D} \in \mathscr{F}$ (a **diagonal conic**), we have that $\triangle ABC$ is self-conjugate in $\mathcal{D}$.
- For any line $\ell \notin (\mathbf{T}A \cap \mathbf{T}B \cap \mathbf{T}C)$ we have that $\ell, \varphi(\ell)$ are isoconjugates in D.

Similarly, any conic satisfying these properties must go through these four points as well.

By duality, we can only address theorems on isoconjugation on points.

Theorem (Barycentric Form Of Isoconjugation)**.** In barycentric coordinates, isoconjugation is a transformation in the form

$$[x : y : z] \to \left[\frac{u}{x} : \frac{v}{y} : \frac{w}{z}\right] = [uyz : vxz : wxy]$$

Proof. It's equivalent to prove that

$$([\triangle PBC] \cdot [\triangle\varphi(P)BC]) : ([\triangle APC] \cdot [\triangle A\varphi(P)C]) : ([\triangle ABP] \cdot [\triangle AB\varphi(P)])$$

is a constant. Let $P_A = AP \cap BC, \varphi(P)_A = A\varphi(P) \cap BC$. By symmetry, it's sufficient to prove that

$$\frac{BP_A}{P_AC} \cdot \frac{B\varphi(P)_A}{\varphi(P)_AC} = \frac{[\triangle ABP] \cdot [\triangle AB\varphi(P)]}{[\triangle APC] \cdot [\triangle A\varphi(P)C]}$$

is a constant. We can just let $U = \varphi(\infty_{BC})$, and then we can rewrite the LHS as

$$\begin{aligned} A(B, C; P, \infty_{BC}) \cdot A(B, C; \varphi(P), \infty_{BC}) &= A(B, C; P, \infty_{BC}) \cdot A(C, B, P, U) \\ &= A(B, C; P, \infty_{BC}) \cdot A(B, C; U, P) \\ &= A(B, C; U, \infty_{BC}), \end{aligned}$$

and the final expression is independent of P.

To prove the converse, i.e. given $[u : v : w]$ such that $uvw \neq 0$, we want to prove that

$$\varphi : [x : y : z] \mapsto \left[\frac{u}{x} : \frac{v}{y} : \frac{w}{z}\right]$$

is an isoconjugation. For any line $\ell = 0x + qy + rz$ through A, we have that its image $\varphi(\ell) = 0x + rwy + qvz$, so $\varphi : \mathbf{T}A \to \mathbf{T}A$ is a projective involution, and thus by symmetry and Proposition 7.4.10, φ is a point isoconjugation. $\square$

In light of this, we define the following:

Definition. For two points / lines $P = [x_P : y_P : z_P], Q = [x_Q : y_Q : z_Q]$, we have that the isoconjugation φ sending P to Q has coefficients

$$[u : v : w] = [x_P x_Q : y_P y_Q : z_P z_Q].$$

We call this "point" the **barycentric product** of P and Q, and we will notate it as $P \times Q$. Similarly, we define the **barycentric quotient**

$$P \div Q = \left[\frac{x_P}{x_Q} : \frac{y_P}{y_Q} : \frac{z_P}{z_Q}\right]$$

and then for any point R, $\varphi^{P\times Q}(R) = P \times Q \div R$.

We call $P = [u : v : w] = \varphi(G)$ the **pole** of the isoconjugation, so we notate this isoconjugation as $\varphi^{P\times Q}$. Under this notation, isogonal conjugation becomes φ^K. Similarly, for an isoconjugation on lines, we define $\varphi(\mathcal{L}_\infty)$ to be the **polar** of this isoconjugation. For example, isotomic **line** conjugation's polar is just $\mathcal{L}_\infty$.

Remark (Type-errors)**.** The barycentric product of two points is not another point — it just looks like one. To get a point out of a sequence of barycentric operations $\times$ and $\div$, the number of products must be equal to the number of quotients. We will formalize this with the notion of "weight" in chapter 12.

Similarly to isogonal conjugation, we have that:

Proposition 7.4.4. Given $\triangle ABC$, let φ be a isoconjugation on points and let ℓ be a line.

(i) If ℓ goes through one of A, B, C, suppose X, then $\varphi(\ell)$ is a line that also passes through X.

(ii) If ℓ does not go through the three vertices, then $\varphi(\ell)$ is a circumconic of $\triangle ABC$.

(iii) With respect to all the vertices A, B, C, then the maps

$$\mathbf{T}X \overset{\varphi}{\to} \mathbf{T}X, \quad XP \to X\varphi(P)$$

are an involution on the pencil of lines through X. (Note that this is well-defined by part (i)).

Proof. (i) Assume WLOG that $X = A$. For two points $P, Q \in \ell$, we want to prove that $A, \varphi(P), \varphi(Q)$ are collinear. Let $\mathcal{D}_0, \mathcal{D}_\infty \in \mathscr{F}$ be two diagonal conics in the pencil of conics given by the isoconjugation. Then $\varphi(P) = \mathfrak{p}_{\mathcal{D}_0}(P) \cap \mathfrak{p}_{\mathcal{D}_\infty}(P), \varphi(Q) = \mathfrak{p}_{\mathcal{D}_0}(Q) \cap \mathfrak{p}_{\mathcal{D}_\infty}(Q)$. Let $R = \ell \cap BC$, then

$$\mathfrak{p}_{\mathcal{D}_0}(A, P; Q, R) = (A, P; Q, R) = \mathfrak{p}_{\mathcal{D}_\infty}(A, P; Q, R).$$

Note that $\mathfrak{p}_{\mathcal{D}_0}(A) = BC = \mathfrak{p}_{\mathcal{D}_\infty}(A)$, so

$$\varphi(P) = \mathfrak{p}_{\mathcal{D}_0}(P) \cap \mathfrak{p}_{\mathcal{D}_\infty}(P), \varphi(Q) = \mathfrak{p}_{\mathcal{D}_0}(Q) \cap \mathfrak{p}_{\mathcal{D}_\infty}(Q), A = \mathfrak{p}_{\mathcal{D}_0}(R) \cap \mathfrak{p}_{\mathcal{D}_\infty}(R)$$

are collinear.

(ii) This is proven in the same way as for isogonal conjugation by a homography (Proposition 7.3.10).

(iii) It is obvious that the order of this automorphism is 2, so we only need to prove that cross-ratio is preserved. Let $\mathcal{D}$ be a diagonal conic in the pencil given by φ, and let $\mathcal{L}$ be a line not passing through a vertex, and let $O = \mathfrak{p}_{\mathcal{D}}(\mathcal{L})$, and let U be a point on $\mathcal{L}$, then since polarity preserves cross-ratio we know that

$$(A, B; C, \varphi(U))_{\varphi(\mathcal{L})} = O(A, B; C, \varphi(U)) = (BC \cap \mathcal{L}, CA \cap \mathcal{L}; AB \cap \mathcal{L}, U).$$

Therefore

$$[XP \mapsto X\varphi(P)] = [S \mapsto XS] \circ [U \mapsto \varphi(U)] \circ [XP \mapsto XP \cap \mathcal{L}]$$

is a projective map.

$\square$

Corollary 7.4.5 (Isoconjugations Give More Isoconjugations)**.** For an arbitrary $\triangle ABC$ and point isoconjugation φ, for two points P, Q, let $R = PQ^\varphi \cap P^\varphi Q, S = PQ \cap P^\varphi Q^\varphi$. Then $S = R^\varphi$.

Proof. By the above listed propositions we know that $A(B,C), A(P,P^\varphi), A(Q,Q^\varphi), A(R,R^\varphi)$ define an involution on the pencil $\mathbf{T}A$. By DDIT we have $A(P,P^\varphi), A(Q,Q^\varphi), A(R,S)$ also define an involution on the pencil $\mathbf{T}A$, so thus $AR^\varphi = AS$. Applying this symmetrically we get $S = R^\varphi$. □

Proposition 7.4.6 (Projective characterization of barycentric product)**.** Let $(P,P^*),(Q,Q^*)$ be any pairs of points and let $R = PQ^* \cap P^*Q$. Then the following statements are equivalent:

(i) $P \times P^* = Q \times Q^*$;

(ii) P,Q,R lie on a common circumconic of $\triangle ABC$

(iii) P^*,Q^*,R^* lie on a common circumconic on $\triangle ABC$;

(iv) There exists a common inconic tangent to the three sides of $\triangle ABC$ and the four sides of the quadrilateral $(PP^*)(QQ^*)$.

(v) There is an involution on the pencil of lines $\mathbf{T}A$ swapping $(AB,AC),(AP,AP^*),(AQ,AQ^*)$ (and similarly for B,C).;

Proof. By the remark, (i) implies an isoconjugation on both pairs, which is equivalent to (v).

Now, let c be the common inconic of $(PP*)(QQ*)$ with BC. Let ℓ_B, ℓ_C be the other tangents of B,C to c. As such, by DDIT there exists an involution

$$(BC,\ell_B),(BP,BP^*),(BQ,BQ^*)$$

so $\ell_B = AB, \ell_C = AC$ is equivalent to both (iv) and (v).

Next, (v) implies (ii), since $(AB,AC),(AP,AP^*),(AQ,AQ^*),(AR,AR^*)$ swap under an involution by (v), and $(BC,BA),(BP,BP^*),(BQ,BQ^*),(BR,BR^*)$ swap under an involution, we have

$$A(C,P;Q,R) = A(B,P^*;Q^*,R^*) = B(A,P^*;Q^*,R^*) = B(C,P;Q,R),$$

and thus P,Q,R and A,B,C are conconic.

Next, (ii) implies (iii) because

$$\begin{aligned} A(P^*,Q^*;B,C) &= A(P,Q;C,B) = R(P,Q;C,B) \\ &= R(Q^*,P^*;C,B) = R(P^*,Q^*;B,C), \end{aligned}$$

so P^*, Q^*, R lie on a common circumconic of $\triangle ABC$. Finally, (iii) implies (v) because

$$A(P,Q;B,C) = R(P,Q;B,C) = R(Q',P',B,C) = A(Q',P';B,C).$$

so there's an involution swapping $(AB,AC),(AP,AP^*),(AQ,AQ^*)$. $\square$

What follows is a result that we'll use in **??**:

Theorem 7.4.7. Given $\triangle ABC$, let $\mathscr{L}$ represent the set of all lines not passing through either of A,B,C. Let $\mathscr{C}$ represent the set of all circumconics of $\triangle ABC$. Then we have the following perfect pairing:

$$\begin{aligned} \mathscr{L}\times\mathscr{C} &\longrightarrow \text{Isoconjugations in } \triangle ABC \\ (\mathcal{L},\mathcal{C}) &\longmapsto \varphi^{\mathcal{L}\times\mathcal{C}} \end{aligned},$$

where $\varphi^{\mathcal{L}\times\mathcal{C}}$ satisfies $\varphi^{\mathcal{L}\times\mathcal{C}}(\mathcal{L}) = \mathcal{C}$. In other words, for any line not passing through a vertex of $\triangle ABC$, we have the following one-to-one correspondence between isoconjugations and circumconics:

$$\begin{aligned} \text{Isoconjugations in } \triangle ABC &\longleftrightarrow \text{Circumconics of } \triangle ABC \\ \varphi &\longmapsto \varphi(\mathcal{L}) \end{aligned}.$$

Proof. If there exists two isoconjugations φ,ψ such that $\varphi(\mathcal{L}) = \psi(\mathcal{L})$, let $P := \mathfrak{t}(L)$ be the trilinear polar of L, then from Corollary 7.4.9 we know that $\varphi(P) = \psi(P)$. For any point Q, we have

$$A(B,C;\varphi(G),\varphi(Q)) = A(C,B;G,Q) = A(B,C;\psi(G),\psi(Q)),$$

so $A,\varphi(Q),\psi(Q)$ are collinear. Analogously we can get that $B,\varphi(Q),\psi(Q)$ and $C,\varphi(Q),\psi(Q)$ are respectively collinear, so $\varphi(Q) = \psi(Q)$. This proves that our map between pairs of lines and circumconics and isoconjugations is injective.

Next, for any circumconic $\mathcal{C}$ and line $\mathcal{L}$, we will give a construction φ such that $\mathcal{C} = \varphi(\mathcal{L})$. For any point P, let $\triangle P_AP_BP_C$ be the $\mathcal{C}$-cevian triangle of P wrt. $\triangle ABC$, then let

$$P_A' = P_A(\mathcal{L}\cap BC)\cap\mathcal{C}, P_B' = P_B(\mathcal{L}\cap CA)\cap\mathcal{C}, P_C' = P_C(\mathcal{L}\cap AB)\cap\mathcal{C}.$$

By Pascal, we get that $P, X := \mathcal{L}\cap BC, Y := P_A'P_B\cap CA, Z := P_A'P_C\cap AB$ are collinear (this will be used in **??**), so

$$\begin{aligned}(A, AP'_A \cap BP'_B; AP'_A \cap BC, P'_A) &\overset{B}{=} (A, P'_B; C, P'_A) \overset{P_B}{=} (A, \mathcal{L} \cap CA; C, Y)\\ &\overset{X}{=} (A, \mathcal{L} \cap AB; B, Z) \overset{P_C}{=} (A, P'_C; B, P'_A)\\ &\overset{C}{=} (A, CP'_C \cap AP'_A; AP'_A \cap BC, P'_A),\end{aligned}$$

and thus AP'_A, BP'_B, CP'_C concur at a point $\varphi(P)$ (this is also known as **Jacobi's Theorem**). Since $[P_A \mapsto P'_A]$ is an involution, from Proposition 7.2.15, we have that

$$[AP \mapsto A\varphi(P)] = [P'_A \mapsto A\varphi(P)] \circ [P_A \mapsto P'_A] \circ [AP \mapsto P_A]$$

is also an involution. Similarily, $[BP \mapsto B\varphi(P)], [CP \mapsto C\varphi(P)]$ is also an involution, so then from Proposition 7.4.10 we can get that φ is a point isoconjugation. $\square$

Remark. This is also immediate by the barycentric equation for a circumconic and a line.

Notably, for a line $\mathcal{L}$, let $\mathcal{C} = \mathcal{L}^\varphi$ be its image under the isoconjugation φ. Then we have that $P \mapsto G^\varphi \div P$ is a map from $\mathcal{L}$ to $\mathcal{C}$.

Multiplying both sides by $\mathcal{L}$, we get a pairing $(L, C) \in \mathcal{L} \times \mathcal{C}$ for which $L \times C = C \times L = G^\varphi$, so if φ sends P to Q, then in a sense

$$P \times Q = \mathcal{L} \times \mathcal{C}.$$

Proposition 7.4.8. Given $\triangle ABC$ and φ a point isoconjugation on $\triangle ABC$, then for an arbitrary point P, $\varphi(P)$ is the perspector of $\triangle ABC$ and $\mathfrak{p}_{\varphi(\mathfrak{t}(P))}(\triangle ABC)$, where $\mathfrak{t}(P)$ is the trilinear polar of P.

Proof. From Proposition 7.4.4,

$$(A\varphi(P), \mathbf{T}_A\varphi(\mathfrak{t}(P)); AB, AC) = A(P, \mathfrak{t}(P) \cap BC; C, B) = -1,$$

so $\mathfrak{p}_{\varphi(\mathfrak{t}(P))}(A) = \mathbf{T}_A\varphi(\mathfrak{t}(P))$ is a corresponding side of the anticevian triangle of $\varphi(P)$. Similarly, we get $\mathfrak{p}_{\varphi(\mathfrak{t}(P))}(B), \mathfrak{p}_{\varphi(\mathfrak{t}(P))}(C)$ are also corresponding sides of the anticevian triangle, so $\varphi(P)$ is the perspector of $\triangle ABC$ and $\mathfrak{p}_{\varphi(\mathfrak{t}(P))}(\triangle ABC)$. $\square$

Choosing P as the centroid G, we get

Corollary 7.4.9. Let φ be an isoconjugation on triangle $\triangle ABC$ with pole as $\varphi(G)$, and let $\triangle^{\varphi(G)}$ be the anticevian triangle of $\varphi(G)$. Then $\varphi(\mathcal{L}_\infty)$ is the circumconic with perspector $\varphi(G)$ (i.e. the inconic of $\triangle^{\varphi(G)}$ through A, B, C).

Proposition 7.4.10. Consider a transformation φ from

$$\varphi : \mathbb{P}^2 \backslash BC \cup CA \cup AB \to \mathbb{P}^2 \backslash BC \cup CA \cup AB$$

that has for all vertices $X \in A, B, C$,

$$X, P_1, P_2 \text{ collinear} \implies X, \varphi(P_1), \varphi(P_2) \text{ collinear}$$

and $[XP \to X\varphi(P)]$ is an involution swapping $(XY, XZ))$, where Y, Z are the other two vertices. Then φ has to be an isoconjugation.

Proof. Obviously we have that $\varphi^2 = \text{id}$, so φ is an involution. Choose some point P_0 that's not one of the vertices of the triangle, or a fixed point of φ. Then note that φ is completely determined by where it sends P_0; for any vertex X, we have

$$X(Y, Z; P_0, P) = X(Z, Y; \varphi(P_0), \varphi(P)), \varphi(P) = B\varphi(P) \cap C\varphi(P),$$

where Y, Z are the two other vertices that aren't X. To prove that this is an isoconjugation, we need to find (at least) two conics $\mathcal{D}_{\prime}, \mathcal{D}_{\infty}$ such that $\triangle ABC$ is self-conjugate in both of these conics and $\varphi(P)$ is the intersection of the polars of P in $\mathcal{D}_{\prime}, \mathcal{D}_{\infty}$ (see Proposition 7.4.2). Since the map on the pencil of lines through X sending $XP \to X\varphi(P)$ is a projective map, we only need to satisfy

$$\varphi(P_0) = \mathfrak{p}_{\mathcal{D}_{\prime}}(P_0) \cap \mathfrak{p}_{\mathcal{D}_{\infty}}(P_0).$$

So how do we find these two conics? Choose $\mathcal{D}_{\prime}$ to be the conic through P_0 and tangent to the line $P_0\varphi(P_0)$ such that $\triangle ABC$ is self-conjugate in $\mathcal{D}_{\prime}$, and choose $\mathcal{D}_{\infty}$ to be defined similarly except passing through $\varphi(P_0)$ instead of P_0. Since $P_0 \neq \varphi(P_0)$, these conics are distinct. Thus since

$$\mathfrak{p}_{\mathcal{D}_{\prime}}(P_0) \cap \mathfrak{p}_{\mathcal{D}_{\infty}}(P_0) = P_0\varphi(P_0) \cap \mathbf{T}_{\mathcal{D}_{\infty}}(\varphi(P_0)) = \varphi(P_0)$$

we are done. $\square$

Remark. In the proof, if P_0 is a fixed point of the isoconjugation ($P_0 = \varphi(P_0)$), then just picking any two diagonal conics in the pencil of conics

$$\mathscr{F} = \{\mathcal{D} \mid P_0, P_0^a, P_0^b, P_0^c \in \mathcal{D}\},$$

where P_0^a, etc are the vertices of the anticevian triangle of P in $\triangle ABC$ actually suffices. (See Proposition 7.4.16).

Definition 7.4.11. Given $\triangle ABC$ and a line $\mathcal{L}$ not passing through the vertices of ABC, let represent the points on $\mathcal{L}$ not on the sides of $\triangle ABC$. Then we define

- For any isoconjugation φ, we define $\mathcal{L}^{\varphi} = \varphi(\mathcal{L})$.
- For any circumconic $\mathcal{C}$, we define $\mathcal{L}_{\mathcal{C}}$ as the isoconjugation obtained by the previously defined bijection between circumconic+lines and point isoconjugations.
- For a line isoconjugation φ, we define $\mathscr{L}^{\varphi}$ as the envelope of $\varphi(\mathbf{T}_{\mathscr{L}})$.
- For an arbitrary inconic c, we define $\mathscr{L}_{\mathcal{C}}$ as the line isoconjugation obtained by the previously defined bijection between inconics + lines and line isoconjugations.

As an example of this notation, the circumcircle Ω of $\triangle ABC$ is notated as $\mathcal{L}_{\infty}^{\varphi^K}$, where φ^K represents isogonal conjugation. For a point isoconjugation φ, we still have one goal we haven't accomplished; which is writing out all possible diagonal conics of isoconjugations. Due to the bijection from Theorem 7.4.7, we can replace φ with a line not passing through the vertices of $\triangle ABC$ and a circumconic $\mathcal{C} = \mathcal{L}^{\varphi}$.

By definition, for a conic $\mathcal{D} \in \mathscr{F}$, we must always have $\mathfrak{p}_{\mathcal{D}}(\mathcal{L}) \in \mathcal{C} = \mathcal{L}^{\varphi}$, else two distinct points on $\mathcal{L}$ would have the same polar. This raises the question of how, for a point $O \neq A, B, C$ on $\mathcal{C}$, we can construct the diagonal conic $\mathcal{D}_O$ such that $\mathcal{L}$ is the polar of O in $\mathcal{D}_O$? (Note that $\mathcal{D}_{\mathcal{O}}$ sends A, B, C, O respectively to $BC, CA, AB, \mathcal{L}$, so $\mathcal{D}_{\mathcal{O}}$ is uniquely.)

In fact, here's a way to construct $\mathcal{D}_O$ (note that $\mathcal{D}_O$ can be possibly imaginary): Let $\triangle O_AO_BO_C$ be the cevian triangle of O in $\triangle ABC$. Consider the involution $\varphi_A \in \mathrm{Aut}(AO)$ such that $\varphi_A(A) = O_A, \varphi_A(O) = AO \cap \mathcal{L}$. Let F_{A1}, F_{A2} be the two fixed points of this involution φ_A, and similarly define $F_{B1}, F_{B2}, F_{C1}, F_{C2}$.

Proposition 7.4.12. $\mathcal{D}_O$ is the common conic through $F_{A1}, F_{A2}, F_{B1}, F_{B2}, F_{C1}, F_{C2}$, and $\triangle ABC$ is self-conjugate in this conic with $\mathfrak{p}_{\mathcal{D}_O}(O) = \mathcal{L}$.

Proof. Take a homography sending $\mathcal{L}$ to $\mathcal{L}_\infty$, to make our notation easier. Then since F_{A1}, F_{A2} are fixed points under this involution, we have

$$(A, O_A; F_{A1}, F_{A2}) = (O, \infty_{AO}; F_{A1}, F_{A2}) = -1.$$

Therefore O is the midpoint of $\overline{F_{A1}F_{A2}}$. Further, F_{A2} is one of the vertices of the anticevian triangle of F_{A1} wrt. $\triangle ABC$, and let's call the other vertices F_{A3}, F_{A4}. Let F'_{A3} be the reflection of F_{A3} across O, then the conic $\mathcal{D}_{OA} := (F_{A1}F_{A2}F_{A3}F_{A4}F'_{A3})$ satisfies the condition of having $\triangle ABC$ as its self-conjugate triangle and having the polar of O as the line at infinity $\mathcal{L}_\infty$ (note that $F'_{A3} \neq F_{A4}$). Similarly, we can define $\mathcal{D}_{OB}, \mathcal{D}_{OC}$ which also satisfy the above characteristics, but by uniqueness of conics through six points, we have $\mathcal{D}_{OA} = \mathcal{D}_{OB} = \mathcal{D}_{OC}$, and thus this is our desired $\mathcal{D}_O$. $\square$

So we can say that the pencil of diagonal conics of φ, $\mathscr{F} = \mathscr{F}_\varphi$ is just

$$\{\mathcal{D}_O \mid O \in \mathcal{L}^\varphi\}.$$

You might be familiar with the concept of cross ratio on a pencil of conics. If not, read Section 7.A. We can define a cross ratio on this pencil here through $\mathcal{L}^\varphi$:

$$(\mathcal{D}_{O_\bullet}) = (\mathcal{D}_{O_1}, \mathcal{D}_{O_2}; \mathcal{D}_{O_3}, \mathcal{D}_{O_4})_{\mathscr{F}} = (O_\bullet)_{\mathcal{L}^\varphi}.$$

To prove that this works as a cross ratio (i.e preserved under homography), we can prove it is invariant wrt. the choice of $\mathcal{L}$ (previously defined as the line at infinity).

Proposition 7.4.13. For any point P not on the sides of triangle ABC, then the map

$$\begin{aligned} \mathscr{F} &\xrightarrow{\mathfrak{p}_-(P)} \mathbf{T}\varphi(P) \\ \mathcal{D} &\longmapsto \mathfrak{p}_\mathcal{D}(P) \end{aligned}$$

is a projective map, irrespective of the choice of $\mathcal{L}$.

Proof. We proceed with projective maps. Let $\mathcal{C} = \mathcal{L}^\varphi$, then by definition we have the projective map

$$\begin{aligned} \mathcal{C} &\xrightarrow{\mathcal{D}_-} \mathscr{F} \\ O &\longmapsto \mathcal{D}_O \end{aligned},$$

We thus want to show that the composition $\mathfrak{p}_-(P) \circ \mathcal{D}_- : O \mapsto \mathfrak{p}_{\mathcal{D}_O}(P)$ is also a projective map. Note that

$$W = \mathfrak{p}_{\mathcal{D}_O}(P) \cap \mathcal{L} = \mathfrak{p}_{\mathcal{D}_O}(OP),$$

so $U = \varphi(W)$ is the second intersection of OP and $\mathcal{C} = \mathcal{L}^\varphi$. Since taking this second intersection is a projective map, we have that $O \to U, \varphi : \mathcal{L} \to \mathcal{C}$ are all projective maps, and we get

$$\mathfrak{p}_P \circ \mathcal{D} = [W \to \mathfrak{p}_{\mathcal{D}_O}(P) = \varphi(P)W] \circ \varphi \circ [O \to U]$$

is a sequence of projective maps, and thus is a projective map itself. $\square$

Proposition 7.4.14. For two different point isoconjugations φ, ψ on triangle $\triangle ABC$, let $\mathscr{F}_\varphi$ and $\mathscr{F}_\psi$ be the pencils of diagonal conics corresponding to these respective isoconjugations. Then the intersection of these two pencils has exactly one element $\mathcal{D}_{\varphi,\psi}$.

Proof. Given a line $\mathcal{L}$ not passing through a vertex, consider the fourth intersection point O of the circumconics $\mathcal{L}^\varphi, \mathcal{L}^\psi$. We know that there exists an element $\mathcal{D}_{\varphi O} \in \mathscr{F}_\varphi, \mathcal{D}_{\psi O} \in \mathscr{F}_\psi$ such that O's polars wrt. the two conics $\mathfrak{p}_{\mathcal{D}_{\varphi O}}(O), \mathfrak{p}_{\mathcal{D}_{\psi O}}(O)$ are both $\mathcal{L}$. Then by uniqueness of O, we know that $\mathcal{D}_{\varphi O} = \mathcal{D}_{\psi O}$, (notate it as $\mathcal{D}_{\varphi,\psi}$). Since a point isoconjugation is determined by two elements in its pencil of diagonal conics Proposition 7.4.2 and since $\varphi \neq \psi$, we get that $\mathscr{F}_\varphi$ and $\mathscr{F}_\psi$'s intersection must be solitary (which proves uniqueness of $\mathcal{D}_{\varphi,\psi}$). □

Under the mapping $\mathcal{D} \mapsto \mathfrak{p}_{\mathcal{D}}(P)$, the family $\mathscr{F}_\varphi$ becomes the set of lines through P^φ, so we can represent conics with lines. For two point isoconjugations φ, ψ, define their "meet" to be the common diagonal conic $\mathcal{D}_{\varphi,\psi} \in \mathscr{F}_\varphi \cap \mathscr{F}_\psi$, which represents the "line" connecting P^φ and P^ψ. Similarly, define the "join" of two diagonal conics $\mathcal{D}_1, \mathcal{D}_2$ to be the isoconjugation that has both of those diagonal conics in its pencil. This pencil is then the lines through the intersection of the "lines" of the conics. Notably, these notions of lines and joining are actually independent from the P chosen. As such, for a point P that's not on any side, we get that three isoconjugations φ, ψ, χ are collinear if and only if $\varphi(P), \psi(P), \chi(P)$ are collinear.

Example 7.4.15. In triangle $\triangle ABC$, for a point P, denote the isogonal conjugate of P in $\triangle ABC$ as P^*. Let P° be the orthologic conjugate of P (see **??**) - we define this as the isoconjugation with pole at H. Prove that if P is on the Euler line of $\triangle ABC$, then H, P^*, P° are collinear, where H is the orthocenter of $\triangle ABC$.

Solution. Let φ be an isoconjugation that sends H to P, then

$$P = \varphi(H), O = H^*, G = H^\circ$$

are collinear, where O and G represent the circumcenter and centroid, since P lies on the Euler line. Therefore,

$$H = \varphi(P), P^*, P^\circ$$

are also collinear. □

7.4.1 Fixed Points of Isoconjugation

What are the fixed points of isogonal conjugation? Clearly, they are I, I^a, I^b, I^c. Note that $I^aI^bI^c$ is the anticevian triangle of I. By taking a suitable homography, we get that if an isoconjugation φ has a fixed point S, then the vertices of its anticevian triangle S^a, S^b, S^c are also fixed points of φ.

Another way of seeing this insight is to know that a point Q is fixed under φ if and only if, for the pencil of conics $\mathscr{F}$ defined by φ, we have

$$Q \in \bigcap_{\mathcal{C} \in \mathscr{F}} \mathcal{C},$$

as then it is self-conjugate in every conic in the pencil.

As such, we can directly get that the maximum number of (real or complex) fixed points of an isoconjugation is 4, since two conics can only intersect at four points.

Let's now look at this from a barycentric point of view. Let G^{φ} be the pole of isoconjugation φ, then denote the **radical transformation** of G^{φ} as the set of points $\mathcal{S} = \{S, S^a, S^b, S^c\}$ such that $G^{\varphi} = P \times P$ for all $P \in \mathcal{S}$. Since S_a, S_b, S_c, etc are obtained from S by negating one of the barycentric coordinates, these are the fixed points of φ as well.

Analogously to Proposition 7.3.11, we can get

Proposition 7.4.16. For point S in $\triangle ABC$, let $\triangle S^a S^b S^c$ be the anticevian triangle of S in $\triangle ABC$, and let $\mathscr{F}$ be the pencil of conics passing through S, S^a, S^b, S^c. For another point $P \notin \{A, B, C\}$, draw lines ℓ_A, ℓ_B, ℓ_C through A, B, C such that AP and ℓ_A are harmonic conjugates in $SS^a, S^b S^c$.

Then ℓ_A, ℓ_B, ℓ_C intersect at a point Q, and for all conics $\mathcal{D} \in \mathscr{F}$, P, Q are conjugate in $\mathcal{D}$.

Thus, there exists an isoconjugation φ that has any point and the vertices of its anticevian triangle as fixed points.

Proof. We only need to prove that for the point Q satisfying

$$(BP, BQ; SS^b, S^c S^a) = (CP, CQ; SS^c, S^a S^b) = -1,$$

it follows that then $(AP, AQ; SS^a, S^b S^c) = -1$. By the same argument as in Proposition 7.3.11, there exists a involution $\psi \in \mathrm{Aut}(PQ)$ that exchanges

$$(SS^a \cap PQ, S^b S^c \cap PQ)$$

which means, P, Q are fixed under ψ, and thus

$$(AP, AQ; SS^a, S^b S^c) = -1.$$

□

Remark. If we treat $\triangle S^a S^b S^c \cup S$ as a complete quadrilateral, then we get that for a complete quadrilateral $\mathcal{Q}$ and a point P, that the polars of P in all circumconics of $\mathcal{Q}$ passes through a fixed point. We call this point the **involutary conjugate** of P in $\mathcal{Q}$ (this is QA-Tf2).

Since this means that $\mathscr{F} = \mathscr{F}_\varphi = \{D \mid S, S^a, S^b, S^c \in \mathcal{D}\}$, it turns out the most natural cross ratio in the pencil of conics $\mathscr{F}$ defined by the isoconjugation previously, is actually just the typical cross ratio defined as the cross ratio of the tangents:

$$(\mathcal{D}_\bullet)_{\mathscr{F}} := (\mathcal{D}_1, \mathcal{D}_2; \mathcal{D}_3, \mathcal{D}_4) = (\mathbf{T}_S\mathcal{D}_i),$$

since $\mathfrak{p}_{\mathcal{D}}(S) = \mathbf{T}_S\mathcal{D}$.

Proposition 7.4.17. Given $\triangle ABC$, let φ be a point isoconjugation such that there exists S, S^a, S^b, S^c as fixed points, then for any line ℓ, $\varphi(\ell)$ is the nine-point conic of quadrangle (S, S^a, S^b, S^c) wrt. ℓ Theorem 6.3.8.

Proof. Let $\mathcal{C}$ be the aforementioned nine-point conic, and let $Q^a, Q^b, Q^c, (Q^a)', (Q^b)', (Q^c)'$ respectively be the intersections of ℓ with $SS^a, SS^b, SS^c, S^bS^c, S^cS^a, S^aS^b$. Let R^a be the harmonic conjugate of Q^a in $\overline{SS^a}$, and similarly define $R^b, R^c, (R^a)', (R^b)', (R^c)'$, then since SS^a is a fixed line and S, S^a are fixed points, we know $R^a = \varphi(Q^a) \in \varphi(\ell)$, and similarly we have $R^b, R^c, (R^a)', (R^b)', (R^c)' \in \varphi(\ell)$, so $\mathcal{C} = \varphi(\ell)$. □

It should be said that we can also directly define nine-point conics like this, to make proving conconicity much easier. Let $\ell = \mathcal{L}_\infty$, then by Corollary 7.4.9 we can get that the nine-point conic of S, S^a, S^b, S^c is an inconic of the anticevian triangle of $\triangle G^\varphi$ with respect to $\triangle ABC$. We will end this section by generalizing this to arbitrary isoconjugations.

Theorem 7.4.18. For complete quadrangle (P_1, P_2, P_3, P_4), the locus of the center of all conics through P_1, P_2, P_3, P_4 is the nine-point conic of quadrangle (P_1, P_2, P_3, P_4).

Under a homography, this becomes

Theorem 7.4.19. For complete quadrangle (P_1, P_2, P_3, P_4) and a line ℓ, the locus of the pole of ℓ in all conics through P_1, P_2, P_3, P_4 is the nine-point conic of quadrangle (P_1, P_2, P_3, P_4) wrt. ℓ.

Proof. Let $\triangle ABC$ be the cevian triangle of quadrangle (P_1, P_2, P_3, P_4), and let φ be the isoconjugation on $\triangle ABC$ such that P_1, P_2, P_3, P_4 are the fixed points of the isoconjugation.

We now show that a point is the pole of ℓ through some circumconic if and only if it is on the nine-point conic.

($\Rightarrow$) Let $\mathcal{C}$ be a conic through P_1, P_2, P_3, P_4, then

$$\varphi(\mathfrak{p}_{\mathcal{C}}((\ell))) \in \mathfrak{p}_{\mathcal{C}}(\mathfrak{p}_{\mathcal{C}}((\ell))) = \ell,$$

and thus from Proposition 7.4.17 we know that $\mathfrak{p}_{\mathcal{C}}((\ell))$ lies on the nine-point conic of ℓ in (P_1, P_2, P_3, P_4).

($\Leftarrow$) Let O be a point on the nine-point conic of (P_1, P_2, P_3, P_4), and take the four points $Q_i \in \overline{OP_i}$ such that

$$(Q_i, P_i; OP_i \cap \ell, O) = -1,$$

and let $\mathcal{C}_i$ be the circumconic of P_1, P_2, P_3, P_4, Q_i, then the polar of O wrt. all the conics $\mathcal{C}_i$ will pass through a fixed point $\varphi(O)$, along with the point $OP_i \cap \ell$. If O is not a pole of ℓ in every $\mathcal{C}_i$, then we have $\varphi(O) = OP_i \cap \ell$, then O, P_1, P_2, P_3, P_4 must be all collinear, a contradiction.

□

The above theorems remain true when dualed and are proven similarly as well.

7.A Revisiting the Cross Ratio

Right now, we have defined the cross ratio on many objects, such as the set of points on a line, the set of lines through a point, and the set of points on a conic, or even the set of tangent lines to a conic, that are invariant under projective transformations. In general, we call these objects **pencils**, and let's rigorize this all into one definition.

Most of what is done here can be extended naturally to the complex line.

Definition 7.A.1. A pencil $(X, (-,-;-,-)_X)$ is a set X with a 4-ary rational function (to define a cross ratio)

$$\begin{aligned} (-,-;-,-)_X : X \times X \times X \times X &\dashrightarrow \mathbb{R} \cup \infty \cong \mathbb{RP}^1 \\ (P_1, P_2, P_3, P_4) &\mapsto (P_1, P_2; P_3, P_4)_X =: (P_\bullet)_X, \end{aligned}$$

such that there exists a bijective function $\varphi : \mathbb{RP}^1 \to X$ that satisfies $(a_\bullet)_X = (\varphi(a_\bullet))$, to the standard cross ratio for not three equal $a_1, a_2, a_3, a_4 \in \mathbb{RP}^1$.

Example 7.A.2. Here's some example of pencils:

- A fixed line and the points on it,
- The set of lines through a fixed point,
- A set of coaxal circles through two fixed points,
- A conic and the points on it,
- A conic and its tangent lines,

- The set of conics through four fixed points,
- The set of conics tangent to four fixed lines "on the same side".

All of these have a naturally defined cross ratio (for the conic ones, it's just the normal cross ratio on the tangents from another point to the four conics).

Remark (Linear combinations of curves)**.** You can think of all pencils as embeddings of $\mathbb{P}^1$. For example, the set of degree n curves in the complex projective plane can be thought to form a $\binom{n+2}{2} - 1$ dimensional projective space ($\binom{n+2}{2}$ coefficients in a homogeneous polynomial, minus one for homogenizing). "Passing through a point" reduces this dimension by 1, and thus the set of conics through four fixed points is a pencil. (the notion of "dimension" in this remark is a bit more complex and out of scope - check the appendix).

Below all of the expressions $(X_i, (-,-;-,-)_{X_i})$ represent pencils. We have already defined

$$A(P_\bullet) = A(P_1, P_2; P_3, P_4) = (AP_1, AP_2; AP_3, AP_4) = (AP_\bullet),$$

and if $\varphi : X_1 \to X_2$ is a projective map, then we define

$$\varphi(P_\bullet) = \varphi(P_1, P_2; P_3, P_4) = (\varphi(P_1), \varphi(P_2); \varphi(P_3), \varphi(P_4)) = (\varphi(P_\bullet))_{X_2}.$$

Definition 7.A.3. We define a map (function) $\varphi : X_1 \to X_2$ to be a **projective map** if for all $P_1, P_2, P_3, P_4 \in X_1$, $(\varphi(P_\bullet))_{X_2} = (P_\bullet)_{X_1}$.

Note that projective maps φ must be bijective. Denote the set of projective maps between two pencils X_1, X_2 as $\text{Hom}(X_1, X_2)$. Then, we let $\text{Aut}(X) := \text{Hom}(X, X)$. As, suchs for two projective transformations ψ, φ, we have

$$((\psi \circ \varphi)(P_\bullet))_{X_3} = (\psi(\varphi(P_\bullet)))_{X_3} = (\varphi(P_\circ))_{X_2} = (P_\bullet)_{X_1},$$

so we get

Proposition 7.A.4 (Composition of Projective Maps)**.** Given pencils X_1, X_2, X_3, if

$$\varphi \in \text{Hom}(X_1, X_2), \psi \in \text{Hom}(X_2, X_3),$$

then $\psi \circ \varphi \in \text{Hom}(X_1, X_3)$.

Proposition 7.A.5. Given pencils X_1, X_2 and map $\varphi \in \text{Hom}(X_1, X_2)$, then there exists an inverse $\varphi^{-1} \in \text{Hom}(X_2, X_1)$.

Proof. Since φ is bijective, φ^{-1} exists. For $P_1, P_2, P_3, P_4 \in X_2$,

$$\begin{aligned}(\varphi^{-1}(P_1), \varphi^{-1}(P_2); \varphi^{-1}(P_3), \varphi^{-1}(P_4)_{X_1} &= (\varphi(\varphi^{-1}(P_1)), \varphi(\varphi^{-1}(P_2)); \varphi(\varphi^{-1}(P_3)), \varphi(\varphi^{-1}(P_4)))_{X_2} \\ &= (P_1, P_2; P_3, P_4)_{X_2},\end{aligned}$$

thus $\varphi^{-1} \in \operatorname{Hom}(X_2, X_1)$. □

Proposition 7.A.6 (Projective Maps Are Determined By Three Points)**.** Given line ℓ and three distinct points P_1, P_2, P_3 on the line, if

$$\varphi : \{P_1, P_2, P_3\} \to \ell$$

such that $\varphi(P_1), \varphi(P_2), \varphi(P_3)$ are distinct, then there exists a unique one-to-one function $\varphi \in \operatorname{Aut}(\ell)$ such that $\varphi(P_i) = \varphi(P_i)$. (In other words, maps from a complex line $\mathbb{CP}^1$ to another complex line are uniquely determined by three points and their images.)

Proof. Pencils are isomorphic to $\mathbb{P}^1$, so we will just prove this for $\mathbb{CP}^1$. Suppose φ sends $P_i \to Q_i$, then define $\tilde{\varphi}$ to be

$$(Q_1, Q_2; Q_3, \tilde{\varphi}(P)) = (P_1, P_2; P_3, P) \implies \widetilde{\varphi}(P) = \frac{aP + b}{cP + d},$$

where a, b, c, d satisfy $ad \neq bc$, then we have $\widetilde{\varphi}(P_i) = \varphi(P_i)$. It remains to show $(\widetilde{\varphi}(Q_\bullet)) = (Q_\bullet)$, which is left to the eager reader.

As such, the group of automorphisms $\operatorname{Aut}(x)$ on a pencil is isomorphic to the group of fractional linear transformations, $\{p \to \frac{ap+b}{cp+d} \mid ad \neq bc\}$ which is also known as $\mathrm{PGL}_2(\mathbb{R})$. □

Proposition 7.A.7. Proposition 7.A.6 holds for two arbitrary pencils X_1, X_2 and three elements $P_1, P_2, P_3 \in X_1$.

Proof. There exists a homomorphism $\psi_i \in \operatorname{Hom}(X_j, \mathbb{P}^1)$ such that $\psi_1(P_i) = \psi_2(\varphi(P_i)))$. Now define $\tilde{\varphi} \coloneqq \psi_2^{-1} \circ \psi_1 \in \operatorname{Hom}(X_1, X_2)$, and note that $\tilde{\varphi}(P_i) = \varphi(P_i)$.

Now, if we define $\chi \coloneqq \tilde{\varphi}^{-1}\varphi \in \operatorname{Aut}(X_1)$, then for $Q \in X_1$,

$$(P_1, P_2; P_3, \chi(Q))_{X_1} = (\chi(P_1), \chi(P_2); \chi(P_3), \chi(Q))_{X_1} = (P_1, P_2; P_3, Q)_{X_1},$$

so χ is the identity and $\tilde{\varphi} = \varphi$. □

Proposition 7.A.8. Automorphisms on an arbitrary pencil are isomorphic to $\mathrm{PGL}_2(\mathbb{R})$ as well.

Proof. Taking $\varphi : X \to \mathbb{P}^1$, it follows that there's a mapping

$$\begin{array}{ccc} \operatorname{Aut}(X) & \xrightarrow{\mathcal{D}_-} & \operatorname{Aut}(\mathbb{P}^1) \\ & & \\ \psi & \longmapsto & \varphi\psi\varphi^{-1} \end{array},$$

as desired. □

Remember that we defined the pencil of lines through a fixed point as $\mathbf{T}P = \{\ell \mid P \in \ell\}$.

We call a bijective transformation on $\mathbb{P}^2$ which preserves lines a **collineation**, and one that also preserves cross-ratio as well, a **homography**. (The terminology "projective transformation" generally refers specifically to homographies, but is slightly ambiguous when used in $\mathbb{CP}^2$, as we will see later.)

Let's now build up the fundamental theorem of projective geometry:

Proposition 7.A.9 (Real Collineations are Homographies)**.** If a bijective function $\varphi : \mathbb{RP}^2 \to \mathbb{RP}^2$ preserves collinearities (so it is a **collineation**), that is

$$P \in QR \iff \varphi(P) \in \varphi(Q)\varphi(R),$$

then φ is a homography.

Proof. Let P_1, P_2, P_3, P_4 be four collinear points, and let Q_1, Q_2, Q_3, Q_4 be four collinear points on a different line. Define $R_i = P_iQ_{i+1} \cap P_{i+1}Q_i$. Then it's known through cross ratio chasing that R_1, R_2, R_3 are collinear if and only if $(P_\bullet) = (Q_\bullet)$. As such, it follows that $(P_\bullet) = (Q_\bullet)$ holds if and only if $\varphi(R_1), \varphi(R_2), \varphi(R_3)$ are collinear if and only if $\varphi(P_\bullet) = \varphi(Q_\bullet)$. It is thus well-defined to define $\tilde{\varphi}\colon \mathbb{P}^1_{\mathbb{R}} \mapsto \mathbb{P}^1_{\mathbb{R}}$:

$$\tilde{\varphi}((P_\bullet)) \coloneqq \varphi(P_\bullet)$$

and note that $0, 1, \infty$ are fixed points of $\tilde{\varphi}$.

Now, take two reals x, y, and let P_1, P_2, P_3, P_4, P_4' be points on a line such that

$$(P_1, P_2; P_3, P_4) = x, \quad (P_1, P_2; P_3, P_4').$$

Then it is possible to find solely through using lines points Q, R on ℓ such that

$$(P_1, P_2; P_3, Q) = xy, \quad (P_1, P_2; P_3, R) = x + y.$$

Then, if L is a line such that $AP_1 \parallel L$ and $P \mapsto AP \cap L$, then we can WLOG $P_1 = \infty, P_2 = 0, P_3 = 1$ which allows us to find Q, R. Thus, $\tilde{\varphi}$ is a field automorphism of the reals, so it must be the identity. $\square$

Proposition 7.A.10 (Real Projective Transformations Are Determined By Four Points)**.** If A, B, C, D are four points with no three collinear and $\varphi : \mathbb{RP}^2 \to \mathbb{RP}^2$ is a projective transformation, then the unique transformation fixing A, B, C, D is the identity transformation.

Proof. By the above, if four points are relabelled as X_1, X_2, X_3, X_4, then $(\varphi(X_\bullet)) = (X_\bullet)$. Taking $O = AD \cap BC$, we get that

$$\varphi(O) = \varphi(AD \cap BC) = AD \cap BC = O.$$

This implies that for any X on AD, that

$$(\varphi(A), \varphi(D), \varphi(O), \varphi(X)) = (A, D; O, X) = (\varphi(A), \varphi(D); \varphi(O), X)$$

so $X = \varphi(X)$, so for any Y,

$$\varphi(BY) \cap AD = \varphi(BY \cap AD) = \varphi(BY \cap AD)$$

so $\varphi(BY) = BY$. Likewise, $\varphi(CY) = CY, \varphi(AY) = AY$, so φ is the identity. $\square$

Remark. Combining Proposition 7.A.9 and Proposition 7.A.10, we can prove that a transformation φ preserves collinearities if and only if it sends the (homogeneous coordinates)

$$\varphi[x : y : z] = [\varphi_{11}x + \varphi_{12}y + \varphi_{13}z : \varphi_{21}x + \varphi_{22}y + \varphi_{23}z : \varphi_{31}x + \varphi_{32}y + \varphi_{33}z],$$

(also, the matrix formed by all the φ_{ij} has to be invertible.) This theorem is called the **Fundamental Theorem of Projective Geometry**. Further, for four points P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 (no three collinear), there exists a projective transformation φ that sends P_i to Q_i.

In complex projective space $\mathbb{CP}^2$, a collineation does not need to be determined by four points. For example take the transformation φ sending $[x : y : z]$ to $[\overline{x} : \overline{y} : \overline{z}]$. This preserves collinearities but does not preserve cross ratio (it complex conjugates it). It also cannot be written as the above matrix expression. The reason for the failure of our previous proof is because of some nontrivial automorphisms of $\mathbb{C}$ (there are no non-identity automorphisms of $\mathbb{R}$), such as $[z \to \overline{z}]$. In fact, the characterization we get for all continuous collineations in $\mathbb{CP}^2$ are compositions of projective transformations and reflections.

Proposition 7.A.11. Given two points P, Q, and $\varphi \in \mathrm{Hom}(\mathbf{T}P, \mathbf{T}Q)$, then the locus of $\ell \cap \varphi(\ell)$ is a (possibly degenerate) conic section. So there exists a conic section $\mathcal{C}$ such that $\mathcal{C} = \{\ell \cap \varphi(\ell) \mid \ell \in \mathbf{T}P\}$.

Proof. Take $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathbf{T}P, R_i = \ell_i \cap \varphi(\ell_i)$, so

$$P(R_\bullet) = (\ell_\bullet) = (\varphi(\ell_\bullet)) = Q(R_\bullet)$$

so $R_4 \in (PQR_1R_2R_3)$. Since R_4 is chosen arbitrarily the result follows. □

Proposition 7.A.12. Given a pencil X and two distinct elements P_1, P_2 in X, the projective involution $\varphi : X \to X$ sending element Q to R such that $(P_1, P_2; Q, R) = k$ for some fixed constant k is an automorphism on X.

Proof. Again, by taking an isomorphism, we only need to prove it for the case where X is a line and P_1, P_2 are two points on the line. Let $\ell \neq X$ be a line through P_1, and let R_1, R_2, R_3, R_4 be four points on line X such that $R_1 = P_1$ and $(R_\bullet) = k$. Then for any point Q, we have $R_2P_2, R_3Q, R_4\varphi(Q)$ are concurrent, and thus $\varphi = [S \to SR_4 \cap X] \circ [Q \to R_3Q \cap R_2P_2]$ is a projective map and in $\mathrm{Aut}(X)$. □

Proposition 7.A.13. Given conic $\mathcal{C}$, line ℓ, and point P on $\mathcal{C}$, let moving line L through P intersect $\mathcal{C}, \ell$ respectively at Q, R. Let A be a point on L such that $(P, Q; R, A)$ is fixed (let it equal k). Then the locus of A is a conic (or degenerate).

Proof. Draw a line K through P, and choose points P_1, P_2, P_3, P_4 on K such that $P_1 = P, P_2 = K \cap \mathcal{C}, P_3 = K \cap \ell$ and $(P_\bullet) = k$, then P_2Q, P_3R, P_4A are concurrent. Consider the sequence of projective maps

$$\varphi := [S \mapsto P_4S] \circ [Q \mapsto P_2Q \cap \ell] \circ [L \mapsto L \cap \mathcal{C}] \in \mathrm{Hom}(\mathbf{T}P, \mathbf{T}P_4)$$

and then from Proposition 7.A.11 we know that $A = L \cap \varphi(L)$'s locus is a conic (or double-cover of a line). □

Proposition 7.A.14 (Generalized Steiner Conic)**.** Given conic $\mathcal{C}$, $\varphi \in \mathrm{Aut}(\mathcal{C})$ if and only if there exists a line ℓ such that for all P, Q in $\mathcal{C}$, $P\varphi(Q) \cap \varphi(P)Q \in \ell$.

Remark. Note that this proves the fact that all involutive automorphisms on a conic are given by the second intersection map for some point. In fact, the line ℓ would just be the polar of that point in $\mathcal{C}$.

Proof. Fix three points A, B, C on conic $\mathcal{C}$. Then apply Pascal's theorem to the hexagon $A\varphi(B)C\varphi(A)B\varphi(C)$ to get that $B\varphi(C) \cap \varphi(B)C, C\varphi(A) \cap \varphi(C)A, A\varphi(B) \cap \varphi(A)B$ are collinear on some line ℓ'. Then consider the sequence of projective maps

$$\varphi' := [R \mapsto AR \cap \mathcal{C}] \circ [P \mapsto P\varphi(A) \cap \ell] \in \mathrm{Aut}(\mathcal{C}),$$

so A, B, C are the fixed points of $\varphi^{-1} \circ \varphi'$.

($\Rightarrow$) Since $\varphi \in \mathrm{Aut}(\mathcal{C})$ and we know the images of A, B, C, we can get $\varphi' = \varphi$. Then for any two $P, Q \in \mathcal{C}$, by applying Pascal's on

$$A\varphi(P)Q\varphi(A)P\varphi(Q),$$

we know that $P\varphi(Q) \cap \varphi(P)Q, Q\varphi(A) \cap \varphi(Q)A, A\varphi(P) \cap \varphi(A)P$ are collinear, and $P\varphi(Q) \cap \varphi(P)Q \in \ell$.

($\Leftarrow$) If $P\varphi(Q) \cap \varphi(P)A \in \ell$, then $\varphi'(P) = \varphi(P)$, and therefore $\varphi \in \mathrm{Aut}(\mathcal{C})$.

□

If you remember Poncelet's porism, this may seem familiar to you.

Proposition 7.A.15. For any conic $\mathcal{C}$ and $\varphi \in \mathrm{Aut}(\mathcal{C})$, there exists another conic $\mathcal{C}'$ such that for all $P \in \mathcal{C}$, $P\varphi(P) \in \mathbb{T}\mathcal{C}'$.

Proof. If φ is the identity, taking $\mathcal{C}'$. Then let A be a point on $\mathcal{C}$ such that $\varphi(A) \neq A$, then by the above characterization we know there exists a line ℓ such that for all other P on $\mathcal{C}$, $A\varphi(P) \cap \varphi(A)P \in \ell$. For a point P on $\mathcal{C}$, by Theorem 6.3.14, we know that $A\varphi(P) \cap P\varphi(A), AP \cap \varphi(A)\varphi(P), \mathfrak{p}_{\mathcal{C}}((A\varphi(A))), \mathfrak{p}_{\mathcal{C}}((P\varphi(P)))$ are collinear, and we also have by chasing harmonics that

$$(A\varphi(P) \cap P\varphi(A), AP \cap \varphi(A)\varphi(P); \mathfrak{p}_{\mathcal{C}}((A\varphi(A))), \mathfrak{p}_{\mathcal{C}}((P\varphi(P)))) = -1.$$

Note that $[AP \mapsto \varphi(A)\varphi(P)] \in \mathrm{Hom}(\mathbf{T}A, \mathbf{T}\varphi(A))$, so we have $AP \cap \varphi(A)\varphi(P)$'s locus is a conic $\mathcal{C}^*$. By setting $P = A$ we can get $\mathfrak{p}_{\mathcal{C}}((A\varphi(A))) \in \mathcal{C}^*$. From Proposition 7.A.13we know that $\mathfrak{p}_{\mathcal{C}}((P\varphi(P)))'s$ locus is a conic $\mathcal{C}^*$, and therefore $P\varphi(P) \in \mathbf{T}\mathcal{C}'$, where $\mathcal{C}' = \mathfrak{p}_{\mathcal{C}}((\mathcal{C}^*))$ is the dual conic of $\mathcal{C}'$. □

Remark. The converse of this theorem does not hold (in general, you get a degree-2 rational map).

It's worth noting that the you can't get all possible inconics $\mathcal{C}'$ by looking at automorphisms (the dimension of automorphisms is three and the dimension of inconics is five). In fact, (non-degenerate) $\mathcal{C}'$ can be obtained from an automorphism of $\mathcal{C}$ if and only if $\mathcal{C}'$ is tangent to $\mathcal{C}$ at two points (possibly imaginary). Note when $\mathcal{C}'$ is degenerate (a point), that this gives you an involution on a conic must pass through a fixed point, which is just the standard Steiner Conic theorem.

7.B Moving Points / The Polynomial Method

Typically, most objects we encounter in geometry can be represented as a sort of polynomial with respect to some other object (for example all the polynomials and birational parametrizations in analytic geometry), so we can actually analyze the degree of these objects and bound them, and then prove things with the

Fundamental Theorem of Algebra by checking only a few cases. Specifically, we will use the fact that if a degree-d polynomial $P(x)$ has $d+1$ roots, then $P(x)=0$.

Definition 7.B.1. Let $\iota : \mathbb{P}^1 \to \mathbb{P}^n$ be a algebraic map, then we can write it as

$$[s:t] \to [x_0(s,t) : x_1(s,t) : \cdots : x_n(s,t)],$$

where x_i are all homogeneous polynomials with the same degree, such that $\gcd(x_0, x_1, \ldots, x_n) = 1$, then we define the **degree** of ι as

$$\deg \iota := \deg x_0 = \deg x_1 = \cdots = \deg x_n.$$

If we consider t as time, then we get the namesake **moving points**.

Since we generally do geometry in the projective plane $\mathbb{P}^2$, for the following section we will only consider $n=2$ and $n=1$. Further, note that all lines and conics can be **parameterized** in terms of $\mathbb{P}^1$, which means that we can find some ι that bijectively maps $\mathbb{P}^1$ to a line / conic. (For an explicit parametrization, check the appendix).

Then for two parametrization $\mathbb{P}^1 \to \mathbb{P}^2$ of two points $\iota_A = [x_A : y_A : z_A], \iota_B = [x_B : y_B : z_B]$ (interpret these as two "moving points" A, B in $\mathbb{CP}^2$), we can write the line AB in terms of points A, B, so it can also be parameterized in terms of $\mathbb{P}^1$, like this:

$$\iota_{AB}^{\vee} := [s:t] \to \begin{bmatrix} y_A z_B - y_B z_A \\ z_A x_B - z_B x_A \\ x_A y_B - x_B y_A \end{bmatrix},$$

which clearly has a degree at most $\deg A + \deg B$.

We can tighten the bound a bit more by factoring out monomials, too:

Proposition 7.B.2 (Zack's Lemma)**.** For two moving points A, B, we have that line AB has degree at most

$$\deg \iota_{AB}^{\vee} \le \deg \iota_A + \deg \iota_B - \#\{P \mid \iota_A(P) = \iota_B(P)\}.$$

Notably, if $\#\{P \mid \iota_A(P) = \iota_B(P)\} \ge \deg \iota_A + \deg \iota_B + 1$, then $\iota_A = \iota_B$.

Proof. Let e_P denote the monomial where $e_P(P) = 0$. Then it follows that

$$\left(\prod_{\iota_A(P) = \iota_B(P)} e_P \right) \mid (y_A z_B - y_B z_A, z_A x_B - z_B x_A, x_A y_B - x_B y_A)$$

so the result follows out by factoring out a monomial for each t corresponding to when the two points overlap. $\square$

Remark. For the rest of this section, if we say "a point A moves on line/conic $\mathcal{K}$ projectively", it means that there is a (rational) parametrization of a point in $\mathbb{P}^2$ in terms of coordinates in $\mathbb{P}^1 \cong \mathbb{R} \cap \infty$ that covers the line/conic **exactly once**.

Note that a point ι covering a degree d curve n times has this parametrization degree of $d \cdot n$, and the converse also holds. This can be proven considering the number of intersections of ι with a general line using Bezout's lemma (see the appendix for more details and proof of the converse).

Since lines are degree 1 and conics are degree 2, a point moving projectively on a line/conic has degree 1 and 2 respectively.

Dually, a line parametrized in terms of $\mathbb{P}^1$, passing through a fixed point, covering the space of lines through that point exactly once also moves projectively and has degree 1.

Since conics are self-dual, a line tangent to a fixed conic moving projectively also has degree 2.

Here's the most important ways to bound degrees of points and lines in purely projective problems.

For the next few propositions, call two moving points $\iota_1 : \mathbb{P}^1 \to X_1, \iota_2 : \mathbb{P}^1 \to X_2$ **projectively related** if $\iota_2 = \varphi \circ \iota_1$ for some projective map $\varphi \in \operatorname{Hom}(X_1, X_2)$.

Proposition 7.B.3 (Projective maps between lines preserve degree)**.** Let two projectively related moving points ι_1, ι_2 move on lines ℓ_1, ℓ_2.Then $\deg \iota_1 = \deg \iota_2$.

Proposition 7.B.4 (Projective maps between conics preserve degree)**.** Let two projectively related moving points ι_1, ι_2 move on conics $\mathcal{C}_1, \mathcal{C}_2$. Then $\deg \iota_1 = \deg \iota_2$.

Proposition 7.B.5 (Conic doubling)**.** Let two projectively related moving points ι_1, ι_2 move on line ℓ and conic $\mathcal{C}$ respectively. Then $\deg \iota_2 = 2 \cdot \deg \iota_1$.

Proposition 7.B.6. If two moving points $\iota_A, \iota_B : \mathbb{P}^1 \to \mathbb{P}^2$ move on a conic $\mathcal{C}$, then the line through them has degree

$$\deg \iota_{AB}^{\vee} = \frac{1}{2}(\deg \iota_A + \deg \iota_B).$$

Proof. Let $\iota\colon \mathcal{C} \to \mathbb{P}^1$ be a projective map. Then by conic doubling, $\iota \circ \iota_A$ and $\iota \circ \iota_B$ have degrees $\frac{\deg \iota_A}{2}$ and $\frac{\deg \iota_B}{2}$, respectively. Let these functions map to $[P_A : Q_A]$ and $[P_B : Q_B]$. The line vanishes whenever $P_A Q_B - P_B Q_A = 0$ because then the moving points coincide. Hence by Zack's lemma the degree is $\deg \iota_A + \deg \iota_B - \deg(P_A Q_B - P_B Q_A) = \frac{1}{2}(\deg \iota_A + \deg \iota_B)$. $\square$

Corollary 7.B.7 (Conic doubling for non-fixed projections)**.** If $\iota_A : \mathbb{P}^1 \to \mathbb{P}^2$ is a moving point on fixed conic $\mathcal{C}$, and $\iota_L : \mathbb{P}^1 \to \mathbb{P}^2$ is a moving line through ι_A, let ι_B be the second intersection of ι_L with $\mathcal{C}$. Then we have

$$\deg \iota_B = 2 \deg \iota_L - \deg \iota_A.$$

These are enough ways to find the degree of most points in purely projective problems.

Theorem 7.B.8. Given three moving points $\iota_1, \iota_2, \iota_3 \colon \mathbb{P}^1 \to \mathbb{P}^2$, if

$$\#\{P \in \mathbb{P}^1 \mid \iota_1(P), \iota_2(P), \iota_3(P) \text{ collinear}\} \geq \deg \iota_1 + \deg \iota_2 + \deg \iota_3 + 1,$$

then for all $P \in \mathbb{P}^1$, $\iota_1(P)$, $\iota_2(P)$, and $\iota_3(P)$ are collinear.

Proof. If we let $\iota_i = [x_i : y_i : z_i]$, then the collinearity condition is equivalent to

$$\det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = 0.$$

The determinant has degree $\deg \iota_1 + \deg \iota_2 + \deg \iota_3$, so if it vanishes at $\deg \iota_1 + \deg \iota_2 + \deg \iota_3 + 1$ values, it hence is identically zero.

This also gives that the area of this triangle has degree $\deg \iota_1 + \deg \iota_2 + \deg \iota_3$. ☐

Remark. If you use a point for Zack's lemma, **you can't use it for your final Fundamental Theorem of Algebra check**, or the left hand side of the above theorem. This is generally the most common reason for moving point solutions to get docked to zero points in olympiads.

Corollary 7.B.9. Given six moving points $\iota_1, \dots, \iota_6 \colon \mathbb{P}^1 \to \mathbb{P}^2$, if

$$\#\{P \in \mathbb{P}^1 \mid \iota_1(P), \dots, \iota_6 \text{ conconic}\} \geq 2(\deg \iota_1 + \cdots + \iota_6) + 1,$$

then for all $P \in \mathbb{P}^1$, $\iota_1(P), \dots, \iota_6(P)$ are conconic.

Proof. Let $X(P) = \iota_1(P)\iota_2(P) \cap \iota_4(P)\iota_5(P)$, $Y(P) = \iota_2(P)\iota_3(P) \cap \iota_5(P)\iota_6(P)$, and $Z(P) = \iota_3(P)\iota_4(P) \cap \iota_6(P)\iota_1(P)$. Then by Pascal's, $\iota_1(P), \dots, \iota_6(P)$ are conconic if and only if $X(P)$, $Y(P)$, and $Z(P)$ are collinear. Note that we have

$$\deg X \leq \deg(\iota_1\iota_2) + \deg(\iota_4\iota_5) \leq \deg \iota_1 + \deg \iota_2 + \deg \iota_4 + \deg \iota_5$$

and similar bounds hold for Y and Z. Hence, the desired result follows from Theorem 7.B.8. ☐

Corollary 7.B.10. Given four moving points $\iota_1, \dots, \iota_4 \colon \mathbb{P}^1 \to \mathbb{P}^2$, if

$$\#\{P \in \mathbb{P}^1 \mid \iota_1(P), \dots, \iota_4 \text{ concyclic}\} \geq 2(\deg \iota_1 + \dots + \iota_4) + 1,$$

then for all $P \in \mathbb{P}^1$, $\iota_1(P), \dots, \iota_4(P)$ are concyclic.

Proof. Apply Corollary 7.B.9 to ι_1 through ι_4 together with the two circle points, which have degree 0 as they are fixed. □

Proposition 7.B.11. For four moving points $\iota_1, \dots, \iota_4 \colon \mathbb{P}^1 \to \mathbb{P}^1$, the cross ratio mapping $\iota_{(1,2;3,4)} \colon \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$\iota_{(1,2;3,4)}(P) = (\iota_1(P), \iota_2(P); \iota_3(P), \iota_4(P))$$

has degree

$$\deg \iota_{(1,2;3,4)} = \sum_{i=1}^{4} (\deg \iota_i - N_i) + 2N,$$

where $N_i = \#\{P \in \mathbb{P}^1 \mid \iota_j(P) \text{ all coincide}, j \neq i\}$ and $N = \#\{P \in \mathbb{P}^1 \mid \iota_j(P) \text{ all coincide}\}$. Also,

$$\deg \iota_{(1,2;3,4)} \leq \deg \iota_1 + \deg \iota_2 + \deg \iota_3 + \deg \iota_4.$$

Proof. Let $\iota_i(P) = [x_i(P) : y_i(P)]$. Then the cross-ratio is given by

$$\iota_{(1,2;3,4)} = [(x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2) : (x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2)].$$

Whenever all ι_i but one coincide, we can cancel out a linear factor from the two terms not involving ι_i. However, if all four ι_i coincide, we have canceled out four linear factors from each coordinate, whereas in reality we can only cancel two. □

Remark. This gives us a way to compute the degrees of angles (for example, angle $\measuredangle BAC$) by computing the degree of the cross ratio $(AB, AC; AI, AJ) \leq \deg B + \deg C$.

Theorem. For a fixed circle ω and a moving point ι, the degree of $\mathbf{Pow}_\omega(\iota)$ is

$$\deg \mathbf{Pow}_\omega(\iota) = 2 \deg \iota - \{\# \text{ of times where } \iota \text{ passes through either circle point}\}.$$

Proof. By taking an arbitrary homography, we can let ω be given by $x^2 + y^2 + bx + cy + d = 0$. Let ι be given by $\iota(P) = [x(P), y(P), z(P)]$. Then

$$\mathbf{Pow}_\omega(\iota(P)) = [x(P)^2 + y(P)^2 + bx(P)z(P) + cy(P)z(P) + dz(P)^2 : z(P)^2].$$

Since both expressions vanish when $\iota(P)$ is a circle point, the desired result follows by Zack's lemma. $\square$

Remark. Any fixed isoconjugation doubles degree in general. This can be seen as to find the isoconjugation of a point ι, we intersect the images of lines $A\iota$ and $B\iota$ after some projective involution on the pencil of lines through A and the pencil of lines through B, which preserves degree. Since $A\iota$ and $B\iota$ both have degree $\deg(\iota)$, their intersection point has degree $2 \cdot \deg(\iota)$.

Conversely, polar reciprocation across a fixed conic preserves degree, as it is the transpose of a matrix multiplication.

We list some other lemmas regarding

Theorem (Rotation Lemma)**.** Let θ be a angle defined as the directed angle between two moving lines ℓ_1, ℓ_2. Given a moving point ι and a moving line ℓ_t through point ι, the rotation of ℓ_t around point ι by θ has degree

$$\deg(A) + \deg(\ell_t) + \deg(\ell_1) + \deg(\ell_2).$$

Corollary (Efficient Cyclic Lemma)**.** For four moving lines $\ell_1, \ell_2, \ell_3, \ell_4$, let ι_1 be the intersection of ℓ_1, ℓ_2, and define $\iota_2, \iota_3, \iota_4$ cyclically. Then $\iota_1, \iota_2, \iota_3, \iota_4$ are always concyclic if

$$\#\{P \in \mathbb{P}^1 \mid \iota_1(P), \ldots, \iota_4(P) \text{ concyclic}\} \geq \deg \ell_1 + \cdots + \ell_4 + 1.$$

Theorem. The degree of reflecting moving point ι across moving line ℓ is degree $2 \cdot \deg \ell + \deg \iota$. This also holds for reflecting a line over a line.

This can be thought of as $2 \cdot \deg(\text{axis of reflection}) + \deg(\text{thing being reflected})$.

Finally, let's derive more lemmas to prove problem statements after computing the degree of every object.

You might have noticed that right now, we can only prove collinearity and conconicity via checking many cases, despite deriving expressions for the degrees of many non-point objects such as areas, cross-ratios/angles, and powers. Here is an extremely important lemma to deal with these, which can also be used for points.

Theorem. For two moving points $\mathbb{P}^1 \to \mathbb{P}^1$, λ_1, λ_2 which move on the same pencil (i.e. parametrize the same copy of $\mathbb{P}^1$ in $\mathbb{P}^2$), then to prove that $\lambda_1 = \lambda_2$ for all t, we have to check $\deg \lambda_1 + \deg \lambda_2 + 1$ values of t.

Proof. We will work directly in $\mathbb{P}^1$ - let $\lambda_1 = [x_1(t) : y_1(t)] = \left[\frac{x_1(t)}{y_1(t)} : 1\right], \lambda_2 = [x_2(t) : y_2(t)] = \left[\frac{x_2(t)}{y_2(t)} : 1\right]$. Then, we have

$$\frac{x_1(t)}{y_1(t)} = \frac{x_2(t)}{y_2(t)} \iff x_1(t)y_2(t) - x_2(t)y_1(t) = 0.$$

This is a homogeneous polynomial of degree

$$\deg \lambda_1 + \deg \lambda_2,$$

so we have to check $\deg \lambda_1 + \deg \lambda_2 + 1$ cases. $\square$

Remark. A common mistake is thinking one needs to check $\max(\deg \lambda_1, \deg \lambda_2) + 1$ cases, which works for polynomials over $\mathbb{R}^1$ but not over $\mathbb{RP}^1$.

A similar theorem holds for $\mathbb{P}^2$:

Theorem. For two moving points $\mathbb{P}^1 \to \mathbb{P}^2$, ι_1, ι_2, to prove that they are equal for all t, we have to check $\deg \iota_1 + \deg \iota_2 + 1$ cases.

Proof. Apply the above proof pairwise on the three coordinates. $\square$

Finally, we give a way to prove a point lies on a line (and dually, a line goes through a point).

Theorem. For a moving point $\mathbb{P}^1 \to \mathbb{P}^2$, ι_1, and a moving line $\mathbb{P}^1 \to \mathbb{P}^2$, ι_ℓ, to prove that $\iota_1 \in \iota_\ell$ for all t, we have to check $\deg \iota_1 + \deg \iota_\ell + 1$ cases.

Proof. Let ι_ℓ be $\begin{bmatrix} X_\ell \\ Y_\ell \\ Z_\ell \end{bmatrix}$ and let ι_1 be $[x_1 : y_1 : z_1]$. Then

$$X_\ell x_1 + Y_\ell y_1 + Z_\ell z_1 = 0$$

is a homogeneous degree $\deg \iota_1 + \deg \iota_\ell$ polynomial, so we need to check $\deg \iota_1 + \deg \iota_\ell + 1$ cases. $\square$

Remark. This can be used to prove a deg-n line passes through a fixed point, and a way to prove a deg-n point moves on a fixed line, by checking $n + 1$ cases.

Theorem. For a moving point $\mathbb{P}^1 \to \mathbb{P}^2$, ι, and a fixed conic $\mathcal{C}$, to prove that $\iota_1 \in \mathcal{C}$ for all t, we have to check $2 \deg \iota + 1$ cases.

Proof. ι will lie on $\mathcal{C}$ iff. $\iota \mathcal{C} \iota^\intercal = 0$. $\iota \mathcal{C}$ has degree $\deg \iota$, so by the previous lemma, we need $2 \deg \iota + 1$ cases. $\square$

Remark. Note that when $\deg \iota$ is odd (let it be $2k + 1$), then by Bezout's, we get $4k + 2$ times where $\deg \iota$ can intersect with $\mathcal{C}$ maximally, which is not enough. When $\deg \iota$ is even (let it be $2k$), then we get that it can intersect $4k$ times unless it is equal to the conic. This theorem is only useful if $\deg \iota \geq 4$.

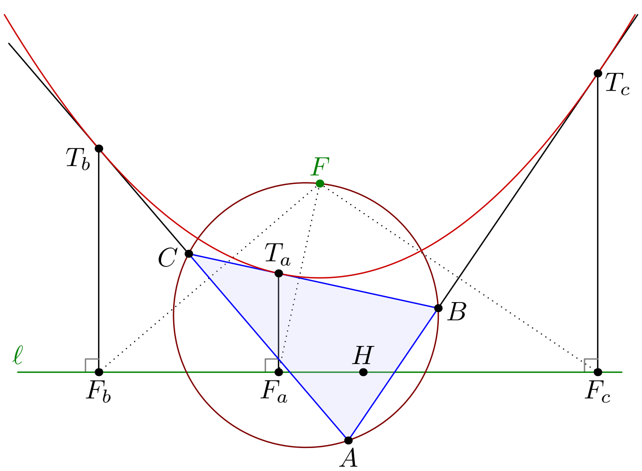
Chapter 8

Circumrectangular Hyperbolas

Why geometry is better than life: in life, you will only meet one special person. In geometry, you will meet multiple special conic sections.

8.1 Special Conics and The Poncelet Point

Proposition 8.1.1. Let $\mathcal{P}$ be a parabola with focus at F, and let ℓ be its directrix. Let $\triangle ABC$ have all three sides tangent to this parabola (similar to an excircle), then $F \in (ABC)$ and ℓ passes through the orthocenter of $\triangle ABC$.



Proof. Reflect F over BC, CA, AB to get F_a, F_b, F_c, and let $\mathcal{P}$ touch BC, CA, AB at T_a, T_b, T_c. Then by the optical property of a parabola we can get that F_a, F_b, F_c respectively are the feet from T_a, T_b, T_c to ℓ. Thus if F_a, F_b, F_c are collinear, then the directrix is just the Steiner line of F wrt. $\triangle ABC$! So it must pass through H. $\square$

Proposition 8.1.2. Let $\mathcal{P}$ be a parabola with focus at F and ℓ as directrix. Let $\triangle ABC$ be a self-conjugate triangle wrt. $\mathcal{P}$, then F lies on the nine-point circle of $\triangle ABC$, and ℓ passes through the circumcenter of $\triangle ABC$.

Proof. Let U be the point at infinity perpendicular to ℓ, then $U \in \mathcal{P}$ (the parabola is tangent to the line at infinity at U) and U is the center of $\mathcal{P}$. Let M_a, M_b, M_c respectively be the midpoints of $\overline{BC}, \overline{CA}, \overline{AB}$, and let AU intersect BC, M_bM_c at P, T respectively. Then we know $(A, T; P, U) = -1$ so we know $T \in \mathcal{P}$. From $M_bM_c \parallel BC$ and the Parallel Chords Theorem Corollary 7.1.13 on T, P, U, we know that M_bM_c is the polar of T wrt. $\mathcal{P}$, so M_bM_c is tangent to $\mathcal{P}$. As such, by symmetry we have that the whole medial triangle is tangent to $\mathcal{P}$. As such, since the circumcenter of the medial circle is the orthocenter, by the previous property on $\triangle M_aM_bM_c$, ℓ goes through the circumcenter of $\triangle ABC$ and F lies on the nine-point circle. $\square$

These are the most well-known theorems for parabolas. Now we will move on to hyperbolas.

Definition 8.1.3. We call a hyperbola $\mathcal{H}$ **rectangular** if its two asymptotes are perpendicular. Equivalently, a rectangular hyperbola is a conic section that has the two circle points as conjugates (polar of one passes through the other).

All non-degenerate rectangular hyperbolas are spirally similar to the solution set of $xy = 1$.

Proposition 8.1.4. Given a non-right triangle $\triangle ABC$, a hyperbolic circumconic (called a **circumhyperbola**) is rectangular iff $\mathcal{H}$ also goes through H.

Proof. We consider isogonal conjugation φ in $\triangle ABC$. Let O be the circumcenter of $\triangle ABC$. Then $H \in \mathcal{H}$ if and only if $\varphi(\mathcal{H})$, which is a line, goes through O. Let $\varphi(\mathcal{H})$ intersect $\varphi(\mathcal{L}_\infty) = (ABC)$ at two points X, Y. Then $O \in \varphi(\mathcal{H})$ if and only if X, Y are antipodes in (ABC), which implies $\varphi(X), \varphi(Y)$ are 90° apart on the line at infinity, so $\mathcal{H}$ is rectangular. $\square$

If we redefine H as the cevapoint Example 7.1.25 of the circle points I and J, then this theorem is just a direct application of Proposition 7.2.23.

We can also quickly solve this with analytic techniques. Since all rectangular hyperbolas are spirally symmetric to $xy = 1$, we can choose a system of coordinates such that $A = (a, a^{-1}), B = (b, b^{-1}), C = (c, c^{-1}) \in \mathcal{H} = \{(x, y) \mid xy = 1\}$. Then the orthocenter H of $\triangle ABC$ is just $(-(abc)^{-1}, -abc)$.

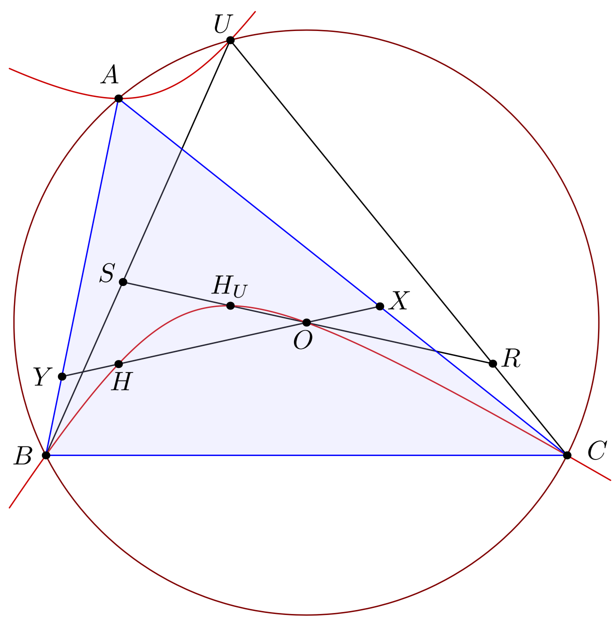
If we have four points A, B, C, D that lie on one rectangular hyperbola $\mathcal{H}$, and we choose H_A, H_B, H_C, H_D

respectively such that A is the orthocenter of $\triangle H_B H_C H_D$, etc, then we have

$$\begin{aligned}(A, B; C, D)_{\mathcal{H}} &= (a, b; c, d)\\ &= (-(bcd)^{-1}, -(cda)^{-1}, -(dab)^{-1}, -(abc)^{-1})\\ &= (H_A, H_B; H_C, H_D)\end{aligned}$$

by projecting from $\infty_{x=0}$ onto the x-axis.

Example 8.1.5. Let $\mathcal{E}$ be the Euler line of a non-equilateral acute triangle $\triangle ABC$. Prove that the Euler line of the triangle formed by lines $CA, AB, \mathcal{E}$ is parallel to BC.



Proof. Let O, H be the circumcenter and orthocenter of $\triangle ABC$. We consider the Jerabek hyperbola, the rectangular hyperbola $\mathcal{H}$ passing through A, B, C, O, H. Note that $\mathcal{H}$ is the image of $\mathcal{E}$ under isogonal conjugation. Let $X = CA \cap \mathcal{E}, Y = AB \cap \mathcal{E}, U$ as the fourth intersection of $\mathcal{H}$ and (ABC), then we have

$$\measuredangle XYA = \measuredangle(\mathcal{E}, AB) = \measuredangle CAU = \measuredangle CBU,$$

$$\measuredangle AXY = \measuredangle(AC, \mathcal{E}) = \measuredangle UAB = \measuredangle UCB,$$

thus $\triangle AYX \overset{+}{\sim} \triangle UBC$. Note that the orthocenter H_U of $\triangle UBC$ must also lie on $\mathcal{H}$, so $\mathcal{H}$ is the isogonal conjugate of the Euler line of $\triangle UBC$ wrt. $\triangle UBC$. Therefore if the Euler line of $\triangle UBC$ intersects CU, UB at R, S, we have $\triangle USR \overset{+}{\sim} \triangle ABC$, so the Euler line $\mathcal{E}'$ of $\triangle AYX$ satisfies

$$\mathcal{E}' = (\measuredangle XY + \measuredangle RS - \measuredangle BC) = (\measuredangle AY + \measuredangle BC - \measuredangle UB) + (\measuredangle US - \measuredangle AB) = \measuredangle BC.$$

□

The following proposition can be thought of as a limiting case of Proposition 8.1.4.

Proposition 8.1.6. Given any right angled triangle $\triangle ABC$, let BC be the hypotenuse and let AD be the altitude from A. Let $\mathcal{H}$ be a circumconic of $\triangle ABC$, then AD is tangent to $\mathcal{H}$ iff $\mathcal{H}$ is a circumrectangular hyperbola.

Proof. We consider isogonal conjugation with φ on $\triangle ABC$, let O be the circumcenter of $\triangle ABC$. Then we have that AD is tangent to $\mathcal{H}$ if and only if the isogonal conjugate of the circumconic $\varphi(\mathcal{H})$, AO, BC are concurrent, which is the same thing as $O \in \varphi(\mathcal{H})$ since it's a right triangle, which finishes. □

Proposition 8.1.7. Let $\mathcal{H}$ be a circumrectangular hyperbola centered at T. Then $\measuredangle \mathfrak{p}_{\mathcal{H}}(P) + \measuredangle TP$ is a constant value. This can also be interpreted as

$$\measuredangle(\mathfrak{p}_{\mathcal{H}}(P_1), \mathfrak{p}_{\mathcal{H}}(P_2)) + \measuredangle P_1TP_2 = 0°,$$

for any two points P_1, P_2.

Proof. Let $U = \infty_{TP}, V = \infty_{\mathfrak{p}_{\mathcal{H}}(P)}$, and then by the Parallel Chords Theorem Corollary 7.1.13 we know that $\mathfrak{p}_{\mathcal{H}}(U) \parallel \mathfrak{p}_{\mathcal{H}}(P)$. Let W_1, W_2 be the two intersections of $\mathcal{H}$ with the line at infinity. Since U and V are conjugate points in $\mathcal{H}$ we know that

$$-1 = (U, V; W_1, W_2) \overset{T}{=} (P, V; W_1, W_2).$$

Since $\measuredangle W_1TW_2 = 90°$, we have that $\angle P_iTV_i$ has angle bisectors TW_1, TW_2. Thus

$$\measuredangle \mathfrak{p}_{\mathcal{H}}(P) + \measuredangle TP = \measuredangle 2 \cdot TW_1 = \measuredangle 2 \cdot TW_2$$

is a constant. □

This has another interpretation in terms of the circle points: since $\triangle TIJ$ is self-conjugate in $\mathcal{H}$, for any two points P, Q we have

$$(\infty_{\mathfrak{p}_{\mathcal{H}}(P)}, \infty_{\mathfrak{p}_{\mathcal{H}}(Q)}; J, I) \overset{\mathfrak{p}_{\mathcal{H}}}{=} T(P, Q; I, J) = (\infty_{TQ}, \infty_{TP}; J, I).$$

Corollary 8.1.8. Let $\mathcal{H}$ be a rectangular hyperbola centered at T, let $\triangle ABC$ be self-conjugate wrt. $\mathcal{H}$. Then T lies on the circumcircle of ABC.

Proof. From Proposition 8.1.7 we get that $\measuredangle BTC = -\measuredangle(CA, AB) = \measuredangle BAC$. □

Corollary 8.1.9. Let $\mathcal{H}$ be a rectangular hyperbola centered at T. Suppose $A, B, C \in \mathcal{H}$, then T lies on the nine-point circle of $\triangle ABC$.

Proof. Let H be the orthocenter of $\triangle ABC$, then $H \in \mathcal{H}$. From Proposition 7.1.8 we know that the cevian triangle of H (which is just the orthic triangle) is self-conjugate in $\mathcal{H}$. So then by Corollary 8.1.8 we know T lies on the circumcircle of the orthic triangle, which is just the nine-point circle. $\square$

We can also re-interpret this in Theorem 7.4.18: T lies on the nine-point conic of the complete quadrangle (A, B, C, H).

Proposition 8.1.10. If BC is a diameter of a rectangular hyperbola $\mathcal{H}$, then given any $A \in \mathcal{H}$, the antipode A^* of A in (ABC) also lies on $\mathcal{H}$. Also, the tangent from A to $\mathcal{H}$, $\mathbf{T}_A\mathcal{H}$ is the A-symmedian of $\triangle BAC$.

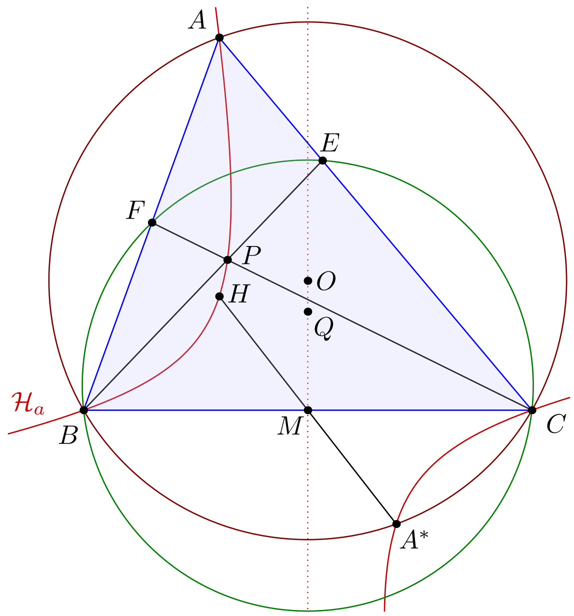
Proof. We consider isogonal conjugation φ in $\triangle ABC$. Let $L = \varphi(\mathcal{H}), L' = \varphi(\mathbf{T}_A\mathcal{H})$, then $\varphi(L \cap L') \in \mathcal{H} \cap \mathbf{T}_A\mathcal{H} = A$, so $L \cap L' \in BC$. Note that the orthocenter H of $\triangle ABC$ is in $\mathcal{H}$, so thus the reflection of H over the midpoint of $\overline{BC}$ lies on $\mathcal{H}$, which is just A^*. So from $\infty_{\perp BC} = \varphi(A^*) \in L$ we know that L is the perpendicular bisector of $\overline{BC}$. Thus $\mathbf{T}_A\mathcal{H}$ is the isogonal conjugate of $L' = A(L \cap BC)$ in $\angle ABC$, and thus, $\mathbf{T}_A\mathcal{H}$ must be the A-symmedian. $\square$

From this characterization we can redefine hyperbolas with a common config:

Example 8.1.11 (First Isogonality Lemma)**.** Let E, F be two points on sides CA, AB of $\triangle ABC$, that also satisfy B, C, E, F concyclic. Let $P = BE \cap CF$, then the isogonal conjugate of P (suppose it's point Q) satisfies

$$\measuredangle CBQ = \measuredangle PBA = \measuredangle EBF = \measuredangle ECF = \measuredangle ACP = \measuredangle QCB$$

which also means that Q lies on the perpendicular bisector of $\overline{BC}$. Therefore the locus of P is the locus of the isogonal conjugate of the perpendicular bisector of $\overline{BC}$, which is just a rectangular hyperbola as it goes through H.



Note that the antipode of A in (ABC), point A^*, also lies on $\mathcal{H}_a$ and $(BC)(HA^*)$ is a parallelogram. Thus the center of $\mathcal{H}_a$ is just the midpoint of $\overline{BC}$, point M. This also implies that BC is a diameter of $\mathcal{H}_a$.

Further, we also get that when we choose a diameter BC of an arbitrary rectangular hyperbola, and pick a point P on that hyperbola, we know $\measuredangle PB + \measuredangle PC$ is always a constant.

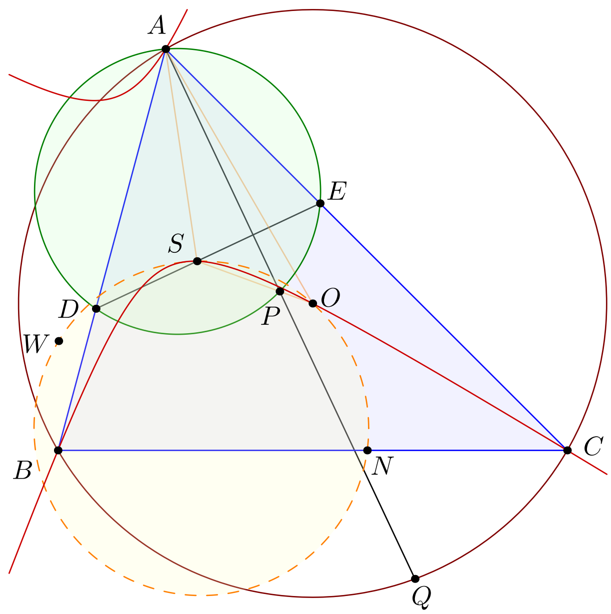
(This can also be understood through the circle points again: since $\triangle MIJ$ is self-conjugate wrt. $\mathcal{H}_a$, from Proposition 7.1.8 we have that $X = BJ \cap CI, Y = BI \cap CJ \in \mathcal{H}$. This tells us that for any point $P \in \mathcal{H}$, we have

$$B(A, P; J, I) = (A, P; X, Y)_{\mathcal{H}} = C(A, P, I, J) = C(P, A; J, I).$$

which finishes by the definition of angles with cross-ratio back in chapter 6.A).

Example 8.1.12 (Taiwan 2022 TST 3 Problem 5)**.** Let $\triangle ABC$ be an acute triangle with circumcenter O and circumcircle Ω. Construct D, E on AB, AC respectively, and let ℓ be the line through A perpendicular to DE. Let ℓ intersect the circumcircle of ADE and Ω again at P, Q respectively. Let OQ and BC intersect at N, and let OP and DE intersect at S. Let W be the orthocenter of $\triangle SAO$.

Prove S, N, O, W are concyclic.



Proof. Since $\measuredangle SWO = \measuredangle OAS$, by the above theorem we only need to prove $\measuredangle SN + \measuredangle SA = \measuredangle ON + \measuredangle OA$, which is the same thing as S, O lying on a rectangular hyperbola with diameter $\overline{AN}$. Since $N = OP \cap BC$, we only need to prove that for a fixed Q, that

- The locus of S is a rectangular hyperbola $\mathcal{H}$;
- $\overline{AN}$ is the diameter of $\mathcal{H}$.

We proceed with moving points. When Q is fixed, $E = D\infty_{\perp AQ} \cap AC, P = AQ \cap \infty_{\measuredangle AQ + \measuredangle(\perp AQ) - \measuredangle AC}$. This tells us that $D \to E, D \to P$ is a projective map. Thus

$$S = OP \cap DE = OP \cap \infty_{\perp AQ} D$$

is on a rectangular hyperbola $\mathcal{H}$ through O, $\infty_{\perp AQ}$. When $D = A$, $S = OA \cap \infty_{\perp AQ} A = A$; when $D = \infty_{AB}$, $S = O\infty_{AQ} \cap \infty_{\perp AQ}\infty_{AB} = \infty_{AQ}$. Thus $\mathcal{H}$ also passes through ∞_{AQ} and $\infty_{\perp AQ}$. Thus H is a rectangular hyperbola.

Next, we need to prove that $N \in \mathcal{H}$. In other words, we need to find a position for D such that $N = OP \cap DE$, so we need to have a $D = AB \cap N\infty_{\perp AQ}$, and thus it remains to prove this when $P = Q$. This is equivalent to Q being the Miquel point of AB, AC, BC, DE. However we actually have B, D, Q, N concyclic, so

$$\measuredangle DNQ = \measuredangle(\perp AQ, OQ) = (A - Q)_\Omega = \measuredangle ABQ = \measuredangle DBQ.$$

So finally, we can say that $\overline{AN}$ is a diameter of $\mathcal{H}$, so we only need to prove that

$$\measuredangle(OA + \measuredangle ON = \measuredangle(\infty_{AQ}A) + \measuredangle(\infty_{AQ}N) = 2AQ,$$

but that is obvious. $\square$

Example 8.1.13. From these properties of hyperbolas, we can prove inversion is actually a point isoconjugation. In reality, if $\mathfrak{J}$ is a inversion with center O and power k, and $\mathfrak{s}$ is reflection across a line ℓ that goes through O, then $\mathfrak{T} := \mathfrak{J} \circ \mathfrak{s}$ is a isoconjugation in $\triangle OIJ$, where I, J are circle points.

We will only consider the case in which the power of inversion is positive. Let Γ be a circle with radius $\sqrt{k}$ and center O. Then $\mathfrak{J}$ is just inversion about this circle. We consider the two intersection points Γ with ℓ, let these points be X, Y.

Define a pencil of conics as

$$\mathscr{F} = \{\mathcal{H} \mid X, Y \in \mathcal{H}, \mathcal{H} \text{ is a rectangular hyperbola centered at } O.\}$$

Since I, J harmonically divide any two perpendicular lines' intersections with the line at infinity, we get that $\triangle OIJ$ is self-conjugate wrt. any $\mathcal{H} \in \mathscr{F}$. For any P, we know that $P^* = \mathfrak{J} \circ \mathfrak{s}(P)$ satisfies $(XY)(PP^*)$ is a harmonic quadrilateral. We only need to prove that for all $\mathcal{H} \in \mathscr{F}$, $\mathfrak{p}_{\mathcal{H}}(P)$ passes through P^*.

Let A, B be the other two intersection points of $\Omega := (XYPP^*)$ and $\mathcal{H}$. From Proposition 8.1.10, the tangents to $\mathcal{H}$ at two points A, B are concurrent with the tangents to Ω from two points X, Y. Let's say they concur at point T. Note that from the harmonic quadrilateral $(XY)(PP^*)$, we also get that T lies on line $L := PP^*$.

We consider the complete quadrangle $\mathfrak{q} = (A, B, X, X)$, where XX is the tangent from X to Ω. We now use DIT on $\mathfrak{q}$ and L to get an involution

$$(P, P^*), (L \cap AB, T), (L \cap AX, L \cap BC).$$

We want to prove that this involution is the involution sending $[Q \to L \cap \mathfrak{p}_{\mathcal{H}}(Q)]$. We already have that

$$L \cap \mathfrak{p}_{\mathcal{H}}(T) = L \cap \mathfrak{p}_{\mathcal{H}}(\mathfrak{p}_{\mathcal{H}}(A) \cap \mathfrak{p}_{\mathcal{H}}(B)) = L \cap AB$$

, so we only need to prove that $\mathfrak{p}_{\mathcal{H}}(L \cap AX), L, BX$ are concurrent, or equivalently, $\mathfrak{p}_{\mathcal{H}}(L), U := \mathfrak{p}_{\mathcal{H}}(AX), L \cap BX$ are collinear. But this is true since

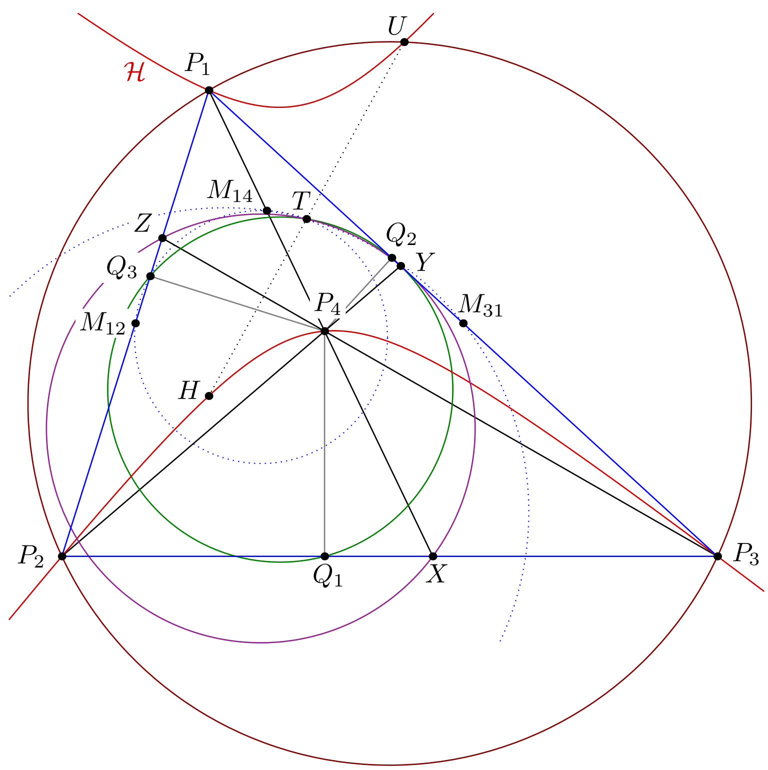
$$U(A, B; C, \mathfrak{p}_{\mathcal{H}}(L)) = (A, B; UX \cap AB, \mathfrak{p}_{\mathcal{H}}(L))$$

$$\overset{\mathfrak{p}_{\mathcal{H}}}{=} (TA, TB; TX, L) = (UA, UB; UX, U(L \cap BX)).$$

Now it remains to show that $\mathscr{F}$ has two other fixed points. We claim that the other two common points of conics in $\mathscr{F}$ are actually just the two points of intersection of the circle of imaginary radius $\sqrt{-k}$ centered at O and the line $O\infty_{\ell_\perp}$. This can be proven algebraically, but it also follows as these are the points fixed under $\mathfrak{T}$. As such, inversions are just isoconjugations in $\triangle OIJ$.

Definition 8.1.14. For a complete quadrangle $\mathfrak{q} = (P_1, P_2, P_3, P_4)$ that isn't an orthocentric system, we define $\mathcal{H}(P_1P_2P_3P_4)$ as the unique rectangular hyperbola through P_1, P_2, P_3, P_4. We call the center of this hyperbola the **Poncelet point** of these four points.

Theorem 8.1.15. Let T be the Poncelet point of complete quadrangle $\mathfrak{q} = (P_1, P_2, P_3, P_4)$. Then T lies on the nine-point circle of every $\triangle P_{i-1}P_iP_{i+1}$, T lies on the pedal circle of P_i wrt. $\triangle P_{i+1}P_{i+2}P_{i+3}$ for all i, and T lies on the circle through the three intersections of diagonals of the quadrangle, (which is just the cevian circle).



Proof. Let $\mathcal{H}$ be a rectangular hyperbola passing through P_1, P_2, P_3, P_4, and let $\triangle XYZ$ be the cevian triangle of $\mathfrak{q}$. Then by Brokard we know that $\triangle XYZ$ is self-polar wrt. $\mathcal{H}$, and thus from *Corollary* 8.1.9 and *Corollary* 8.1.8 we know that T lies on the nine-point circle of $\triangle P_{i-1}P_iP_{i+1}$ and the cevian circle.

Let $\triangle Q_1Q_2Q_3$ be the pedal triangle of P_4 wrt. $\triangle P_1P_2P_3$, and let M_{ij} be the midpoint of P_iP_j. From T

lying on $(M_{14}Q_2M_{31})$ and $(M_{14}Q_3M_{12})$ we can get

$$\begin{aligned}\measuredangle Q_2TQ_3 &= \measuredangle Q_2TM_{14} + \measuredangle M_{14}TQ_3 = \measuredangle Q_2M_{31}M_{14} + \measuredangle M_{14}M_{12}Q_3 \\ &= \measuredangle Q_2P_3P_4 + \measuredangle P_4P_3Q_3 = \measuredangle Q_2Q_1P_4 + \measuredangle P_4Q_1Q_3 = \measuredangle Q_2Q_1Q_3.\end{aligned}$$

and thus T lies on $(Q_1Q_2Q_3)$, do this cyclically. □

Definition 8.1.16. Given $\triangle ABC$, and point P that's not the orthocenter or a vertex, define $\widehat{P}$ to be the point such that

$$\measuredangle BPC + \measuredangle B\widehat{P}C = \measuredangle CPA + \measuredangle C\widehat{P}A = \measuredangle APB + \measuredangle A\widehat{P}B = 0^\circ.$$

We call $\widehat{P}$ to be the **antigonal conjugate** of P.

The proof of uniqueness and existence is just angle chasing. Note that this is not an isoconjugation.

Proposition 8.1.17. Given $\triangle ABC$, if $\widehat{P}$ is the antigonal conjugate of P wrt. $\triangle ABC$, then the midpoint of P and $\widehat{P}$ is the Poncelet point of (A, B, C, P).

Proof. We do phantom-points. Let P^* be the antipode of P in the rectangular hyperbola through A, B, C, P. Then P^* is the reflection of P across the center of this hyperbola, so we have

$$\measuredangle AP + \measuredangle(AP^*) = \measuredangle BP + \measuredangle(BP^*) = \measuredangle CP + \measuredangle(CP^*).$$

This tells us that

$$\measuredangle BPC + \measuredangle BP^*C = \measuredangle CPA + \measuredangle CP^*A = \measuredangle APB + \measuredangle AP^*B = 0^\circ,$$

so by the uniqueness of antigonal conjugation we know that $P^* = \widehat{P}$, so their center is thus the center of this hyperbola and it's also the Poncelet point of (A, B, C, P). □

Example 8.1.18 (2018 G4)**.** A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

Proof. When T is the orthocenter of $\triangle ABC$, we note that $A = A_2, B = B_2, C = C_2$. So the point of concurrency isn't defined well wrt. H. This reminds us of antigonal conjugation.

As such, let us guess that the concurrency point is the antigonal conjugate T^* of T. Note the midpoint of T and T^* lies on T's pedal circle in ABC since it's the Poncelet point of (A, B, C, T), so the antigonal conjugate of T lies on the circumcircle of $A_1B_1C_1$ by homothety. It remains to show that A, A_2, T^* are collinear.

Since $\measuredangle CB_1A = -\measuredangle CTA = \measuredangle CT^*A$, T^* lies on (AB_1C). Now since C is the circumcenter of $\triangle A_1B_1T$,

$$\measuredangle B_1T^*A = \measuredangle B_1CA = \measuredangle B_1A_1T = \measuredangle B_1T^*A_2,$$

so we're done. $\square$

Here's a "better" characterization of antigonal conjugation.

Proposition 8.1.19.

$$\widehat{P} = \varphi^K \circ \mathfrak{J}_{(ABC)} \circ \varphi^K(P).$$

where $\phi^K, \mathfrak{J}$ represent isogonal conjugation and inversion.

Proof. Let $\mathcal{H}$ be the rectangular hyperbola through $ABCP$. Let W_1, W_2 be the two points at infinity along $\mathcal{H}$. Then since $\widehat{P}, P$ are antigonal conjugates, and since $\mathbf{T}_P\mathcal{H} \cap \mathbf{T}_{\widehat{P}}\mathcal{H} \in \mathcal{L}_\infty = W_1W_2$, we know that $(P\widehat{P})(W_1W_2)$ is a harmonic quadrilateral on $\mathcal{H}$.

Note that this also implies W_1, W_2's isogonal conjugates wrt. $\triangle ABC$ (let them be W_1^*, W_2^*) are the two intersections of $P^*\widehat{P^*}$ wrt. (ABC). Thus

$$(P, \widehat{P}; W_1, W_2)_{\mathcal{H}} = (P^*, \widehat{P^*}; W_1^*, W_2^*) = -1,$$

so $P^*, \widehat{P^*}$ are inverses wrt. (ABC). $\square$

We will revisit antigonal conjugation in **??**.

Proposition 8.1.20. Given $\triangle ABC$, let O be the circumcenter of $\triangle ABC$. Let P and Q be isogonal conjugates in $\triangle ABC$. Let T be the Poncelet point of A, B, C, P. Then T is the anti-Steiner point of line OQ wrt. the medial triangle of $\triangle ABC$.

Proof. Let $\triangle M_aM_bM_c$ be the medial triangle of $\triangle ABC$. Let $\mathcal{S}$ be the Steiner line of $\triangle M_aM_bM_c$ wrt. T, and let $\mathcal{H}$ be the circumrectangular hyperbola of A, B, C, P. Let U be the antipode of H in $\mathcal{H}$, then we know that U lies on (ABC).

Now we consider isogonal conjugation φ on $\triangle ABC$. We know that $\varphi(U)$ is just ∞_{OQ}. Note that the isogonal conjugate (in the medial triangle) of T's antipode wrt. $(M_aM_bM_c)$ is just the point at infinity along the Steiner line, and since O is the orthocenter of $\triangle M_aM_bM_c$, we know that the Steiner line has to be OQ. $\square$

A similar property also holds for pedal circles, in fact:

Proposition 8.1.21. Given $\triangle ABC$, let O be its circumcenter and let P, Q be two isogonal conjugates in $\triangle ABC$. Let T be the Poncelet point of (A, B, C, P), then the Steiner line of T in the pedal circle of P is parallel to OQ.

Proof. Let $\triangle P_aP_bP_c$ be the pedal triangle of P wrt. $\triangle ABC$, and let $\triangle M_aM_bM_c$ be the medial triangle of $\triangle ABC$. From Proposition 8.1.20 and by Proposition 1.4.5 we can instead show that

$$\measuredangle P_aP_b + \measuredangle P_aP_c - \measuredangle P_aT = \measuredangle M_aM_b + \measuredangle M_aM_c - \measuredangle M_aT.$$

Note that (P_aM_aT) is actually just the nine-point circle of $\triangle PBC$. Thus

$$\begin{aligned}\measuredangle P_aTM_a &= \measuredangle PB + \measuredangle PC - 2\measuredangle BC \\ &= (\measuredangle BC + \measuredangle BA - (\perp P_cP_a)) + (\measuredangle CA + \measuredangle CB - (\perp P_aP_b)) - 2\measuredangle BC \\ &= \measuredangle M_aM_b + \measuredangle M_aM_c - \measuredangle P_aP_b - \measuredangle P_aP_c.\end{aligned}$$

□

Note that the nine-point circle and the pedal circle of any point P always intersect in the Poncelet point of (A, B, C, P). We can actually calculate this angle of intersection.

Theorem 8.1.22. Given $\triangle ABC$ and P, the angle between P's pedal circle and the nine-point circle is just

$$\perp \sum(\measuredangle AP - \measuredangle BC) = 90^\circ + \measuredangle(BC + CA + AB, AP + BP + CP).$$

Proof. Let $\triangle P_aP_bP_c$ be the pedal triangle of P. Let $\triangle M_aM_bM_c$ be the medial triangle of $\triangle ABC$. Let T be the Poncelet point of (A, B, C, P). Then we know that T lies on the nine-point circles of $\triangle PCA, \triangle PAB$.

Let ℓ_P, ℓ_M respectively be the tangent from T to the pedal circle and the nine-point circle. Then we have

$$\ell_P = \measuredangle TP_b + \measuredangle TP_c - \measuredangle P_bP_c, \ell_M = \measuredangle TM_b + \measuredangle TM_c - \measuredangle M_bM_c.$$

By the proof of Proposition 8.1.21, we have

$$\measuredangle P_bTM_b = \measuredangle PC + \measuredangle PA - 2\measuredangle CA, \measuredangle P_cTM_c = \measuredangle PA + \measuredangle PB - 2\measuredangle AB,$$

and therefore

$$\begin{aligned}\ell_M - \ell_P &= \measuredangle P_bTM_b + \measuredangle P_cTM_c + \measuredangle P_bP_c - \measuredangle M_bM_c\\ &= (2\measuredangle PA + \measuredangle PB + \measuredangle PC - 2\measuredangle CA - 2\measuredangle AB) + (\measuredangle AB - \measuredangle AC - (\perp \measuredangle AP)) - \measuredangle BC\\ &=\perp (\measuredangle PA + \measuredangle PB + \measuredangle PC - \measuredangle BC - \measuredangle CA - \measuredangle AB).\end{aligned}$$

□

We also have a pretty easy corollary.

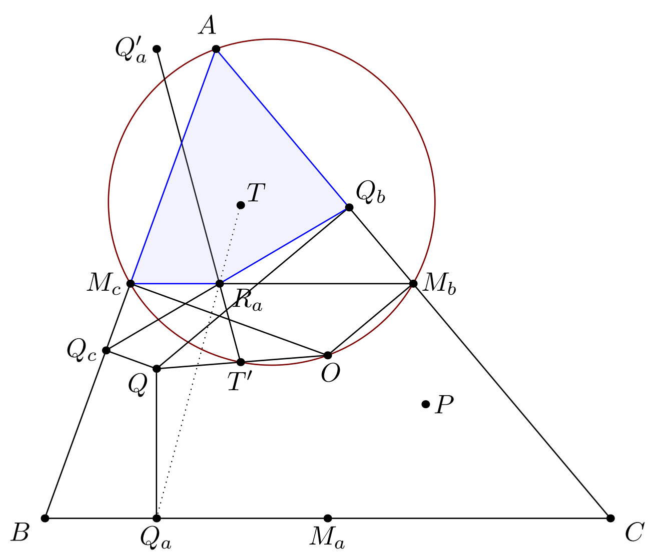
Theorem 8.1.23. Given any $\triangle ABC$ and a circumrectangular hyperbola $\mathcal{H}$, let P, Q be two points on $\mathcal{H}$. Then the angle between the pedal circles of P, Q (at the center of $\mathcal{H}$) is just

$$\measuredangle(AQ + BQ + CQ, AP + BP + CP).$$

8.1.1 Fontené's Theorems

Like Newton, Fontené also invented three theorems, but they're more complicated.

Theorem 8.1.24 (Fontené I)**.** Let $\triangle M_aM_bM_c$ be the medial triangle of $\triangle ABC$. For any two isogonal conjugates (P, Q), let T be the Poncelet point of (A, B, C, P). Let $\triangle Q_aQ_bQ_c$ be the pedal triangle of Q wrt. $\triangle ABC$. Let $R_a = Q_bQ_c \cap M_bM_c$, etc. Then Q_aR_a, Q_bR_b, Q_cR_c concur at T.



Proof. Let T', Q'_a respectively be the reflections of T, Q_a across M_bM_c, then $T' \in (AO)$ and $T' \in \overline{OQ}$ by Proposition 8.1.20. Thus we have $\measuredangle AT'Q = 90°$, which also implies that T' is the Miquel point of

(CA, AB, M_bM_c, Q_bQ_c). Thus from

$$\measuredangle R_aT'Q_b = \measuredangle R_aM_bA = \measuredangle Q'_aAQ_b = \measuredangle Q'_aT'Q_b,$$

we can get $T' \in R_aQ'_a$, so by reversing the reflection we get $T \in Q_aR_a$. $\square$

Corollary 8.1.25. The orthocenter of $\triangle R_aR_bR_c$ is the circumcenter of $\triangle Q_aQ_bQ_c$ and is also the midpoint of $\overline{PQ}$.

Proof. Note that $\triangle R_aR_bR_c$ is the cevian triangle of T wrt. $\triangle Q_aQ_bQ_c$, so $\triangle R_aR_bR_c$ is self-conjugate wrt. $(Q_aQ_bQ_c)$. Thus by Brokard the orthocenter of $\triangle R_aR_bR_c$ is the circumcenter of $(Q_aQ_bQ_c)$. $\square$

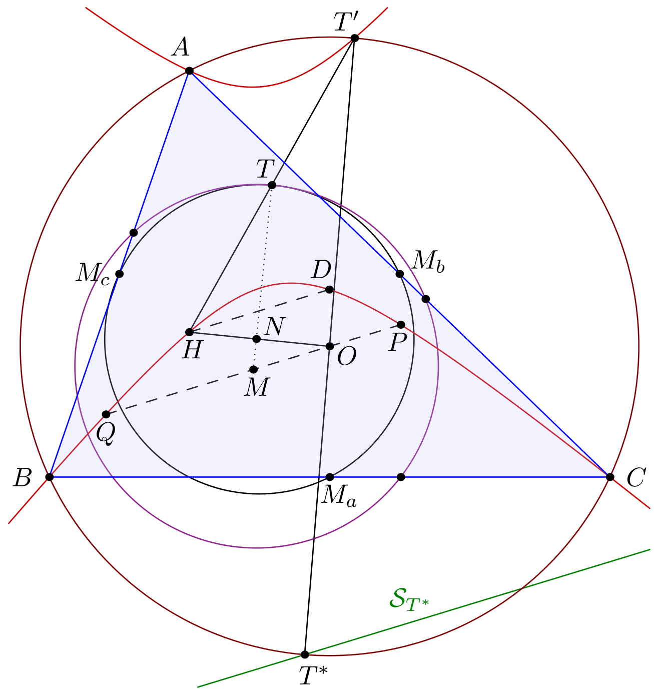
Theorem 8.1.26 (Fontené II)**.** Given an arbitrary $\triangle ABC$, and ℓ as a line through the circumcenter, let Q be a moving point on ℓ. Then the pedal circle of Q passes through a fixed point on the nine-point circle of $\triangle ABC$. Additionally, this fixed point is the Poncelet point of (A, B, C, P) where P is the isogonal conjugate of Q.

Proof. Note that P and Q have the same pedal circle, which thus must pass through the Poncelet point of (A, B, C, P). But this is also the anti-Steiner point of OQ wrt. the medial triangle, so we are done. $\square$

Finally this is the third one. This has many wrong proofs on the internet.

Theorem 8.1.27 (Fontené III)**.** Given $\triangle ABC$, let P, Q be isogonal conjugates. Then the pedal circle of P is tangent to the nine-point circle of $\triangle ABC$ if and only if PQ goes through the circumcenter of $\triangle ABC$.

Proof. Let O, H, N respectively be the circumcenter, orthocenter, and nine-point center of $\triangle ABC$. Let T_1 be the Poncelet point of (A, B, C, P), and let T_2 be the Poncelet point of (A, B, C, Q).



(i) Only if: If P's pedal circle is tangent to the nine-point circle, then $T_1 = T_2$. Thus P, Q lie on one circumrectangular hyperbola of $\triangle ABC$ (which also goes through H), so just take its isogonal conjugate and get a line through O. (ii) If: Since $O \in PQ$, we have $T_1 = T_2$ also, call it just T. These two circles being tangent is equivalent to M, N, T collinear, where M is the midpoint of PQ. Let the reflection of H over T be T', and let $\mathcal{H}$ be the conic through A, B, C, P, Q. Then $H, T' \in \mathcal{H}$ and T is the center of $\mathcal{H}$. Let T''s antipode in (ABC) be T^*, and let $\mathcal{S}_X$ be the Steiner line of X in $\triangle ABC$. Then by Proposition 1.4.5, $\mathcal{S}_{T^*} \parallel PQ$, so

$$(T'A, T'B; T'C, T'O) = (A, B; C, T^*) = (\mathcal{S}_A, \mathcal{S}_B; \mathcal{S}_C, \mathcal{S}_{T^*}) = (HA, HB; HC, H\infty_{PQ}).$$

Thus $D := H\infty_{PQ} \cap OT' \in \mathcal{H}$. From the Parallel Chords Theorem Corollary 7.1.13 we know that the midpoint of $\overline{HD}$ lies on TM, so when OT' is dilated by $\frac{1}{2}$ around H we get $N \in TM$. □

Combining this with Theorem 8.1.22, we can get a much more powerful result.

Corollary 8.1.28. Given a fixed $\triangle ABC$, for an arbitrary point P, then the following statements are equivalent:

- $\sum \measuredangle(AP, BC) = 90^\circ$;
- The pedal circle of P is tangent to the nine-point circle;
- If P, Q are isogonal conjugates then PQ goes through the circumcenter.

In reality, the locus of isogonal conjugates P, Q such that the circumcenter lies on PQ is a cubic called the McCay cubic. We will revisit this later.

Practice Problems

Problem 1. Let $\triangle ABC$ be self-conjugate wrt. rectangular hyperbola $\mathcal{H}$. Prove that I, I^a, I^b, I^c lie on $\mathcal{H}$.

8.2 Feuerbach

This is the most commonly seen named hyperbola, and can, unsuprisingly, prove Feuerbach's theorem.

Theorem 8.2.1 (Feuerbach's)**.** The nine-point circle is always tangent to the incircle.

Of course, we already proved this back in Example 3.3.4, but here's a direct proof of the machinery we've just established.

Proof. Follows by applying Corollary 8.1.28 to $\sum \measuredangle(AI, BC) = 90°$. □

We return to the Feuerbach point:

Definition 8.2.2. For an arbitrary $\triangle ABC$, let the tangency point of the incircle and the nine-point circle be the **Feuerbach point** of $\triangle ABC$. Similarly, we can define A, B, C-Feuerbach points for the A, B, C-excircle tangency points.

Remark. Note that a lot of theorems that hold for the incircle also hold for the three excircles in a slightly modified way (maybe with flipping around signed distances). We call this **extraversion**, and for the rest of this section we will only prove stuff for the incircle Feuerbach point.

Let I, G, O, H, N, Fe be the incenter, centroid, circumcenter, orthocenter, nine-point center, and Feuerbach point as usual.

Since the Poncelet point of (A, B, C, P) lies on the intersection of the pedal circle of P and the nine-point circle, we have

Proposition 8.2.3. Fe is the Poncelet point of (A, B, C, I).

Definition 8.2.4. We call the rectangular hyperbola through A, B, C, I the **Feuerbach hyperbola**.

Now we give some basic properties of the Feuerbach hyperbola.

Proposition 8.2.5. The Feuerbach hyperbola $\mathcal{H}_{Fe}$ is the isogonal conjugate of line $\overline{OI}$.

Corollary 8.2.6. Let X_{104} be the antipode of H in the Feuerbach hyperbola. Then this is the fourth intersection of $\mathcal{H}_{Fe}$ with Ω. Thus X_{104} is also the isogonal conjugate of ∞_{OI}, so we will call it ∞_{OI}^*.

Proposition 8.2.7. Let F_I, F_O be the feet from I, O to BC, let $E = OI \cap BC$. Let I_E, O_E be the foot from E to AI, AO. Then $Fe = F_I I_E \cap F_O O_E$.

Actually, we can extend this:

Proposition 8.2.8. Let P be an arbitrary point, and let Q be its isogonal conjugate. Let T be the Poncelet point of (A, B, C, Q). Let T_P, T_O be the feet from P, O to BC, and let E be the intersection point of OP and BC. Let P_E, O_E respectively be the feet from E to AP, AO, then

$$\triangle APO \overset{+}{\sim} \triangle TT_PT_O$$

and $T = T_PP_E \cap T_OO_E$.

Proof. Let T', T'_P, T'_O be the reflections of T, T_P, T_O over the midline parallel to BC. Then by the proof of Fontené I, we know that $T' \in OP$ and A, P, T', T'_P and A, O, T', T'_O are both concyclic. Thus

$$\measuredangle OPA = \measuredangle T'T'_PA = \measuredangle T_OT_PT, \quad \measuredangle AOP = \measuredangle AT'_OT' = \measuredangle TT_OT_P$$

so we have $\triangle APO \overset{+}{\sim} \triangle TT_PT_O$. Note that P, T_P, E, P_E and O, T_O, E, O_E are respectively concyclic, so we have

$$\measuredangle TT_PE = \measuredangle APO = \measuredangle P_EPE = \measuredangle P_ET_PE$$

so $T \in T_PP_E$. Similarly $T \in T_OO_E$. □

From the above proof we can derive some more results:

Proposition 8.2.9. Let OP intersect BC, CA, AB at D, E, F. Then

$$\measuredangle ATD = \measuredangle BTE = \measuredangle CTF = 90^\circ.$$

Proof. Define T' similarly as the reflection of T across the BC-midline. Then T' is the foot from A to OP, so T' in (AD), and since the midpoint of AD lies on the BC-midline, we have $T \in (AD)$. Similarly we have $T \in (BE), (CF)$. □

So we can actually say that OP is the orthotransversal of T wrt. $\triangle ABC$. We have a pretty good understanding of orthotransversals of points on the nine-point circle now! The proof of Proposition 8.2.9 can also give us some more results:

Corollary 8.2.10. The line HT is the Steiner line of complete quadrilateral $\triangle ABC \cup OP$.

Proof. H is the orthocenter of $\triangle ABC$ and thus lies on the Steiner line, T lies on the line due to the above proposition. □

We go back to incenter configurations: from Theorem 7.1.16 we can get

Corollary 8.2.11. Let K be a line and let $\triangle K^aK^bK^c$ be the anticevian triangle of K wrt. $\triangle ABC$. Let $\triangle M_aM_bM_c$ be the medial triangle of $\triangle ABC$, then we have lines

$$K, \mathcal{L}_\infty, K^a, K^b, K^c, M_bM_c, M_cM_a, M_aM_b$$

are all tangent to a common parabola.

Corollary 8.2.12. Let $\triangle DEF$ be the intouch triangle and let X, Y, Z respectively be

$$EF \cap BC, FD \cap CA, DE \cap AB,$$

then the quadrilaterals formed by $\triangle DEF \cup \overline{XYZ} \cup \triangle M_aM_bM_c$ have a common Miquel point, and further, this Miquel point is Fe, and OI is the Steiner line.

Proof. First, by applying the above lemma we get a common parabola $\mathcal{P}$ which by Proposition 8.1.1 passes through $Fe = (DEF) \cap (M_aM_bM_c)$, which must be the focus of $\mathcal{P}$. By the above tangencies we get Fe is the common Miquel point.

Then O is the orthocenter of $\triangle M_aM_bM_c$ and I is the orthocenter of $\triangle DEF$ which finishes. □

Let's revisit the Feuerbach hyperbola. Note that the insimilicenter of the circumcircle Ω and the incircle ω is X_{55} and the exsimilicenter is X_{56}, and that these are the isogonal conjugates of the Gergonne point Ge and the Nagel point Na. Thus we have:

Proposition 8.2.13. $Ge, Na \in \mathcal{H}_{Fe}$ and

$$(H, I; Na, Ge)_{\mathcal{H}_{Fe}} = -1.$$

Proposition 8.2.14. OI is tangent to $\mathcal{H}_{Fe}$.

We can extend this to other isoconjugations, too.

Proposition 8.2.15. Let φ be a point isoconjugation in $\triangle ABC$, and let S be its fixed point. Then for any non-degenerate conic $\mathcal{C}$ through A, B, C, S, then $\varphi(\mathcal{C})$ is just the tangent from S to $\mathcal{C}$.

Proof. Suppose $\varphi(\mathcal{C})$ isn't tangent to $\mathcal{C}$, and intersects it at S, P. Then we have

$$\varphi(P) \in \varphi(\mathcal{C} \cap \varphi(\mathcal{C})) \subset \varphi(\mathcal{C}) \cap \mathcal{C} = \{P, S\}.$$

However S is a fixed point, so $\varphi(P) = P$. Since $\mathcal{C}$ is non-degenerate, and the fixed points of φ are S and the anticevian triangle of S, this results in a contradiction. □

The dual of this theorem also holds, and we can use it to get Proposition 5.6.3 immediately.

Corollary 8.2.16. Let L be the reflection of H across O (the de Longchamps point). Then I, Ge, L are collinear.

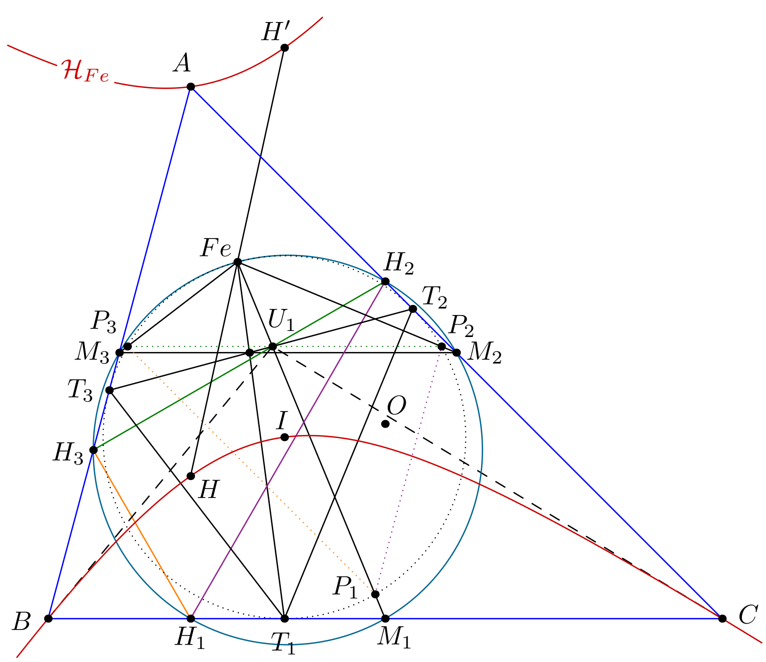
Proof. Note that I, Na, G are collinear (Nagel line), so we have

$$(IH, IO; IG, IL) = -1 = (H, I; Na, Ge)_{\mathcal{H}_{Fe}} = (IH, IO; IG, I(IGe \cap \mathcal{E})),$$

so I, Ge, L are collinear. □

(In fact, this is just a special case of Liang-Zelich).

Example 8.2.17 (IMO 2000/6)**.** Let AH_1, BH_2, CH_3 respectively be the altitude in $\triangle ABC$. Let the intouch points on BC, CA, AB be T_1, T_2, T_3 respectively. Consider the reflections of H_2H_3, H_3H_1, H_1H_2 across T_2T_3, T_3T_1, T_1T_2. Prove that the triangle formed by these three lines has vertices on the incircle of $\triangle ABC$.



Proof. Let $\triangle M_1M_2M_3$ be the median triangle where M_1 is the midpoint of BC.

We consider the Feuerbach hyperbola of $\triangle ABC$, denote it as $\mathcal{H}_{Fe}$. Note that H_2H_3, T_2T_3 are the polars of H_1, T_1 in $\mathcal{H}_{Fe}$. This implies that $U_1 := H_2H_3 \cap T_2T_3$ is the pole of $BC = H_1T_1$ wrt. $\mathcal{H}_{Fe}$.

Let ℓ_i be the reflection of $H_{i+1}H_{i+2}$ across $T_{i+1}T_{i+2}$, then we have B, C, H_2, H_3 concyclic, so as such

$$\measuredangle(\ell_1) = 2\measuredangle(T_2T_3) - \measuredangle(H_2H_3) = (\measuredangle AB + \measuredangle AC) - (\measuredangle CA + \measuredangle AB - \measuredangle BC) = \measuredangle BC,$$

and ℓ_1 is parallel to BC.

Since BU_1, CU_1 are both tangent to $\mathcal{H}_{Fe}$, we have that the line $\ell_1 := U_1\infty_{BC}$ is the polar of M_1 with respect to $\mathcal{H}_{Fe}$. As such, we get that $\ell_2 \cap \ell_3$ is the pole of M_2M_3, let it be P_1.

So we have now identified the vertices of the triangle formed by these three lines! We prove P_1 lies on the incircle, and the rest follow by symmetry.

By Fontené I Theorem 8.1.24, T_1Fe, M_2M_3, T_2T_3 are concurrent. Take the polar of this and we get that $\infty_{T_2T_3}, P_1, T_1$ are collinear. By the Parallel Chords Theorem Corollary 7.1.13, Fe, P_1, M_1 are collinear. Thus P_1 is the intersection of lines $T_1\infty_{T_2T_3}$ and FeM_1.

Let P_1' be the second intersection of FeM_1 with the incircle. Then from Proposition 8.2.8 we have

$$\measuredangle M_1T_1P_1' = \measuredangle T_1FeM_1 = \measuredangle IAO = \measuredangle(BC, \perp AI) = \measuredangle(M_1T_1, T_2T_3),$$

so we have $P_1' = FeM_1 \cap T_1\infty_{T_2T_3} = P_1$, so P_1 must lie on the incircle. $\square$

Since $O \times H = Na \times Na^*$ (barycentric product) and $ONa^* \cap HNa = \infty_{OI}$, by *Corollary* 7.4.5 we have:

Proposition 8.2.18. ∞_{OI}^* is $ONa \cap HNa^*$.

Practice Problems

Problem 1. Prove the four Feuerbach points are concyclic.

Problem 2. Let X be the A-mixtouch point. Let Fe be the Feuerbach point. Prove $AX \perp BC$ iff. A, Fe, X are collinear.

Problem 3 (2015 All-Taiwan P3)**.** Let $\triangle DEF$ be the intouch triangle, let $\triangle D_tE_tF_t$ be the image of $\triangle DEF$ under some homothety from I. Prove that AD_t, BE_t, CF_t concur at one point I_t, and that the locus of I_t is the Feuerbach hyperbola.

Problem 4. Given two points O_1, O_2, let Ω_1, Ω_2 respectively be two circles centered at O_1, O_2 with radius $\overline{O_1O_2}$. Choose $A \in \Omega_1$ outside of Ω_2, and draw the two tangents from A to Ω_2, let these be lines AT_b, AT_c.

Suppose these tangents intersect Ω_1 at B, C. Let the orthocenter of $\triangle ABC$ be H, and let D be the reflection of H across BC, and let OD intersect BC at E. If M, F respectively are the midpoints of BC, AH, prove that T_bT_c is tangent to (DFM).

Problem 5. Let Γ be the circumcircle of $\triangle ABC$, and let it have circumcenter O. Let I^a be the A-excenter. Let OI^a intersect Γ at K, and let the feet from K to CA, AB be E, F. Prove that EF intersects OI^a on the $A-$excircle.

Problem 6 (2017 CMO)**.** Let $(O), (I)$ respectively be the circumcircle and incircle of $\triangle ABC$. Let the tangents to (O) at B, C intersect at L. Let (I) touch BC at D. Let Y be the foot from A to BC, and let AO and BC intersect at X. Let OI intersect (O) at P, Q. Prove that P, Q, X, Y are concyclic if and only if A, D, L are collinear.

Problem 7. Let ℓ be the orthotransversal of P, and let $\mathcal{H}$ be the rectangular hyperbola through A, B, C, P. Prove the tangent from P to $\mathcal{H}$ is perpendicular to ℓ.

8.3 Kiepert

Before discussing this hyperbola, we first revisit the Fermat and isodynamic points. We will also define some notation.

Definition 8.3.1. In $\triangle ABC$, define the Fermat points F_1, F_2 as

$$\measuredangle BF_iC = \measuredangle CF_iA = \measuredangle AF_iB = -i \cdot 60°.$$

We call F_1 the **first Fermat point** and we call F_2 the **second Fermat point**. (These are X_{13}, X_{14} respectively.)

Definition 8.3.2. In $\triangle ABC$, define the two isodynamic points S_1 and S_2 as the two common intersection points of the Apollonian circles with foci B, C through A, foci C, A through B, and foci A, B through C. (For a proof of existence, see Section 3.4). Define S_1 to be the point inside (ABC) and define S_2 to be the point outside (ABC). We call S_1 the **first isodynamic point** and we call S_2 the **second isodynamic point**. (These are X_{15}, X_{16}.)

Let's revisit the alternate definitions given back in *Proposition* 3.4.9.

Definition 8.3.3. Given $\triangle ABC$, construct equilateral triangles on BC, CA, AB either all inside (2) or all outside (1) of the triangle, let these triangles be $\triangle A_{1(2)}BC$, $\triangle B_{1(2)}CA$, and $\triangle C_{1(2)}AB$. Then $AA_{1(2)}$, $BB_{1(2)}$, and $CC_{1(2)}$ intersect in the first and second Fermat points respectively.

Now time for the hyperbola.

Proposition 8.3.4 (Kiepert's hyperbola)**.** A, B, C, F_1, F_2 lie on one rectangular hyperbola $\mathcal{H}_K$, and $\overline{F_1F_2}$ is a diameter of this hyperbola.

Proof. Note that $\measuredangle BF_1C + \measuredangle BF_2C = 0°$ (cyclically). Thus F_1, F_2 are antigonal conjugates. So by Proposition 8.1.17 they must lie on one rectangular hyperbola. □

Of course, we call this hyperbola the **Kiepert hyperbola** of $\triangle ABC$. Since F_i, S_i are isogonal conjugates, we have that the rectangular hyperbola $\mathcal{H}_K$ is just the isogonal conjugate of the line S_1S_2, also called the **Brocard axis** of $\triangle ABC$.

(We will further investigate the Brocard axis in Definition 8.3.1).

Proposition 8.3.5. For any angle θ, choose $A_\theta, B_\theta, C_\theta$ such that

$$\measuredangle A_\theta BC = \measuredangle BCA_\theta = \measuredangle B_\theta CA = \measuredangle CAB_\theta = \measuredangle C_\theta AB = \measuredangle ABC_\theta = \theta,$$

we have $AA_\theta, BB_\theta, CC_\theta$ concur at a point K_θ. The locus of K_θ as θ varies gives the Kiepert hyperbola $\mathcal{H}_K$.

Proof. Let $K_\theta = BB_\theta \cap CC_\theta$. When $\theta = \pm 60°$, we have the two Fermat points; when $\theta = 90°$, K_θ is just H. So thus the locus of K_θ is a conic through B, C, F_1, F_2, H (degree can be bounded at max. 2). However this is just the Kiepert hyperbola so we are done, and it follows by symmetry that AA_θ also passes through this. □

Notably when $\theta = 0°$, G is the centroid. This means that O, K lie on the Brocard axis.

Proposition 8.3.6. For any θ, we have $(K_\theta, K_{-\theta}; H, G)_{\mathcal{H}_K} = -1$.

Proof. Note that A_0 is the midpoint of BC, so it's also the midpoint of $A_\theta A_{-\theta}$, and $A_{90°} = \infty_{\perp BC}$, so we have

$$(K_\theta, K_{-\theta}; H, G)_{\mathcal{H}_K} \stackrel{A}{=} (A_\theta, A_{-\theta}; A_0, A_{90°}) = -1.$$

□

When we set $\theta = 60°$, we get that $(F_1F_2)(HG)$ is a harmonic quadrilateral on $\mathcal{H}_K$. Additionally,

Corollary 8.3.7. The line F_1F_2 bisects GH.

Proof. Since $(F_1F_2)(HG)$ is a harmonic quadrilateral on $\mathcal{H}_K$, we have

$$\mathfrak{p}_{\mathcal{H}_K}(GH) = \mathbf{T}_G\mathcal{H}_K \cap \mathbf{T}_H\mathcal{H}_K \in F_1F_2$$

and F_1F_2 passes through the center of $\mathcal{H}_K$, so by Corollary 7.1.14 we have F_1F_2 bisects GH. □

The following theorems will use Liang-Zelich to prove theorems back in Chapter 5. You should probably wait until you read that section before coming back.

Let $\mathcal{E}$ be the Euler line of $\triangle ABC$.

Proposition 8.3.8. Let $\theta \in [-\pi/2, \pi/2)$, we have

$$t(A_\theta) = t(B_\theta) = t(C_\theta) = -\frac{1}{2\cos 2\theta}.$$

Proof. By symmetry, we only need to prove this for A_θ. Let $\triangle O_aO_bO_c$ be the Carnot triangle of $\triangle ABC$ wrt. A_θ, let $\triangle O_AO_BO_C$ be the image of $\triangle O_aO_bO_c$ under a homothety of $-2\cos 2\theta$ from A_θ. Then O_A is the orthocenter of $\triangle A_\theta BC$. Since $O_AO_B \parallel O_aO_b \perp A_\theta C \perp O_AB$ we know that $O_A \in BO_B$, and similarly we have $O_A \in CO_C$. Thus we have AO_A, BO_B, CO_C concur at O_A. Thus

$$t(A_\theta) = -\frac{1}{2\cos 2\theta}.$$

□

Let $A_\theta^*, B_\theta^*, C_\theta^*, K_\theta^*$ respectively be the isogonal conjugates of $A_\theta, B_\theta, C_\theta, K_\theta$. By the previous theorems we have A, A_θ, K_θ, A, B_θ, C_θ^*, A, B_θ^*, C_θ and $A, A_\theta^*, K_\theta^*$ collinear (and same for the cyclic permutations), so we actually have 12 collinearities!

Theorem 8.3.9. Using the same notation from Proposition 8.3.5, we have

$$t(K_\theta) = -\frac{1}{2\cos 2\theta}.$$

In particular, $t(F_i) = 1$ (where F_i is either Fermat point).

Proof. Let t_0 be the claimed value, and let $T \in \mathcal{E}$ such that $t(T) = t_0$, then we have $T \in B_\theta B_\theta^* \cap C_\theta C_\theta^*$ by Liang Zelich. Note that the perspectrix of $\triangle BB_\theta^*C_\theta$ and $\triangle CC_\theta^*B_\theta$ is $AA_\theta^*K_\theta^*$, so by Desargues' theorem we know that $BC, B_\theta C_\theta, B_\theta^*C_\theta^*$ are concurrent. Since $B = K_\theta B_\theta \cap K_\theta^*B_\theta^*, C = C_\theta K_\theta \cap C_\theta^*K_\theta^*$, by Desargues's and the fact that $B_\theta C_\theta \cap B_\theta^*C_\theta^* \in BC$, we know that triangles $\triangle K_\theta B_\theta C_\theta$ and $\triangle K_\theta^*B_\theta^*C_\theta^*$ are perspective, or just $T \in K_\theta K_\theta^*$, therefore $t(K_\theta) = t_0$ as well. □

Proposition 8.3.10. For three angles α, β, γ, $K_\alpha, K_\beta, K_\gamma^*$ are collinear if and only if

$$\alpha + \beta + \gamma = 0°.$$

Proof. Let AK_α^* intersect $\mathcal{H}_K$ at K_{α_A}, then we have $\alpha_A + \alpha = \measuredangle BAC$. By the proof of Theorem 8.3.9 we also have $K_\alpha K_{\alpha_A} \cap K_\alpha^* K_{\alpha_A}^* = K_\alpha K_{\alpha_A} \cap OK \in BC$. Similarly define α_B, α_C so it follows that $K_\alpha K_{\alpha_B} \cap OK \in CA$,

$K_\alpha K_{\alpha_C} \cap OK \in AB$. Let OK intersect BC, CA, AB respectively at D, E, F. Since $K^*_{\gamma'} = K_\alpha K_\beta \cap OK$, then

$$\begin{aligned}(D, E; F, K^*_{\gamma'})_{OK} &\overset{K_\alpha}{=} (K_{\alpha_A}, K_{\alpha_B}; K_{\alpha_C}, K_\beta)_{\mathcal{H}_K} \overset{A}{=} (A_{\alpha_A}, A_{\alpha_B}; A_{\alpha_C}, A_\beta) \\ &= (A_{-\measuredangle BAC}, A_{-\measuredangle CBA}; A_{-\measuredangle ACB}, A_{-\alpha-\beta}) \overset{A}{=} (A, B; C, K_{-\alpha-\beta})_{\mathcal{H}_K} \\ &= (D, E; F, K^*_{-\alpha-\beta})\end{aligned}$$

so $\alpha + \beta + \gamma' = 0°$, and thus $K_\alpha, K_\beta, K^*_\gamma$ are collinear iff. $\alpha + \beta + \gamma = 0°$. $\square$

Remark. Once you learn more about cubics, this is really just an example of the group law of Kiepert hyperbola union Brocard axis.

By choosing $\alpha = \beta = \theta, \gamma = -2\theta$, we get that

Corollary 8.3.11. $K_\theta K^*_{-2\theta}$ is tangent to $\mathcal{H}_K$ at K_θ.

Corollary 8.3.12. For any θ, we have $K^*_{-2\theta} = \mathfrak{p}_{\mathcal{H}_K}(K_\theta K_{\theta+90°})$. (For example, K, the symmedian point, is the pole of the Euler line wrt. $\mathcal{H}_K$.)

In other words, we can now know the polar of any point on line OK wrt. $\mathcal{H}_K$! Combining the previous characteristics, we get this crucially important theorem.

Theorem 8.3.13 (The Essence of the Kiepert Hyperbola)**.** Let N be the nine-point center of $\triangle ABC$, let $T_\theta = K_\theta K^*_\theta \cap K_{-\theta} K^*_{-\theta}, P_\theta := K_\theta K^*_{-2\theta} \cap K_{-\theta} K^*_{2\theta}$, then for any angle θ, we have:

- $G = K_\theta K^*_{-\theta} \cap K^*_\theta K_{-\theta}$;
- $K = K_\theta K_{-\theta} \cap K^*_\theta K^*_{-\theta}$;
- $N \in K_\theta K_{\theta+90°}$;
- $T_\theta \in \mathcal{E}$;
- $P_\theta \in \mathcal{E}$.

Proof. (i) and (ii) are trivial by applying Proposition 8.3.10 for $\alpha = 0, \beta = \pm\theta$ and $a = \pm\theta, b = \mp\theta$. They are also equivalent by DDIT.

For (iii), since $\mathfrak{p}_{\mathcal{H}_K}(K_\theta K_{\theta+90°}) = K_{2\theta} \in OK$, follows by the above, we have $\mathfrak{p}_{\mathcal{H}_K}(OK) \in K_\theta K_{\theta+90°}$. Let $\theta = 0°$, then we have $\mathfrak{p}_{\mathcal{H}_K}(OK) \in \mathcal{E}$, but since $O \in \mathcal{E}$, we have

$$(G, H; O, \mathfrak{p}_{\mathcal{H}_K}(OK)) = -1,$$

so $N = \mathfrak{p}_{\mathcal{H}_K}(OK)$.

For (iv), note that
$$t(K_\theta) = -\frac{1}{2\cos(2\theta)} = -\frac{1}{2\cos(-2\theta)} = t(K_{-\theta}),$$
so $T_\theta = \mathcal{E} \cap K_\theta K_\theta^* \in K_{-\theta}K_{-\theta}^*$.

Finally, for (v), since $K_{\pm\theta}K_{\mp 2\theta}^* = \mathbf{T}_{K_{\pm\theta}}\mathcal{H}_K$, and $K_\theta H K_{-\theta} G$ is harmonic on $\mathcal{H}_K$, we have $P_\theta = \mathfrak{p}_{\mathcal{H}_K}(K_\theta K_{-\theta}) \in GH$. $\square$

This trivializes many of our previous results!

Example 8.3.14 (Essence of the Fermat and Isodynamic points)**.** In the previous theorem, set θ to $\pm 60°$. Thus we have

- $G = F_1S_2 \cap F_2S_1, K = F_1F_2 \cap S_1S_2, F_1S_1 \parallel F_2S_2 \parallel \mathcal{E}$;
- F_1, F_2, O, N concyclic (we call this the **Lester circle**);
- GK is the symmedian of $\triangle GF_1F_2$.

Proof. The first line follows by the above.

For the Lester circle, by power of a point and Corollary 8.3.7 it remains to show that (GF_1F_2) is tangent to the Euler line, which is true by Proposition 8.1.7.

The final result then follows since K is the polar of the Euler line wrt $\mathcal{H}_K$. $\square$

We can also get that F_iS_i is tangent to $\mathcal{H}_K$.

Example 8.3.15. Some other important points on the Kiepert hyperbola are the **Vecten points**, $V_1 = K_{45°}, V_2 = K_{-45°}$. This suprisingly actually comes up, in 2001 ISL G1! We have

- O is the pole of V_1V_2 wrt. $\mathcal{H}_K$.
- $V_1V_2 = NK$;
- $H = V_1V_1^* \cap V_2V_2^*$

Note that the map sending T_θ to P_θ is an automorphism of $\mathcal{E}$, so in theory we could calculate the positions of these points.

Proposition 8.3.16. For an angle θ, we have $T_\theta G = 2 \cdot GP_\theta$, and thus
$$OP_\theta = 2\cos 2\theta \cdot P_\theta N.$$

If we set $Q_\theta = K_\theta K_{-\theta} \cap \mathcal{E}$, then we have $[P_\theta \to Q_\theta] \in \mathrm{Aut}(\mathcal{E})$, so we have

Proposition 8.3.17. For any θ, $3 \cdot NQ_\theta = 2\cos 2\theta \cdot Q_\theta O$.

8.3.1 Brocard, Humpty, Dumpty

This is a slight tangent, but let's touch on the Brocard points.

Proposition 8.3.18. In a triangle $\triangle ABC$, there exists two unique points Br_1, Br_2 such that

$$\measuredangle BABr_1 = \measuredangle CBBr_1 = \measuredangle ACBr_1, \measuredangle Br_2AC = \measuredangle Br_2BA = \measuredangle Br_2CB.$$

We call these points the **first Brocard point** and **second Brocard point**. These points are isogonal conjugates too, obviously.

Proof. Suppose there exists a Br_1 satisfying these conditions. Then from $\measuredangle CBBr_1 = \measuredangle ACBr_1$ we know that (BBr_1C) and CA are tangent. Similarly, we know (CBr_1A) and AB are tangent too, and (ABr_1B) and BC are tangent. Thus, if we let Ω_{A1} to be a circle through B, C tangent to CA, and similarly define Ω_{B1}, Ω_{C1}. Then if these three circles concur, Br_1 is just their concurrency point. Since

$$(B-C)_{\Omega_{A1}} + (C-A)_{\Omega_{B1}} + (A-B)_{\Omega_{C1}} = \measuredangle BCA + \measuredangle CAB + \measuredangle ABC = 0^\circ,$$

we have they concur. Similarly we can prove that Br_2 also exists. □

We can also re-interpret this stuff with Trig Ceva: we want to find a angle θ such that

$$\frac{\sin\theta}{\sin(\alpha-\theta)} \cdot \frac{\sin\theta}{\sin(\beta-\theta)} \cdot \frac{\sin\theta}{\sin(\gamma-\theta)} = 1,$$

where α, β, γ are the angles of the triangle. We call this angle θ the **Brocard angle** of $\triangle ABC$.

When $\theta = 0^\circ$, the LHS is just 0. When $\theta = \min \alpha, \beta, \gamma$, the LHS is positive infinity. Thus, by the Intermediate Value Theorem, θ is between $0, \min \alpha, \beta, \gamma$. In other words, Br_1, Br_2 both lie inside $\triangle ABC$.

Additionally, the circles we defined in the proof $\Omega_{Ai}, \Omega_{Bi}, \Omega_{Ci}$'s pairwise intersections have names.

Definition 8.3.19. Let P_A be the second intersection point of Ω_{B2}, Ω_{C1}, and let Q_A be the second intersection point of Ω_{B1}, Ω_{C2}. We call P_A, Q_A, the **A-Humpty** and **A-Dumpty** points. Similarly, we can define P_B, P_C, Q_B, Q_C for B, C too.

From

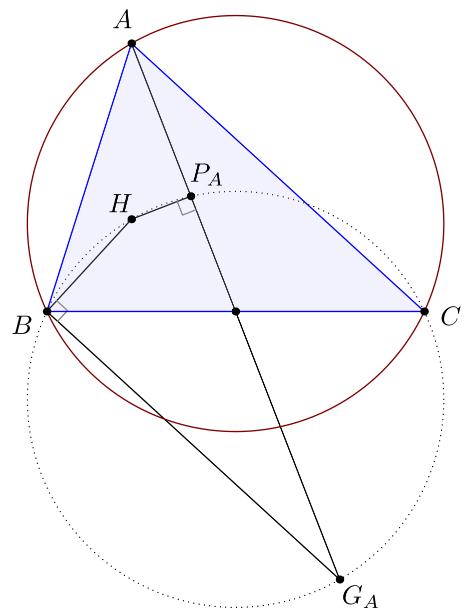
$$\measuredangle CP_AA + \measuredangle CQ_AA = \measuredangle BCA + \measuredangle CAB + \measuredangle CBA$$

$$\measuredangle AP_AB + \measuredangle AQ_AB = \measuredangle ABC + \measuredangle CAB = \measuredangle ACB$$

by Proposition 1.3.6 we get

Proposition 8.3.20. The A-Humpty point and $A-$Dumpty point are isogonal conjugates.

Proposition 8.3.21. Let H, G, P_A be the orthocenter, centroid, and A-Humpty points. Then B, C, H, P_A are concyclic and P_A is also the foot from H to the A-median.



Proof. From

$$(C-B)_{(BHC)} + (A-C)_{\Omega_{B2}} + (B-A)_{\Omega_{C1}} = (-\measuredangle BAC) + \measuredangle BCA + \measuredangle ABC = 0^\circ,$$

we have $P_A \in (BHC)$. Let G_A be a point such that $(AG_A)(BC)$ is a parallelogram, then G_A lies on (BHC) and A, G, G_A are collinear. From

$$\measuredangle BP_AG_A = \measuredangle BCG_A = \measuredangle CBA = \measuredangle BP_AA,$$

we get that A, P_A, G_A are collinear. Combining this with $\measuredangle HP_AG_A = \measuredangle HBG_A = 90^\circ$, we get that P_A is the foot from H to $P_AG_A = AG$. $\square$

Remark. Another fast proof of concyclicity is inversion across the Polar Circle.

This foot property is also why the A-Humpty point is also sometimes called the **HM-point**.

Proposition 8.3.22. Let O, K, Q_A be the circumcenter, symmedian point, and $A-$Dumpty point. Then B, C, O, Q_A are concyclic and Q_A is the foot from O to AK.

Proof. It's basically the same proof as from before. Since

$$(C-B)_{(BOC)} + (A-C)_{\Omega_{B1}} + (B-A)_{\Omega_{C2}} = 2 \cdot \measuredangle BAC + \measuredangle CAB + \measuredangle CAB = 0^\circ,$$

we have $Q_A \in (BOC)$. Let K_A be the pole of BC. Then $K_A \in (BOC)$ and from A, K, K_A collinear, and

$$\measuredangle BQ_AK_A = \measuredangle BCK_A = \measuredangle BAC = \measuredangle BQ_AA,$$

we have A, Q_A, K_A collinear. Combining this with $\measuredangle OQ_AK_A = \measuredangle OBK_A = 90^\circ$, we get that Q_A is the foot from O to $Q_AK_A = AK$. □

Remark. The Dumpty point is also called the "midpoint of symmedian chord point", for obvious reasons.

Theorem 8.3.23. Let O, K, Br_1, Br_2 be the circumcenter, symmedian point, and the two Brocard points. Let $Br_A = BBr_1 \cap CBr_2, Br_B = CBr_1 \cap ABr_2, Br_C = ABr_1 \cap BBr_2$, and let Q_A, Q_B, Q_C respectively be the A, B, C-Dumpty points. Then $O, K, Br_1, Br_2, Br_A, Br_B, Br_C, Q_A, Q_B, Q_C$ are concyclic with diameter OK. We call this the **Brocard circle** Γ_{Br} and we call OK the **Brocard axis**.

Proof. Since ABr_1, ABr_2 are isogonal lines in $\measuredangle BAC$, we have

$$\measuredangle Br_ABC = \measuredangle Br_1BC = \measuredangle Br_1AB = \measuredangle CABr_2 = \measuredangle BCBr_2 = \measuredangle BCBr_A,$$

so Br_A lies on the perpendicular bisector of BC. Similarly Br_B lies on the perpendicular bisector of CA and Br_C lies on the perpendicular bisector of AB. Then since $Br_1 \in \Omega_{C1}, Br_2 \in \Omega_{B2}$, we get

$$\measuredangle Br_BBr_1Br_C = \measuredangle BAC = \measuredangle Br_BOBr_C = \measuredangle CAB,$$

so we have $O, Br_1, Br_2, Br_B, Br_C, Br_A$ are all concyclic on the Brocard circle Γ_{Br}. Since Q_A is the second intersection of Ω_{B1} and Ω_{C2} we have

$$\begin{aligned}\measuredangle Br_1Q_ABr_2 &= \measuredangle Br_1Q_AA + \measuredangle AQ_ABr_2 = \measuredangle Br_1CA + \measuredangle ABBr_2 \\ &= \measuredangle Br_1BC + \measuredangle BCBr_2 = \measuredangle Br_1Br_ABr_2,\end{aligned}$$

so $Q_A \in \Gamma_{Br}$. Similarly $Q_B, Q_C \in \Gamma_{Br}$. Finally, Proposition 8.3.22 tells us that Q_B, Q_C are just the feet from O to BK, CK, so O, K, Q_B, Q_C are concyclic and the diameter is OK. □

Corollary 8.3.24. OK is the perpendicular bisector of Br_1Br_2.

Proof. Continuing from above, since Br_A lies on the perpendicular bisector of BC,

$$\begin{aligned}\measuredangle Br_2Br_1O = \measuredangle Br_2Br_AO = \measuredangle(CBr_A, \perp BC)\\ = \measuredangle(\perp BC, BBr_A) = \measuredangle OBr_ABr_1 = \measuredangle OBr_2Br_1,\end{aligned}$$

so $\triangle OBr_1Br_2$ is an isosceles triangle with top vertex O. Since O is the antipode of K in the Brocard circle, OK is the perp. bisector of $\overline{Br_1Br_2}$. $\square$

Proposition 8.3.25. Let $\triangle Br_{1a}Br_{1b}Br_{1c}, \triangle Br_{2a}Br_{2b}Br_{2c}$ be the pedal triangles of Br_1 and Br_2. Then

$$\triangle Br_{1c}Br_{1a}Br_{1b} \overset{+}{\cong} \triangle Br_{2b}Br_{2c}Br_{2a} \overset{+}{\sim} \triangle ABC.$$

Proof. We have

$$\measuredangle Br_{1b}Br_{1a}Br_{1c} = \measuredangle ACBr_1 + \measuredangle Br_1BA = \measuredangle CBBr_1 + \measuredangle Br_1BA = \measuredangle CBA.$$

Similarly, $\measuredangle Br_{1c}Br_{1b}Br_{1a} = \measuredangle ACB, \measuredangle Br_{1a}Br_{1c}Br_{1b} = \measuredangle BAC$. which implies $\triangle Br_{1c}Br_{1a}Br_{1b} \overset{+}{\sim} \triangle ABC$, and we also have $\triangle Br_{2b}Br_{2c}Br_{2a} \overset{+}{\sim} \triangle ABC$ by symmetry. The proof of congruency is because the two triangles share a circumcircle. $\square$

Proposition 8.3.26. Keeping the notation from above, let Br_1^* be the isogonal conjugate of Br_1 wrt. $\triangle Br_{1a}Br_{1b}Br_{1c}$, and let Br_2^* be the isogonal conjugate of Br_2 in $\triangle Br_{2a}Br_{2b}Br_{2c}$. Then

- Br_1^*, Br_2^* lie on the circle (Br_1Br_2);
- $K = Br_1Br_2^* \cap Br_2Br_1^*$.

Proof. It's obvious that the spiral-similarity center between $\triangle Br_{1c}Br_{1a}Br_{1b}$ and $\triangle Br_{2b}Br_{2c}Br_{2a}$ is their common circumcenter, O_{Br}, which is also the midpoint of $\overline{Br_1Br_2}$, since those two are antipodes in their common pedal circle. Let $\mathfrak{r}$ be this spiral similarity at O_{Br}. Since $\triangle Br_{1c}Br_{1a}Br_{1b}$'s spiral similarity center with $\triangle ABC$ is just Br_1, we have that Br_1^* is the second Brocard point of $\triangle Br_{1c}Br_{1a}Br_{1b}$. Thus $\mathfrak{r}(Br_1^*) = Br_2$ and $\mathfrak{r}(Br_1) = Br_2^*$. This tells us that

$$O_{Br}Br_1 = O_{Br}Br_2 = O_{Br}Br_1^* = O_{Br}Br_2^*,$$

and Br_1^*, Br_2^* lie on (Br_1Br_2). From

$$\begin{aligned}\measuredangle(Br_1Br_1^*, Br_1Br_2) = \measuredangle(Br_{1b}Br_{1c}, CA) = 90^\circ + \measuredangle BABr_1\\ = \measuredangle OBr_aBr_1 = 90^\circ + \measuredangle KBr_2Br_1,\end{aligned}$$

we can get $Br_2Br_1^* \perp Br_1Br_1^* \perp KBr_2$, so $K \in Br_2Br_1^*$. Similarly, we also have $K \in Br_1Br_2^*$. □

Proposition 8.3.27. Let $\mathcal{C}_{Br}$ be the inellipse of $\triangle ABC$ with Br_1, Br_2 as foci. We call this the **Brocard inellipse** of $\triangle ABC$. Then $\mathcal{C}_{Br}$ touches the sides of $\triangle ABC$ at the vertices of the cevian triangle of the symmedian point.

Proof. We trig-bash. Let D, E, F respectively be the touchpoints of $\mathcal{C}_{Br}$ with BC, CA, AB. From Corollary 7.3.5, since Br_1, Br_2 lie inside $\triangle ABC$, we don't need directed lengths and we have

$$\frac{BD}{DC} = \frac{CA}{AB} \cdot \frac{BBr_1}{Br_1C} \cdot \frac{BBr_2}{Br_2C}.$$

We also have

$$\begin{aligned}\frac{BBr_1}{Br_1C} &= \frac{AB \cdot |\sin\measuredangle BABr_1/\sin\measuredangle ABr_1B|}{CA \cdot |\sin\measuredangle Br_1AC/\sin\measuredangle CBr_1A|} \\ &= \frac{AB}{CA} \cdot \left|\frac{\sin\measuredangle BABr_1}{\sin\measuredangle Br_1AC}\right| \cdot \left|\frac{\sin\measuredangle CAB}{\sin\measuredangle ABC}\right| = \frac{AB \cdot BC}{CA^2} \cdot \left|\frac{\sin\measuredangle BABr_1}{\sin\measuredangle Br_1AC}\right|.\end{aligned}$$

And similarly, we have

$$\frac{BBr_2}{Br_2C} = \frac{AB^2}{BC \cdot CA} \cdot \left|\frac{\sin\measuredangle BABr_2}{\sin\measuredangle Br_2AC}\right|.$$

Thus

$$\frac{BD}{DC} = \frac{CA}{AB} \cdot \frac{AB \cdot BC}{CA^2} \cdot \left|\frac{\sin\measuredangle BABr_1}{\sin\measuredangle Br_1AC}\right| \cdot \frac{AB^2}{BC \cdot CA} \cdot \left|\frac{\sin\measuredangle BABr_2}{\sin\measuredangle Br_2AC}\right| = \left(\frac{AB}{CA}\right)^2,$$

so $D = AK \cap BC$. By the same logic, we have $E = BK \cap CA, F = CK \cap AB$. □

The Brocard inellipse $\mathcal{C}_{Br}$ is actually the envelope of projective maps on a circle.

Proposition 8.3.28. Consider the projective map $\varphi : \Omega = (ABC) \to \Omega$ such that $\varphi(A) = B, \varphi(B) = C, \varphi(C) = A$, then the envelope of

$$\{P\varphi(P) \mid P \in \Omega\}$$

is the Brocard inellipse.

Proof. From Proposition 7.A.15, the envelope of $\{P\varphi(P) \mid P \in \Omega\}$ is a conic $\mathcal{C}$. By setting $P = A, B, C$ we can get that this is an inconic of $\triangle ABC$. Let $\triangle K_AK_BK_C$ be the circumcevian triangle of the symmedian point wrt. $\triangle ABC$. Then we have

$$\begin{aligned}(B, K_B; C, A) &= -1 = (A, K_A; B, C) \\ &= (\varphi(A), \varphi(K_A); \varphi(B), \varphi(C)) = (B, \varphi(K_A); C, A),\end{aligned}$$

so we have $\varphi(K_A) = K_B, \varphi(K_B) = K_C, \varphi(K_C) = K_A$.

To prove that $\mathcal{C}$ is the desired inellipse, we know a conic is fully determined by six tangent lines by Brianchon's theorem. Three of these lines are simply the sides of the triangle, so we only need to find three more. Thus we can just prove that K_BK_C, K_CK_A, K_AK_B are tangent to $\mathcal{C}_{Br}$. By symmetry, we only need to prove that K_BK_C is tangent to $\mathcal{C}_{Br}$. Let S, T respectively be the intersection points of K_BK_C and CA, AB. Let $\triangle DEF$ be the $\mathcal{C}$-touchpoint triangle wrt. $\triangle ABC$, then we have

$$(A, E; C, S) \stackrel{K_B}{=} (A, B; C, K_C) = -1 = (C, A; B, K_B) \stackrel{K_C}{=} (F, A; B, T).$$

Since AF, EA respectively are tangent to $\mathcal{C}_{Br}$ at F, E, and BC is also obviously tangent, we have $ST = K_BK_C$ is also tangent to $\mathcal{C}_{Br}$. □

Since $\varphi^3(A) = A, \varphi^3(B) = B, \varphi^3(C) = C$, we have that φ^3 has three fixed points and is thus the identity transformation. Thus for any point $P \in \Omega$, $\mathcal{C}_{Br}$ is tangent to the triangle $\triangle_P = P\varphi(P)\varphi^2(P)$. Thus from the definition of $\triangle_P$ we can get that $\varphi(\triangle_P) = \triangle_P$, as it is cyclic with period 3. This tells us that $\mathcal{C}_{Br}$ has to be the inellipse with the two Brocard points as foci.

Proposition 8.3.29. Keeping the previous notation, for a point $P \in \Omega$, for triangle $\triangle_P = \triangle P\varphi(P)\varphi^2(P)$, its circumcenter, symmedian point, first Brocard point, and second Brocard point all coincide with the corresponding points of $\triangle ABC$, and as P moves, the Brocard angle $\measuredangle\varphi(P)PBr_1$ is constant. At this point, the Brocard inellipse of $\triangle_P$ touches its three sides at the vertices of the cevian triangle of the symmedian point.

Proof. Obviously the circumcenter of $\triangle_P$ is just O. We define the involution $\psi : \Omega \to \Omega$ on the circumcircle as $\psi(Q) = QBr_1 \cap \Omega$, and the cross-ratio preserving transformation $\Phi = \psi \cdot \varphi^{-1}$. Then we have

$$(\Phi(A) - A)_\Omega = (\Phi(B) - B)_\Omega = (\Phi(C) - C)_\Omega = \theta := \measuredangle BABr_1,$$

so thus for all $Q \in \Omega$ we have $(\Phi(Q) - Q)_\Omega = \theta$. Set $Q = A_P = P, B_P = \phi(P), C_P = \phi^2(P)$. Then we get

$$\measuredangle B_PA_PBr_1 = \measuredangle B_PA_P\Phi(B_P) = \theta,$$

$$\measuredangle C_PB_PBr_1 = \measuredangle C_PB_P\Phi(C_P) = \theta,$$

$$\measuredangle A_PC_PBr_1 = \measuredangle A_PC_P\Phi(A_P) = \theta,$$

so Br_1 is the first Brocard point of $\triangle_P$. Similarly, Br_2 is the second Brocard point of $\triangle_P$. Finally note that $\triangle_P$'s symmmedian point is the antipode of O in (OBr_1Br_2), and is thus also the symmedian point of the reference triangle. So since $\mathcal{C}_{Br}(\triangle_P) = \mathcal{C}_{Br}$, and the $\mathcal{C}_{Br}$-touchpoints in $\triangle_P$ is the cevian triangle of the symmedian point of $\triangle ABC$, it is also the symmedian point of $\triangle_P$. $\square$

Practice Problems

Problem 1 (τ)**.** Choose three points from the set A, B, C, F_1, F_2. Prove the Euler line of this triangle goes through G.

Problem 2. Let O_A, O_B, O_C respectively be the circumcenters of $\triangle S_iBC, \triangle AS_iC, \triangle ABS_i$. Prove that the lines AO_A, BO_B, CO_C are concurrent on OK.

Problem 3. Let N be the nine-point circle of $\triangle ABC$, and let I^a, I^b, I^c be the three excircles. Prove that N and the midpoints of $\overline{F_1S_1}, \overline{F_2S_2}$ lie on a common circumrectangular hyperbola of $\triangle I^aI^bI^c$.

Problem 4. Let $\triangle H_aH_bH_c, \triangle M_aM_bM_c$ respectively be the orthic and medial triangles. Prove that for a point P, the isogonal conjugate of P wrt. the orthic triangle lies on the line connecting the isogonal and isotomic conjugates of P wrt. the medial triangle.

Problem 5. Let K_θ be a point on the Kiepert hyperbola with corresponding angle θ. Let $\triangle XYZ$ be the circumcevian triangle of P wrt. the pedal triangle of P wrt. $\triangle ABC$. Then prove that K_θ lies on the Kiepert hyperbola of $\triangle XYZ$, with corresponding angle $-\theta$.

Problem 6. Prove that a quadrilateral $ABCD$ has Brocard points (that is, points P, Q such that $\measuredangle BAP = \measuredangle CBP = \measuredangle DCP = \measuredangle ADP, \measuredangle QAD = \measuredangle QBA = \measuredangle QCB = \measuredangle QDC$) if and only if $ABCD$ is a harmonic cyclic quadrilateral.

8.4 Jerabek

We already know that for $\triangle ABC$ and a point P, we can define some points, lines, conics etc. with relation to P. In previous sections, we defined the orthotransversal and the trilinear polar, let's see some nice properties of them.

Proposition 8.4.1. Given $\triangle ABC$ and a point P, let $\mathcal{O}_P, \mathfrak{t}(P)$ respectively be the orthotransversal and trilinear polar of P. Let $\mathcal{S}_P$ be the polar of P wrt. its pedal circle ω (we will also call this $\mathfrak{p}_\omega(P)$), then $\mathcal{O}_P, \mathfrak{t}(P), \mathcal{S}_P$ are concurrent.

Proof. Let Ω be an arbitrary circle centered at P, and define $A^* = \mathfrak{p}_\Omega(BC)$. Similarly define B^*, C^*. Then by using polarity, $\mathfrak{p}_\Omega(\mathcal{O}_P)$ is the orthocenter of $\triangle A^*B^*C^*$, and $\mathfrak{p}_\Omega(\mathfrak{t}(P))$ is the centroid of $\triangle A^*B^*C^*$. Note that $\omega, \Omega, (A^*B^*C^*)$ are coaxal by inverting about ω.

Let O^* be the circumcenter of $\triangle A^*B^*C^*$ and let Γ be the circumcircle, and we consider OP's intersections with ω, Ω, Γ. Let the two intersections of OP with $\mathcal{C}$ be $\mathcal{C}_1, \mathcal{C}_2$ Since inversion preserves cross-ratio, we have that

$$(P, \mathcal{S}_P \cap OP; \omega_1, \omega_2) \overset{\Omega}{=} (\infty, \mathfrak{p}_\Omega(\mathcal{S}_P); \Gamma_1, \Gamma_2)$$

so $\mathfrak{p}_\Omega(\mathcal{S}_P)$ is the circumcenter of $\triangle A^*B^*C^*$ too. Thus the three poles are collinear by the Euler line, so $\mathcal{O}_P, \mathfrak{t}(P), \mathcal{S}_P$ are concurrent. □

When P lies on ABC, $\mathcal{S}_P$ is just the Steiner line of P. At this time we have:

Proposition 8.4.2. Given $\triangle ABC$ and P on (ABC), then the orthotransversal of P, the trilinear polar of P, and the Steiner line of P are concurrent.

We now look at polarity wrt. a circle centered at P. There's some nice properties about this too:

Proposition 8.4.3. Given $\triangle ABC$ and P on its circumcircle, let Ω be a circle centered at P. Then

- $\mathfrak{p}_\Omega(\triangle ABC) \,\bar{\sim}\, \triangle ABC$ and P lies on its circumcircle.
- If (Q, Q^*) are two isogonal conjugates in $\triangle ABC$, choose Q' such that $\mathfrak{p}_\Omega(\triangle ABC) \cup Q' \,\bar{\sim}\, \triangle ABC \cup Q$, then $Q' \in \mathfrak{p}_\Omega(Q^*)$.

Proof. (i) follows by angle chasing. For (ii), we consider the composition of a reflection and a spiral similarity φ such that $\varphi(A') = A, \varphi(B') = B, \varphi(C') = C$. Let $\mathcal{C}$ be the diagonal conic centered at P (under isogonal conjugation) through I, I_A, I_B, I_C, which is possible because P is on the circumcircle. Now look at the point-to-point transformation $\Phi := \mathfrak{p}_\mathcal{C} \circ \varphi \circ \mathfrak{p}_\Omega$. Obviously, A, B, C, P are fixed points of Φ, and Φ is a bijective transformation that preserves incidence relations and cross ratio. Thus from Proposition 7.A.10, and four fixed points, we get that this transformation is just the identity transformation! Thus

$$\varphi(\mathfrak{p}_\Omega(Q^*)) = \mathfrak{p}_\mathcal{C}(Q^*)$$

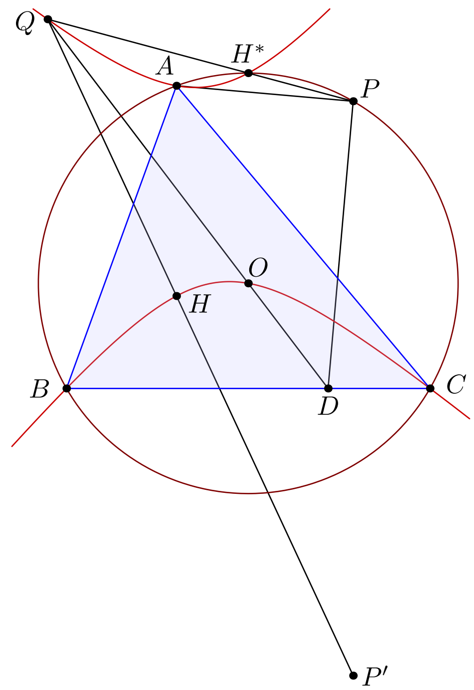
and from properties of isogonal conjugation we know that $Q \in \mathfrak{p}_\mathcal{C}(Q^*)$, so $Q' \in \varphi^{-1}(\mathfrak{p}_\mathcal{C}(Q^*)) = \mathfrak{p}_\Omega(Q^*)$. □

Then modifying the proof of Proposition 8.4.1, we have

Corollary 8.4.4. Given $\triangle ABC$ and P on its circumcircle, the orthotransversal, trilinear polar, and Steiner line of P respectively pass through the circumcenter, symmedian point, and orthocenter of $\triangle ABC$.

Let's consider the locus of the concurrency point of these three lines as P moves.

Proposition 8.4.5. Given $\triangle ABC$ and a moving point P on its circumcircle, the locus of the concurrency point Q of the orthotransversal, trilinear polar, and Steiner line is a circumrectangular hyperbola. Additionally, PQ goes through a fixed point on (ABC).



Proof. Let O, H be the circumcenter and orthocenter. For a point $P \in (ABC)$, choose D on BC such that $\measuredangle APD = 90^\circ$. Let A', P' be the reflections of A, P across BC. Then

$$(OA, OB; OC, OD) = O(A, B; C, P) = (HA', HB; HC, HP'),$$

so $Q = OD \cap HP' \in \mathcal{H}$, where $\mathcal{H}$ is the hyperbola through A, B, C, O, H (by symmetry OD is the orthotransversal of P). Let H^* be the fourth intersection point of $\mathcal{H}$ with (ABC). Then

$$(H^*A, H^*B; H^*C, H^*P) = (HA', HB; HC, HP') = (HA, HB; HC, HQ) = (H^*A, H^*B; H^*C, H^*Q),$$

so $H^* \in PQ$. $\square$

Definition 8.4.6. We call this hyperbola the **Jerabek hyperbola** of $\triangle ABC$. The fixed point is just the fourth intersection point of the Jerabek hyperbola with (ABC).

Immediately we get the following proposition:

Proposition 8.4.7. The Jerabek hyperbola is the isogonal conjugate of the Euler line.

Example 8.4.8. The isotomic conjugate X_{69} of H lies on the Jerabek hyperbola.

Practice Problems

Problem 1. Let $\mathcal{E}$ intersect CA, AB at E, F. Prove the Euler line of $\triangle AEF$ is parallel to BC.

(This is a duplicate of a problem example from earlier)

Problem 2. Given $\triangle ABC$ and a point P, let $\mathcal{O}_P, \mathfrak{t}(P)$ respectively be the orthotransversal and trilinear polar of P. Let $\triangle Q_aQ_bQ_c$ be the pedal triangle of the isogonal conjugate of P. If H', G' are the orthocenter and centroid of $\triangle Q_aQ_bQ_c$, prove:

- $QH' \perp \mathcal{O}_P$,
- $QG' \perp \mathcal{T}_P$.

Problem 3. Let P be a point on the Euler line of $\triangle ABC$. Let H be the orthocenter of $\triangle ABC$, and let Q be the isogonal conjugate of P. Prove that HQ is tangent to the conic through A, B, C, H, P.

Problem 4. Let N be the nine-point center of $\triangle ABC$, and let Ko be the Kosnita point, that is, the isogonal conjugate of N. Prove NKo passes through the anti-Steiner point of the Euler line of $\triangle ABC$ (we call this the **Euler reflection point**, X_{110}), and prove that this point is the antipode of the fourth intersection of the Jerabek hyperbola with (ABC).

Chapter 9

Perfect Six-Point Sets and The Isoptic Cubic

9.1 Perfect Six-Point Sets

For a thorough understanding of the isoptic cubic, we first introduce a very useful tool for quadrilateral geometry - the **perfect six-point set**. This tool was developed by Chinese geometers Ye Zhonghao and Shan Zun (or maybe p_square on AoPS?) (it was originally named the "Perfect Hexagon"), and they wrote a poem.

So what is a perfect six-point set?

Definition 9.1.1. We call $(AD)(BE)(CF)$ a **perfect six-point set** if $(BE)(CF)$, $(CF)(AD)$, $(AD)(BE)$ share the same Miquel Point M. In this case, M is referred as the Miquel point of $(AD)(BE)(CF)$.

More generally, we call the $2n$ points $(X_1Y_1)\ldots(X_nY_n)$ a **perfect $2n$-point set** if there exists a point M such that for all i, j, the Miquel point of $(X_iY_i)(X_jY_j)$ is M. In particular, for any perfect six-point set with Miquel point M, $(AD)(BE)(CF)(M\infty)$ is a perfect eight-point set, where ∞ is any point at infinity.

From this definition we know that a perfect six-point set $(AD)(BE)(CF)$ remains one under reflections, translations, homotheties and rotations.

If $M \notin \mathcal{L}_\infty$, then by Example 8.1.13 there exists a negative inversion (the composition of an inversion and a reflection) φ on points of $\triangle MIJ$ (where I, J are the circle points). sending A, B, C to D, E, F respectively. This is called the **Clawson-Schmidt transformation** (the C-S transformation) on $(AD)(BE)(CF)$. If $M \in \mathcal{L}_\infty$, then in reality M can be any point on $\mathcal{L}_\infty$, and $(BE)(CF)$, $(CF)(AD)$, $(AD)(BE)$ are all parallelograms.

That is, $(AD)(BE)(CF)$ is a hexagon with parallel opposite sides, and in this case we simply denote M as ∞. Then, the C-S transformation on $(AD)(BE)(CF)$ is simply a reflection about their common midpoint. Thus, given five points B, C, D, E, F, there exists a unique point A such that $(AD)(BE)(CF)$ is a perfect six-point set (for defining all six point sets, we work in $\mathbb{CP}^1$ or points at infinity are the same point ∞, we are working in $\mathbb{CP}^1$).

Example 9.1.2. Let $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$ be a complete quadrilateral and $A_{ij} := \ell_i \cap \ell_j$ be the vertices of $\mathcal{Q}$. Then, $(A_{23}A_{14})(A_{31}A_{24})(A_{12}A_{34})$ is a perfect six-point set.

Proposition 9.1.3. A six-point set $(AD)(BE)(CF)$ is perfect if and only if

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \cdot \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = -1 \in \mathscr{S}^{ab} \tag{1}$$

where -1 represents any reflection about a point and notation is taken from Example 6.A.1. If $W \in \mathcal{L}_\infty \setminus \{I, J\}$, $X, Y \notin \mathcal{L}_\infty$, we define $\frac{\overrightarrow{XW}}{\overrightarrow{WY}}$ as -1.

(Note: $\overrightarrow{XY}$ represents the signed length, defined as $\overrightarrow{XY} = -\overrightarrow{YX}$ and $\left|\overrightarrow{XY}\right| = \overline{XY}$.)

In the special case of 9.1.2, this property is just the familiar Menelaus Theorem.

Proof. From the existence and uniqueness, we only need to prove that when $(AD)(BE)(CF)$ is a perfect six-point set,

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \circ \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \circ \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = -1.$$

Let M be the Miquel point of $(AD)(BE)(CF)$. Then,

$$\frac{\overrightarrow{CE}}{\overrightarrow{BF}} = \frac{\overrightarrow{MC}}{\overrightarrow{MB}}, \quad \frac{\overrightarrow{AF}}{\overrightarrow{CD}} = \frac{\overrightarrow{MA}}{\overrightarrow{MC}}, \quad \frac{\overrightarrow{BD}}{\overrightarrow{AE}} = \frac{\overrightarrow{MB}}{\overrightarrow{MA}}.$$

This means that

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \circ \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \circ \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = \left(-\frac{\overrightarrow{MC}}{\overrightarrow{MB}}\right) \circ \left(-\frac{\overrightarrow{MA}}{\overrightarrow{MC}}\right) \circ \left(-\frac{\overrightarrow{MB}}{\overrightarrow{MA}}\right) = -1$$

as desired $\square$

Proposition 9.1.4. The hexagon $(AD)(BE)(CF)$ is a perfect six-point set if and only if there exists two points X_+, X_- such that $(AD)(X_+X_-), (BE)(X_+X_-), (CF)(X_+X_-)$ are all harmonic quadrilaterals.

Proof.

($\Rightarrow$) Let M be the Miquel point and let X_+ and X_- be the fixed points of Clawson-Schmidt conjugation. Then this implies immediately that $(IA, ID; IX_+, IX_-) = -1$, and the holds for J. This implies that $(AD)(X_+)(X_-)$ is harmonic cyclic, which finishes by symmetry.

($\Leftarrow$) For the backwards direction, let M be the midpoint of X_+X_-. Since $(X_+, X_-; M, X_+X_- \cap \mathcal{L}_\infty) = 1$, we have that X_- is one of the vertices of the anticevian triangle of X_+ wrt. $\triangle MIJ$. Thus there exists a point isoconjugation φ in $\triangle MIJ$ such that X_+, X_- are fixed, since Clawson-Schmidt conjugation is an isoconjugation in this triangle. Since

$$(IA, I\varphi(A); X_+, X_-) = -1 = (IA, ID; IX_+, IX_-),$$

(and similarly for J), we have $\varphi(A) = D$ and thus $\varphi(B) = E, \varphi(C) = F$. This isoconjugation corresponds then to a negative inversion about (X_+X_-) as desired.

□

Proposition 9.1.5. Perfect six-point sets are preserved under inversion: if $(AD)(BE)(CF)$ is a perfect six-point set, then for any inversion $\mathfrak{J}$ we have that $(A^{\mathfrak{J}}D^{\mathfrak{J}})(B^{\mathfrak{J}}E^{\mathfrak{J}})(C^{\mathfrak{J}}F^{\mathfrak{J}})$ is also a perfect six-point set.

Proof. Let M, φ respectively be the Miquel point and Clawson-Schmidt conjugation. We can view $\mathfrak{J}$ as just some point isoconjugation in $\triangle OIJ$, where O is the center of the inversion.

To prove $(A^{\mathfrak{J}}D^{\mathfrak{J}})(B^{\mathfrak{J}}E^{\mathfrak{J}})(C^{\mathfrak{J}}F^{\mathfrak{J}})$ is a perfect six-point set, we just need to prove there exists a point X and a isoconjugation on $\triangle XIJ$ such that $A^{\mathfrak{J}} \leftrightarrow D^{\mathfrak{J}}, B^{\mathfrak{J}} \leftrightarrow E^{\mathfrak{J}}, C^{\mathfrak{J}} \leftrightarrow F^{\mathfrak{J}}$. This isoconjugation is just $\mathfrak{J} \circ \varphi \circ \mathfrak{J}$, and X is just $\mathfrak{J}(\varphi(O))$: we note that $\mathfrak{J} \circ \varphi \circ \mathfrak{J}$ is restricted to an isoconjugation by being a involution on $\mathbf{T}I, \mathbf{T}J$, and exchanging $\mathcal{L}_\infty$ and X. □

Example 9.1.6. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let AC and BD intersect at P. Then $(AC)(BD)(OP)$ is a perfect six-point set.

Proof. We consider inversion about O. We know that P is sent to the Miquel point M and O is sent to ∞, and A, B, C, D are fixed. However $(AC)(BD)(\infty M)$ is a perfect six-point set, so we're done. □

Example 9.1.7. Let $ABCDEF$ be a cyclic hexagon. Then $(AD)(BE)(CF)$ is a perfect six-point set if and only if AD, BE, CF are concurrent.

Proof. Let $P = CF \cap AD, P' = AD \cap BE$, then by the above example we know that $(CF)(AD)(OP)$ and $(AD)(BE)(OP')$ are both perfect six-point sets. Thus $(AD)(BE)(CF)$ is a perfect six-point set if and only if $P = P'$, or AD, BE, CF concurrent. □

Proposition 9.1.8. If $(AD)(BE)(CF)$ is a perfect six-point set and the six points are not all concyclic, then

$$(DEF), (DBC), (AEC), (ABF)$$

are concurrent at P, and

$$(ABC), (AEF), (DBF), (DEC)$$

are concurrent at Q. Further, $(AD)(BE)(CF)(PQ)$ is a perfect eight-point set.

Proof. Let the second intersection of (AEC) and (ABF) be point P. We consider the inversion $\mathfrak{J}$ centered at P. Then obviously $A^{\mathfrak{J}}, E^{\mathfrak{J}}, C^{\mathfrak{J}}$ are collinear and $A^{\mathfrak{J}}, B^{\mathfrak{J}}, F^{\mathfrak{J}}$ are also collinear. Let $(D')^{\mathfrak{J}} = B^{\mathfrak{J}}C^{\mathfrak{J}} \cap E^{\mathfrak{J}}F^{\mathfrak{J}}$, then $(A^{\mathfrak{J}}(D')^{\mathfrak{J}})(B^{\mathfrak{J}}E^{\mathfrak{J}})(C^{\mathfrak{J}}F^{\mathfrak{J}})$ is a perfect six-point set. Thus from $(AD)(BE)(CF)$ being a perfect six-point set, we get that $D = (D')^{\mathfrak{J}\circ\mathfrak{J}} = D'$, so $(DEF), (DBC)$ both pass through P.

By Miquel's theorem, $(A^{\mathfrak{J}}B^{\mathfrak{J}}C^{\mathfrak{J}})$, $(A^{\mathfrak{J}}E^{\mathfrak{J}}F^{\mathfrak{J}})$, $(D^{\mathfrak{J}}B^{\mathfrak{J}}F^{\mathfrak{J}})$, and $(D^{\mathfrak{J}}E^{\mathfrak{J}}C^{\mathfrak{J}})$ go through the Miquel point M of $(B^{\mathfrak{J}}E^{\mathfrak{J}})(C^{\mathfrak{J}}F^{\mathfrak{J}})$. Thus (ABC), (AEF), (DBF), and (DEC) are concurrent at a point Q. Therefore $(A^{\mathfrak{J}}D^{\mathfrak{J}})(B^{\mathfrak{J}}E^{\mathfrak{J}})(C^{\mathfrak{J}}F^{\mathfrak{J}})(M\infty)$ is a perfect eight-point set, so by inverting $(AD)(BE)(CF)(QP)$ is also a perfect eight-point set. $\square$

We call this kind of perfect eight-point set a **cyclic perfect eight-point set**. Notably, by definition, $(BE)(CF)(PQ)(AD)$ is also a perfect eight-point set: these four sets of points are symmetric.

Here's a nice example of this:

Example 9.1.9. Let P_A, P_B, P_C be the reflections of arbitrary point P across BC, CA, AB. Then the four circles $(BP_AC), (CP_BA), (AP_CB), (P_AP_BP_C)$ are concurrent at the antigonal conjugate of P Definition 8.1.16.

Proof. By applying Proposition 9.1.3 we obtaining that $(AP_A)(BP_B)(CP_C)$ is a six point set and thus the result follows. $\square$

Proposition 9.1.10. If $(AD)(BE)(CF)(PQ)$ is a cyclic perfect eight-point set, then the midpoints of AD, BE, CF, PQ are concyclic.

Proof. Let M be the Miquel point of $(AD)(BE)(CF)(PQ)$, and let M_A, M_B, M_C, M_P respectively be the midpoints of segments AD, BE, CF, PQ. Since the isogonal conjugate of the Miquel point is the point at infinity along the Newton line Theorem 7.3.9, we have

$$\begin{aligned}\measuredangle M_CM_A + \measuredangle M_BM_P &= (\measuredangle AC + \measuredangle AF - \measuredangle AM) + (\measuredangle PB + \measuredangle PE - \measuredangle PM) \\ &= (A + B + C + P)_{(ABCP)} + (A + E + F + P)_{(AEFP)} - \measuredangle AM - \measuredangle PM.\end{aligned}$$

In the above expression, note that the right hand is symmetric in swap $(B, E), (C, F)$, so it is also equal to $\measuredangle M_AM_B + \measuredangle M_CM_P$. $\square$

Example 9.1.11. In Example 9.1.9, the center of the circle is N.

Proof. Let M_A, M_B, M_C, be the midpoints of AP_A, BP_B, CP_C respectively. Let O_A be the reflection of O over BC. Since $AH \parallel OO_A$ and $AH = OO_A$, it follows that the midpoint of AO_A is N. As such,

$$\overline{NM_A} = \frac{1}{2}\overline{O_A P_A} = \frac{1}{2}\overline{OP}$$

which shows the result. $\square$

Proposition 9.1.12 (Fixing the Broken Mirror)**.** Let $AFBDCE$ be a hexagon, and let D, E, F's reflections over BC, CA, AB be D', E', F'. Then $\triangle DEF \overset{-}{\sim} \triangle D'E'F'$ if and only if $(AD)(BE)(CF)$ is a perfect six-point set.

Proof. Let E'', F'' respectively be the reflections of E', F' across BC. Then $\triangle DE''F'' \overset{-}{\sim} \triangle D'E'F$, and therefore

$$\triangle DEF \overset{-}{\sim} \triangle D'E'F' \iff \triangle DEF \overset{+}{\sim} \triangle DE''F'' \iff \triangle DEE'' \overset{+}{\sim} \triangle DFF''.$$

Let O be the circumcenter of $\triangle ABC$, then we can check that

$$\triangle CEE'' \overset{+}{\sim} \triangle OAB, \triangle BFF'' \overset{+}{\sim} \triangle OAC.$$

As such, this tells us that

$$\frac{\overrightarrow{BA}}{\overrightarrow{AC}} \circ \frac{\overrightarrow{CE}}{\overrightarrow{ED}} \circ \frac{\overrightarrow{DF}}{\overrightarrow{FB}} = -\frac{\overrightarrow{CE} \circ \overrightarrow{AB}/\overrightarrow{OA}}{\overrightarrow{BF} \circ \overrightarrow{AC}/\overrightarrow{OA}} \circ \frac{\overrightarrow{DF}}{\overrightarrow{DE}} = -\frac{\overrightarrow{DF}/\overrightarrow{FF''}}{\overrightarrow{DE}/\overrightarrow{EE''}}.$$

Thus $(AD)(BE)(CF)$ is a perfect six-point set if and only if $\triangle DEE'' \overset{+}{\sim} \triangle DFF''$. $\square$

Proposition 9.1.13. If $(AD)(BE)(CF)$ is a perfect six-point set, then there exists a point P such that

$$\triangle PBC \overset{+}{\sim} \triangle AFE, \triangle APC \overset{+}{\sim} \triangle FBD, \triangle ABP \overset{+}{\sim} \triangle EDC.$$

Proof. Let M be the Miquel point of $(AD)(BE)(CF)$, and choose P such that $\triangle PBC \overset{+}{\sim} \triangle AFE$. Then $\triangle MBC \overset{+}{\sim} \triangle MFE$ tells us that $\triangle PBC \cap M \overset{+}{\sim} \triangle AFE \cap M$. Thus $\triangle MAP \overset{+}{\sim} \triangle MFB$, and with $\triangle MCA \overset{+}{\sim} \triangle MDF$, we now have $\triangle APC \cup M \overset{+}{\sim} \triangle FBD \cup M$. By repeating this, we have $\triangle ABP \cup M \overset{+}{\sim} \triangle EDC \cup M$. $\square$

We call P the **internal point** of $\triangle DEF$ wrt. $(AD)(BE)(CF)$ (note that by symmetry internal points exist for the other seven triangles too).

Practice Problems

Problem 1. Let M be the Miquel point of perfect six-point set $(A_1A_4)(A_2A_5)(A_3A_6)$, and let M_i be the foot from M to A_iA_{i+1}. Prove that M_1M_4, M_2M_5, M_3M_6 are concurrent.

Problem 2. Let P be the internal point of $\triangle DEF$ wrt. $(AD)(BE)(CF)$, and let Q be the isogonal conjugate of P wrt. $\triangle DEF$. Let D', E', F' respectively be the reflections of D, E, F across BC, CA, AB. Prove that $\triangle DEF \cup Q \overset{-}{\sim} \triangle D'E'F' \cup Q$.

9.2 Perfection of Isogonal Conjugation

So why do we consider the perfect six-point set? Let's first consider a complete quadrilateral $\mathcal{Q}$ composed of lines $(\ell_1, \ell_2, \ell_3, \ell_4)$. Let $\triangle ABC = \triangle \ell_1\ell_2\ell_3$, and let A^*, B^*, C^* be the intersections of ℓ_1, ℓ_2, ℓ_3 with ℓ_4 such that AA^*, BB^*, CC^* are the three diagonals of $\mathcal{Q}$.

Our goal is to characterize the locus of points with an isogonal conjugate in $\mathcal{Q}$. Define

$$\begin{aligned}\mathcal{K}(\mathcal{Q}) &= \{P \mid P \text{ has an isogonal conjugate in } \mathcal{Q}\}\\ &= \{P \mid \text{there exists a } P^* \text{ such that for all triangles in } \mathcal{Q}, P \times P^* = I \times J\ \}.\end{aligned}$$

Since $(AA^*)(BB^*)(CC^*)$ is a perfect six-point set, define a Clawson-Schmidt conjugation sending $A \leftrightarrow A^*, B \leftrightarrow B^*, C \leftrightarrow C^*$. Call this $\varphi_{\mathcal{Q}}$.

Proposition 9.2.1. If (P, P^*) are isogonal conjugates in complete quadrilateral $(BB^*)(CC^*)$, then $(BB^*)(CC^*)(PP^*)$ is a perfect six-point set.

Proof. Let P' be the point such that $(BB^*)(CC^*)(PP')$ is a perfect six-point set. Then note that since $(AA^*)(BB^*)(CC^*)$ is a perfect six-point set, we have $(AA^*)(BB^*)(CC^*)(PP')$ is a perfect eight-point set. Since P has an isogonal conjugate, we have $\measuredangle PB + \measuredangle PB^* = \measuredangle PC + \measuredangle PC^*$ by Proposition 1.3.14. Since

$$\frac{\overrightarrow{BP'}}{\overrightarrow{P'C}} \circ \frac{\overrightarrow{CB^*}}{\overrightarrow{B^*P}} \circ \frac{\overrightarrow{PC^*}}{\overrightarrow{C^*B}} = -1,$$

we get

$$\begin{aligned}\measuredangle BP'C &= \measuredangle CB^* - \measuredangle B^*P + \measuredangle PC^* - \measuredangle C^*B = \measuredangle CA - \measuredangle AB + \measuredangle PB - \measuredangle PC\\ &= \measuredangle CA - \measuredangle AB + (\measuredangle AB + \measuredangle BC - \measuredangle BP^*) - (\measuredangle BC + \measuredangle CA - \measuredangle CP^*)\\ &= \measuredangle BP^*C\end{aligned}$$

which implies that $P' = P^*$ by symmetry. $\square$

Proposition 9.2.2. If (P, P^*) are isogonal conjugates in $\mathcal{Q}$, then the midpoint of PP^* lies on the Newton line τ.

Proof. Let $\mathcal{C}$ be a inscribed conic in $\mathcal{Q}$, with P, P^* as foci (this must exist since they are isogonal conjugates). Then PP^*'s center is just the center of this conic. However by Newton II, Theorem 6.3.13 the center of $\mathcal{C}$ lies on τ. $\square$

So we have two characteristics for isogonal conjugates (P, P^*). We claim that in reality, these characteristics are also sufficient!

Proposition 9.2.3. Let $(BB^*)(CC^*)$ be a quadrilateral $\mathcal{Q}$ that's not a parallelogram. If $(BB^*)(CC^*)(PP^*)$ is a perfect six-point set, and the midpoints of BB^*, CC^*, PP^* are collinear, then (P, P^*) are isogonal conjugates in $(BB^*)(CC^*)$.

Proof. Since the midpoint of PP^* lies on the line through the midpoints of BB^*, CC^*, from Proposition 6.A.4 there exists an conic $\mathcal{C}$ with center as the midpoint of PP^*. Then the four foci of $\mathcal{C}$ form two pairs of isogonal conjugates $(Q, Q^*), (R, R^*)$ such that PP^*, QQ^*, RR^* have a common midpoint and that $I = QR^* \cap Q^*R, J = QR \cap Q^*R^*$. At this point we know that $(BB^*)(CC^*)(QQ^*), (BB^*)(CC^*)(RR^*)$ are all complete six-point sets. Thus

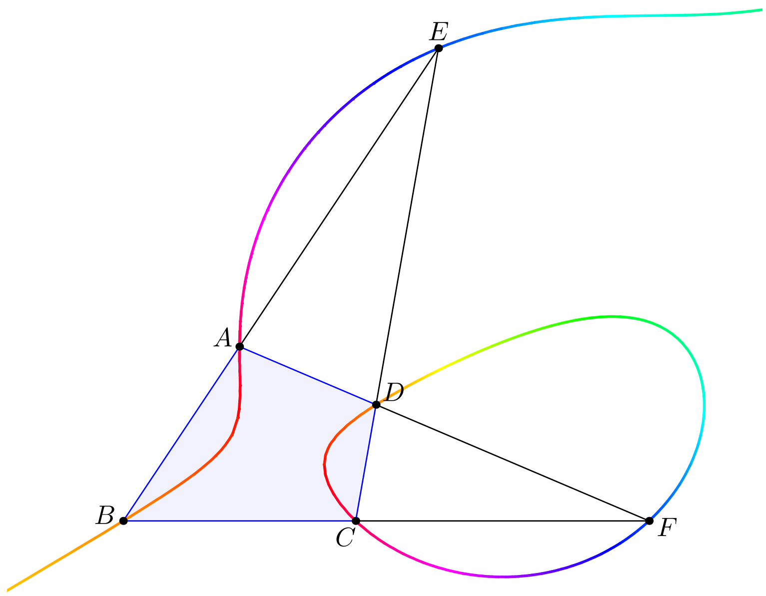
$$(BB^*)(CC^*)(PP^*)(QQ^*)(RR^*)$$

is a complete ten-point set. We proceed by contradiction. If $\{(P, P^*)\} \neq \{(Q, Q^*)\}, \{(R, R^*)\}$, let $M \notin \mathcal{L}_\infty$ be the Miquel point of $\mathcal{Q}$, which is also the Miquel point of $(PP^*)(QQ^*)(RR^*)$. Since $PQ \parallel P^*Q^*$, it follows that $\measuredangle MPQ = \measuredangle MQ^*P^*$ tells us that $MP \parallel MQ^*$, but since $M \notin \mathcal{L}_\infty$, this means that $M \in PQ^*$. Similarly we also get that $M \in P^*Q$, however $PQ^* \parallel P^*Q$, contradiction. $\square$

So we can say that for a quadrilateral $\mathcal{Q}$, the locus of points with isogonal conjugates $\mathcal{K}$ is completely determined by a given Clawson-Schmidt conjugation and a Newton line.

Theorem 9.2.4 (QL-Cu1, the Isoptic Cubic)**.** Let τ be the Newton line, and let φ be the Clawson-Schmidt conjugation which exchanges $A \leftrightarrow A^*$, etc. opposite Then it follows that

$$\mathcal{K}(Q) \setminus \mathcal{L}_\infty = \{P \mid \text{midpoint of } P\varphi(P) \in \tau\}$$



(in this diagram, a pair of isogonal conjugates are marked with the same color.)

By bashing this out, we get that this is a degree-3 condition on the position of P, or in other words:

Proposition 9.2.5. $\mathcal{K}$ is a cubic.

We call $\mathcal{K}$ the **isoptic cubic** of $\mathcal{Q}$, since any point on $\mathcal{K}$ "sees" the two sides of the quadrilateral with the same angle, by Proposition 1.3.14. This definition gets us that the isoptic cubic of a complete quadrilateral goes through its six vertices, the Miquel point, the point at infinity along the Newton line, and the two circle points. Thus $\mathcal{K}$ is actually a **circular cubic**, a cubic that passes through the two circle points.

Remark. This gets us that the real asymptote of the isoptic cubic is the Newton line under a 2 homothety from the Miquel point.

Let's notate the isoptic cubic of quadrilateral $\mathcal{Q}$ as $\mathcal{K}(\mathcal{Q})$.

If $ABDE$ is a parallelogram, we get something very nice.

Proposition 9.2.6. If $(BB^*)(CC^*)$ is a parallelogram, then the isoptic cubic is just a rectangular hyperbola $\mathcal{H}$ through its vertices and the line at infinity.

Proof. It's obvious that the line at infinity is in the isoptic cubic since it's a parallelogram. So we only have to prove $P \in \mathcal{K} \backslash \mathcal{L}_\infty \iff P \in \mathcal{H} \backslash \mathcal{L}_\infty$, so we just have to prove that all points on this rectangular hyperbola satisfy

$$\measuredangle BPC + \measuredangle B^*PC^* = 0^\circ.$$

Choose Q such that $\triangle BCQ \overset{+}{\sim} \triangle C^*B^*P$, then $CQ \parallel B^*P$, and

$$\begin{aligned}\measuredangle BPC + \measuredangle B^*PC^* = 0^\circ &\iff \measuredangle BPC + \measuredangle CQB = 0^\circ \\ &\iff Q \in (PBC) \\ &\iff \measuredangle CB^*P = \measuredangle PQC = \measuredangle PBC.\end{aligned}$$

Thus, from Proposition 8.1.10 we have that P lies on a rectangular hyperbola centered at the midpoint of this parallelogram passing through B, B^*, C, C^*. □

To sum the above up, we get that if (P, P^*) are isogonal conjugates in $(BB^*)(CC^*)$, then (B, B^*) are isogonal conjugates in $(CC^*)(PP^*)$, and (C, C^*) are isogonal conjugates in $(PP^*)(BB^*)$. We call these six points $(PP^*)(BB^*)(CC^*)$ a **perfect isogonal six-point set.**

Definition 9.2.7. For a complete quadrilateral $\mathcal{Q}$, define $\mathcal{K}(Q)$ to be the locus of points with an isogonal conjugate in $\mathcal{Q}$. Define $\tau(\mathcal{Q})$ to be the Newton line of $\mathcal{Q}$, and define $\varphi_{\mathcal{Q}}$ to be the Clawson-Schmidt conjugation swapping opposite vertices of the complete quadrilateral.

The following results give us a lot of small useful lemmas about isogonal conjugates.

Proposition 9.2.8. If $(P, P^*), (Q, Q^*)$ are isogonal conjugates in complete quadrilateral $\mathcal{Q}$, then $\mathcal{K}(\mathcal{Q}) = \mathcal{K}((PP^*)(QQ^*))$.

Proof. This is easy when the Miquel point of $\mathcal{Q}$ lies on the line at infinity, so assume it's not on the line at infinity. Since $(BB^*)(CC^*)(PP^*)$ and $(BB^*)(CC^*)(QQ^*)$ are perfect six-point sets, $(P, P^*), (Q, Q^*)$ are swapped by $\varphi_{\mathcal{Q}}$. Since the midpoints of PP^*, QQ^* lie on the Newton line, we are done by Theorem 9.2.4. □

In other words, we can consider the perfect $2n$ point set $(X_1Y_1)(X_2Y_2)\dots(X_nY_n)$ such that all (X_i, Y_i) is a pair of isogonal conjugates in $(X_1Y_1)(X_2Y_2)$, then for any i, j, k, (X_k, Y_k) are isogonal conjugates in $(X_iY_i)(X_jY_j)$! Thus the roles of everything in this is symmetric and we call it a **perfect isogonal $2n$-point set.**

Corollary 9.2.9. Let $(P, P^*), (Q, Q^*)$ be a pair of isogonal conjugates in $\triangle ABC$. If (R, R^*) are isogonal conjugates in $(PP^*)(QQ^*)$, then (R, R^*) are also isogonal conjugates in $\triangle ABC$.

Proof. Since $(P, P^*), (Q, Q^*)$ are isogonal conjugates wrt. $\triangle ABC$, there exists two inconics $\mathcal{C}_P, \mathcal{C}_Q$ with foci at P, P^* and Q, Q^*. These conics have three common tangent lines as the three sides of the triangle, let ℓ be their fourth common tangent line. Then $(P, P^*), (Q, Q^*)$ are isogonal conjugates in complete quadrilateral $\triangle ABC \cap \ell$, so by the previous lemma we get that (R, R^*) are isogonal conjugates in $\triangle ABC \cap \ell$, so they are also isogonal conjugates in $\triangle ABC$. □

Remark. Taking the "fourth common tangent" is a very common trick to turn a problem about triangles to a problem about complete quadrilaterals.

Note that this gives a fast proof of Corollary 7.4.5 in the case of isogonal conjugation. By setting $(R, R^*) = (PQ^* \cap P^*Q, PQ \cap P^*Q^*)$, we get

Corollary 9.2.10. Let $(P, P^*), (Q, Q^*)$ be isogonal conjugates in $\triangle ABC$, then $(PQ^* \cap P^*Q, PQ \cap P^*Q^*)$ are also isogonal conjugates in $\triangle ABC$.

Corollary 9.2.11. Let $(P, P^*), (Q, Q^*)$ be isogonal conjugates in $\triangle ABC$. Then the Miquel point of the complete quadrilateral $\mathcal{Q} = (PP^*)(QQ^*)$ lies on (ABC), and the isogonal conjugate of the Miquel point M on $\triangle ABC$ is the point at infinity along the Newton line $\infty_{\tau(\mathcal{Q})}$.

Proof. In Corollary 9.2.9 just set (R, R^*) to be $(M, \infty_{\tau(\mathcal{Q})})$. Since $\infty_{\tau(\mathcal{Q})} \in \mathcal{L}_\infty, M \in (ABC)$. □

Remark. There's actually a really elementary proof of the fact that the Miquel point M lies on (ABC). Let X be a point such that $MAX \overset{+}{\sim} MPQ \overset{+}{\sim} MQ^*P^*$. Then $\triangle APQ^* \overset{+}{\sim} \triangle XQP^*$ tells us that

$$\measuredangle QXP^* = \measuredangle PAQ^* = \measuredangle QAP^*,$$

so A, P^*, Q, X are concyclic. As such,

$$\measuredangle AX = \measuredangle AQ + \measuredangle P^*X - \measuredangle(P^*Q).$$

In the same logic, choose Y such that $\triangle MBY \overset{+}{\sim} \triangle MPQ \overset{+}{\sim} \triangle MQ^*P^*$, we have

$$\measuredangle BY = \measuredangle BQ + \measuredangle P^*Y - \measuredangle(P^*Q).$$

Then, since $\triangle ABQ^* \overset{+}{\sim} \triangle XYP^*$,

$$\measuredangle BY - \measuredangle AX = \measuredangle P^*Y - \measuredangle P^*X + \measuredangle AQB = \measuredangle AQ^*B + \measuredangle AQB = \measuredangle ACB,$$

so $AX \cap BY \in (ABC)$, so since $\triangle MAX \overset{+}{\sim} \triangle MBY$ this implies $M \in (ABC)$.

Corollary 9.2.12. Let (P, P^*) be isogonal conjugates in $\triangle ABC$, and let I either be the incenter or one of the excenters. Then there exists a point $M \in (ABC)$ such that $\triangle MPI \overset{+}{\sim} \triangle MIP^*$. Also, if we let X be the midpoint of PP^*, then the isogonal conjugate of M is ∞_{IX}.

Proof. Let M be the Miquel point of $(PP^*)(II)$, then $\triangle MPI \overset{+}{\sim} \triangle MIP^*$. Since the Newton line is IX, the isogonal conjugate of M is ∞_{IX}. □

Do you still remember the cyclic perfect eight-point set Proposition 9.1.8? Let's see if we can define nice isogonal conjugates on it.

Proposition 9.2.13. Let (P, P^*) be isogonal conjugates in $(BB^*)(CC^*)$, and let

$$Q = (PBC) \cap (PB^*C^*) \cap (P^*BC^*) \cap (P^*B^*C), Q = (P^*B^*C^*) \cap (P^*BC) \cap (PB^*C) \cap (PBC^*),$$

then (Q, Q^*) are isogonal conjugates in $(BB^*)(CC^*)$.

Proof. Note that $(BB^*)(CC^*)(PP^*)(QQ^*)$ is a perfect eight-point set. Since the midpoints of segments BB^*, CC^*, PP^*, QQ^* are cyclic, but since the midpoints of BB^*, CC^*, PP^* are collinear on the Newton line, so the midpoint of QQ^* also lies on the Newton line as desired. □

Example 9.2.14 (IMO 2018/6)**.** Let $ABCD$ be a convex quadrilateral such that $AB \cdot CD = BC \cdot DA$, and let X be inside $ABCD$ such that

$$\angle XAB = \angle XCD, \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180°$.

Proof. This problem is equivalent to proving that X lies on the isoptic cubic of $(AC)(BD)$. Let P, Q be points such that

$$P \in (AXB) \cap (CXD), Q \in (BXC) \cap (DXA),$$

and choose Y such that $(AC)(BD)(XY)(PQ)$ is a cyclic perfect eight-point set. So now we just need to prove that (P, Q) are isogonal conjugates in $(AC)(BD)$. From

$$\measuredangle XPB = \measuredangle XAB = \measuredangle XCD = \measuredangle XPD,$$

we know that B, P, D are collinear, and similarly A, Q, C are collinear. If we can now choose two points P', Q' on BD, AC such that P', Q' are isogonal conjugates in $(AC)(BD)$, then we have that

$$X' = (AP'B) \cap (BQ'C) \cap (CP'D) \cap (DQ'A)$$

which finishes.

Let's now find (P', Q'). In fact, we can prove that AC's isogonal line ℓ_A wrt. $\angle BAD$, $AC's$ isogonal line ℓ_C wrt. $\angle DCB$, and line BC concur: let P'_A, P'_C, Z respectively be the intersection points of ℓ_A, ℓ_C, AC

with BD. Then

$$\frac{BP'_A}{P'_AD} = \frac{AB}{DA} \cdot \frac{\sin \measuredangle BAP'_A}{\sin \measuredangle P'_AAD} = \frac{AB}{DA} \cdot \frac{\sin \measuredangle ZAD}{\sin \measuredangle BAZ} = \frac{AB^2}{DA^2} \cdot \frac{ZD}{BZ}$$
$$\frac{BP'_C}{P'_CD} = \frac{CB}{DC} \cdot \frac{\sin \measuredangle BCP'_C}{\sin \measuredangle P'_CCD} = \frac{CB}{DC} \cdot \frac{\sin \measuredangle ZCD}{\sin \measuredangle BCZ} = \frac{CB^2}{DZ^2} \cdot \frac{ZD}{BZ}$$

Then, $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA}$ implies that the above two expressions are equal, and therefore $P'_A = P'_C = P'$. Define Q' similarly on AC. By isogonal lines, we see that (P', Q') are a pair of isogonal conjugates in $(AC)(BD)$, which □

From Proposition 9.2.13, we have

Proposition 9.2.15. Let (P, P^*) be isogonal conjugates in complete quadrilateral $(BB^*)(CC^*)$. Let O, O_P, O_B, O_C respectively be the circumcenters of

$$\triangle PBC, \triangle PB^*C^*, \triangle P^*BC^*, \triangle P^*B^*C$$

and let O^*, O_P^*, O_B^*, O_C^* respectively be the circumcenters of

$$\triangle P^*B^*C^*, \triangle P^*BC, \triangle PB^*C, \triangle PBC^*$$

then $(OO^*)(O_PO_P^*)(O_BO_B^*)(O_CO_C^*)$ is a cyclic perfect eight-point set, with common Miquel point with $(BB^*)(CC^*)$, and the Newton lines of the eight-point set and the quadrilateral are perpendicular.

Proof. Let (PBC), (PB^*C^*), (P^*BC^*), and (P^*B^*C) concur at Q, and let $(P^*B^*C^*)$, (P^*BC), (PB^*C), and (PBC^*) concur at Q^*. Then by Proposition 9.2.13 (Q, Q^*) are isogonal conjugates in $(BB^*)(CC^*)$. First note that

$$\measuredangle QBO = 90^\circ - \measuredangle BPQ = 90^\circ - \measuredangle Q^*PB^* = \measuredangle B^*Q^*O_B^*,$$

so $\triangle OBQ \overset{+}{\sim} \triangle O_B^*Q^*B^*$. From $\triangle MBQ \overset{+}{\sim} \triangle MQ^*B^*$ we know that

$$\triangle MOO_B^* \overset{+}{\sim} \triangle MBQ^* \overset{+}{\sim} \triangle MQB^*.$$

Similarly, we have

$$\triangle MO^*O_B \overset{+}{\sim} \triangle MB^*Q \overset{+}{\sim} \triangle MQ^*B,$$

so the Miquel point of $(OO^*)(O_BO_B^*)$ is M. Do this symmetrically and we get that $(OO^*)(O_PO_P^*)(O_BO_B^*)$ $(O_CO_C^*)$ is a perfect eight-point set, with Miquel point M.

$$\measuredangle O_BO + \measuredangle O_BO^* = \perp BQ + \perp C^*P^* = \perp BP^* + \perp C^*Q = \measuredangle O_BO_P^* + \measuredangle O_BO_P,$$

we have that O_B has an isogonal conjugate wrt. $(OO^*)(O_PO_P^*)$, and since $(OO^*)(O_PO_P^*)(O_BO_B^*)$ is a perfect six-point set, its isogonal conjugate is forced to be O_B^*. Doing the same for O_P, O_C gives $(OO^*)(O_PO_P^*)(O_BO_B^*)(O_CO_C^*)$ is a perfect isogonal eight-point set. By some more angle-chasing, since

$$\measuredangle O_BO + \measuredangle O_CO_P = \perp BQ + \perp B^*Q = \perp CQ + \perp C^*Q = \measuredangle O_CO + \measuredangle O_BO_P$$

we get that $OO_PO_BO_C$ is concyclic. By symmetry we get that $(OO^*)(O_PO_P^*)(O_BO_B^*)(O_CO_C^*)$ is a cyclic perfect isogonal eight-point set.

Finally, to prove the Newton lines τ, τ_O are perpendicular, from $\triangle MOO_B^* \overset{+}{\sim} \triangle MBQ^* \overset{+}{\sim} \triangle MQB^*$ we have

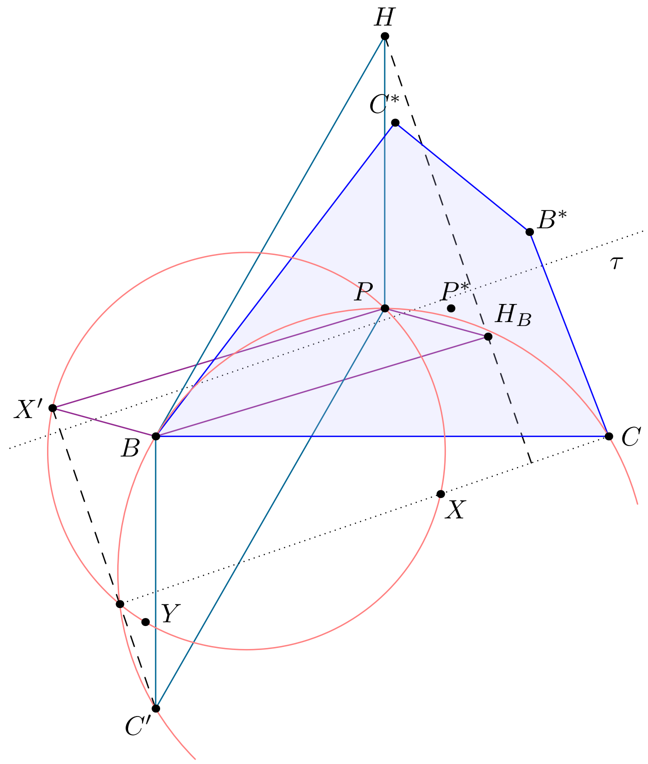
$$\tau = \measuredangle BQ + \measuredangle BQ^* - \measuredangle BM = \perp OO_B + OO_B^* - \measuredangle OM = \tau_O + 90^\circ,$$

so $\tau \perp \tau_O$. $\square$

The above proof implies that if we take this operation twice on a cyclic eight point set, it becomes equivalent to taking a homothety of each point through M.

The following section isn't super related to perfect six-point sets, but it's still interesting.

Proposition 9.2.16 (Generalized Steiner line)**.** Let (P, P^*) be isogonal conjugates in $(BB^*)(CC^*)$, then the orthocenters H, H_P, H_B, H_C of $\triangle PBC, \triangle PB^*C^*, \triangle P^*BC^*, \triangle P^*B^*C$ are collinear and are perpendicular to the Newton line τ.



Proof. By symmetry, we only need to prove that HH_B is perpendicular to τ. We consider the transformation $C^* \to P$, and let this send $P^* \to X, B \to Y$. Let C', X' respectively be the antipodes of C, X in $(PBC), (XYP)$. Then note that both of $(BP)(C'H), (BP)(X'H_B)$ are parallelograms, so $C'X' \parallel HH_B$. From

$$\measuredangle CC'P = \measuredangle CBP = \measuredangle P^*BC^* = \measuredangle XYP = \measuredangle XX'P$$

and $\measuredangle C'PC = \measuredangle X'PX = 90^\circ$ we can get $\triangle C'CP \overset{+}{\sim} \triangle X'XP$, or in other words $\triangle CXP \overset{+}{\sim} \triangle C'X'P$, so $CX \perp C'X'$. Since the midpoints of CC^*, XC^* both lie on τ, we have $HH_B \parallel C'X' \perp \tau$. $\square$

At this point we can consider the four other triangles $\triangle P^*B^*C^*, \triangle P^*BC, \triangle PB^*C, \triangle PBC^*$, to get four more points H^*, H_P^*, H_B^*, H_C^* are collinear. You might wonder when these eight orthocenters are all collinear.

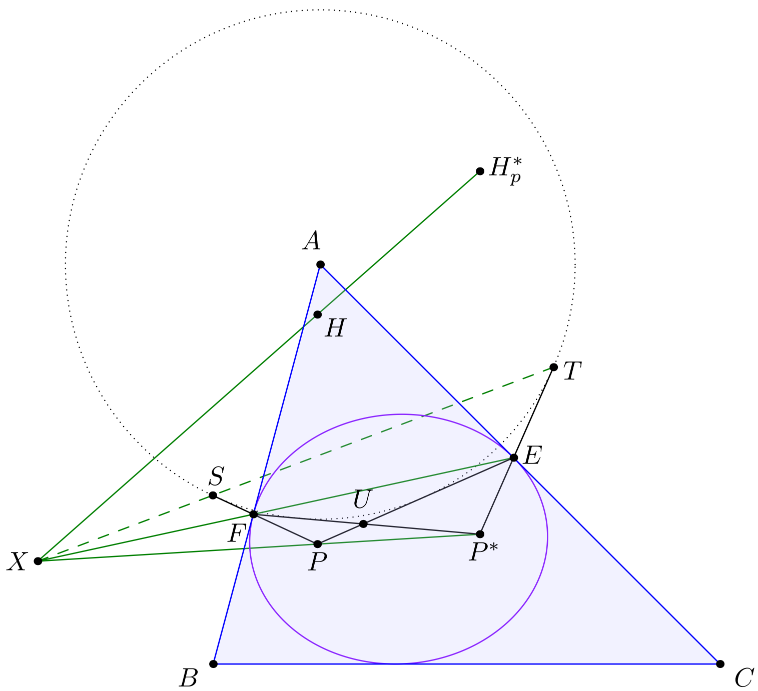
Proposition 9.2.17. Let (P, P^*) be isogonal conjugates in $(BB^*)(CC^*)$, then the orthocenters of

$$\triangle PBC, \triangle PB^*C^*, \triangle P^*BC^*, \triangle P^*BC^*, \triangle P^*B^*C, \triangle P^*B^*C^*, \triangle P^*BC, \triangle PB^*C, \triangle PBC^*$$

are all collinear on line ℓ if and only if PP^*, BB^*, CC^* are concurrent at a point X. Further, $X \in \ell$.

We first prove this lemma:

Lemma 9.2.18. Let P, P^* be isogonal conjugates in $\triangle ABC$, and let H and H_P^* be the orthocenters of $\triangle PBC$ and $\triangle P^*BC$. Let the inellipse with foci at P, P^* touch CA, AB at E, F. Then PP^*, EF, HH_P^* are collinear.



Proof. Let X be the intersection point of PP^* and HH_P^*. Then from $PH \parallel P^*H_P^*$ and P, P^* isogonal conjugates wrt. $(CF)(AB)$, we have

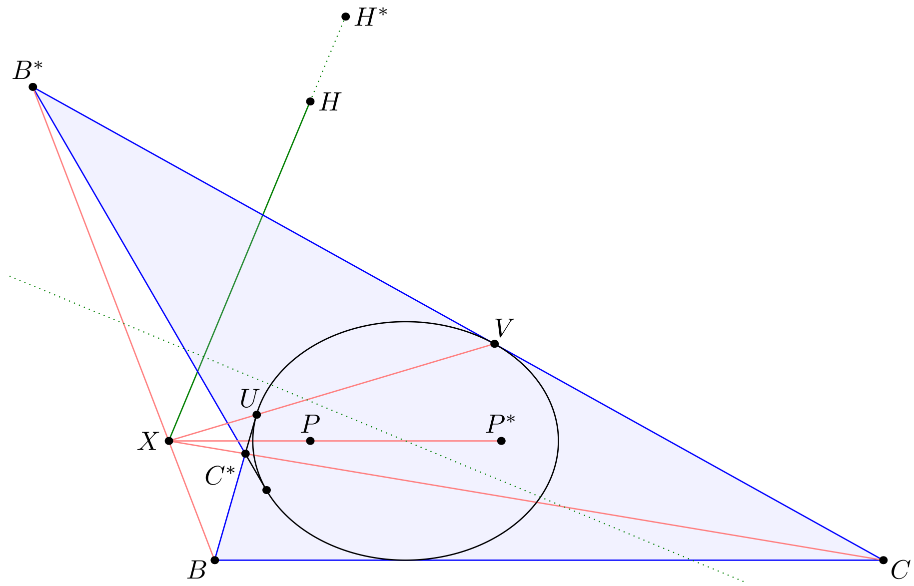
$$\frac{PX}{P^*X} = \frac{PH}{P^*H_P^*} = \frac{\overline{BC}\cot\measuredangle BPC}{\overline{BC}\cot\measuredangle BP^*C} = \frac{\cot\measuredangle APF}{\cot\measuredangle AP^*F}.$$

Note that A is the center of an excircle of to $(EF)(PP^*)$. Let S, T be the tangency points on PE, P^*F, and let U be the intersection point of PE, P^*F. By Newton III Theorem 6.3.14, PP^*, EF, ST are concurrent. Finally, from

$$\frac{PX}{XP^*}\cdot\frac{P^*T}{TU}\cdot\frac{US}{SP} = -\frac{\cot\measuredangle APF}{\cot\measuredangle AP^*F}\cdot\frac{\tan\measuredangle P^*AT}{\tan\measuredangle TAU}\cdot\frac{\tan\measuredangle UAS}{\tan\measuredangle SAP} = -1$$

so by Menelaus we have that X, T, S are collinear. $\square$

Back to the original:



Proof of 9.2.17. From Proposition 9.2.16, the eight orthocenters are collinear if and only if the orthocenter H of $\triangle PBC$ and the orthocenter H_P^* of $\triangle P^*BC$ lie on a line ℓ perpendicular to the Newton line τ. Thus, when we fix B, C, P, P^* (then define B^*, C^* as the envelope of conics with foci P, P^* that are tangent to BC), B^*, C^* under $HH_P^* \perp \tau$ is unique (unless the other tangents at B, C form a parallelogram, however then it's trivial). So we just need to prove ($\Leftarrow$), and

♠: Given fixed B, C, P, P^*, there exists B^*, C^* such that $(PP^*)(BB^*)(CC^*)$ form a perfect isogonal six-point set and PP^*, BB^*, CC^* are concurrent.

Let's just construct ♠. Let $\mathcal{C}$ be a conic with foci P, P^* and tangent to BC, let T_B, T_C be the tangents to $\mathcal{C}$ from B, C that aren't BC, and let these touch C at U, V respectively. Let $X = PP^* \cap UV, B^* = BX \cap T_c, C^* = CX \cap T_B$. By Newton III Theorem 6.3.14, B^*C^* is tangent to $\mathcal{C}$, so $(PP^*)(BB^*)(CC^*)$ is a perfect isogonal six-point set and PP^*, BB^*, CC^* concur at X.

Now we prove ($\Leftarrow$). We prove that H, H_P^*, X are collinear, and the rest follows similarly. By Newton III Theorem 6.3.14, if U, V respectively are the tangency points of BC^*, CB^* with $\mathcal{C}$, then PP^*, BB^*, CC^*, UV concur at X. Then by Lemma 9.2.18, PP^*, UV, HH_P^* are concurrent. $\square$

9.2.1 Taming the Isoptic Dragon

Let's return to looking at the isoptic cubic $\mathcal{K}$. Following precedent set in 6.A.1, let's consider for a second what happens when we replace the circle points I, J with two random points $\mathbf{W} = \{W_+, W_-\}$, then we can define the $\mathbf{W}$-isoconjugate cubic as:

$$\mathcal{K}^{\mathbf{W}} = (\mathcal{Q}) = \mathbf{W}^{W_+ \times W_-}(\mathcal{Q}) = \{P \mid \exists P^* \text{ such that on all } \triangle \subseteq \mathcal{Q}, P \times P^* = W_+ \times W_-\}.$$

We can convert all of the above results to **W**-versions. We can define a **W**-Miquel point, M, as the common intersection point of the four conics through some 3 points of the quadrilateral, W_+, and W_-. Similarly we can define a **W**-Newton line τ as the line through the harmonic conjugates of the intersections of each diagonal $\mathcal{L} = W^+W^-$ wrt to the diagonal (this is also the QL-Tf2 conjugate).

Finally, we can define **W**-Clawson-Schmidt conjugation as the point isoconjugation on $\triangle MW_+W_-$ that switches the vertices of $\mathcal{Q}$.

Dually, we define a **W**-line conjugate cubic on complete quadrangle q as

$$k^{\mathbf{W}}(q) = k^{w_+ \times w_-}(q) = \{\ell \mid \exists \ell^* \text{ such that on all } \triangle \subseteq q, \ell \times \ell^* = w_+ \times w_-\}.$$

Since we know any two pairs of isogonal conjugates to determine an isoptic cubic $\mathcal{K}$, let's just throw away the whole quadrilateral $\mathcal{Q}$ and see what we can do. Note that we can still find a Newton line and Miquel point, a Clawson-Schmidt conjugation, more pairs of isogonal conjugates, etc.

Definition 9.2.19. A cubic curve $\mathcal{K}$ is an isoptic cubic if there exists a complete quadrilateral $\mathcal{Q}$ such that $\mathcal{K} = \mathcal{K}(\mathcal{Q})$.

The first key topic we will address is inversion on the isoptic cubic:

Proposition 9.2.20. Let A be a point on the isoptic cubic $\mathcal{K}$. Then the inversion $\mathcal{K}^{\mathfrak{J}}$ of $\mathcal{K}$ across A is still some isoptic cubic. Furthermore, a pair $(P, P^{\mathfrak{J}})$ of isogonal conjugates on $\mathcal{K}$ remains a pair after being inverted.

Proof. Let (B, B^*) be a pair of isogonal conjugates on $\mathcal{K}$, let $C = AB^* \cap A^*B$, and let $C^* = AB \cap A^*B^*$. Then by Isoconjugations Give More Isoconjugations we have that (C, C^*) are isogonal conjugates. Further, we have

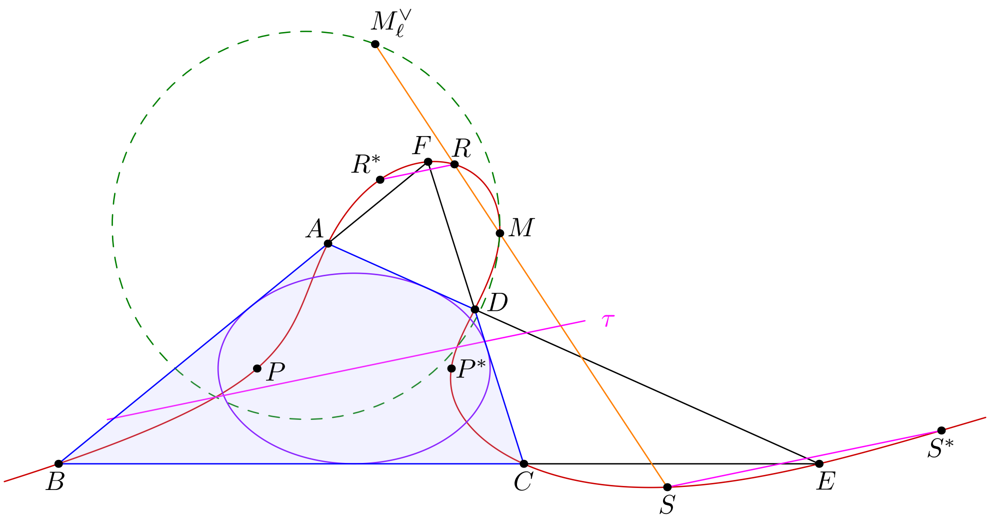
$$\begin{aligned}
P \in \mathcal{K} &\iff \measuredangle PB + \measuredangle PB^* = \measuredangle PC + \measuredangle PC^* \\
&\iff (\measuredangle AP + \measuredangle AB - \measuredangle P^{\mathfrak{J}}B^{\mathfrak{J}}) + (\measuredangle AP + \measuredangle AB^* - \measuredangle P^{\mathfrak{J}}(B^*)^{\mathfrak{J}}) \\
&= (\measuredangle AP + \measuredangle AC - P^{\mathfrak{J}}C^{\mathfrak{J}}) + (\measuredangle AP + \measuredangle AC^* - \measuredangle P^{\mathfrak{J}}(C^*)^{\mathfrak{J}} \\
&\iff \measuredangle P^{\mathfrak{J}}B^{\mathfrak{J}} + \measuredangle P^{\mathfrak{J}}(B^*)^{\mathfrak{J}} = \measuredangle P^{\mathfrak{J}}C^{\mathfrak{J}} + \measuredangle P^{\mathfrak{J}}(C^*)^{\mathfrak{J}} \\
&\iff \measuredangle P^{\mathfrak{J}} \in \mathcal{K}((B^{\mathfrak{J}}(B^*)^{\mathfrak{J}})(C^{\mathfrak{J}}(C^*)^{\mathfrak{J}})).
\end{aligned}$$

Thus $\mathcal{K}^{\mathfrak{J}}$ is also the isoptic cubic of $(B^{\mathfrak{J}}(B^*)^{\mathfrak{J}})(C^{\mathfrak{J}}(C^*)^{\mathfrak{J}})$.

Another note: If P^* is the isogonal conjugate of P on the isoptic cubic, then $(PP^*)(BB^*)(CC^*)$ being a perfect six-point set tells us that $(P^{\mathfrak{J}}(P^*)^{\mathfrak{J}})(B^{\mathfrak{J}}(B^*)^{\mathfrak{J}})(C^{\mathfrak{J}}(C^*)^{\mathfrak{J}})$ is also a perfect six-point set. $\square$

In reality we can actually also define polarity on a cubic! There are two ways to define polarity on a cubic, which we will explore in chapter Chapter 11, one involving a line and one involving polar conics. One day, I (Li4) was bored and randomly decided to take the polar conic of the Miquel point wrt. the isoptic cubic, and found the following:

Proposition 9.2.21. Let M be the Miquel point of the isoptic cubic $\mathcal{K}$. Let line ℓ be a moving line through M that intersects the cubic at R, S. Let $M_\ell^\vee$ be the harmonic conjugate of M in segment R, S. Then the locus of $M_\ell^\vee$ is a circle.



Proof. Let R^*, S^* respectively be the isogonal conjugates of R, S in $\mathcal{K}$. Then $\varphi_{\mathcal{K}}(M_\ell^\vee)$ is the midpoint of $\overline{R^*S^*}$. Since RS and R^*S^* intersect at M, it follows that $RS^* \cap R^*S$ is $\infty_{\tau(\mathcal{K})}$, so $RS^* \parallel R^*S \parallel \tau = \tau(\mathcal{K})$. However the midpoints of RR^* and SS^* both lie on τ, so the midpoint of R^*S^* also lies on τ, so the locus of $M^\vee$ is $\varphi_{\mathcal{K}}(\tau)$. $\square$

Similarly, the polar conic of ∞_τ is a rectangular hyperbola. With the above, we can show the following:

Theorem 9.2.22 (Telv Cohl)**.** Let M be the Miquel point of complete quadrilateral $\mathcal{K}$, let a line through M intersect the cubic again at points R, S. Then the circles with diameter RS are coaxal.

Proof. Keep using the above notation.

We proved the midpoint of RS will always lie on τ, so all we need to prove is that there is a fixed point with constant power wrt. all of these circles.

If the Miquel point lies on the Newton line, then all of these circles are concentric at M and have common radical axis as the line at infinity.

Else, we can actually just explicitly construct this point. Let O be the center of the Clawson-Schmidt conjugate of the Newton line $\varphi(\tau)$. Since M and $M^\vee$ both lie on $\varphi(\tau)$, (RS) is orthogonal to it. (Alternatively, just note that the C-S conjugate of (RS) is a circle with the center on the Newton line). So this implies the power of O wrt. (RS) is just the radius of $\varphi(\tau)$, which is fixed. $\square$

As such, the common radical axis is just the perpendicular from O to the Newton line. We call this axis of $\mathcal{K}$ the **Telvcohl-axis** (Li4: I randomly made this up) or the **TC-axis** of $\mathcal{K}$.

If all of these circles all intersect on two common points on the TC-axis (of course, for some $\mathcal{K}$ these points are not real, but in general the two points are real for all $\mathcal{K}$ made out of two real parts), then these two points have a property that you can probably guess:

Proposition 9.2.23. Continuing the notation of Theorem 9.2.22, let U, V be the common points of all (RS). Then U, V are a pair of isogonal conjugates in $\mathcal{Q}$.

Proof. Let R^*, S^* respectively be the isogonal conjugates of R, S in $\mathcal{K}$. Then

$$\measuredangle RUS + \measuredangle R^*US^* = 90^\circ + 90^\circ = 0^\circ,$$

so $U \in \mathcal{K}$, and similarly $V \in \mathcal{K}$. From

$$\measuredangle RUS + \measuredangle RVS = 90^\circ + 90^\circ = 0^\circ = \measuredangle R\infty_\tau S,$$
$$\measuredangle R^*US^* + R^*VS^* = 90^\circ + 90^\circ = 0^\circ = \measuredangle R^*\infty_\tau S^*$$

we get by C-S conjugation that (U, V) are isogonal conjugates in $\mathcal{Q}$. $\square$

Proposition 9.2.24. Let $\mathcal{M}, \mathcal{I}$ respectively be the Miquel point and TC-axis, then M's foot V onto $\mathcal{I}$ lies on the isoptic cubic $\mathcal{K}$.

Proof. Let $M\infty_{\tau(\mathcal{K})}$ intersect $\mathcal{K}$ again at V', then we know that the circle with diameter $\overline{V'\infty_{\tau(\mathcal{K})}}$ is orthogonal to all the other circles with diameter (RS), however the circle $(V'\infty_{\tau(\mathcal{K})})$ is just a line through V' perpendicular to τ, and since $\mathcal{I} \perp \tau$, $V = V' \in \mathcal{K}$. $\square$

We call the point V the **vanishing point** and will revisit it in Chapter 11. Currently we'll just mention one property of it.

Proposition 9.2.25. Let V be the vanishing point of isoptic cubic $\mathcal{K}$, let Y be the foot from V to the Newton line τ of $\mathcal{K}$, then for any $P, Q \in \mathcal{K}$, (P, Q) is a pair of isogonal conjugates if and only if the orthocenters of $\triangle PVY, \triangle QVY$ are reflections across τ.

Practice Problems

Problem 1. Let I be the incenter of triangle $\triangle ABC$, let H_A be the foot from A to BC, let A^* be the antipode of A in (ABC). Prove that $\measuredangle BIH_a = \measuredangle A^*IC$.

Problem 2 (Taiwan 2018 3J M3)**.** Let I be the incenter of triangle ABC, and ℓ be the perpendicular bisector of AI. Suppose that P is on the circumcircle of triangle ABC, and line AP and ℓ intersect at point Q. Point R is on ℓ such that $\angle IPR = 90°$.Suppose that line IQ and the midsegment of ABC that is parallel to BC intersect at M. Show that $\angle AMR = 90°$.

Problem 3 (Taiwan 2019 2J M6)**.** Let ABC be a triangle with incenter I and A-excenter J. Let $\overline{AA'}$ be the diameter of the circumcircle of $\triangle ABC$, and let H_1 and H_2 be the orthocenters of $\triangle BIA'$ and $\triangle CJA'$, respectively.

Prove that the line H_1H_2 is parallel to the line BC.

Problem 4 (Tau)**.** Let I, O, N be the incenter, circumcenter, and nine-point center. Prove that if $IN \parallel BC$, then $\measuredangle AIO = 135°$.

Problem 5 (Taiwan 2023 IJ I2-G)**.** In isosceles trapezoid $ABCD$ with AD parallel to BC, let Ω be its circumcircle. Let X be the reflection of D across BC, let Q be a point on arc BC of Ω, let P be the intersection of DQ and BC. Point E satisfies EQ parallel to PX and EQ bisects angle $\angle BEC$. Prove EQ also bisects $\angle AEP$.

Problem 6. Let I be the incenter of $\triangle ABC$, let ω be its incircle. Point P lies on ω such that its isogonal conjugate Q lies outside of ω. Let $\triangle QEF$ be the triangle formed by Q's two tangents to ω and P's one tangent to ω. Prove that $\triangle QEF$'s Q-mixtilinear touchpoint S lies on (ABC).

9.3 Oblique Strophoids

When a complete quadrilateral has an incircle (or an excircle), we typically have some nice properties. Perfect six-point sets are no exception; let's see what cool things we get. First, since (I, I) are a pair of isogonal conjugates in $\mathcal{Q}$, we have

Proposition 9.3.1. I is a fixed point of Clawson-Schmidt conjugation.

In other words, quadrilateral Q's Newton line has fixed point I. Through the theory of perfect six-point sets, we can easily prove Proposition 4.3.5 which we previously spent too much effort on.

Proposition 9.3.2. Let M be the Miquel point of tangential quadrilateral $(BB^*)(CC^*)$, let J be the reflection of the incenter I across M, then $(IJ)(BB^*), (IJ)(CC^*)$ are all harmonic quadrilaterals.

Proof. I, J are both fixed points of Clawson-Schmidt conjugation, so from Proposition 9.1.4 we know that $(IJ)(BB^*), (IJ)(CC^*)$ are all harmonic quadrilaterals. □

Proposition 9.3.3. The isoptic cubic $\mathcal{K}$ of a tangential quadrilateral is special: it's a nodal cubic (self-intersecting). Furthermore, it's also the image of a rectangular hyperbola that was inverted around the incircle ω. We call $\mathcal{K}$ a **oblique strophoid**, and we call I the **node** of this strophoid.

Proof. Let $(BB^*)(CC^*)$ be our tangential quadrilateral. Let $\mathfrak{J}$ represent inversion around ω. Then $(B^{\mathfrak{J}}(B^*)^{\mathfrak{J}})$ $(C^{\mathfrak{J}}(C^*)^{\mathfrak{J}})$ is a parallelogram, so I lies in $\mathcal{K}$ and the isoptic cubic is the union of a rectangular hyperbola and the line at infinity, by Proposition 9.2.20 and Proposition 9.2.6. Thus

$$\mathcal{K} = (\mathcal{H} \cap \mathcal{L}_\infty)^{\mathfrak{J}} = \mathcal{H}^{\mathfrak{J}} \cap I = \mathcal{H}.$$

as desired. □

Remark. Sometimes we call the Miquel point of the tangential quadrilateral the **focus** of the isoptic cubic/strophoid. This is because of an alternative definition of a strophoid: take M varying on a fixed line through O, and let the circle with center M through O intersect at A and B. Then the locus of A, B is the strophoid. (This condition is interpreted much more naturally with what is called a "group law").

Corollary 9.3.4. Keeping the previous notation, for a pair of isogonal conjugates (P, P^*) in $\mathcal{K}$, when inverted about I, they will be antipodes on the rectangular hyperbola.

Proof. Let M be the Miquel point of $\mathcal{K}$, and fix two pairs of isogonal conjugates $(B, B^*), (C, C^*)$, let J be the reflection of M across I. Then from Proposition 9.3.2, we get that $(IJ)(PP^*), (IJ)(BB^*), (IJ)(CC^*)$ are harmonic quadrilaterals, so since inversion preserves cross ratios, J^* is the common midpoint of segments $\overline{P^{\mathfrak{J}}(P^*)^{\mathfrak{J}}}, \overline{B^{\mathfrak{J}}(B^*)^{\mathfrak{J}}}, \overline{C^{\mathfrak{J}}(C^*)^{\mathfrak{J}}}$. This is just the center of the rectangular hyperbola, so $P^{\mathfrak{J}}, (P^*)^{\mathfrak{J}}$ are antipodes. □

Corollary 9.3.5. Let (P, P^*) be a pair of isogonal conjugates in $\triangle ABC$, let ω be the incircle of $\triangle ABC$, and let $\mathfrak{J}$ represent inversion about ω. Then $A^{\mathfrak{J}}, B^{\mathfrak{J}}, C^{\mathfrak{J}}, P^{\mathfrak{J}}, (P^*)^{\mathfrak{J}}, I$ lie on a common rectangular hyperbola.

Proof. We prove that A, B, C, P, P^* lie on a common strophoid. Let $\mathcal{C}$ be a conic with P, P^* as foci that's tangent to $\triangle ABC$, let ℓ be the fourth common tangent of $\mathcal{C}$ and ω. Then $(P, P^*), (I, I)$ are a pair of isogonal conjugates in $\mathcal{Q} = \triangle ABC \cup \ell$, so the isoptic cubic of this quadrilateral $\mathcal{K}(\mathcal{Q})$ is a strophoid through A, B, C, P, P^*, I with I as its node. Therefore by inverting across ω we have that $A^{\mathfrak{J}}, B^{\mathfrak{J}}, C^{\mathfrak{J}}, P^{\mathfrak{J}}, (P^*)^{\mathfrak{J}}, I$ lie on a common rectangular hyperbola. □

By properties of rectangular hyperbolas, we can get the following result:

Corollary 9.3.6. Let M be the focus of strophoid $\mathcal{K}$, and suppose a line ℓ through M intersects the strophoid again at points R, S. Then $\measuredangle RIS = 90°$.

Proof. Obviously, the points $R^{\mathfrak{I}}, S^{\mathfrak{I}}$ on $\mathcal{H}$ satisfy $I, M^{\mathfrak{I}}, R^{\mathfrak{I}}, S^{\mathfrak{I}}$ concyclic. But since $I, M^{\mathfrak{I}}$ are antipodes on $\mathcal{H}$, by Proposition 8.1.10 we have that $\overline{R^{\mathfrak{I}}S^{\mathfrak{I}}}$ is the diameter of circle $(IM^{\mathfrak{I}}R^{\mathfrak{I}}S^{\mathfrak{I}})$, so $\measuredangle RIS = 90°$. □

Corollary 9.3.7. If a pair of isogonal conjugates (P, P^*) has line PP^* intersect $\mathcal{K}$ again at V_P, then $IV_P \perp PP^*$.

Proof. Obviously the three points $P^{\mathfrak{I}}, (P^*)^{\mathfrak{I}}, V_P^{\mathfrak{I}}$ on $\mathcal{H}$ satisfy that $IP^{\mathfrak{I}}, (P^*)^{\mathfrak{I}}V_P^{\mathfrak{I}})$ is cyclic on some circle Ω, and $P^{\mathfrak{I}}, (P^*)^{\mathfrak{I}}$ are a pair of antipodes on $\mathcal{H}$, so $IV_P^{\mathfrak{I}}$ is a diameter of circle Ω, or $IV_P \perp PP^*$. □

Remember the TC-axis? Theorem 9.2.22, this is just really setting (U, V) to (I, I). So basically, in the strophoid case:

Proposition 9.3.8. Let M, τ respectively be the Miquel point and Newton line of strophoid $\mathcal{K}$. Let line ℓ through M intersect $\mathcal{K}$ again at points R, S, then the circle with diameters RS are all coaxal on the perpendicular line from I to τ, which is also the tangent from I to circle (RS).

Proof. From Theorem 9.2.22 all that we really need to prove is that this perpendicular is tangent to (RS). Since by the earlier proof, the midpoint N of RS lies on the Newton line τ, we have $NI \parallel \tau$, so it has to be tangent. □

Remark. This shows in fact that the strophoid K in fact is generated by the earlier definition with node I, fixed line τ, and focus M.

If V is the vanishing point of strophoid $\mathcal{K}$, then it's just the foot from I to $M\infty_\tau$.

Like how the isoptic cubic when inverted about a point on it remains isoptic, we expect the same for be true for strophoids:

Proposition 9.3.9. Let A be a point on strophoid $\mathcal{K}$ that isn't the node I, then the inverse of $\mathcal{K}$ across A is still some strophoid, possibly rotated.

Proof. Let $K^{\mathfrak{I}}, I^{\mathfrak{I}}$ be the inverses of $\mathfrak{K}, \mathfrak{I}$ under the inversion at A. By Proposition 3.1.16, the inversion of $K^{\mathfrak{I}}$ about $I^{\mathfrak{I}}$ is equal to the inversion of K around I plus some reflection, or a rectangular hyperbola. Uninverting gives the result. □

Since (I, I) are always a pair of isogonal conjugates for all 4 component triangles of a complete tangential quadrilateral, (using the notation in subsection Proposition 9.3.3), $\mathcal{K}(\mathcal{Q})$ is just $\mathcal{K}^{I^2}(\mathcal{Q})$. Let's generalize this to arbitrary circle points as well.

Theorem 9.3.10. Complete quadrilateral $\mathcal{Q}$'s W^2-isoconjugate cubic $\mathcal{K}^{W^2}(\mathcal{Q})$ is the image of (BB^*CC^*W) under an isoconjugation on $\triangle AA^*W$ that swaps $B \times B^* = C \times C^*$, where $A = BC^* \cap B^*C, A* = BC \cap B^*C^*$.

Proof. Let $Q = B \times B^* \div P = C \times C^* \div P$. Then by the conics from Proposition 7.4.6, we have

$$B(W, Q; C, C^*) = B(W, BQ \cap B^*P; A^*, A) = P(W, B^*; A^*, A),$$
$$B^*(W, Q; C, C^*) = B^*(W, B^*Q \cap BP; A, A^*) = P(W, B, A, A^*$$

Thus Q lies on conic (BB^*CC^*W) if and only if $P(W, W), P(A, A^*), P(B, B^*)$ define a pencil involution on $\mathbf{T}P$, so by Proposition 1.3.14 this is equivalent to $P \in \mathcal{K}^{W^2}(\mathcal{Q})$. $\square$

Practice Problems

Problem 1. Let acute triangle $\triangle ABC$ have circumcenter O, and incenter I. Let D, E, F respectively be the midpoints of arcs BC, CA, AB, let S be the intersection of BO and FD, let T be the intersection of CO and DE, let H be the orthocenter of OEF. Prove that OI, HD, ST are concurrent.

Problem 2. Let strophoid $\mathcal{K}$ have node I, prove that the envelope of the perpendicular to the moving line PI at I for P on the strophoid is a conic.

Problem 3. Let tangential quadrilateral $ABCD$ have incircle ω, let AB and CD intersect at E, let AD and BC intersect at F, let AC and BD intersect at T, let (ATP) and (CTD) intersect again at P, let (ATD) and (BTC) intersect again at Q. Prove that if E, F, P, Q are concyclic, then I also lies on this circle.

9.4 Duality - The Isohaptic Locus of Isotomic Conjugation

We know that the "dual" of isogonal conjugate points are isotomic transversal lines (see Section 1.5), so let's see if we can also define the dual of the isoptic cubic in a similar manner.

Given complete quadrangle $q = (A, B, C, D)$, we define the **isohaptic curve** of q as the envelope of

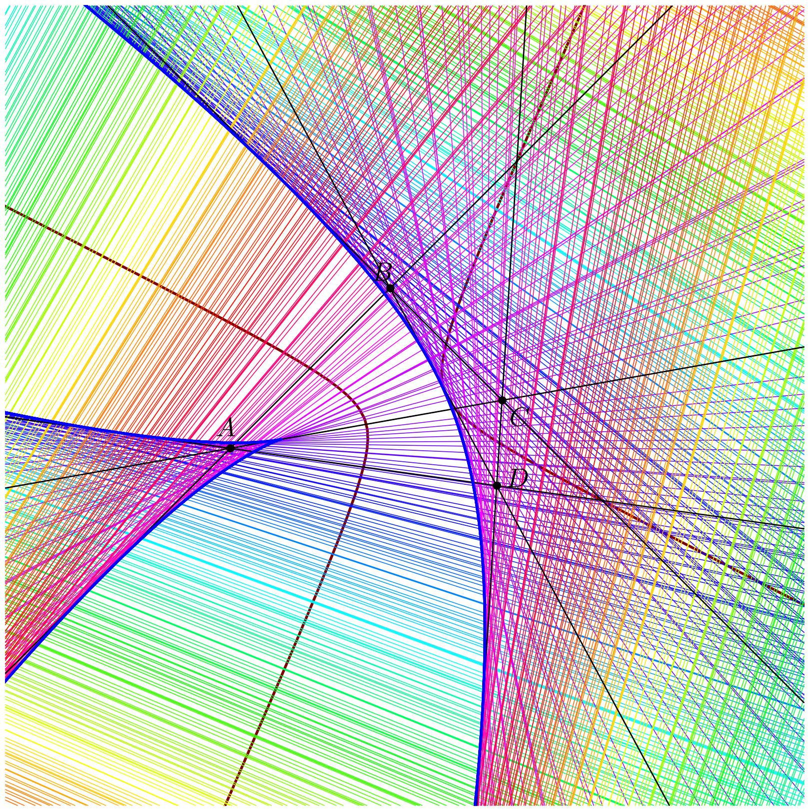
$$k(q) = \{\ell \mid \ell \text{ has an isotomic transversal in } q\}.$$

Note that $\mathcal{L}_\infty \in k(q)$ since it is invariant under isotomic transversal conjugation. So (by the notation back in Proposition 9.2.1), we represent it as $k^{\mathcal{L}_\infty^2}(q)$, therefore we can define it as the projective dual of a strophoid.

Remark. By Plücker's formulas, we obtain that the isohaptic curve is a quartic with three cusps (projectively equivalent to a deltoid and cardioid, which we will explore in the next chapter).

For convenience, let's just consider the affine part of the isohaptic curve

$$k^\circ(q) = k(q) \setminus \{\mathcal{L}_\infty\}$$



This lets us return to Proposition 1.5.5:

Proposition 9.4.1. Let $q = (A, B, C, D)$. For a line $\ell \neq \mathcal{L}_\infty$, let P_{XY} be $XY \cap \ell$. Then ℓ has an isotomic transversal wrt. q if and only if the midpoints of $\overline{P_{BC}P_{AD}}, \overline{P_{CA}P_{BD}}, \overline{P_{AB}P_{CD}}$ coincide.

For all lines ℓ with isotomic transversals, let M_ℓ be this point. As such, the DIT involution wrt q, and any such line ℓ is reflecting across M_ℓ.

Proposition 9.4.2. M_ℓ lies on the nine-point conic $\mathcal{C}_q$ Theorem 6.3.8 of q wrt $\mathcal{L}_\infty$. Conversely, for any M on the nine-point conic, there exists a unique line $\ell \in k^\circ(q)$ such that $M = M_\ell$.

Proof. Let $\triangle X_A X_B X_C$ be the cevian triangle of q, and consider the point isoconjugation φ on it such that the vertices of q are fixed (dual of QL-Tf2). By Proposition 7.4.17, $\mathcal{C}_q = \varphi(\mathcal{L}_\infty)$. For any circumconic $\mathcal{C}$ of q, Proposition 7.2.20 tells us that M_ℓ, ∞_ℓ are conjugate points wrt. this circumconic, so $\varphi(M_\ell) = \infty_\ell$, thus $M_\ell \in \varphi(\mathcal{L}_\infty) = \mathcal{C}_q$.

Now for the other direction. Let M be a point on $\mathcal{C}_q$, and choose line ℓ such that M is the midpoint of $P_{BC} = BC \cap \ell$ and $P_{AD} = AD \cap \ell$ (in other words, choose $\ell = MU$ where U is a point at infinity such that $(\infty_B C, \infty_A D; \infty_{X_A M}, U) = -1$). By taking the polar of M and Proposition 7.2.10, it follows that M is the fixed point of the DIT-given involution defined by ℓ cutting q. Thus M is also the midpoint of $\overline{P_{CA}P_{BD}}, \overline{P_{AB}P_{CD}}$, so $M = M_\ell$. ☐

Example 9.4.3. If q is an orthocentric system (example: D is the orthocenter of $\triangle ABC$), then the nine-point conic of $\triangle ABC$ is just the nine-point circle. For a point M on the nine-point circle, let P be the reflection of D, the orthocenter, across M. Then M's corresponding ℓ is just P's Simson line wrt. $\triangle ABC$.

Since M simultaneously is the midpoint of PD and $\overline{P_{BC}P_{AD}}$, $PP_{BC} \parallel P_{AD}D = AD$, so P_{BC} is the intersection of the line through P parallel to AD and line BC, which is just the foot from P to BC. Thus symmetrically we have that P's Simson line $\overline{P_{BC}P_{CA}P_{AB}} = \ell$.

So in the case of an orthocentric system, the isohaptic curve is just the envelope formed by all of the Simson lines of $\triangle ABC$. Since it's an orthocentric system, we also have that it's the envelope formed by all the Simson lines of $\triangle BCD$, etc. We will see in Section 10.2 that this is actually a deltoid, so we get that the dual of a strophoid is actually equivalent to a deltoid.

Let ℓ_* be the isotomic transversal of line ℓ wrt. q. Define M_{ℓ_*} analogously.

Proposition 9.4.4. M_ℓ, M_{ℓ_*} are antipodes on the nine-point conic $\mathcal{C}_q$.

Proof. Since we already know that M_ℓ, M_{ℓ_*} already lie on $\mathcal{C}_q$, we only need to prove that the midpoint of segment $M_\ell M_{\ell_*}$ is the center of $\mathcal{C}_q$. By definition,

$$\begin{aligned}\frac{1}{2}(M_\ell + M_{\ell_*}) &= \frac{1}{4}(P_{BC} + P_{AD}) + \frac{1}{4}(P_{BC*} + P_{AD*}) \\ &= \frac{1}{4}(P_{BC} + P_{BC*}) + \frac{1}{4}(P_{AD} + P_{AD*} = \frac{1}{2}(M_{BC} + M_{AD}),\end{aligned}$$

where M_{BC}, M_{AD} are the midpoints of $\overline{BC}$ and $\overline{AD}$, so the midpoint of $\overline{M_{BC}M_{AD}}$ is just the midpoint of $\overline{M_\ell M_{\ell_*}}$, which is also the center of $\mathcal{C}_q$ (the centroid of q). ☐

In the example of an orthocentric system in Example 9.4.3, ℓ, ℓ_* are just the Simson lines of antipodes on (ABC) P, P^*, so we should have:

Proposition 9.4.5. ℓ, ℓ_* intersect on the nine-point conic of $\mathcal{C}_q$.

Proof. Let $X = AD \cap BC \in \mathcal{C}_q, U = M_\ell \infty_{AD} \cap M_{\ell_*}\infty_{BC}, V = M_\ell \infty_{BC} \cap M_{\ell_*}\infty_{AD}$. Consider applying converse Pascal's on hexagon $M_\ell U M_{\ell_*} M_{AD} X M_{BC}$, which since $(M_{BC}M_{AD})(M_\ell M_{\ell_*})$ is a parallelogram, gives that

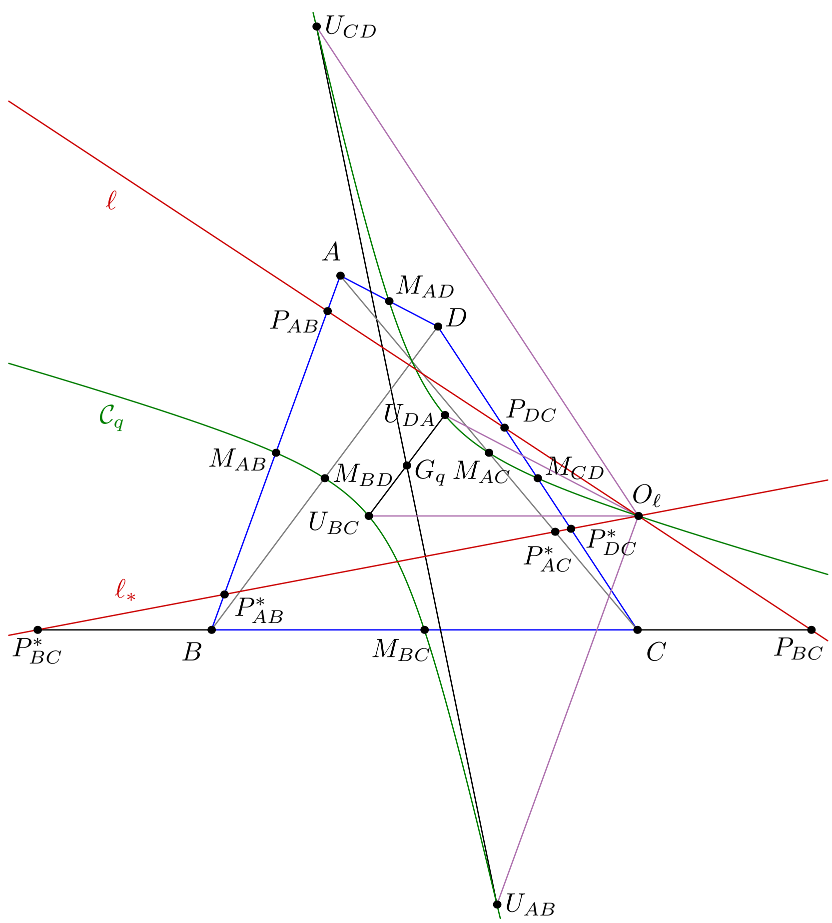
$$\infty_{AD} = M_\ell U \cap M_{AD}X, \infty_{BC} = UM_{\ell_*} \cap XM_{BC}, \infty_{M_{BC}M_\ell} = M_{\ell_*}M_{AD} \cap M_{BC}M_\ell$$

which all line on the line at infinity, so U lies on $\mathcal{C}_q$. Similarly, we get that V lies on $\mathcal{C}_q$. As such,

$$\begin{aligned}(\ell, M_\ell X; M_\ell U, M_\ell V) &= X(\infty_\ell, P_{AD}, P_{BC}) = -1 \\ &= X(\infty_{\ell_*}, M_{\ell_*}; P_{AD*}, P_{BC*}) = (\ell_*, M_{\ell_*}X; M_{\ell_*}U, M_{\ell_*}V)\end{aligned}$$

so $\ell \cap \ell_*$ lies on $\mathcal{C}_q$. ☐

Proposition 9.4.6. $A, B, C, D, \infty_\ell, \infty_{\ell_*}$ are conconic.



Proof. By DIT, we only need to show that $(\infty_\ell, \infty_{\ell_*})$ is also a pair under the DIT involution φ wrt $\mathcal{L}_\infty$ and q. As such, we want to show $(\infty_\ell, \infty_{\ell_*}), (\infty_{CA}, \infty_{BD}), (\infty_{AB}, \infty_{CD})$ are pairs under φ.

Let $O_\ell \in \mathcal{C}_q$ be the intersection of ℓ, ℓ_*, then by projecting through O_ℓ, it suffices to show

$$M_\ell M_{\ell_*}, U_{CA}U_{BD}, U_{AB}U_{CD}$$

concurring, where U_{XY} is the other intersection of $O_\ell \infty_{XY}$ with $\mathcal{C}_q$.

So we only need to prove that $\overline{U_{CA}U_{BD}}, \overline{U_{AB}U_{CD}}$ are both diameters of $\mathcal{C}_q$ to show that these lines concur at the center G_q of $\mathcal{C}_q$. Note that

$$(U_{CA}, M_{BC}; M_{CA}, M_{AB}) \overset{\infty_{CA}}{\overline{\wedge}} (O_\ell, M_{AB}; AC \cap BD\, M_{CA}) \overset{\infty_{BD}}{\overline{\wedge}} (U_{BD}, M_{AD}; M_{BD}, M_{CD}),$$

Since $M_{BC}M_{AD}, M_{CA}M_{BD}, M_{AB}M_{CD}$ concur at G_q, we know that $U_{CA}U_{BD}$ also pass through G_q, and we finish by symmetry. $\square$

Let this conic be $\mathcal{C}_\ell$, then since DIT on ℓ and q has ℓ as a fixed point, we get that ℓ is an asymptote of $\mathcal{C}_\ell$. Similarly ℓ_* is also an asymptote of $\mathcal{C}_\ell$. So:

Proposition 9.4.7. $\mathcal{C}_\ell$ is a hyperbola with asymptotes ℓ and ℓ_*.

From these properties, we get that $\mathscr{k}^\circ(q)$ is defined by a projective map $\varphi : \mathcal{C}_q \to \mathcal{L}_\infty$ which maps $M_\ell \mapsto \infty_\ell$, where φ satisfies for all diameters $\overline{MM^*}$ of $\mathcal{C}_q$, $M\varphi(M) \cap M^*\varphi(M^*) \in \mathcal{C}_q$. In the previous example of complete quadrangle $\triangle ABC \cup H$, $\mathcal{C}_q$ is the nine-point circle of $\triangle ABC$, and φ sends point P on the nine-point circle to $\infty_{\perp(A+B+C-\mathfrak{h}_{H,2}(P))}$ (the point at infinity parallel to the Simson line).

Note that φ is defined by two distinct pair of points M_1, M_2, so similarly to isoptic cubics, we can get:

Proposition 9.4.8. If $(K, K_*), (L, L_*) \neq (\mathcal{L}_\infty, \mathcal{L}_\infty)$ are two pairs of isogonal lines in complete quadrangle q, then

$$\mathscr{k}^\circ(q) = \mathscr{k}^\circ((KK_*)(LL_*)).$$

In this case, the notation $(KK_*)(LL_*)$ means complete quadrilateral $(K \cap L, L \cap K_*, K_* \cap L_*, L_* \cap K)$.

We can also investigate isotomic transversal conjugation under a projective transformation sending $\mathcal{L}_\infty$ to some line $\mathcal{L}$, and redefining isotomic conjugation of point X on segment P_1P_2 with harmonic conjugate of $P_1P_2 \cap \mathcal{L}$. Denote this new harmonic conjugation involution on $P_1P_2 \to P_1P_2$ as ψ_ℓ. Note that the asymptotes of the isohaptic cubic become the tangents ℓ, ℓ_* to $\mathcal{C}_\ell$ at the intersection of the polar of $\mathcal{L}$ with $\mathcal{C}_\ell$.

Finally, since $\mathscr{k}(q)$ is the dual of a strophoid, we can find some more properties of $\mathcal{K}^{W^2}(\mathcal{Q})$ (strophoid under projective transformation). Let $\mathcal{C}$ be the W-nine-line conic (see Chapter 6) of $\mathcal{Q}$, let L_{ij} be the other tangent from A_{ij} to $\mathcal{C}$ that's not $A_{ij}A_{kl}$. Then there exists a projective map $\varphi : \mathbf{T}W \to \mathbf{T}C$ such that $\varphi(WA_{ij}) = L_{ij}$, and

$$\mathcal{K}^{W^2}(\mathcal{Q}) = \{\ell \cap \varphi(\ell) \mid \ell \in \mathbf{T}W\}$$

is thus a cubic tangent to $\mathcal{C}$ three times. When $\mathcal{Q}$ is the incircle or an excircle, $\mathcal{C}$ is a parabola with I's reflection over M as the focus and the Newton line τ as the directrix.

Chapter 10

Secrets of the Complete Quadrilateral

10.1 Steiner's Hidden Deltoid

10.1.1 Preliminaries

Let $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$ be a complete quadrilateral, so as in chapter 4, we define:

- Six vertices: $A_{ij} = \ell_i \cap \ell_j$
- Four triangles: $\triangle_i := \triangle \ell_{i+1}\ell_{i+2}\ell_{i+3}$
- Three diagonals and the corresponding vertices: $A_{ij}A_{kl}$ and (A_{ij}, A_{kl}).
- Three quadrilaterals: $A_{ij}A_{jk}A_{kl}A_{li}$, including convex quadrilateral, concave quadrilateral and self-intersecting quadrilaterals (from a graphical point of view)
- Diagonal triangle (Cevian triangle): δ, a triangle formed by the extension of three diagonal segment.

And by now I believe you have heard of a few properties of these special quadrilateral objects:

Proposition 10.1.1 (QL-P1, Miquel Point)**.** The circumcircle of the triangles $\triangle_1, \triangle_2, \triangle_3, \triangle_4$ are concurrent at the point $M = M(\mathcal{Q})$.

Proposition 10.1.2 (QL-L1, Newton line)**.** The midpoint of the three diagonal segments $\overline{A_{23}A_{14}}$, $\overline{A_{31}A_{24}}$, and $\overline{A_{12}A_{34}}$ are collinear. These points are on the line $\tau = \tau(\mathcal{Q})$.

Proposition 10.1.3 (QL-Ci3, Miquel circle)**.** Five points M, O_1, O_2, O_3, O_4 are concyclic, where O_i is the circumcenter of $\triangle_i$.

Proposition 10.1.4 (QL-L2, orthocenters line, Steiner line)**.** Let H_i be the orthocenter of $\triangle_i$, then the four points H_1, H_2, H_3, H_4 lies on a line $\mathcal{S}$.

Proposition 10.1.5 (QL-Co1)**.** Let $\mathcal{P}$ the the parabola with focus M and directrix $\mathcal{S}$. Then $\mathcal{P}$ is tangent to the four lines in $\mathcal{Q}$.

Proposition 10.1.6 (QL-Tf1, Clawson–Schmidt Conjugate)**.** There exists exactly one transformation $\varphi_{\mathcal{Q}}$ which is a composition of inversion and reflection at the Miquel point such that $A_{23} \leftrightarrow A_{14}, A_{31} \leftrightarrow A_{24}, A_{12} \leftrightarrow A_{34}$.

(Don't worry, this is a run of the mill quadrilateral property as well.)

Proposition 10.1.7 (QL-Cu1)**.** The isoptic cubic of the complete quadrilateral $\mathcal{Q}$ is

$$\mathcal{K} = \{P | \infty^{\vee}_{P\varphi(P)} \in \tau(\mathcal{Q})\}$$

Using what we have learned about isoconjugations, we can also define:

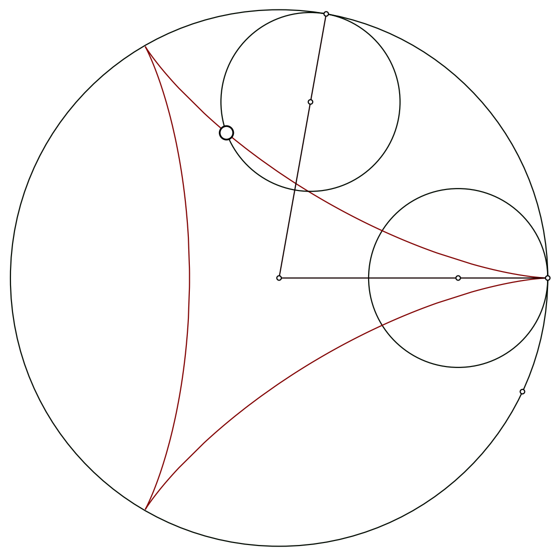
Proposition 10.1.8 (QL-Tf2, QL-Line Isoconjugate)**.** Consider the line isoconjugation in the cevian triangle δ of $\mathcal{Q}$ that fixes the four lines ℓ_i of the quadrilateral. Then the QL-Tf2 transformation of L is $L^{\vee} = \varphi(L)$.

Remark. A non-isoconjugation way to define this transformation is by intersecting a line ℓ with all the diagonals of the quadrilateral, then the harmonic conjugate of the three intersection points wrt. their diagonals' endpoints are collinear on line $\varphi(\ell)$. This motivates the notation $L^{\vee}$.

10.2 Deltoids

In order to thoroughly understand what we are doing, let's first actually define what a deltoid is:

Definition 10.2.1. A **deltoid** is a curve made by the locus of a point on a circle rolling around the inside of a circle three times its diameter.



To let the reader have a clue as to what direction we will be going, here's an important result we will prove in Theorem 10.2.5: for a moving point P on (ABC), the envelope of all the Simson lines formed by P wrt. $\triangle ABC$ forms a deltoid.

Proposition 10.2.2. Let $\triangle ABC$ be an equilateral triangle, and (I) be its incircle. Then the isogonal conjugate of (I) in $\triangle ABC$ is the deltoid with A, B, C as vertices.

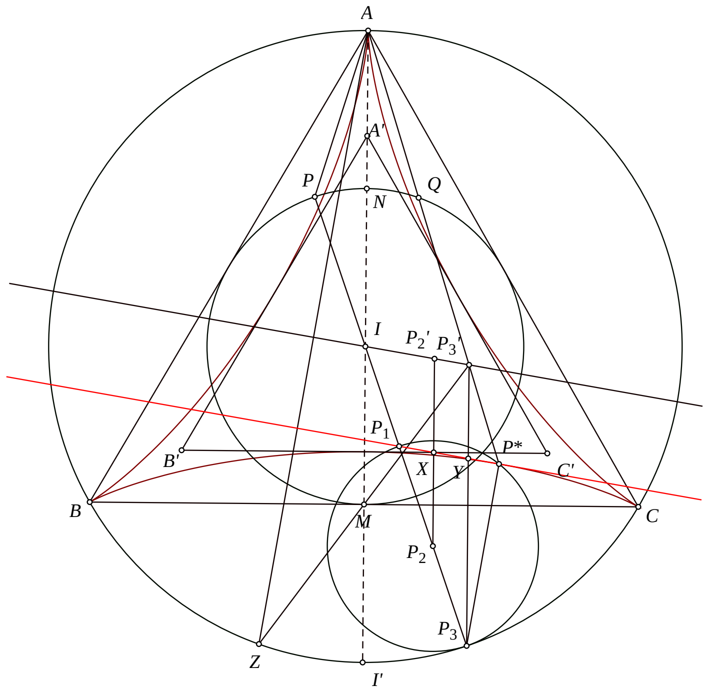
Proof. Let P be a point on (I), let the image of P through the homothety with center I, and respective ratios $-\frac{2}{3}, -\frac{4}{3}, -2$ be P_1, P_2, P_3, and let the image of A, B, C through the homothety with center I, ratio $\frac{2}{3}$ be A', B', C'. Finally let $\mathcal{S}$ be the Simson line of P_2 with $\triangle A'B'C'$.

Claim. The foot P^* of P_3 on $\mathcal{S}$ is the isogonal conjugate of P with $\triangle ABC$.

Proof of claim. Let Q be the reflection of P through AI. Then from symmetry, we only need to proof A, P^*, Q are collinear. Let the reflection of P_2 through $B'C'$ be P_2', and let the reflection of P_3 through BC be P_3'. Then I, P_2', P_3' are collinear and the line containing these points is the Steiner line of P_2 with $\triangle A'B'C'$. Let X, Y respectively be the intersection of P_2P_2', P_3P_3' with $\mathcal{S}$, and Z be the reflection of P_3 through AI. Then I is the centroid of $\triangle AP_3'Z$, so Q is the midpoint of AP_3'.
Now, let M, N respectively be the midpoint of $\overline{BC}, \overline{AI}$. Angle chasing gives that $\triangle P_3P^*Y$ is similar to $\triangle MQN$, so:

$$\frac{P_3Y}{YP_3'} = \frac{2P_2X}{XP_2'} = 2 = \frac{MN}{NA}$$

Thus we get $P^*P_3' \parallel AQ$, that is, A, P^*, Q are collinear. □



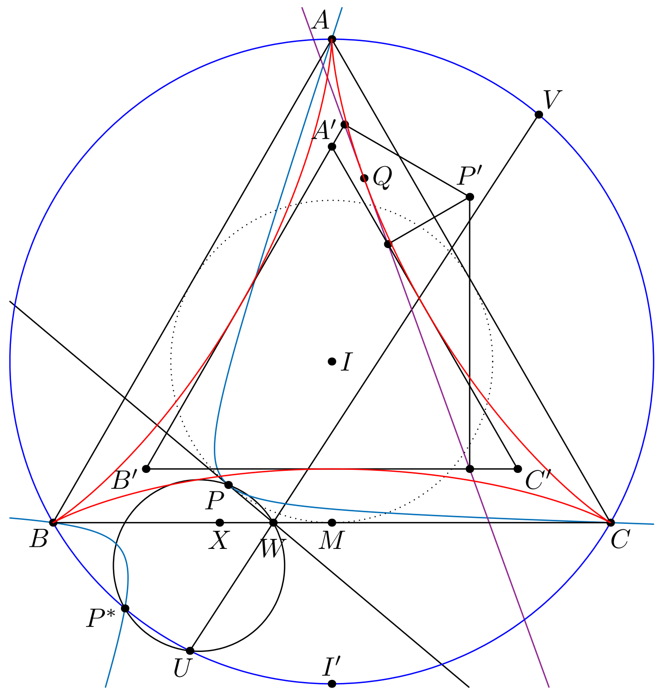
Returning to the original proof, P^* lies on $(\overline{P_1P_3})$ (its radius is 1/3 the radius of (I)). Let I' be the reflection of I through BC, then:

$$\measuredangle P_3P_1P^* = \measuredangle P_3IA + \measuredangle I'IP_3' = 3\measuredangle P_3I'A$$

Therefore P^* is on the deltoid with A, B, C as vertices.

By rewriting the above proof, we can obtain that every point is on the deltoid with A, B, C as vertices of the inscribed circle. □

Proposition 10.2.3. Let $\triangle ABC$ be an equilateral triangle, with (I) as the incircle, P is a point on (I), Q is the isogonal conjugate of P in $\triangle ABC$. Let the image of A, B, C through the homothety with center I, ratio 2/3 respectively be A', B', C', the image of P through the homothety with center I ratio $-2/3$ be P'. Then the Simson line of P' with $\triangle A'B'C'$ is the tangent line of Q to the deltoid with A, B, C as vertices.



Proof. By the previous claim, we have that Q is on $\mathcal{S}$. Taking an isogonal conjugation, we want to show the isogonal conjugation of $\mathcal{S}$ is tangent to (I) at P.

Let the image of P through the homothety at I with ratio 2 be P^*, then the isogonal conjugate of P^* in $\triangle ABC$ is on $\mathcal{S}$, so the isogonal conjugation of $\mathcal{S}$ is a conic $\mathcal{C}$ passing through A, B, C, P, P^*. Let the second intersection of AP, P^*P with (ABC) be U, V, the tangent line of (I) going through P intersects BC at W, M is the midpoint of $\overline{BC}$, I' is the reflection of I through BC. Then:

$$\measuredangle P^*WP = \measuredangle PWI = \measuredangle PMI = \measuredangle P^*I'A = \measuredangle P^*UP$$

therefore P, P^*, U, W are concyclic, so

$$\measuredangle VUW = \measuredangle VUP^* + \measuredangle P^*UW = 180^\circ$$

so U, V, W are collinear. Now let AP intersects BC at X, then

$$P(A, B; C, W) = (X, B; C, W) = U(A, B; C, V) = P^*(A, B; C, P)$$

gives that PW is the tangent line of $\mathcal{C}$ at P as desired. $\square$

Corollary 10.2.4. Let ω be a circle. For any point P on ω, draw a line $\ell_P \in TP$, such that

$$\ell_P + (P)_\omega$$

is a fixed value, that is, for all $P_1, P_2 \in \omega$,

$$\measuredangle(\ell_{P_1}, \ell_{P_2}) = \measuredangle P_2AP_1$$

where A is any point on ω. Then the envelope of $\{\ell_P | P \in \omega\}$ is a deltoid.

Proof. Let Ω be the image of ω through the homothety at the center O of ω ratio 2, a point A on ω such that $\ell_A = OA$, A' is the reflection of O through A, and take $B', C' \in \Omega$ such that $\triangle A'B'C'$ is an equilateral triangle. For any point P' on Ω, let $\mathcal{S}'_P$ be the Simson line of P' with $\triangle A'B'C'$, then $\mathcal{S}'_P$ goes through the midpoint P of $\overline{OP'}$. Note that OA is the Simson line of A' with $\triangle A'B'C'$ so

$$\measuredangle(\mathcal{S}'_P, \ell_P) = \measuredangle(\mathcal{S}'_P, \mathcal{S}_A) + \measuredangle(\ell'_A, \ell_P) = \measuredangle A'B'P' + \measuredangle PBA = 0^\circ$$

where B is the midpoint of $\overline{OB'}$. As such, $\mathcal{S}'_P = \ell_P$, so from Proposition 10.2.3 we know that the envelope of $\{\ell_P | P \in \omega\}$ is the deltoid. □

With these properties we have established, the theorem becomes very simple.

Theorem 10.2.5 (Steiner's deltoid theorem)**.** Let $\triangle ABC$ be any triangle, and for each point $P \in (ABC)$, let $\mathcal{S}_P$ be the Simson line of P with $\triangle ABC$, then the envelope of $\{\mathcal{S}_P \mid P \in (ABC)\}$ is a deltoid.

Proof. Let H be the orthocenter of $\triangle ABC$, (N) be the nine-point circle of $\triangle ABC$, and for any point $P_1, P_2 \in (N)$, let P'_i be the reflection of H through P_i, then $P_i \in \mathcal{S}_{P'_i}$ and

$$\measuredangle(\mathcal{S}_{P'_1}, \mathcal{S}_{P'_2}) = \measuredangle P'_2AP'_1 = -\measuredangle P_1XP_2$$

where $X \in (N)$. Therefore, from Corollary 10.2.4 we get that the envelope of $\{\mathcal{S}_P \mid P \in (ABC)\}$ is a deltoid. □

Definition 10.2.6. With the same labelling as the previous theorem, we call the deltoid of $\triangle ABC$ as $\mathcal{D}$, and the circumcircle of the three vertices in $\mathcal{D}$ the Steiner circle.

From the proof above, we can deduce the following result:

Proposition 10.2.7. Let $\triangle ABC$ be any triangle, $H, (N), \mathcal{D}$ respectively be the orthocenter, nine-point circle and the deltoid of $\triangle ABC$. Let P be an arbitrary point on (ABC), M be the midpoint of $\overline{HP}$. Let the Simson line $\mathcal{S}$ of P with $\triangle ABC$ intersects (N) at $Q \neq M$, then $\mathcal{S}$ is tangent to $\mathcal{D}$ at the point that is the reflection of Q through M.

Proof. The point P^* in Proposition 10.2.2 is equal to the reflection of Q over M. □

10.2.1 Rotate it!

We proved above that the envelope of the Simson line is a deltoid. In fact, we can even define the α-Simson line (Li4: I named this too):

Definition 10.2.8. Let $\triangle ABC$ be any triangle and an angle α. For any point $P \in (ABC)$, let D, E, F respectively be points on BC, CA, AB such that

$$\measuredangle(PD, BC) = \measuredangle(PE, CA) = \measuredangle(PF, AB) = \alpha(PE, CA) = \alpha$$

Then D, E, F are collinear by **??**, and this is called the α-Simson line of P wrt $\triangle ABC$, denoted as $\mathcal{S}_{\alpha,P}$.

When P is fixed, the envelope of $\mathcal{S}_{\alpha,P}$ is a parabola with focus P, and when $\alpha = 0°$, $\mathcal{S}_{\alpha,P}$ is the infinity line. Similarly to the original Simson line, we also have the following angular relation:

Proposition 10.2.9. For any $\triangle ABC$ and angles α_1, α_2, then for all $P_1, P_2 \in (ABC)$,

$$\measuredangle(\mathcal{S}_{\alpha_1,P_1}, \mathcal{S}_{\alpha_2,P_2}) = \alpha_1 - \alpha_2 - \measuredangle P_1AP_2$$

Proof. Simply calculate the angle. In fact,

$$\begin{aligned}\measuredangle(\mathcal{S}_{P_1,\alpha_1}, \mathcal{S}_{P_2,\alpha_2}) &= \measuredangle(\mathcal{S}_{P_1,\alpha_1}, \mathcal{S}_{P_1,\alpha_2}) + \measuredangle(\mathcal{S}_{P_1,\alpha_2}, \mathcal{S}_{P_2,\alpha_2})\\ &= (\alpha_1 - \alpha_2) - \measuredangle P_1AP_2.\end{aligned}$$

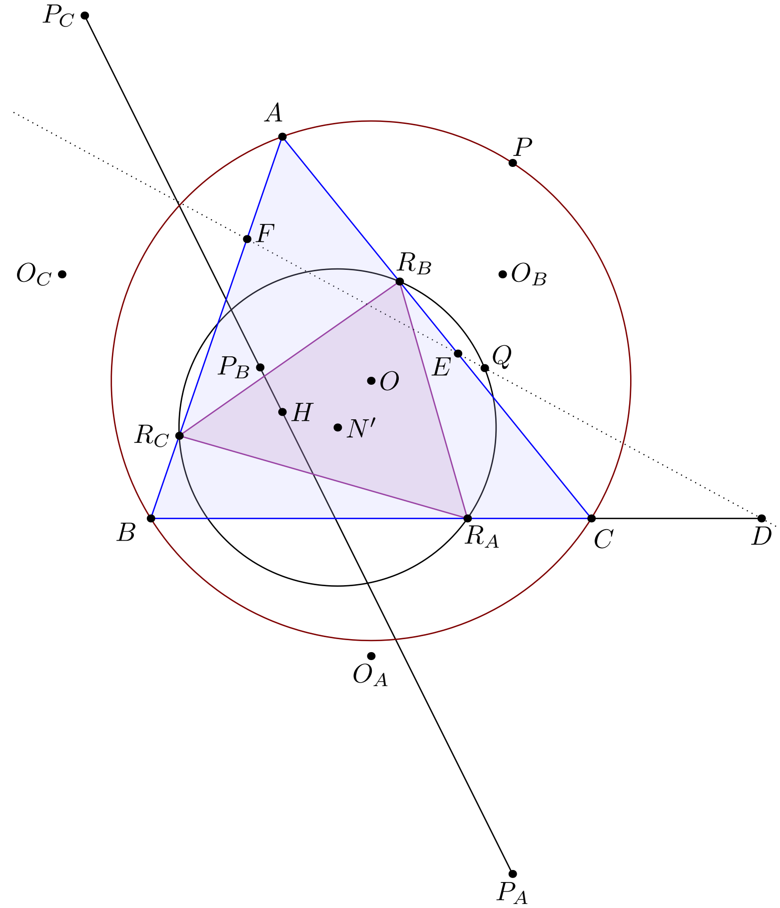
□

Looking at the title of this section, you will wonder if the envelope of all α-Simson line is also a deltoid. If $\alpha = 0°$, all α-Simson line concides to the line at infinity, so we do not consider this situation.

Theorem 10.2.10. Given a $\triangle ABC$ with an angle $\alpha \neq 0°$, then the envelope of $\{\mathcal{S}_{\alpha,P} \mid P \in (ABC)\}$ is a deltoid.

Unfortunately, we cannot directly apply Corollary 10.2.4 because we do not have yet the "α-Steiner theorem". So we need the following proposition:

Proposition 10.2.11. Given a $\triangle ABC$ with an angle $\alpha \neq 0°$, H is the orthocenter of $\triangle ABC$, then there exists a circle (N') and a spiral similarity transformation $\mathfrak{S}$ centered at H such that $\mathfrak{S}((ABC)) = (N')$ and for any point $P \in (ABC)$, $\mathfrak{S}(P)$ lies on the α-Simson line of P wrt $\triangle ABC$.



Proof. Let O be the circumcenter of $\triangle ABC$, and define R_A, R_B, R_C on BC, CA, AB such that

$$\measuredangle(OR_A, BC) = \measuredangle(OR_B, CA) = \measuredangle(OR_C, AB) = \alpha$$

It's easy to get that $\triangle R_AR_BR_C \cup O \overset{+}{\sim} \triangle ABC \cup H$. Let $(N') = (R_AR_BR_c)$, and for any point P, denote $\mathfrak{S}(P)$ as the point Q such that $\triangle HPQ \overset{+}{\sim} \triangle HON'$.

Let O_A, O_B, O_C respectively be the reflection of O through BC, CA, AB. Note that H is the circumcenter of $\triangle O_AO_BO_C$ so

$$\triangle OO_AR_A \overset{+}{\sim} \triangle OO_BR_B \overset{+}{\sim} \triangle OO_CR_C \overset{+}{\sim} \triangle OHN'$$

so $\triangle O_AO_BO_C \cup H \overset{+}{\sim} \triangle R_AR_BR_C \overset{+}{\sim} N'$, and N' lies on the perpendicular bisector of $\overline{OH}$. For any point

$P \in (ABC)$, let $Q = \mathfrak{S}(P)$, from $\triangle HPQ \overset{+}{\sim} \triangle HON'$, we have

$$\frac{\overline{N'Q}}{\overline{OP}} = \frac{\overline{HN'}}{\overline{HO}} = \frac{\overline{ON'}}{\overline{OH}} = \frac{\overline{N'R_A}}{\overline{HO_A}}$$

and (ABC) has the same radius as $(O_AO_BO_C)$, so $\overline{N'Q} = \overline{N'R_A}$, that is, $Q \in (N')$. Therefore we have $\mathfrak{S}((ABC)) = (N')$. Let $\mathcal{S}_{\alpha,P}$ be the α-Simson line of P wrt $\triangle ABC$, and D, E, F respectively be the intersection of $\mathcal{S}_{\alpha,P}$ with BC, CA, AB, let P_A, P_B, P_C be the reflection of P through BC, CA, AB. From Steiner's theorem, we know that P_A, P_B, P_C are collinear on a line that goes through H. Therefore,

$$\triangle PDP_A \overset{+}{\sim} \triangle PEP_B \overset{+}{\sim} \triangle PFP_C \overset{+}{\sim} \triangle OR_AO_A \overset{+}{\sim} \triangle ON'H \overset{+}{\sim} \triangle PQH$$

As such, we have $P_AP_BP_C \cup H \overset{+}{\sim} DEF \cup Q$, so $Q \in \mathcal{S}_{\alpha,P}$. □

Remark. This is also exercise 5 from Section 1.4

After having this property, the proof of Theorem 10.2.10 is the same as the proof of the original theorem Theorem 10.2.5. Now we define analogously:

Definition 10.2.12. With the same notation as Theorem 10.2.10, we denote $\mathcal{D}_\alpha$ as the α-Steiner deltoid of $\triangle ABC$.

All deltoids that is tangent to $\triangle ABC$ must be some α-Steiner deltoid. Then our question is how to get α given any three tangent lines on the deltoid and the triangle formed by those lines. To solve this, we have to go deeper into the proof of Proposition 10.2.11.

Proposition 10.2.13. Using the same notation as Proposition 10.2.11, let S be isogonal conjugate of $\infty_{\mathcal{S}_{\alpha,P}}$ wrt $\triangle ABC$, then

$$\triangle ABC \cup S \overset{+}{\sim} \triangle R_AR_BR_C \cup \mathfrak{S}(P)$$

and the rotation angle is $90° + \alpha$.

Proof. Simple angle chasing. In fact, let $A^* = \mathfrak{S}^{-1}(R_A), B^* = \mathfrak{S}^{-1}(R_B)$, then

$$\measuredangle R_BR_A\mathfrak{S}(P) = \measuredangle B^*A^*P = -\measuredangle(\mathcal{S}_{\alpha,B^*}, \mathcal{S}_{\alpha,P}) = \measuredangle(\mathcal{S}_{\alpha,P}, CA) = \measuredangle BAS$$

and

$$\measuredangle(R_BR_C, BC) = \measuredangle(\perp OR_A, BC) = 90° + \alpha.$$

□

Finally, we prove the most important theorem of this section:

Theorem 10.2.14. Given a $\triangle ABC$ with an angle $\alpha \neq 0°$, let $\mathcal{D}_\alpha$ be the α-Steiner deltoid of $\triangle ABC$. Let P_1, P_2, P_3 be three points on $\Omega = (ABC)$, $\mathcal{S}_i = \mathcal{S}_{P_i,\alpha}$. Let

$$\beta = P_1 + P_2 + P_3 - A - B - C - 2\alpha \ (\Omega).$$

(i) When $\beta = 0°$, $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are concurrent.

(ii) When $\beta \neq 0°$, $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ aren't concurrent, and $\mathcal{D}_\alpha$ is the β-Steiner deltoid of the triangle formed by $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$.

Proof. With the same notation as Proposition 10.2.11, let S_i be the isogonal conjugate of $\infty_{\mathcal{S}_i}$ wrt $\triangle ABC$, T_i is the intersection of $\mathcal{S}_{i-1}$ and $\mathcal{S}_{i+1}$, and H' be the orthocenter of $\mathfrak{S}(\triangle P_1P_2P_3)$ and $\mathfrak{S}(P_i) = Q_i$. Let $\Omega' = (N')$. Then, from Proposition 1.4.5 and Proposition 10.2.13, we have

$$\begin{aligned}
\beta &= \sum_i P_i - \sum_{\text{cyc}} A - 2\alpha = \sum_{\text{cyc}} \mathcal{S}_{A,\alpha} - \sum_i \mathcal{S}_i - 2\alpha \ (ABC) \\
&= \sum \measuredangle(O_BO_C, \mathcal{S}_{P_1,\alpha}) + \alpha = \sum_i Q_i - \sum_{\text{cyc}} R_A + \alpha \quad (\Omega') \\
&= Q_1 - R_A + Q_2Q_3 - R_BR_C + \alpha \quad ((N')) \\
&= S_1 - A + Q_2Q_3 - BC + 90° \quad ((ABC)) \\
&= \measuredangle(\perp Q_2Q_3, \mathcal{S}_1)
\end{aligned}$$

Note that

$$\measuredangle Q_{i-1}T_iQ_{i+1} = \measuredangle(\mathcal{S}_{i-1}, \mathcal{S}_{i+1}) = -\measuredangle Q_{i-1}Q_iQ_{i+1}$$

Therefore $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are concurrent if and only if $T_1 = T_2 = T_3 = H'$, or if and only if $H' \in \mathcal{S}_1$, which is equivalent to

$$\beta = \measuredangle(\perp Q_2Q_3, \mathcal{S}_1) = 0$$

This proves (i), now le'ts prove (ii). Assume that $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ aren't concurrent. Let $\mathcal{D}_a$ be the β'-deltoid of $\triangle T_1T_2T_3$, then we can take the proof of Proposition 10.2.11 with $T_1T_2T_3$ as ABC and $Q_1Q_2Q_3$ as $R_1R_2R_3$, H' is the circumcenter of $\triangle T_1T_2T_3$, where $\beta' = \measuredangle(H'Q_1, \mathcal{S}_1)$. So

$$\beta' = \measuredangle(\perp Q_2Q_3, \mathcal{S}_1) = \beta$$

which suffices by symmetry. □

Taking $\alpha = 90°$ gives this famous corollary (Exercise 3 of section 1.4):

Corollary 10.2.15. Given a $\triangle ABC$, P_1, P_2, P_3 are three points on (ABC), $\mathcal{S}_i$ is the Simson line of P_i wrt $\triangle ABC$, then $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are concurrent if and only if

$$A + B + C = P_1 + P_2 + P_3 \quad ((ABC))$$

Practice Problems

Problem 1. Assume the Steiner deltoid of $\triangle ABC$ intersects BC, CA, AB at D, E, F. Then AD, BE, CF are concurrent at the isotomic conjugate of the orthocenter of $\triangle ABC$.

Problem 2. Let S_1, S_2, S_3 be the vertices of the Steiner deltoid of $\triangle ABC$. Prove that: (formal sum)

$$3 \cdot S_1S_2 = BC + CA + AB$$

10.3 The Kantor-Hervey Point

We've finished discussing the Steiner deltoid, so let's try to stuff a complete quadrilateral into this.

Proposition 10.3.1. Let complete quadrilateral $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$ have the four circumcenters O_1, O_2, O_3, and O_4, and let H'_i be the orthocenter of $O_{i+1}O_{i+2}O_{i+3}$, then O_i, H'_i are conconic and $H'_i \in \ell_i$.

Proof. Let M be the Miquel point of $\mathcal{Q}$, then note that ℓ_i is the Steiner line of M wrt. $\triangle O_{i+1}O_{i+2}O_{i+3}$, and thus $H'_i \in \ell_i$. Also H'_i just lies on the rectangular hyperbola through O_1, O_2, O_3, and O_4. □

Corollary 10.3.2 (QL-P5, the Clawson point)**.** The midpoints of $\overline{O_iH'_i}$ coincide.

Proof. This is just the Poncelet point of $O_1O_2O_3O_4$. □

Corollary 10.3.3 (QL-Ci4, the Hervey circle)**.** The orthocenters H'_1, H'_2, H'_3, H'_4 are concyclic.

Proof. Reflect circle $(O_1O_2O_3O_4)$ over the Poncelet point of $O_1O_2O_3O_4$, then we get a circle through H'_1, H'_2, H'_3, H'_4 as desired. □

Theorem 10.3.4 (QL-Qu2, the Kantor-Hervey deltoid)**.** There exists a deltoid Δ tangent to all four sides of $\mathcal{Q}$.

We can obtain this from the α-Steiner deltoid (in reality, Δ is the α-Steiner deltoid of $\triangle \ell_j\ell_k\ell_l$, where the angle

$$\alpha = \measuredangle(M(\ell_i \cap \ell_j), \ell_j) = \measuredangle(\ell_i, \tau).$$

(τ is the Newton line) This gives us some motivation for our proof:

Proof. Consider the transformation $\varphi = \varphi \circ \mathfrak{s}_T$, where $\mathfrak{s}_T$ is reflection across the Clawson point Corollary 10.3.2 and φ is Clawson-Schmidt conjugation in $\mathcal{Q}$. For a point $P \in (H_1'H_2'H_3'H_4')$, define L_P to be the perpendicular from P to $M\varphi(P)$, then by simple angle chasing and Corollary 10.2.4 we get the envelope of L_P is a deltoid Δ. Note that $\ell_i = L_{H_i'}$, so Δ is tangent to the four sides of $\mathcal{Q}$. □

Since this deltoid is unique, we get the converse of this proof, and we can also see that:

Corollary 10.3.5. The Kantor-Hervey deltoid $\triangle$ of quadrilateral $\mathcal{Q}$ is tangent to the Hervey circle of $\mathcal{Q}$.

Proposition 10.3.6 (QL-P3, the Kantor-Hervey point)**.** Let θ be the center of the Kantor-Hervey deltoid, then θ lies on the four perpendicular bisectors of the segment between the orthocenter and circumcenter of the component triangles of $\mathcal{Q}$.

Proof. Since $\triangle$ is an α_i deltoid with respect to $\triangle \ell_j \ell_k \ell_l$, it follows by the proof of Proposition 10.2.11 that θ lies on the perpendicular bisector of O_iH_i. □

Proposition 10.3.7 (QL-P2, the Morley point)**.** Let Mo be the foot from θ to the Steiner line $\mathcal{S}$. Let N_1, N_2, N_3, N_4 be the nine-point centers of each component triangle of the quadrilateral. Then the lines passing through N_i perpendicular to ℓ_i concur at Mo.

Proof. Continuing the previous notation, note that $\theta N_i \perp O_iH_i$, so θ, H_i, N_i, Mo are concyclic. We have

$$\begin{aligned}\measuredangle(N_iMo, \ell_i) &= \measuredangle N_iMoH_i + \measuredangle(\mathcal{S}_{\mathcal{Q}}, \ell_i) = \measuredangle N_i\theta H_i + \measuredangle(\tau, \ell_i) + 90^\circ \\ &= \alpha_i - \alpha_i + 90^\circ = 90^\circ,\end{aligned}$$

where τ is the Newton line of $\mathcal{Q}$. Thus $N_iMo \perp \ell_i$. □

There are also proofs of all of this without deltoids.

Practice Problems

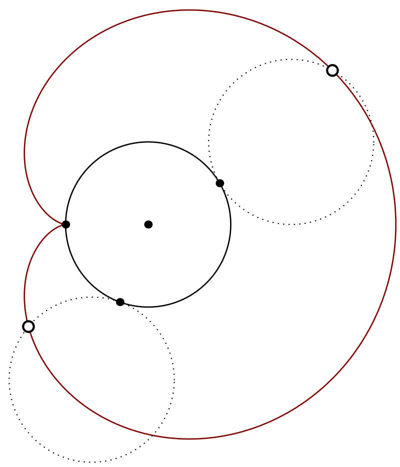
Problem 1. Let ℓ be the reflection of the Newton line of quadrilateral $\mathcal{Q}$ over its Miquel point. Prove that ℓ is tangent to the Kantor-Hervey deltoid of $\mathcal{Q}$.

10.4 Morley's Beautiful Cardioids

After talking so long about deltoids, everyone should be at least a little interested in cardioids, right?

Remark. (The original joke is the word for "deltoid" is the same as the word for "tricuspid valve curve", and "cardioid" is "heart curve", in Chinese.)

Definition 10.4.1. A **cardioid** $\mathbb{K}$ is the curve made by the locus of a point on a circle Γ' that's rolling around another congruent circle Γ. We call the center of Γ the center of the cardioid $\mathbb{K}$. Additionally, each cardioid has one **cusp**, where it's not differentiable.



This definition is useless. So we first re-define the cardioid as such:

Proposition 10.4.2. A cardioid is the image of the inverse of a parabola around its focus. (Similarly, inverting a cardioid at its cusp gets you a parabola.).

Proof. When we define $\mathbb{K}$ as the locus of rolling Γ' around Γ, let Y be the cusp of $\mathbb{K}$. Then by the definition of rolling without slipping, the cardioid is the locus of reflection of Y over all tangent lines in $\mathbf{T}\Gamma$. After inverting this property with respect to Y, we get that the new locus is just the locus of centers of circles going through Y and tangent to the image of Γ' (which is just a fixed line). However, this locus is just a parabola with Y as focus and the image of Γ' as the directrix. □

Proposition 10.4.3. Let Y be the cusp of the cardioid $\mathbb{K}$, formed by rolling Γ' around Γ. Then for every point P on $\mathbb{K}$, the perpendicular bisector of PY is tangent to Γ.

Proof. This follows by reflecting over the perpendicular bisector, since the arc lengths formed by rolling on both circles are the same, by the definition of rolling without slipping. □

Now let us first review Morley's trisector theorem Theorem 1.6.1:

Theorem 10.4.4. For any triangle ABC (labelled counterclockwisely) draw the angular trisector ℓ_A^{Bi} through A of $\angle ABC$, with $i = -1, 0, 1$, satisfying ℓ_A^{B0} is located in $\angle BAC$, and

$$\measuredangle(\ell_A^{B0}, CA) = 2\measuredangle(AB, \ell_A^{Bi}), \quad \measuredangle(\ell_A^{Bi}, \ell_A^{Bj}) = (j - i) \cdot 60^\circ$$

ℓ_A^{Ci} is the isogonal conjugate of ℓ_A^{Bi} in $\angle BAC$, define $\ell_B^{Ci}, \ell_B^{Ai}, \ell_C^{Ai}, \ell_C^{Bi}$ similarly. Define $a_{ij} = \ell_B^{Ci} \cap \ell_C^{Bj}$, define b_{ij}, c_{ij} similarly. Then for all $(i,j,k) \in \{-1,0,1\}^3$ such that $3 \nmid 1+i+j+k$, $\triangle a_{jk}b_{ki}c_{ij}$ is an equilateral triangle, and

$$\measuredangle b_{ki}c_{ij} + \measuredangle BC = \measuredangle a_{jk}B + \measuredangle a_{jk}C$$

The rest are similar.

We also define the first, second and third Morley triangles as $\triangle_0 = \triangle a_{00}b_{00}c_{00}$, $\triangle_1 = \triangle a_{11}b_{11}c_{11}$, and $\triangle_2 = \triangle a_{(-1)(-1)}b_{(-1)(-1)}c_{(-1)(-1)}$. Define

$$\Lambda^{\triangle ABC} = \bigcup (b_{ii}c_{ii} \cup c_{ii}a_{ii} \cup a_{ii}b_{ii})$$

as the union of all edges of $\triangle_i$, and

$$\lambda_A^{\triangle ABC} = \{a_{ij} | (i,j) \in \{-1,0,1\}^2\}$$

Similarly, we also define $\lambda_B^{\triangle ABC}, \lambda_C^{\triangle ABC}$, $\lambda = \lambda_A \cup \lambda_B \cup \lambda_C$ are all the vertices.

Theorem 10.4.5. Given a complete quadrilateral $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$, let the four triangles in $\mathcal{Q}$ be $\triangle_1, \triangle_2, \triangle_3$, and $\triangle_4$. Let $\lambda_i = \lambda^i, \Lambda_i = \lambda^{\triangle i}$ and $A_{ij} = \ell_i \cap \ell_j$. If $L_i \subset \Lambda_i, L_j \subset \Lambda_j$ are two lines such that $L_i \cap \lambda_{A_{jk}}^{\triangle_i} = \{P_i, Q_i\}$ and $L_j \cap \lambda_{A_{ki}}^{\triangle_j} = \{P_j, Q_j\}$ (which exist by Proposition 1.6.3) such that $P_iP_j \cap Q_iQ_j = A_{kl}$, then

$$L_i \cap L_j \in (A_{jl}P_i \cap A_{il}P_j)(A_{jl}Q_i \cap A_{il}Q_j) \cap (A_{jk}P_i \cap A_{ik}P_j)(A_{jl}Q_i \cap A_{il}Q_j)$$

with i,j,k,l pairwise distinct. In other words, $L_i \cap L_j \in \bigcap \Lambda_m$.

Proof. Observe that:

$$(A_{jl}P_i \cap A_{il}P_j)(A_{jl}Q_i \cap A_{il}Q_j) \subset \Lambda_k$$

Since $\triangle A_{jl}P_iQ_i$ and $\triangle A_{il}P_jQ_j$ are perspective wrt A_kl, by Desargues's theorem,

$$L_i \cap L_j \in (A_{jl}P_i \cap A_{il}P_j)(A_{jl}Q_i \cap A_{il}Q_j) \subset \Lambda_k,$$

Similarly

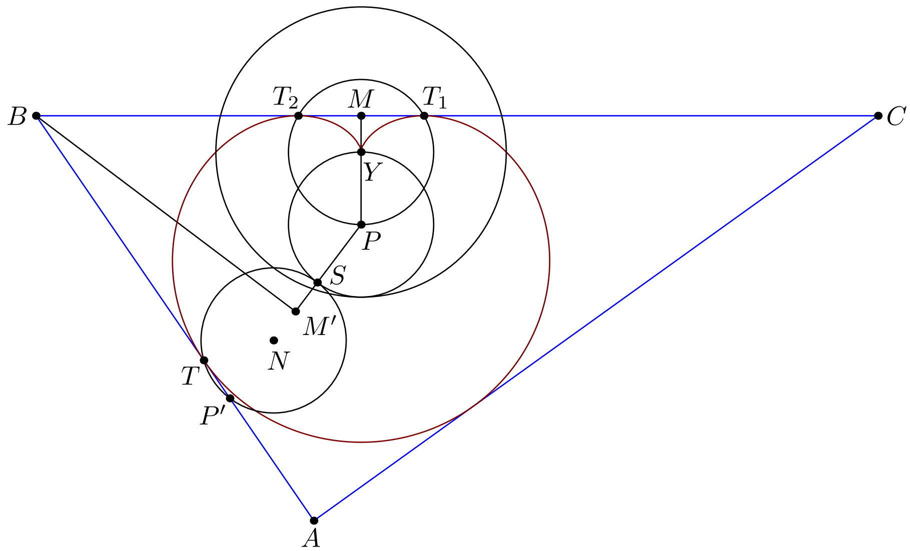
$$L_i \cap L_j \in (A_{jk}P_i \cap A_{ik}P_j)(A_{jk}Q_i \cap A_{ik}Q_j) \subset \Lambda_l,$$

so $L_i \cap L_j \in \bigcap \Lambda_m$. $\square$

Corollary 10.4.6. Using the same notation with Theorem 10.4.5, $|\bigcap \Lambda_i| \geq 27$ (counted with multiplicity).

Proof. We know that the number of combinations of (L_i, Lj) satisfying $\{P_i, Q_i\}$and $\{P_j, Q_j\}$ wrt the A_{kl} is 27 (given i, j, k, l), so $|\bigcap \Lambda_i| \geq 27$. □

Proposition 10.4.7. If all three sides of $\triangle ABC$ are tangent to a cardioid $\mathbb{K}$, and $\mathbb{K}$ and BC are tangent at two points, then the center P of $\mathbb{K}$ is $P \in \lambda_A$. In the other hand, if $P \in \lambda_A$, then there exists a cardioid $\mathbb{K}$ that is tangent to all three sides of $\triangle ABC$, and tangent to BC at two points.



Proof. Let $\mathbb{K}$ be tangent to BC at T_1, T_2, and tangent to AB at T, and let the cusp be Y. Then we can verify that $\triangle PT_1T_2$ is an equilateral triangle with center Y. There exists a point S such that the circle with center S tangent to (P) goes through T. Let P' be the image of P through the homothety with center S, ratio -2. Since T is the reflection of F over the tangent at S and the tangent to $\mathbb{K}$ at T is parallel to the angle bisector of TP_1S, it follows that P' lies on AB.

Let the perpendicular bisector of $\overline{PP'}$ intersect $P'T$ at B', and let M, M', N respectively be the midpoints of $\overline{T_1T_2}, \overline{PP'}, \overline{SP'}$. Note that

$$\measuredangle MPB' = \measuredangle YPS + \measuredangle P'PB' = \measuredangle SNT + \measuredangle TP'S = \measuredangle B'PP'$$

so $\triangle B'PM \stackrel{-}{\sim} \triangle B'P'M'$ since $PM = PM'$. As such, this gives that $\measuredangle B'MP = 90°$, so $B' = B$ and

$$3 \cdot \measuredangle CBP = \measuredangle CBP + \measuredangle PBM' + \measuredangle M'BP' = \measuredangle CBA$$

Similarly, $3 \cdot \measuredangle PCB = \measuredangle ACB$, and thus $P \in \lambda_A$.

Conversely, if $P \in \lambda_a$, take two points T_1, T_2 on BC such that $\triangle PT_1T_2$ is an equilateral triangle, and then write the proof above in reverse. The uniqueness is because P and the cusp are unique. □

I (Li4) haven't found a short proof for the following theorem. I hope you can find a shorter one for me (?). Although the proof looks long, in fact the idea is not difficult.

Theorem 10.4.8. If a cardioid $\mathbb{K}$ is tangent to all three sides of $\triangle ABC$, then the center of $\mathbb{K}$ lies on $\Lambda^{\triangle ABC}$. On the other hand, if $P \in \Lambda^{\triangle ABC}$, then there is a unique cardioid centered at P and is tangent to all three sides of $\triangle ABC$.

Proof. Let L be a line that is tangent to $\mathbb{K}$ and intersects $\mathbb{K}$ at two points. Let D, E, F be the respective intersections of L with BC, CA, AB. Then from Proposition 10.4.7 we know that $P \in \lambda_A^{\triangle AEF} \cap \lambda_B^{\triangle BFD} \cap \lambda_C^{\triangle CDE}$. Let U, V respectively be two points on $(PFD), (PDE)$ such that $\measuredangle UDP = \measuredangle VDP = 60°$, then $PU \subset \Lambda^{\triangle BFD}, PV \subset \Lambda^{\triangle CDE}$ and D, U, V are collinear, so from Theorem 10.4.5 and Proposition 1.3.6 we know that $P \in \Lambda^{\triangle ABC}$.

On the other hand, if $P \in \Lambda^{\triangle ABC}$, assume $P \in \ell \subset \Lambda^{\triangle ABC}$, let

$$\{U_1, U_2\} = \ell \cap \lambda_B^{\triangle ABC}, \quad \{V_1, V_2\} = \ell \cap \lambda_C^{\triangle ABC},$$

so (AU_i, AV_i) are the paired angular trisectors of $\angle BAC$. Take two points S_1, T_1 on AU_1, AV_1 respectively, such that $\triangle PS_1T_1$ is an equilateral triangle and $\measuredangle PS_1T_1 = \measuredangle U_1AU_2$. Let

$$S_2 = PT_1 \cap AU_2, T_2 = PS_1 \cap AV_2$$

Then it is immediate that $S_2, T_2 \in (AS_1T_1)$, so $\triangle PS_2T_2$ is also an equilateral triangle. Define

$$E = CA \cap S_1S_2, F = AB \cap T_1T_2$$

It's easy to see that the second intersection G, H of (AS_1T_1) with CA, AB respectively are the reflection of P wrt S_1S_2, T_1T_2. Therefore, ES_1, FT_1 are the angle bisector of $\angle AEP, \angle PFA$ respectively, so $\triangle PS_1T_1$ is a Morley triangle of $\triangle AEF$. Similarly, $\triangle PS_2T_2$ is also a Morley triangle of $\triangle AEF$.

Let D be intersection of EF and BC, $W = PS_1 \cap AS_2, X = PT_1 \cap AT_2$, note that $P \in \Lambda^{\triangle ABC} \cap \Lambda^{\triangle AEF}$ and

$$P = U_1U_2 \cap S_1W = V_1V_2 \cap T_1X, A = U_1S_1 \cap U_2W = V_1T_1 \cap V_2X$$

So from Theorem 10.4.5 we get $P \in \lambda_B^{\triangle BFD} \cap \lambda_C^{\triangle CDE}$, and then from Proposition 10.4.7 we know that there exists cardioids $\mathbb{K}_A, \mathbb{K}_B, \mathbb{K}_C$ with center P such that $\mathbb{K}_A$ is tangent to all three sides of $\triangle AEF$ and is tangent to EF at two points, similarly with $\mathbb{K}_B, \mathbb{K}_C$. Since D, E, F are collinear, we get $\mathbb{K}_A = \mathbb{K}_B = \mathbb{K}_C$, and are tangent to $\triangle ABC$.

Finally, suppose there are two cardioids $\mathbb{K}_1, \mathbb{K}_2$ centered at P and tangent to all three sides of $\triangle ABC$. Let

L_i be the tangent line of $\mathbb{K}_i$ and intersects $\mathbb{K}_i$ at two points. E_i and F_i are the intersection points of L_i with CA and AB respectively, then

$$P \in \lambda_A^{\triangle AE_1F_1} \cap \lambda_A^{\triangle AE_2F_2}.$$

Take two points Y_i, Z_i on AU_1, AV_1 respectively such that $\triangle AY_iZ_i$ is a Morley triangle of $\triangle AE_iF_i$. Note that

$$\triangle PY_1Z_1 \overset{+}{\sim} \triangle PY_2Z_2$$

So $\triangle PY_1Z_1 = \triangle PY_2Z_2$, which gives $E_1 = E_2, F_1 = F_2$. This implies $\mathbb{K}_1 = \mathbb{K}_2$, which proves the uniqueness. $\square$

Proposition 10.4.9 (QL-27Qu1, Morley's Multiple Cardioids)**.** There are at least 27 cardioids that is tangent to four lines of a complete quadrilateral $\mathcal{Q}$ (counted with multiplicity).

Proof. From Corollary 10.4.6, we know $|\bigcap \Lambda_i \geq 27|$ (counted with multiplicity), Let $P \in \bigcap \Lambda_i$, then from Theorem 10.4.8 we know that there is a cardioid $\mathbb{K}_i$ centered at P that is tangent to all three sides of $\triangle_i$. For any distinct i, j, by Theorem 10.4.8 the last part of the proof is that $\mathbb{K}_i = \mathbb{K}_j$, so there is a unique cardioid $\mathbb{K}$ centered at P that is tangent to all four sides of $\mathcal{Q}$. $\square$

Practice Problems

Problem 1. Let $\mathbb{K}$ be a cardioid with cusp Y. An arbitrary line through Y intersects $\mathbb{K}$ at A, B. Prove that the locus of the midpoint of $\overline{AB}$ is a circle.

Problem 2. With the same notation as Theorem 1.6.1, prove that: For all $(i, j, k) \in \{-1, 0, 1\}^3$ such that $3 \nmid 1 + i + j + k$,

(i) $\triangle a_{jk}b_{ki}c_{ij}$ is perspective with $\triangle a_{(j-1)(k-1)}b_{(k-1)(i-1}c_{(i-1)(j-1)}$ wrt center P_{ijk} (Note that not all equilateral triangle are positively similar);

(ii) The isogonal conjugate of P_{ijk} wrt. $\triangle ABC$ is the perspector of $\triangle a_{jk}b_{ki}c_{ij}$ and $\triangle ABC$.

Problem 3. With the same notation as Theorem 1.6.1, prove that:

(i) For all $i \in \{-1, 0, 1\}, A \in (a_{ii}a_{(i+1)(i+2)}a_{(i+2)(i+1)})$;

(ii) $(a_{11}a_{23}a_{32}), (a_{22}a_{31}a_{13}), (a_{33}a_{12}a_{21})$ are coaxial.

Problem 4 (QL-Qu1, Morley's Mono Cardioid)**.** Let the Miquel point of the complete quadrilateral $\mathcal{Q}$ be M, and the Miquel circle be $\mathbb{M}$. Consider all circles passing through M and whose center is located on $\mathbb{M}$. Prove that the envelope of these circles is a cardioid $\mathbb{K}$

Problem 5 (QL-Cu2)**.** Let the four triangles of the complete quadrilateral $\mathcal{Q}$ be $\triangle_1, \triangle_2, \triangle_3, \triangle_4$. Prove that: the 27 common intersection points of Λ_i defined by Theorem 10.4.5 are located on the same cubic curve.

Problem 6. Prove that the Morley triangle of $\triangle ABC$ is homothetic to the vertices of its Steiner deltoid.

Chapter 11

Basic Cubic Theory with Liang and Zelich

Definition 11.0.1. In the complex projective plane $\mathbb{CP}^2$, the set of roots $[x : y : z]$ (or the zero locus) of a degree d polynomial $F(x, y, z)$ form a degree d curve $\mathcal{V} = \mathcal{V}(F)$. We say that V is **non-singular** if every point on it has a unique tangent. Formally, this is equivalent to saying

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

never occurs.

A classical result in algebraic geometry is:

Theorem 11.0.2 (Bezout's)**.** A degree d curve intersects a degree e curve at de points in $\mathbb{CP}^2$ (up to multiplicity). (The proof is algebraic and is given in the appendix.)

11.1 The Group Law

In order to talk about the group law, it's necessary to use a theorem that is incredibly important regarding cubic curves:

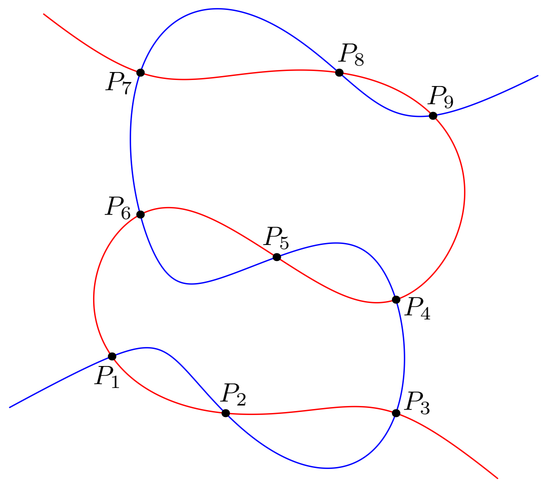
Theorem 11.1.1 (Cayley-Bacharach)**.** If two cubics $\mathcal{K}_0, \mathcal{K}_\infty$ (including factorizable cubics), intersect at points $P_1, \ldots, P_9$ and any five of $P_1, \ldots, P_8$ are not collinear, then we have that

(i) Every cubic curve $\mathcal{K}$ passes through the 8 points $P_1, \ldots, P_8$ can be written as a linear combination $\mathcal{K}_0 + t\mathcal{K}_\infty$, particularly, $P_9 \in \mathcal{K}$;

(ii) And

$$P_9(P_5, P_6; P_7, P_8) = (P_5, P_6; P_7, P_8)_{P_1P_2P_3P_4}$$

Here, we work with the linear form / equation corresponding to each cubic, and the cross ratio on the right-hand side is (TODO conic cross ratio (link revisiting cross ratio section)).



Proof. We first consider some more degenerate cases (i).

(a) First, suppose that among $P_1, \ldots, P_8$, there are four collinear points. WLOG let these points be $P_1, \ldots, P_4$ on some line ℓ. Then suppose that $\mathcal{K}_0, \mathcal{K}_\infty, \mathcal{K}$ all go through these points. Since four are collinear, we can factor out the linear form / polynomial corresponding to ℓ out of these cubics, so let $\mathcal{K}_0 = \ell \cup \mathcal{C}_0, \mathcal{K}_\infty = \ell \cup \mathcal{C}_\infty, \mathcal{K} = \ell \cup \mathcal{C}$, where $\mathcal{C}_0, \mathcal{C}_\infty, \mathcal{C}$ are cubics that pass through $P_5, \cdots, P_8$. Then since 5 points determine a conic, there exists a t such that $\mathcal{C} = \mathcal{C}_0 + t\mathcal{C}_\infty$ and thus $\mathcal{K} = \mathcal{K}_0 + t\mathcal{K}_\infty$.

(b) Likewise, suppose WLOG that we have seven points in $P_1, \ldots, P_8$ that are conconic, assume WLOG that it is $P_1, \ldots, P_7$. Then let the three cubics through these points be $\mathcal{K}_0 = \ell_0 \cup \mathcal{C}, \mathcal{K}_\infty = \ell_\infty \cup \mathcal{C}, \mathcal{K} = \ell \cup \mathcal{C}$, where $\ell_0, \ell_\infty, \ell$ are lines through P_8. Then we can find a t such that

$$\mathcal{K} = \ell \cup \mathcal{C} = (\ell_0 + t\ell_\infty) \cup \mathcal{C} = \mathcal{K}_0 + t\mathcal{K}_\infty.$$

(c) Now, suppose that some three points P_1, P_2, P_3 are collinear on some line ℓ. Then there exists a $t \neq 0$ (WLOG swapping $\mathcal{K}_0, \mathcal{K}_\infty$ if necessary) such that $\ell \subseteq \mathcal{K}_t$ or $\mathcal{K}_t = \ell \cup (P_4P_5P_6P_7P_8)$. Similarly, there exists a $t' \neq 0$ such that $\mathcal{K}_0 + t'\mathcal{K} = \mathcal{K}_t$, so we can express $\mathcal{K}$ as an appropriate linear combination.

(d) If some six points $P_1, \ldots, P_6$ are conconic, then we can similarly find $t, t' \neq 0$ such that $\mathcal{K}_t = \mathcal{K}_0 + t'\mathcal{K} = \mathcal{C} \cup P_7P_8$, and we finish as before.

Now suppose $P_1, \ldots, P_8$ are in general position, and thus not in the above cases. Suppose that there exists a $\mathcal{K}$ through these points such that $\mathcal{K}$ is not of the form $\mathcal{K}_0 + t\mathcal{K}_\infty$. Then we can choose s, t such that $\mathcal{K}' = s\mathcal{K} + \mathcal{K}_0 + t\mathcal{K}_\infty$ and $\overline{P_1P_2} \subseteq \mathcal{K}'$. This implies then that $\mathcal{K}' = \overline{P_1P_2} \cup \mathcal{C}$, but we know that $P_3, \ldots, P_8$ are not conconic, giving a contradiction.

We now prove (ii). We first find another characterization for a pencil of conics. Take an arbitrary point X. Now, let $\{\mathcal{C}_t = \mathcal{C}_0 + t\mathcal{C}_\infty\}$ be the pencil of conics through P_1, P_2, P_3, P_4 and have $\mathcal{C}_{t_j}$ pass through P_j for $j = 5, 6, 7$. Then, if we let $\{\ell_t = \ell_0 + t\ell_\infty\}$ is the pencil of lines through X, have ℓ_{t_j} pass through P_j for $j = 5, 6, 7$, with t_j being the same by scaling. Now, we may check algebraically that the locus

$$\mathcal{K}_X = \bigcup_t \mathcal{C}_t \cap \ell_t$$

is a cubic through $P_1, P_2, \ldots, P_7$ and X. Fix the conic

$$\mathcal{C} = \{X \mid X(P_5, P_6; P_7, P_8) = (P_5, P_6; P_7, P_8)_{P_1P_2P_3P_4}\}$$

through P_5, P_6, P_7, P_8. Now if X lies on $\mathcal{C}$ then taking t_8 satisfying

$$(t_5, t_6; t_7, t_8) = X(P_5, P_6; P_7, P_8) = (P_5, P_6; P_7, P_8)_{P_1P_2P_3P_4}$$

gives that $P_8 \in \mathcal{K}_X$ by cross ratios. Varying X along $\mathcal{C}$ gives a family of conics $\{\mathcal{K}_X\}_{X \in \mathcal{C}}$ through $P_1, P_2, \ldots P_8$.

We now show the converse, or that any conic through $P_1, P_2, \ldots, P_8$ in fact lies in this family for some $X \in \mathcal{C}$. Let Q_j be the sixth intersection of $\mathcal{C}_j = (P_1P_2P_3P_4P_j)$ with $\mathcal{K}$, and X_j the third intersection of $\ell_j = P_jQ_j$ with $\mathcal{K}$. Now, for distinct $5 \le j \le 8$, we have that the cubics $\ell_j \cup \mathcal{C}_k$ and $\ell_k \cup \mathcal{C}_j$ intersect at $P_1, P_2, \ldots, P_4, P_j, P_k, Q_j, Q_k, \ell_j \cap \ell_k$. By part (1), this means that $\ell_j \cap \ell_k \in \mathcal{K} \implies X_j = X_k$, so let $X_5 = X_6 = \cdots = X_8$. Now, $\mathcal{K}_X$ and $\mathcal{K}$ intersect at $P_1, P_2, \ldots, P_7, Q_5, Q_6, Q_7, X$, so they are equal by Bezout's. Now take t_8 to satisfy

$$(t_5, t_6; t_7, t_8) = (\ell_5, \ell_6; \ell_7, \ell_8)$$

Since $\ell_8 \cap \mathcal{C}_{t_8} = \ell_8 \cap \mathcal{C}_{t_8} = \{P_8, Q_8\}$ by definition, we get that $\mathcal{C}_{t_8} = \mathcal{C}_8$, so

$$(\ell_5, ell_6; \ell_7, \ell_8) = (t_5, t_6; , t_7, t_8) = (\mathcal{C}_5, \mathcal{C}_6; \mathcal{C}_7, \mathcal{C}_8)$$

and $X \in \mathcal{C}$.

Now take $X_0, X_\infty \in \mathcal{C}$ such that $\mathcal{K}_0 = \mathcal{K}_{X_0}, \mathcal{K}_\infty = \mathcal{K}_{X_\infty}$. Now, if we define $\mathcal{C}'$ in the same way as $\mathcal{C}$ with P_8 swapped for P_9, then we obtain in the identical way a family $\{\mathcal{K}_X\}_{X \in \mathcal{C}'}$ through $P_1, P_2, \ldots, P_7, P_9$, so

$X_0, X_\infty \in \mathcal{C}'$ and thus

$$\mathcal{C} = (P_5P_6P_7X_0X_\infty) = \mathcal{C}' \implies P_9 \in \mathcal{C}$$

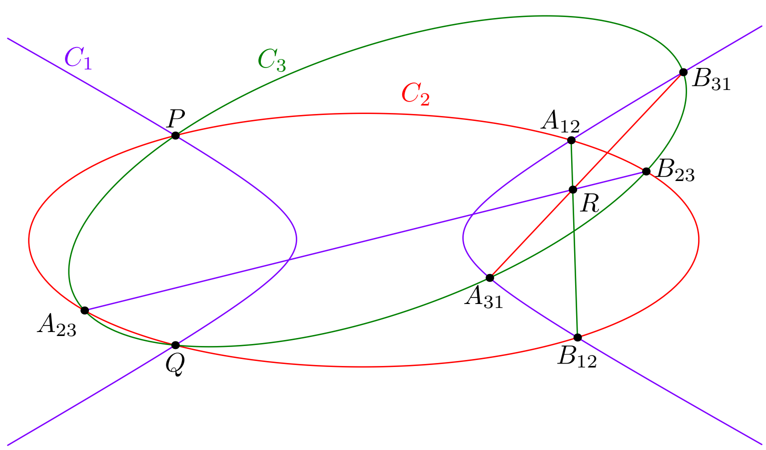
which implies the result. $\square$

Cayley-Bacharach allows us to construct more points on a cubic curve given some on known points, as 3 straight lines are a degenerate cubic. A great example of this is Miquel's Theorem:

Example 11.1.2. For a complete quadrilateral in the form $Q = (\ell_1, \ell_2, \ell_3, \ell_4)$, let $A_{ij} = \ell_i \cap \ell_j$ and let I, J be the circle points and let $\mathcal{K}_i = \ell_i \cup (A_{kl}A_{lj}A_{jk}IJ)$. Then K_i must pass through the 8 points $A_{ij}, \infty_{\pm i}$. So $\mathcal{K}_i$ must all pass through a common ninth point we call this point M. Notice that $(A_{kl}A_{lj}A_{jk}IJ)$ already intersects each of ℓ_j, ℓ_k, ℓ_l at two points, so the circumcircles of the four triangles in Q all intersect at M.

Another corollary of Cayley-Bacharach is the following:

Theorem 11.1.3 (Three Conics theorem)**.** Let three non-degenerate conics (degree-2 curves) C_1, C_2, C_3 share exactly two points P, Q. If C_i and C_j intersect at two other points A_{ij}, B_{ij} then $\overline{A_{23}B_{23}}, \overline{A_{31}B_{31}}, \overline{A_{12}B_{12}}$ concur.



Proof. Let $R = \overline{A_{31}B_{31}} \cap \overline{A_{12}B_{12}}$. Note that the cubics $C_2 \cup \overline{A_{31}B_{31}}$ and $C_3 \cup \overline{A_{12}B_{12}}$ pass through the points $A_{23}, A_{31}, A_{12}, B_{23}, B_{31}, B_{12}, P, Q, R$. Since $C_1 \cup \overline{A_{23}B_{23}}$ passes through all of those points but R, by Cayley-Bacharach, we get that $R \in C_1 \cup \overline{A_{23}B_{23}}$. If $R \in C_1$, then it follows that $R \in \{A_{31}.B_{31}\} \cap \{A_{12}, B_{12}\}$ we get that C_1, C_2, C_3 intersect at a point R separate from P, Q contradicting our claim, thus, $R \in \overline{A_{23}B_{23}}$. $\square$

Of course, this proposition can be proved with just the cross ratio, or taking a homography sending P, Q to the circle points (which turns it into the existence of a radical center).

Definition 11.1.4. Given a non-singular cubic curve $\mathcal{K}$ and a point O that lies on it, we define the **addition** of two points P, Q on $\mathcal{K}$ as follows: Let X be the third intersection of $\overline{PQ}$ and $\mathcal{K}$. We define $P + Q$ as the third intersection of $\overline{OX}$ and $\mathcal{K}$. (A line through two of the same points is a tangent, and tangencies are counted with multiplicity).

We now show this induces a group $(\mathcal{K}, O)$: The identity element of this addition is O and commutativity holds immediately. Let S be the third intersection of the tangent to $\mathcal{K}$ at O with $\mathcal{K}$, (note that because the line is a tangent, O is counted twice). Then the additive inverse of a point $P \in \mathcal{K}$ is the third intersection of $\overline{PS}$ with $\mathcal{K}$.

We will now prove associativity. Take $P, Q, R \in \mathcal{K}$, and let X, Y, Z be the third intersections of $\overline{PQ}, \overline{QR}, \overline{(P+Q)R}$ with $\mathcal{K}$ respectively. Then, $O, X, P+Q$; $O, Y, Q+R$; and $O, Z, (P+Q)+R$ are respectively collinear. Consider the nine intersection points of $PQ \cup OY \cup (P+Q)R$ with $\mathcal{K}$. They are

$$P, Q, X, O, Y, Q+R, P+Q, R, Z$$

Notice that $OR \cup OX \cup \overline{(Q+R)P}$ passes through eight of the points (except Z), so by Cayley-Bacharach, it must go through the ninth. In other words, Z is the third intersection of $\overline{(Q+R)P}$ and $\mathcal{K}$. So $P+(Q+R)$ is the third intersection of OZ and $\mathcal{K}$. Thus $P+(Q+R) = (P+Q)+R$.

Note that it is necessary to assume that $\mathcal{K}$ is non-singular to guarantee the existence and uniqueness of the tangent. Equivalently, we may only perform group law on the non-singular part of $\mathcal{K}$, which we denote as $\mathcal{K}^\circ$.

Proposition 11.1.5. For the group $(\mathcal{K}, O)$, let L be the third intersection of the tangent to $\mathcal{K}$ at O with $\mathcal{K}$. Then three points $P, Q, R \in \mathcal{K}$ are collinear iff. $P+Q+R = L$

Proof. If P, Q, R are collinear, then $P+Q$ is the third intersection of $\overline{OR}$ with $\mathcal{K}$. Thus $(P+Q)+R$ is the third intersection of OO (the tangent to $\mathcal{K}$ at O) with $\mathcal{K}$, or $(P+Q)+R = L$.

Now, if we assume $P+Q+R = L$, then $P+Q = L-R$ is the third intersection of OR with $\mathcal{K}$, thus P, Q, R are collinear. □

Proposition 11.1.6. For any six points $P_1, P_2, P_3, P_4, P_5, P_6$ lying on a cubic curve $\mathcal{K}$, they are conconic if and only if

$$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 2 \cdot L$$

where L is the third intersection of the tangent to $\mathcal{K}$ at O.

Proof. Let Q_1, Q_2, Q_3 be the third intersections of P_1P_4, P_2P_5, P_3P_6 with $\mathcal{K}$. Thus by (TODO Cayley-Bacharach (TODO 11.1.1)), $P_1, P_2, P_3, P_4, P_5, P_6$ lie on a common conic section if and only if Q_1, Q_2, Q_3 are collinear. Because

$$P_1 + P_4 + Q_1 = P_2 + P_5 + Q_2 = P_3 + P_6 + Q_3 = L,$$

the condition Q_1, Q_2, Q_3 is equivalent to

$$\begin{aligned} Q_1 + Q_2 + Q_3 &= 3 \cdot L - (P_1 + P_2 + P_3 + P_4 + P_5 + P_6) \\ &= L \iff P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 2 \cdot L \end{aligned}$$

□

Proposition 11.1.7. For any point A on the cubic curve $\mathcal{K}$, define $i_A : \mathcal{K} \to \mathcal{K}$ be the map that sends any P to the third intersection of AP and $\mathcal{K}$. Then for any point $B \in \mathcal{K}$ satisfying $2 \cdot A = 2 \cdot B$ (the tangent at A intersects the tangent at B on the cubic), we get

(a) For $P, Q \in \mathcal{K}$ such that B, P, Q are collinear, $B, i_A(P), i_A(Q)$ are also collinear. Then we obtain the mapping

$$\begin{aligned} \mathbf{T}B &\xrightarrow{\varphi} \mathbf{T}B \\ BP &\longmapsto Bi_A(P) \end{aligned}$$

(b) φ is a projective involution.

Proof. We first prove (a), since $B + P + Q = L$, it follows that

$$\begin{aligned} B + i_A(P) + i_A(Q) &= B + (L - A - P) + (L - A - Q) \\ &= 2L + (2B - 2A) - (B + P + Q) = L \end{aligned}$$

Thus $B, i_A(P), i_A(Q)$ are collinear. For (b), for two points $P, Q \in \mathcal{K}$, we have

$$P + i_A(P) + Q + i_A(Q) + 2 \cdot B = 2 \cdot (L - A) + 2 \cdot A = 2 \cdot L,$$

Therefore the conic $\mathcal{C} = (Pi_A(P)Qi_A(Q)B)$ has a common tangent with $\mathcal{K}$ at B. Let X be the second intersection of line BA with $\mathcal{C}$. Then, since involutions are conics are second intersection maps, we get a projective involution on $\mathcal{C}$ such that $(B, X), (P, i_A(P)), (Q, i_A(Q))$ are pairs under this involution. This tells us that there exists a projective involution on the pencil of lines $\mathbf{T}B$ such that $(\mathbf{T}_B\mathcal{C} = \mathbf{T}_B\mathcal{K}, BA), (BP, Bi_A(P) = \varphi(BP)), (BQ, Bi_A(Q) = \varphi(BQ))$ are involutive pairs. Since these pairs hold for all Q, φ must be a projective involution. □

Now let's address the isoptic cubic (see: (TODO 9.2.4)) as an example. Let M, τ, φ denote the Miquel point, the Newton line, and Clawson-Schmidt conjugation on $\mathcal{K}$. Assume that Clawson-Schmidt conjugation has no fixed points (or else at the fixed points, there are two tangents to $\mathcal{K}$ at the fixed points). Let M be the zero element O of the group law, and let's see what happens.

For a pair of isogonal conjugates (P, P^*) on the isoptic cubic, we know that by DDIT, $P\infty_\tau \cap P^*M \in \mathcal{K}$, so we have $P + \infty_\tau = P^* = \varphi(P)$. In other words, on $\mathcal{K}$, φ is just equivalent to adding ∞_τ. Since φ is an involution, we have $2 \cdot \infty_\tau = O$.

Let $I, J \in \mathcal{K}$ be the two circle points. Let ∞_τ be the third intersection of IJ with $\mathcal{K}$. Since $I + J$ is the vanishing point T of $\mathcal{K}$(see (TODO 9.2.24)), we have that for four points P_1, P_2, P_3, P_4 on $\mathcal{K}$, they are cyclic if and only if

$$P_1 + P_2 + P_3 + P_4 + T = 2 \cdot S.$$

Because M under Clawson-Schmidt conjugation is sent to ∞_τ (we pick this specific infinity point by the tangent), we have $\varphi(S) = T$, and thus $2 \cdot S = 2 \cdot (T + \infty_\tau) = 2 \cdot T$. In other words, we can simplify the above expression as four points are cyclic if and only if

$$P_1 + P_2 + P_3 + P_4 = I + J = T.$$

A hard to prove but very essential theorem on cubics:

Theorem 11.1.8. If a non-singular cubic $\mathcal{K}$ is defined on $\mathbb{C}$, there exists a complex number $\tau, \Im\tau > 0$, we have (as Riemann surfaces and groups),

$$(\mathcal{K}, O) \cong {}^{\mathbb{C}}\!\big/_{\mathbb{Z}\tau + \mathbb{Z}}.$$

This belongs properly in Riemann surface theory as a consequence of the uniformization theorem, so we will not prove it here. However, we can use this to get a corollary.

Corollary 11.1.9. In a group law, define a **m-torsion** point as a point P such that $\underbrace{P + P + \cdots + P}_{m \text{ times}} = O$.

Then, we have

$$(\mathcal{K}, O)[m] := \{P \in \mathcal{K} \mid \underbrace{P + P + \cdots + P}_{m \text{ times}} = O\} \cong (\mathbb{Z}/m\mathbb{Z})^2.$$

11.1.1 Rematch with the Dragon

Let $\mathcal{Q} = (\ell_1, \ell_2, \ell_3, \ell_4)$ be a complete quadrilateral. Our goal is to address the isoptic cubic $\mathcal{K}$ (TODO 9.2) through these tools we developed on cubics. As before, define the notation $A = A_{23}, B = A_{31}, C = A_{12}, A^* = A_{14}, B^* = A_{24}, C^* = A_{34}$ such that $\triangle A^*B^*C^*$ are collinear and "cut" $\triangle ABC$.

This is completely projective, so we can use this to prove some old theorems. For example, (TODO 1.3.14):

Proposition 11.1.10. For a point P not on the line at infinity, $P \in \mathcal{K}$ if and only if

$$\measuredangle PB + \measuredangle PB^* = \measuredangle PC + \measuredangle PC^*.$$

Proof. First, by DDIT, we have that $\measuredangle PB + \measuredangle PB^* = \measuredangle PC + \measuredangle PC^*$ if and only if $\{P(A_{ij}, A_{kl})\}$, $P(I, J)$ all swap under a pencil involution.

Let P^* be the isogonal conjugate of P in $\triangle ABC$, and let $Q = PJ \cap P^*I$. If $P \in \mathcal{K}$, then by ($TODO$7.4.6) we have that P, Q, I lie on a common circumconic of $\triangle ABC$. But ($TODO$7.4.6) also tells us that on $\triangle PQI$, we have

$$A \times A^* = B \times B^* = C \times C^*.$$

In other words, $P(A_{ij}, A_{kl}), P(I, J) = P(I, Q)$ define a projective involution on $\mathbf{T}P$.

For the other direction, if $P(A_{ij}, A_{kl})$, $P(I, J)$ define a projective involution on $\mathbf{T}P$, consider isogonal conjugation on $\triangle PQI$, $\varphi = A \times A^*$. Then since A, B, C, P, Q, I are conconic, by applying (TODO 7.4.6), we get that on $\triangle AB^*C^*$, $P \times P^* = I \times J$. So $P \in \mathcal{K}$. □

This is equivalent to

$$P(A_{12}, A_{13}; J, I) = P(A_{24}, A_{34}; J, I),$$

and we can also try to represent $\mathcal{K}$ as the locus of both intersection points of conics in these two pencils:

$$\mathcal{C}_t := \{P \mid P(A_{12}, A_{13}; J, I) = t\},$$

$$\mathcal{C}'_t := \{P \mid P(A_{24}, A_{34}; J, I) = t\}.$$

where t is some constant.

To get this as a cubic, note that when $t = 1$, since then the locus of these two pencils' intersection points is just $\mathcal{L}_\infty \cup A_{14}$, so $\mathcal{K}$ becomes a cubic by factoring out the line at infinity. Since $\mathcal{K}$ passes through I, J by this definition, it is also a circular cubic. (You can also prove this by Miquel's theorem and Cayley-Bacharach.)

Theorem 11.1.11 (Fundamental Theorem of the Isoptic Cubic)**.** For two pairs of isogonal conjugates $(P, P^*), (Q, Q^*)$ in quadrilateral $\mathcal{Q}$, we have

$$\mathcal{K}(\mathcal{Q}) = \mathcal{K}((PP^*)(QQ^*)).$$

Proof. First let's just set one of these pairs of isogonal conjugates to be (A, A^*). We want to prove $\mathcal{K}(\mathcal{Q}) = \mathcal{K}((AA^*)(PP^*))$. From

$$BA + BA^* = BP + BP^*,$$

we have that B lies on the isoptic cubic of $(AA^*)(PP^*)$, and thus B^* lies on it too. Similarly, C, C^* all lie on this isoptic cubic as well. This gives us that $\mathcal{K}(\mathcal{Q})$ and $\mathcal{K}((AA^*)(PP^*))$ all go through the ten points $A, A^*, B, B^*, C, C^*, P, P^*, I, J$. Thus these are the same cubic!

Note that this also gives us that $Q, Q^* \in \mathcal{K}((AA^*)(PP^*))$. So by the exact same proof we also have

$$\mathcal{K}(\mathcal{Q}) = \mathcal{K}((AA^*)(PP^*)) = \mathcal{K}((PP^*)(QQ^*)). \qquad \square$$

This also gives a new proof for the fact that (M, ∞_τ) are isogonal conjugates.

Proposition 11.1.12. Let M^* be the isogonal conjugate of the Miquel point M in $\mathcal{Q}$. Then M^* is the intersection of $\mathcal{L}_\infty$ and $\mathcal{L}_\infty^\vee = \tau$.

Proof. We do a very big tethered MMP (see 7.A). Fix $\triangle ABC$ and A^*, and animate M on $(ABCIJ)$. Let $\triangle \mathcal{L}_\infty^a \mathcal{L}_\infty^b \mathcal{L}_\infty^c$ be the anticevian triangle of $\mathcal{L}_\infty$ wrt. $\triangle ABC$, and let X, Y, Z be the intersections of AA^*, BB^*, CC^* with $\mathcal{L}_\infty^a, \mathcal{L}_\infty^b, \mathcal{L}_\infty^c$. Then X, Y, Z all lie on $\mathcal{L}_\infty^\vee$, by cross-ratio. Thus

$$M \to (CA^*IJM) \to B^* \to Y \to \mathcal{L}_\infty^\vee \to \mathcal{L}_\infty \cap \mathcal{L}_\infty^\vee$$

is a projective map (In other words, $\mathcal{L}_\infty \cap \mathcal{L}_\infty^\vee$ moves with degree 2). So by checking three cases, $M = B, C, I, J$, we have that $\mathcal{L}_\infty \cap \mathcal{L}_\infty^\vee$ is just $\infty_{CA}, \infty_{AB}, J, I$, respectively. Thus $M^* \in \infty_\tau$. $\square$

Note that from (TODO 7.4.6), on $\triangle MIJ$, we have that Clawson-Schmidt conjugation on $\mathcal{Q}$ is

$$\varphi := A \times A^* = B \times B^* = C \times C^*.$$

We discover that for a point P on the isoptic cubic, $\varphi(P)$ and the isogonal conjugate of P, P^* have the following relation:

$$\measuredangle A_{ij}\varphi(P)A_{ik} = \measuredangle PA_{kl}M + \measuredangle MA_{jl}P = \measuredangle(\ell_j, \ell_k) - \measuredangle A_{ij}PA_{ik} = \measuredangle A_{ij}P^*A_{ik},$$

where the first equality comes from the involution given by DIT where $\mathcal{L}_\infty$ cuts the quadrilaterals $(M, \varphi(P), A_{kl}, A_{ij}\varphi(P) \cap A_{kl}P)$ and $(M, \varphi(P), A_{jl}, A_{ik}\varphi(P) \cap A_{jl}P)$. The final equality is directly due to $P, I, J, PJ \cap P^*I, P^*J \cap PI$ lying on a common circumconic of $\triangle A_{jk}A_{ki}A_{ij}$.

Specifically, this tells us that for any two pairs of isogonal conjugates $(P, P^*), (Q, Q^*)$, $(PP^*)(QQ^*)$'s Clawson-Schmidt conjugation is the exact same as the Clawson-Schmidt conjugation of the original quadrilateral, and thus these two quadrilaterals have the same Miquel point! (This can also be directly proven by group-law on the common isoptic cubic.) Since these quadrilaterals and quadrilateral $(AA^*)(PP^*)$ have a common isoptic cubic, their third intersection with $\mathcal{L}_\infty$, the point at infinity, ∞_τ is the same, so they have the same Newton line as well!

Thus ∞_{PP^*}'s harmonic conjugate on $\overline{PP^*}$, $\infty^\vee_{PP^*}$ lies on $\tau = \mathcal{L}^\vee_\infty = \infty^\vee_{AA^*}\infty_\tau$.

This completely gives us the characterization of $\mathcal{K}$ given in (9.2.4)!!

Theorem 11.1.13. The isoptic cubic $\mathcal{K}$'s affine part(remove points at infinity) is just

$$\widetilde{\mathcal{K}}(\mathcal{Q}) := \left\{P \mid \tau, \mathcal{L}_\infty \text{ harmonically divide } \overline{P\varphi(P)}\right\}.$$

Proof. We already know that $\mathcal{K} \setminus \mathcal{L}_\infty$ lies in this set, and since $\widetilde{\mathcal{K}(\mathcal{Q})}$ is also a (affine) cubic, it is also the affine part of $\mathcal{K}$. $\square$

Practice Problems

Problem 1. Prove the following theorems with Cayley-Bacharach:

- Pappus and Pascal's theorems,
- Miquel's theorem,
- (Conic) Reim's theorem,
- Radical axis concurrence
- Brokard's theorem,
- Humpty point, H, B, and C concyclic.

Problem 2 (Four Conics Theorem)**.** Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be three conics. Let $A_{ij}, B_{ij}, P_{ij}, Q_{ij}$ be the four intersection points of $\mathcal{C}_i, \mathcal{C}_j$. If $P_{23}, P_{31}, P_{12}, Q_{23}, Q_{31}, Q_{12}$ are conconic, prove: $A_{23}B_{23}, A_{31}B_{31}, A_{12}B_{12}$ are concurrent.

Problem 3. Let $\mathcal{K}$ be a cubic, and let the two tangents from $A \in \mathcal{K}$ intersect $\mathcal{K}$ again at B, C. Let S be the third intersection point of BC and $\mathcal{K}$. Prove that A, S's respective tangents to $\mathcal{K}$ intersect also on $\mathcal{K}$.

Problem 4 (2024 USA TSTST Problem 8)**.** Let ABC be a scalene triangle, and let D be a point on side BC satisfying $\angle BAD = \angle DAC$. Suppose that X and Y are points inside ABC such that triangles ABX and ACY are similar and quadrilaterals $ACDX$ and $ABDY$ are cyclic. Let lines BX and CY meet at S and lines BY and CX meet at T. Prove that lines DS and AT are parallel.

11.2 The Path of Polarity

Work in homogeneous coordinates.

For a degree d curve $\mathcal{V}$ and a point $P \in \mathcal{V} = [x : y : z]$, let $\frac{\partial V}{\partial x}$ be the partial derivative of $\mathcal{V}$ wrt. x, and let $\frac{\partial V}{\partial x}(P)$ be this partial derivative evaluated at P.

Then, the tangent from P to $\mathcal{V}$ is given by the set of points $[p : q : r]$ that lie on line

$$p\frac{\partial V}{\partial x}(P) + q\frac{\partial V}{\partial y}(P) + r\frac{\partial V}{\partial z}(P) = 0.$$

This follows from applying Euler's homogeneous function theorem, since $\mathcal{V}(P) = k \cdot (p\frac{\partial V}{\partial x}(P) + q\frac{\partial V}{\partial y}(P) + r\frac{\partial V}{\partial z}(P)) = 0$.

Note that this definition of tangents from $P \in \mathcal{V}$ via partial derivatives still defines some line that doesn't pass through P for $P \notin \mathcal{V}$.

We have that for a conic $\mathcal{C}$ and a point $Q = [q_x : q_y : q_z]$, that there are two points P on the conic $\mathcal{C}$ such that

$$q_x\frac{\partial C}{\partial x}(P) + q_y\frac{\partial C}{\partial y}(P) + q_z\frac{\partial C}{\partial z}(P) = 0$$

In other words, the tangent line from P passes through Q.

Further, we know that for a conic $\mathcal{C}$, all points $P \in \mathbb{P}^2$ **not necessarily on** $\mathcal{C}$ that satisfy

$$q_x\frac{\partial C}{\partial x}(P) + q_y\frac{\partial C}{\partial y}(P) + q_z\frac{\partial C}{\partial z}(P) = 0$$

will all lie on a degree-1 curve (line), which is just the well-known polar of P in $\mathcal{C}$, since it passes through the two points on $\mathcal{C}$ which have their tangents to $\mathcal{C}$ go through P.

In reality, we can generalize this to higher degree curves.

Definition 11.2.1. Given a homogeneous degree d curve $\mathcal{V}$ and a point $Q = [q_x : q_y : q_z]$, we can define the **polar curve** of $\mathcal{V}$ wrt. Q, notated as $\partial_Q\mathcal{V}$, as the set of points $P = [p_x : p_y : p_z]$ such that

$$q_x\frac{\partial V}{\partial x}(P) + q_y\frac{\partial V}{\partial y}(P) + q_z\frac{\partial V}{\partial z}(P) = 0$$

This will give a curve of degree $d-1$, since $\frac{\partial V}{\partial x}(P)$ is a homogeneous polynomial of degree $d-1$ and q_x is just a constant.

For two distinct points P and Q, we have $\partial_P\partial_Q\mathcal{V} = \partial_Q\partial_P\mathcal{V}$, by the swapping of partial derivatives. Polar curves have a very good characterization:

Proposition 11.2.2. Given a degree d curve $\mathcal{V}$, and given any line ℓ passing through P, let $Q_1, \dots, Q_d$ be the intersections of ℓ and $\mathcal{V}$. Then for any point $M \in \ell$, $M \in \partial_P \mathcal{V}$ if and only if

(i) $M = P \in \{Q_1, \dots, Q_d\}$,

(ii) M occurs twice or more in $\{Q_1, \dots, Q_d\}$ (i.e. is a repeated root/tangency), or

(iii) $\frac{d}{PM} = \sum_{i=1}^{d} \frac{1}{Q_i M}$.

Remark. If you don't know multivariable calculus, interpret this as the degree $d-1$ curve through all (possibly complex) points on $\mathcal{V}$ such that their tangents pass through P.

Proof. If we parameterize ℓ as $\{P_t = [p+tu : q+tv : r+tw] \mid t \in \mathbb{P}^1\}$ and let t_i be the coordinate of Q_i, then

$$f(t) = F(p+tu, q+tv, r+tw) = c\prod_{i=1}^{d}(t-t_i)$$

for some constant c. Then

$$\left(u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}\right)(P_t) = \frac{\partial f}{\partial t}(t) = \sum_{i=1}^{d}\frac{f(t)}{t-t_i}.$$

Since F is homogeneous of degree d, by Euler's formula

$$\begin{aligned} d \cdot f(t) &= \left((p+tu)\frac{\partial F}{\partial x} + (q+tv)\frac{\partial F}{\partial y} + (r+tw)\frac{\partial F}{\partial z}\right)(P_t) \\ &= \left(p\frac{\partial F}{\partial x} + q\frac{\partial F}{\partial y} + r\frac{\partial F}{\partial z}\right)(P_t) + \sum_{i=1}^{d}\frac{tf(t)}{t-t_i}. \end{aligned}$$

Thus $M \in \partial_P \mathcal{V}$ if and only if

$$d \cdot f(t) = \sum_{i=1}^{d}\frac{tf(t)}{t-t_i}.$$

If $f(t) = 0$, then $t = t_i$ for some i, and hence

- When $t_i = 0$, the above equation obviously holds, i.e. (i);
- When $t_i \neq 0$, this is equivalent to $(t-t_i)^2 \mid f(t)$, i.e. (ii).

If $f(t) \neq 0$, then this is equivalent to

$$\frac{d}{PM} = \frac{d}{t} = \sum_{i=1}^{d}\frac{1}{t-t_i} = \sum_{i=1}^{d}\frac{1}{Q_i M},$$

i.e. (iii). $\square$

Remark. The most interesting use of this theorem is in the case if $\mathcal{V}$ is a cubic and P lies on $\mathcal{V}$. Then we define $\mathcal{V}$ as the locus of the harmonic conjugate of P in the other two intersections of any line through P with the cubic. In this case, we call $\mathcal{V}$ the **polar conic** of P. An important property of the polar conic is that it is tangent to the cubic at P. Note that if the cubic is singular, take a line through P passing through the singularity to get that the polar conic must also pass through the singularity.

Corollary 11.2.3. Given any projective transformation φ, for any degree d curve $\mathcal{V}$ and a point P,

$$\partial_{\varphi(P)}\varphi(\mathcal{V}) = \varphi(\partial_P \mathcal{V}).$$

In other words, the polar curve is projectively equivariant.

Proof. Cases (i) and (ii) are clearly equivariant, so we only need to show that

$$\frac{d}{PM} = \sum_{i=1}^{d} \frac{1}{Q_i M} \implies \frac{d}{\varphi(P)\varphi(M)} = \sum_{i=1}^{d} \frac{1}{\varphi(Q_i)\varphi(M)}.$$

If all the Q_i are equal to P, then the left and right equalities are always true (and $\ell \subseteq \partial_P \mathcal{V}$). Otherwise, WLOG suppose $Q_1 \neq P$. The left side can then be rewritten as

$$0 = \sum_{i=1}^{d} \frac{PQ_i}{Q_i M} = \frac{PQ_1}{Q_1 M}\left(1 + \sum_{i=2}^{d} (P, M; Q_i, Q_1)\right)$$

and then

$$\sum_{i=2}^{d} (\varphi(P), \varphi(M); \varphi(Q_i), \varphi(Q_1)) = \sum_{i=2}^{d} (P, M; Q_i, Q_1) = -1.$$

Symmetrically, this can be rewritten as

$$\frac{d}{\varphi(P)\varphi(M)} = \sum_{i=1}^{d} \frac{1}{\varphi(Q_i)\varphi(M)}.$$

□

Through (11.2.2), we find that $\partial_P \mathcal{V}$ in the case of $d = 2$ is the polar of P with respect to the conic $\mathcal{V}$. Note that when $d > 2$, we can take a polar more than once.

Definition 11.2.4. The ***k*th polar** of a degree d curve $\mathcal{V}$ wrt. a point P is defined as

$$\mathfrak{p}_{\mathcal{V}}^{k}(P) := \partial_P^{d-k} \mathcal{V} = \underbrace{\partial_P \ldots \partial_P}_{d-k \text{ times}} \mathcal{V}.$$

Remark. Important: Note that the algebraic degree of the kth polar of a point in a degree d curve is k.

Proposition 11.2.5 (Self-Conjugate Points are Self-Conjugate in Any Degree)**.** For a degree d curve $\mathcal{V}$ and a point P, the following statements are equivalent:

- $P \in \mathfrak{p}_{\mathcal{V}}^k(P)$ for one k of $0 \le k \le d$;
- $P \in \mathfrak{p}_{\mathcal{V}}^k(P)$ for all of $0 \le k \le d$.

(Note how this gives you traditional properties of polars in conics.)

Proof. We only need to prove that $P \in \mathfrak{p}_{\mathcal{V}}^k(P)$ if and only if $P \in \mathfrak{p}_{\mathcal{V}}^{k-1}(P)$ and then we can win by induction. Let $F^{(k)}$ be defined as the expression of $\mathfrak{p}_{\mathcal{V}}^k(P)$ if we set $P = [p : q : r]$. Then the homogenized degree of $F^{(k)}$ is just k, so

$$\begin{aligned} F^{(k-1)}(P) &= ()\,(P) \\ &= ()\,(P) = (k \cdot F^{(k)})(P). \end{aligned}$$

Thus $P \in \mathfrak{p}_{\mathcal{V}}^k(P)$ if and only if $P \in \mathfrak{p}_{\mathcal{V}}^{k-1}(P)$. $\square$

Since the zero locus of a degree-0 polynomial F is either the empty set $\varnothing$ (when $F \neq 0$) or the whole plane $\mathbb{P}^2$ (when $F = 0$), thus:

Corollary 11.2.6. For a degree d curve $\mathcal{V}$ and a point P,

$$\mathfrak{p}_{\mathcal{V}}^d = \begin{cases} \varnothing & \text{if } P \notin \mathcal{V}, \\ \mathbb{P}^2 & \text{if } P \in \mathcal{V}. \end{cases}$$

Because there are only two cases, we can use $\neq 0$ to represent the case where it is $\varnothing$ and $= 0$ for the case where it is $\mathbb{P}^2$. So we may write

$$\mathfrak{p}_{\mathcal{V}}^d(P) = 0 \quad \Longleftrightarrow \quad P \in \mathcal{V}.$$

Finally, we arrive at

Corollary 11.2.7 (Polar duality; generalization of La Hire)**.** For a degree d curve $\mathcal{V}$, any two points P, Q and an integer $0 \le k \le d$,

$$P \in \mathfrak{p}_{\mathcal{V}}^k(Q) \quad \Longleftrightarrow \quad Q \in \mathfrak{p}_{\mathcal{V}}^{d-k}(P)$$

Proof. The left side is equal to $\partial_P^k \partial_Q^k \mathcal{V} = 0$. Since we are working with continuous polynomials, by Clairaut's theorem, we can swap the partials and this is equivalent to $\partial_P^{d-k} \partial_Q^k(\mathcal{V}) = 0$. $\square$

In the special case for cubics when $n = 3$, if a line ℓ intersects a cubic curve $\mathcal{K}$ at three points P, Q, R, then ℓ and $\mathfrak{p}^1_{\mathcal{K}}(P)$ meet again at M (distinct from P), which satisfies

$$\begin{aligned}\frac{3}{PM} = \frac{1}{PM} + \frac{1}{QM} + \frac{1}{RM} \quad &\iff \quad \frac{2}{PM} = \frac{1}{QM} + \frac{1}{RM} \\ &\iff \quad (P, M; Q, R) = -1 \\ &\iff \quad \frac{2}{QR} = \frac{1}{PR} + \frac{1}{MR}\end{aligned}$$

Thus, $R \in \partial_Q \partial_P \mathcal{K}$. Equivalently, we obtain:

Proposition 11.2.8. Three collinear points P, Q, R on a cubic curve $\mathcal{K}$ satisfy

$$\partial_P \partial_Q \partial_R \mathcal{K} = 0.$$

Definition 11.2.9. Given a cubic curve $\mathcal{K}$. For a moving point P on some line ℓ,

- P's polar conic $\mathfrak{p}^2_{\mathcal{K}}(P)$ will pass through four fixed points S_1, S_2, S_3, S_4. The set of points $\{S_1, S_2, S_3, S_4\}$ will be called the **pole** of ℓ w.r.t $\mathcal{K}$, denoted $\mathfrak{p}_0\mathcal{K}(\ell)$.
- The envelope of P's polar line $\mathfrak{p}^2_{\mathcal{K}}(P)$ is a conic. We call this the **poloconic** of ℓ w.r.t $\mathcal{K}$, denoted $\mathfrak{p}_2\mathcal{K}(\ell)$. (Do not confuse this with polar conics of a point.)

Of course, we must have by extended La Hire's that

Proposition 11.2.10. Given a cubic curve $\mathcal{K}$, then for any line ℓ and any point P,

$$P \in \mathfrak{p}^0_{\mathcal{K}}(\ell) \quad \iff \quad \ell = \mathfrak{p}^1_{\mathcal{K}}(P)$$

Proposition 11.2.11. Given a cubic $\mathcal{K}$ and an arbitrary line ℓ, then there exists an isoconjugation φ on some triangle $\triangle ABC$ such that the set of polar conics of points P on ℓ, $\{\mathfrak{p}^2_{\mathcal{K}}(P) \mid P \in \ell\}$, is the pencil of diagonal conics of φ. Also $\mathfrak{p}^2_{\mathcal{K}}(\ell) = \varphi(\ell)$.

Proposition 11.2.12. Given a cubic $\mathcal{K}$ and an arbitrary line ℓ, then $\mathfrak{p}^2_{\mathcal{K}}(\ell)$ is the nine-point conic of K wrt. $\mathfrak{p}^0_{\mathcal{K}}(\ell)$.

11.2.1 Triangles as Degenerate Cubics

Let $\triangle = \triangle ABC$. Then we can also view $\triangle$ as a cubic curve in the barycentric coordinate system, i.e., $xyz = 0$.

Proposition 11.2.13. For any point P:

(i) $\mathfrak{p}^1_{\triangle}(P)$ is the trilinear polar of P wrt. $\triangle$, $\mathfrak{t}(P)$;

(ii) $\mathfrak{p}^2_{\triangle}(P)$ is the circumconic of $\triangle$ with perspector P, $\mathfrak{c}(P)$.

Proof. (i) Let $P_a = AP \cap BC$; let Q be the intersection of AP with the polar line $\mathfrak{p}^1_{\triangle}(P)$. Then Q divides Q and $\mathfrak{p}^2_{\triangle}(P)$ into thirds. Therefore,

$$\frac{3}{QP} = \frac{2}{AP} + \frac{1}{P_aP} \quad \Longrightarrow \quad 2 \cdot \frac{QP}{AP} + \frac{QP}{P_aP} = 3 \quad \Longrightarrow \quad 2 \cdot \frac{QA}{AP} + \frac{QP_a}{P_aP} = 0 \implies (Q, P; P_a, A) = -2.$$

Let $P_b = BP \cap CA$, and let $P_a^\vee, P_b^\vee$ be the intersections of $\mathfrak{t}(P)$ with BC and CA, respectively. Then we have:

$$(AP \cap \mathfrak{t}(P), P; P_a, A) = P_a^\vee(P_b^\vee, P_b; C, A), \cdot P_b(P_a^\vee, P; P_a, A) \qquad = -1 \cdot (1 - (P_a^\vee, P_a; B, C)) = -2.$$

Thus, by symmetry, $\mathfrak{p}^1_{\triangle}(P) = \mathfrak{t}(P)$.

(ii) Let M be the second intersection of AP with $\mathfrak{p}^2_{\triangle}(P)$. Similar to the previous calculation, we have:

$$\frac{3}{PM} = \frac{2}{AM} + \frac{1}{P_aM} \quad \Longrightarrow \quad (P, M; P_a, A) = -2.$$

Let P_A be the second intersection of AP with $\mathfrak{c}(P)$, and let P^a be the intersection of the tangents to $\mathfrak{c}(P)$ at B and C (in other words, the vertex opposite to A in the anticevian triangle of P). Then we also have:

$$(P, P_A; P_a, A) = (P, P^a; P_a, A) \cdot (P^a, P_A; P_a, A) \qquad = -1 \cdot (1 - (P^a, P_a; P_A, A)) = -2.$$

$\square$

Thus, by symmetry, $\mathfrak{p}^2_{\triangle}(P) = \mathfrak{c}(P)$.

From this, we can rewrite (12.3.23) as:

Proposition 11.2.14. Let $S = P * Q$ be the cevapoint of P and Q. Then the following holds:

$$\partial_P \partial_Q \triangle = \partial_S^2 \triangle.$$

11.2.2 The Dragon's Defeat

Let $\mathcal{K}$ be an isoptic cubic. We know that $\mathcal{K}$ is also a circular cubic. For a point $P \in \mathcal{K}$, denote P^* as the isogonal conjugate of P (also in $\mathcal{K}$).

Definition 11.2.15. For a point P, we define $\triangle_P$ as the double derivative operator $\partial_P\partial_{P^*}$, and define the P-vanishing point V_P as the third intersection of PP^* with $\mathcal{K}$.

Since the isogonal conjugate of the Miquel point is the point at infinity along the Newton line, we have $V_M = V_{\infty_\tau}$. The M-vanishing point is just the well-known vanishing point $V_\mathcal{K}$ of $\mathcal{K}$, defined in chapter 9.

Proposition 11.2.16. For any two pairs of isogonal conjugates $(Q, Q^*), (R, R^*)$, let $P = QR^* \cap Q^*R, P^* = QR \cap Q^*R^*$. Then $\triangle_P\mathcal{K}, QQ^*, RR^*$ are concurrent.

Proof. Let $\mathcal{C}_P = \partial_P\mathcal{K}, S = \mathfrak{p}_{\mathcal{C}_P}(Q^*R^*), S^* = \mathfrak{p}_{\mathcal{C}_P}(QR)$. From $(TODO 11.2.8)$, we have that Q, R^* are conjugates in conic $\mathcal{C}_P$, and thus we have $\mathfrak{p}_{\mathcal{C}_P}(Q) = R^*S^*$. By the same logic we have $\mathfrak{p}_{\mathcal{C}_P}(R) = S^*Q^*, \mathfrak{p}_{\mathcal{C}_P}(Q^*) = RS, \mathfrak{p}_{\mathcal{C}_P}(R^*) = SQ$. Thus the polar of $\triangle QRS$ is just $\triangle Q^*R^*S^*$. Combining this with Problem 2 in (TODO 7.1) (or just DDIT) we have that QQ^*, RR^*, SS^* are concurrent. Further we know that

$$\mathfrak{p}_{\mathcal{C}_P}(P^*) = \mathfrak{p}_{\mathcal{C}_P}(QR \cap Q^*R^*) = SS^*.$$

□

Corollary 11.2.17. For a point $P \in \mathcal{K}\backslash I, J$, we have $\triangle_P\mathcal{K} = V_P\infty_{\perp PP^*}$. For I, J, we have that $\triangle_{I,J}\mathcal{K} = \tau$.

Proof. From $(TODO 11.2.8)$, V_P already lies on $\triangle_P\mathcal{K}$. Therefore we only need to prove $S := QQ^* \cap RR^*$'s foot (call it V) onto PP^* is just V_P. Since $(Q, Q^*; S, QQ^* \cap PP^*) = -1$ and $SV \perp PP^*$, we have

$$\measuredangle(VQ) + \measuredangle(VQ^*) = 2 \cdot \measuredangle(PP^*) = \measuredangle(VP) + \measuredangle(VP^*),$$

so V has to lie on $\mathcal{K}$. For the two circle points, we can just prove for all Q, $S = QQ^* \cap RR^*$ lies on τ. This is trivial, since from

$$(Q, Q^*; IJ \cap QQ^*, S) = -1,$$

S is the midpoint of QQ^*, and thus lies on τ. □

Corollary 11.2.18. Borrowing notation from $(TODO 11.2.16)$, we have that $\triangle_P\mathcal{K}, \triangle_Q\mathcal{K}, \triangle_R\mathcal{K}$ concur at the orthocenter of the diagonal triangle δ_Q of the complete quadrilateral $\mathcal{Q} = \triangle PQR \cup P^*Q^*R^*$.

If the quadrilateral isn't tangential, then the isoptic cubic isn't singular (otherwise it has a double-point at the incenter).

Proposition 11.2.19. Let P be a moving point on $\mathcal{K}$, then $V_P + 2P$ (addition in the group law of $\mathcal{K}$) is constant. Thus,

$$\begin{aligned}(\mathcal{K}, M) &\longrightarrow (\mathcal{K}, V_\mathcal{K})\\ P &\longmapsto V_P\end{aligned}$$

is a group homomorphism.

Proof. Since for any two pairs of isogonal conjugates $(P, P^*), (Q, Q^*)$ we have that $P + Q^* = P^* + Q$, so

$$\begin{aligned} V_P + 2P &= (V_P + P + P^*) + (P - P^*) \\ &= (V_Q + Q + Q^*) + (Q - Q^*) = V_Q + 2Q. \end{aligned}$$

□

This tells us that for any point V_0 on the isoptic cubic, there exists $4 = 2^2$ points P_1, P_1^*, P_2, P_2^* such that $V_0 = V_{P_1} = V_{P_2}$. In reality, they are just the intersections of the lines ℓ_1, ℓ_2 through V_0 such that

$$2\measuredangle(\ell_1) = 2\measuredangle(\ell_2) = \measuredangle(V_0M) + \measuredangle(\tau)$$

with the isoptic cubic. Since $\ell_1 = P_1P_1^* \perp \ell_2 = P_2P_2^*$, we have

Proposition 11.2.20. The line $\triangle_{P_1}\mathcal{K}$'s three intersection points with the isoptic cubic is just V_0, P_2, P_2^*.

Proposition 11.2.21. Let V_0's isogonal conjugate be V_0^*, then the poloconic $\mathfrak{p}^2_{\mathcal{K}}(V_0^*)$ of V_0^* wrt. $\mathcal{K}$ is just $(V_0^*PP^*QQ^*)$.

Proof. We want to prove $\mathfrak{p}^2_{\mathcal{K}}(V_0^*) = (V_0^*PP^*QQ^*)$, from symmetry we only need to prove $P \in \mathfrak{p}^2_{\mathcal{K}}(V_0^*)$. By duality, we have that this is equivalent to $V_0^* = V_P^* \in \mathfrak{p}^1_{\mathcal{K}}(P)$. Since P, P^*, V_P are collinear and

$$P + P + V_P^* = P + P^* + V_P,$$

V_P^* lies on P's tangent to $\mathcal{K}$, $\mathfrak{p}^1_{\mathcal{K}}(P)$. □

Proposition 11.2.22. For a point $V_0 \in \mathcal{K}$, let S_1, S_2 respectively be the second intersections of the previously defined ℓ_1, ℓ_2 with the polar conic $\partial_{V_0}\mathcal{K}$. Then $\triangle_{V_0}\mathcal{K} = S_1S_2$. Let N be the pole of $V_0^*\infty_{\perp V_0V_0^*}$ wrt. $\mathfrak{p}^2_{\mathcal{K}}(V_0)$, (which is also the midpoint of $\overline{S_1S_2}$). Then V_0N is perpendicular to $\mathfrak{p}^1_{\mathcal{K}}(V_0)$.

Proof. Let P_i, P_i^* be the two other intersections of $\mathcal{K}$ with V_0S_i. Then $V_0^*P_i, V_0^*P_i^*$ and $\mathcal{K}$'s third intersection points with $\mathcal{K}$ are P_i, P_i^* (in other words, $V_0^*P_i$ and $V_0^*P_i^*$ are tangent to $\mathcal{K}$). Therefore combining

$$(P_i, P_i^*; V_0, S_i) = -1$$

and (TODO 11.2.16) tells us that $S_i \in \triangle_{V_0}\mathcal{K}$. □

Let's look at polar conics again. Consider the involution φ on $\mathcal{C}_{V_0} = \partial_{V_0}\mathcal{K}$. We have that $\varphi(U) = \infty_{\perp V_0 U} \cap \mathcal{C}_{V_0}$. Our goal is to prove that N is the center of (the projection from the conic to itself corresponding) to the involution, i.e. $N = \bigcap_U U\varphi(U)$. Note that the second intersections R_+, R_- of V_0I, V_0J (circle points) with $\mathcal{C}_{V_0}$ are the fixed points of the involution φ. Thus, we get that $N = \mathfrak{p}_{\mathcal{C}_{V_0}}(V_0^*\infty_{\perp V_0 V_0^*})$ must be the intersection of the tangent lines from R_+, R_- to the conic $\mathcal{C}_{V_0}$. Equivalently, we get that $R_+, R_- \in V_0^*\infty_{\perp V_0 V_0^*}$. If we let $Q_+, Q_- = V_0I \cap V_0^*J, V_0J \cap V_0^*I$, then

$$(V_0, R_+; I; Q_+) = (V_0, R_-; J; Q_-) = -1$$

tells us that $R_+, R_- \in V_0^*(IJ \cap Q_+Q_-)$. Through

$$(I, J; IJ \cap V_0V_0^*; IJ \cap Q_+Q_-) = -1,$$

we get that $IJ \cap Q_+Q_- = \infty_{\perp V_0 V_0^*}$, and therefore N is the center of the involution φ. Finally, since N lies on $V_0\varphi(V_0)$, we have $V_0N = V_0\varphi(V_0)$ is perpendicular to the tangent $\partial^2_{V_0}\mathcal{K}$ of V_0 to $\mathcal{C}_{V_0}$.

Practice Problems

Problem 1.

- Let τ be the Newton line of a complete quadrilateral $\mathcal{Q}$. Prove that $\mathfrak{p}^2_{\mathcal{K}(Q)}(\infty_\tau)$ is a rectangular hyperbola.
- Prove that for all points P on the Newton line, $\mathfrak{p}^2_{\mathcal{K}(Q)}(P)$ is a rectangular hyperbola.

Problem 2. Consider the set of all points P such that their isogonal conjugate Q in $\triangle ABC$ has PQ parallel to BC. Prove that the midpoint of all P, Q in this set lies on a rectangular hyperbola, and locate its center.

Problem 3 (2009 ISL G4)**.** Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of $\overline{AB}$ and $\overline{CD}$ are G and H, respectively. Show that $\overline{EF}$ is tangent at E to the circle through the points E, G and H,

11.3 Special Isocubics

Let's address the most important cubics for all triangle geometry.

Definition 11.3.1. For a point isoconjugation φ, we call a cubic $\mathcal{K}$ an **isocubic** if $\varphi(\mathcal{K}) = \mathcal{K}$.

To prove these cubics exist, recall the barycentric definition of isoconjugations. Suppose

$$\varphi[x:y:z] = \left[\frac{p}{x} : \frac{q}{y} : \frac{r}{z}\right],$$

You (as in: Wolfram-Alpha) can prove that there are two ways for a cubic $\mathcal{K}$ to satisfy being self-isoconjugate:

$$ux(ry^2 - qz^2) + vy(pz^2 - rx^2) + wz(qx^2 - py^2) = 0 \tag{11.1}$$

$$ux(ry^2 + qz^2) + vy(pz^2 + rx^2) + wz(qx^2 + py^2) + kxyz = 0. \tag{11.2}$$

We will address only the first equation right now. Denote these cubics as $p\mathcal{K}$.

Definition 11.3.2. For $\triangle ABC$ and an isoconjugation φ, then for any point P, we can define a special isocubic with P as **pivot**, defined as

$$\mathcal{K}^p_\varphi(P) = \{Q \mid P, Q, \varphi(Q) \text{ collinear.}\}.$$

Then we call $\mathcal{K}^p_\varphi(P)$ a **pivotal** cubic.

Let $P = [u:v:w]$. Then obviously, we have:

Proposition 11.3.3. For a point P, we have that $A, B, C, P, \varphi(P)$ and the four fixed points S, S^a, S^b, S^c of the isoconjugation φ (S^a, S^b, S^c are just anticevian triangle vertices of S) all lie on $\mathcal{K}^p_\varphi$.

Definition 11.3.4. Let φ be an isoconjugation in $\triangle ABC$, let $\mathcal{E}$ be the Euler line of $\triangle ABC$, and let $G, O, N, L \in \mathcal{E}$ be the centroid, circumcenter, nine-point center, and de Longchamps point of $\triangle ABC$. We define the five cubics

$$\mathcal{K}^p_\varphi(\infty_\mathcal{E}), \mathcal{K}^p_\varphi(G), \mathcal{K}^p_\varphi(O), \mathcal{K}^p_\varphi(N), \mathcal{K}^p_\varphi(L)$$

respectively as the **Neuberg, Thomson, McCay, Napoleon-Feuerbach, Darboux** cubics. (These are K001–K005 in the Catalogue of Triangle Cubics).

(So the weird condition given by Fontené III is actually just a property of the McCay cubic.)

Proposition 11.3.5. For a cubic $\mathcal{K}$ and four points on it, A, B, C, P, if we define a group law such that $2 \cdot A = 2 \cdot B = 2 \cdot C = 2 \cdot P$, then $\mathcal{K}$ is an isopivotal cubic on $\triangle ABC$ with P as pivot.

Proof. Consider the third intersection map: the map $i_P : \mathcal{K} \to \mathcal{K}$, sending points $Q \in \mathcal{K}$ to the third intersection of PQ with $\mathcal{K}$. By (TODO 11.1.7), the transformation $\varphi_A \colon AQ \to Ai_P(Q)$ is a well-defined projective involution. (TODO 11.1.9) tells us that the 2-torsion points,

$$(\mathcal{K}, P)[2] := \{Q \in \mathcal{K} \mid 2 \cdot Q = O\} \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

so since A, B, C, P are all 2-torsion, it's isomorphic to the Klein 4-group. Thus, by properties of this group, we have

$$C + A = B + P, A + B = C + P,$$

(which means that $i_P(B), i_P(C)$ respectively lie on CA, AB.) This tells us that φ_A is a projective involution sending AB to AC. Define φ_B, φ_C as $BQ \to Bi_P(Q), CQ \to Ci_P(Q)$, then by the same logic, these are also projective involutions exchanging their respective sides. Thus we have that $i_P(Q)$ has to be an isoconjugation! Since by definition we have $P, Q, i_P(Q)$ collinear, $\mathcal{K}$ is a isopivotal cubic with pivot at P. $\square$

Now let's consider the cases in (TODO 11.3.1) where $k \neq 0$, define these cubics as $\mathcal{K}_\varphi^{n,k}(P)$. Note that we can rewrite the bary formula as

$$\left(\frac{x}{u} + \frac{y}{v} + \frac{z}{w}\right)\left(\frac{p}{ux} + \frac{q}{vy} + \frac{r}{wz}\right) = \frac{p}{u^2} + \frac{q}{v^2} + \frac{r}{w^2} - \frac{k}{uvw},$$

Writing the formula in this form, we can see the equations of $\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0$, the trilinear polar of $P = [u : v : w]$ wrt. $\triangle ABC$, and $\frac{p}{ux} + \frac{w}{vy} + \frac{r}{wz} = 0$, the formula for circumconics of $\triangle ABC$ with perspector $\varphi(P)$ $\mathfrak{c}(\varphi(P)) = \varphi(\mathfrak{t}(P))$, (i.e isoconjugate of trilinear polar) inside it. So we can actually define $\mathcal{K}_\varphi^{n,k}(P)$ as the locus of $U = [x : y : z]$ such that the barycentric equations of $\mathfrak{t}(U) \cdot \varphi(\mathfrak{t}(U))$ always multiply to a constant.

Proposition 11.3.6. For a point P and its trilinear polar $\mathfrak{t}(P)$, its intersections with the three sides at $P_a^\vee, P_b^\vee, P_c^\vee$ all lie on some $\mathcal{K}_\varphi^{n,k}$.

We call P the **root** of this cubic (note that P most likely does not lie on the cubic). In other words, the root of the cubic is the trilinear pole of the line through the 3rd intersections of the intersections of $\mathcal{K}_\varphi^{n,k}$.

Theorem 11.3.7. An arbitrary isocubic $\mathcal{K}_\varphi^{n,k}(P)$ can be considered as the locus of points Q for some circumconic $\mathcal{C} := C_\varphi^{n,k}(P)$ such that $Q, \varphi(Q)$ are conjugate in $\mathcal{C}$.

Proof. Recall from section (TODO 6.2.1) that a conic $\mathcal{C}$ has the equation $\mathbf{x}^T M_{\mathcal{C}} \mathbf{x} = 0$ for a unique symmetric matrix $M_{\mathcal{C}}$. We will also use that two points X, Y are conjugate in $\mathcal{C}$ if and only if their coordinates $\mathbf{x}, \mathbf{y}$ satisfy $\mathbf{x}^T M_{\mathcal{C}} \mathbf{y} = 0$ (see (TODO 14.2.4))

If we express the equation for $\mathcal{K}_\varphi^{n,k}$ as above and divide by xyz on both sides, we get

$$u\left(y \cdot \frac{r}{z} + z \cdot \frac{q}{y}\right) + v\left(z \cdot \frac{p}{x} + x \cdot \frac{r}{z}\right) + w\left(x \cdot \frac{q}{y} + y \cdot \frac{p}{x}\right) + k = 0.$$

Then $Q = [x : y : z]$ and $\varphi(Q) = [p/x : q/y : r/z]$ lying on $\mathcal{K}_\varphi^{n,k}$ are conjugate in a conic $\mathcal{C}$ when

$$M_{\mathcal{C}} = \begin{pmatrix} C_1 & w & v \\ w & C_2 & u \\ v & u & C_3 \end{pmatrix}$$

for C_1, C_2, C_3 such that $C_1 p + C_2 q + C_3 r = k$. $\square$

Note that given the isocubic, this $\mathcal{C}$ isn't unique, so we can actually just set $\mathcal{C}$ to be a circle Γ (possibly a imaginary circle or a line). This is also equivalent to all of the circles $(\overline{Q\varphi(Q)})$ having a common radical center (at the center of Γ, as they are all orthogonal to Γ).

So let's redefine our cubic $\mathcal{K}_\varphi^{n,k}(P)$ as $\mathcal{K}_\varphi^{n}(\Gamma)$. In other words, $\mathcal{K}_\varphi^{n,k}$ is just the locus of points such that $P, \varphi(P)$ are conjugate wrt. a fixed circle Γ.

However Γ isn't always unique, for instance if $\odot(\overline{Q\varphi(Q)})$ were all coaxial. So in reality we have

Proposition 11.3.8. $\mathfrak{p}_{\mathcal{C}}(A), \mathfrak{p}_{\mathcal{C}}(B), \mathfrak{p}_{\mathcal{C}}(C)$ respectively intersect BC, CA, AB at $P_a^\vee, P_b^\vee, P_c^\vee$.

This is because $A, B, C \in \mathcal{K}_\varphi^{n,k}$. Note that by varying the constant k in the barycentric representation of $\mathcal{K}$, we can produce a net (i.e. dimension 2 space) of isocubics passing through A, B, C, D, E, F. A easy example of these nonpivotal cubics is the isoptic cubic (on "quadrilateral" $\triangle ABC \cup \ell$, since it's invariant under isogonal conjugation but has no center. (Additionally, any circular pivotal cubic must have its pivot at the line at infinity, which is obviously impossible for the isoptic cubic). Here, D, E, F are collinear on ℓ, and we have that for two isogonal conjugates P, P^*, that the center of (PP^*) lies on the Newton line τ, so the radical axis is just $\infty_{\perp\tau}$, and $\Gamma = \tau$. Similar to the isoptic cubic, we have:

Proposition 11.3.9. Given $\triangle ABC$, an isoconjugation φ and a circle Γ, if P, Q are two points on $\mathcal{K}_\varphi^n(\Gamma)$, then $R = PQ \cap \varphi(P)\varphi(Q) \in \mathcal{K}_\varphi^n(\Gamma)$.

Proof. From (TODO 7.4.5) we know that $\varphi(R) = P\varphi(Q) \cap \varphi(P)Q$, and therefore we have the three circles $\omega_P := (P\varphi(P)), \omega_Q := (Q\varphi(Q)), \omega_R := (R\varphi(R))$ are coaxial (Steiner line), and the center of Γ lies on the radical axis of ω_P, ω_Q, so it also lies on the radical axis of ω_P, ω_R, so $R \in \mathcal{K}_\varphi^n(\Gamma)$. $\square$

When φ is isogonal conjugation, things become nicer. We first look at a elementary theorem:

Theorem 11.3.10. Given a fixed $\triangle ABC$, let $(P, P^*), (Q, Q^*)$ be a pair of isogonal conjugates in $\triangle ABC$, and let $\omega_P = (\overline{PP^*}), \omega_Q = (\overline{QQ^*})$, let Ω_P, Ω_Q respectively be the pedal circles of P, Q wrt. $\triangle ABC$. Then the radical axis of ω_P, ω_Q is the image of the radical axis of Ω_P, Ω_Q under a 2 homothety at H.

Corollary 11.3.11. Given $\triangle ABC$, and (P, P^*) and (Q, Q^*) as a pair of isogonal conjugates, let $R = PQ \cap P^*Q^*$, then P, Q, R's pedal circles wrt. $\triangle ABC$ are coaxal.

Let's reinterpret this with $\mathcal{K}_\varphi^{n,k}$.

Proposition 11.3.12. Given $\triangle ABC$ and a circle $\Gamma \not\supset \mathcal{L}_\infty$, let φ be an isoconjugation in $\triangle ABC$, let Γ' be the 1/2 homothety of Γ at H. Then $P \in \mathcal{K}_\varphi^n(\Gamma)$ if and only if P's pedal circle is orthogonal with Γ'.

However from a while ago (Chapter 8?) we actually found a way to calculate the angle between a point's pedal circle and the nine-point circle, so:

Corollary 11.3.13. For an arbitrary $\triangle ABC$ and isoconjugation φ, then for a point P, the following statements are equivalent:

- $P \in \mathcal{K}_\varphi^n((ABC))$,
- P's pedal circle wrt. $\triangle ABC$ is orthogonal to the nine-point circle of $\triangle ABC$,
- $\measuredangle(AP, BC) + \measuredangle(BP, CA) + \measuredangle(CP, AB) = 0°$.

We call this isocubic $\mathcal{K}_\varphi^n((ABC))$ the **Kjp** cubic of $\triangle ABC$ (it is K024 in the Catalogue of Triangle Cubics). The locus of points satisfying $\measuredangle(AP, BC) + \measuredangle(BP, CA) + \measuredangle(CP, AB) = 90°$ is the McCay cubic. In reality, the locus of points satisfying $\measuredangle(AP, BC) + \measuredangle(BP, CA) + \measuredangle(CP, AB) = \theta$ is a cubic $\mathcal{K}_\theta$, and we also have $\varphi(\mathcal{K}_\theta) = \mathcal{K}_{-\theta}$.

Practice Problems

Problem 1. For a given $\triangle ABC$ and a angle θ, prove that the locus of points satisfying $\measuredangle BAP + \measuredangle ACP + \measuredangle CBP = \theta$ is a cubic.

11.4 Liang-Zelich

Can this theorem really allow for interstellar travel? Let's see.

Definition 11.4.1. Given $\triangle ABC$ with orthocenter H and circumcenter O, let P, Q be a pair of isogonal conjugates. Define

$$t(P) = t(P, \triangle ABC) = \frac{TO}{TH}$$

where $T = PQ \cap OH$.

There are a bunch of points for which we cannot define t: these are exactly A, B, C, I^x, O, H, where I^x are either the incenter or an excenter. We will put these in a set $\mathcal{Z}$, and for the subsequent proposition we'll assume that P and its isogonal conjugate both do not belong in $\mathcal{Z}$. Then, we have that

Proposition 11.4.2. If Q is the isogonal conjugate of P in $\triangle ABC$, then $t(P) = t(Q)$.

If φ denotes isoconjugation in $\triangle ABC$, then for each $t = t_0$, we know that the loci of $t(P) = t_0$ is $\mathcal{K}^p_\varphi(T)$, where T is the point on the Euler line OH satisfiying $t(T) = t_0$.

Definition 11.4.3 (Generalized Euler line)**.** Given $\triangle ABC$, a point P and a constant x, define the (x, P)-**Euler line** as the line connecting $H\mathfrak{h}_{P,1/x}(O)$, where $\mathfrak{h}_{P,1/x}$ is the homothety centered at P with factor $1/x$.

Theorem 11.4.4 (Liang-Zelich)**.** Given $\triangle ABC$, a constant $t_0 \neq 0, \infty$ and any point P, let P_a, P_b, and P_c be the reflection of P about BC, CA, and AB respectively. Let O_a, O_b, and O_c be the circumcenters of $\triangle BPC$, $\triangle CPA$, and $\triangle APB$ respectively. Then the following are equivalent:

(i) $t(P) = t_0$.

(ii) $\triangle P_aP_bP_c$ dilated from P with a factor of t_0 is perspective with $\triangle ABC$.

(iii) $\triangle O_aO_bO_c$ (also known as the **Carnot triangle** of $\triangle ABC$) dilated from P with a factor of t_0^{-1} is perspective with $\triangle ABC$.

(iv) The (t_0, P)-Euler lines of $\triangle ABC, \triangle BPC, \triangle CPA, \triangle APB$ are concurrent.

When $t_0 = 0$, (i), (iii) and (iv) are equivalent. When $t_0 = \infty$, (i) and (ii) are equivalent.

Let's postpone the proof for now. We can deduce the following facts:

Proposition 11.4.5. Given $\triangle ABC$ and a point P, then for the following triangles XYZ, $t(P, \triangle XYZ) = t(P, \triangle ABC)$:

(i) P is the Carnot triangle of $\triangle ABC$.

(ii) P is the pedal triangle of $\triangle ABC$.

(iii) P is the anti-pedal triangle of $\triangle ABC$.

(iv) P is the circumcevian triangle.

Proof. We'll prove them in order:

(i) Let $t_0 = t(P, \triangle ABC)$. Recall that from (TODO 11.4.4) $\triangle O_aO_bO_c$ dilated by a factor of $1/t_0$ from P is perspective with P, so P reflected about the three sides of $\triangle O_aO_bO_c$ is just ABC. Thus the isogonal conjugate of P in $\triangle O_aO_bO_c$ is just O.

(ii) Let Q be the isogonal conjugate of P wrt. $\triangle ABC$, let $\triangle O'_aO'_bO'_c$ be the Carnot triangle of Q wrt. $\triangle ABC$, let $\triangle P_aP_bP_c$ be the pedal triangle of P wrt. $\triangle ABC$. Since $\triangle P_aP_bP_c \cup P \overset{+}{\sim} \triangle O'_aO'_bO'_c \cup O$, so from (i),

$$\begin{aligned} t(P, \triangle P_aP_bP_c) &= t(O, \triangle O'_aO'_bO'_c) = t(Q, \triangle O'_aO'_bO'_c) \\ &= t(Q, \triangle ABC) = t(P, \triangle ABC) \end{aligned}$$

(iii) Trivial from part (ii).

(iv) Let $\triangle DEF$ be the circumcevian triangle of P wrt $\triangle ABC$, let $\triangle P_aP_bP_c$ be the pedal triangle of P wrt. $\triangle ABC$, let P^* be the isogonal conjugate of P in triangle $\triangle P_aP_bP_c$, then by angle-chasing we have $\triangle DEF \cup P \overset{+}{\sim} \triangle P_aP_bP_c \cup P^*$, so from part (ii) we have

$$t(P, \triangle DEF) = t(P^*, \triangle P_aP_bP_c) = t(P, \triangle P_aP_bP_c) = t(P, \triangle ABC). \qquad \square$$

Remark. Normally people call just part (ii) the Liang-Zelich theorem, however this is actually just a small part of it. Also, for circumcevian triangle we can just invert; inversion at P preserves the value of t.

This is the main theorem, but in its proof we will use some very interesting sub-theorems. Let's see an important one:

Theorem 11.4.6 (Strong Sondat's Theorem)**.** Given a line L, for two triangles $\triangle A_1B_1C_1, \triangle A_2B_2C_2$ that don't have L as their perspectrix, the locus of P satisfying $A_2(L\cap A_1P), B_2(L\cap B_1P), C_2(L\cap C_1P)$ concurrent is the union of a circumconic of $\triangle A_1B_1C_1$ and line L.

Proof. We use the method of tethered moving points. Let $X_1 = B_1(A_2B_2 \cap L) \cap C_1(A_2C_2 \cap L)$, and cyclically define Y_1, Z_1. Since

$$B_1Y_1 \cap C_1Z_1, C_1X_1 \cap A_1Z_1, A_Y \cap BX \in L,$$

we have by converse Pascal (TODO 6.3.1) that $A_1, B_1, C_1, X_1, Y_1, Z_1$ lie on one conic. Let this conic be $\mathcal{C}_1$. Define the same things on the other triangle and get $\mathcal{C}_2, X_2, Y_2, Z_2$.

By Pascal on $A_1A_1C_1X_1B_1Z_1$, we can get that if

$$A_1^* = \mathbf{T}A_1\mathcal{C}_1 \cap B_1X_1, C_1^* = A_1C_1 \cap B_1Z_1,$$

then $\triangle A_1^*B_1C_1^*$ and $\triangle A_2B_2C_2$ have perspectrix as L.

Consider the projective map φ such that

$$A_1^* \mapsto A_2, B_1 \mapsto B_2, C_1^* \mapsto C_2.$$

From $A_1C_1^* \cap X_2C_2, A_1B_1 \cap X_2B_2 \in L$, we know that $\varphi(A_1) = X_2$, so

$$\varphi(\mathbf{T}A_1\mathcal{C}_1) = \varphi(A_1A_1^*) = A_2X_2.$$

Let P be a point on $\mathcal{C}_1$, let $Q_A = A_1P \cap L$, and similarly define Q_B, Q_C. Then

$$A_1(P, X_1; Y_1, Z_1) = A_2(Q_A, A_2; B_2, C_2).$$

Thus

$$A_2(Q_A, A_2; B_2, C_2) = B_2(Q_B, A_2; B_2, C_2) = C_2(Q_C, A_2; B_2, C_2).$$

So A_2Q_A, B_2Q_B, C_2Q_C concur at $Q \in \mathcal{C}_2$.

Now we prove the converse. Let's prove that all P satisfying the statement must lie on $\mathcal{C}_1 \cup L$. We proceed by MMP. Consider a point $P' \in AP$, and using the above definitions, define Q'_A, Q'_B, Q'_C, then $B_2Q'_B \mapsto C_2Q'_C$ is a projective map. Since $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ do not have L as their perspectrix, $B_2Q'_B \cap C_2Q'_C$ moves on a non-degenerate conic that intersects $A_2Q'_A = A_2Q_A$ at two points. These two points are formed when we choose a P' such that $A_2Q'_A, B_2Q'_B, C_2Q'_C$ are concurrent. When $P' = Q_A$, obviously $A_2Q'_A, B_2Q'_B, C_2Q'_C$ concur at Q_A, so we have checked enough cases and we have that when $P' \neq Q_A$, P' has to be P. □

When we set L as the line at infinity, this theorem just becomes

Corollary 11.4.7. For two non-homothetic triangles $\triangle A_1B_1C_1, \triangle A_2B_2C_2$, the locus of points P such that the line through A_2 parallel to A_1P, the line through B_2 parallel to B_1P, the line through C_2 parallel to C_1P concur is the union of a circumconic of $\triangle A_1B_1C_1$ and the line at infinity.

Directly this gives us:

Corollary 11.4.8 (Sondat's Theorem)**.** If $\triangle A_1B_1C_1, \triangle A_2B_2C_2$ are orthologic and perspective, then their perspector lies on the line connecting the two centers of orthology.

Proof. If $\triangle A_1B_1C_1, \triangle A_2B_2C_2$ are homothetic, then this is obvious, so let's assume they're not homothetic.

Let Q be the perspector of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$, and let P_1 be the center of orthology of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ (such that $A_1P_1 \perp B_2C_2$, etc). Similarly define P_2.

By Strong Sondat's, let $\mathcal{C}_i \cup \mathcal{L}_\infty$ be the locus of points P such that $A_{3-i}\infty_{A_iP}, B_{3-i}\infty_{B_iP}, C_{3-i}\infty_{C_iP}$ concur. Let $Q = \mathcal{C}_1 \cap \mathcal{C}_2$. Then by the construction in the proof of Strong Sondat's theorem, we get that $D := B_1\infty_{A_2B_2} \cap C_1\infty_{C_2A_2} \in \mathcal{C}_1$, and also D is the orthocenter of $\triangle P_1B_1C_1$. So from $A_1D \parallel T_{A_2}\mathcal{C}_2$ we know

$$Q(A_1, B_1; C_1, P_1) = D(A_1, B_1; C_1, P_1) = A_2(A_2, B_2; C_2, P_2) = Q(A_2, B_2; C_2, P_2).$$

Thus $QP_1 = QP_2$, or $Q \in P_1P_2$. □

Corollary 11.4.9. Let P_1, P_2 be the two centers of orthology, then P_1P_2 is perpendicular to the perspectrix of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$.

Let's return to points that lie on the pivotal isogonal cubic $\mathcal{K}_{t_0} := \mathcal{K}_\varphi(T)$. From (TODO 11.3.3) we have that $T, \varphi(T)$, the incenter and the three excenters all lie on this pivotal isogonal cubic. However, we can get some more free points, by the Liang-Zelich theorem!

Proposition 11.4.10. Given $\triangle ABC$ and a constant t_0, let $\triangle DEF$ be the orthic triangle of triangle $\triangle ABC$. Choose a point H_a on AD such that $AH_a/DH_a = 2t_0$, similarly define H_b, H_c. Then $H_a, H_b, H_c \in \mathcal{K}_{t_0}$.

Proof. Let H_a's reflections across BC, CA, AB be H_{aa}, H_{ab}, H_{ac}. Then we know that the image of H_{aa} under a homothety by t_0 from H_a is A, so the image of $\triangle H_{aa}H_{ab}H_{ac}$ under a homothety of t_0 from H_a is perspective to $\triangle ABC$. Then from part (ii) of Liang-Zelich, $H_a \in \mathcal{K}_{t_0}$. Similarly we have $H_b, H_c \in \mathcal{K}_{t_0}$. □

If you want to use the above two proofs in part (iii) of Liang-Zelich, you will discover a point T, where the other two points don't necessarily exist.

Proposition 11.4.11. Let $\mathcal{K}_\varphi^p(L)$ be the Darboux cubic of $\triangle ABC$, then $\mathcal{K}_\varphi^p(L)$ is symmetric across O.

Proof. Note that $t(L) = 1/2$, then from the equivalency of parts (i) and (ii) in (TODO 11.4.4), a point $P \in \mathcal{K}_\varphi^p(L)$ if and only if P's pedal triangle wrt. $\triangle ABC$ is perspective to $\triangle ABC$. This lets us redefine the Darboux cubic!

Let P' be the reflection of P across O, for P on the Darboux cubic. Let $\triangle P_aP_bP_c, \triangle P'_aP'_bP'_c$ respectively be the pedal triangles of P, P' wrt. $\triangle ABC$. Then their corresponding vertices are reflections over the midpoints of their sides (i.e isotomic). Thus the two perspectors of these pedal triangles wrt. $\triangle ABC$ are also isotomic conjugates (isoconjugate with G as pole), so P, P' both lie on $\mathcal{K}_\varphi^p(L)$. □

11.4.1 Proof

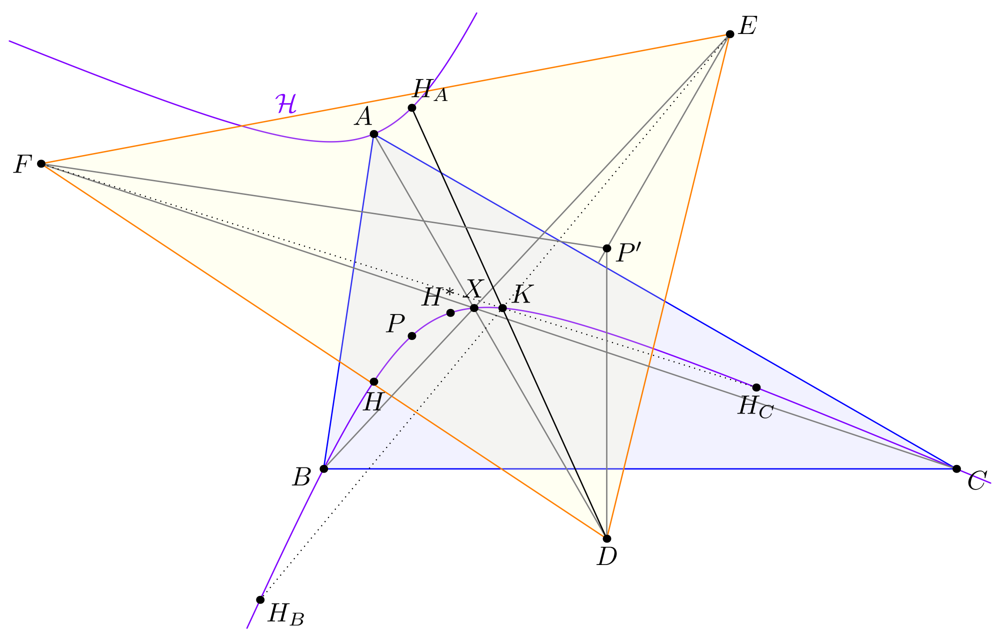
Most of this proof is taken from Liang and Zelich's original paper at here. Note that the t_0 of (i) is unique, so to prove that (i) and the others are equivalent, we can actually just prove that in the condition of (i), there exists a t_0 such that it holds, and also that there exists a t_0 such that (ii), (iii), (iv) hold.

If we let $\triangle P'_aP'_bP'_c$ be the image of the homothety from P of $\triangle P_aP_bP_c$, then the locus of $CP'_c \cap AP'_a$ is a conic $\mathcal{C}_B$ through C, A, H, P, where H is the orthocenter of $\triangle ABC$. Similarly, the locus of $AP'_a \cap BP'_b$ is a conic $\mathcal{C}_C$ through A, B, H, P. Since $\mathcal{C}_C, \mathcal{C}_B$ already have three intersection points A, H, P, they will have a fourth (real) intersection point X. Here X is actually just the perspector of some $\triangle P'_aP'_bP'_c$ with $\triangle ABC$, so we have proved existence for (ii).

Theorem 11.4.12. Given a triangle $\triangle ABC$ and a point P, let H_A, H_B, and H_C be the orthocenters of $\triangle BPC$, $\triangle CPA$, and $\triangle APB$. Consider a triangle $\triangle DEF$ such that $\triangle DEF$ is orthologic and perspective to $\triangle ABC$, and consider the center of orthology P between $\triangle ABC$ wrt. $\triangle DEF$ (order of the two centers of orthology is defined in the previous sections). Let P' be the other center of orthology. Then we have

- $\triangle DEF$ and $\triangle H_A H_B H_C$ are orthologic and perspective.
- If we define J is the perspector of $\triangle DEF$ and $\triangle H_A H_B H_C$ and H, H' as the orthocenters of $\triangle ABC$ and $\triangle DEF$, then we have $J = HP' \cap H'P$ lies on circumconic of $\triangle DEF$, conic $(DEFHP')$.

Proof. it's time to spam pascal For part (i), let X be the perspector of $\triangle ABC$ and DEF. Then (TODO 11.4.6) tells us that X lies on the circumrectangular hyperbola $\mathcal{H} = (ABCHP)$, along with the points $H_A, H_B, H_C \in \mathcal{H}$.



Thus $\triangle AH_BH_C$'s orthocenter (let it be H^*) lies on $\mathcal{H}$ and satisfies

$$H_BH^* \perp H_CA \perp BP, H_CH^* \perp AH_B \perp CP.$$

Therefore H^* is both the orthocenter of $\triangle H_ABH_C$ and also the center of orthology of $\triangle H_AH_BH_C$ wrt. $\triangle DEF$.

Let K be the second intersection point of line H_AD with $\mathcal{H}$. By Pascal on the hexagon $CXAH_CKH_A$ we have that

$$CX \cap H_CK, XA \cap KH_A = D, AH_C \cap H_AC = \infty_{\perp BP} = \infty_{DF}$$

are collinear, so $F \in H_C K$. Similarly we have $E \in H_B K$, so K is the perspector of $\triangle DEF$ and $\triangle H_A H_B H_C$.

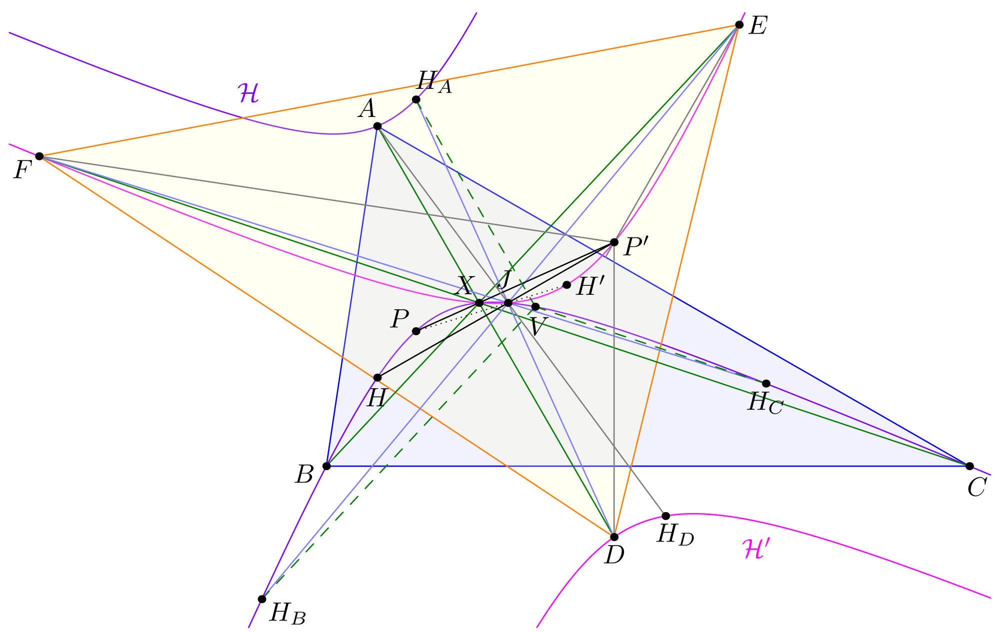
For part (ii), consider the hexagon $JHAXPH_A$. By Pascal we get that

$$JH \cap XP, HA \cap PH_A = \infty_{\perp BC}, AX \cap H_A J = D$$

are collinear. From Sondat's theorem((TODO 11.4.8)), we have that $D\infty_{\perp BC} \cap XP = P'$, so we have $J \in HP'$. Let V be the second intersection point of line $H_A\infty_{AX}$ with hyperbola $\mathcal{H}$. By Pascal on the hexagon $AXBH_AVH_B$ we have that

$$AX \cap H_A V = \infty_{AX}, XB \cap VH_B, BH_A \cap H_B A = \infty_{\perp CP}$$

are collinear, and thus $BX \parallel VH_B$. Similarly we can get that $CX \parallel VH_C$, which just means that $H_A\infty_{DX}, H_B\infty_{EX}, H_C\infty_{FX}$ are concurrent.



From (TODO 11.4.6) and the fact that $\triangle DEF, \triangle H_A H_B H_C$ are orthologic, we can get that J lies on the rectangular hyperbola $\mathcal{H}' = (DEFXP')$, since

$$\begin{aligned} P(A, H_A; H_B, H, C) &= P'(\infty_{\perp EF}, D; E, F) \\ &= J(P'\infty_{\perp EF} \cap \mathcal{H}', D; E, F) \\ &= J(P'\infty_{\perp EF} \cap \mathcal{H}', H_A; H_B, H_C), \end{aligned}$$

so we get the orthocenter of $\triangle P'EF$, $H_D = P'\infty_{\perp EF} \cap \mathcal{H}'$ lies on line AJ. Let H' be the orthocenter of

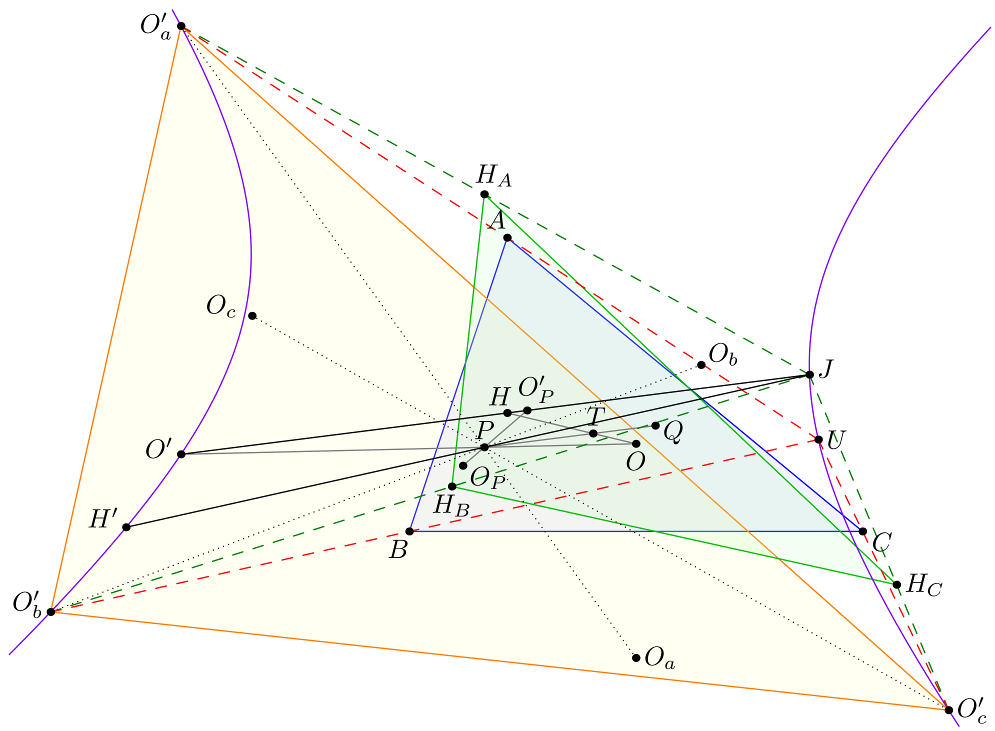
$\triangle DEF$. We finish by Pascal on $JH'DXP'H_D$, to get that

$$JH' \cap XP, H'D \cap P'H_D = \infty_{\perp EF}, DX \cap H_D J = A$$

are collinear. Then from Sondat's theorem, we have that $A\infty_{\perp EF} \cap XP' = P$, so $J \in H'P$. $\square$

Now we have enough machinery to start the actual proof of Liang-Zelich.

First we prove (i) implies (iii). Let $\triangle O'_a O'_b O'_c$ be the Carnot triangle of P wrt. $\triangle ABC$ under a t_0^{-1} homothety at P, let O_P be the circumcenter of $\triangle O_a O_b O_c$, let O', O'_P respectively also be the images of O, O_P under a t_0^{-1} homothety at P.



Since $PQ \parallel OO_P \parallel O'O'_P$, we only need to prove that $\triangle ABC$ and $\triangle O'_a O'_b O'_c$ are perspective, to get that O', H, O'_P are collinear. Let's use (TODO 11.4.12) and choose $\triangle O'_a O'_b O'_c$ as our $\triangle DEF$. We get that $\triangle H_A H_B H_C$ and $\triangle O'_a O'_b O'_c$ are perspective. Note that the center of orthology of $\triangle O'_a O'_b O'_c$ wrt. $\triangle ABC$ is O', and the center of orthology of $\triangle ABC$ wrt. $\triangle O'_a O'_b O'_c$ is P, so thus $J = HO' \cap H'P \in \mathcal{C} = (O'_a O'_b O'_c O' H')$ is the perspector of $\triangle H_A H_B H_C$ and $\triangle O'_a O'_b O'_c$, where H' is the orthocenter of $\triangle O'_a O'_b O'_c$.

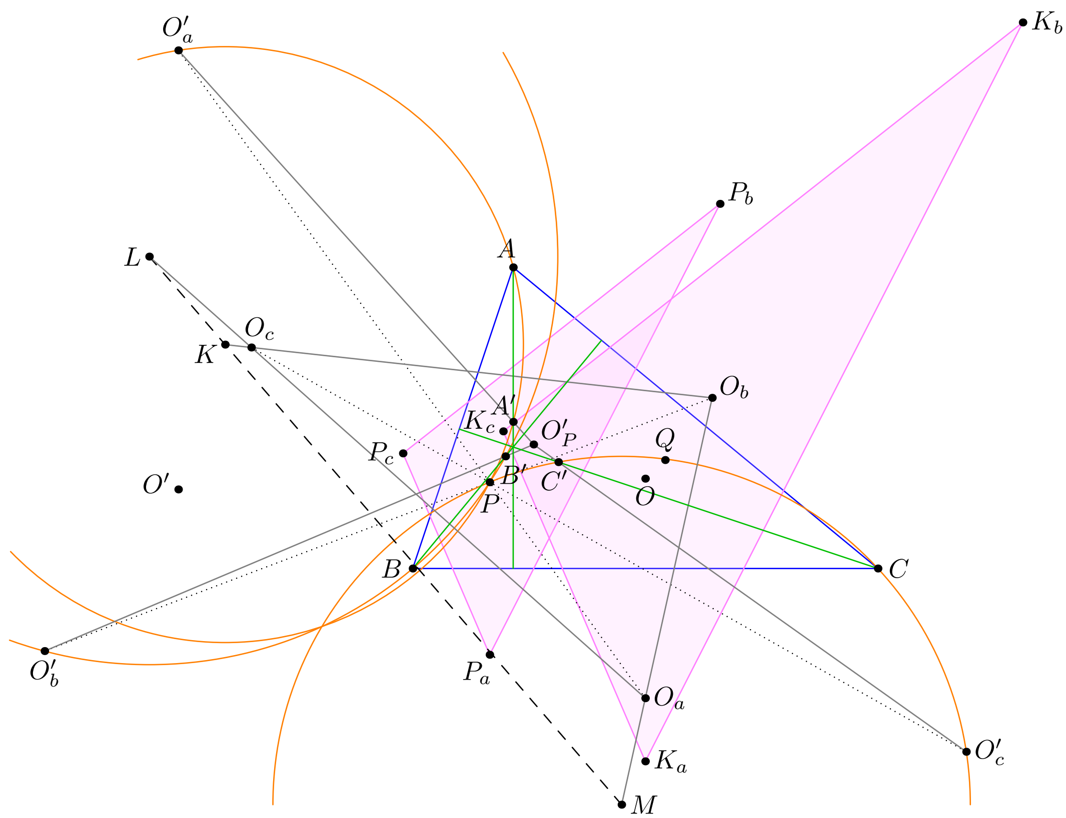
Since (P, O') are isogonal conjugates in $\triangle O'_a O'_b O'_c$ (opposite centers of orthology), we have that $H'P \cap O'O'_P$ is the isogonal conjugate of $H'O' \cap PO'_P$ by DDIT, and thus $H'P \cap O'O'_P$ lies on $\mathcal{C}$. Thus J is simultaneously the second intersections of $O'H$ and $O'O'_P$ with $\mathcal{C}$, so O', H, O'_P are collinear and (i) implies (iii). (We could have also seen in the proof that $O'a = AU \cap PO_a, O'_b = BU \cap PO_b, O'_c = CU \cap PO_c$, where $U = OP \cap HQ$.)

Now let's prove that (i) and (iii) imply (iv). Let $\triangle O'_aO'_bO'_c$ be defined similarly to the previous proof. Setting $\triangle DEF$ in (TODO 11.4.12) to $\triangle O'_aO'_bO'_c$ gets us that the (t_0, P) Euler line of $\triangle BPC, \triangle CPA, \triangle APB$ (which are just lines $O'_aH_A, O'_bH_B, O'_cH_c$) concur at a point J, and $J \in HO'$. Then from (i) we have that HO' is the (t_0, P) Euler line of $\triangle ABC$, so all four (t_0, P) Euler lines concur.

Finally, let's prove that (i) and (iii) imply (ii). We first prove that $\Gamma_A = (APO'_a), \Gamma_B = (BPO'_b), \Gamma_C = (CPO'_c)$ are coaxial. Let $A' = O'_PO'_a \cap AH, B' = O'_PO'_b \cap BH, C' = O'_PO'_c \cap CH$, then

$$\begin{aligned}\measuredangle(PO'_a) - \measuredangle(A'O'_a) &= \measuredangle(PO_a) - \measuredangle(O_PO_a)\\ &= (\measuredangle(PB) + \measuredangle(PC) - \measuredangle(AH)) - (\measuredangle(O_aO_b) + \measuredangle(O_aO_c) - \measuredangle(AP))\\ &= \measuredangle(PA) - \measuredangle(A'A)\end{aligned}$$

so $A' \in \Gamma_A$. Similarly we have $B' \in \Gamma_B, C' \in \Gamma_C$.



From $O'O'_a \parallel OO_a \parallel AH$ and O', H, O'_P collinear, we have

$$\frac{O'_PA'}{O'_PO'_a} = \frac{O'_PH}{O'_PO'}.$$

Calculating this ratio symmetrically we just get that $\triangle A'B'C'$ and $\triangle O'_aO'_bO'_c$ are homothetic with center O'_P. This tells us that

$$O'_PO'_a \cdot O'_PA' = O'_PO'_b \cdot O'_PB' = O'_PO'_c \cdot O'_PC',$$

so O'_P is the radical center of the three circles $\Gamma_A, \Gamma_B, \Gamma_C$. Thus they are coaxial with common rad-ax PO'_P.

Let $\triangle O_AO_BO_C$ be the Carnot triangle of P wrt. $\triangle O'_aO'_bO'_c$, then $\triangle O_AO_BO_C$ is also obviously the Carnot triangle of P wrt. $\triangle O_aO_bO_c$ under a t_0 homothety at P. Note that the centers of the circles $\Gamma_A, \Gamma_B, \Gamma_C$ respectively are $O_bO_c \cap O_BO_C, O_cO_a \cap O_CO_A, O_aO_b \cap O_AO_B$, therefore since $\Gamma_A, \Gamma_B, \Gamma_C$ are coaxial and by Desargues' theorem, we have that $\triangle O_aO_bO_c$ and $\triangle O_AO_BO_C$ are perspective. This tells us that $t(P, \triangle ABC) = t(P, \triangle O_aO_bO_c)$.

Let $\triangle K_aK_bK_c$ be the Carnot triangle of Q wrt. $\triangle ABC$, then by the same logic we have $t(Q, \triangle ABC) = t(Q, \triangle K_aK_bK_c)$. Since $\triangle P_aP_bP_c \cup P \overset{+}{\sim} \triangle K_aK_bK_c \cup O$, we have

$$\begin{aligned} t(P, \triangle P_aP_bP_c) &= t(O, \triangle K_aK_bK_c) = t(Q, \triangle K_aK_bK_c) \\ &= t(Q, \triangle ABC) = t(P, \triangle ABC). \end{aligned}$$

Thus P's Carnot triangle wrt. $\triangle P_aP_bP_c$ is just $\triangle ABC$, and $\triangle ABC$ under a homothety of t_0^{-1} at P is perspective with $\triangle P_aP_bP_c$. In other words, $\triangle P_aP_bP_c$ under a homothety of t_0 at P is perspective with $\triangle ABC$.

Practice Problems

Problem 1. Given $\triangle ABC$, let A_1 be the reflection of A across BC, and similarly define B_1, C_1. Let $A_2 = BC_1 \cap CB_1$. Prove that A_1A_2 is parallel to the Euler line of $\triangle ABC$.

Problem 2. Given $\triangle ABC$, let B^*, C^* respectively be the reflections of B, C across CA, AB. Let I^b, I^c be the B, C excenters. Let $P = I^bC^* \cap I^cB^*, A', B', C'$ respectively be the reflections of P across $BC < CA, AB$. Prove that AA', BB', CC' are concurrent.

Problem 3. Let I be the incenter and let N_9 be the nine-point center. Let $\triangle XYZ$ be the incentral triangle (cevian triangle of I), and let ℓ be the Euler line of $\triangle XYZ$. Prove that N_9 lies on ℓ.

Problem 4 (Taiwan 2017 TST 3 M3)**.** For triangle $\triangle ABC$ with circumcircle Γ, let A' be the antipode of A. Construct equilateral triangle BCd, such that A, D lie on opposite sides of BC. Let the line perpendicular to $A'D$ through A intersect CA, AB at E, F. Construct isosceles triangle with base EF and top angle $30°$ ETF, such that A, T lie on opposite side of EF.

Problem 5 (USA TST 2016)**.** Let acute triangle $\triangle ABC$ be non-isosceles and let P be a internal point. Let A_1, B_1, C_1 be the feet from P onto the sides. Find all points P such that AA_1, BB_1, CC_1 are concurrent and

$$\measuredangle PAB + \measuredangle PBC + \measuredangle PCA = 90^\circ.$$

Chapter 12

Special Lines

12.1 Conjugate Conics

Our goal for this section is to generalize orthotransversals.

Proposition 12.1.1. Given a $\triangle ABC$, a point $P \notin \{A, B, C\}$ and angles α, β and γ such that

$$(\alpha, \beta, \gamma) \neq (\measuredangle(BC, AP), \measuredangle(CA, BP), \measuredangle(AB, CP)),$$

there exists a circumconic $\mathcal{C}^{\sharp}_{P,\alpha,\beta,\gamma}$ such that the following property holds:

For any point $Q \notin \mathcal{L}_\infty$, construct points $D \in BC$, $E \in CA$, $F \in AB$ such that

$$\measuredangle(QD, AP) = \alpha,\ \measuredangle(QE, BP) = \beta,\ \measuredangle(QF, CP) = \gamma.$$

Then $Q \in \mathcal{C}^{\sharp}_{P,\alpha,\beta,\gamma}$ if and only if D, E, and F lie on a line L.

Proof. Construct $\triangle XYZ$ such that $A \in YZ$, $B \in ZX$, $C \in XY$ such that

$$\measuredangle(YZ, AP) = \alpha,\ \measuredangle(ZX, BP) = \beta,\ \measuredangle(XY, CP) = \gamma.$$

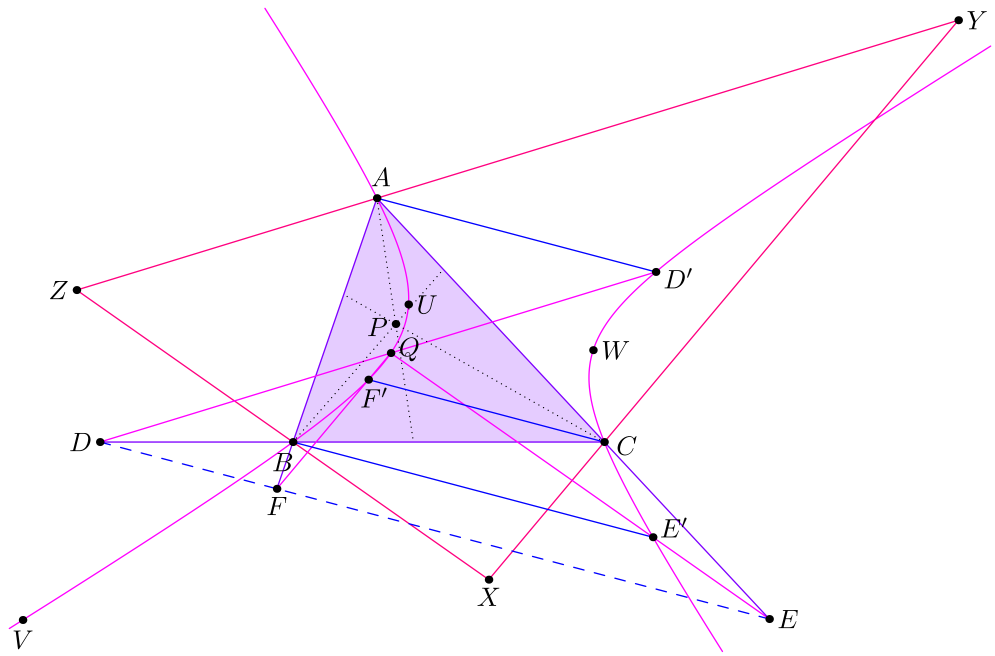
Additionally, construct point U such that $BXCU$ is a parallelogram; construct V and W similarly. Note that since $BW \cap CV, CU \cap AW, AV \cap BU \in \mathcal{L}_\infty$, by the converse of Pascal's theorem A, B, C, U, V, W lie on a conic.

Claim. We can choose $\mathcal{C}^{\sharp}_{P,\alpha,\beta,\gamma}$ to be this conic.

Proof. Choose a $Q \in \mathcal{C}^\sharp_{P,\alpha,\beta,\gamma}$ and construct points $D \in BC$, $E \in CA$, $F \in AB$ such that

$$\measuredangle(QD, AP) = \alpha,\ \measuredangle(QE, BP) = \beta,\ \measuredangle(QF, CP) = \gamma$$

so that $QD \parallel YZ$, $QE \parallel ZX$, $QF \parallel XY$.



Furthermore, construct point D' on $\mathcal{C}^\sharp_{P,\alpha,\beta,\gamma}$ such that Q, D, D' are collinear; construct E' and F' similarly. By Pascal's Theorem (TODO 6.3.1) on $\mathcal{C}^\sharp_{P,\alpha,\beta,\gamma)}$, E, F, $BE' \cap CF'$ are collinear and cyclically we have the other two collinearities with D, E, etc. So we only need to prove that AD', BE', CF' are concurrent. We proceed by tethered MMP. We know that when $Q \in \{U, V, W\}$, the said proposition $\mathscr{P}$ holds (all of the concurrency points lie on $\mathcal{L}_\infty$). Since $Q \mapsto D'$, E', F' is a projective map (TODO 7.2.15), $\mathscr{P}$ is purely projectively defined on $\mathcal{C}^\sharp_{P,\alpha,\beta,\gamma}$, and thus since we checked the three cases U, V, W, our result holds for all $Q \in \mathcal{C}^\sharp_{P,\alpha,\beta,\gamma}$.

Now let's prove necessity/uniqueness. Suppose there exists $Q \notin \mathcal{C}^\sharp_{P,\alpha,\beta,\gamma)}$, but D, E, F are collinear. Then for any chord $\overline{RS}$ through Q (where R, S are points on the conic $\mathcal{C}^\sharp_{P,\alpha,\beta,\gamma)}$), we have that D, E, F, D_R, E_R, F_R, D_S, E_S, F_S respectively are collinear (D_R, D_S is the point D corresponding to R, S respectively on the conic, etc with E_R, F_R). Therefore, for any $T \in \overline{RS}$, D_T, E_T, F_T are collinear, by tethered MMP, since this is three cases. Further, since $\overline{RS}$ is defined arbitrarily, for all T, D_T, E_T, F_T are collinear. But this is blatantly false (i.e set T to be some point very close to A, then D_T is in some very small region on segment BC, but E_T and F_T can be whereever, based on the direction T approaches A from.). $\square$

$\square$

Definition 12.1.2. Given a triangle $\triangle ABC$, a point P and angles α, β and γ satisfying (TODO 12.1.1):

- We call the above defined conic $\mathcal{C}^\sharp_{P,\alpha,\beta,\gamma}$ the **α, β, γ-conjugate conic** of P wrt. $\triangle ABC$.
- If $Q \in \mathcal{C}^\sharp_{P,\alpha,\beta,\gamma}$, we notate the above defined line L as $\mathcal{T}_Q(\triangle ABC, P, \alpha, \beta, \gamma)$ and we call it the **$(P, \alpha, \beta, \gamma)$-transversal** of Q wrt. $\triangle ABC$.
- If $\alpha = \beta = \gamma$, we simplify this notation $\mathcal{C}^\sharp_{P,\alpha,\beta,\gamma}$ to $\mathcal{C}^\sharp_{P,\alpha}$, and we call the above conic the **α-conjugate conic** of $\triangle ABC$ wrt. P. We will also shorten the $\mathcal{T}_Q(\triangle ABC, P, \alpha, \alpha, \alpha)$ transversal of a point Q on $\mathcal{C}^\sharp_{P,\alpha}$ as simply $\mathcal{T}_Q(\triangle ABC, P, \alpha)$.
- When $\alpha = \beta = \gamma = 90^\circ$, define $\mathcal{O}_Q(\triangle ABC, P) = \mathcal{T}_Q(\triangle ABC, P, 90^\circ)$ as the **P-orthotransversal** of Q wrt. $\triangle ABC$.
- When $\alpha = \beta = \gamma = 90^\circ$ and $P = Q$, define $\mathcal{O}(\triangle ABC, P) = \mathcal{O}_P(\triangle ABC, P)$, which is just the orthotransversal of P wrt. $\triangle ABC$.

From the proof of the construction in (TODO 12.1.1), we can see that if $\tilde{P}$ is the (triangle) Miquel point of $\triangle XYZ$ with X, Y, Z on the sides of $\triangle ABC$, then $\mathcal{C}^\sharp_{P,\alpha,\beta,\gamma} = \mathcal{C}_{\tilde{P},\tilde{\alpha}}$ for an angle $\tilde{\alpha} = \measuredangle(YZ, A\tilde{P})$.

In addition to conjugate conics, we can also dually define anticonjugate curves.

Definition 12.1.3. Given $\triangle ABC$, a point Q, and three angles α, β, γ, we define

$$\mathcal{C}^\flat_{Q,\alpha,\beta,\gamma} := \{P \mid Q \in \mathcal{C}^\sharp_{P,\alpha,\beta,\gamma}\}$$

as the **α, β, γ-anticonjugate curve** of Q wrt. $\triangle ABC$.

Note that we haven't found what these curves are yet. In fact, they're also conics! We have

Proposition 12.1.4. Given $\triangle ABC$, a point $Q \notin \{A, B, C\}$ and an angle α, where $(Q, \alpha) \neq (H, 90^\circ)$ (H is orthocenter), then

$$\mathcal{C}^\flat_{Q,\alpha} = \mathcal{C}^\flat_{Q,\alpha,\alpha,\alpha} = \mathcal{C}^\sharp_{Q,-\alpha}.$$

Proof. Construct point U such that $(BC)(QU)$ is a parallelogram. Similarly, define V, W cyclically. Construct X such that

$$\measuredangle UBX = \measuredangle UCX = \alpha.$$

Similarly, define Y, Z. From the construction of $\mathcal{C}^\sharp_{Q,-\alpha}$, A, B, C, X, Y, Z all lie on $\mathcal{C}^\sharp_{Q,-\alpha}$. $\square$

Claim. The conic $\mathcal{C}^\flat_{Q,\alpha} = \mathcal{C}^\sharp_{Q,-\alpha}$ through A, B, C, X, Y, Z satisfies the conditions.

Proof of Claim. Let P be a point on $\mathcal{C}^\flat_{Q,\alpha}$. We hope to prove $Q \in \mathcal{C}^\sharp_{P,\alpha}$. Recall the construction of $\mathcal{C}^\sharp_{P,\alpha}$ from before, choose triangle $\triangle X'Y'Z'$ such that P is the Miquel point of $\triangle ABC$ wrt. $\triangle X'Y'Z'$ and

$$\measuredangle(Y'Z', AP) = \measuredangle(Z'X', BP) = \measuredangle(X'Y', CP) = \alpha.$$

Choose U' such that $BX'CU'$ is a parallelogram. Similarily define V', W'. Thus $\mathcal{C}^\sharp_{P,\alpha}$ is a conic passing through A, B, C, U', V', W'. Note that $[BP \mapsto BX'], [CP \mapsto CX']$ are both projective maps, so X' moves on a fixed conic. Therefore we know U' also moves on a fixed conic $\mathcal{C}_U$. Similarily define $\mathcal{C}_V$ and $\mathcal{C}_W$. When $P = X$, we have $U' = Q$, so $Q \in \mathcal{C}_U$. Analogously, $Q \in \mathcal{C}_V, \mathcal{C}_W$. Thus, by the converse of Pascal's theorem, $Q \in \mathcal{C}^\sharp_{P,\alpha}$ iff.

$$BW' \cap CV', CQ \cap AW', AV' \cap BQ$$

are collinear. We proceed by moving points on the assertion these three points are collinear. We have the cases $P = X, Y, Z$ easily. Note when $BW' \cap CV' \in \mathcal{L}_\infty$, then

$$[P \mapsto \overline{(CQ \cap AW')(AV' \cap BQ)}]$$

is a projective map from points on $\mathcal{C}^\flat_{Q,\alpha}$ to tangent lines to some conic $\mathcal{C}'$ which is tangent to BQ and CQ. When $P = X$, we get $\mathcal{C}'$ and $\mathcal{L}_\infty$ are tangent, so thus for any $P \in \mathcal{C}^\flat_{Q,\alpha}$, our assertion is satisfied.

Now we prove the converse. Suppose $P \notin \mathcal{C}^\flat_{Q,\alpha}$, but $Q \in \mathcal{C}^\sharp_{P,\alpha}$. For any chord $\overline{RS}$ on $\mathcal{C}^\flat_{Q,\alpha}$ passing through P, we have $Q \in \mathcal{C}^\sharp_{P,\alpha} \cap \mathcal{C}^\sharp_{R,\alpha} \cap \mathcal{C}^\sharp_{S,\alpha}$. Therefore for any $T \in RS, Q \in \mathcal{C}^\sharp_{T,\alpha}$. Since chord RS was defined arbitrarily, for all T, $Q \in \mathcal{C}^\sharp_{T,\alpha}$. By choosing $T \in BC, CA, AB$, we get

$$\measuredangle(BC, AQ) = \measuredangle(CA, BQ) = \measuredangle(AB, CQ) = \alpha$$

which only holds if $(Q, \alpha) = (H, 90^\circ)$. $\square$

Remark. In the proof we can see that if we choose X, Y, Z such that

$$\measuredangle VAY = \measuredangle WAZ = \alpha, \measuredangle WBZ = \measuredangle UBX = \beta, \measuredangle UCX = \measuredangle VCY = \gamma,$$

and A, B, C, X, Y, Z are conconic (equivalently, $BZ \cap CY, CX \cap AZ, AY \cap BX$ are collinear), then $\mathcal{C}^\flat_{Q,\alpha,\beta,\gamma}$ is also a conic. However this is not sufficient to prove, as we haven't used the definition of $\mathcal{C}^\flat$ as the preimage of $C^\sharp$ yet.

Let's work on relating $\mathcal{C}^\sharp$ and $\mathcal{C}^\flat$.

Proposition 12.1.5. For a point $P \notin \{A, B, C, H\}$,

- For any angle α, $\mathcal{C}^{\sharp}_{P,\alpha} = \mathcal{C}^{\flat}_{P,-\alpha}$;
- Let $\widehat{P}$ be the antigonal conjugate of P (see (TODO 8.1.16)), then

$$\{A, B, C, \widehat{P}\} = (\bigcap_{\alpha} \mathcal{C}^{\sharp}_{P,\alpha}) \cap (\bigcap_{\alpha} \mathcal{C}^{\flat}_{P,\alpha});$$

- Let $\mathscr{F}$ be the pencil of conics through $A, B, C, \widehat{P}$, then the map

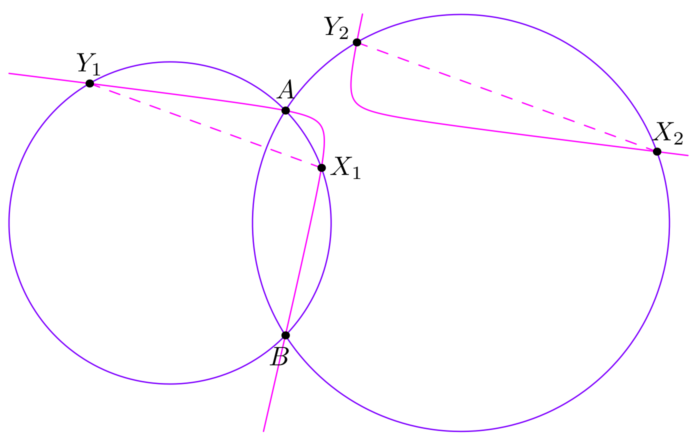
$$\begin{aligned} \mathbb{R}^{\circ}/180^{\circ} &\to \mathscr{F} \\ \alpha &\mapsto \mathcal{C}^{\sharp}_{P,\alpha} \end{aligned}$$

  from all angles α to the pencil of conics is a projective map.

Proof. □

We also get an extension of Reim's theorem (TODO 0.1.14). (Reim's theorem is the special case when this conic is two lines.)

Corollary 12.1.6 (Conic Reim's)**.** Let Ω_1, Ω_2 be two circles that intersect at A, B, and let X_1, Y_1 be two points on Ω_1, and let X_2, Y_2 be two points on Ω_2. Then A, B, X_1, Y_1, X_2, Y_2 are conconic if and only if $X_1Y_1 \parallel X_2Y_2$.



Remark. This is easily proven by Cayley-Bacharach (TODO 11), on the three cubics made of $\Omega_1 \cup X_2Y_2$, $\Omega_2 \cup X_1Y_1$, $(ABX_1Y_1X_2Y_2) \cup \mathcal{L}_\infty$ (the common points are $A, B, X_1, Y_1, X_2, Y_2, I, J, \infty_{X_1Y_1}$ where I, J are the circle points.)

Proposition 12.1.7. Given a triangle $\triangle ABC$ and a circumconic $\mathcal{C}$, we have the mapping

$$\mathbb{R}^{\circ}/180^{\circ} \longrightarrow \{(P_\alpha, \alpha) \mid \mathcal{C} = \mathcal{C}_{P_\alpha,\alpha}\}$$

$$\alpha \longmapsto (P_\alpha, \alpha),$$

(i.e for every angle α, there exists a point P such that $\mathcal{C}$ is $\mathcal{C}_{P_\alpha,\alpha}$.) As angle α varies, the locus of the isogonal conjugate of P_α (let it be P_α^*) is a circle through the circumcenter, and $\alpha \mapsto P_\alpha^*$ is a projective map. This circle is also the image of the isogonal conjugate of the circumconic $\mathcal{C}$, inverted across the circumcircle (ABC).

Proof. For a point $\widehat{P}$ on $\mathcal{C}$, let P be the antigonal conjugate of $\widehat{P}$. Then $P, \widehat{P}$ and $\widehat{P}^*, P^*$ are pairwise inverses over the circumcircle (ABC) by the other definition of antigonal conjugate (TODO 8.1.19), so we have that the locus of P^* is a circle through O. Similarly to the construction in the proof of (TODO 12.1.1), let U, V, W respectively be the 2nd intersections of $(B\widehat{P}C), (C\widehat{P}A), (A\widehat{P}B)$ with $\mathcal{C}$, and let X, Y, Z be the reflections of U, V, W respectively across the midpoints of $\overline{BC}, \overline{CA}, \overline{AB}$. Then P is the Miquel point of triangle $\triangle ABC$ wrt. $\triangle XYZ$.

Claim. $\measuredangle(YZ, AP) = \measuredangle(OP^*, \varphi^K(\mathcal{C})) + 90°$.

Proof of Claim. Let U^* be the isogonal conjugate of U in $\triangle ABC$. Then $\varphi^K(\mathcal{C}) = \widehat{P}^* U^*$, so we have

$$\begin{aligned}\measuredangle(OP^*, \varphi^K(\mathcal{C})) &= \measuredangle O\widehat{P}^*B + \measuredangle B\widehat{P}^*U^* = \measuredangle P^*BO + (U^* - B)_{(B\widehat{P}^*C} \\ &=_\perp (BP - CA) + (U^* - B)_{(B\widehat{P}^*C)},\end{aligned}$$

but

$$\begin{aligned}\measuredangle(YZ, AP) &= \measuredangle XBP = \measuredangle BP - \measuredangle CU = \measuredangle BP - \measuredangle CA - \measuredangle CB + \measuredangle CU^* \\ &= (\measuredangle BP - \measuredangle CA) + (U^* - B)_{(B\widehat{P}^*C)},\end{aligned}$$

so our claim is proven. ☐

what does this sentence say? This choice of α exists and is unique, so $\alpha \mapsto P_\alpha^*$ is a projective map. ☐

Proposition 12.1.8. Given $\triangle ABC$ and a point $P \notin \{A, B, C, H\}$, let $\widehat{P}$ be the antigonal conjugate of P in $\triangle ABC$, let $\widehat{P}^*$ be the isogonal conjugate of the antigonal conjugate of P in $\triangle ABC$, then the envelope of $\mathcal{T}_{\widehat{P}}(\triangle ABC, P, \alpha)$ is a conic tangent to the sides of $\triangle ABC$ and with $\widehat{P}, \widehat{P}^*$ as foci.

Proof. Let E, F respectively be the intersection points of the transversal $\mathcal{T}_{\widehat{P}}(\triangle ABC, P, \alpha)$ with CA and AB. Then,

$$\measuredangle E\widehat{P}F = \measuredangle(\widehat{P}E, BP) + \measuredangle BPC + \measuredangle(CP, \widehat{P}F) = \measuredangle C\widehat{P}B$$

so $\widehat{P}$ has an isogonal conjugate in quadrilateral $(BE)(CF)$ by (TODO 1.3.14), and therefore the ellipse with foci $\widehat{P}, \widehat{P}^*$ is always tangent to line EF as well, so it must be the envelope. $\square$

In reality, we can also use this proof to get part (ii) of (TODO 12.1.5).

Now let's see what $\mathcal{C}^\sharp$ and $\mathcal{C}^\flat$ we get at special points and angles.

(1) $\alpha = 0°$:

Proposition 12.1.9. For an arbitrary point P, $\mathcal{C}^\sharp_{P,0°} = \mathcal{C}^\flat_{P,0°}$ is the nine-point conic $\mathcal{C}$ of P with respect to $\triangle ABC$ under a $2\times$ homothety at P. Notably, if we let $\triangle P'_A P'_B P'_C$ be the image of P's cevian triangle under a $2\times$ dilation, then $P'_A, P'_B, P'_C \in \mathcal{C}^\sharp_{P,0°} = \mathcal{C}^\flat_{P,0°}$.

In other words, for any circumconic $\mathcal{C}$, if we choose P such that $\mathcal{C}$ is the image of the nine-point conic of (A, B, C, P) under a $2\times$ homothety from P, (i.e set P as the anticomplement $O^{\mathfrak{J}}_{\mathcal{C}}$ of the center $O_{\mathcal{C}}$ of $\mathcal{C}$), then $P = P_{0°}$. Here, the proof of (TODO 12.1.7) tells us that the image of P^* after inversion around (ABC) is the foot of O onto the line $\varphi^K(\mathcal{C})$.

For a point Q on $\mathcal{C}$, consider the set of $(P, 0°)$-transversals $\mathcal{T}_{\widehat{P}}(\triangle ABC, P, 0°)$. In reality, this set (plus the line at infinity $\mathcal{L}_\infty$) is actually the isohaptic locus (TODO 9.4.3) of complete quadrangle (A, B, C, P)! So we can think of them as a generalization of the Simson line, denote it as $\mathcal{S}^{\mathcal{C}}_Q$. A simple corollary is:

Corollary 12.1.10. The line $\mathcal{S}^{\mathcal{C}}_Q = \mathcal{T}_{\widehat{P}}(\triangle ABC, P, 0°)$ bisects segment $\overline{PQ}$.

Let's now look at the case where $P = H$, the orthocenter. (2) $P = H$:

Proposition 12.1.11. Given $\triangle ABC$ and an angle $\alpha \neq 90°$, then $\mathcal{C}^\sharp_{H,\alpha} = (ABC)$.

And now the Simson line is just the transversal.

Corollary 12.1.12 (Simson Line)**.** Given a triangle $\triangle ABC$, P lies on (ABC) if and only if the feet from P to the sides of $\triangle ABC$ are collinear.

These are just some easy angle calculations. (3) $\alpha = 90°$: In this special case, we actually just get rectangular hyperbolas.

Proposition 12.1.13. For any point P, $\mathcal{C}^\sharp_{P,90°} = \mathcal{C}^\flat_{P,90°}$ is a rectangular hyperbola through P, and for any point $Q \in \mathcal{C}^\sharp_{P,90°} = \mathcal{C}^\flat_{P,90°}$, $PQ \perp \mathcal{O}_Q(\triangle ABC, P)$.

Proof. Following the notation of (TODO 12.1.1), we have that U, V, W respectively are the orthocenters of $\triangle PBC, \triangle PCA, \triangle PAB$, and therefore the rectangular hyperbola through P also passes through U, V, W, implying that $\mathcal{C}^\sharp_{P,90^\circ} = \mathcal{C}^\flat P, 90^\circ = \mathcal{H}_P$. Let Q be a point on $\mathcal{C}^\sharp_{P,90^\circ}$, H_B, H_C respectively are the orthocenters of $\triangle BPQ, \triangle CPQ$. By Pascal (TODO 6.3.1), we have that

$$H_BQ \cap CA, H_CQ \cap AB, BH_B \cap CH_C$$

are collinear, and also $BH_B \parallel CH_C \perp PQ$, so we have that $\mathcal{O}_Q(\triangle ABC, P) = EF \perp PQ$. □

Corollary 12.1.14. The orthotransversal of a point P in $\triangle ABC$ is tangent to the rectangular hyperbola $(ABCHP)$.

12.1.1 Orthologic Conjugates

With all of the setup we did before, let's investigate a new isoconjugation. We will call this **orthologic conjugation** (newly coined, unaware of a English name for this specific isoconjugation). This is actually a really good example of the fixed points of isoconjugations, because this isoconjugation might not have (real) fixed points.

Remark. Specifically, when $\triangle ABC$ is obtuse, since then one of the barycentric coordinates for H is negative.

Definition 12.1.15. In $\triangle ABC$, we define the **orthologic conjugate** of P as the point P° as the trilinear pole of $\mathcal{O}_H(\triangle ABC, P)$, where H is the orthocenter.

Since H lies on $\mathcal{C}^\sharp_{P,90^\circ}$ (TODO 12.1.13), this is well-defined. From (TODO 12.1.13), we can easily get:

Proposition 12.1.16. For a point P, HP is perpendicular to the trilinear polar of P°.

Proposition 12.1.17. The map from $[P \mapsto P^\circ]$ is actually just the isoconjugation with pole H, φ_H.

From (TODO 7.4.10) and the definition of trilinear polars, we need to prove that the map $[P \mapsto P^\circ]$ satisfies

(i) $H^\circ = G$, where G is the centroid.

(ii) $P \notin BC \cup CA \cup AB \implies P^\circ \notin BC \cup CA \cup AB$,

(iii) For any vertex $X \in \{A, B, C\}$, X, P_1, P_2collinear $\implies X, P_1^\circ, P_2^\circ$ collinear,

(iv) For any vertex $X \in \{A, B, C\}$, $[XP \mapsto XP^\circ]$ is a projective involution.

(i) $\mathcal{O}_H(\triangle ABC, H) = \mathcal{L}_\infty$, and the trilinear polar of $\mathcal{L}_\infty$ is G.

(ii) If $P^\circ \in BC \cup CA \cup AB$, assume $P^\circ \in BC$, then the trilinear polar of P° is the tangent T to BC. But this implies $BP \perp CH$ and $CP \perp BH$, which means $P = A$.

(iii) If A, P_1, P_2 are collinear, then $\mathcal{O}_H(\triangle ABC, P_1)$ and $\mathcal{O}_H(\triangle ABC, P_2)$ intersect at D on BC. Therefore, P_1°, P_2° satisfy

$$A(P_1^\circ, D; B, C) = -1 = A(P_2^\circ, D; B, C),$$

which means A, P_1, P_2° are collinear.

(iv) Let's first prove it's a projective map:

$$X(P_\bullet) = H(\mathcal{O}_H(\triangle ABC, P_\bullet) \cap YZ) = (\mathcal{O}_H(\triangle ABC, P_\bullet) \cap YZ) = X(P_\bullet^\circ)$$

since reflection across the midpoint of YZ preserves cross-ratio.

Now let's push on all the properties we know about isoconjugations onto this isoconjugation.

Corollary 12.1.18.

- The map from $[P \mapsto P^\circ]$ sends any line not through a vertex to a circumconic of $\triangle ABC$.
- Since G and H swap, the orthologic image of the Euler line is the Kiepert hyperbola (TODO 8.3.4).
- Let $\triangle H^a H^b H^c$ be the anticevian triangle of H wrt. $\triangle ABC$, then $\varphi(\mathcal{L}_\infty)$ is an inconic of $\triangle H^a H^b H^c$.

12.2 Zhang Zhihuan's Permutation Line

Zhang Zhihuan, Prince of geo, uses the permutation line to nuke concyclicity in geo problems.

The following section was co-authored by him.

Proposition 12.2.1. Let $\triangle ABC$ have circumconic $\mathcal{C}$ and point P be an arbitrary point in the plane. Let $\triangle P_A P_B P_C$ be its $\mathcal{C}$-cevian triangle and let X be a point in $\mathcal{C}$. Define

$$X_{P,A} = XP_A \cap BC, X_{P,B} = XP_B \cap AC, X_{P,C} = XP_C \cap AB,$$

then $P, X_{P,A}, X_{P,B}, X_{P,C}$ are collinear.

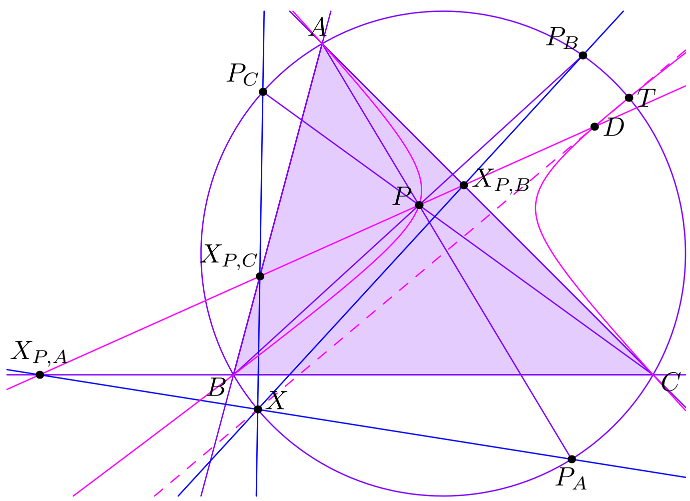
Proof. By Pascal's on BCP_CXP_AA, we get that $BC \cap XP_A = X_{P,A}, CP_C \cap P_AA = P, P_CX \cap BA = X_{P,C}$ line on a line.

By a similar Pascal's on CAP_AXP_BB, the result follows. □

Proposition 12.2.2. The map $\mathbf{Per}_P^{\mathcal{C}}(X) \colon \mathcal{C} \to PX_{P,A}$ from $\mathcal{C}$ to the pencil through P is a projective map.

Proof. Follows by considering the projective map $X \to X_{P,A} \to PX_{P,A}$. □

Proposition 12.2.3. Take $\triangle ABC$ with circumconic $\mathcal{C}$ and let P be a point in the plane. Let $\mathcal{C}'$ be another conic through A, B, C, P, and let T be the fourth intersection point of $\mathcal{C}'$ and $\mathcal{C}$. Let X be a point on $\mathcal{C}$; then $\mathbf{Per}_P^{\mathcal{C}}(X)$ intersects TX on $\mathcal{C}'$.



Proof. Let D be the second intersection of $\mathbf{Per}_P^{\mathcal{C}}(X)$ and $\mathcal{C}'$. Since taking a permutation line is a projective map, we have

$$\begin{aligned} T(A,B;C,X) &= (A,B;C,X)_{\mathcal{C}} = (\mathbf{Per}_P^{\mathcal{C}}(A), \mathbf{Per}_P^{\mathcal{C}}(B); \mathbf{Per}_P^{\mathcal{C}}(C), \mathbf{Per}_P^{\mathcal{C}}(X)) \\ &= (AP, BP; CP, \mathbf{Per}_P^{\mathcal{C}}(X)) = (A,B;C,D)_{\mathcal{T}} = T(A,B;C,D) \end{aligned}$$

As such, T, X, D are collinear as desired. $\square$

Proposition 12.2.4. Take $\triangle ABC$ with circumconic $\mathcal{C}$, two points P, Q in the plane, and point $X \in \mathcal{C}$.

Then $Z = \mathbf{Per}_P^{\mathcal{C}}(X) \cap \mathbf{Per}_Q^{\mathcal{C}}(X)$ lies on $\mathcal{T} = (ABCPQ)$. We will call Z $\mathbf{Li}_{P,Q}^{\mathcal{C}}(X)$ (the **P,Q-Li conjugate** of X).

Proof. Let $P_A = AP \cap \mathcal{C}, Q_A = AQ \cap \mathcal{C}$. This is effectively combining **??** for two values of P, Q, as $\mathbf{Per}_P^{\mathcal{C}}(X) \cap \mathcal{T}$ and $\mathbf{Per}_Q^{\mathcal{C}}(X) \cap \mathcal{T}$ both lie on TX.

The fact that Z lies on $\mathcal{T}$ can also be proven by noting that

$$\begin{aligned} P(A,Z;B,C) &= (AP \cap BC, XP_A \cap BC; B, C) \stackrel{P_A}{=} (A,X;B,C)_{\mathcal{C}} \\ &\stackrel{Q_A}{=} (AQ \cap BC, XQ_A \cap BC; B, C) = Q(A,Z;B,C). \end{aligned}$$

Thus we are done by the definition of a conic in (TODO 6.2.6). $\square$

Remark 12.2.5. Here's a bad way to remember that $\mathbf{Li}^{\mathcal{C}}_{\mathcal{D}}$ lies on $\mathcal{D}$. The *Li*ttle Dipper is a constellation, so the point lies on the dipper or lower conic, which is $\mathcal{D}$.

Proposition 12.2.6. Let T be the fourth intersection of two circumconics $\mathcal{C}, \mathcal{C}'$. Then for any $X \in \mathcal{C}$, we have that $T, X, \mathbf{Li}^{\mathcal{C}}_{\mathcal{C}'}$ collinear.

Remark 12.2.7. This gives us another lens to look at the fact that $\mathbf{Li}^{\mathcal{C}}_{\mathcal{C}'}(X)$ lies on TX – this simply follows because $\mathbf{Per}^{\mathcal{C}}_{T}(X) = TX$!

We also have more projective maps:

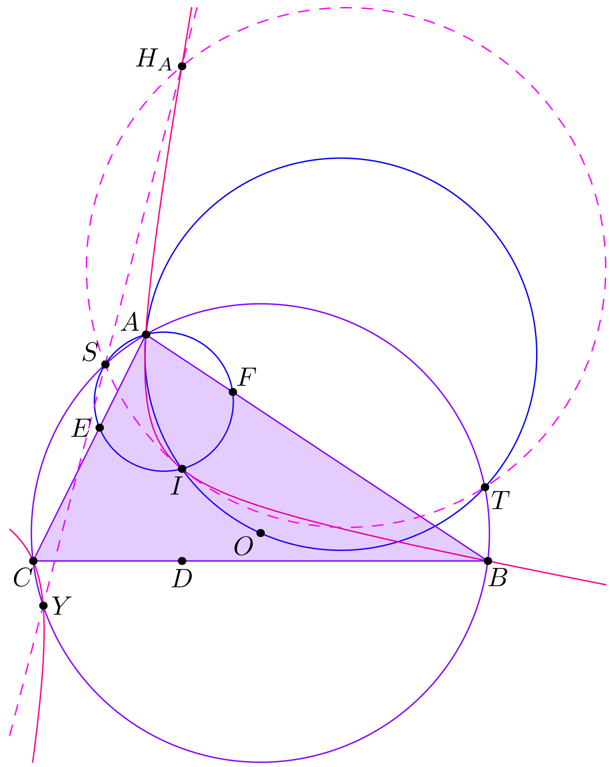
Proposition 12.2.8. Let $\triangle ABC$ and some point X have circumconic $\mathcal{C}$ through A, B, C, X. Then the map from another circumconic $\mathcal{C}'$ to the pencil of lines through the $\mathcal{C}, \mathcal{C}'$-Li conjugate

$$\begin{aligned} \mathcal{C}' &\to \mathbf{T}(\mathbf{Li}^{\mathcal{C}}_{\mathcal{C}'}(X)) \\ P &\mapsto \mathbf{Per}^{\mathcal{C}}_{P}(X) \end{aligned}$$

is projective.

Proof. Follows since $\mathbf{Per}^{\mathcal{C}}_{P}(X) = P\mathbf{Li}^{\mathcal{C}}_{\mathcal{C}'}(X)$. $\square$

Example 12.2.9. Let $\triangle ABC$ have incenter I, circumcenter O, and intouch triangle $\triangle DEF$. Let H_A be the orthocenter of $\triangle BIC$. Let $S = (AEF) \cap (ABC)$ and let $T = (AIO) \cap (ABC)$ other than A. Show that T, H_A, I, S are concyclic.



Proof. Let $\mathcal{H}_{Fe} = (ABCHIH_A)$ be the Feuerbach hyperbola of $\triangle ABC$.

Then it follows that $\mathbf{Per}_I(S) = ID = IH_A$ which implies that $\mathbf{Li}_{\mathcal{H}_{Fe}}(S) = H_A$.

Let U be the isogonal conjugate of ∞_{OI}, which is also the fourth intersection of (ABC) and $\mathcal{H}_{Fe}$. It then follows that U, S, H_A collinear by (TODO 12.2.5).

As such, it follows that

$$\measuredangle H_AST = \measuredangle UST = \measuredangle UAT = \measuredangle UAO + \measuredangle OAT = \measuredangle(H_AI, OI) + \measuredangle(OIT) = \measuredangle H_AIT$$

as desired. $\square$

A bit of review on isoconjugations: for a line ℓ not passing through any of the vertices, we know that a (point) isoconjugation wrt. $\triangle ABC$ sends ℓ to a circumconic bijectively(see (TODO 7.4.7)) (denote this point isoconjugation as φ). Let $\varphi(\ell)$ represent this circumconic. When φ is isogonal conjugation, then when $\ell = \mathcal{L}_\infty$, $\varphi(\mathcal{L}_\infty)$ is the circumcircle.

Proposition 12.2.10. Let φ be a point isoconjugation on $\triangle ABC$, and let $\mathscr{F}$ be its corresponding pencil of conics (TODO 7.4.1). For a circumconic $\mathcal{C}$ and a point on this circumconic X, let another conic $\mathcal{D} = \mathcal{D}_X \in \mathscr{F}$

satisfy that the polar of X wrt $\mathcal{D}$, $\mathfrak{p}_{\mathcal{D}}X = \varphi(\mathcal{C})$. Then for any point P, we have

$$\mathbf{Per}^{\mathcal{C}}_{P}(X) = \mathfrak{p}_{\mathcal{D}}\varphi(P).$$

Proof. Let P_A be the second intersection of AP with $\mathcal{C}$, $X_A = BC \cap XP_A$. We will prove that $X_A \in \mathfrak{p}_{\mathcal{D}}\varphi(P)$. Note that

$$\varphi(P_A) = \ell \cap \mathfrak{p}_{\mathcal{D}}P_A = \mathfrak{p}_{\mathcal{D}}XP_A,$$

and therefore

$$\mathfrak{p}_{\mathcal{D}}X_A = \mathfrak{p}_{\mathcal{D}BC}\mathfrak{p}_{\mathcal{D}}XP_A = A\varphi(P_A) = A\varphi(P),$$

and therefore $X_A \in \mathfrak{p}_{\mathcal{D}}\varphi(P)$. ◻

Through this perspective, we have a different way to express **Li**.

Corollary 12.2.11. Extending the above notation, for two circumconics $\mathcal{C}, \mathcal{C}'$,

$$\mathbf{Li}^{\mathcal{C}}_{\mathcal{C}'}(X) = \bigcap_{P \in \mathcal{C}'} \mathbf{Per}^{\mathcal{C}}_{P}(X) = \mathfrak{p}_{\mathcal{D}}(\varphi(\mathcal{C}')).$$

Proposition 12.2.12. Given $\triangle ABC$, two points P, Q, a circumconic $\mathcal{C}$, and a point X on $\mathcal{C}$, let $\mathbf{Li} = \mathbf{Li}^{\mathcal{C}}_{P,Q}(X)$, then

$$\mathbf{Per}^{\mathcal{C}}_{P \times Q \div \mathbf{Li}}(X) = PQ.$$

In other words, $P \times Q = \mathbf{Li}^{\mathcal{C}}_{P,Q}(X) \times (\mathbf{Per}^{C}_{-}(X))^{-1}(PQ)$.

Proof. Let $\varphi = \varphi^{P \times Q}$, then $\varphi(\mathbf{Li}) = P \times Q \div \mathbf{Li}$. Therefore

$$\mathbf{Per}^{\mathcal{C}}_{P \times Q \div \mathbf{Li}}(X) = \mathfrak{p}_{\mathcal{D}}(\varphi(P \times Q \div \mathbf{Li})) = \mathfrak{p}_{\mathcal{D}}(\mathbf{Li}) = \varphi((ABCPQ)) = PQ.$$

◻

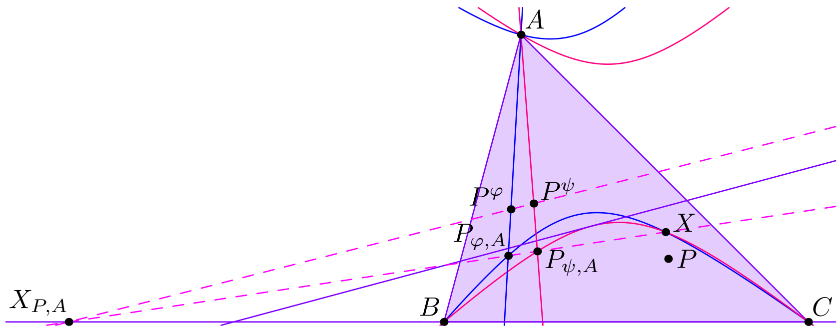
Here's a theorem that holds when we have two isoconjugations.

Theorem 12.2.13 (Fundamental Theorem of Permutation Lines)**.** For a $\triangle ABC$, if we have two isoconjugations φ, ψ, then for a line ℓ, let X be the fourth intersection of ℓ^{φ} and ℓ^{ψ}. Let P be a point in the plane.

Define $P_{\varphi,A} = A\varphi(P) \cap \ell^{\varphi}$ and define $P_{\psi,A}$ similarly.

Then it follows that $X, P_{\varphi,A}, P_{\psi,A}$ are collinear. (i) Furthermore,

$$\varphi(P)\psi(P) = \mathbf{Per}^{\ell^{\varphi}}_{\varphi(P)}(X) = \mathbf{Per}^{\ell^{\psi}}_{\psi(P)}(X). \text{ (ii)}$$



Proof. Note that

$$A(A, P_{\varphi,A}; B, C)_{\ell^\varphi} \stackrel{\varphi}{=} A(\ell \cap BC, P; B, C) \stackrel{\psi}{=} A(A, P_{\psi,A}; B, C)_{\ell^\psi}.$$

It thus follows that $X(A, P_{\varphi,A}; B, C) = X(A, P_{\psi,A}; B, C)$, giving the concurrency.

Now, let $XP_{\varphi,A}P_{\psi,A}$ intersect BC at $X_{P,A}$. Define $X_{P,B}, X_{P,C}$ simlarly. The second claim now just follows as

$$X_{P,A}\varphi(P) = \mathbf{Per}^{\ell^\varphi}_{\varphi(P)}(X) = X_{P,A}X_{P,B}X_{P,C} = \mathbf{Per}^{\ell^\psi}_{\psi(P)}(X) = X_{P,A}\psi(P)$$

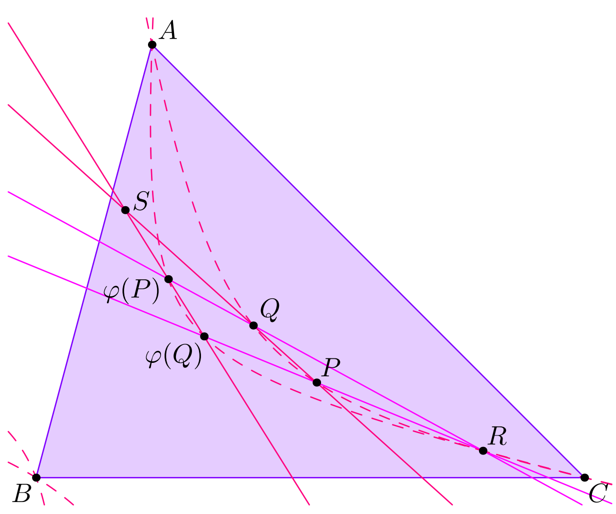
as both permutation lines pass through $X_{P,A}, X_{P,B}, X_{P,C}$.

An alternate way to prove (ii) is by considering the set of diagonal conics $\mathcal{D} = \mathcal{D}_{\varphi,\psi} \in \mathscr{F}_\varphi \cap \mathscr{F}_\psi$, then $\mathfrak{p}_{\mathcal{D}}(X) = \varphi(X)\psi(X) = \ell$ and $\mathfrak{p}_{\mathcal{D}}(P) = \varphi(P)\psi(P)$, so therefore

$$\mathbf{Per}^{\ell^\varphi}_{\varphi(P)}(X) = \mathfrak{p}_{\mathcal{D}}(P) = \varphi(P)\psi(P).$$

Similarly, this is also equal to $\mathbf{Per}^{\ell^\psi}_{\psi(P)}$. $\square$

Example 12.2.14 (Isoconjugates Lead to More Isoconjugates)**.** For $\triangle ABC$, let φ be a isoconjugation and let P and Q be points. If $P\varphi(Q) \cap \varphi(P)Q = R$, $PQ \cap \varphi(P)\varphi(Q) = S$, then R and S are φ-conjugates.



Proof. Let $\mathcal{L}$ be an arbitrary line.

Define an isoconjugation ψ that maps P to Q and let $\mathcal{L}^{\varphi}$ intersect $\mathcal{L}^{\psi}$ at X as a fourth intersection.

Then, by Theorem 12.2.13 it follows that

$$\mathbf{Per}_P^{\mathcal{L}^{\psi}}(X) = P\varphi(Q) = \mathbf{Per}_{\varphi(Q)}^{\mathcal{L}^{\varphi}}(X),\ \mathbf{Per}_Q^{\mathcal{L}^{\psi}}(X) = Q\varphi(P) = \mathbf{Per}_{\varphi(P)}^{\mathcal{L}^{\varphi}}(X)$$

As such, it follows that $R = \mathbf{Per}_P^{\mathcal{L}^{\psi}}(X) \cap \mathbf{Per}_Q^{\mathcal{L}^{\psi}}(X) = \mathbf{Li}_{P,Q}^{\mathcal{L}^{\psi}}(X)$. By the same logic, $R = \mathbf{Li}_{\varphi(P),\varphi(Q)}^{\mathcal{L}^{\varphi}}(X)$. It then follows that $\varphi(R)$ lies on both PQ and $\varphi(P)\varphi(Q)$ and is thus S. $\square$

Let's look at the special cases of permutation lines for setting the circumconic as $\Omega = (ABC)$. Then the permutation lines we get with common triangle centers are all very familiar lines!

Example 12.2.15. Let O be the circumcenter of $\triangle ABC$, and let X be a point on the circumcircle $\Omega = (ABC)$. Then $\mathbf{Per}_O^{\Omega}(X)$ is the orthotransversal of X, $\mathcal{O}(X)$.

Example 12.2.16. Let H be the orthocenter of $\triangle ABC$, and let X be a point on the circumcircle $\Omega = (ABC)$. Then $\mathbf{Per}_H^{\Omega}(X)$ is the Steiner line of X, $\mathcal{S}_X$.

Example 12.2.17. Let K be the symmedian point of $\triangle ABC$, and let X be a point on the circumcircle $\Omega = (ABC)$. Then $\mathbf{Per}_K^{\Omega}(X)$ is the trilinear polar $\mathfrak{t}(()X)$ of X.

Proof. The first one follows as if A' is the antipode of A, then $\angle AXA' = 90^\circ$.

The second one follows as if H_A, X_A are the reflections of H, X over BC, then the lines BC, H_AX, HX_A concur on BC.

For the third one, note that it remains to show that the polar of $AX \cap BC, XD, BC$ concur.

This then follows as

$$(A, D; B, C) \stackrel{X}{=} (AX \cap BC, XD \cap BC; B, C) = -1.$$

which finishes. $\square$

Following these examples, we will simplify our notation for the special case when the circumconic is just the circumcircle, and we will write just $\mathbf{Per}_P(X)$ for $\mathbf{Per}_P^{\Omega}(X)$. We will call $\mathbf{Per}_P(X)$ the P-permutation line of X.

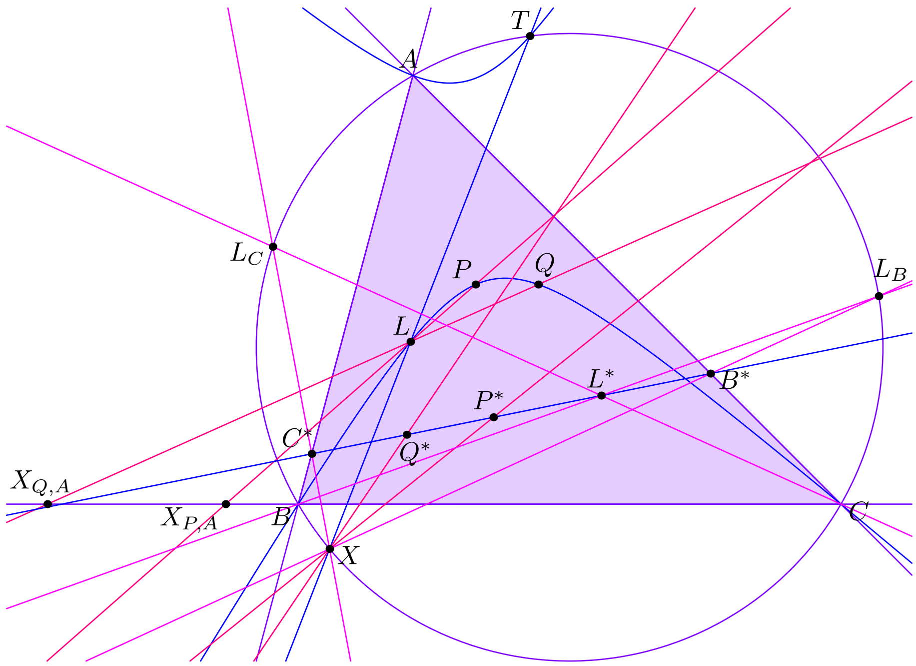
Proposition 12.2.18. For $\triangle ABC$ with point X on Ω and arbitrary P and Q with isogonal conjugates P^* and Q^*, we have that

$$\measuredangle(\mathbf{Per}_P(X), \mathbf{Per}_Q(X)) = \measuredangle Q^*XP^*$$

Notably, when $Q \in BC$ this becomes $\measuredangle(\mathbf{Per}_P(X), BC) = AXP^*$.

This is cleaner when written in line arguments, as just

$$\mathbf{Per}_P(X) + P^*X = (A + B + C + X)_\Omega.$$



Proof. Let $\mathcal{D}$ be a diagonal conic with center X. Note that since X is on the circumcircle and $\mathcal{D}$ passes through I, I^a, I^b, I^c, $\mathcal{D}$ is a rectangular hyperbola. Thus,

$$\mathbf{Per}_P(X) + P^*X = \mathfrak{p}_{\mathcal{D}}(P^*) + P^*X = \mathfrak{p}_{\mathcal{D}}(A) + AX = (A + B + C + X)_\Omega.$$

□

Remark 12.2.19. The specialized case when $Q \in BC$ has a specialized proof.

Let $P_A = AP \cap \Omega, P_A^* = AP^* \cap \Omega, D = AP \cap BC, X_{P,A} = XP_A \cap BC$. Note that $P_AP_A^* \parallel BC$.

As such, since $\measuredangle X_{P,A}P_AD = \measuredangle XP_AA = \measuredangle XP'_AD$ and $\measuredangle DX_{P,A}P_A = \measuredangle XP_AP'_A = \measuredangle XAP'_A$, it follows that $\triangle X_{P,A}P_AD \overset{-}{\sim} \triangle AP_A^*X$.

Define E on $P_AX_{P,A}$ such that $DE \parallel PX_{P,A}$. Then it is a well known lemma (see second spoiler at https://artofproblemsolving.com/community/c284651h1512821) that

$$\frac{AP^*}{P^*P_A^*} = \frac{PD}{DP_A} = \frac{X_{P,A}E}{EP_A}$$

It thus follows that $\triangle X_{P,A}ED \,\bar{\sim}\, \triangle AP^*X$, hence it follows that

$$\measuredangle AXP^* = \measuredangle EDX_{P,A} = \measuredangle(\mathbf{Per}_P(X), BC),$$

as desired.

Note that the first proof also allows us to extend this result to circumconics that correspond to isoconjugations that aren't isogonal conjugation.

12.3 The Barycentric Product

We have previously introduced the barycentric product (quotient) at the end of (TODO 7.4): given a triangle $\triangle = \triangle ABC$,

$$[x_1 : y_1 : z_1] \times [x_2 : y_2 : z_2] := [x_1x_2 : y_1y_2 : z_1z_2],$$
$$[x_1 : y_1 : z_1] \div [x_2 : y_2 : z_2] := \left[\frac{x_1}{x_2} : \frac{y_1}{y_2} : \frac{z_1}{z_2}\right]$$

We know that the isogonal conjugate of a point P in Δ is $K \times G \div P$, where G and K are respectively the centroid $[1 : 1 : 1]$ and the symmedian point $[a^2 : b^2 : c^2]$. Notice that there is no need to multiply by G again despite all coordinates being 1; the reason we do this is that multiplying by G again makes the left and right side "non-homogenous" (since barycentric coordinates can be arbitrarily scaled), and leads to notational confusion, especially when we transform triangle $\triangle ABC$. As another example, for a point P its isotomic conjugate in Δ is $G \times G \div P$.

Remark (Clarification on weight)**.** We define the **weight** of a point/expression as a way to formalize this "homogeneity", define it analogously to the degree of a polynomial. For example, the weight of $G = [1 : 1 : 1]$ is 1, but the weight of the fixed ratio $[1 : 1]$ is 0. This is so if we scale the barycentric coordinate both sides scale by the same amount. Barycentric quotient of two points returns a ratio (weight 0 point), and barycentric product of two points returns a "isoconjugation" (weight 2 point). A projective transformation acts on the plane (weight 1 points): $\varphi[x : y : z] = [ux : vy : wz]$ for some $[u : v : w]$ in $\mathbb{P}^2$. So we can extend it to any weight of points via

$$\varphi([x : y : z], k) := ([u^kx : v^ky : w^kz], k)$$

so that barycentric quotient and product are preserved. In particular, weight 0 points are preserved under this transformation. Also, the weight of both sides of a equality of barycentric products/quotients of points must be the same, to ensure we don't make a type-error.

More precisely, we consider the set

$$\mathbb{P}^2 \times \mathbb{Z} = \{([x : y : z], k) \mid [x : y : z] \in \mathbb{P}^2, k \in \mathbb{Z}\}.$$

The multiplication and division above are defined as

$$([x_1 : y_1 : z_1], k_1) \times ([x_2 : y_2 : z_2], k_2) := ([x_1x_2 : y_1y_2 : z_1z_2], k_1 + k_2),$$
$$([x_1 : y_1 : z_1], k_1) \div ([x_2 : y_2 : z_2], k_2) := \left(\left[\frac{x_1}{x_2} : \frac{y_1}{y_2} : \frac{z_1}{z_2}\right], k_1 - k_2\right)$$

and a point P in the plane has weight 1, i.e. $(P, 1)$.

Example 12.3.1.

- An isoconjugation on points φ is a transformation of the form

$$([x : y : z], 1) \longmapsto \left(\left[\frac{u}{x} : \frac{v}{y} : \frac{w}{z}\right], 1\right),$$

so if we let $\varphi = ([u : v : w], 2)$, then we have $\varphi(P) = \varphi \div P$. Thus an isoconjugation φ has weight 2.

- A projective transformation φ fixing A, B, and C is of the form

$$([x : y : z], 1) \mapsto ([ux : vy : wz], 1),$$

so if we let $\varphi = ([u : v : w], 0)$, then $\varphi(P) = \varphi \times P$. We sometimes call a point with weight 0 a **proportion**. In particular, the identity transformation is multiplication by $([1 : 1 : 1], 0)$, which we abbreviate as **1**.

In barycentric coordinates, the complement is

$$[u : v : w]^{\mathsf{C}} = [v + w : w + u : u + v],$$

and the anticomplement is

$$[u : v : w]^{\mathsf{J}} = [v + w - u : w + u - v : u + v - w].$$

In order for these operations to map points with weight 1 to points with weight 1, we assume for this purpose that both operations preserve weight, i.e.

$$([u : v : w], k)^{\mathsf{C}} = ([v + w : w + u : u + v], k),$$
$$([u : v : w], k)^{\mathsf{J}} = ([v + w - u : w + u - v : u + v - w], k).$$

Notice that these two transformations are associated with a choice of the centroid G (or equivalently the line at infinity $\mathcal{L}_\infty$) at weight 1, so in order to make them projectively invariant we should make them weight 0. One of the most basic collinearities comes from complementing: $G, P, P^{\mathtt{C}}$ collinear.

Proposition 12.3.2. For any proportion $r = ([u : v : w], 0)$,

$$r \times r^{\mathtt{C}} = \left(r^{-1}\right)^{\mathtt{C}}.$$

Proof. You know how to do algebra, right? □

If we take $r = P \div G$, we have

$$P \times P^{\mathtt{C}} = G^2 \times (G \div P)^{\mathtt{C}} = G \times (G^2 \div P)^{\mathtt{C}}.$$

Similarly, if we define

$$[u : v : w]^{\mathtt{D}} = [v - w : w - u : u - v] = [w - v : u - w : v - u],$$

then $r \times r^{\mathtt{D}} = \left(r^{-1}\right)^{\mathtt{D}}$.

This $^{\mathtt{D}}$ operation actually has a geometric meaning: the point $P^{\mathtt{D}}$ is the pole $\mathfrak{p}_{\mathcal{S}t}(GP)$ of GP in the Steiner circumellipse $\mathcal{S}t$, called the **difference**, and is undefined when $P = G$ (as the pole is undefined). Since G is the center of this ellipse, every $P^{\mathtt{D}}$ is on the line at infinity. Since $P, P^{\mathtt{C}}, P^{\reflectbox{$\mathtt{C}$}}, G$ are collinear, and every difference point lies on the lie at infinity, we have $P^{\mathtt{D}} = P^{\mathtt{CD}} = P^{\mathtt{DC}} = P^{\reflectbox{$\mathtt{C}$}\mathtt{D}} = P^{\mathtt{D}\reflectbox{$\mathtt{C}$}}$. In other words, $P^{\mathtt{D}}$ is the inverse of $P^{\mathtt{C}}$ and also the inverse of $P^{\reflectbox{$\mathtt{C}$}}$.

Definition 12.3.3. For any point X, let $\triangle X_aX_bX_c$ and $\triangle X^aX^bX^c$ be the cevian and anticevian triangles, respectively. We define the **X-complement transformation** $(-)^{\mathtt{C}_X}$ as the homography that sends A, B, C, X to X_a, X_b, X_c, X respectively, and the **X-anticomplement transformation** $(-)^{\reflectbox{$\mathtt{C}$}_X}$ as the homography that sends A, B, C, X to X^a, X^b, X^c, X respectively.

In particular, the G-complement and G-anticomplement transformations are the ordinary complement and anticomplement, respectively.

Similarly, let $\mathfrak{c}(X)$ be the circumconic with X as the perspector wrt. ABC. Then we define the **X-difference transformation** $(-)^{\mathtt{D}_X}$ of a point P as the pole of PX in $\mathfrak{c}(X)$.

Note that X, $\triangle X_aX_bX_c$, $\triangle X^aX^bX^c$, and $\mathfrak{c}(X)$ play the role of G, $\triangle G_aG_bG_c$, $\triangle G^aG^bG^c$, and $\mathcal{S}t = \mathfrak{c}(G)$

under the projective transformation $\varphi = X \div G$, so

$$
\begin{aligned}
(-)^{\mathsf{C}_X} &= \varphi \circ (-)^{\mathsf{C}} \circ \varphi^{-1}, \\
(-)^{\reflectbox{C}_X} &= \varphi \circ (-)^{\reflectbox{C}} \circ \varphi^{-1}, \\
(-)^{\mathsf{D}_X} &= \varphi \circ (-)^{\mathsf{D}} \circ \varphi^{-1}.
\end{aligned}
$$

and thus we get:

Proposition 12.3.4. Fix a point X. For any point P,

$$
\begin{aligned}
P^{\mathsf{C}_X} &= X \times (P \div X)^{\mathsf{C}}, \\
P^{\reflectbox{C}_X} &= X \times (P \div X)^{\reflectbox{C}}, \\
P^{\mathsf{D}_X} &= X \times (P \div X)^{\mathsf{D}}.
\end{aligned}
$$

More generally, for a weight k object $\mathbf{P}$,

$$
\begin{aligned}
\mathbf{P}^{\mathsf{C}_X} &= X^k \times (\mathbf{P} \div X^k)^{\mathsf{C}}, \\
\mathbf{P}^{\reflectbox{C}_X} &= X^k \times (\mathbf{P} \div X^k)^{\reflectbox{C}}, \\
\mathbf{P}^{\mathsf{D}_X} &= X^k \times (\mathbf{P} \div X^k)^{\mathsf{D}}.
\end{aligned}
$$

By (TODO 7.4.6), we have

Proposition 12.3.5. Given four points P, P^*, Q, Q^* such that $P \times P^* = Q \times Q^*$, $P, Q, PQ^* \cap P^*Q$ all lie on a common circumconic of $\triangle$.

Example 12.3.6. Let $G = X_2$, $K = X_6$, $Ge = X_7$, $Mt = X_9$, $Ge^* = X_{55}$, and $I' = X_{75}$. (These are the centroid, symmedian point, Gergonne point, Mittenpunkt, insimilicenter of the circumcircle and incircle, and the isotomic conjugate of the incenter respectively). Prove: GGe^*, KGe, and MtI' are concurrent.

Solution. Since I, G, and Na are collinear (Nagel line), by taking the isotomic conjugate I', G, and Ge lie on a circumconic $\mathcal{C}$ of $\triangle$. Since $G \times K = Ge \times Ge^*$, we get that GGe^* and KGe intersect at a point on $\mathcal{C}$. It thus suffices to show that $K \times I' = Ge \times Mt$, which follows from

$$
\begin{aligned}
K \times I' &= K \times (G^2 \div I) = (K \times G \div I) \times G = I \times G \\
&= (G \div Ge)^{\mathsf{C}} \times G^2 = (Ge \div G) \times G \times (Ge \div G)^{\mathsf{C}} \times G = Ge \times Mt.
\end{aligned}
$$

□

Let us recall the definition of the crosspoint, (TODO 7.1.17):

Definition 12.3.7. For any two points P and Q, the crosspoint $P \pitchfork Q$ is the pole of line PQ in the conic $(\triangle PQ)$.

Proposition 12.3.8. Let $R = P \pitchfork Q$ be the crosspoint of P and Q in $\triangle$. Then

$$R = P \times (P \div Q)^{\mathsf{C}} = Q \times (Q \div P)^{\mathsf{C}}.$$

Proof. Apply a projective transformation sending $\{A, B, C, Q\} \mapsto \{A, B, C, G\}$. Then it suffices to prove the case when $Q = G$, which is true since

$$R = G \times (G \div P)^{\mathsf{C}}.$$

□

Example 12.3.9. For any $\triangle$, the symmedian point K is the crosspoint of the centroid G and orthocenter H. Then we have

$$G \pitchfork H = H \times (H \div G)^{\mathsf{C}} = H \times O \div G,$$
$$G \pitchfork H = G \times (G \div H)^{\mathsf{C}} = G \times (H' \div G)^{\mathsf{C}}$$

So K is the isogonal conjugate of G and the complement of the isotomic conjugate H' of H.

Example 12.3.10. $X_{55} = I \pitchfork Mt$:

$$\begin{aligned} I \pitchfork Mt &= I \times (I \div Mt)^{\mathsf{C}} = I \times (Ge \div G)^{\mathsf{C}} \\ &= I \times (Mt \div G) = I \times (I \div Ge) = X_{55}. \end{aligned}$$

Next we recall the definition of the cevapoint (TODO 7.1.23):

Definition 12.3.11. Let $\triangle P^a P^b P^c$ and $\triangle Q^a Q^b Q^c$ be the anticevian triangles of P and Q. Then the cevapoint $P \star Q$ is the pole of PQ in conic $(PP^aP^bP^cQQ^aQ^bQ^c)$.

We define the ceva conjugate of S and Q as the unique point $P = S/Q$ such that $S = P * Q$.

Through the definition of crosspoints, we also get that $P \star Q$ is simultaneously the crosspoint of P and Q wrt. both triangles $\triangle P^a P^b P^c$ and $\triangle Q^a Q^b Q^c$.

Proposition 12.3.12. If $S = P * Q$ is the cevapoint of P and Q, then

$$S = Q \div (P \div Q)^{\mathsf{C}} = P \div (Q \div P)^{\mathsf{C}}.$$

Proof. Similarly to (TODO xiooix), assume that $Q = G$. Then S is the crosspoint of P and G with respect to $\triangle A^{\reflectbox{C}} B^{\reflectbox{C}} C^{\reflectbox{C}}$. Thus,

$$S^{\mathsf{C}} = P^{\mathsf{C}} \pitchfork G = G \times (G \div P^{\mathsf{C}})^{\mathsf{C}} = (G^2 \div P^{\mathsf{C}})^{\mathsf{C}},$$

so $S = G^2 \div P^{\mathsf{C}} = Q \div (P \div Q)^{\mathsf{C}}$. $\square$

Corollary 12.3.13. Let R and S be the crosspoint and cevapoint of P and Q, respectively. Then $P \times Q = R \times S$.

Corollary 12.3.14. Let $P = S/Q$ be the ceva conjugate of S and Q. Then

$$P = Q \times (Q \div S)^{\reflectbox{C}}.$$

Of course, we can also define the cross conjugate of R and Q as the unique point P satisfying $R = P \pitchfork Q$, then

$$P = Q \div (R \div Q)^{\reflectbox{C}}.$$

We denote P by $R \Psi Q$.

Proposition 12.3.15. For points P and Q,

$$P \times Q = (P/Q) \times (Q \Psi P).$$

In this section, we will use $\mathfrak{t}(P), \mathfrak{t}(\ell)$ as the set of points on the trilinear pole/polar of P, ℓ respectively, and $\mathfrak{c}(P)$ as the set of points on the circumconic $\mathcal{C}$ with perspector P, and $\mathfrak{c}(\mathcal{C})$ as the perspector of circumconic $\mathcal{C}$.

Define $t(P) \times c(Q)$ as the image of $t(P)$ under the isoconjugation $P \times Q$ is $c(Q)$, and vice versa.

Since $\mathfrak{t}(), \mathfrak{c}()$ are projectively equivariant with $\triangle ABC$, we have

Proposition 12.3.16. $\mathfrak{t}(P) \div P$ and $\mathfrak{c}(P) \div P$ are constant for all P. Specifically, we can choose P to be G, to get that $\mathfrak{t}(P) \div P = \mathcal{L}_\infty \div G = \mathfrak{t}(\mathbf{1}), \mathfrak{c}(P) \div P = \mathcal{S}\sqcup \div G = \mathfrak{c}(\mathbf{1})$.

Proposition 12.3.17 ((TODO [)7.4.8). reformulated] For two points P, Q, $P \times Q = \mathfrak{c}(P) \times \mathfrak{t}(Q)$. In other words, for an isoconjugation φ,

$$\varphi(\mathfrak{c}(P)) = \mathfrak{t}(\varphi(P)), \varphi(\mathfrak{t}(Q)) = \mathfrak{c}(\varphi(Q)).$$

Corollary 12.3.18. For two points P, Q, $Q \in \mathfrak{c}(P)$ if and only if $P \in \mathfrak{t}(Q)$.

Proof. The morally correct way to approach this is with extended La Hire's on the cubic $\triangle ABC$; Q is on the polar conic of P if and only if P is on the polar line of Q.

Let $\varphi = P \times Q$, then $\mathfrak{t}(Q) = \varphi(\mathfrak{c}(P))$. Therefore

$$P \in \mathfrak{t}(Q) = \varphi(\mathfrak{c}(P)) \iff Q = \varphi(P) \in \mathfrak{c}(P).$$

□

A diagonal conic $\mathcal{D}$ on $\triangle ABC$ is a conic such that $\triangle ABC$ is self-conjugate in $\mathcal{D}$. For any three points P, Q, R, these points lie on a common diagonal conic if and only if the isoconjugations represented by P^2, Q^2, R^2 all use the same diagonal conic, implying that the points P^2, Q^2, R^2 lie on a line $\mathfrak{t}(\varphi)$ (see (TODO 7.4.14) for further explanation and definition), where φ is some isoconjugation. Therefore we can always write a diagonal conic $\mathcal{D}$ as $\sqrt{\mathfrak{t}(\varphi)}$. Equivalently, $\varphi = \mathfrak{t}(\mathcal{D}^2)$, and the diagonal conic $\mathcal{D}_{P,Q}$ passing through P, Q is $\sqrt{P^2 \times Q^2}$.

Example 12.3.19. The polar circle $\mathcal{D}_\infty$, i.e. the diagonal conic through the two circle points, is just $\sqrt{\mathfrak{t}(G \times H)}$, where G, H are centroid and orthocenter.

Proposition 12.3.20. For any point P and a diagonal conic $\mathcal{D} = \sqrt{\mathfrak{t}(\varphi)}$,

$$\mathfrak{t}(\mathfrak{p}_{\mathcal{D}}(P)) = \mathfrak{p}_{\mathcal{D}}(\mathfrak{t}(P)) = \varphi(P) = \mathfrak{t}(\mathcal{D}^2) \div P.$$

Proof. First we prove that $\mathfrak{p}_{\mathcal{D}}(P) = \mathcal{D}^2 \div P$, which is true by the definition of isoconjugations, we have that for a point $Q \in \mathcal{D}$ that $Q^2 \div P$ lies on $\mathfrak{p}_{\mathcal{D}}(P)$. If $Q = \mathfrak{p}_{\mathcal{D}}(\mathfrak{t}(P))$, then $P = \mathfrak{t}(\mathfrak{p}_{\mathcal{D}}(Q)) = \varphi(Q)$. Therefore $\mathfrak{p}_{\mathcal{D}}(\mathfrak{t}(P)) = \varphi(P)$. □

Proposition 12.3.21. The line $\mathfrak{t}(P)$ is tangent to the diagonal conic $\mathcal{D} = \sqrt{\mathfrak{t}(\varphi)}$ if and only if $\varphi \in \mathfrak{t}(P^2)$. In other words, $P^2 \in \mathfrak{c}(\varphi)$.

Proof. By (TODO 12.3.20), $\mathfrak{t}(P)$ and $D = \sqrt{\mathfrak{t}(\varphi)}$ are tangent if and only if $\mathfrak{p}_{\mathcal{D}}(\mathfrak{t}(P)) = \varphi(P)$ lies on $\mathfrak{t}(P)$. This is equivalent to $\varphi \in \mathfrak{t}(P) \times P = \mathfrak{t}(P^2)$. □

Proposition 12.3.22. For a line $\mathfrak{t}(Q)$ and a point P, $\mathfrak{t}(Q)$'s image under P-complementing is

$$\mathfrak{t}(Q)^{\complement_P} = \mathfrak{t}(P \div (P \div Q)^{\daleth}),$$

and under P-anticomplementing is

$$\mathfrak{t}(Q)^{\daleth_P} = \mathfrak{t}(P \div (P \div Q)^{\complement}).$$

If we let $\varphi = P^2$, then we can write the above expression as

$$(-)^{\mathsf{C}_P} \circ \mathfrak{t} \circ \varphi = \mathfrak{t} \circ \varphi \circ (-)^{\mathsf{J}_P}$$

Proof. Consider the isoconjugation $\varphi = P^2$ and the diagonal conic $\mathcal{D} = \sqrt{\mathfrak{t}(\varphi)}$. From (TODO 12.3.20), for any point R we have

$$\mathfrak{t}(\varphi(R)) = \mathfrak{p}_{\mathcal{D}}(R).$$

Note that $P \div (P \div Q)^{\mathsf{J}} = P^2 \div (P^2 \div Q)^{\mathsf{J}_P} = \varphi(\varphi(Q)^{\mathsf{J}_P})$, so if we let $R = \varphi(Q)$, the first statement that we want to prove can be rewritten as

$$\mathfrak{p}_{\mathcal{D}}(R)^{\mathsf{C}_P} = \mathfrak{t}(\varphi(R))^{\mathsf{C}_P} = \mathfrak{t}(\varphi(R^{\mathsf{J}_P})) = \mathfrak{p}_{\mathcal{D}}(R^{\mathsf{J}_P}.$$

Since P, R, R^{J_P} are collinear, we have that the three lines $\mathfrak{p}_{\mathcal{D}}(P) = \mathfrak{t}(\varphi(P)) = \mathfrak{t}(P), \mathfrak{p}_{\mathcal{D}}(R), \mathfrak{p}_{\mathcal{D}}(R^{\mathsf{J}_P})$ concur at a point $W = \mathfrak{p}_{\mathcal{D}}(PR)$.

Let X be the intersection of PR with $\mathfrak{t}(P)$. Then $\mathfrak{p}_{\mathcal{D}}(X) = WP$. Therefore

$$\begin{aligned}(\mathfrak{t}(P), WP; \mathfrak{p}_{\mathcal{D}}(R), \mathfrak{p}_{\mathcal{D}}(R)^{\mathsf{C}_P}) = -2 = (P, X; R^{\mathsf{J}_P}, R)\\ \overset{\mathfrak{p}_{\mathcal{D}}}{=} (\mathfrak{t}(P), WP; \mathfrak{p}_{\mathcal{D}}(R), \mathfrak{p}_{\mathcal{D}}(R^{\mathsf{J}_P})).\end{aligned}$$

This tells us that $\mathfrak{p}_{\mathcal{D}}(R)^{\mathsf{C}_P} = \mathfrak{p}_{\mathcal{D}}(R^{\mathsf{J}_P})$. By substituting in R^{J_P} for R, we get that $\mathfrak{p}_{\mathcal{D}}(R)^{\mathsf{J}_P} = \mathfrak{p}_{\mathcal{D}}(R^{\mathsf{C}_P})$ also. Writing this back into (TODO 12.3.20), we get

$$\mathfrak{t}(Q)^{\mathsf{J}_P} = \mathfrak{p}_{\mathcal{D}}(\varphi(Q))^{\mathsf{J}_P} = \mathfrak{p}_{\mathcal{D}}(\varphi(Q)^{\mathsf{C}_P}) = \mathfrak{t}(P \div (P \div Q)^{\mathsf{C}}).$$

□

Proposition 12.3.23. Let $S = P \star Q$ be the cevapoint of P and Q in $\triangle ABC$. Then the polar of P wrt. $\mathfrak{c}(Q)$ is $\mathfrak{t}(S)$. In other words,

$$\mathfrak{t}(\mathfrak{p}_{\mathfrak{c}(Q)}(P)) = P \star Q.$$

Proof. Let $\triangle Q^aQ^bQ^c$ be the anticevian triangle of Q. Then $\mathfrak{c}(Q)$ is an inconic of $\triangle Q^aQ^bQ^c$, and S is the crosspoint of P and Q wrt. $\triangle Q^aQ^bQ^c$. Let D be the intersection of $\mathfrak{p}_{\mathfrak{c}(Q)}(P)$ wrt. BC. Then $\mathfrak{p}_{\mathfrak{c}(Q)}(D) = Q^aP$, which intersects AS on BC. Do this cyclically, and the trilinear polar of S is

$$\mathfrak{p}_{\mathfrak{c}(Q)}(P).$$

□

Corollary 12.3.24. For two points S, Q,

$$\mathfrak{p}_{\mathfrak{c}(Q)}(\mathfrak{t}(S)) = S/Q.$$

Corollary 12.3.25. If $\mathcal{C}$ is the image of line $\ell = \mathfrak{t}(U)$ under isoconjugation φ, then

$$\mathfrak{t}(\mathfrak{p}_{\mathcal{C}}((P))) = \varphi(U \times (U \div \varphi(P))^{\mathsf{C}}),$$

$$\mathfrak{p}_{\mathcal{C}}((\mathfrak{t}(P))) = \varphi(U \div (\varphi(P) \div U)^{\mathsf{\reflectbox{C}}}).$$

Proof. Let $Q = \varphi(U) = \varphi \div U$. Then we have

$$\mathfrak{t}(\mathfrak{p}_{\mathcal{C}}((P))) = Q \div (P \div Q)^{\mathsf{C}} = \varphi \div U \div (U \div \varphi(P))^{\mathsf{C}} = \varphi(U \times (U \div \varphi(P))^{\mathsf{C}}).$$

The second line (which is just ceva conjugation) is just rewriting the first line backwards. □

Theorem 12.3.26. For two points P, Q,

$$(P \pitchfork Q) \div \mathfrak{t}(PQ) = ((P \star Q) \div \mathfrak{t}(PQ))^{\mathsf{C}}.$$

Proof. Let $R = P \pitchfork Q, S = P \star Q, U = \mathfrak{t}(PQ), \varphi = P \times Q = R \times S$. Then $\mathcal{C} := \varphi(PQ) = (ABCPQ)$, and then we know by (TODO 12.3.25) that

$$\begin{aligned} R &= \mathfrak{p}_{\mathcal{C}}((\mathfrak{t}(U))) = \varphi(U \div (\varphi(U) \div U)^{\mathsf{\reflectbox{C}}}) = (\varphi \div U) \times (\varphi \div U^2)^{\mathsf{\reflectbox{C}}}, \\ S &= \varphi \div R = U \div (\varphi \div U^2)^{\mathsf{\reflectbox{C}}}. \end{aligned}$$

So if we let $r = \varphi \div U^2$, then we have

$$R \div U = r \times r^{\mathsf{\reflectbox{C}}} = r^{\mathsf{\reflectbox{C}}} \times r^{\mathsf{\reflectbox{C}}\mathsf{C}} = ((r^{\mathsf{\reflectbox{C}}})^{-1})^{\mathsf{C}} = (S \div U)^{\mathsf{C}}.$$

Specially, if we let $P, Q = I, J$, the circle points, then $I \pitchfork J = O, I \star J = H, \mathfrak{t}(IJ) = G$. Therefore, we proved that $O = H^{\mathsf{C}}$. □

Corollary 12.3.27. For line ℓ and a circumconic $\mathcal{C}$, if $U = \mathfrak{t}(\ell), R = \mathfrak{p}_{\mathcal{C}}((\ell))$, then

$$(R \div U) \times (R \div U)^{\mathsf{\reflectbox{C}}} = \ell \times \mathcal{C} \div U^2.$$

Proof. Just let $\{P, Q\} = \ell \cap \mathcal{C}$ in the previous. □

Proposition 12.3.28. The intersection of the trilinear polars of two points P, Q is

$$T = \mathfrak{t}(P) \cap \mathfrak{t}(Q) = P \times (P \div Q)^{\mathtt{D}} = Q \times (Q \div P)^{\mathtt{D}}.$$

Proof. From (TODO 12.3.24) and (TODO 12.3.14), we have

$$\mathfrak{t}(P) = \mathfrak{p}_{\mathfrak{c}(Q)}(P/Q) = \mathfrak{p}_{\mathfrak{c}(Q)}(Q \times (Q \div P)^{\reflectbox{$\mathtt{D}$}}),$$

and therefore

$$\mathfrak{t}(P) \cap \mathfrak{t}(Q) = Q \times (Q \div P)^{\reflectbox{$\mathtt{D}$}\mathtt{D}} = Q \times (Q \div P)^{\mathtt{D}}.$$

□

Proposition 12.3.29. The trilinear pole of the line through P, Q is just

$$U = \mathfrak{t}(PQ) = Q \div (P \div Q)^{\mathtt{D}} = P \div (Q \div P)^{\mathtt{D}}.$$

This can be applied to prove that three points P, Q, R are collinear if and only if

$$(Q \div P)^{\mathtt{D}} = (R \div P)^{\mathtt{D}}.$$

Proof. From (TODO 12.3.23) and (TODO 12.3.12), we have

$$\begin{aligned} \mathfrak{t}(PQ) &= \mathfrak{t}(\mathfrak{p}_{\mathfrak{c}(Q)}((P \div Q)^{\mathtt{D}} \times Q))) = ((P \div Q)^{\mathtt{D}} \times Q) \star Q \\ &= Q \div (P \div Q)^{\mathtt{DC}} = Q \div (P \div Q)^{\mathtt{D}}. \end{aligned}$$

□

Corollary 12.3.30. Let $T = \mathfrak{t}(P) \cap \mathfrak{t}(Q), U = \mathfrak{t}(PQ)$. Then $P \times Q = T \times U$.

When P, Q are the circle points I, J,

$$\begin{aligned} T &= \mathfrak{t}(I) \cap \mathfrak{t}(J) = \mathfrak{c}(\triangle IJ) = K \\ U &= \mathfrak{t}(PQ) = \mathfrak{t}(\mathcal{L}_\infty) = G, \end{aligned}$$

where K, G are the symmedian point and centroid. This proves that the symmedian point and centroid are isogonal conjugates. Note that we can also similarly define dually, for two lines p, q, the **crossline** and **cevaline** of p and q. Define the crossline $p \pitchfork q$ to be the polar of $p \cap q$ in the inconic tangent to all of the five lines $\triangle, p, q$, and define the cevaline $p \star q$ to be the polar of $p \cap q$ in the diagonal conic $\mathcal{D}_{p,q}$ defined as the diagonal conic tangent to both lines p, q.

Proposition 12.3.31. Let p, q, r, s respectively be the trilinear polars of P, Q, R, S wrt. $\triangle$. Then

- $R = P \pitchfork Q$ if and only if $r = p \star q$;
- $S = P \star Q$ if and only if $s = p \pitchfork q$.

Proof. Let $\varphi = \mathfrak{t}(P^2) \cap \mathfrak{t}(Q^2) = \mathfrak{c}(\triangle P^2 Q^2)$ be the isoconjugation with diagonal conic $\mathcal{D}_{p,q} = \sqrt{\mathfrak{t}(\varphi)}$ such that p, q are tangent. Then

$$\mathfrak{t}(p \star q) = \mathfrak{t}(\mathfrak{p}_{\mathcal{D}}(p \cap q)) = \varphi(p \cap q).$$

Since $\varphi = P^2 \times (P^2 \div Q^2)^{\mathtt{D}}$, we have $p \cap q = P \times (P \div Q)^{\mathtt{D}}$, therefore

$$\varphi(p \cap q) = P \times (P \div Q)^{\mathtt{C}} = R.$$

The proof for (ii) is just the dual of (i).

□

(TODO 12.3.30) has a natural generalization to lines t, u:

$$\begin{aligned} T = \mathfrak{t}(P) \cap \mathfrak{t}(Q) &\iff t = \mathfrak{t}(p \cap q) \\ U = \mathfrak{t}(PQ) &\iff u = \mathfrak{t}(p)\mathfrak{t}(q). \end{aligned}$$

Consider the point isoconjugation φ on the cevian triangle of S such that $\{A, B, C, S\} := \triangle \cup S$ are the four fixed points of φ. Then we have that $(-)^{\mathtt{Ɔ}}_S \circ \varphi \circ (-)^{\mathtt{C}}_S$ is the point isoconjugation S^2 on $\triangle$. Therefore for any point Q,

$$\begin{aligned} Q^{\varphi} &= (S^2 \div Q^{\mathtt{Ɔ}_S})^{\mathtt{C}_S} = S \times (((Q \div S)^{\mathtt{Ɔ}})^{-1})^{\mathtt{C}} \\ &= S \times (Q \div S)^{\mathtt{Ɔ}} \times (Q \div S) = Q \times (Q \div P)^{\mathtt{Ɔ}} = S/Q, \end{aligned}$$

in other words,

Proposition 12.3.32. Two points P, Q are exchanged by an isoconjugation that has $\triangle \cup S$ as its fixed points if and only if $S = P \star Q$.

Dually, consider the line isoconjugation φ on the cevian triangle of line r such that BC, CA, AB, r are the fixed lines of this isoconjugation (this trasformation known as QL-Tf2). The dual of ($TODO$12.3.32) tells us that two lines p, q are swapped under this isoconjugation if and only if $r = p \star q$. By using (TODO 12.3.31), we can get

Proposition 12.3.33. Two lines $\mathfrak{t}(P), \mathfrak{t}(Q)$ are exchanged (lines?) under the (line isoconjugation??) with $\triangle \cup \mathfrak{t}(R)$ as fixed (lines?) if and only if $R = P \pitchfork Q$.

Example 12.3.34. Consider the Newton line τ of the complete quadrilateral $\triangle \cup \mathfrak{t}(R)$. We know that Ql-Tf2 of the Newton line is the line at infinity $\mathcal{L}_\infty$. Therefore

$$\tau = \mathfrak{t}(R \Psi \mathfrak{t}(\mathcal{L}_\infty)) = \mathfrak{t}(R \Psi G),$$

where G is the centroid.

Example 12.3.35. Let E, F respectively be the B, C-intouch points of incircle ω in $\triangle ABC$. Suppose we now want to analyze the second real intersection point of (AEF) with the circumcircle, M. We can analyze this as such: note that $EF = \mathfrak{t}(Ge^a)$ (Ge^a is A-extraversion of Gergonne pt), and therefore the Newton line of quadrilateral $\triangle \cup \overline{EF}$ is just $\mathfrak{t}(Ge^a \Psi G)$, where

$$Ge^a \Psi G = G \div (Ge^a \div G)^{\reflectbox{\texttt{C}}} = [1] \div \left[\frac{-1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right]^{\reflectbox{\texttt{C}}}.$$

But M is just the Miquel point of $\triangle \cup EF$, so its isogonal conjugate wrt. $\triangle$ is $\infty_\tau = \mathfrak{t}(G) \cap \mathfrak{t}(Ge^a \Psi G)$, and therefore

$$\begin{aligned}
M &= [a^2] \div (G \times (G \div (Ge^a \Psi G))^{\mathtt{D}}) \\
&= [a^2] \div \left[\frac{-1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right]^{\mathtt{D}} \\
&= [a^2] \div [a-b-c : c+a-b : a+b-c] \times [a-b-c : c+a-b : a+b-c]^{\mathtt{D}} \\
&= \left[\frac{a^2}{(b-c)(b+c-a)} : \frac{b}{c+a-b} : \frac{-c}{a+b-c}\right].
\end{aligned}$$

In more normal words, $\triangle \cup \mathfrak{t}(R)$'s Miquel point is $[a^2] \div (R \div G)^{\mathtt{D}}$.

12.3.1 Permutation Line Revisited

Theorem 12.3.36. For any point P and a point X on a circumconic $\mathcal{C}$,

$$\mathbf{Per}_P^{\mathcal{C}}(X) = P \times X \div \mathcal{C} = \mathfrak{t}(P \times X \div \mathfrak{c}(\mathcal{C})),$$

Advancing this a bit, if we define $\mathbf{Per}_P^{\mathcal{C}}(X)$ as the line through

$$P_A X_A \cap BC, P_B X_B \cap CA, P_C X_C \cap AB$$

(where $\triangle P_AP_BP_C, \triangle X_AX_BX_C$ are the $\mathcal{C}$-cyclocevian triangles of P, X respectively), then

$$\mathbf{Per}_P^{\mathcal{C}}(X) = P \times X \div \mathcal{C} = \mathfrak{t}(P \times X \div \mathfrak{c}(\mathcal{C})).$$

Proof. Let point $\mathbf{Per}_a = P_AX_A \cap BC$. We only need to prove that $\mathbf{Per}_a$ lies on $P \times X \div \mathcal{C}$, which is equivalent to proving that the image of $A\mathbf{Per}_a$ after applying the isoconjugation from $P \times X$, $\varphi = \varphi^{P\times X}$ is tangent to $\mathcal{C}$ at A. But this is true, since

$$\begin{aligned}(AB, AC; \mathbf{T}_A\mathcal{C}, AX) &= P_A(B, C; A, X_A) = (B, C; AP \cap BC, \mathbf{Per}_a)\\ &= A(B, C; P, \mathbf{Per}_a) = (AC, AB; AX, \varphi(A\mathbf{Per}_a))\\ &= (AB, AC; \varphi(A\mathbf{Per}_a), AX).\end{aligned}$$

□

Example 12.3.37. For a point X on the circumcircle Ω, the orthotransversal $\mathcal{O}(X)$ of X in $\triangle ABC$ is

$$\mathcal{O}(X) = \mathbf{Per}_O^{\Omega}(X) = \mathfrak{t}(O \times X \div K),$$

and the Steiner line is

$$\mathcal{O}(X) = \mathbf{Per}_H^{\Omega}(X) = \mathfrak{t}(O \times X \div K).$$

Since

$$\begin{aligned}\mathbf{Li}_{P,Q}^{\mathcal{C}} &= \mathbf{Per}_P^{\mathcal{C}}(X) \cap \mathbf{Per}_Q^{\mathcal{C}}(X)\\ &= \mathfrak{t}(P \times X \div \mathfrak{c}(C)) \cup \mathfrak{t}(Q \times X \div \mathfrak{c}(C))\\ &= (\mathfrak{t}(P) \cap \mathfrak{t}(Q)) \times X \div \mathfrak{c}(\mathcal{C}) = \mathfrak{c}(ABCPQ) \times X \div \mathfrak{c}(\mathcal{C}),\end{aligned}$$

so we get

Theorem 12.3.38. For any three points X, Y, Z, $\mathbf{Li}_{\mathfrak{c}(Z)}^{\mathfrak{c}(Y)}(X) = Z \times X \div Y$.

Example 12.3.39. We will prove (TODO 12.2.5) with this theory: If we let T be the fourth intersection of $\mathfrak{c}(Y)$ and $\mathfrak{c}(Z)$, then $T, X, \mathbf{Li}_{\mathfrak{c}(Z)}^{\mathfrak{c}(Y)}(X)$ are collinear.

Proof. Because $T = \mathfrak{c}(Y) \cup \mathfrak{c}(Z) = \mathfrak{t}(YZ) = Y \div (Z \div Y)^{\mathsf{D}}$, so $T, X, \mathbf{Li}_{\mathfrak{c}(Z)}^{\mathfrak{c}(Y)}(X) = Z \times X \div Y$ are collinear if and only if $Y \div X, (Z \div Y)^{\mathsf{D}}, (Z \div Y) \times (Z \div Y)^{\mathsf{D}} = (Y \div Z)^{\mathsf{D}}$ are collinear. However this is just because the final two points lie on $\mathfrak{t}(\mathbf{1})$, and the first one also lies on $\mathfrak{t}(X) \div X = \mathfrak{t}(\mathbf{1})$! □

12.3.2 Radical and Square Transformations

In the view of analytical geometry, sometimes we see an expression for a point has every term have only squares a^2, b^2, c^2. In this case we can rewrite it as $a^2 = u, b^2 = v, c^2 = w$. So can we actually just rephrase our whole problem in terms of u, v, w? By doing this, three points X, Y, Z collinear becomes A, B, C, X', Y', Z' conconic.

So how do we interpret this purely geometrically? Let's define a new geometrical framework, by redefining the two circle points $I_i^{\mathsf{s}}, J^{\mathsf{s}}$ as the two intersections of $\mathfrak{c}(K^2 \div G)$ with $\mathcal{L}_\infty$ (K here is the symmedian point.), while fixing the line at infinity. We can reinterpret

$$K^2 \div G = \mathfrak{c}(\triangle I_i^{\mathsf{s}} J^{\mathsf{s}})$$

as $K = \mathfrak{c}(\triangle IJ)$. Since $(K^2 \div G) \times G = K^2$, (in normal geometry), we get that the equivalent of the symmedian point in this new geometry is the point K such that $K \times G = K$, which is just the incenter or the three excenters (any point and its extraversions is the same under this new geometry).

This lets us define more operations in barycentric coordinates, the **radical transformation** $(-)^{\mathsf{r}}$. By definition, $(I^{\mathsf{s}})^{\mathsf{r}} = I$ and similarly for J.

Example 12.3.40. Since $G^{\mathsf{r}} = \mathfrak{t}(\mathcal{L}_\infty) = G, K^{\mathsf{r}} = I$, then the Nagel point $Na = I^{\mathsf{Ɔ}} = (K^{\mathsf{Ɔ}})^{\mathsf{r}} = (H')^{\mathsf{r}}$ where H' is the isotomic conjugate of H. Therefore we have $H^{\mathsf{r}} = (G^2 \div H')^{\mathsf{r}} = G^2 \div Na = Ge$. Further, $O^{\mathsf{r}} = Ge^{\mathsf{C}} = Mt$. This gives us the following table of points and their transformations:

P	G	O	H	K	H'
P^{r}	G	Mt	Ge	I	Na

Inversely, we can change a, b, c to u^2, v^2, w^2 (the geometrical interpretation of this is defining two new circle points, $I^{\mathsf{r}}, J^{\mathsf{r}}$ as the two intersections of $\mathfrak{c}(I)$ with $\mathcal{L}_\infty$). We call this the **square transformation** and we will notate it as $(-)^{\mathsf{s}}$. s, r are inverses of each other (this requires proof but is left to the reader).

In full generality, if we want to let $u = u(a, b, c), v = v(a, b, c), w = w(a, b, c)$, then this defines two new circle points $I^{\text{new}}, J^{\text{new}}$ as the two intersections of $\mathfrak{c}(P)$ and $\mathcal{L}_\infty$, where $P = [u(a, b, c)^2 : v(a, b, c)^2 : w(a, b, c)^2]$.

Chapter 13

The Worst Of X_n

The objective of this chapter is to introduce, for $1 \leq n \leq 100$, the triangle centers X_n. Recall that:

- X_1 is the incenter I, the point where the angle bisectors concur;
- X_2 is the centroid G, the point where the medians concur;
- X_3 is the circumcenter O, the point where the perpendicular bisectors of the sides concur;
- X_4 is the orthocenter H, the point where the altitudes concur;
- X_5 is the center of the nine-point circle, N;
- X_6 is the symmedian point K, the isogonal conjugate of G.
- X_7 is the Gergonne point Ge, the perspective center of the reference triangle and the contact triangle;
- X_8 is the Nagel point Na, the perspective center of the reference triangle and the extouch triangle;
- X_9 is the Mittenpunkt Mt, the perspective center of the excentral triangle and the medial triangle;
- X_{10} is the Spieker center Sp, the anticomplement of I;
- X_{11} is the Feuerbach point Fe, the point where the incircle ω and the nine point circle ε are tangent;
- X_{13} is the first Fermat point F_1, which satisfies

$$\measuredangle BF_1C = \measuredangle CF_1A = \measuredangle AF_2B = 120^\circ;$$

- X_{14} is the second Fermat point F_2, which satisfies

$$\measuredangle BF_2C = \measuredangle CF_2A = \measuredangle AF_2B = 60^\circ;$$

- X_{15} is the first isodynamic point S_1, the isogonal conjugate of F_1;
- X_{16} is the second isodynamic point S_2, the isogonal conjugate of F_2;
- X_{19} is the Clawson point $C\ell$, the perspective center of the excentral triangle and the orthic triangle;
- X_{20} is the de Longchamps point L, the reflection of H over O;
- X_{21} is the Schiffler point Sc, the concurrency point of the Euler lines of the triangles $\triangle ABC$, $\triangle IBC$, $\triangle AIC$, $\triangle ABI$;
- X_{25} is the homothetic center of the orthic and tangential triangles;
- X_{33} is the perspector of the orthic and intangents triangles;
- X_{40} is the Bevan point Be, the reflection of I over O;
- X_{54} is the Kosnita point Ko, the isogonal conjugate of N;
- X_{55} is the insimilicenter of Ω and ω Ge^*;
- X_{56} is the exsimilicenter of Ω and ω Na^*;
- X_{65} is the orthocenter of the intouch triangle $Sc*$;
- X_{69} is the isotomic conjugate of H, H';
- X_{104} is the isogonal conjugate of ∞_{OI}, ∞_{OI}^*.

Additionally we can define $\triangle_P = \triangle P_aP_bP_c$ representing the cevian triangle, $\triangle^P$ representing the anticevian triangle. Then we let X_{nP}, X_n^P represent X_n in the triangles $\triangle_P$ and $\triangle^P$. For example, O_H is the circumcenter of $\triangle_H = \triangle H_aH_bH_c$, that is, N. For $\triangle_H$, we always select X_1, or I_H, and similarly $\triangle^K$, $I^K = O$.

First let us see a commonly used theorem:

Theorem 13.0.1. For $17 \le n \le 53$, (X_n, X_{n+44}) is a pair of isogonal conjugates.

For example,

Proposition 13.0.2. The isogonal conjugate of X_{25} is H'.

Proof. By definition

$$X_{25} = H/K = K \times (K \div H)^{\text{Ↄ}} = K \times (O \div G)^{\text{Ↄ}} = K \times H \div G.$$

Therefore $G \times K \div X_{25} = G^2 \div H = H'$. $\square$

So we will write $(H')^*$ as X_{25}.

13.1 X_1 - related points

If we define the incenter $I = X_1$ as the fixed point of the isoconjugation $\infty_i \times \infty_{-i}$, then actually there are four choices for I (the other three being the excenters). Some points depend on this selection, like the Feuerbach point $Fe = X_{11}$, while others do not, like the centroid $G = X_2$.

Proposition 13.1.1. The three points I, O, K lie on a (non-circum) rectangular hyperbola. This hyperbola is called the Stammler hyperbola of $\triangle ABC$, and is denoted $\mathcal{D}_S$.

Proof. Consider the hyperbola $\mathcal{D}$ through O and K, we only need to prove that D is a rectangular hyperbola. Because O and K are the anticevian points of $\triangle^K$, that is, the incenter (or excenter) and Gergonne point of the triangle $\mathbf{T}_\triangle\Omega = \triangle(\mathbf{T}_A\Omega)(\mathbf{T}_B\Omega)(\mathbf{T}_C\Omega)$, $\mathcal{D}$ is the Feuerbach hyperbola of the tangential triangle, and is thus rectangular. □

Because the line through the circumcenter of $\mathbf{T}_\triangle\Omega$ and O is the Euler line $\mathcal{E}$, we have:

Corollary 13.1.2. The Euler line $\mathcal{E}$ is tangent to $\mathcal{D}_S$ at O.

Lemma 13.1.3. For any point P,

- The crosspoint of I and P, $I \pitchfork P$, and the cevapoint $I \star P^*$, are isogonal conjugates.
- The cross conjugate of P and I, $P \,\Psi\, I$, and the ceva conjugate P/I, are isogonal conjugates.

Proof. Because $I \div P = P^* \div I,\ P \div I = I \div P^*$, we have

$$(I \pitchfork P) \times (I \star P^*) = (I \times (I \div P)^{\complement}) \times (I \div (P^* \div I)^{\complement}) = I^2$$
$$(P \,\Psi\, I) \times (P^*/I) = (I \div (P \div I)^{\reflectbox{$\complement$}}) \times (I \times (I \div P^*)^{\reflectbox{$\complement$}}) = I^2.$$ □

Of course, if I was replaced with the fixed point of any isogonal conjugation φ, and P^* was replaced with P^φ, then the above proof would still be valid.

13.1.1 What points are on OI and $\mathcal{H}_{Fe}$?

When $n = 1, 35, 36, 40, 46, 55, 56, 57, 65,\ X_n \in OI$ and is related to the selection of X_1.

When $n = 1, 7, 8, 9, 21, 79, 80, 84, 90,\ X_n \in \mathcal{H}_{Fe}$ and is related to the selection of X_1.

Out of these points, we have already introduced $n = 1, 3, 4, 7, 8, 9, 21, 40, 55, 56, 65$.

Proposition 13.1.4. The quadrilateral $(IMt)(NaSc)$ is a harmonic quadrilateral on $\mathcal{H}_{Fe}$.

Proof. This is just because (by (TODO 5.6.8))

$$(I, Mt; Na, Sc)_{\mathcal{H}_{Fe}} = H(I, Mt; Na, Sc) = (I, Be; \infty_{OI}, O) = -1. \qquad \square$$

Proposition 13.1.5. Under the barycentric product, $I \times Na = G \times Mt = Sp \times Sc$. Then $I \times Sp = Na \times Sc^*$.

Proof. $I \times Na = G \times Mt$ is, under the square transformation, equivalent to $K \times H' = G \times O$, which is clearly true. Under the isogonal conjugation $I \times Na$,

$$(Na, I \times Na \div Sp; Mt, I) = (I, Sp; G, Na) = -1 = (Na, Sc; Mt, I),$$

thus $I \times Na = Sp \times Sc$. $\square$

Through (TODO 5.6.5), we have

Proposition 13.1.6. The Schiffler point Sc is actually just the cevapoint $I \star O$.

X_{57}, X_{84}

- X_{57} is the isogonal conjugate of $Mt = X_9$, Mt^*.
- X_{84} is the isogonal conjugate of $Be = X_{40}$, Be^*.

Proposition 13.1.7. Point Mt is on $\mathcal{H}_{Fe}$, and

$$(I, Mt; Na, Sc)_{\mathcal{H}_{Fe}} = -1,$$

so Mt^* lies on OI and satisfies $(I, Mt^*; Na^*, Sc^*) = -1$.

Proof. By (TODO 5.4.5), if we let A be the orthocenter of $\triangle BIC$, N_a be the second intersection of AI with Ω, then

$$H_A(I, Mt; Na, Sc) = (I, I^a; \infty_{AI}, N_a) = -1.$$

And this is symmetric in A, B, C, so $Mt \in (INaScH_AH_BH_C) = \mathcal{H}_{Fe}$. $\square$

Given the definition of Mt, we know that it is just the ceva conjugate of G and I, G/I, and so

$$I \times (I \div G)^{\text{ↄ}} = I \times Na = I \times G \div Ge.$$

Then

$$Mt^* = I \times Ge \div G = I \times (Mt \div G)^{\mathfrak{I}} = I \times (I \div Ge)^{\mathfrak{I}} = Ge/I,$$

which gives us:

Proposition 13.1.8. The point Mt^* is the homothetic center of the excentral triangle $\triangle I^a I^b I^c$ and the contact triangle $\triangle DEF$.

Corollary 13.1.9. The points G, Ge, Mt, Mt^* are collinear.

Proof. We already know that G, Ge, Mt^* are collinear ((TODO 5.4.1)), and Ge, Mt, Mt^* are collinear because $\triangle DEF$, $\triangle I^a I^b I^c$ are the symmedian points of the respective triangles [is this right?] □

Proposition 13.1.10. We have $Be = OMt^* \cap HMt, Be^* = OMt \cap HMt*$.

Proof. By (TODO 7.4.5), we just need to prove $Be = OMt^* \cap HMt$. $Be \in OMt^*$ is just (TODO 13.1.7), $Be \in HMt$ is just (TODO 5.6.8). □

$X_{35}, X_{36}, X_{79}, X_{80}$

- X_{36} is the inverse of I about Ω, $I^{\mathfrak{I}}$;
- X_{35} is the harmonic conjugate of $I^{\mathfrak{I}}$ on OI;
- X_{79} is the isogonal conjugate of X_{35};
- X_{80} is the isogonal conjugate of X_{36}.

The most famous of these four points should be X_{80}, because we have by (TODO 8.1.19):

Proposition 13.1.11. The point X_{80} is the antigonal conjugate of I, so we denote X_{80} by $\hat{I}$.

Corollary 13.1.12. We have that I, N, Fe, $Fe^\vee$, $\hat{I}$ are collinear.

And our first thought upon seeing X_{36} should be (TODO 5.5.6), because it tells us:

Proposition 13.1.13. The three lines $Fe^\vee X_{35}$, $FeI^{\mathfrak{I}}$, IX_{79} are parallel to the Euler line $\mathcal{E}$, which is to say, all pass through $\infty_{\mathcal{E}} = X_{30}$.

Proof. We already have that Fe, $\infty_{\mathcal{E}}$, $I^{\mathfrak{I}}$ are collinear. Thus

$$\infty_{\mathcal{E}}(Fe, Fe^\vee; I, N) = -1 = \infty_{\mathcal{E}}(I^{\mathfrak{I}}, X_{35}; I, O)$$

gives us that $Fe^\vee = X_{13}$, $\infty_{\mathcal{E}}$, X_{35} are also collinear.

Finally, by (TODO 13.1.11), (TODO 13.1.12),

$$\begin{aligned}I(O, H; X_{79}, \hat{I}) &= (I, H; X_{79}, \hat{I})_{\mathcal{H}_{Fe}}\\ &= (I, O; X_{35}, I^{\mathfrak{J}}) = -1 = (H, O; \infty_{\mathcal{E}}, N),\end{aligned}$$

and moreover I, $\infty_{\mathcal{E}}$, X_{79} are collinear. □

Proposition 13.1.14. We have

$$(I, X_{35}; O, Ge^*) = (I, I^{\mathfrak{J}}; O, Na^*) = -1.$$

Proof. Because $(I, O; X_{35}, I^{\mathfrak{J}}) = (I, O; Ge^*, Na^*) = -1$, we only need to prove that $(I, I^{\mathfrak{J}}; O, Na^*) = -1$. But this is equivalent to $(I, \hat{I}; H, Na) = -1$, and this is true because the tangents at I, $\hat{I}$ to $\mathcal{H}_{Fe}$ are both parallel to $OI \parallel HNa$. □

Proposition 13.1.15. The Schiffler point $Sc = X_{21}$ is the intersection of $X_{35}\hat{I}$ and $I^{\mathfrak{J}}X_{79}$, and its isogonal conjugate $Sc^* = X_{65}$ is the intersection of $OI = X_{35}I^{\mathfrak{J}}$ and $X_{79}\hat{I}$.

Proof. Because

$$(I, H; X_{79}, \hat{I})_{\mathcal{H}_{Fe}} = (I, O; X_{35}, I^{\mathfrak{J}}) = -1,$$

$X_{79}I^{\mathfrak{J}}$ passes through the polar of IH in $\mathcal{H}_{Fe}$, Sc^*, but we know already that $Sc^* \in OI$. □

Proposition 13.1.16. The points Sp, Sc, X_{35}, $\hat{I}$ are collinear.

Proof. We show that Sp, Sc, X_{35} are collinear: by (TODO 13.1.14),

$$Sc(I, Sp; G; Na) = -1 = Sc(I, X_{35}; O, Ge^*),$$

since we only need Na, Sc, Ge^* collinear, which is (TODO 5.6.19). □

Proposition 13.1.17. The points $I^{\mathfrak{J}}$, $\hat{I}$, ∞^*_{OI} are collinear.

Proof. We have $(I\hat{I})(HNa)$ is a complete quadrilateral on $\mathcal{H}_{Fe}$, so from (TODO 13.1.14) and (TODO 8.2.18),

$$\infty^*_{OI}(I, \hat{I}; H, Na) = -1 = (I, I^{\mathfrak{J}}; Na^*, O) = \infty^*_{OI}(I, I^{\mathfrak{J}}; H, Na),$$

and $I^{\mathfrak{J}}$, $\hat{I}$, ∞^*_{OI} are collinear. □

13.1.2 X_{46}, X_{90}

- X_{46} is the ceva conjugate of H and I, H/I;
- X_{90} is the isogonal conjugate of H/I.

Because the pole of X_{46} in the diagonal conic with I will pass through $X_{90} = X_{46}^*$ and H, there is:

Proposition 13.1.18. The points H, X_{46}, X_{90} are collinear.

Because

$$H/I = I \times (I \div H)^{\text{Ↄ}} = I \times (O \div I)^{\text{Ↄ}}$$

Proposition 13.1.19. The point X_{46} is the reflection of O over I, and notably, X_{46} lies on OI.

On the other hand, it turns out that:

Proposition 13.1.20. The point Sc^* is the I-complement of O.

Proof. Because Sc^* is the crosspoint of I and H, we have

$$Sc^* = I \times (I \div H)^{\text{Ↄ}} = I \times (O \div I)^{\text{Ↄ}}.$$

□

Of course, this could also be proven by (TODO 13.1.6), because:

Proposition 13.1.21. The points I, O harmonically divide X_{46}, Sc^*, so

$$(I, H; X_{90}, Sc)_{\mathcal{H}_{Fe}} = -1$$

By (TODO 13.1.3), we have $X_{90} = H^* \Psi I = O \Psi I$, and the polar of O in $\mathcal{H}_{Fe}$ is IX_{90}.

Proposition 13.1.22. We have $X_{46} = I^{\mathfrak{J}}Be \cap \hat{I}Be^*$, $X_{90} = I^{\mathfrak{J}}Be^* \cap \hat{I}Be$.

Proof. By (TODO 7.4.5), we just need to prove that $Be = I^{\mathfrak{J}}X_{46} \cap \hat{I}X_{90}$. Obviously, $Be \in OI = I^{\mathfrak{J}}X_{46}$.

Because the tangent to $\mathcal{H}_{Fe}$ from X_{90} passes through O, we have

$$\hat{I}(I, X_{90}; \infty_{\mathcal{E}}, O) = (I, X_{90}; \hat{I}, \hat{I}O \cap \mathcal{H}_{Fe})_{\mathcal{H}_{Fe}} = -1,$$

and so $\hat{I}X_{90}$ passes through Be. □

Proposition 13.1.23. We have $X_{46} = MtX_{79} \cap Mt^*X_{35}$, $X_{90} = MtX_{35} \cap Mt^*X_{79}$.

Proof. By (TODO 7.4.5), we just need to prove that $X_{35} = MtX_{90} \cap Mt^*X_{46}$. Obviously, $X_{35} \in OI = Mt^*X_{46}$.

By (TODO 13.1.10), O, Mt, Be^* are collinear, so

$$(I, X_{90}; Mt, Be^*) = -1.$$

With (TODO 13.1.22), this tells us

$$X_{90}(I, O; Mt, I^{\mathfrak{J}}) = (I, X_{90}; Mt, Be^*)_{\mathcal{H}_{Fe}} = -1,$$

and so Mt, X_{35}, X_{90} are collinear. □

13.1.3 IG and $(IG)^*$?

When $n = 1,\ 8,\ 10,\ 42,\ 43,\ 78$, $X_n \in IG$ depends on the choice of X_1.

When $n = 1,\ 34,\ 56,\ 58,\ 86,\ 87$, $X_n \in (IG)^*$ depends on the choice of X_1.

Proposition 13.1.24. The point $Sp = I^{\mathsf{C}}$ lies on the Kiepert hyperbola, $\mathcal{H}_K$.

Proof. Consider the diagonal conic $\mathcal{D}$ of I, G. Because I^a, I^b, I^c lie on $\mathcal{D}$, $\mathcal{D}$ is a diagonal conic; it passes through the centroid H^{C} of the anticevian triangle of G, $\triangle A^{\mathsf{C}}B^{\mathsf{C}}C^{\mathsf{C}}$. So $\mathcal{D}^{\mathsf{C}}$ is a circumconic of $\triangle$ which passes through G, H, Sp, that is, $\mathcal{H}_K$. □

Corollary 13.1.25. The points I, G, L lie on one diagonal conic.

Proof. Because $Sp = I^{\mathsf{C}}$, $G = G^{\mathsf{C}}$, $H = L^{\mathsf{C}}$ share a circumconic. □

- X_{58} is the isogonal conjugate of Sp, Sp^*.

Because G, H, Sp lie on one circumconic, we also have:

Proposition 13.1.26. The point Sp^* lies on the Brokard line OK.

Proposition 13.1.27. The points I, Sc, Sp^* lie on a line, which is the anticomplement of $SpSc^*$.

Proof. Consider the hexagon $III'Sc^*Sp(I')^{\mathsf{C}}$ on $(\triangle ISp)$. By Pascal's Theorem,

$$\mathbf{T}_I(\triangle ISp) \cap Sc^*Sp,\ II' \cap Sp(I')^{\mathsf{C}},\ I'Sc^* \cap (I')^{\mathsf{C}}I$$

are collinear. Also, $II' = (Sp(I')^{\complement})^{\reflectbox{$\complement$}}$ by (TODO 13.1.52), or $I'Sc^* = ((I')^{\complement}I)^{\reflectbox{$\complement$}}$. So, $IScSp^* = \mathbf{T}_I(\triangle ISp) \parallel Sc^*Sp$. Because $I = Sp^{\reflectbox{$\complement$}}$, we have $IScSp^* = (SpSc^*)^{\reflectbox{$\complement$}}$. □

- X_{42} is the crosspoint of I and K, $I \pitchfork K$;
- X_{43} is the ceva conjugate of K and I, K/I;
- X_{86} is the cevapoint of I and G, $I \star G$;
- X_{87} is the cross conjugate of G and I, $G \Psi I$.

By (TODO 13.1.3), (X_{42}, X_{86}), (X_{43}, X_{87}) are isogonal conjugates. Because IX_{42} is tangent to $(\triangle IK)$, we have $IX_{42} = (\triangle IK)^* = IG$. Because IX_{43} is the trilinear polar of K in $(\triangle^I X_{43})$, we have $K^* = G$.

13.1.4 IH and $(IH)^*$?

When $n = 1,\ 33,\ 34,\ 73$, $X_n \in IH$ depends on the choice of X_1.

When $n = 1,\ 29,\ 77,\ 78$, $X_n \in (IH)^*$ depends on the choice of X_1.

- X_{33} is the perspective center of the intouch triangle and the orthic triangle;
- X_{34} is the harmonic conjugate of X_{33} on $\overline{IH}$;
- X_{77} is the isogonal conjugate of X_{33};
- X_{78} is the isogonal conjugate of X_{34}.

Proposition 13.1.28. The cevapoint of $C\ell \star X_{33}$ is just H.

Proof. Notice that the homothety centered about A which sends I to I^a also sends T_a to T_a', the reflection over I, so

$$(AH_a, BC; H_aC\ell, H_aX_{33}) = H_a(A, AI \cap BC; T_a', T_a) = A(H_a, I; T_a', T_a) = -1.$$

So by symmetry $C\ell \star X_{33} = H$. □

Corollary 13.1.29. Under the barycentric product,

$$I \times H = Ge \times X_{33} = Na \times X_{34},$$

$$I \div H = X_{77} \div Ge = X_{78} \div Na.$$

Proof. By (TODO 13.1.41), $C\ell = I \times H \div G$, so

$$\begin{aligned} X_3 3 &= (I \times H \div G) \times (I \times H \div G \div H)^{\supset} \\ &= I \times H \div X \times (Na \div G) = I \times H \div Ge. \end{aligned}$$

Because $(H, I; X_{33}, X_{34}) = -1$, $I \times H \div X_{34}$ lies on $\mathcal{H}_{Fe}$ and satisfies

$$(I, H; Ge, I \times H \div X_{34})_{\mathcal{H}_{Fe}} = -1,$$

so $I \times H = Na \times X_{34}$.

And the second equality follows from taking the isogonal conjugate of the first $\square$

Proposition 13.1.30. The points I, Ge, X_{77}, and the points I, Na, X_{78} are collinear.

Proof. Because I, H, Ge lie on one circumrectangular hyperbola, taking an isoconjugation about $I \times Ge$ gives that Ge, X_{77}, I are collinear. Similarly, I, H, Na lie on one circumrectangular hyperbola, so taking an isoconjugation about $I \times Na$ gives that Na, X_{78}, I are collinear. $\square$

13.1.5 IN and $(IN)^*$?

When $n = 1,\ 11,\ 12,\ 80$, $X_n \in IN$ depends on the choice of X_1.

When $n = 1,\ 36,\ 59,\ 60$, $X_n \in (IN)^*$ depends on the choice of X_1.

- X_{12} is the harmonic conjugate of Fe in $\overline{IN}$, $Fe^{\vee}$.
- X_{59} is the isogonal conjugate of Fe, Fe^*;
- X_{90} is the isogonal conjugate of $Fe^{\vee}$, $Fe^{\vee *}$.

13.1.6 IK and $(IK)^*$?

When $n = 1,\ 37,\ 45,\ 72$, $X_n \in IK$ depends on the choice of X_1.

When $n = 1,\ 28,\ 81,\ 89$, $X_n \in (IK)^*$ depends on the choice of X_1.

- X_{37} is the crosspoint of I and G, $I \pitchfork G$;
- X_{81} is the isogonal conjugate of X_{37}.

Obviously, we have

$$I \pitchfork G = G \times (G \div I)^{\mathsf{C}},$$

so we will just use $(I')^{\mathsf{C}}$ instead of X_{37}. Also,

$$X_{81} = I^2 \div (I \times (I \div G)^{\mathsf{C}}) = I \div (K \div I)^{\mathsf{C}} = I \star K.$$

Proposition 13.1.31. The point $(I')^{\mathsf{C}}$ lies on IK.

Proof. Because IK is tangent to $(IK)^* = (\triangle IG)$, it passes through $I \pitchfork G = (I')^{\mathsf{C}}$. □

Because $r \times r^{\mathsf{C}} = (r^{-1})^{\mathsf{C}}$, it follows that:

Proposition 13.1.32. The barycentric product $G \times (I')^{\mathsf{C}} = I \times Sp$.

Because, for any point P, the points P, P', P^{C}, $(P')^{\mathsf{C}}$ lie on one circumconic, it follows that:

Proposition 13.1.33. The points I, Sp, $(I')^{\mathsf{C}}$, I' lie on one circumconic, and

$$(I, Sp; (I')^{\mathsf{C}}, Sc^*)_{\mathcal{C}} = -1.$$

Then, I, $(I')^*$, Sp^*, $I \star K$ are collinear and

$$(I, Sp^*; I \star K, Sc) = -1.$$

Proof. We only need to prove that $(I, Sp; (I')^{\mathsf{C}}, Sc^*)_{\mathcal{C}} = -1$. Taking an isoconjugation about $I \times Sp$, it follows from (TODO 13.1.32) and (TODO 13.1.5) that the proposition is equivalent to $(Sp, I; G, Na) = -1$, which is obviously true. □

There is another way to look at this: because ISp^* is tangent to $(\triangle ISp)$, then by (TODO 13.1.43),

$$(I, Sp; (I')^{\mathsf{C}}, Sc^*)_{\mathcal{C}} = I(Sp^*, Sp; (I')^{\mathsf{C}}, Sc^*) = (C\ell^*, G; Mt; Mt^*) = -1.$$

Corollary 13.1.34. The point $(I')^{\mathsf{C}}$ is the harmonic conjugate of K on $\overline{IMt}$.

Proof. This is because

$$(I, Mt; (I')^{\mathsf{C}}, K) = C\ell(I, Sp; (I')^{\mathsf{C}}, Mt^*) = -1.$$

□

Corollary 13.1.35. The points $C\ell$, $(H')^*$, $(I')^*$ are collinear.

Proof. Because

$$C\ell(I, Sp; (I')^{\complement}, Mt^*) = -1 = (I, Be; Ge^*, Mt^*) = C\ell(I, Sp; Ge^*, Mt^*),$$

it follows that $C\ell$, $(H')^*$, $(I')^{\complement}$ are collinear. □

13.1.7 Euler line and $\mathcal{H}_J$?

When $n = 21,\ 27,\ 28,\ 29$, $X_n \in \mathcal{E}$ depends on the choice of X_1.

When $n = 65,\ 71,\ 72,\ 73$, $X_n \in \mathcal{H}_J$ depends on the choice of X_1.

- X_{27} is the cevapoint of H and $C\ell$, $H \star C\ell$;
- X_{28} is the barycentric product of $C\ell$ and $(H')^*$, $C\ell \times (H')^*$;
- X_{29} is the cevapoint of I and H, $I \star H$.

We first prove that these three points lie on $\mathcal{E}$: by (TODO 13.1.41),

$$H \star C\ell = H \div ((I \times H \div G) \div H)^{\complement} = H \div (Sp \div G) = Sp^\circ,$$

and we know $\mathcal{E}\deg = (\triangle G^\circ H^\circ) = (\triangle H^\circ G^\circ) = \mathcal{K}_K \ni Sp$.

Notice that

$$\begin{aligned} C\ell \star (H')^* &= (I \times H \div G) \div ((H \times K \div G) \div (I \times H \div G))^{\complement} \\ &= I \times H \div G \div (I \div G)^{\complement} = I \times H \div Sp. \end{aligned}$$

Because $C\ell \star (H')^* \in \mathcal{E}$ if and only if $Sp \in I \times H \div \mathcal{E} = (\triangle IC\ell)$, this is just because I, Sp^*, $C\ell^*$ are collinear.

Finally, because the polar of H about the diagonal conic $\mathcal{D}$ of I, H, $\mathfrak{p}_{\mathcal{D}}(H)$, passes through the isogonal conjugate of H, O, and $I \star H$, we have $I \star H \in \mathcal{E}$.

Proposition 13.1.36. The point X_{73} is on IH.

Proof. Because $X_{73} = I \pitchfork O$, the tangent to $(\triangle IO)$ from I, $(\triangle IO)^* = IH$, passes through X_{73}. □

Consider the cross-ratio preserving map $\varphi : \mathcal{E} \to OI,\ P \to X_{73}L^* \cap OI$. We have $O^\varphi = I,\ H^\varphi = O,\ Sc^\varphi = Sc^*$. So for any point $P \in \mathcal{E}$,

$$\frac{R}{2(R+r)} \cdot \frac{HP}{PO} = (O, H; Sc, P) = (O, P; Sc^*, \varphi(P)) = -\frac{r}{R+r} \cdot \frac{OP^\varphi}{P^\varphi I},$$

which is to say

$$\frac{OP^\varphi}{P^\varphi I} = -\frac{R}{2r} \cdot \frac{HP}{PO}.$$

If $P = N$, we have

$$\frac{ON^\varphi}{N^\varphi I} = -\frac{R}{2r} \quad \Longrightarrow \quad N^\varphi = I^{\mathfrak{Z}}.$$

If $P = \infty_{\mathcal{E}}$, we have

$$\frac{O\infty_{\mathcal{E}}^\varphi}{\infty_{\mathcal{E}}^\varphi I} = -\frac{R}{2r} \quad \Longrightarrow \quad \infty_{\mathcal{E}}^\varphi = X_{35}.$$

If $P = G$, we have

$$\frac{OG^\varphi}{G^\varphi I} = -\frac{R}{2r} \quad \Longrightarrow \quad G^\varphi = Na^*.$$

If $P = L$, we have

$$\frac{OL^\varphi}{L^\varphi I} = -\frac{R}{2r} \quad \Longrightarrow \quad L^\varphi = Ge^*.$$

To put all of these results together:

Proposition 13.1.37. We have $X_{73} = IH \cap I^{\mathfrak{Z}}Ko \cap \infty_E X_{35} \cap KNa^* \cap Ge^*L^*$.

Proposition 13.1.38. The three points $I \pitchfork K$, sc^*, X_{73} are collinear.

Proof. Because K, H, O lie on one circumconic, it follows that $I \div K$, $I \div H$, $I \div O$ are collinear, and it follows that $I \pitchfork K$, $Sc^* = I \pitchfork H$, $X_{73} = I \pitchfork O$ are also collinear. □

13.1.8 Other related points

The points which did not appear above, but which are related to X_1, are:

19, 27, 31, 38, 41, 44, 47, 48, 63, 75, 82, 85, 88, 91, 92, 100.

Recall the Clawson point:

- The Clawson point X_{19}, also named $C\ell$, is the homothetic center of the extangent triangle $\triangle T'_A T'_b T'_c$ and the orthic triangle $\triangle H_a H_b H_c$.

Proposition 13.1.39. The four points $K,\ C\ell, X_{34},\ Sc^*$ are collinear.

Proof. If we let $X = Sc^*H' \cap HMt$, then by (TODO 5.8.8) we have

$$Sc^*(O, H; K, H') = (O, H; K, H')_{\mathcal{H}_J} = (H, O; G, (H')^*),$$
$$Sc^*(O, H; C\ell, H') = (Be, H, C\ell, X) = (H, Be; X, C\ell) = Ge^*(H, O; X, (H')^*).$$

Thus $K, C\ell, Sc^*$ are collinear if and only if G lies on Ge^*X.

By (TODO 5.6.19) and (TODO 5.6.22), $Sc^*H' = NaGe$ is the complement of IMt, so therefore X is the reflection of Mt across Sp, so the ratios given by (TODO 5.6.8) give us

$$\begin{aligned}\frac{HX}{XBe} \cdot \frac{BeGe^*}{Ge^*O} \cdot \frac{OG}{GH} &= \frac{BeMt}{MtH} \cdot \left(\frac{2R+r}{R}\right) \cdot \frac{1}{2} \\ &= \frac{2R}{2R+r} \cdot \left(\frac{2R+r}{R}\right) \cdot \frac{1}{2} = -1,\end{aligned}$$

so X, Ge^*, G are collinear.

By (TODO 5.6.18),

$$C\ell(I, H, ; X_{33}, X_{34}) = -1 = C\ell(I, Be; Ge^*, Sc^*),$$

so $C\ell, X_{34}, Sc^*$ are collinear. □

Proposition 13.1.40. The five points $K, Mt, C\ell, Ge^*, Mt^*$ lie on a common circumconic and the quadrilateral $(KC\ell)(MtMt^*)$ is a harmonic quadrilateral.

Proof. Note that the image of line GGe under isotomic conjugation is the circumconic $(ABCGNa)$, and since G is its own isotomic conjugate, GGe is tangent to $(ABCGNa)$ at G. We know that $GGeMtMt^*$ are collinear, so take the isogonal conjugate of this statement to get that KNa^* is tangent to $\mathcal{C} := (ABCKGe^*)$. So to prove that $C\ell$ lies on $\mathcal{C}$, we only need to prove that $(KMtC\ell Ge^*Mt^*)$ is tangent to KNa^*, but this is true by applying (TODO 5.6.18) and (TODO 13.1.4), with

$$\begin{aligned}Ge^*(K, C\ell; Mt, Mt^*) &= (KGe^* \cap HMt, C\ell; Mt, Be) \overset{K}{=} (Ge^*, Sc^*; I, Be) = -1; \\ &= K(Na^*, C\ell; Mt, Mt^*) = (Na^*, Sc^*; I, Mt^*) = -1.\end{aligned}$$

This also tells us that $(KC\ell)(MtMt^*)$ is a harmonic quadrilateral on $\mathcal{C}$. □

Proposition 13.1.41. In barycentric coordinates, $C\ell \times G = I \times H$.

Proof. Suppose $\widetilde{C\ell} = I \times H \div G, \widetilde{Sp} = IG \cap H\widetilde{C\ell}$. Then we have that $G, H, \widetilde{Sp}$ lie on a common circumconic (namely, the Kiepert hyperbola $\mathcal{H}_K$). This tells us that $\widetilde{Sp}$ is the second intersection of line IG with $\mathcal{H}_K$ (by (TODO 13.1.24) this is Sp). Thus $\widetilde{Cl}$ lies on $H\widetilde{Sp} = HMt$. We do a similar thing for the isogonal conjugate of $\widetilde{C\ell}^* = I \times G \div H = Ge \times Mt \div H$, so H, Ge, Mt since lie on a common circumconic as well, we get that $\widetilde{Cl}^* \in GeMt = GeMt^*$. But from (TODO 13.1.40) we know that $C\ell^* \in GeMt^*$. Therefore $\widetilde{C\ell}$ is the second intersection of HMt with $\overline{GeMt^*}^* = (ABCGe^*Mt)$, which is $C\ell$. □

Corollary 13.1.42. $I, H, C\ell$ lie on a common diagonal conic.

Proof. The I-complements of these three points are

$$I, I \times (H \div I)^{\complement} = I \times (I \div O)^{\complement} = I \pitchfork O, I \times (C\ell \div I)^{\complement} = I \times (H \div G)^{\complement}.$$

Therefore $I, H, C\ell$ lie on a common diagonal conic if and only if $I, I \pitchfork O, I \times O \div G$ lie on a common circumconic. Consider the isoconjugation given by the barycentric product $I \times O$; this is equivalent to the fact that $O, Sc = I \star O, G$ are collinear. □

- X_{63} is the isogonal conjugate of the Clawson point, we will call it $C\ell^*$.

In barycentric coordinates,

$$C\ell^* = K \times G \div (I \times H \div G) = I \times G \div H.$$

From (TODO 13.1.40), we directly get:

Proposition 13.1.43. $C\ell^*$ lies on GGe and

$$(G, C\ell^*; Mt, Mt^*) = -1.$$

Proposition 13.1.44. I, Sc, Sp^*, Cl^* are collinear.

Proof. Note that $I = I^*, K = G^*, Na^*, Sp^*$ lie on a common circumconic $(ABCIK)$ and

$$(I, Sp^*; G, Na)_{(IG)^*} = (I, Sp; G, Na) = -1.$$

Since IG is tangent to $(IG)^*$ at I, from (TODO 13.1.43),

$$I(G, Sp^*; Mt, Mt^*) = I(G, Sp^*; K, Na^*) = -1 = (G, Cl^*; Mt, Mt^*),$$

giving us that $I, Sp^*, C\ell^*$ are collinear. Similarly, since IMt^* is tangent to $\mathcal{H}_{Fe}$, from (TODO 13.1.4) and (TODO 13.1.43),

$$I(G, Sc; Mt, Mt^*) = (Na, Sc; Mt, I)_{\mathcal{H}_{Fe}}) = -1 = (G, C\ell^*; Mt, Mt^*),$$

so $I, Sc, C\ell^*$ are collinear. $\square$

Proposition 13.1.45. $I, G, C\ell^*$ lie on a common diagonal conic.

Proof. Note that these three points are just the three points in (TODO 13.1.42) under a projective transformation $G \div H$. $\square$

Since $I \times G = H \times C\ell^*$ and $Sc = ICl^* \cap GH$, we have $IH \cap GC\ell^* = I \times G \div Sc$. From (TODO 13.1.5),

$$I \times G \div Sc = G \times Sp \div Na.$$

In ETC this point is X_{226}. What is its actual characterization?

Proposition 13.1.46. X_{226} is actually the complement $(C\ell^*)^{\complement}$ of $C\ell^*$.

Proof. From (TODO 13.1.45), $Sp = I^{\complement}, G = G^{\complement}, (Cl^*)^{\complement}$ lie on a common circumconic. Since $X_{226} = G \times Sp \div Na$ also lies on $(\triangle GSp) = \overline{GSp}^{G\times Sp}$, we only need to prove that $X_{226}, G, C\ell^*$ are collinear. Take the isoconjugation given by $I \times G$; this is equivalent to proving that Sc, I, H lie on a common circumconic, however this is just the Feuerbach hyperbola. $\square$

Corollary 13.1.47. Na, L (= de Longchamps point)$, C\ell^*$ are collinear.

Proof. Since $(Cl^*)^{\complement} \in IH$, we have $Cl^* \in \overline{IH}^{\text{\reflectbox{$\complement$}}} = NaL$. $\square$

Proposition 13.1.48. $Sp, X_{46}, C\ell^*$ are collinear.

Proof. Since $IH \parallel NaLC\ell^*$, we have that the line through the midpoint Sp of segment INa and the midpoint O of segment HL is also parallel to them. Since

$$(I, X_{46}; O, Sc^*) = -1,$$

so we only need to prove that $Sp(I, Cl^*; O, Sc^*) = -1$, but

$$Sp(I, C\ell^*; O, Sc^*) = (G, C\ell^*; GC\ell^* \cap OSp, (C\ell^*)^{\complement}) \stackrel{\infty_{IH}}{=} (G, Na; Sp, I) = -1.$$

$\square$

- X_{75} is the isotomic conjugate I' of the incenter I;
- X_{31} is the isogonal conjugate $(I')^*$ of the isotomic conjugate I' of the incenter I.

Since I, Ge, Na lie on a common circumconic $\mathcal{H}_{Fe}$, by taking isotomic conjugation we get

Proposition 13.1.49. Ge, Na, I' are collinear.

Proposition 13.1.50. $I, (I')^*, C\ell^*$ are collinear.

Proof. Since $(I')^* = I \times K \div G$, we have $I, (I')^*, C\ell^* = I \times G \div H$ are collinear if and only if G, K, H' are collinear, which we have proven back in (TODO 5.6.21). $\square$

Again, we can take the isotomic conjugate of the above result to get

Corollary 13.1.51. $I, C\ell, I'$ lie on a common circumconic.

Proposition 13.1.52. Na, Sc^*, I' all lie on the line anticomplement of $\overline{II'}^{\mathsf{C}}$.

Proof. From (TODO 13.1.33),

$$I'(I, Sp; (I')^{\mathsf{C}}, Sc^*) = -1 = I'(I, Sp; G, Na),$$

and therefore $I'Sc^* = I'Na$. Since $\overline{NaI'}^{\mathsf{C}} = \overline{II'}^{\mathsf{C}}$, we get $\overline{NaSc^*I'} = \overline{II'}^{\mathsf{C}^{\reflectbox{$\scriptstyle\mathsf{C}$}}}$. $\square$

- X_{48} is the crosspoint of I and $C\ell^*$, $I \pitchfork C\ell^*$;
- X_{92} is the cevapoint of I and $C\ell$, $I * C\ell$;
- X_{91} is the cross conjugate of X_{48} and X_{92}, $X_{48} \Psi X_{92}$;
- X_{47} is the isogonal conjugate of X_{91}.

Since $C\ell = I \times H \div G$, we have

$$I \Psi C\ell^* = I \times (I \div C\ell^*)^{\mathsf{C}} = I \times (H \div G)^{\mathsf{C}} = I \times O \div G,$$

$$I \star C\ell = I \div (C\ell \div I)^{\mathsf{C}} = I \div (H \div G)^{\mathsf{C}} = I \times G \div O.$$

This tells us that (X_{48}, X_{92}) are a pair of isogonal conjugates, and that (X_{63}, X_{92}) are also a pair of isogonal conjugates.

Note that since I is its own isogonal conjugate, $\overline{IC\ell}$ is tangent to $(\triangle IC\ell^*)$, and thus:

Proposition 13.1.53. $I, C\ell, X_{48}$ are collinear.

Proposition 13.1.54. H, Na, X_{92} are collinear.

Proof. Let X be the second intersection of line HX_{92} with the diagonal conic $\mathcal{D}$ through $I, H, C\ell$. Then $(IC\ell)(HX)$ is harmonic on $\mathcal{D}$ by the tangencies. Since the Euler line $\mathcal{E}$ is the tangent to $\mathcal{D}$ at H (note that $X_{27} = I \star H \in \mathcal{E}$), so

$$H(I, Sp; G, X_{92}) = H(I, C\ell; G, X) = -1 = (I, Sp; G, Na),$$

so by Prism Lemma H, Na, X_{92} are collinear. □

From (TODO 13.1.42), IX_{29}, IX_{92} both are the tangents from I to the diagonal conic through $I, H, C\ell$. Thus we also have:

Proposition 13.1.55. I, X_{29}, X_{92} are collinear.

Similarly, $C\ell X_{27}, C\ell X_{92}$ are both the tangents from I to the diagonal conic, so we also have

Proposition 13.1.56. $C\ell, X_{27}, X_{92}$ are collinear.

Proposition 13.1.57. X_{47} is the X_{92}-complement of X_{91}.

Proof. This is because

$$X_{47} = X_{48} \times X_{92} \div X_{91} \qquad = (X_{91} \times (X_{91} \div X_{92})^{\mathsf{C}}) \times X_{92} \div X_{91} = X_{92} \times (X_{91} \div X_{92})^{\mathsf{C}}.$$

□

Proposition 13.1.58. $X_{91} = I \times H \div X_{24}$, and therefore $X_{47} = I \times X_{24} \div H$.

Proof. First, we have

$$X_{91} = X_{92} \div (X_{48} \div X_{92})^{\reflectbox{C}} = I \times G \div O \div (O^2 \div G^2)^{\reflectbox{C}}.$$

Thus we only need to prove

$$H \times (N \div H)^{\reflectbox{C}} = X_{24} = K \times (O^2 \div G^2)^{\reflectbox{C}},$$

in other words

$$(N \div H)^{\reflectbox{C}} = O \div G \times (O^2 \div G^2)^{\reflectbox{C}}.$$

In reality, we prove: □

Lemma 13.1.59. For all ratios (weight-0) r,

$$(r^{\mathsf{C}} \div r^{\reflectbox{C}})^{\reflectbox{C}} = r \times (r^2)^{\reflectbox{C}}.$$

Proof of Lemma. Let $r = I \div G$, which is just

$$(Sp \div Na)^{\complement} = I \times H' \div G^2 = I \div H,$$

which is just (TODO 13.1.46) (note that $C\ell^* = I \times G \div H, X_{226} = Sp \times G \div Na$). □

Choosing $r = O \div G$ in the above lemma solves our problem.

Since X_{24} lies on the Euler line, we immediately get:

Corollary 13.1.60. X_{47} lies on line $IC\ell^* = I \times \mathcal{E} \div H$.

Proposition 13.1.61. The three points Sc, X_{24}, X_{47} lie on a common circumconic, and thus Sc^*, X_{68}, X_{91} are collinear.

Proof. We only need to prove that $Sc \times X_{47} \div X_{24}$ lies on $ISc = IC\ell^* = IX_{47}$ see (TODO 13.1.43). However (TODO 13.1.58) tells us that

$$Sc \times X_{47} \div X_{24} = I \times Sc \div H.$$

Thus we only need to prove that I, H, Sc lie on a common circumconic, which is just $\mathcal{H}_{Fe}$. □

Proposition 13.1.62. X_{47} is the intersection of $X_{33}X_{90}$ and $X_{34}X_{46}$.

13.2 X_1 - unrelated points

13.2.1 Points on the Euler line

When $n = 2, 3, 4, 5, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$ X_n lies on the Euler line.

- X_{22} is the perspector of the tangential triangle and the circumcevian triangle of G.

Let $\triangle K^aK^bK^c$ be the tangential triangle, let $\triangle K_aK_bK_c$ be the cevian triangle of K. let $\triangle_G^\Omega = \triangle G_a^\Omega G_b^\Omega G_c^\Omega$ be the circumcevian triangle of G, and let $\triangle_K^\Omega = \triangle K_a^\Omega K_b^\Omega K_c^\Omega$ be the circumcevian triangle of K, then the corresponding vertices G_a^Ω, K_a^Ω in the two triangles are reflections across the perpendicular bisector of $\overline{BC}$ by isogonal conjugation. Since the perpendicular bisector of BC goes through K^a, therefore $K^aG_a^\Omega$ intersects $\overline{BC}$ at the isotomic conjugate of K_a (call it K_a'). Therefore AK_a' passes through the isotomic conjugate of K', and by symmetry we get:

Proposition 13.2.1. X_{22} is the ceva conjugate K'/K of K' and K.

This tells us that

$$X_{22} = K \times (K \div K')^{\mathfrak{J}} = K \times (K^2 \div G^2)^{\mathfrak{J}}.$$

Proposition 13.2.2. X_{22} lies on the Euler line and

$$(G, X_{22}; O, (H')*) = -1.$$

Proof. X_{22}'s image under the radical transformation is

$$I \times (I^2 \div G^2)^{\mathfrak{J}} = I \times H' \div G = I \times G \div H = C\ell^*,$$

but $\mathcal{E} = GO$ under the radical transformation is GMt. Therefore from (TODO 13.1.43), $X_{22} \in \mathcal{E}$ and

$$(G, X_{22}; O, K^2 \div O) = -1.$$

Finally, note that $(H')^* = K \times G \div (G^2 \div H) = K \times H \div G = K^2 \div O$, $\square$

- X_{23} is the far-out point, defined as the inversion of G wrt. the circumcircle Ω. We will call it $G^{\mathfrak{J}}$.
- X_{67} is the isotomic conjugate of $G^{\mathfrak{J}}$, and is also the antipode of K on $\mathcal{H}_J$. These two points are basically useless.

- X_{24} is the perspector of $\triangle ABC$ and the orthic triangle of the orthic triangle.
- X_{51} is the centroid of the orthic triangle, G_H.
- X_{52} is the orthocenter of the orthic triangle, H_H.
- X_{53} is the symmedian point of the orthic triangle, K_H.
- X_{68} is the isogonal conjugate of X_{24}.
- X_{95} is the isogonal conjugate G_H^* of G_H.
- X_{96} is the isogonal conjugate H_H^* of H_H.
- X_{97} is the isogonal conjugate K_H^* of K_H.

Proposition 13.2.3. X_{24} lies on the Euler line $\mathcal{E}$.

Proof. Consider X_{24} in reference to the orthic triangle; since H is its incenter (call it I_H), we have that X_{24} is X_{46} wrt. the orthic triangle, in other words $X_24 = H_H/I_H$. Therefore from (TODO 13.1.19), $X_{24} \in O_H I_H = NH = \mathcal{E}$. □

(TODO 13.1.19) also tells us that X_{24} is the $H = I_H$-anticomplement wrt. the orthic triangle of $N = O_H$. Switching our reference triangle, this means that there exists a projective transformation that sends A, B, C, H, X_{24} to H_a, H_b, H_c, H, N, so X_{24} is the H-anticomplement of N, $H \times (N \div H)^{\complement}$. In the proof of (TODO 13.1.58) we can also see that this is equal to $K \times (O^2 \div G^2)^{\complement}$.

Proposition 13.2.4. N, K, X_{24}, H_H lie on a common circumconic.

Proof. If we switch our reference triangle to the orthic triangle, then this is equivalent to proving that O, Mt, O's I-anticomplement $I \times (O \div I)^{\complement}$ and H lie on a common circumconic of the excentral triangle. Take the I-complement of this statement, this is equivalent to proving that $Sc^*, K, I \Psi O, O$ lie on a common circumconic of the reference triangle, which is just $\mathcal{H}_J$. □

Corollary 13.2.5. We have $X_{24} = GN \cap KKo, X_{68} = GKo \cap KN$.

Proposition 13.2.6. We have $X_{24} = X_{33}X_{35} \cap X_{34}X_{36}$.

Proof. From $(I, O; X_{35}, I^{\mathfrak{I}}) = (I, H; X_{33}, X_{34} = -1$, and therefore we only need to prove that X_{24}, X_{34}, X_{36} are collinear. This is true by Pascal on the hexagon $OHX_{73}KoKSc^*$ on the Jerabek hyperbola $\mathcal{H}_J$, where

$$X_{34} = HX_{73} \cap KSc^*, X_{36} = X_{73}Ko \cap Sc^*O, X_{24} = OH \cap KoK$$

are collinear. □

From (TODO 5.6.8), we get $O = Be_H, K = Mt_H, H_H$ are collinear, implying:

Proposition 13.2.7. H_H lies on the Brocard axis.

From (TODO 5.4.6), we can get

Proposition 13.2.8. $H = I_H, K_H, K = Mt_H$ are collinear.

Obviously, we have $G_H = H \Psi K, H_H = H \Psi X_{24}$.

- X_{26} is the circumcenter O^K of the tangential triangle.

By applying (TODO 5.2.3) on the tangential triangle, we get that O^K lies on the Euler line.

13.2.2 $\mathcal{H}_J$ points

$n = 3, 4, 6, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74.$

13.2.3 Brocard axis OK points

$n = 3, 6, 15, 16, 32, 39, 50, 52, 58, 61, 62.$

13.2.4 Kiepert hyperbola points

$n = 2, 4, 13, 14, 76, 83, 94, 96, 98.$

13.2.5 Others

We'll do these later.

Part III

Appendices

Chapter

Homogeneous and Barycentric Coordinates

These coordinate systems are used extensively in part II of this book. However, they are never properly introduced and just left as assumed knowledge, so we will actually introduce them. These are useful for rigorously proving many of the remarks in part II.

Ideally, read this section along with part II.

A small amount of linear algebra and MV knowledge will be assumed.

We define **homogeneous coordinates** as a way to represent projective spaces, which can be defined over either the reals or complexes. In this book, we will primarily work with four projective spaces only.

- $\mathbb{RP}^1$ is the real projective line, defined as $\mathbb{R} \cup \infty$ (there is only one point at infinity, and no distinction between positive and negative infinity)
- $\mathbb{CP}^1$ is the complex projective line defined as $\mathbb{C} \cup \infty$ (there is no distinction between complex infinities).
- $\mathbb{RP}^2$ is the real projective plane, defined as $\mathbb{R}^2 \cup \mathcal{L}_\infty$.
- $\mathbb{CP}^2$ is the complex projective plane, defined as $\mathbb{C}^2 \cup \mathcal{L}_\infty$.

Remark 14.0.1. There is weird terminology between $\mathbb{CP}^1$ and $\mathbb{CP}^2$. Although the dimension of $\mathbb{CP}^1$ is 1 over the complex numbers, it is often called the "complex projective plane" in other sources (as $\mathbb{C}$ has dimension 2 if you interpret it as a vector space $\{z = a + bi\}$ the reals), which is easily confused with $\mathbb{CP}^2$.

In this book, we will always refer to $\mathbb{CP}^1$ as the "complex projective line" to avoid this confusion, but be careful with other sources. Additionally, the "complex projective plane" will always refer to $\mathbb{CP}^2$.

Also, we will refer to $\mathbb{C}$ as the "complex line".

Define the **affine** part of a projective space as the part of the space without the points at infinity.

Now we can define a homogeneous coordinate system for each of these spaces.

Theorem 14.0.2 (Homogeneous coordinates for projective spaces)**.** Each element of a $\mathbb{P}^1$ space can be represented as 2-tuples of values $[a : b]$ either in $\mathbb{R}$ or $\mathbb{C}$ such that $[a : b] = [ka : kb]$ for some scale factor k, excluding the tuple $[0 : 0]$ (i.e tuples representing a common ratio are glued together) Similarly, each element of a $\mathbb{P}^2$ space can be represented as 3-tuples of values $[a : b : c]$ either in $\mathbb{R}$ or $\mathbb{C}$ such that $[a : b : c] = [ka : kb : kc]$ for some scale factor k, excluding the tuple $[0 : 0 : 0]$.

Proof. We can arbitrarily identify the affine part of the $\mathbb{P}^1$ with tuples in the form $x \mapsto [x : 1]$ and identify the point at infinity with tuples in the form $\infty \mapsto \{[x : 0] \mid x \neq 0\}$ (which are all equivalent under scaling). Note that we can scale any tuple $[x : y]$ down to one of these forms.

Similarly, we can identify the affine part of the $\mathbb{P}^2$ space with tuples in the form $\{[x : y : 1]\}$, and the line at infinity (which is the same as $\mathbb{P}^1$) with tuples in the form $\{[x : y : 0\}$. □

Out of these four, $\mathbb{RP}^1$ and $\mathbb{CP}^1$ are the simplest, as they are one-dimensional and don't have a structure of "collinearity". These are primarily discussed in chapter 7.A.

Theorem 14.0.3 (Automorphisms of $\mathbb{P}^1$)**.** All maps from $\mathbb{CP}^1$ to itself that preserve cross-ratio between any four elements are easily categorized if we separate $\mathbb{C}$ into its real and complex parts to get the "complex plane" (we apologize for this terminology). Then these maps are just compositions of inversions/reflections and spiral similarities in this plane. By considering only the real axis, we get that all maps from $\mathbb{RP}^1$ (represented as a line) to itself are compositions of inversions, reflections, and translations as well.

Proof. The converse of this is proved in Chapter 3. For the rest, see 6.A for information on circle points. □

For the rest of the appendix, we will work in the spaces $\mathbb{RP}^2$ and $\mathbb{CP}^2$.

Remark. If you are familiar with vector spaces, we define the **projectivization** of a vector space $\mathcal{V}$. Let $\mathcal{V} \setminus \{0\}$ represent the set of nonzero vectors in $\mathcal{V}$, then we define the projectivization $P(\mathcal{V})$ as the space we get if we set all collinear vectors equal. For example, the projectivization of $\mathbb{R}^3$ is $\mathbb{RP}^2$. Linear transformations on V, after this projectivization, will become equivalent homographies of $P(\mathcal{V})$.

14.1 Linear Algebra

Here we define the familiar structures of collinearity and concurrency rigorously in projective space.

Definition 14.1.1 (Lines)**.** A line is defined as a vertical matrix $\ell = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$.

Definition 14.1.2 (Incidence)**.** A point $P = [x : y : z]$ lies on a line $\ell = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ iff. $P\ell = 0$. Dually, a line ℓ goes through a point P iff. $P\ell = 0$.

Theorem 14.1.3 (Join and Meet)**.** The line connecting two points $P_a = [x : y : z]$ and $P_b = [u : v : w]$ is the line $\ell = \begin{bmatrix} yw - vz \\ zu - xw \\ xv - yu \end{bmatrix}$. Dually, the intersection of two lines $\ell_1 = \begin{bmatrix} A \\ B \\ C \end{bmatrix}, \ell_2 = \begin{bmatrix} D \\ E \\ F \end{bmatrix}$ is the point $P = [BF - CE : CD - AF : AE - BD]$.

Proof. Cramer's rule / cross product motivates this, but if you don't know it, just plug it in and bash out the two equations. □

Theorem 14.1.4 (Collinearity and Concurrence)**.** Three points $P_a = [x : y : z], P_b = [u : v : w], P_c = [p : q : r]$ are collinear iff. their determinant

$$\det \begin{bmatrix} x & y & z \\ u & v & w \\ p & q & r \end{bmatrix} = 0$$

Dually, three lines $\ell_1 = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix}$, $\ell_2 = \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix}$, and $\ell_3 = \begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix}$ are concurrent if and only if their determinant

$$\det \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = 0$$

Proof. P_a, P_b, P_c are only collinear if there exists a line ℓ such that $P_a\ell = 0, P_b\ell = 0, P_c\ell = 0$. In other words, the equivalent vectors in $\mathbb{R}^3$ lie on a plane orthogonal in $\mathbb{R}^3$, implying their determinant is zero. The dual statement is analogous by taking the transpose. □

Definition 14.1.5 (Projective maps and Homographies)**.** Given homogeneous matrix $\mathbf{M}$

$$\mathbf{M} := \begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix},$$

define a **homography** as a map sending the point $P := [x : y : z]$ to $P\mathbf{M}$. Given homogeneous matrix $\mathbf{N}$

$$\mathbf{N} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

define a **projective map** as a map sending the point $P := [x : y]$ to $P\mathbf{N}$.

Theorem 14.1.6. This matrix definition corresponds to the standard definition of homographies as cross-ratio preserving collineations, given by the Fundamental Theorem of Projective Geometry (TODO 7.A).

Proof. For $\mathbb{RP}^2$, the proof is easy, as note that this preserves collinearity by multiplicativity of determinants. [ADD IN CP2 SECTION] □

Example 14.1.7 (Pappus's Theorem)**.** For nine points $\begin{matrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{matrix}$ in either $\mathbb{RP}^2$ or $\mathbb{CP}^2$, if we have P_{11}, P_{12}, P_{13}, P_{13}, P_{22}, P_{31}, P_{31}, P_{32}, P_{33}, P_{11}, P_{21}, P_{32}, P_{13}, P_{23}, P_{33}, P_{12}, P_{21}, P_{31},, P_{12}, P_{23}, P_{33}, P_{11}, P_{22}, P_{33} are collinear, then P_{21}, P_{22}, P_{23} are also collinear.

Proof. Assign our projective frame $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]$ to the points $P_{13}, P_{21}, P_{33}, P_{11}$. Then the line between P_{13}, P_{12}, P_{11} is $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, the line between P_{11}, P_{22}, P_{33} is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, the line between P_{11}, P_{21}, P_{32} is

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Therefore, we can express points P_{12}, P_{22}, P_{32} as $[p : 1 : 1], [1 : q : 1], [1 : 1 : r]$. We can express the lines $P_{12}P_{21}, P_{13}P_{22}, P_{33}P_{32}$ as $\begin{bmatrix} -1 \\ p \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ q \end{bmatrix}$, $\begin{bmatrix} r \\ 0 \\ -1 \end{bmatrix}$ Since it is given that these three lines are concurrent, we have $\det \begin{bmatrix} -1 & 0 & r \\ p & -1 & 0 \\ 0 & q & -1 \end{bmatrix} = 0 = (-1 + rpq) \implies rpq = 1.$

By a similar argument we get that $P_{13}P_{32}, P_{12}P_{33}, P_{21}P_{22}$ are concurrent at P_{23} iff. $rqp = 1$.

□

Example 14.1.8 (Desargues's Theorem)**.** Given two triangles $\triangle A_1B_1C_1, \triangle A_2B_2C_2$, we have that $B_1C_1 \cap B_2C_2, C_1A_1 \cap C_2A_2, A_1B_1 \cap A_2B_2$ are collinear iff. A_1A_2, B_1B_2, C_1C_2 are concurrent (let's say at P).

Proof. Let's first prove the "only if" direction. Set our projective frame $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1] = A_1, B_1, C_1, P$. Since A_2 lies on $A_1P = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, we have that A_2 is in the form $[x : p : p]$ or $[x : p : p]$. Similarly, we have that B_2 is in the form $[q : y : q]$ and C_2 is in the form $[r : r : z]$. We can get the six lines

- $A_1B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- $B_1C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- $C_1A_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
- $A_2B_2 = \begin{bmatrix} p(q-y) \\ q(p-x) \\ xy - pq \end{bmatrix}$
- $B_2C_2 = \begin{bmatrix} q(r-z) \\ r(q-y) \\ yz - qr \end{bmatrix}$
- $C_2A_2 = \begin{bmatrix} r(p-x) \\ p(r-z) \\ zx - rp \end{bmatrix}$

Then line A_1A_2 is just $\ell_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Let B_2, C_2 have coordinates $[a : b : c], [d : e : f]$, then the lines B_1B_2

and C_1C_2 are just $\ell_2 = \begin{bmatrix} c \\ 0 \\ -a \end{bmatrix}$, $\ell_3 \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$. We are given that

$$\det[\ell_1, \ell_2, \ell_3] = 0$$

The lines B_1C_1, C_1A_1, A_1B_1 are just $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and the lines B_2C_2, C_2A_2, A_2B_2 are just $\square$

Remark. Note that the proof of Pappus's theorem relies on the commutativity of multiplication in $\mathbb{R}$ or $\mathbb{C}$. We can actually define projective planes over non-commutative scalars (called a **division ring**) in which Pappus's theorem does not hold. Similarily, note that Desargues's theorem relies on the associativity of multiplication in $\mathbb{R}$ or $\mathbb{C}$. We can also define projective planes over non-associative scalars in which Desargues's theorem does not hold.

These special projective planes are beyond the scope of this book.

Remark 14.1.9 (Barycentric coordinates)**.** If you didn't know barycentric coord-bashing, congratulations, you learned it! Barycentric coordinates are just homogeneous coordinates, except we take a suitable projective frame that sends $[1:0:0], [0:1:0], [0:0:1], [1:1:1]$ to A, B, C, G, and all our theory still holds. Note that the trilinear polar of a point $P = [x:y:z]$ is just $\begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$. We will call the set of points $[x:y:z]$ that lie on this line $\mathfrak{t}(P)$, and will use extensively in chapter 12. (you can prove this by just bashing out the definition).

14.2 Advanced Things

14.2.1 Conics

This section is basically just expanding on 6.3's proof of conics as degree-2 curves, and tying it into chapter 11.

A conic is defined in homogeneous coordinates as the implicit curve

$$F(x:y:z) := Ax^2 + By^2 + Cz^2 + Dyz + Exz + Fxy = 0$$

.

We define the symmetric matrix of a conic as the 3×3 matrix of double partial derivatives of F (Hessian matrix)

$$\nabla^2 F(x:y:z) = \begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial z \partial x} & \frac{\partial^2 F}{\partial z \partial y} & \frac{\partial^2 F}{\partial z^2} \end{bmatrix} = \begin{bmatrix} 2A & F & E \\ F & 2B & D \\ E & D & 2C \end{bmatrix}$$

Symmetric matrices are actually extremely convenient in homogeneous coordinates. This is because for an implicit curve $C(x:y:z) = 0$ we have that for a point $[p:q:r]$ on the curve, the only requirement for a line through $[p:q:r]$ to be tangent is that it matches the gradient

$$\nabla C = \begin{bmatrix} \frac{\partial C}{\partial x} \\ \frac{\partial C}{\partial y} \\ \frac{\partial C}{\partial z} \end{bmatrix}$$

at $[p:q:r]$.

Theorem 14.2.1 (Euler's homogeneous function theorem)**.** Conveniently, the line $\begin{bmatrix} \frac{\partial C}{\partial x} \\ \frac{\partial C}{\partial y} \\ \frac{\partial C}{\partial z} \end{bmatrix}$ works (i.e. if evaluated at $[p:q:r]$, then the line goes through $[p:q:r]$).

This is a special case of the the full theorem, which states that for an implicit homogeneous function

$$F(x,y,z) = k\left(x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z}\right)$$

where k is a scale factor that we can ignore by the magic of homogeneous coordinates. (Proving this is beyond the scope of this text.) However, Euler's theorem gives us a few surprising corollaries:

Corollary 14.2.2. The set of points $P = [x:y:z]$ on conic $\mathbf{M}$ is just the solutions to $P\mathbf{M}P^\intercal = 0$. Dually, the set of lines ℓ tangent to this conic is just the solutions to $\ell^\intercal \mathbf{M} \ell = 0$

Proof. Apply Euler's theorem twice. □ write it out

Corollary 14.2.3 (Poles and polars)**.** For a point $P = [x:y:z]$ and conic $F(x,y,z)$ with corresponding symmetric matrix $\mathbf{M}$, we have that the polar of P is $(P\mathbf{M})^\intercal = \mathbf{M}P^\intercal$.

Proof. Let $(P\mathbf{M})^\intercal = \mathbf{M}^\intercal P^\intercal$ intersect $\mathbf{M}$ at R, S, then we have that R, S satisfy $R\mathbf{M}^\intercal P^\intercal = 0$, and analogously for S. Note that by Euler's theorem since $\mathbf{M}$ is a matrix of second derivatives, we get that the tangent at R is $(R\mathbf{M})^\intercal$. So we are trying to prove that $P(R\mathbf{M})^\intercal = 0$, i.e. that P lies on the tangent at R. Since $(R\mathbf{M})^\intercal = \mathbf{M}^\intercal R^\intercal$, this is equivalent to proving $P\mathbf{M}^\intercal R^\intercal = 0$. Now just take the transpose of the left and right side, and we are done, as $\mathbf{M} = \mathbf{M}^\intercal$ since it's symmetric. □

Remark. For P inside $F(x,y,z)$ (i.e. any real line through P intersects $F(x,y,z)$ at two nondegenerate real points), R, S are complex points that are complex conjugates in $\mathbb{C}$, implying the line through them is real.

Theorem 14.2.4. Two points P, Q are conjugate wrt. conic $F(x,y,z)$ with corresponding matrix $\mathbf{M}$ if and only if $Q\mathbf{M}P^\intercal = 0$.

Proof. We have $Q\mathbf{M}P^\intercal = 0$, so Q lies on the polar of P. To prove P lies on the polar of Q, just take the transpose to get $P\mathbf{M}Q^\intercal = 0$. ☐

Here is a very useful theorem:

Theorem 14.2.5 (Codimension of the space of conics through n points is $5-n$)**.** The space of conics through n points is isomorphic to $\mathbb{P}^{5-n}$.

We will prove this later using the Veronese map in 14.3.

14.2.2 Barycentric Coordinates

All of this theory also obviously holds for barycentric coordinates. Let's look at some special triangle conics in bary.

Theorem 14.2.6. A circumconic $\mathcal{C}(x:y:z)=0$ is a conic with symmetric matrix in barycentric coordinates in the form

$$\mathbf{C} = \begin{bmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{bmatrix}$$

Proof. This obviously satisfies $[1:0:0] \begin{bmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$, etc. ☐

Remark. It's easy to see with the previous results that the barycentric coordinates for the perspector of the circumconic $\mathcal{C}(x:y:z) := pyz + qxz + rxy = 0$ is just $P = [p:q:r]$. Define $\mathfrak{c}(P)$ as the set of points on the circumconic with perspector $[x:y:z]$.

In barycentric coordinates, a diagonal conic is really just a conic with a diagonal symmetric matrix, justifying the name.

Theorem 14.2.7. The symmetric matrix in barycentric coordinates of a diagonal conic $\mathcal{D}$ is

$$\mathbf{D} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Therefore, the space of diagonal conics is a **net** isomorphic to $\mathbb{P}^2$.

Proof. Since $\triangle ABC$ is self-polar in a diagonal conic, we have that the polar of $[1:0:0]$ is line $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, implying that

$$\mathbf{D} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In other words, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are the eigenvectors of $\mathbf{D}$, implying that it is a diagonal matrix. □

Example 14.2.8. A diagonal conic is a circumconic of a point P and its anticevian triangle $\triangle P_aP_bP_c = [-x:y:z], \ldots$.

Proof. Let P have nonzero coordinates $[x:y:z]$. Therefore, if we have

$$[x:y:z]\mathbf{D} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

, by anticevian triangle we also have

$$[-x:y:z]\mathbf{D} \begin{bmatrix} -x \\ y \\ z \end{bmatrix} = 0$$

$$[x:-y:z]\mathbf{D} \begin{bmatrix} x \\ -y \\ z \end{bmatrix} = 0$$

$$[x : y : -z]\mathbf{D}\begin{bmatrix} x \\ y \\ -z \end{bmatrix} = 0$$

. Let $\mathbf{D}_{12}$ be the element in the first row and second column, then we have $\mathbf{D}_{12}x^2 = 0 \implies \mathbf{D}_{12} = 0$. Apply for all off-diagonal elements, and we are done. $\square$

14.3 Algebraic Geometry

Let's move away from linear algebra and just look at implicit homogeneous polynomial functions $F(x : y : z) = 0$.

Definition 14.3.1. Define a curve of degree n to be the set of solutions to $F(x : y : z) = 0$ for irreducible polynomials F.

Let's see two theorems that are useful for studying general curves.

Theorem 14.3.2 (Linear combinations)**.** For two implicit homogeneous curves of the same degree d, $F(x : y : z)$ and $G(x : y : z)$, that intersect at d^2 points, then $rF(x : y : z) + sG(x : y : z) = 0$ defines another curve of degree d through the d^2 points.

Define the **Veronese map** ν_n of degree n from $\mathbb{P}^2$ to $\mathbb{P}^{n(n+3)/2}$ as mapping $[x : y : z]$ to the point whose coordinates are all possible monomials $x^a y^b z^c$ for $a + b + c = n$ of degree n in x, y, and z, i.e.

$$\left[x^n : x^{n-1}y : x^{n-1}z : x^{n-2}y^2 : x^{n-2}yz : x^{n-2}z^2 : \cdots : y^n : y^{n-1}z : y^{n-2}z^2 : \cdots : yz^{n-1} : z^n\right].$$

To see the utility of this transformation, let's look at the degree 2 Veronese map as an example, from $\mathbb{P}^2$ to $\mathbb{P}^5$, sending $[x : y : z] \to [x^2 : y^2 : z^2 : yz : xz : xy]$.

The set of points in $\mathbb{P}^5$ that correspond to some $[x : y : z]$ in $\mathbb{P}^2$ before applying the map is going to be just the surface formed in $\mathbb{P}^5$ by the image of all points in the projective plane after the map, we call this image the **Veronese surface**.

Define a **hyperplane** in $\mathbb{P}^5$ as the set of points $[a : b : c : d : e : f]$ satisfying $Aa+Bb+Cc+Dd+Ee+Ff = 0$ for some A, B, C, D, E, F (we will also define this for higher dimensional projective spaces as well similarly).

Example 14.3.3. Let $\mathcal{C}$ be the set of points in $\mathbb{P}^5$ corresponding to the intersection of some hyperplane with the Veronese surface. Then the set of preimages in $\mathbb{P}^2$ of the points in $\mathcal{C}$ is a conic section. In other words, six points in $\mathbb{P}^2$ are conconic if and only if their images on the Veronese surface lie on a common plane in $\mathbb{P}^5$ (which follows from unpacking the definitions).

Obviously, this generalizes to higher degree Veronese maps as well.

In a slight abuse of notation, we transpose it and use this as a (vertical) “vector” in the following theorem for more concise notation:

Theorem 14.3.4 (Generalized Collinearity). Points $P_1, P_2, \ldots, P_{n(n+3)/2}$ lie on a (possibly degenerate) degree n curve if and only if $\nu_n(P_1), \nu_n(P_2), \ldots, \nu_n\left(P_{n(n+3)/2}\right)$ lie on a hyperplane in $\mathbb{P}^{n(n+3)/2}$, i.e.

$$\det\begin{bmatrix}\nu_n(P_1) & \nu_n(P_2) & \cdots & \nu_n(P_{n(n+3)/2})\end{bmatrix} = 0.$$

Proof. If the $\nu_n(P_i)$ lie on a hyperplane, they satisfy a common linear equation in their coordinates, i.e. the monomials of degree n in x, y, and z. Therefore, the P_i satisfy a common homogeneous polynomial of degree n, so lie on a degree n curve.

Similarly, if the P_i lie on a curve of degree n, we get a linear equation in the coordinates of the $\nu_n(P_i)$ implying they lie on a hyperplane in $\mathbb{P}^{n(n+3)/2}$. □

Definition 14.3.5. Define a **rational map** φ to be a map between two curves $\mathcal{F} := F(x : y : z), \mathcal{G} := G(x : y : z)$ (treated as point sets), and for point P on $\mathcal{F}$

$$\varphi : \mathcal{F} \to \mathcal{G}$$

$$\varphi(P) \mapsto [x(P) : y(P) : z(P)] \in \mathcal{G}$$

where $x(P), y(P), z(P)$ are rational functions of P. Note that not all rational maps are necessarily one-to-one.

Definition 14.3.6. An **isomorphism** of $\mathbb{P}^2$ is a rational map ϕ which is everywhere regular (i.e. well defined, not $[0 : 0 : 0]$) and an inverse which is also an everywhere regular rational map. A **birational map** is a rational map with an inverse which is a rational map (though they may not be everywhere regular, i.e. at a singular point). We call two curves **birationally equivalent** if there is a birational map between them.

A **pencil** is an algebraic curve $\mathcal{C}$ that is isomorphic to $\mathbb{P}^1$. A **genus-0 curve** is an algebraic curve birationally equivalent to $\mathbb{P}^1$.

Example 14.3.7. Let $\mathcal{C}$ be the curve in $\mathbb{P}^2$ given by $y^2z = x^3 - x^2z$. Consider a line ℓ represented as $\begin{bmatrix}-t\\ 1\\ 0\end{bmatrix}$: $y = tx$. We claim that this line will intersect $\mathcal{C}$ at exactly two points for all t. Substitute it in, then we get $y^2z = x^3 - x^2z$, implying there's a double root at $x = 0$ (and thus $y = t \cdot 0 = 0$), factor it out to get $t^2z = x - z$, implying another intersection is at $x = t^2z + z$. Thus we get the second intersection point at $[t^2z + z : t^3z + tz : z]$, which is a rational function of t.

Then the function $f\colon \mathcal{C} \to \mathbb{P}^1$ given by $[x : y : z] \mapsto [x : y]$ is rational and has inverse given by $[s : t] \mapsto [t^2 s + s^3 : t^3 + ts^2 : s^3]$. Thus $\mathcal{C}$ is a genus-0 curve. However, it is not a pencil because f is not defined at $[0 : 0 : 1]$.

Theorem 14.3.8. A birational map between two [smooth?] curves in the projective plane is bijective and everywhere regular.

Proof. □

Now we go from studying curves to functions on curves.

Lemma 14.3.9. Let P and Q be two relatively prime polynomials. Then there are finitely many values of t for which $P(x) + tQ(x)$ has a repeated root.

Proof. Suppose that there are infinitely many t such that $P(x) + tQ(x)$ has a repeated root. Let x_t be the repeated root. Then $P'(x_t) + tQ'(x_t) = 0$, so $P(x_t)Q'(x_t) - P'(x_t)Q(x_t) = 0$. If $x_t = x_s$ for distinct s, t, then $P(x_t) = Q(x_t) = 0$, which is impossible. So all x_t are distinct. Thus $PQ' - P'Q$ has infinitely many roots, implying $PQ' = P'Q$. This is impossible. □

Here, let's provide a rigorization of the idea of counting the number of times a locus is traversed to get the degree of a point, used back in chapter 7.B.

Theorem 14.3.10 (AG degree equals degree)**.** Let ϕ be a rational map from $\mathbb{P}^1$ to itself that is n-to-1 at all but finitely many points. Then ϕ is given by $\phi([x : y]) = [P(x, y) : Q(x, y)]$ for degree n homogeneous polynomials P and Q.

Proof. Let $\phi([x : y]) = [P(x, y) : Q(x, y)]$ where P and Q are coprime. Then for all but finitely many values of k, $P(x/y) - kQ(x/y)$ has no repeated roots. Thus almost all points have $\deg(P)$ preimages. Thus $n = \deg P$. □

Theorem 14.3.11 (Rational curve parameterization)**.** Let $\mathcal{C}$ be a genus-0 plane curve with degree d. Then:

(i) There exists a parameterization of $\mathcal{C}$ of degree d in terms of $\mathbb{P}^1$, $\iota(t) = [P_0(t) : Q_0(t) : R_0(t)]$.

(ii) Any other parameterization has degree nd for some integer n and takes the form $t \mapsto [a(t)^d P_0(b(t)/a(t)) : \cdots]$.

(iii) A parameterization of degree nd covers a generic point n times.

Proof. By Bézout's, $\mathcal{C}$ intersects a generic line d times, while a parameterization κ intersects a generic line $\deg \kappa$ times. Thus $d \mid \deg \kappa$ and $\frac{\deg \kappa}{d}$ is the number of times κ covers a generic point.

Let $\phi \colon \mathbb{P}^1 \to \mathcal{C}$ be a birational map. Then ϕ is a parameterization. Since a generic point on $\mathcal{C}$ is covered once, ϕ must be a degree d parameterization.

Let κ be a degree nd parameterization. Then κ is n-to-1, so $\phi^{-1} \circ \kappa$ is a n-to-1 map of $\mathbb{P}^1$ into itself. The desired result follows from (TODO xooks). □

Example 14.3.12. For a conic $\mathcal{C}$, a point $P \notin \mathcal{C}$, and a line ℓ, we can define a rational map φ from $\mathcal{C} \to \ell$, by sending $Q \in \mathcal{C} \mapsto \overline{PQ} \cap \ell$. This covers a generic point in ℓ twice, so it has degree 2. Note that this map is not birational, as it is not even bijective!

Theorem 14.3.13 (Parametrization of conics and cubics)**.** Every conic is a pencil, and every singular cubic is genus-0.

Proof. For conics, project through a fixed point on the conic onto a fixed line isomorphic to $\mathbb{P}^1$.

For cubics, project through the singular point. □

Now let's see how to develop the theory of Section 7.A in the setting of algebraic geometry.

Theorem 14.3.14 (Birational implies Projective)**.** If ϕ is a rational map from $\mathbb{P}^1$ to itself that is 1-to-1 at all but finitely many points, ϕ is a homography and hence preserves the cross-ratio.

Proof. Trivial by (TODO ag degree equals degree). □

The importance of this theorem is that it is usually easier to show that a map is birational, while it is more useful geometrically to know that a map is projective. To show that a map is birational, it suffices to show that it and its inverse are rational. This is usually very easy because generally any geometric construction that doesn't require choosing between roots (for example taking a specific tangent from a point to a conic) is rational.

Example 14.3.15 (Generalized DIT)**.** Let $n > 1$ be a positive integer and let $P_1, P_2, \dots P_{\frac{n(n+3)}{2}-1}$ be points such that the space of degree n curves through them forms a pencil and $n-2$ are collinear on line ℓ. Let $\mathcal{K}$ be a degree n curve through all of these points that intersects ℓ again not in any of the P_i at points Q_1, Q_2. Then there is an involution on ℓ that swaps Q_1, Q_2, for all $\mathcal{K}$.

Proof. The map $Q_1 \mapsto \mathcal{K}$ can be expressed as a rational function by Generalized Collinearity. Then the intersection of $\mathcal{K}$ and ℓ is a polynomial whose coefficients are rational in Q_1 with $n-1$ roots that are also rational in Q_1. So the nth intersection is rational in Q_1 by Vieta. So $Q_1 \mapsto Q_2$ is rational. Since it is an involution, it is birational and hence projective. □

Definition 14.3.16 (Moving Curves)**.** Define a **degree-d curve moving with degree k** as the preimage of a degree-k moving hyperplane in $\mathbb{P}_{\frac{d(d+3)}{2}}$ after the degree d Veronese map.

Theorem 14.3.17. For moving points $P_1, P_2, P_3, \dots, P_{\frac{d(d+3)}{2}}$, the degree d curve passing through them has degree at most $d \cdot \left(\sum_i^{\frac{d(d+3)}{2}} \deg P_i\right)$.

Proof. Let Q be your curve after the Veronese map. Then Q satisfies coplanarity, so take the determinant of the square matrix formed by

$$\det \begin{bmatrix} \text{coordinates of } P_1 \text{ in the Veronese} \\ \text{coordinates of } P_2 \text{ in the Veronese} \\ \vdots \\ P_{n-1} \\ P_n \\ Q \end{bmatrix} = 0,$$

□

Theorem 14.3.18 (Moving Curve Intersection)**.** Let $\mathcal{A}, \mathcal{B}$ be two curves of degrees a, b moving with degrees d_1, d_2. Then all intersection points of $\mathcal{A} \cap \mathcal{B}$ as the parameter varies, will lie on a fixed algebraic curve with degree of degree at most $ad_2 + bd_1$.

Proof. Write the two curves as bi-homogeneous polynomials in $[x : y : z]$ and $[s : t]$ where $[s : t]$ is your parameterization variable. Then, take a resultant in your parameterization variable to get a polynomial of degree $ad_2 + bd_1$. □

For a pencil $\mathcal{C}$, let $\phi\colon \mathcal{C} \to \mathbb{P}^1$ be an isomorphism. Then $\mathcal{C}$ has a cross-ratio given by $(a, b; c, d)_{\mathcal{C}} = (\phi(a), \phi(b); \phi(c), \phi(d))$.

Theorem 14.3.19. The cross-ratio is well-defined and preserved under birational maps.

Proof. Let $\mathcal{C}$ be a pencil and ϕ, ψ be two equivalences from $\mathcal{C}$ to $\mathbb{P}^1$. Then apply Birational implies Projective to $\phi \circ \psi^{-1}$ to obtain that the cross-ratios associated to ϕ and ψ agree.

Let $\mathcal{C}, \mathcal{D}$ be pencils and let ϕ be an equivalence from $\mathcal{C}$ to $\mathbb{P}^1$, ψ be an equivalence from $\mathcal{D}$ to $\mathbb{P}^1$, and $\chi\colon \mathcal{C} \to \mathcal{D}$ be a rational map. Then apply Birational implies Projective to $\psi \circ \chi \circ \phi^{-1}$ to obtain that χ preserves the cross-ratio. □

14.3.1 Isoconjugations

Definition 14.3.20. For this section, an **isoconjugation** is a quadratic birational involution from $\mathbb{P}^2$ to itself.

Our aim is to show that this definition coincides with the definition given in Section 7.4.

Let X be a sufficiently generic point, and let φ be some isoconjugation. Let $\mathcal{K}$ be the isopivotal cubic with pivot X (defined with our new definition of isoconjugation). Note that $\mathcal{K}$ contains X, $\varphi(X)$, and all singular (i.e. vertices - they map to $[0:0:0]$) and fixed points of φ. Define a group law $(\mathcal{K}, O)$ where O, the identity in the group law, is any inflection point of $\mathcal{K}$.

By definition of an isopivotal cubic, the isoconjugation φ on $\mathcal{K}$ is given by

$$P \xmapsto{\varphi} -P - X.$$

Hence $\varphi(X) = -2X$.

Theorem 14.3.21. We have in the group law that

- If Q is a fixed point of φ, then $2Q = -X$,
- Let Q^A, Q^B, Q^C be the other three fixed points of φ, and set $A = QQ^A \cap Q^BQ^C$, etc. Then A lies on $\mathcal{K}$ and $Q^A = Q + A - O$, etc,
- $2A, 2B$, and $2C$ are all equal to $2X$.

Proof. If Q is a fixed point, then $Q = -Q - X$, implying $2Q = -X$.

The group $\{0, Q^A - Q, Q^B - Q, Q^C - Q\}$ is the 2-torsion group of $\mathcal{K}$, and is hence isomorphic to the Klein four-group. So $Q^A - Q = Q^B - Q + Q^C - Q$, or $Q + Q^A = Q^B + Q^C$. Thus QQ^A and Q^BQ^C intersect on $\mathcal{K}$ at $A = -Q - Q^A$. Doubling gives $2A = -2Q - 2Q^A = 2X$.

□

Lemma 14.3.22. For any point $P \in QQ^A$, $\varphi(P) \in QQ^A$.

Proof. We use moving points. Since $\varphi(P)$ is degree 2, it suffices to check three cases. Since Q and Q^A are fixed points of φ, they work. By group law we have that A, Q, Q^A collinear, and $QQ^A \cap BC$ and A are conjugates. Thus A also works, proving the result. □

Now this implies $\varphi(A)$ is on QQ^A, and similarly $\varphi(A) \in Q^BQ^C$. Thus either $\varphi(A) = A$ or φ is singular at A. The former is impossible as we have already determined all fixed points, so φ is not regular at A.

Lemma 14.3.23. For a point $P \in AB$, $\varphi(P) = C$.

Proof. We use moving points. The desired condition is degree 2, so it suffices to check 3 cases. Cases $P = A, B$ work by degeneracy (Since φ is singular at A, plugging A into φ gives $[0:0:0]$ which satisfies every homogeneous constraint). Finally, note that $\varphi(AB \cap CQ) = C$ by group law. □

Lemma 14.3.24. There exists an involution φ_A on $\mathbf{T}A$ that fixes QQ^A and Q^BQ^C and swaps AB and AC.

Proof. DDIT. □

Lemma 14.3.25. For a point $P \neq A$, $\varphi(P) \in \varphi_A(AP)$.

Proof. We use moving points. The map $P \mapsto \varphi(P)$ has degree 2, and $\varphi_A(AP)$ has degree 1. Hence the desired condition has degree 3. Note that the desired condition holds when $P \in Q^BQ^C, AQ, AB, AC$. □

Theorem 14.3.26. Let φ be an isoconjugation and let Q, Q^A, Q^B, Q^C be as defined above. Then for any conic $\mathcal{C}$ through Q, Q^A, Q^B, Q^C, $\varphi(P)$ and P are conjugate with respect to $\mathcal{C}$.

Proof. We use moving points on $\mathcal{C}$. The space of conics through $\mathcal{C}$ is a pencil, and the desired condition is linear in $\mathcal{C}$ by Theorem 14.2.4. Hence it suffices to prove it for two conics in this pencil. Let $\mathcal{C}$ be the degenerate conic $AQ \cup Q^BQ^C$. Then by Lemma 14.3.24, we have $A(P, \varphi(P); Q, Q^B) = -1$. Hence if $\ell = P\varphi(P)$, then $(P, \varphi(P); \ell \cap AQ, \ell \cap Q^BQ^C) = -1$. Thus P and $\varphi(P)$ are conjugate in $\mathcal{C}$. Similarly, $\mathcal{C} = BQ \cup Q^AQ^C$ also works. □

14.3.2 Cubic Group Law

Theorem 14.3.27 (Generalized Cayley-Bacharach)**.** Let $\mathcal{K}$ be a cubic given by $K(x, y, z) = 0$, and let $\mathcal{C}, \mathcal{D}$ be degree $d \geq 3$ curves not containing $\mathcal{K}$ given by $C(x, y, z) = 0$ and $D(x, y, z) = 0$. Suppose that $\mathcal{K}, \mathcal{C}, \mathcal{D}$ share $3d - 1$ points of intersection. Then they all pass through a $3d$th common intersection.

Proof. We proceed by induction on d. If $d = 3$, the result follows from standard Cayley-Bacharch. Otherwise, pick a line ℓ through two intersection points of $\mathcal{K}$ and $\mathcal{C}$. The condition that $F(x, y, z)K(x, y, z) + uC(x, y, z) + vD(x, y, z) = 0$ on ℓ is $d - 1$ linear constraints. Note that $\{K, xK, \ldots, x^{d-3}K, C, D\}$ spans a space of dimension d, so when restricted to ℓ there must be a linear dependency. Then we have $G(x)K(x, y, z) + sC(x, y, z) + tD(x, y, z) = 0$ on ℓ. Since K does not divide C or D and GK is not identically zero on ℓ, both s and t are nonzero. If $CK + sC = tD$ vanishes identically on $\mathbb{P}^2$, the desired result follows. Hence suppose otherwise. Let C' be $G(x)K(x, y, z) + sC(x, y, z) + tD(x, y, z)$. Now C' vanishes identically on ℓ. Set $C''(x, y, z) = C'(x, y, z)$ divided by the equation for ℓ.

Similarly, we can form a linear combination D' of $\{K, xK, \ldots, x^{d-3}K, C'', D\}$ that vanishes identically on ℓ, where the coefficients of C'' and D are nonzero. If this combination vanishes identically, the desired result follows. Otherwise, let D'' be D' divided by the equation for ℓ. The desired result follows from the inductive hypothesis on $\mathcal{C}''$ and $\mathcal{D}''$. □

Theorem 14.3.28. For points $P_1, P_2, \ldots, P_{3d}$ lying on cubic $\mathcal{K}$, if L is the third intersection of the tangent to $\mathcal{K}$ at O then $P_1, P_2, \ldots, P_{3d}$ lie on a degree d curve not containing $\mathcal{K}$ if and only if

$$P_1 + P_2 + \cdots + P_{3d} = dL.$$

Proof. □

For a cubic $\mathcal{K}$, define a **divisor** on $\mathcal{K}$ as a formal linear combination of points (i.e. an element of the free abelian group). For a meromorphic complex function f, let $\operatorname{div} f$ (call this the divisor of f on $\mathcal{K}$) be the formal sum of zeroes minus sum of poles counted with multiplicity. Let $\operatorname{Pic} \mathcal{K}$ be the group of divisors modulo divisors of rational functions. Set $\operatorname{Pic}^0 \mathcal{K}$ to be the subgroup of divisors whose coefficients sum to zero.

Example 14.3.29. Three points A, B, C are collinear iff. $A+B+C = 2L+M$, implying $A+B+C-2L-M = 0$, so the formal sum $A + B + C - 2L - M$ is an element of $\operatorname{Pic}^0 \mathcal{K}$.

Theorem 14.3.30. Let $(\mathcal{K}, O)$ be a cubic. Then the map $u : \mathcal{K} \to \operatorname{Pic}^0(\mathcal{K})$ sending $P \mapsto P - O$ is a group isomorphism.

Proof. We first show that it is a homomorphism. If P and Q are two points on $\mathcal{K}$, let R be the third intersection of PQ with $\mathcal{K}$ and let S be their sum. Then we have to show that $P + Q - S - O$ is the divisor of a rational function. Let f_1 be the equation of line PQ and let f_2 be the equation of line OS. Then $\operatorname{div} f_1/f_2 = P + Q - S - O$.

Next we show that it is injective. Suppose that $\operatorname{div} f = P - Q$ for distinct points P, Q. Write $f = G/H$ where G and H are homogeneous polynomials on $\mathbb{P}^2$ of the same degree. Then generalized Cayley-Bacharach on $\mathcal{K}$ and the curves $G = 0, H = 0$ gives a contradiction.

Next we show that it is surjective. For techincal simplicity, suppose that O is an inflection point. This does not matter since changing identity is trivial. Let D be a degree 0 divisor. We want to show that D is equivalent to a divisor of the form $P - O$.

Suppose that we have D in the form

$$D = p_1 + p_2 + \cdots - q_1 - q_2 - \cdots + nO.$$

Let f_1 be the equation of the tangent at O and let f_2 be the equation of the line through q_1 and O. Then adding $\operatorname{div} f_2/f_1$ cancels out the $-q_1$, changes coefficients of O, and adds a new positive term. So we can suppose that there are no q_i.

Let f_1 be the equation of line p_1p_2 and let f_2 be the equation of line OS, where S is sum of p_1 and p_2. Then adding $\operatorname{div} f_2/f_1$ cancels out $p_1 + p_2$ and adds a new positive point, and changes the coefficient of n. The net effect is to reduce the number of p_i by 1. Hence we can suppose there is only one p_i.

Then D takes the form $D = P + nO$. Since $\deg D = 0$, we have $n = -1$, so we have the desired form. □

Now we can prove a very important theorem:

Theorem 14.3.31. Let $(\mathcal{K}_1, O_1)$ and $(\mathcal{K}_2, O_2)$ be cubics with an identity element O_i. Let $f\colon \mathcal{K}_1 \to \mathcal{K}_2$ be a rational map that sends O_1 to O_2. Then f is a group homomorphism.

Proof. It suffices to show that f induces an homomorphism on Pic^0. Let D_1, D_2, D_3 be divisors on $\mathcal{K}_1$ with $\operatorname{div} g = D_1 + D_2 - D_3$. Let h be the rational function on $\mathcal{K}_2$ given by $h(P) = \prod_{f(Q)=P} g(Q)$. Then $\operatorname{div} h = f(D_1) + f(D_2) - f(D_3)$. □

The rest of this section is devoted to giving an algebraic proof of (TODO Corollary 11.1.9). Let $[m]$ denote multiplication by m. Given a rational function f from $(\mathcal{K}, O)$ to itself that fixes O, let the degree $\deg f$ be the number of times f covers a generic point. Define the **dual isogeny** by

$$\hat{f}(x) = \sum_{f(y)=x} y,$$

where y takes values in an algebraically closed field and solutions are counted with appropriate multiplicity. This is also a rational function, because it is symmetric in the roots of $f(y) = x$.

Theorem 14.3.32. For a rational function f, we have $f \circ \hat{f} = \hat{f} \circ f = [\deg f]$.

Proof. The equality of the first and third follows from definitions and the fact that f is a homomorphism. The equality of the second and third follows from $[m] \circ f = f \circ \hat{f} \circ f = (f \circ \hat{f}) \circ f$. Either $f = 0$, in which case the desired result holds, or f is surjective, in which case $[m] \circ f = (f \circ \hat{f}) \circ f$ implies $[m] = f \circ \hat{f}$. □

Theorem 14.3.33. We have $\widehat{\varphi + \psi} = \hat{\varphi} + \hat{\psi}$.

Proof. Let (x_1, y_1) and (x_2, y_2) be two sets of coordinates on $\mathcal{K}$. Then $\varphi(P)$, $\psi(P)$, and $(\varphi+\psi)(P)$ are points of $\mathcal{K}$ over the base field $\mathbb{K}[x_1, x_2]$, where P is a point of $\mathcal{K}$ over $\mathbb{K}[x_1, x_2]$. Since addition doesn't depend on the base field, we have $\varphi(P) + \psi(P) - (\varphi + \psi)(P) = 0$ in $\operatorname{Pic}^0(\mathcal{K})$. Thus there exists some rational function $f_P(Q)$ such that $\operatorname{div} f_P = \varphi(P) + \psi(P) - (\varphi + \psi)(P)$. Now we switch around the variables and let x_2, y_2 be fixed and consider f as a function of P. Then we can calculate that $\operatorname{div} f = \hat{\varphi}(Q) + \hat{\psi}(Q) - \widehat{(\varphi+\psi)}(Q)$ plus some constant. So $\hat{\varphi}(Q) + \hat{\psi}(Q) - \widehat{(\varphi+\psi)}(Q)$ is constant in the group law. But when Q is the identity, it is the identity. Thus $\widehat{(\varphi+\psi)}(Q) = \hat{\varphi}(Q) + \hat{\psi}(Q)$. $\square$

Thus by induction, we get that $\widehat{[m]} = [m]$. Thus $[\deg[m]] = [m]\widehat{[m]} = [m][m] = [m^2]$, so $\deg[m] = m^2$. Now the preimage of a generic point P under $[m]$ consists of one preimage Q plus every element of $\ker[m]$. Thus $|\ker[m]| = m^2$. For prime $m = p$, this implies $\ker[p] \cong (\mathbb{Z}/p\mathbb{Z})^2$. For prime power $m = p^k$, the standard decomposition of $\ker[p^k]$ must have exactly two components, and each has order dividing p^k. Since their orders multiply to p^{2k}, we have $\ker[p^k] \cong (\mathbb{Z}/p^k\mathbb{Z})^2$. Finally, $\ker[m]$ must contain subgroups $(\mathbb{Z}/p^k\mathbb{Z})^2$ for every prime power $p^k \mid m$ and has order m^2, so $\ker[m] \cong (\mathbb{Z}/m\mathbb{Z})^2$.