

HIDDEN CONNECTIONS BETWEEN WONDERFUL ELEMENTS IN THE GEOMETRY OF A TRIANGLE

Part 1: Mixtilinear incircles

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This four-part research aims to investigate particular unique objects in the triangle geometry and their relationships. Specifically, in this paper I focus on mixtilinear incircles (hereinafter may be referred to as mixtilinears), which trace their origins to the work of Leon Bankoff [1] and have been lately researched by Evan Chen [2], Jafet Baca [3], and more. In the next two papers, I will extend this work by exploring why-points¹ and cuddling circles², while also relating them to each other and mixtilinear incircles. I will derive these relations sometimes explicitly, but often through other unique points such as the sharky-devil point³. Furthermore, only those properties of the “transitional” points that are directly linked to researched objects will be studied. This means that I will not be going in-depth on things that are not the focus of the research, without necessity. Finally, the fourth paper will be an extension of the previous ones and will introduce one special point that is closely related to all three: mixtilinears, why-points, and cuddling circles.

In this part, I will investigate the most common approaches to mixtilinear incircles, including, but not limited to, inversion and homothety, and demonstrate a lot of various configurations. I will also introduce the so-called sharky-devil point and show its relation to mixtilinears. The paper ends with a crucial corollary about this point that will in particular be used to connect mixtilinears to why-points and cuddling circles in the following parts of this research.

Keywords – mixtilinear incircle, mixti-point, triplemixti-point, mixti-sidepoint, mixti-tangent, mixti-miquel-point, sharky-devil point.

¹This is a slang name that likely appeared somewhere in the vastness of AoPS threads.

²I could not find anyone naming this circle before, so the term “*cuddling circle*” is coined by me.

³This name was also proposed on AoPS by user aops29 in his post, <https://aops.com/community/c946900h1911664>.

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1 Preliminaries

Throughout the entire research, I will be referring to the lengths of the respective sides and semi-perimeter of $\triangle ABC$ as a, b, c, p , and to its respective angles as $2\alpha, 2\beta, 2\gamma$. Additionally, $P-X$ will denote any object X defined through a point P , and, unless otherwise specified, this object will relate to $\triangle ABC$, e.g., the A -excircle.

A few other specifications:

- XY denotes the *line* XY , while \overline{XY} denotes the *segment* XY .
- The circle passing through X, Y, Z will be denoted by (XYZ) (to expand, (O) implies a zero-radius circle centred at O , and (XY) implies a circle with the diameter XY).
- The power of a point P wrt circle ω will be denoted by $pow(P, \omega)$.
- $(A, B; C, D)$ denotes a **directed** (i.e., in a vector form) *cross-ratio* of points A, B, C, D .
- ∞ denotes a point at infinity.
- " $(!)$ X " notation means that we have to prove the fact X .

2 Definition

A *mixtilinear incircle* is a circle tangent to the two sides of a triangle and internally to its circumcircle.

Now we can begin with a setup for the upcoming problems. We are given $\triangle ABC$. Its A -mixtilinear touches \widehat{BC} not containing A at T_A and the sides AB, AC at C_1, B_1 , respectively. I will call T_A a *mixti-point*, or *A-mixtipoint*.

3 Inversion approach

Denote a composition of the inversion around A , with $R = \sqrt{AB \cdot AC}$, and the symmetry over the bisector of $\angle BAC$ by $\varphi(X)$ for any object X (a point, a line, a polygon, a circle, an angle, etc). Unless otherwise specified, I will denote the images of the objects under $\varphi(X)$ with primes, i.e., $\varphi(X) = X'$. Let us see what this transformation does to the A -mixtilinear.

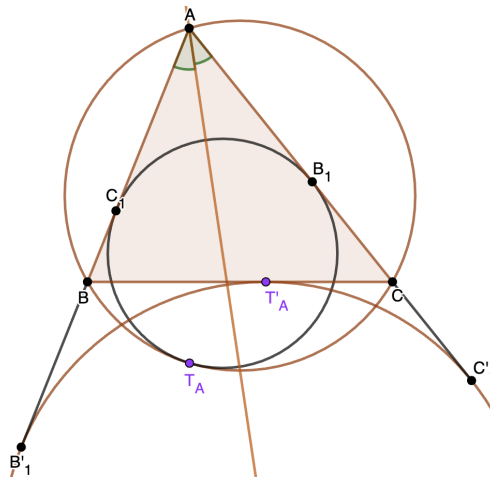


Figure 1: Mixtilinear under inversion

It is clear that $\varphi(B) = C$ and $\varphi(C) = B$. Thus, $\varphi(AB) = AC$, $\varphi(AC) = AB$, $\varphi(BC) = (ABC)$, so the A -mixtilinear maps to the A -excircle. It means that B'_1, C'_1, T'_A are the points of tangency of the latter circle with the lines AB, AC, BC , respectively. Hence, the transformation φ could be a powerful tool in many problems involving mixtilinears since it allows working with less intricate objects — the excircles. Let me demonstrate this with a few examples below. I will also provide a synthetic solution without inversion for each. I begin with two key lemmas.

3.1 Shooting lemma [4]

Problem. Let W_B be the midpoint of \widehat{AC} not containing B . Prove that points T_A, B_1, W_B are collinear.

Proof. Here I will use a single inversion around A (choosing points T_A, B_1, C, W_B as the inversion centres will also work). Then $C'W'_B = C'A'$ and $C'T'_A = C'B'_1$ as tangents, so $A'W'_B B'_1 T'_A$ is cyclic. This immediately yields that points T_A, B_1, W_B are collinear as desired.

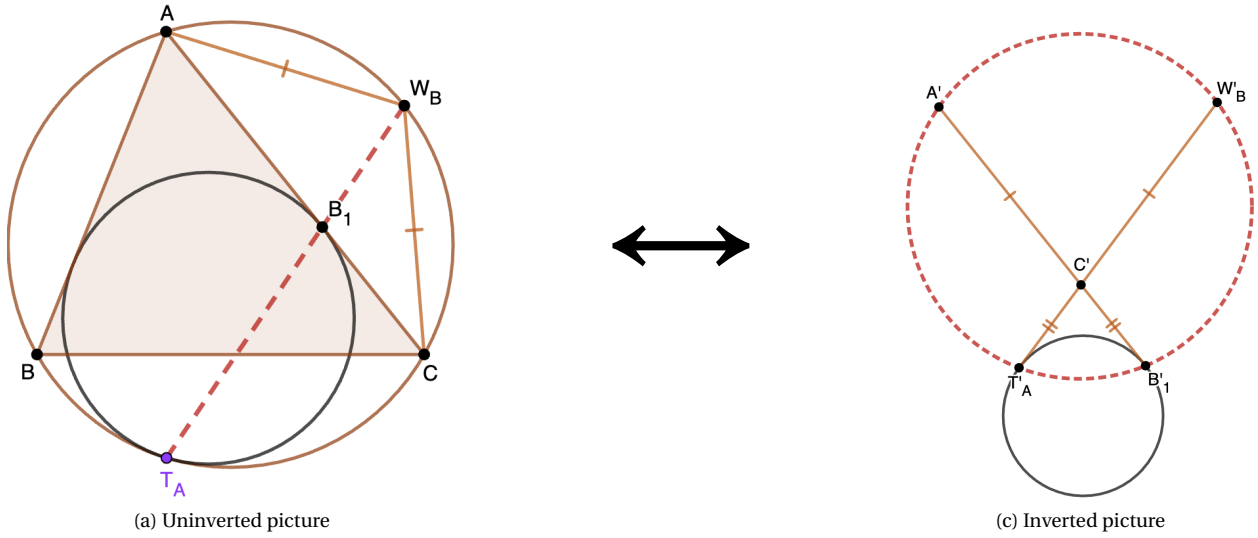


Figure 2: Shooting lemma under inversion

Alternative proof. Let the common tangent of the A -mixtilinear and (ABC) intersect AC at B_2 . It suffices to show that $T_A B_1$ bisects $\angle AT_A C$, which is true as

$$\angle B_1 T_A A = \angle T_A B_1 B_2 - \angle T_A A B_1 = \angle B_2 T_A B_1 - \angle B_2 T_A C = \angle C T_A B_1.$$

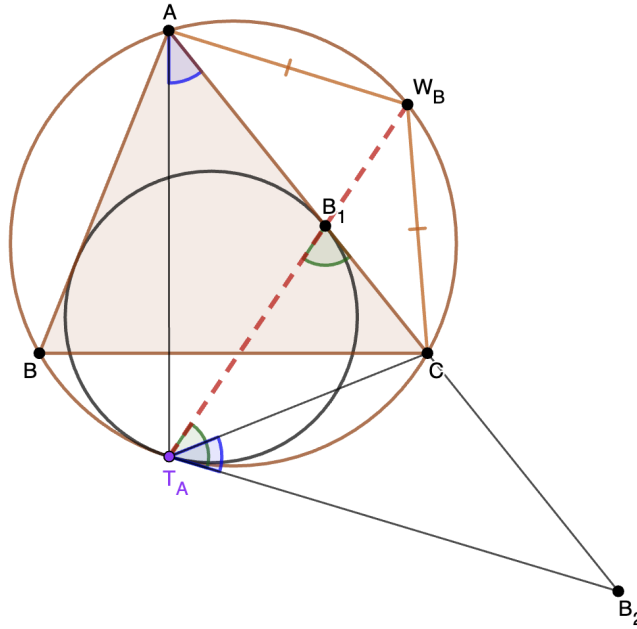


Figure 3: Synthetic proof of the shooting lemma

q.e.d.

3.2 Varrier's lemma [5]

Problem. Let I be the incentre of $\triangle ABC$. Prove that points B_1, I, C_1 are collinear.

Proof. Let us first quickly prove a famous fact that $\varphi(I) = I_A$ where I_A is the A -excentre. Indeed,

$$\angle AIB = 180^\circ - \alpha - \beta = 90^\circ + \gamma = \angle ACI_A,$$

from which $\triangle AIB \sim \triangle ACI_A$ and thus $AI \cdot AI_A = AB \cdot AC$ as desired. So, we are left to prove that A, B'_1, I_A, C'_1 are concyclic, which is trivial.

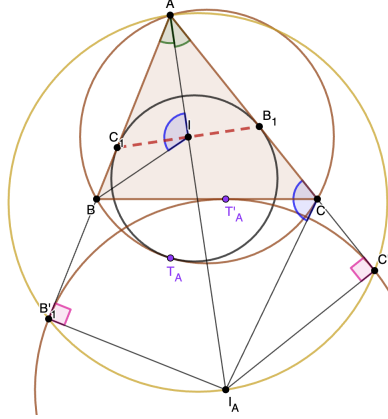


Figure 4: Varrier's lemma and inversion

Alternative proof. This proof is considered standard. By shooting lemma, $T_A W_B$ bisects $\angle AT_A C$, thus

$$\angle AT_A W_B = \angle W_B T_A C = \angle W_B A C,$$

and therefore $W_B A^2 = W_B B_1 \cdot W_B T_A$, or

$$pow(W_B, (A)) = pow(W_B, (T_A B_1 C_1)) \quad (1).$$

By analogy, if we define W_C similarly to W_B ,

$$pow(W_C, (A)) = pow(W_C, (T_A B_1 C_1)) \quad (2).$$

Combining (1) and (2), we conclude that $W_B W_C$ is the radical axis of (A) and $(T_A B_1 C_1)$, hence it bisects AC_1 and AB_1 at, say, H and J . By the trillium theorem, $W_C I = W_C A$ and $W_B I = W_B A$, so it follows that point I is the reflection of A over HJ and hence lies on $B_1 C_1$ as desired.

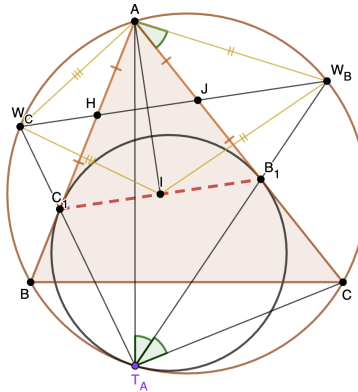


Figure 5: Synthetic proof of Varrier's lemma

q.e.d.

Now with the two crucial lemmas proven, we can proceed.

3.3 Relation to nagelian ⁴ (EGMO 2013/5)

Problem. Let the A -excircle touch the side BC at N . Prove that $\angle BAT_A = \angle CAN$.

Proof. This follows directly from the fact that $\varphi(T_A) = N$ which was shown at the beginning of this section.

Alternative proof. It suffices to show that $\triangle AT_A B \sim \triangle ACN$, which will follow if $\frac{BT_A}{AT_A} = \frac{NC}{AC}$. Note that

$$\frac{NC}{AC} = \frac{p-b}{b} (*).$$

Since $T_A C$ bisects $\angle AT_A B$, we obtain that

$$\frac{BT_A}{AT_A} = \frac{BC_1}{AC_1} = \frac{c-AC_1}{AC_1} (**).$$

Let E a foot of the perpendicular from I on AB . Then

$$AC_1 = \frac{AI^2}{AE} = \frac{AI^2}{p-a} = \frac{IE^2 + AE^2}{p-a} = \frac{r^2 + (p-a)^2}{p-a} = \frac{\frac{(p-a)(p-b)(p-c)}{p} + (p-a)^2}{p-a} = \frac{(p-b)(p-c) + p(p-a)}{p-a} = \frac{bc}{p} (***) .$$

Combining (*), (**), (***) leaves us to check that

$$\frac{p-b}{b} = \frac{c - \frac{bc}{p}}{\frac{bc}{p}},$$

which is true.

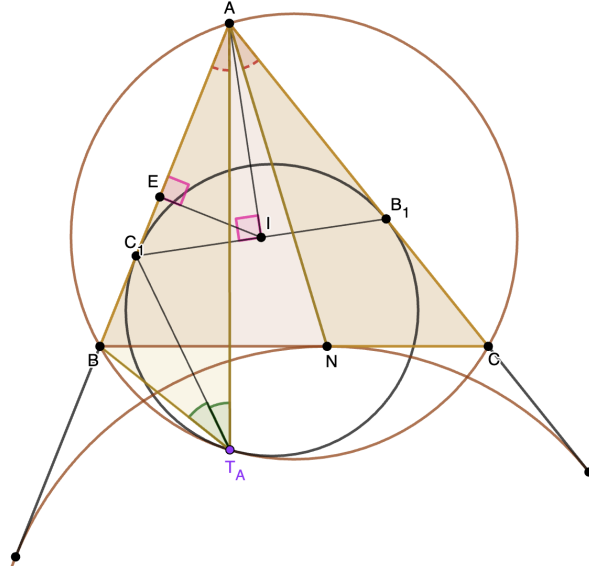


Figure 6: Relation to nagelian

q.e.d.

3.4 Mixticlcity

Problem. Prove that points I, C_1, B, T_A are concyclic.

Proof. After applying φ , we must prove that I_A, C'_1, C, N are concyclic, which is trivial ⁵.

Alternative proof. Let the incircle of ABC touch the side BC at D , and let $(BC_1 I)$ intersect BC at $L \neq B$. Then, it suffices to show that $\angle BLT_A = \angle BC_1 T_A$, or $\triangle AT_A C_1 \sim \triangle CT_A L$. As $T_A C_1$ bisects $\angle AT_A B$ and, from 5.1, $\triangle AT_A B \sim \triangle CT_A D$, we are left to show that $T_A L$ bisects $\angle DT_A C$, or $\frac{T_A D}{T_A C} = \frac{LD}{LC}$. As

$$\angle ILB = \angle IC_1 A = \angle IB_1 A,$$

⁴A *nagelian* is a cevian connecting a vertex of a triangle with the point of tangency of the respective excircle with the opposite side. All three nagelians of the triangle intersect at its Nagel point.

⁵Furthermore, since inversion preserves cross-ratios and $I_A C'_1 CN$ is clearly harmonic, we can conclude that $IC_1 B T_A$ is harmonic as well.

we infer that $\triangle ILC = \triangle IB_1C$. Using that, as proven in 3.3, $AC_1 = \frac{bc}{p}$, we have

$$(!) \frac{T_AD}{T_AC} = \frac{LD}{LC} \iff (!) \frac{T_AB}{T_AA} = \frac{LD}{LC} \iff (!) \frac{BC_1}{C_1A} = \frac{LD}{CB_1} \iff (!) \frac{c-AC_1}{AC_1} = \frac{CD-CB_1}{CB_1} \iff (!) \frac{c}{AC_1} = \frac{CD}{CB_1} \iff (!) \frac{p}{b} = \frac{p-c}{b-\frac{bc}{p}},$$

which is true.

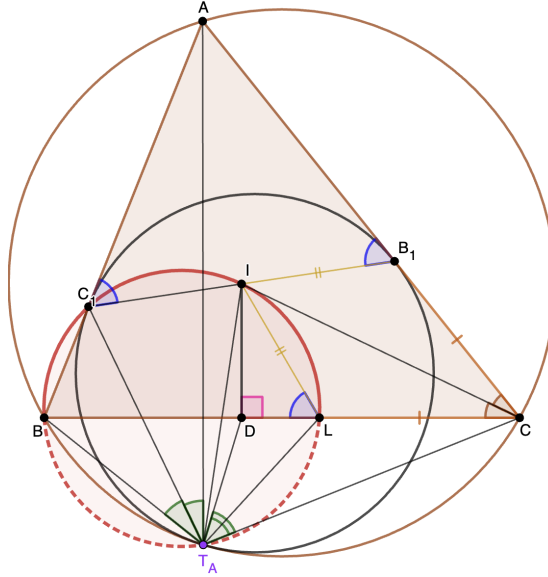


Figure 7: Mixticlcity

q.e.d.

3.5 Mixtiarity

Problem. Let V be the midpoint of \widehat{BAC} . Prove that points V, I, T_A are collinear.

Proof. Let V_0 be a foot of the external bisector of $\angle BAC$. I will first show that $\varphi(V) = V_0$. Indeed, V_0, A, V are clearly collinear, and we have that $\triangle AV_0B \sim \triangle ACV$ and thus $VA \cdot VV_0 = AB \cdot AC$ as desired. Therefore, we must prove that A, V_0, I_A, N are concyclic, which is clear as

$$\angle V_0NI_A = 90^\circ = \angle V_0AI_A.$$

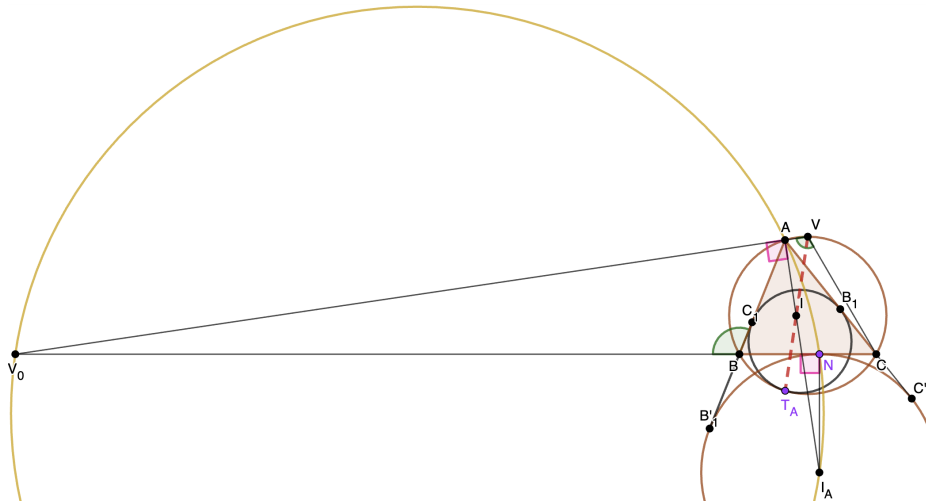


Figure 8: Mixtiarity under inversion

Alternative proof. By 3.4, IC_1BT_A is cyclic, and similarly IB_1CT_A is cyclic too. Hence,

$$\angle IT_AB = \angle IC_1A = \angle IB_1A = \angle IT_AC,$$

so we may conclude.

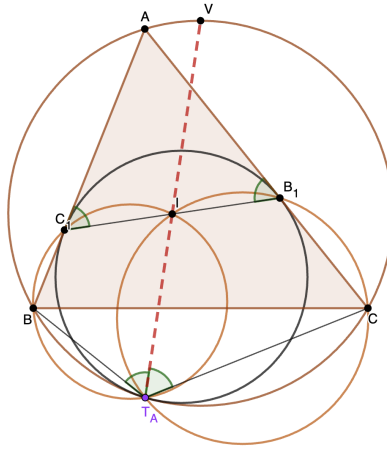


Figure 9: Synthetic proof of mixtiarity

q.e.d.

4 Homothety approach

Since the mixtilinear incircle is defined through tangencies, it is reasonable to assume that it is related to homothety. In this subsection, we will see how different homotheties are the keys to connect the A -mixtilinear to many circles in $\triangle ABC$.

4.1 Mixti-circumcircle homothety

Problem. Let AD intersect the A -mixtilinear at Q s.t. it doesn't lie on \overline{AD} . Prove that points T_A , Q , W_A are collinear (where W_A is defined similarly to W_B and W_C).

Proof. Let T_AB , T_AA , T_AC secondarily intersect the A -mixtilinear at W , U , Z , respectively. Since T_AC_1 bisects $\angle AT_AB$ and T_AB_1 bisects $\angle AT_AC$, we have C_1 is the midpoint of \widehat{UW} and B_1 is the midpoint of \widehat{UZ} .

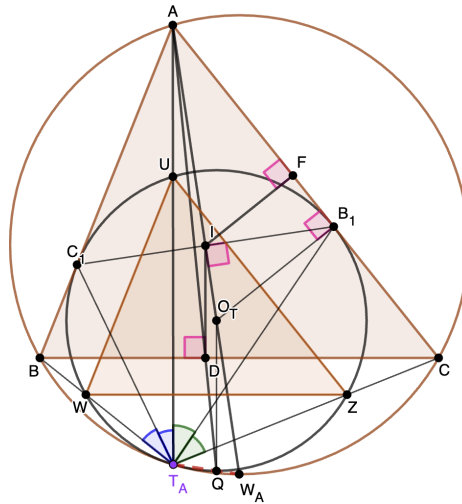


Figure 10: Mixti-circumcircle homothety

Therefore, $UW \parallel AB$ and $UZ \parallel AC$. Hence,

$$\frac{T_A W}{W B} = \frac{T_A U}{U A} = \frac{T_A Z}{Z C},$$

so $WZ \parallel BC$. Thus, $\triangle ABC \sim \triangle UWZ$ with point T_A being their centre of spiral similarity. Therefore, it suffices to show that Q is the midpoint of \widehat{WZ} , as in this case, the mentioned homothety would take Q to W_A and the desired would become obvious. So, let O_T be centre of the A -mixtilinear, then it obviously lies on AI . Also, let the incircle of $\triangle ABC$ touch the side AC at F . We have

$$(!) ID \parallel O_T Q \Leftrightarrow (!) \frac{ID}{O_T Q} = \frac{AI}{AO_T} \Leftrightarrow (!) \frac{IF}{O_T B_1} = \frac{AI}{AO_T} \quad ^6,$$

which is true because $\triangle AFI \sim \triangle AB_1 O_T$.

q.e.d.

4.2 Mixti-excircle homothety (USA TST 2016/2)

Problem. Let AT_A intersect the A -excircle at R s.t. it lies on $\overline{AT_A}$, and let K be a foot of the internal bisector AK in $\triangle ABC$. Prove that points K, R, T_A, W_A are concyclic.

Proof. Notice that by the law of sines, $AK = c \cdot \frac{\sin(2\alpha+\gamma)}{\sin 2\beta}$ and $AW_a = b \cdot \frac{\sin 2\beta}{\sin(2\alpha+\gamma)}$. So, again using that, from 3.3, $AC_1 = \frac{bc}{p}$, and clearly $AB'_1 = p$, we obtain that

$$AK \cdot AW_A = bc = AC_1 \cdot AB'_1,$$

so $B'_1 C_1 K W_A$ is cyclic **(a)**. Consider homothety $H_A^{R=\frac{AB'_1}{AC_1}}$ that takes the A -mixtilinear to the A -excircle, and let AT_A intersect the latter circle at R_0 . It is not difficult to see that $C_1 \rightarrow B'_1$ and $U \rightarrow R$, which implies $C_1 U \parallel B'_1 R$, so

$$\angle C_1 T_A R = \angle R R_0 G_C = \angle C_1 B'_1 R,$$

and therefore $C_1 R T_A B'_1$ is cyclic too **(b)**. Now combining **(a)** and **(b)**, we get that

$$AK \cdot AW_A = AC_1 \cdot AB'_1 = AR \cdot AT_A,$$

hence $KRT_A W_A$ is cyclic as desired.

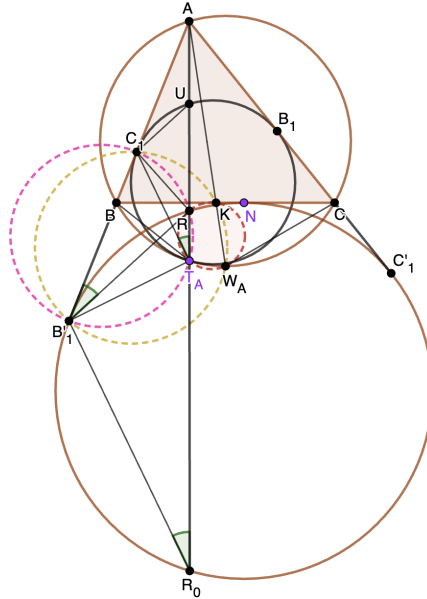


Figure 11: Mixti-excircle homothety

q.e.d.

⁶Continuing this, it is not hard to obtain that $r = r_A \cdot \cos(\alpha)$ where r_A denotes the radius of the A -mixtilinear. This result was one of the first about mixtilinears and was obtained again by Leon Bankoff [6].

4.3 Triplemixti-point and the line OI

Problem. Define T_B and T_C similarly to T_A , and let O be the circumcentre of $\triangle ABC$. Prove that the lines AT_A , BT_B , CT_C , OI are concurrent at a point that I will call a triplemixti-point.

Proof. Let T be the centre of a homothety taking the incircle of $\triangle ABC$ to its circumcircle. Then T clearly lies on the line OI . Now consider the two latter circles and the A -mixtilinear. The centres of homothety of each two of them are A , T_A , T , and two of these homotheties (centred at $(T$ and $T_A)$) have negative coefficients. Therefore, we can apply the Monge's theorem and conclude that points A , T , T_A must be collinear. Similarly, B , T , T_B are collinear and C , T , T_C are collinear, as desired ⁷.

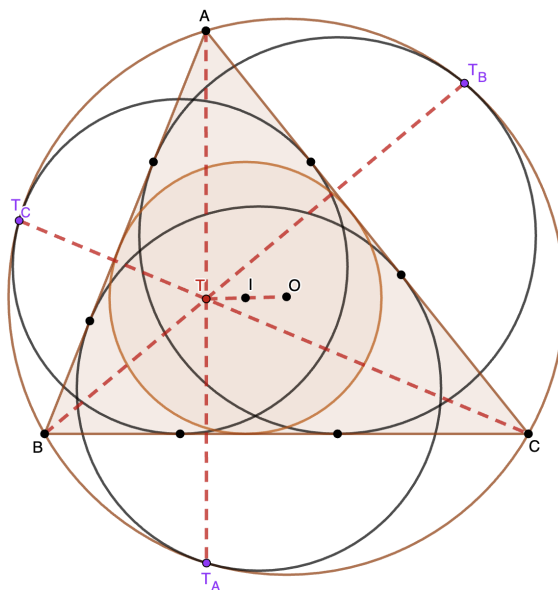


Figure 12: Triplemixti-point and the line OI

Alternative proof. By 3.4, we know that the lines AW_A , BW_B , CW_C are isogonal to the respective nagelians and therefore will meet at the point isogonal to the Nagel point in $\triangle ABC$ as desired. This point is denoted by $X(56)$ in the Encyclopedia of Triangle Centers [7].

q.e.d.

5 Other methods

Sometimes, a problem has nothing to do with homothety. Even the inversion around A might be useless if the statement, e.g., involves some angle Y unrelated to the point A , turning $\varphi(Y)$ into something unclear and too intricate to work with. In these cases, we must resort to more diverse approaches, which might often involve using similar triangles and tinkering with ratios, as I did in a couple of previous problems. Let me once again illustrate this with a few examples below.

5.1 Mixtigonality

Problem. Prove that $\angle AT_AB = \angle DT_AC$.

Proof. It suffices to show that $\triangle AT_AB \sim \triangle CT_AD$, or that $\frac{AT_A}{AB} = \frac{CT_A}{CD}$, or $\frac{CT_A}{AT_A} = \frac{CD}{AB}$. Using a similar approach to 3.3, we have that

$$\frac{CT_A}{AT_A} = \frac{b-AB_1}{AB_1} = \frac{b-AC_1}{AC_1} = \frac{b-\frac{bc}{p}}{\frac{bc}{p}} = \frac{p-c}{c} = \frac{CD}{AB},$$

⁷Since we are having a look at all three mixtilinears, I cannot but mention a beautiful fact that the six points of tangencies with the sides all lie on a conic. To prove this, we first have to show that the opposite sides of the formed hexagon are parallel (this is not hard if we simply calculate some ratios as in 3.3), and then the desired will follow from the reverse of Pascal's theorem for conics.

so we may conclude.

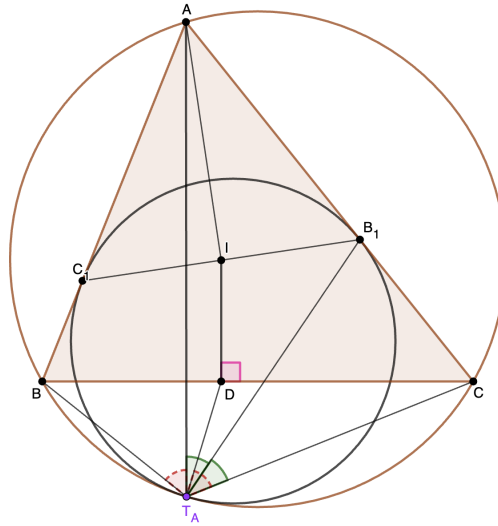


Figure 13: Mixtigonality

q.e.d.

5.2 Relation to A -trapezoid point

Problem. Let A_0 be the reflection of A over the perpendicular bisector of \overline{BC} . Prove that points T_A , D , A_0 are collinear.

Proof. By 5.1, we have $\angle DT_A C = \angle AT_A B$, and from 3.5 — that $\angle CT_A V = \angle BT_A V$. Hence, we have that $T_A V$ bisects $\angle AT_A D$, then the desired is true by symmetry as $\widehat{VA} = \widehat{VA_0}$.

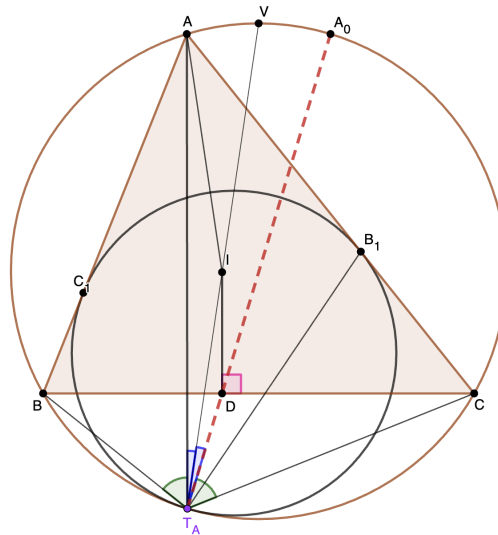


Figure 14: Relation to A -trapezoid point

q.e.d.

5.3 Cross-ratios burst

Problem. Let the tangents to (ABC) through W_B and W_C meet at S . Prove that points S, A, T_A are collinear.

Proof. It suffices to show $AW_BW_CT_A$ is harmonic. It is not hard to see that $AV \perp AI$ and $W_BW_C \perp AI$, so $AV \parallel W_BW_C$. By the trillium theorem, we know that

$$W_CI = W_CA = W_BV \text{ and } W_BI = W_BA = W_CV,$$

so VW_BIW_C is the parallelogram. Now let IV and W_BW_C meet at M , then M is the midpoint of $\overline{W_BW_C}$. It remains to project points A, W_B, T_A, W_C from point V onto W_BW_C and obtain that

$$(A, T_A; W_B, W_C) = (\infty, M; W_B, W_C) = -1,$$

which solves the problem.

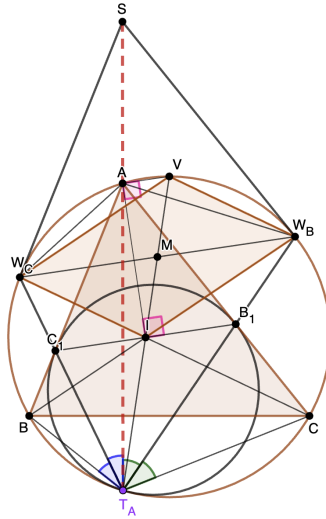


Figure 15: Cross-ratios burst

Alternative proof. This proof appears very unexpectedly but immediately solves the problem. In fact, since $SW_C \parallel AC_1$, $SW_B \parallel AB_1$, $W_BW_C \parallel B_1C_1$, then from the Desargues's theorem, as $SW_C \cap AC_1$, $SW_B \cap AB_1$, $W_BW_C \cap B_1C_1$ lie on the infinitely distant line, we conclude that the lines $W_C C_1$, SA , $W_B B_1$ are concurrent as desired.

q.e.d.

5.4 More isogonals

Problem. Let B_1C_1 and AT_A meet at P . Prove that $\angle BPC_1 = \angle CPB_1$.

Proof. Let AB and CP meet at Q , AC and BP meet at R . Since $\angle AC_1B_1 = \angle AB_1C_1$ and $\angle BPQ = \angle CPR$, we conclude that

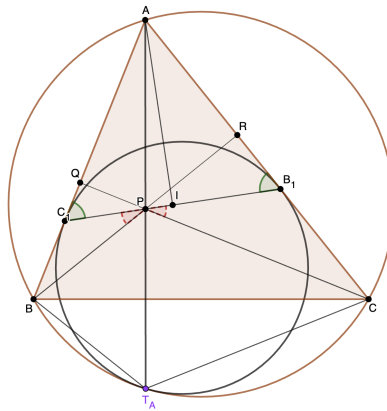


Figure 16: More isogonals

$\triangle BPQ \sim \triangle CPR$, which yields the desired equality.
q.e.d.

5.5 A-sidepoint

Problem. Let BC and the tangent through A to (ABC) meet at A' . Prove that points A' , A , D , T_A are concyclic.

Proof. Remember that, by 5.2, T_A , D , A_0 are collinear. Since

$$\angle ABA' = \angle AA_0C \text{ and } \angle A'AB = \angle ACB = \angle CAA_0,$$

we have $\triangle ABA' \sim \triangle ACA_0$, so the desired follows from

$$\angle AA'D = \angle ACA_0 = \angle AT_A A_0 = \angle AT_A D.$$

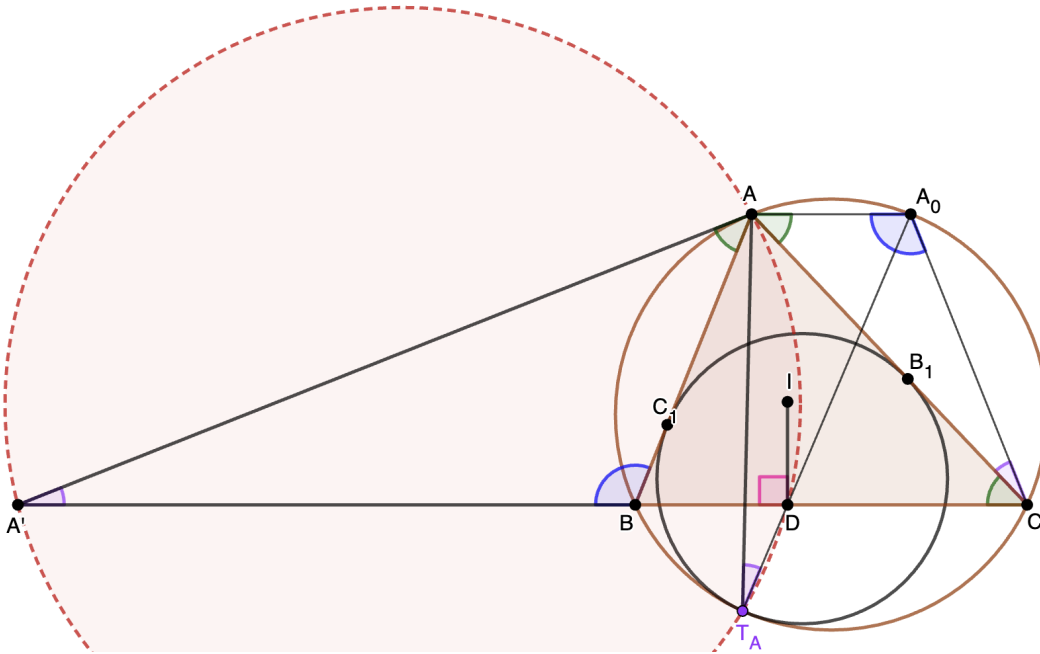


Figure 17: A-sidepoint

q.e.d.

6 Mixti-sidepoint

Let B_1C_1 and BC meet at X , which we will call a mixti-sidepoint. This point is very natural and might occur in many of the problems (not necessarily involving mixtilinears). As we will see, it also has a lot to do with A -sharkydevil point and with the objects from the next two parts of this research. Therefore, let us investigate the properties of the mixti-sidepoint and what it has to do with radical centres.

6.1 Doublemixtiarity

Problem. Prove that points X , T_A , W_A are collinear.

Proof. Consider circles (BIC) and $(IT_A W_A)$. By the trillium theorem again, it follows that W_A is the circumcentre of (BIC) , so since $\angle W_A IX = 90^\circ$, we easily conclude (BIC) is tangent to B_1C_1 . Note that, as proven in 3.4, IC_1BT_A is cyclic, so we obtain that

$$\angle C_1IT_A = 180^\circ - \angle C_1BT_A = \angle IW_AT_A,$$

hence $(IT_A W_A)$ is tangent to $B_1 C_1$ as well. Thus, X is the radical centre of (ABC) , (BIC) , $(IT_A W_A)$, so we may conclude.

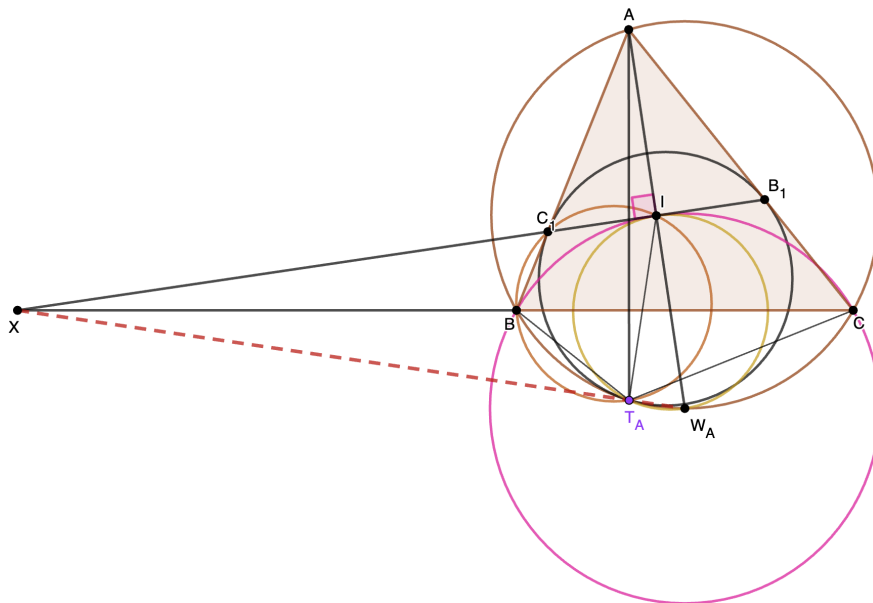


Figure 18: Doublemixtiarity

q.e.d.

6.2 More radical axes

Problem. Prove that points K , D , T_A , W_A are concyclic.

Proof. By 6.1, $(IT_A W_A)$ touches XI at I . The fact that $\angle IDK = 90^\circ$ implies that (IDK) also touches XI at I . Thus, the desired becomes obvious as

$$XT_A \cdot XW_A = XI^2 = XD \cdot XK.$$

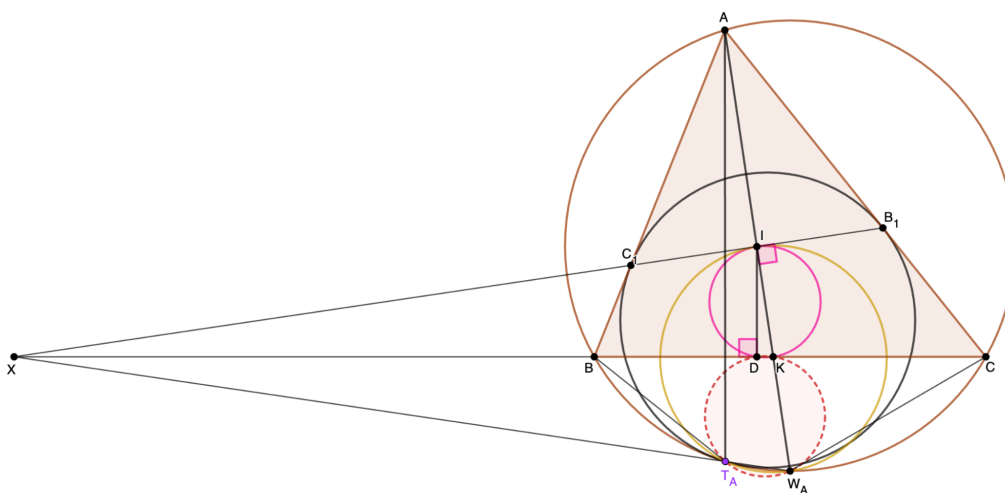


Figure 19: More radical axes

q.e.d.

6.3 Mixti-tangent

Problem. Let the common tangent of (ABC) and the A -mixtilinear meet BC at T_0 ⁸, and IT_A and BC at P_0 . Prove that points X, P, P_0, T_A lie on the circle centred at T_0 .

Proof. Note that since

$$\angle IDX = 90^\circ = \angle IT_A X,$$

we have that $XIDT_A$ is cyclic. From **5.2**, we also know that T_AI bisects $\angle AT_AD$, therefore

$$\angle P_0XP = \angle DT_AI = \angle PT_AP,$$

so $XP P_0 T_A$ is also cyclic as desired. Now notice that

$$\angle P_0PI = 90^\circ = \angle IDP_0,$$

so we have one more cyclic $PIDP_0$. Then,

$$\angle PDT_0 = \angle PIP_0 = \angle XDT_A = \angle CDA_0 = \angle AA_0T_A = \angle PT_AT_0,$$

from which $T_0PP_0T_A$ is cyclic as well and DT_0 bisects $\angle PDT_A$, thus we infer $T_0P = T_0T_A$. And since

$$\angle T_0PX = \angle DT_0P - \angle T_0XP = \angle DT_4P - \angle T_0XP = \angle T_0XP,$$

we finally conclude that $T_0X = T_0P$, from which the desired follows.

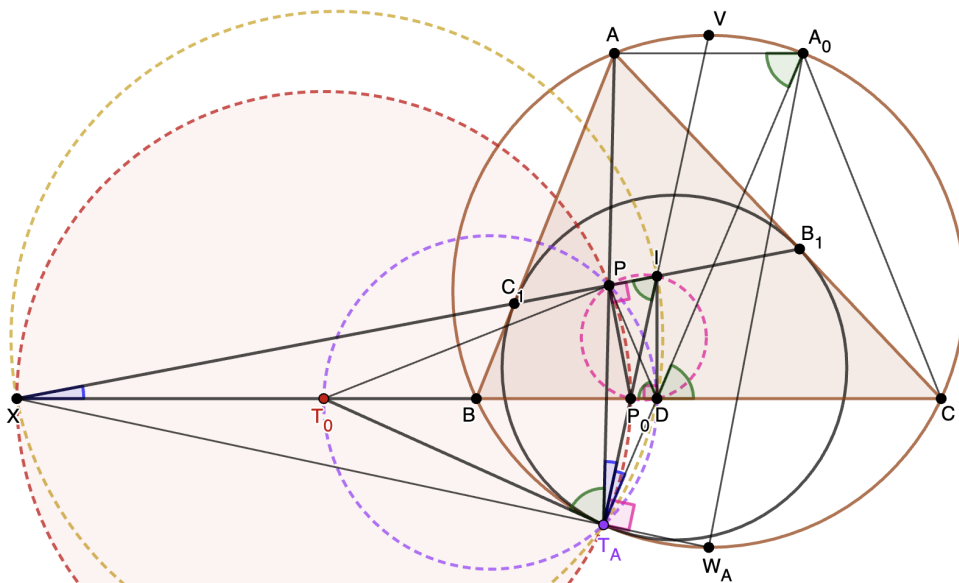


Figure 20: Mixti-tangent

q.e.d.

6.4 Relation to A -sharkydevilpoint

Now, I will introduce one more unique point. Let (AI) and (ABC) intersect at $X_S \neq A$. This point is called *A-sharkydevil point* and will often be used in future problems. A crucial corollary about this point will be mentioned in the last subsection. But for now, let us link this point to the mixtilinear.

⁸Honourable mention: if we denote the intersection point of B -mixitangent with AC and of C -mixitangent with AB , then these two points will be collinear with T_0 .

Problem. Prove that points X_S , X , A are collinear.

Proof. We know that $\angle IX_SA = 90^\circ$, so $(AX_S I)$ is tangent to $B_1 C_1$ at I . By **6.1**, $(IT_A W_A)$ is also tangent to $B_1 C_1$, so we can conclude that the lines AX_S , $B_1 C_1$, $W_A T_A$ will meet at X as radical axes of (ABC) , $(AX_S I)$, $(IT_A W_A)$, so we may conclude.

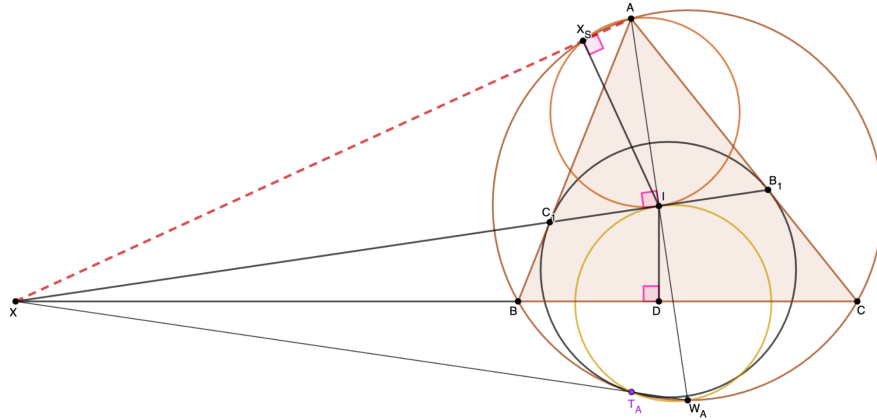


Figure 21: Relation to A -sharkydevilpoint

q.e.d.

7 Mixti-miquelpoint

Denote the Miquel point of BC_1B_1C by M_T . Let us have a look at some properties of this point.

7.1 Mixti-inversion

Problem. Let AT_A intersect the A -mixtilinear at $P_1 \neq A$. Prove that points M_T, P_1, I, O_T, T_A are concyclic.

Proof. As the title suggests, consider inversion around the A -mixtilinear. Then $A \leftrightarrow I$, so since A, P_1, T_A lie on one line, we conclude that $P_1 IO_T T_A$ is cyclic. We are left to prove that M_T lies on this circle, which will follow if

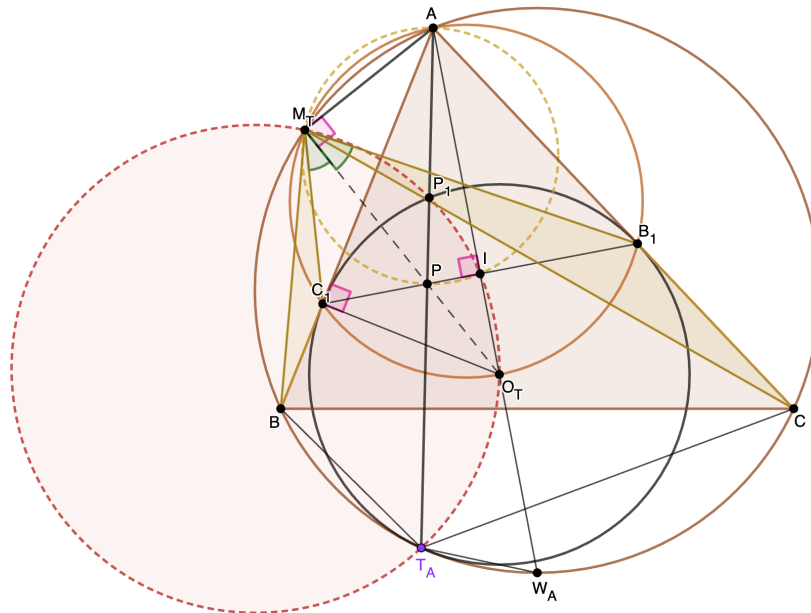


Figure 22: Mixti-inversion

we show that $M_T \leftrightarrow P$. Let us first show that O_T, P, M_T are collinear, or that $M_T P$ bisects $\angle C_1 M_T B_1$. There clearly exists a spiral similarity around A that takes $\triangle M_T B_1 C$ to $\triangle M_T C_1 B$, so we have

$$(!) \frac{M_T C_1}{M_T B_1} = \frac{P C_1}{P B_1} \Leftrightarrow (!) \frac{B C_1}{C B_1} = \frac{\sin \angle C_1 A P}{\sin \angle B_1 A P} \Leftrightarrow (!) \frac{B C_1}{C B_1} = \frac{B T_A}{C T_A}.$$

We know from 3.3 that $B C_1 = c - \frac{bc}{p}$ and $C B_1 = b - \frac{bc}{p}$. Also from 5.1, it follows that

$$\frac{B T_A}{C T_A} = \frac{B T_A}{A T_A} \cdot \frac{A T_A}{C T_A} = \frac{p-b}{b} \cdot \frac{c}{p-c},$$

so

$$(!) \frac{c - \frac{bc}{p}}{b - \frac{bc}{p}} = \frac{p-b}{b} \cdot \frac{c}{p-c},$$

which is true. Now notice that

$$\angle P M_T A = \angle O_T M_T A = 90^\circ = \angle A I P,$$

from which $M_T A I P$ is cyclic. And we may conclude after

$$O_T P \cdot O_T M_T = O_T I \cdot O_T A = O_T C_1^2.$$

q.e.d.

7.2 Relation to mixti-sidepoint (BxMO 2023/3)

Problem. Prove that point P_0 is the orthocentre of $\triangle X M_T W_A$.

Proof. Let M_{BC} be the midpoint of \overline{BC} . Then as $\angle X M_{BC} W_A = 90^\circ = \angle X I W_A$, we get that $X I M_{BC} W_A$ is cyclic, so it suffices to show that $X M_T M_{BC} W_A$ is cyclic too. Since M_T is the Miquel point of $BC_1 B_1 B$, we have that $X B C_1 M_T$ is cyclic. So we may conclude after

$$\angle M_T X I = \angle M_T B C_1 = \angle M_T W_A A.$$

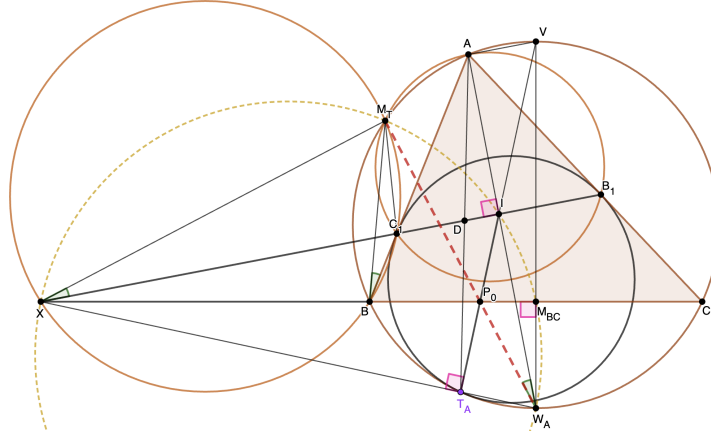


Figure 23: Relation to mixti-sidepoint

q.e.d.

7.3 Sharky-devil point, again

Problem. Prove that points M_T, X_S, D, P_0 are concyclic.

Proof. I have just proved in 7.2 that $\angle X M_T P_0 = 90^\circ$, hence $X M_T P_0 T_A$ is cyclic. It is well-known that X_S, D, W_A are collinear (I will also prove this in the third part of the research). So the desired follows after

$$\angle M_T X_S D = \angle M_T X_S W_A = \angle M_T T_A X = \angle M_T P_0 X.$$

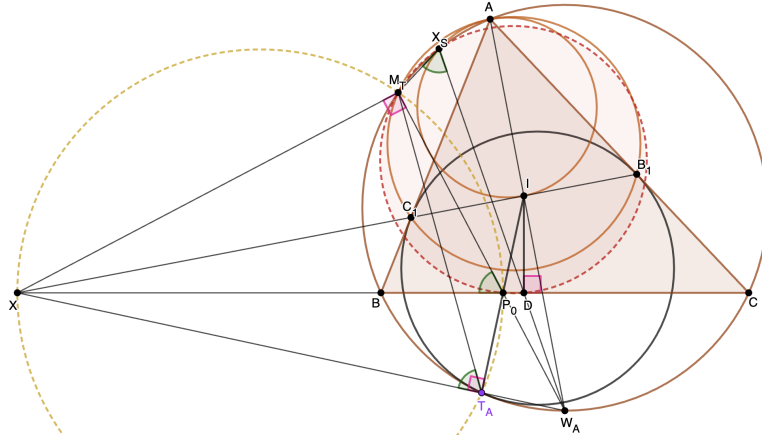


Figure 24: Sharky-devil point, again

q.e.d.

8 A point on \widehat{BC}

Choose point Y on \widehat{BC} not containing A . It turns out that even an arbitrary point is well-related to the mixtilinears. Let us investigate.

8.1 The first wonderful mixticyclicity (Iranian MO 1997/4a, ISL 1999/G8)

Problem. Denote the incentres of $\triangle ABY$ and $\triangle ACY$ by I_1 and I_2 , respectively. Prove that points T_A , Y , I_1 , I_2 are concyclic.

Proof. Obviously, W_C and W_B lie on YI_1 and YI_2 , respectively. So it suffices to show that $\angle T_A I_1 Y = \angle T_A I_2 Y$, which will immediately follow if $\triangle T_A W_C I_1 \sim \triangle T_A W_B I_2$ or $\frac{W_C I_1}{W_C T_A} = \frac{W_B I_2}{W_B T_A}$. Once again by the trillium theorem, we have $W_C I_1 = W_C A$ and $W_B I_2 = W_B A$. Then we are left to show that

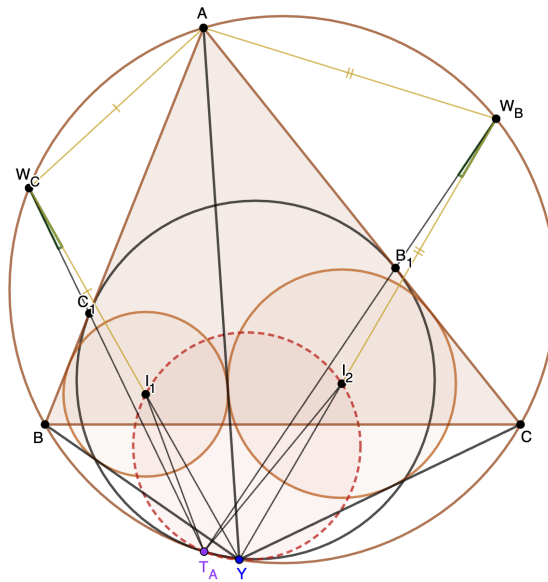


Figure 25: The first wonderful mixticyclicity

$$\frac{W_C A}{W_C T_A} = \frac{W_B A}{W_B T_A} \text{ or } (W_C, W_B; A, T_A) = -1,$$

which is clear since, from 5.1, $AW_B T_A W_C$ is harmonic.

q.e.d.

8.2 The second wonderful mixticity (Taiwan TST 2014/3)

Problem. Let the tangents from point Y to the incircle of $\triangle ABC$ meet BC at Y_1 and Y_2 . Prove that points T_A, Y, Y_1, Y_2 are concyclic.

Proof. We will first prove the following lemma and claim.

Lemma. Given $\triangle ABC$ and its altitudes AA_0, BB_0, CC_0 intersecting at H . Let K be the midpoint of \overline{AH} . Let X be an arbitrary point on the nine-point circle of $\triangle ABC$ and let O be the circumcentre of (ABC) . A line perpendicular to OX intersects (ABC) at P and Q . Prove that the nine-point circle of $\triangle PAQ$ passes through K .

Proof. First let M and N be the midpoints of \overline{AQ} and \overline{AP} , respectively. Now consider homothety around A under which $M \rightarrow Q, N \rightarrow P, K \rightarrow H$. Let $X \rightarrow A'$. It suffices to show that $PHQA'$ is cyclic. Let H' be reflection of H over the point X . Since H is the centre of homothety taking the nine-point circle of $\triangle ABC$ to its circumcircle, we infer that H' lies on (ABC) . It is clear that $PHQH'$ and $PAQA'$ are parallelograms. Therefore,

$$\angle PHQ = \angle PH'Q = 180^\circ - \angle PAQ = 180^\circ - \angle PA'Q,$$

so we may conclude.

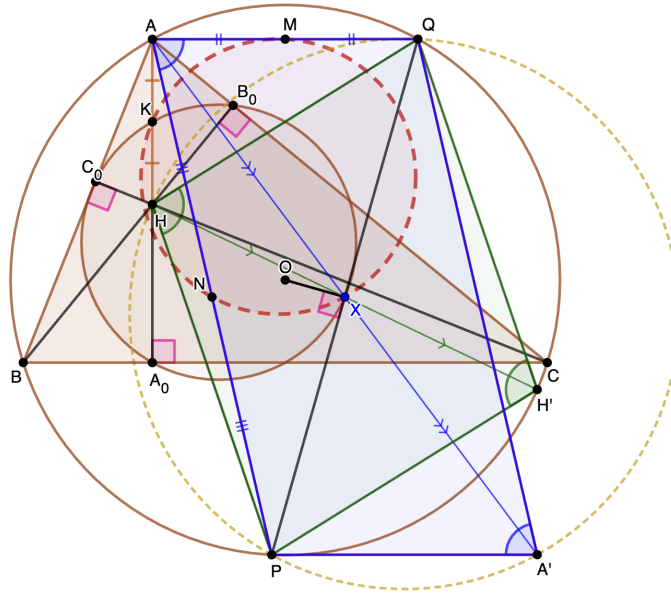


Figure 26: The second wonderful mixticity lemma

Claim. If we let the perpendicular from D to EF and line IT_A intersect at T_I , then T_I is an invert of T_A around the incircle of $\triangle ABC$.

Proof. Note that

$$\angle DEF = \alpha + \beta \text{ and } \angle EDF = \beta + \gamma,$$

so it is easy to calculate that $\angle IDT_I = \beta - \gamma$. From 5.2, we get that $T_A I$ bisects $\angle BT_A C$ and $\angle AT_A D$, thus

$$\angle AT_A B = \angle DT_A C = 2\gamma.$$

Therefore,

$$\angle IT_A D = \frac{\angle AT_A D}{2} = \frac{\angle BT_A C - 4\gamma}{2} = \beta - \gamma.$$

So, $\angle IT_A D = \angle ID T_I$, yielding $ID^2 = IT_I \cdot IT_A$, from which the desired follows.

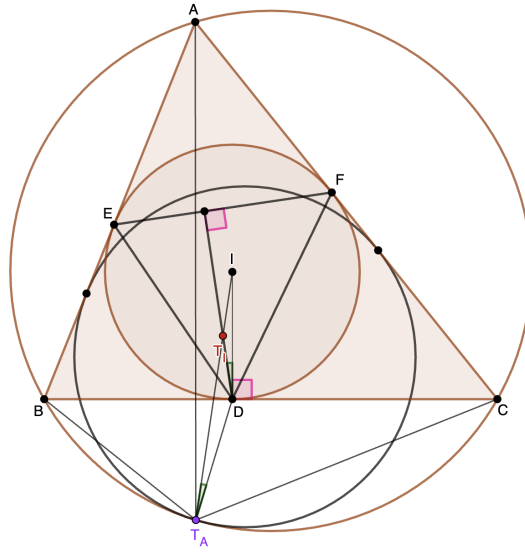


Figure 27: The second wonderful mixticyclicity claim

Now let us get back to the original problem. Let $Y Y_1$ and $Y Y_2$ touch the incircle of $\triangle ABC$ at Y' and Y'' , respectively. Let M_1, M_2, M_3 be the midpoints of $Y'D, Y'Y'', Y''D$, respectively. Consider inversion around the incircle. Notice it takes (ABC) to the nine-point circle of $\triangle DEF$, so T_I lies on it and $Y \leftrightarrow M_3, Y_1 \leftrightarrow M_1, Y_2 \leftrightarrow M_2$. Therefore, it suffices to show that $M_1 T_I M_2 M_3$ is cyclic, but it follows by *lemma*.

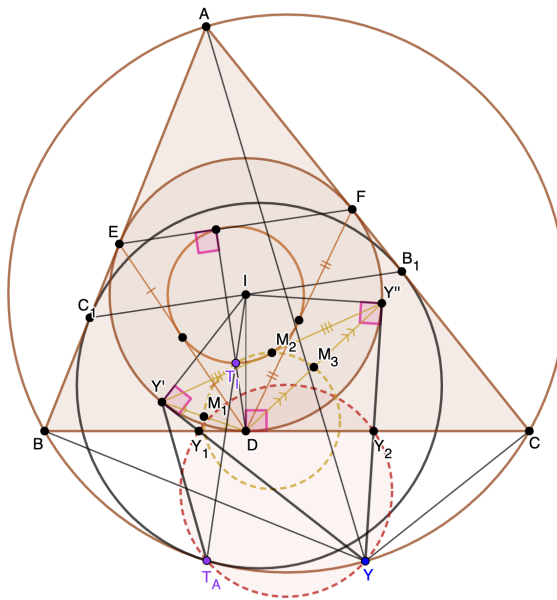


Figure 28: The second wonderful mixticyclicity

q.e.d.

8.3 Sharky-devil point corollary

We will end this part with an important corollary about the sharky-devil point, which we will use in future parts of the research.

Problem. Prove that the invert of the A -sharkydevil point is a foot of the altitude from D to EF .

Proof. As we defined X_S to be the intersection point of (AI) and (ABC) , we conclude, from **8.2**, that X_S will map to the intersection point X'_S of the nine-point circle of $\triangle DEF$ and EF under the mentioned inversion around the incircle. So, it also lies on the altitude dropped from D in $\triangle DEF$ as desired.

q.e.d.

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