Lecture Notes (25th Jan, 2026)

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In this lecture, we study the properties of tangent circles.

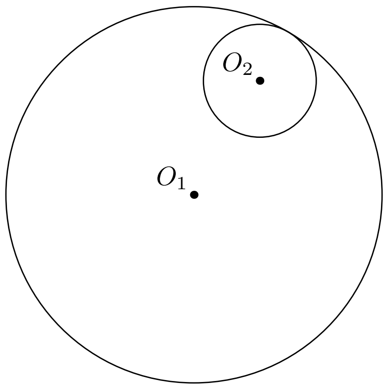
§1 Internally Tangent Circles

We would like to study about configurations that deal with tangent circles. When we say two circles are tangent, it means that these circles only have one point in common. A pair of circles could be either **internally** tangent or **externally** tangent.

Definition 1.1. A pair of circles Γ and ω are

1. **internally tangent**, if and only if Γ and ω share a single point and the center of the smaller circle is contained inside the larger circle.
2. **externally tangent**, if and only if Γ and ω share a single point and the center of either of the circles lie outside the other circle.

We would like to focus on pair of circles that are internally tangent for now and study their properties. Let's start with the simplest configuration with two internaly tangent circles.

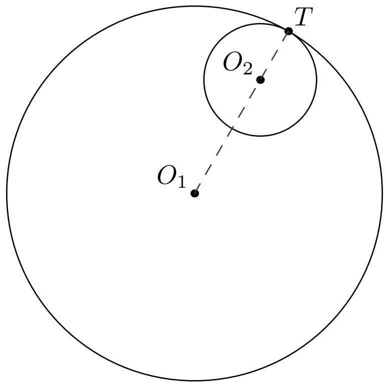


Let Γ and ω be two circles centered at O_1 and O_2, with radius R and r, where $R > r$. Suppose that Γ and ω are internally tangent to each other.

§1.1 Homothetic Mapping

Proposition 1.2

Suppose T is the internal tangency point of Γ and ω, then the points O_1, O_2 and T are collinear.

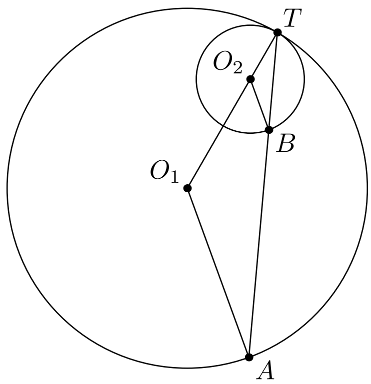


Proof. Consider a homothety at point T that maps ω to Γ. Observe that under this homothetic transformation, we map the point O_2 to O_1. Hence, the points O_1, O_2 and T must be collinear. □

As a consequence of the homothetic mapping, we have the following result

Proposition 1.3

Choose a point B on ω. Suppose TB meets Γ again at A. Then $\overline{O_1A} \parallel \overline{O_2B}$.



Proof. Consider a homothety at point T that maps ω to Γ. Under this homothety, the point B is mapped to A. Hence, $\triangle TO_2B \sim \triangle TO_1A \implies \angle TBO_2 = \angle TAO_1 \implies \overline{O_1A} \parallel \overline{O_2B}$. □

Corollary 1.4

Let T be the point of internal tangency of two circles Γ and ω with radii R and r, where $R > r$. Choose a point B on ω and let TB meet Γ at A. Then

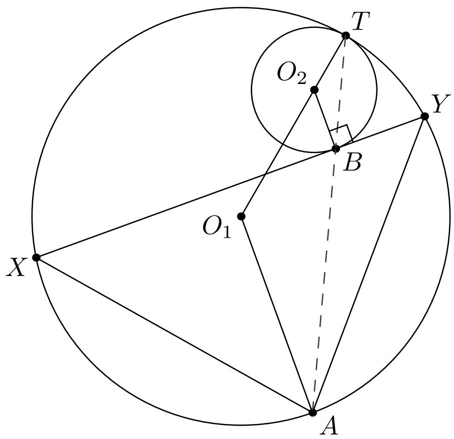
$$\frac{\overline{TB}}{\overline{TA}} = \frac{r}{R}$$

The proof for the above result immediately follows from the fact that $\triangle TO_2B \sim \triangle TO_1A$. Now, we move onto a very celebrated result by Archimedes which appears very frequently in geometry configurations.

§1.2 Archimedes' Lemma

Theorem 1.5 (Archimedes' Lemma)
Let Γ and ω be two circles centered at O_1 and O_2. Suppose that these circles are internally tangent at the point T. Let $\overline{XY}$ be the chord of Γ such that $\overline{XY}$ is tangent to ω at point B. Let A be the midpoint of the arc XY that does not contain T. Then

1. points T, B and A are collinear.
2. $\overline{AB} \cdot \overline{AT} = \overline{AX}^2 = \overline{AY}^2$



Proof. Consider a homothety at point T that sends ω to Γ. Suppose that this homothety sends point B to A', where A' lies on Γ. Since $\overline{XY}$ is tangent to ω at B, therefore this homothety maps XY to a line ℓ passing through A' that is tangent to Γ. Since ℓ is the image under a homothetic transformation of $XY \implies \overline{XY} \parallel \ell$. Therefore,

$$\angle YXA' = \angle\left(\overline{A'X}, \ell\right) = \angle XYA'$$

This implies that $\triangle XYA'$ is isosceles $\implies$ A' is the midpoint of the arc XY not containing T. Thus $A \equiv A'$, proving that T, B and A are collinear.

For the second part, we shall show that $\overline{AX}$ is tangent to $\odot(TBX)$ at X. This is easy to establish since,

$$\angle XTB = \angle XTA = \angle XYA = \angle AXY = \angle AXB$$

Using the power of a point theorem, we get that

$$\overline{AX}^2 = \overline{AB} \cdot \overline{AT}$$

Since $\overline{AX} = \overline{AY}$, which implies the relation. □

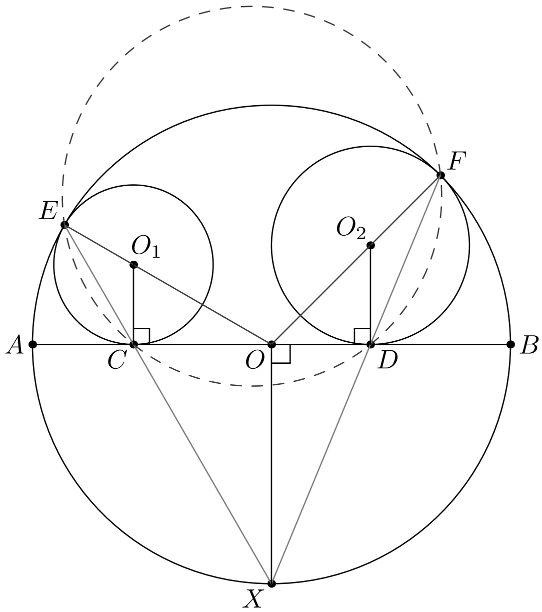
Exercise 1.6 (Russia 2001)**.** In the above configuration, show that the circumradius of $\triangle ABY$ is a constant that does not depend upon the position of point B.

Let's look at some nice results that revolve around the **Archimedes' Lemma**.

§1.3 Examples

Problem 1.7

Let h be a semicircle with diameter AB. The two circles k_1 and k_2, $k_1 \neq k_2$, touch the segment AB at the points C and D, respectively, and the semicircle h fom the inside at the points E and F, respectively. Prove that the four points C, D, E and F lie on a circle.



Proof. Suppose X is the midpoint of arc AB not containing E. Then X lies on lines EC and FD by archimedes' lemma. Since,

$$\overline{XC} \cdot \overline{XE} = \overline{AX}^2 = \overline{XD} \cdot \overline{XF}$$

Therefore, by the converse of power of a point theorem $\implies$ points C, D, E and F lie on a circle. □

§1.4 Exercises

Exercise 1.8 (INMO 2019). Let AB be the diameter of a circle Γ and let C be a point on Γ different from A and B. Let D be the foot of perpendicular from C on to AB.Let K be a point on the segment CD such that AC is equal to the semi perimeter of ADK.Show that the excircle of ADK opposite A is tangent to Γ.

Exercise 1.9 (RMO 2019). Given a circle τ, let P be a point in its interior, and let l be a line through P. Construct with proof using ruler and compass, all circles which pass through P, are tangent to τ and whose center lies on line l.

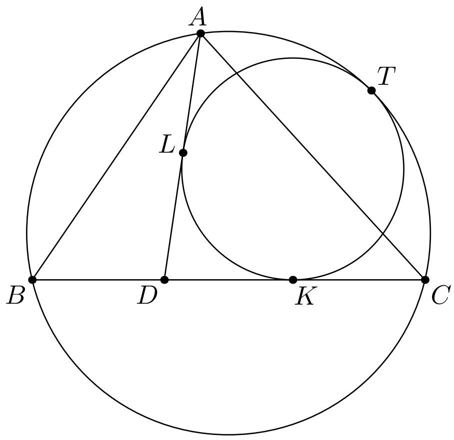
Exercise 1.10 (RMO 2017). Let Ω be a circle with a chord AB which is not a diameter. Γ_1 be a circle on one side of AB such that it is tangent to AB at C and internally tangent to Ω at D. Likewise, let Γ_2 be a circle on the other side of AB such that it is tangent to AB at E and internally tangent to Ω at F. Suppose the line DC intersects Ω at $X \neq D$ and the line FE intersects Ω at $Y \neq F$. Prove that XY is a diameter of Ω .

§2 Curvilinear Incircles

Let's move to something more complicated and miraculous.

> **Definition 2.1.** Given $\triangle ABC$ and a point D on $\overline{BC}$, a circle ω is called the **curvilinear incircle** of $\triangle ABC$ if ω is tangent to sides $\overline{AD}$ and $\overline{BC}$, and is internally tangent to $\odot(ABC)$.

Curvilinear incircles are a natural extension of the archimedes' lemma. Essentially, we are choosing another point on the outer circle and adding more tangents.

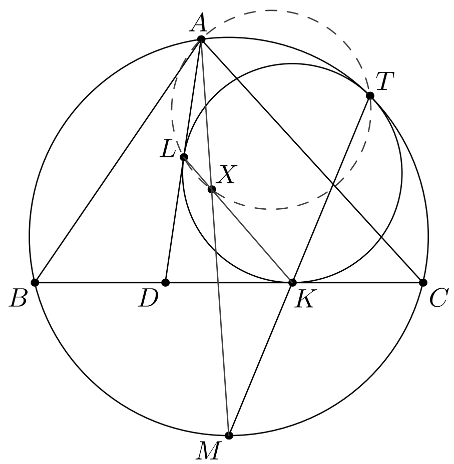


Let's look at some properties of the **curvilinear incircles**.

§2.1 More Circles!

Proposition 2.2

Given $\triangle ABC$ and a point D on the $\overline{BC}$. Suppose ω is the curvilinear incircle of $\triangle ABC$ tangent to $\overline{AD}$ and $\overline{BC}$ at L and K, and tangent to $\odot(ABC)$ at T. Let M be the midpoint of arc BC not containing A. Suppose $\overline{AM}$ intersects $\overline{KL}$ at X. Then the points A, L, X and T are concyclic.



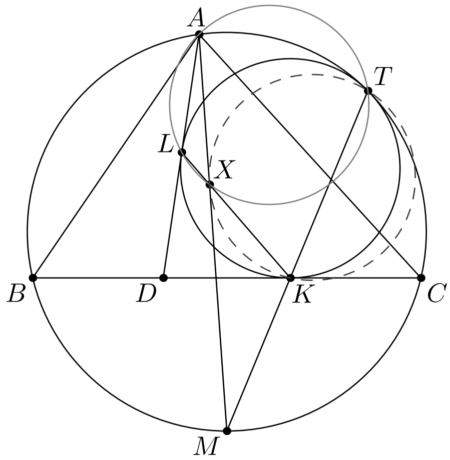
Proof. By archimedes' lemma, we know that M lies on $\overline{KT}$. To show that the points A, L, X and T are concyclic, we just need to angle chase

$$\angle XLT = \angle KLT = \angle CKT = \angle MBT = \angle MAT = \angle XAT$$

where, $\angle CKT = \angle MBT$ holds because, $\overline{TM}$ is the angle bisector of $\angle BTC \implies \triangle TBM \sim \triangle TKC$. Therefore, $\angle XLT = \angle XAT$, which implies that the four points are concyclic. □

Proposition 2.3

Given $\triangle ABC$ and a point D on the $\overline{BC}$. Suppose ω is the curvilinear incircle of $\triangle ABC$ tangent to $\overline{AD}$ and $\overline{BC}$ at L and K, and tangent to $\odot(ABC)$ at T. Let M be the midpoint of arc BC not containing A. Suppose $\overline{AM}$ intersects $\overline{KL}$ at X. Then, $\overline{MX}$ is tangent to $\odot(XKT)$ at point X.



Proof. Effectively, we just want to show that $\angle MXK = \angle MTX$. Fortunately, this is just straightforward angle chasing

$$\angle MTX = \angle MTL - \angle XTL = \angle DLK - \angle XAL = \angle AXL = \angle MKX$$

which proves that $\overline{MX}$ is tangent to $\odot(XKT)$ at X. □

§2.2 Introducing the Incenter

Proposition 2.4 (Sawayama's Theorem)

Show that the point X is the **Incenter** of $\triangle ABC$.

Proof. Observe that $\overline{MB} = \overline{MC} \implies \overline{AM}$ is the angle bisector of $\angle BAC$. Since,

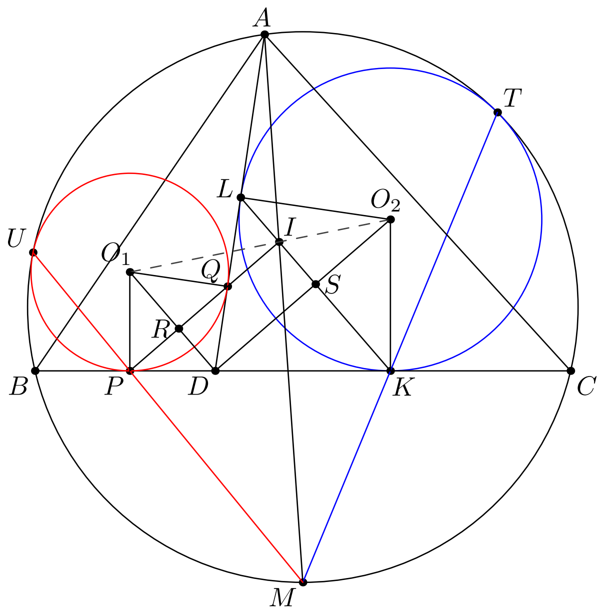
$$\overline{MX}^2 = \overline{MK} \cdot \overline{MT} = \overline{MB}^2 = \overline{MC}^2$$

Hence by the incenter/excenter lemma, we get that X is the incenter of $\triangle ABC$. □

It's very surprising how the incenter appears in this configuration. Something even more interesting occurs when we add the other curvilinear incircle of the cevian $\overline{AD}$ to the diagram, which leads to a reknowned result by *Victor Thébault*.

§2.3 Thébault's Theorem

Theorem 2.5 (Thébault's Theorem)
Given $\triangle ABC$ and a point D on $\overline{BC}$, let ω_1 and ω_2 be the two curvilinear incircles of $\triangle ABC$ tangent to the cevian $\overline{AD}$. Suppose O_1 and O_2 are the centers of the two curvilinear incircles and I is the incenter of $\triangle ABC$, then points O_1, I and O_2 are collinear.



Proof. Suppose $\odot(O_1)$ touches $\overline{BC}$ at P and $\overline{AD}$ at Q, and $\odot(O_2)$ touches $\overline{BC}$ at K and $\overline{AD}$ at L. Let $\overline{PQ}$ intersect $\overline{O_1D}$ at R and $\overline{KL}$ intersect $\overline{O_2D}$ at S.

Since $\odot(O_1)$ and $\odot(O_2)$ are the A-curvilinear incircles of cevian $\overline{AD}$, hence PQ intersects LK at I. It's easy to that the quadrilaterals PO_1QD and LDO_2K are kites, which is because,

$$\angle O_1PD = \angle O_1QD = 90^\circ \ \& \ \angle O_2LD = \angle O_2KD = 90^\circ$$

and $DP = QD$, $DK = DL$. Further, this implies that DO_1 and DO_2 are the angle bisectors of $\angle PDQ$ and $\angle KDL$. Therefore,

$$\angle O_1DO_2 = \angle O_1DQ + \angle LDO_2 = \frac{1}{2}(\angle PDQ + \angle KDL) = 90^\circ$$

Since, $PQ \perp O_1D$ and $KL \perp O_2D \implies \overline{PQ} \parallel \overline{O_2D}$ and $\overline{KL} \parallel \overline{O_1D}$. Hence, $RDSI$ is a rectangle and $\angle PIK = 90^\circ$. Suppose $\angle DO_2K = \alpha$, then

$$\angle O_1PQ = \angle O_1DP = \angle DKL = \angle DO_2K = \alpha$$

Therefore,

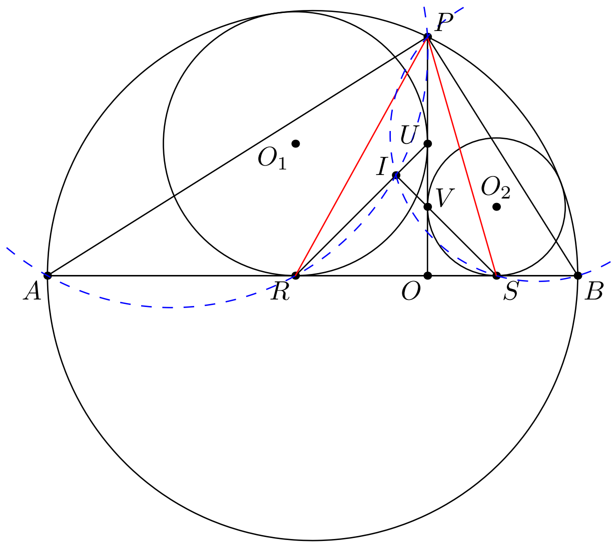
$$\begin{aligned}\frac{O_1R}{RI} &= \frac{O_1P\sin\alpha}{DS} = \frac{O_1D\sin^2\alpha}{DS} \\ &= \frac{O_1D\sin^2\alpha}{DK\sin\alpha} = \frac{O_1D\sin^2\alpha}{O_2D\sin^2\alpha} = \frac{O_1D}{O_2D}\end{aligned}$$

and as a result, we have $\triangle O_1RI \sim \triangle O_1DO_2 \implies I$ lies on $\overline{O_1O_2}$. $\square$

§2.4 Examples

Problem 2.6

Let O be any point on the diameter AB of a circle ω. Let the perpendicular to AB at O meet ω at P. Suppose that the incircles of the curvilinear triangles AOP and BOP meet AB at R and S respectively. Prove that $\angle RPS$ is independent of the position of O.



Proof. Since $\overline{IR}$ and $\overline{IS}$ are parallel to the angle bisectors of $\angle PAO = 90°$ which means that $\triangle IRS$ is right-angled isosceles. Since $\angle API = \angle BPI = 45° \implies PARI$ and $BSIP$ are cyclic quadrilaterals. Hence,

$$\begin{aligned}\angle RPS &= 180° - \angle PRS - \angle PSR \\ &= 90° - \angle PRI - \angle PSI \\ &= 90° - \tfrac{1}{2}(\angle PAB + \angle PBA) \\ &= 45°\end{aligned}$$

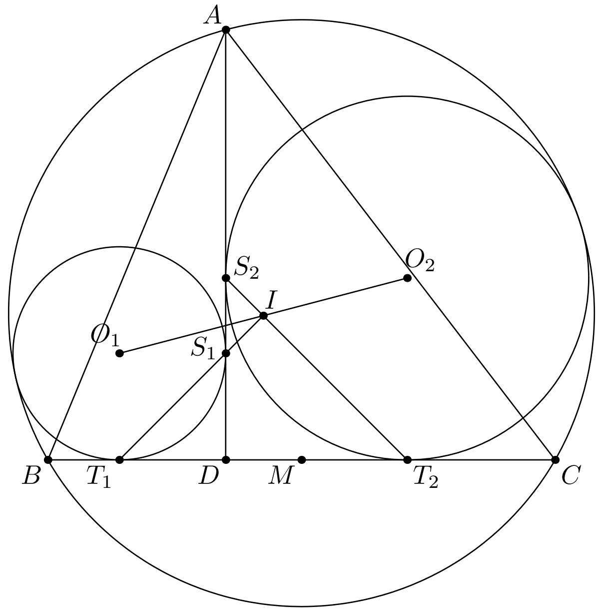
which is independent of the position of O. □

Problem 2.7 (Romania TST 2006)

Let ABC be an acute triangle with $AB \neq AC$. Let D be the foot of the altitude from A and ω the circumcircle of the triangle. Let ω_1 be the circle tangent to AD, BD and ω. Let ω_2 be the circle tangent to AD, CD and ω. Let ℓ be the interior common tangent to both ω_1 and ω_2, different from AD. Prove that ℓ passes through the midpoint of BC if and only if $2BC = AB + AC$.

Proof. Suppose I is the incenter of $\triangle ABC$, O_1, O_2 are the centers of ω_1 and ω_2 and they touch $\overline{BC}$ and $\overline{AD}$ at points T_1, S_1 and T_2, S_2 respectively. Let M be the midpoint of $\overline{BC}$.

By sawayama's theorem, we know that T_1S_1 and T_2S_2 pass through I and they are parallel to angle bisectors of $\angle ADB$. Since $\angle ADB = 90° \implies \triangle IT_1T_2$ is right-angled



isosceles. Let the internal common tangent of ω_1 and ω_2 meet $\overline{BC}$ at P. Since, $\angle O_1DO_2 = 90°$ and $\angle O_1PO_2 = 90°$, we get that O_1DPO_2 is cyclic. By thebault's theorem, we get that I lies on $\overline{O_1O_2}$. Since $\triangle IT_1T_2$ is right-angled isosceles, hence if the incircle touches $\overline{BC}$ at X, then X must be the midpoint of $\overline{T_1T_2} \implies I$ must be the midpoint of $\overline{O_1O_2}$. Since $\angle O_1DO_2 = 90° \implies I$ is the center of $\odot(O_1DPO_2) \implies X$ is the midpoint of $\overline{DP} \implies \overline{DT_1} = \overline{PT_2}$. Using the fact that $\cos\left(\frac{B}{2}\right) = \sqrt{\frac{s(s-b)}{ca}}$, and

$$\overline{DX} = (s-b) - c\cos B \ \& \ \overline{XP} = (s-c) - \frac{a}{2}$$

we get P is the midpoint of $\overline{BC}$ if and only if $2a = b + c$. □

§2.5 Exercises

Exercise 2.8. Two circles Ω and Γ are internally tangent at the point B. The chord AC of Γ is tangent to Ω at the point L, and the segments AB and BC intersect Ω at the points M and N. Let M_1 and N_1 be the reflections of M and N about the line BL; and let M_2 and N_2 be the reflections of M and N about the line AC. The lines M_1M_2 and N_1N_2 intersect at the point K. Prove that the lines BK and AC are perpendicular.

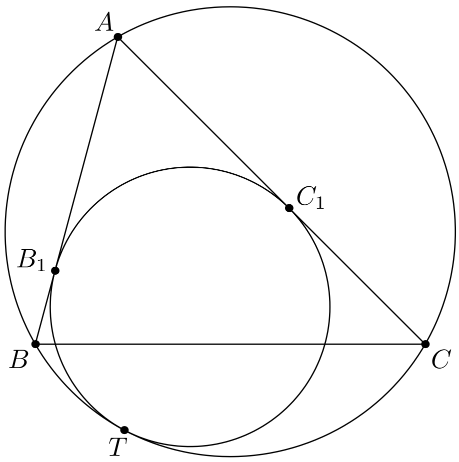
Exercise 2.9 (Bulgaria 2005)**.** Consider two circles k_1, k_2 touching externally at point T. a line touches k_2 at point X and intersects k_1 at points A and B. Let S be the second intersection point of k_1 with the line XT . On the arc $\widehat{TS}$ not containing A and B is chosen a point C . Let CY be the tangent line to k_2 with $Y \in k_2$, such that the segment CY does not intersect the segment ST . If $I = XY \cap SC$. Prove that :

1. the points C, T, Y, I are concyclic.
2. I is the excenter of triangle ABC with respect to the side BC.

§3 Mixtilinear Incircles

In the previous section, we looked at circles tangent to cevians called the curvilinear incircles. A special case arises when the point D coincides with a vertex of the triangle.

Definition 3.1. Given $\triangle ABC$, the circle internally tangent to $\odot(ABC)$ and sides $\overline{AB}$ and $\overline{AC}$ is known as the A-**mixtilinear incircle** of $\triangle ABC$.



In a triangle, there are three mixtilinear incircles. Each opposite to a vertex of the triangle. Since mixtilinear incircles are a special case of curvilinear incircles, hence the properties discussed in the previous sections apply to the mixtilinear incircles too!

§3.1 Immediate Properties

Proposition 3.2

Given $\triangle ABC$ and the A-mixtilinear incircle ω_A that touches the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1, and the circle $\odot(ABC)$ at T. Suppose E and F are the midpoints of the arc AC not containing B and arc AB not containing C, then the points

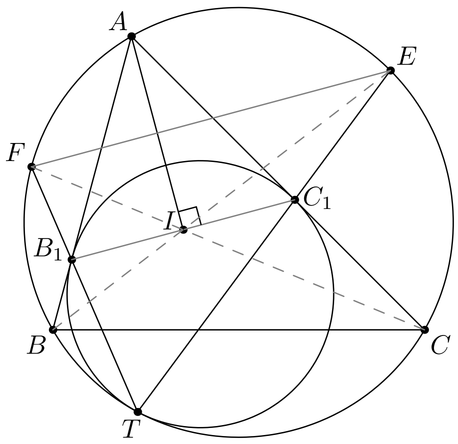
1. T, B_1, F are collinear.
2. T, C_1, E are collinear.

Proof. Immediate application of archimedes' lemma. □

Proposition 3.3 (Verrier's Lemma)

Given $\triangle ABC$ and the A-mixtilinear incircle ω_A that touches the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1, and the circle $\odot(ABC)$ at T. If I is the incenter of $\triangle ABC$ then I lies on the line $\overline{B_1C_1}$.

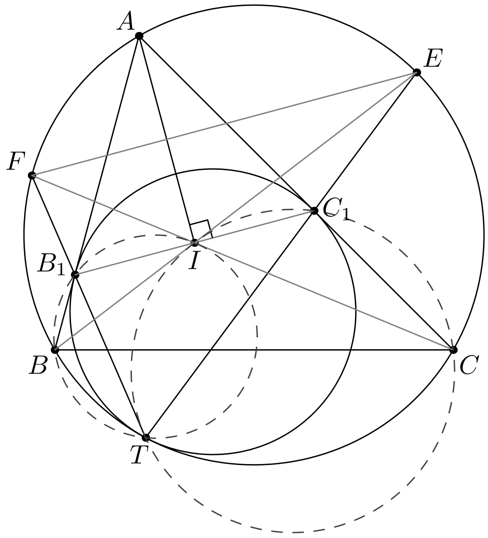
Proof. Since ω_A is also the C-curvilinear incircle of the cevian $\overline{CA}$, hence the result follows from sawayama's theorem. □



§3.2 Some More Properties

Proposition 3.4

Given $\triangle ABC$ and the A-mixtilinear incircle ω_A that touches the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1, and the circle $\odot(ABC)$ at T. Let I be the incenter of $\triangle ABC$. Then IB_1BT and IC_1CT are cyclic quadrilaterals.



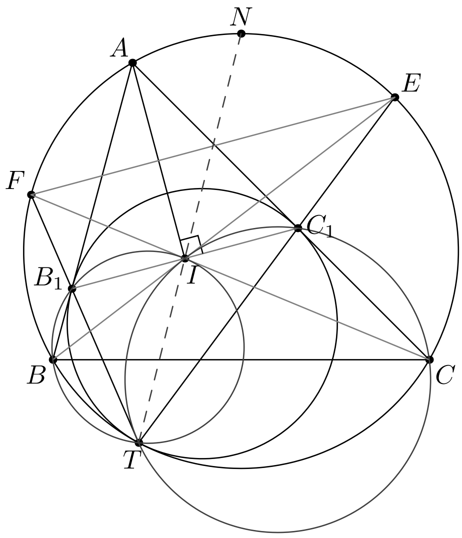
Proof. Converse of reim's theorem applied on $\overline{B_1I} \parallel \overline{EF}$ and $\overline{C_1I} \parallel \overline{EF}$ implies that IB_1BT and IC_1CT are cyclic. $\square$

Proposition 3.5

Given $\triangle ABC$ and the A-mixtilinear incircle ω_A that touches the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1, and the circle $\odot(ABC)$ at T. Let I be the incenter of $\triangle ABC$. Suppose N is the midpoint of arc BAC, then TI passes through N.

Proof. Since AB_1 and AC_1 are tangents drawn from A to $\omega_A \implies \triangle AB_1C_1$ is isosceles. Hence,

$$\angle BTI = \angle AB_1I = \angle AC_1I = \angle ITC$$



So $\overline{TI}$ is the angle bisector of $\angle BTC \implies TI$ passes through the midpoint of the arc BAC. $\square$

Remark 3.6. There are lot of angle bisectors in this configuration. For example,

1. $\overline{TB_1}$ is the T-angle bisector of $\triangle ATB$.
2. $\overline{TC_1}$ is the T-angle bisector of $\triangle ATC$.

Also $\overline{TA}$ is the T-symmedian in $\triangle TB_1C_1$. All of this makes the configuration a really nice treasure trove for angle bisector theorem and ratio lemma applications.

§3.3 Isogonal Lines

Proposition 3.7

Given $\triangle ABC$ and the A-mixtilinear incircle ω_A that touches the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1, and the circle $\odot(ABC)$ at T. Let I be the incenter of $\triangle ABC$. Then lines TA and TI are isogonal with respect to $\angle FTE$.

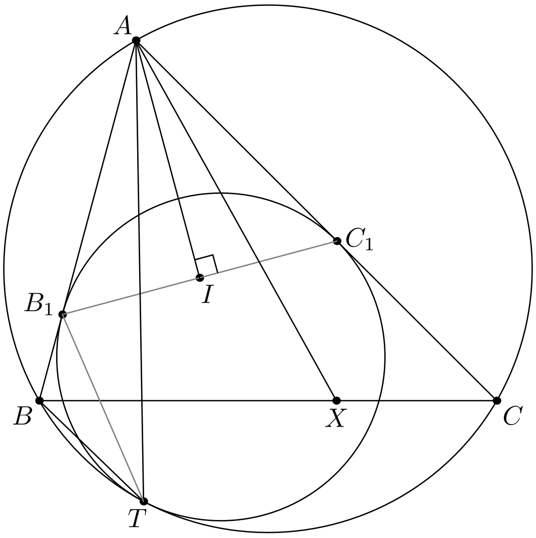
Proof. Effectively, we want to show that $\angle ATF = \angle ITE$. However,

$$\angle ATF = \angle ACF = \angle ICC_1 = \angle ITC_1 = \angle ITE$$

which implies the isogonal condition. $\square$

Proposition 3.8

Given $\triangle ABC$ and the A-mixtilinear incircle ω_A that touches the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1, and the circle $\odot(ABC)$ at T. Let I be the incenter of $\triangle ABC$ and X be the point where the A-excircle touches $\overline{BC}$. Then $\overline{AX}$ and $\overline{AT}$ are isogonal with respect to $\angle BAC$.



Proof. We will show that $\triangle ATB \sim \triangle ACX$. We already have that $\angle ATB = \angle ACX$. Hence it only remains to establish that,

$$\frac{\overline{AT}}{\overline{TB}} = \frac{\overline{AC}}{\overline{CX}}$$

Using the relation $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}$, we get

$$\begin{aligned}\frac{\overline{AT}}{\overline{TB}} = \frac{\overline{BB_1}}{\overline{AB_1}} &= \frac{\overline{AB} - \overline{AB_1}}{\overline{AB_1}} \\ &= \frac{\overline{AB}}{\overline{AB_1}} - 1 = \frac{c\cos^2\left(\frac{A}{2}\right)}{s-a} - 1 \\ &= \frac{c\left(\frac{s(s-a)}{bc}\right)}{s(s-a)} - 1 = \frac{s-b}{b} = \frac{\overline{AC}}{\overline{CX}}\end{aligned}$$

and thus, $\triangle ATB \sim \triangle ACX \implies AT$ and AX are isogonal with respect to $\angle BAC$. □

Proposition 3.9

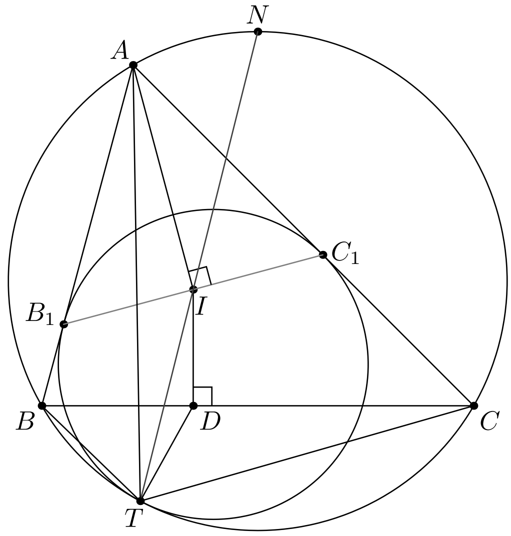
Given $\triangle ABC$ and the A-mixtilinear incircle ω_A that touches the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1, and the circle $\odot(ABC)$ at T. Let I be the incenter of $\triangle ABC$ and D be the point where the incircle of $\triangle ABC$ touches $\overline{BC}$. Then TA and TD are isogonal with respect to $\angle BTC$.

Proof. Suppose X is the point where the A-excircle touches $\overline{BC}$. Since $\angle BAX = \angle TAC \implies \triangle ATC \sim \triangle ABX$. So, we have

$$\frac{\overline{AT}}{\overline{TC}} = \frac{\overline{AB}}{\overline{BX}} = \frac{\overline{AB}}{\overline{CD}}$$

This can be written as

$$\frac{\overline{AT}}{\overline{AB}} = \frac{\overline{TC}}{\overline{CD}}$$



and using the fact that $\angle BAT = \angle DCT$, we can claim by the SAS similarity criterion that $\triangle ATB \sim \triangle CTD$. This implies that

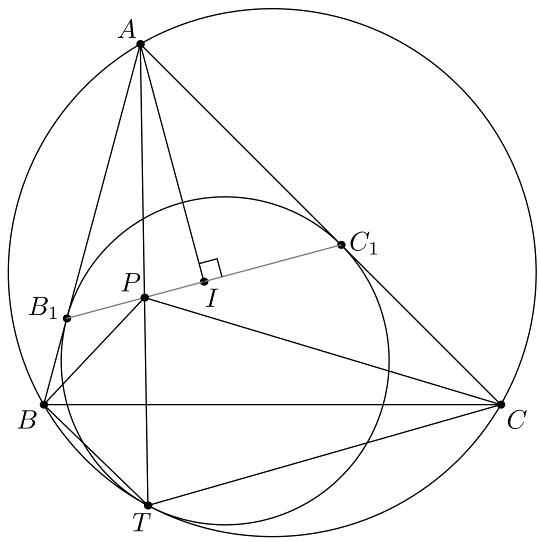
$$\angle BTA = \angle DTC$$

or in other words, lines TA and TD are isogonal with respect to $\angle BTC$. $\square$

Proposition 3.10

Given $\triangle ABC$ and the A-mixtilinear incircle ω_A that touches the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1, and the circle $\odot(ABC)$ at T. Let I be the incenter of $\triangle ABC$ and P be the intersection of $\overline{AT}$ and $\overline{B_1C_1}$. Then

$$\angle BPB_1 = \angle CPC_1$$



Proof. We will show that $\triangle BPB_1 \sim \triangle CPC_1$. Since,

$$\frac{\overline{B_1P}}{\overline{C_1P}} = \frac{\sin \angle BAT}{\sin \angle CAT} = \frac{\overline{BT}}{\overline{CT}} = \frac{\frac{\overline{BB_1}}{\overline{AB_1}} \cdot \overline{AT}}{\frac{\overline{CC_1}}{\overline{C_1A}} \cdot \overline{AT}} = \frac{\overline{BB_1}}{\overline{CC_1}}$$

Since $\angle BB_1P = \angle CC_1P \implies \triangle BPB_1 \sim \triangle CPC_1$ by SAS similarity criterion. $\square$

§3.3.1 Exercises

Exercise 3.11. In **Proposition 3.10**, if CP is extended to intersect AB at Q and BP is extended to intersect AC at R, then $BQRC$ is cyclic.

Exercise 3.12. Given $\triangle ABC$, let D be the point where the incircle of $\triangle ABC$ touches $\overline{BC}$ and T be the point where the A-mixtilinear incircle touches $\odot(ABC)$. Suppose the tangent to $\odot(ABC)$ at A intersects $\overline{BC}$ at X, then show that $AXTD$ is a cyclic quadrilateral.

Exercise 3.13. Given $\triangle ABC$ and its A-mixtilinear incircle ω_A. Suppose ω_A touches $\odot(ABC)$ at T and sides $\overline{AB}$, $\overline{AC}$ at B_1, C_1. Let M be the midpoint of arc BC not containing A. Show that B_1C_1, BC and TM are concurrent.

Exercise 3.14. Given $\triangle ABC$, its incenter I and its A-mixtilinear incircle ω_A. Let the incircle touch $\overline{BC}$ at D and ω_A touch $\odot(ABC)$ at T and the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1. Suppose the angle bisector of $\angle BAC$ intersects $\overline{BC}$ and $\odot(ABC)$ at K and M. Then show that $KDTM$ is a cyclic quadrilateral.

Exercise 3.15. Given $\triangle ABC$, its incenter I and its A-mixtilinear incircle ω_A. Let the incircle touch $\overline{BC}$ at D and ω_A touch $\odot(ABC)$ at T and the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1. Suppose B_1C_1 intersects BC at X, then show that $IXTD$ is a cyclic quadrilateral.

Exercise 3.16. Given $\triangle ABC$, its incenter I and its A-mixtilinear incircle ω_A. Let the incircle touch $\overline{BC}$ at D and ω_A touch $\odot(ABC)$ at T and the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1. Suppose B_1C_1 intersects BC at X, AT intersects $\overline{B_1C_1}$ at P and $\overline{TI}$ intersects $\overline{BC}$ at Q. Then show that $IPQD$ and $PXTQ$ are cyclic quadrilaterals.

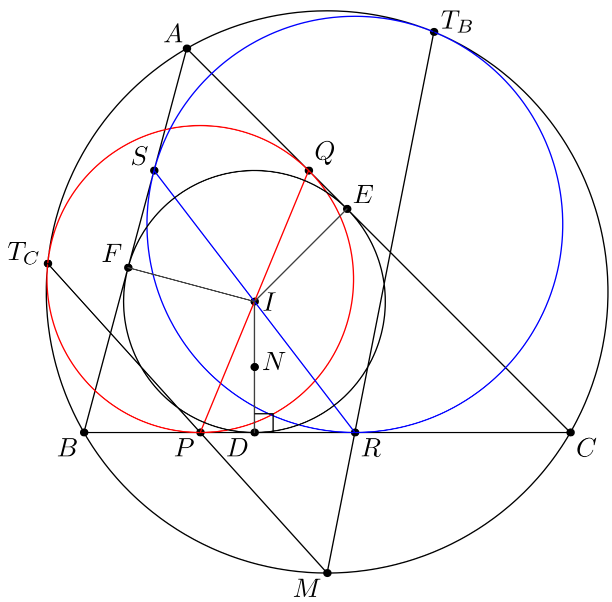
Exercise 3.17. Given $\triangle ABC$, its incenter I and its A-mixtilinear incircle ω_A. Let the incircle touch $\overline{BC}$ at D and ω_A touch $\odot(ABC)$ at T and the sides $\overline{AB}$ and $\overline{AC}$ at B_1 and C_1. Suppose the tangent to $\odot(ABC)$ at T intersects BC at Y, AT intersects $\overline{B_1C_1}$ at P and $\overline{TI}$ intersects $\overline{BC}$ at Q. Then show that $YPDT$ is cyclic and Y is the center of $\odot(PQT)$.

§3.4 Multiple Mixtilinear Incircles

Now let's add multiple mixtilinear incircles to the configuration. We use the notation ω_X to denote the X-mixtilinear incircle (the mixtilinear incircle opposite to vertex X).

Proposition 3.18

Given $\triangle ABC$ and its incenter I, let the incircle touche $\overline{BC}$ at D, M be the midpoint of arc BC not containing A and N be midpoint of $\overline{ID}$. Then MN is the radical axis of ω_B and ω_C.



Proof. Suppose the incircle touches the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ at D, E and F, and ω_B touches the sides $\overline{BC}$ and $\overline{AB}$ at R and S, and ω_C touches the sides $\overline{BC}$ and $\overline{AC}$ at P and Q. Due to archimedes' lemma, we have that T_CP and T_BR pass through M and

$$\overline{MP} \cdot \overline{MT_C} = \overline{MB}^2 = \overline{MR} \cdot \overline{MT_B}$$

Hence, M has an equal power with respect to both the circles ω_B and ω_C. Since the points P, D, E and Q are the tangency points of common tangents of the incircle and ω_C $\implies$ their radical axis is the midline of the isosceles trapezium $PQED$ and hence passes through N. Similarly, the radical axis of the incircle and ω_B passes through $N \implies N$ is the radical center of ω_B, ω_C and incircle and thus, N also has an equal power with respect to ω_B and ω_C, implying that MN is indeed the radical axis of ω_B and ω_C. $\square$

§3.4.1 Exercises

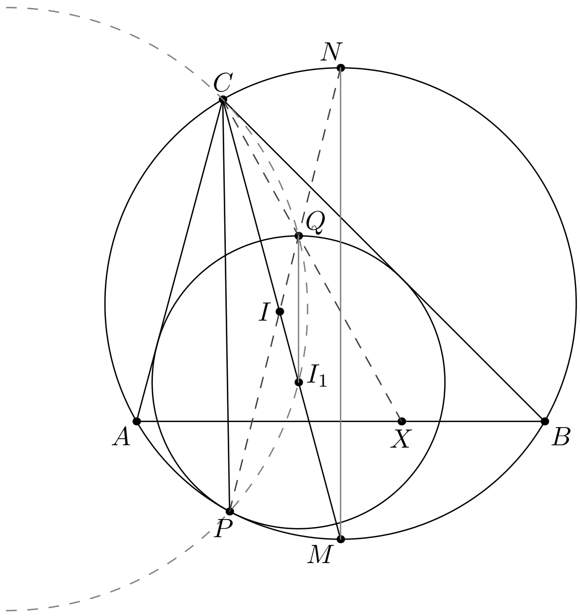
Exercise 3.19. Suppose ω_A touches $\odot(ABC)$ at T_A and ω_B and ω_C touch $\overline{BC}$ at D and E and $\odot(ABC)$ at T_B and T_C. If M is the midpoint of arc BC not containing A, then

1. T_CDET_B is a cyclic quadrilateral. If T_AM intersects BC at T,
2. $\overline{IT_A} \perp \overline{TM}$.
3. $\overline{TI} \perp \overline{AM}$.
4. T_CTT_AD is a cyclic quadrilateral.
5. $EDTM$ is a cyclic quadrilateral.

§3.5 Examples

Problem 3.20 (EGMO 2013)

Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q. Prove that $\angle ACP = \angle QCB$.



Proof. Since Q lies such that the tangent to ω at Q is parallel to $\overline{AB}$, hence there exists a homothety at P that maps ω to Ω under which Q is mapped to the midpoint of arc ACB, let's say N. Also suppose I is the incenter, I_1 is center of ω and M is the midpoint of arc AB not containing C.

We know that I_1 lies on the line $\overline{CIM}$. Since $\overline{I_1Q} \parallel \overline{MN}$, therefore by converse of reim's theorem $\implies$ PI_1QC is cyclic. Since, $\overline{PI_1} = \overline{QI_1} \implies \overline{CI_1}$ is the angle bisector of $\angle PCQ$, or in other words $\angle PCI = \angle QCI \implies \angle ACP = \angle QCB$. $\square$

Problem 3.21 (IMO 2019)

Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides $BC, CA,$ and AB at $D, E,$ and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q. Prove that lines DI and PQ meet on the line through A perpendicular to AI.

Proof. Let T be the A-mixtilinear touch point with $\odot(ABC)$.

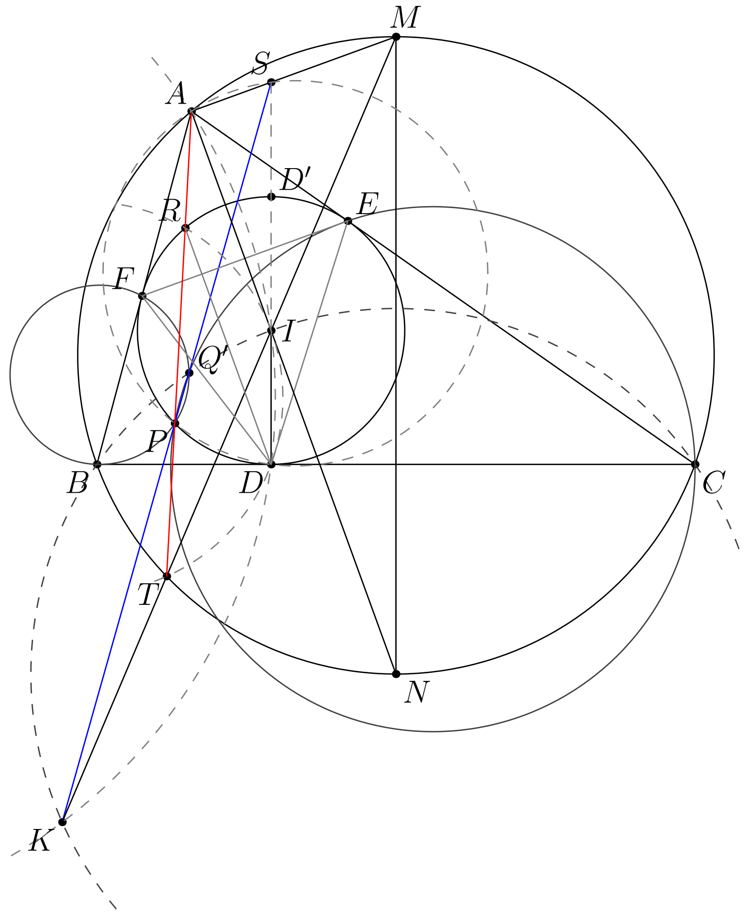
Claim 3.22. T lies on the line AR.

Proof. Suppose $\overline{AT}$ cuts the incircle at P' and R' where P' is closer to T than R'. Since $\overline{TA}$ and $\overline{TD}$ are isogonal with respect to $\angle BTC$ and $\overline{BI}$ is the angle bisector of $\angle BTC$ $\implies$ P' is the reflection of D over $\overline{TI}$. Let M be the midpoint of arc BAC and N be the midpoint of arc BC not containing A. Then

$$\angle DR'P' = \tfrac{1}{2}\angle PID = \angle DIT = \angle NMT = \angle IAT$$

Hence, $\overline{R'D} \parallel \overline{AI}$. But $\overline{AI} \perp \overline{EF} \implies \overline{R'D} \perp \overline{EF} \implies R' = R$ and $P' = P$. $\square$

Define S as the intersection of DI and AM. We want to show that PQ passes through S to solve the problem. Suppose D' is the reflection of D over I. Since $\overline{DR}$ and $\overline{DD'}$ are isogonal with respect to $\angle EDF \implies \overline{RD'} \parallel \overline{EF} \parallel \overline{AM}$. Therefore, by converse of



reim's theorem we get that $ASDP$ is a cyclic quadrilateral. We can further show that $RIDT$ is a cyclic quadrilateral too.

$$\angle RTI = \angle ATM = \angle ANM = \angle DIM = \angle RDI$$

which implies that $RIDT$ is a cyclic quadrilateral. Now, define K as the intersection of line TI with $\odot(BIC)$.

Claim 3.23. $AIDK$ is a cyclic quadrilateral.

Proof. Since $\angle ITN = 90^\circ$ and using the fact that N is the center of $\odot(BIC)$, we get that T is the midpoint of $\overline{IK}$. Suppose I_A is the A-excenter of $\triangle ABC \implies \overline{II_A}$ is the diameter of $\odot(BIC)$ and therefore, $\angle IKI_A = 90^\circ = \angle I_AAM \implies AMI_AK$ is cyclic too. Applying radical axis theorem on $\odot(AMI_AK)$, $\odot(ABC)$ and $\odot(BIC)$, we get that MA, BC and I_AK are concurrent. Hence, $AIDK$ is a cyclic quadrilateral. □

Using the fact that $\overline{TK} = \overline{TI}$ and previously established angle relation $\angle DIT = \angle IAT$, we get that

$$\overline{TP} \cdot \overline{TA} = \overline{TI}^2 = \overline{TK}^2$$

Hence $\overline{TK}$ is tangent to $\odot(APK)$ at K. This implies that

$$\angle KPT = \angle AKT = \angle ADI = \angle ADS = \angle APS$$

which implies that K lies on PS. The final claim is showing that Q lies on the line $\overline{KPS}$ too. Suppose Q' is the intersection of $\overline{SK}$ and $\odot(BIC)$. Then,

$$\angle BFP = \angle FEP = \angle FED - \angle PED = \angle BID - \angle KID = \angle BIK = \angle BQ'P$$

So $BFQ'P$ is cyclic and similarly, $CEQ'P$ is cyclic too. Therefore Q' must be Q and we get Q lies on $\overline{SP}$ □

§3.6 Exercises

Exercise 3.24 (USA TST 2016)**.** Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at E and F. The circumcircle of $\triangle DEF$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq F$. Prove that line AT passes through either S_1 or S_2.

Exercise 3.25 (IMO Shortlist 1999)**.** Given a triangle ABC. The points A, B, C divide the circumcircle Ω of the triangle ABC into three arcs BC, CA, AB. Let X be a variable point on the arc AB, and let O_1 and O_2 be the incenters of the triangles CAX and CBX. Prove that the circumcircle of the triangle XO_1O_2 intersects the circle Ω in a fixed point.

Exercise 3.26 (IMO Shortlist 2016)**.** Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Exercise 3.27 (IMO Shortlist 2017)**.** In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

§4 Practice Problems

Exercise 4.1 (Japan 2009)**.** Let Γ be a circumcircle. A circle with center O touches to line segment BC at P and touches the arc BC of Γ which doesn't have A at Q. If $\angle BAO = \angle CAO$, then prove that $\angle PAO = \angle QAO$.

Exercise 4.2 (ELMO Shortlist 2012)**.** Circles Ω and ω are internally tangent at point C. Chord AB of Ω is tangent to ω at E, where E is the midpoint of AB. Another circle, ω_1 is tangent to Ω, ω, and AB at D, Z, and F respectively. Rays CD and AB meet at P. If M is the midpoint of major arc AB, show that $\tan \angle ZEP = \frac{PE}{CM}$.

Exercise 4.3 (EGMO 2018)**.** Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q. Prove that $\angle ABP = \angle QBC$.

Exercise 4.4 (IMO Shortlist 1992)**.** Two circles touch externally at a point I. The two circles lie inside a large circle and both touch it. The chord BC of the large circle touches both smaller circles (not at I). The common tangent to the two smaller circles at the point I meets the large circle at a point A, where the points A and I are on the same side of the chord BC. Show that the point I is the incenter of triangle ABC.

Exercise 4.5 (Romania 1997)**.** Let ABC be a triangle, D be a point on side BC, and let $\mathcal{O}$ be the circumcircle of triangle ABC. Show that the circles tangent to $\mathcal{O}, AD, BD$ and to $\mathcal{O}, AD, DC$ are tangent to each other if and only if $\angle BAD = \angle CAD$.

Exercise 4.6 (USAMO 2017)**.** Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

Exercise 4.7 (ELMO Shortlist 2017)**.** Let ABC be an acute triangle with incenter I and circumcircle ω. Suppose a circle ω_B is tangent to BA, BC, and internally tangent to ω at B_1, while a circle ω_C is tangent to CA, CB, and internally tangent to ω at C_1. If B_2, C_2 are the points opposite to B, C on ω, respectively, and X denotes the intersection of B_1C_2, B_2C_1, prove that $XA = XI$.

Exercise 4.8 (IMO Shortlist 1999)**.** Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

Exercise 4.9 (IMO Shortlist 2014)**.** Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V , respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y . Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

Exercise 4.10 (Taiwan TST 2014)**.** Let M be any point on the circumcircle of triangle ABC. Suppose the tangents from M to the incircle meet BC at two points X_1 and X_2. Prove that the circumcircle of triangle MX_1X_2 intersects the circumcircle of ABC again at the tangency point of the A-mixtilinear incircle.

Exercise 4.11 (Taiwan TST 2015)**.** Let O be the circumcircle of the triangle ABC. Two circles O_1, O_2 are tangent to each of the circle O and the rays $\overrightarrow{AB}, \overrightarrow{AC}$, with O_1 interior to O, O_2 exterior to O. The common tangent of O, O_1 and the common tangent of O, O_2 intersect at the point X. Let M be the midpoint of the arc BC (not containing the point A) on the circle O, and the segment $\overline{AA'}$ be the diameter of O. Prove that X, M, and A' are collinear.

Exercise 4.12 (Sharygin 2021)**.** Let ABC be a scalene triangle, AM be the median through A, and ω be the incircle. Let ω touch BC at point T and segment AT meet ω for the second time at point S. Let δ be the triangle formed by lines AM and BC and the tangent to ω at S. Prove that the incircle of triangle δ is tangent to the circumcircle of triangle ABC.

Exercise 4.13. Let ABC be an acute scalene triangle with orthocentre H and circumcircle Γ. Let M be the midpoint of BC. Lines BH and CH meet sides AC and AB at points B_1 and C_1 respectively. The lines through M perpendicular to AB and AC respectively meet line B_1C_1 at points B_2 and C_2 respectively. The perpendicular bisectors of BC and B_2C_2 meet at point T. Prove that lines TA and MH meet at a point on Γ.