Permutation Line

Li4 + permutation-chang

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Made with evan.sty.

Original can be found at https://permutation-chang.github.io/Permutationline/, some changes are made for clarity.

§1 Permutation Lines

§1.1 Introduction

Zhang Zhihuan, Prince of geo, uses the permutation line to nuke geo problems.

Definition 1.1. A **circumconic** of a triangle $\triangle ABC$ is a conic going through the triangle's vertices.

Here, the **fourth intersection** of two circumconics refers to the point they intersect at that isn't A, B, or C.

Definition 1.2. The $\mathcal{C}$-**cevian triangle** is the circumcevian triangle for general circumconic $\mathcal{C}$.

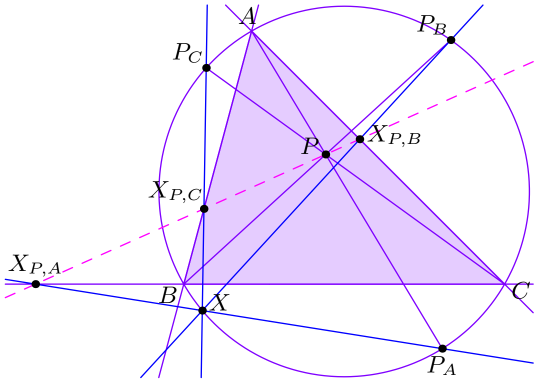
(In other words, project $\triangle ABC$ through a point P back onto the circumconic).

Proposition 1.3

Let $\triangle ABC$ have circumconic $\mathcal{C}$ and point P be an arbitrary point in the plane. Let $\triangle P_AP_BP_C$ be its $\mathcal{C}$-cevian triangle and $X \in \mathcal{C}$. Define

$$X_{P,A} = XP_A \cap BC, X_{P,B} = XP_B \cap AC, X_{P,C} = XP_C \cap AB,$$

then $P, X_{P,A}, X_{P,B}, X_{P,C}$ are collinear.



Proof. By Pascal's on BCP_CXP_AA, we get that $BC \cap XP_A = X_{P,A}, CP_C \cap P_AA = P, P_CX \cap BA = X_{P,C}$ line on a line.

By a similar Pascal's on CAP_AXP_BB, the result follows. □

Definition 1.4. For a variable X with respect to a fixed $\mathcal{C}$ and $\triangle ABC$, we call this line $PX_{P,A}X_{P,B}X_{P,C}$ the P-**permutation line** (originally (張志煥線)) and denote it as $\mathbf{Per}_P^{\mathcal{C}}(X)$.

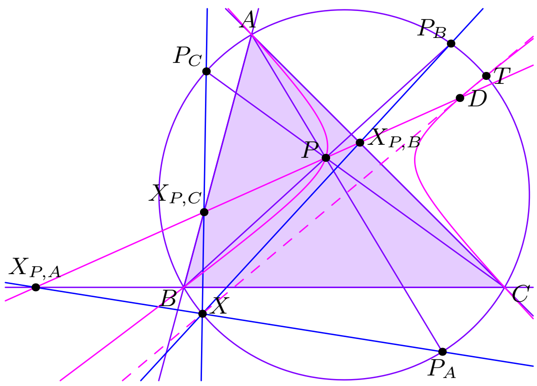
Proposition 1.5

The map $\mathbf{Per}_P^{\mathcal{C}}(X) \colon \mathcal{C} \to PX_{P,A}$ from $\mathcal{C}$ to the pencil through P is projective.

Proof. Follows by considering the projective map $X \to X_{P,A} \to PX_{P,A}$. □

Proposition 1.6

Take $\triangle ABC$ with circumconic $\mathcal{C}$ and fix points P, X, $T \in \mathcal{C}$. Then the other intersection D of $\mathbf{Per}_P^{\mathcal{C}}(X)$ and $\mathcal{T} = (ABCPT)$ lies on TX.



Proof. Note that by Proposition 1.5 taking a permutation line is a projective map.

Thne we get that

$$\begin{aligned} T(A,B;C,X) &= (A,B;C,X)_{\mathcal{C}} = (\mathbf{Per}_P^{\mathcal{C}}(A),\mathbf{Per}_P^{\mathcal{C}}(B);\mathbf{Per}_P^{\mathcal{C}}(C),\mathbf{Per}_P^{\mathcal{C}}(X)) \\ &= (AP,BP;CP,\mathbf{Per}_P^{\mathcal{C}}(X)) = (A,B;C,D)_{\mathcal{T}} = T(A,B;C,D) \end{aligned}$$

As such, T, X, D are collinear as desired. □

Proposition 1.7

Take $\triangle ABC$ with circumconic $\mathcal{C}$, and points P, Q with $X \in \mathcal{C}$.

Then $Z = \mathbf{Per}_P^{\mathcal{C}}(X) \cap \mathbf{Per}_Q^{\mathcal{C}}(X)$ lies on $\mathcal{T} = (ABCPQ)$. Furthermore, if T is the fourth intersection of $\mathcal{T}$ and $\mathcal{C}$, then T, X, Z are collinear.

Proof. This is effectively combining Proposition 1.6 for two values of P, Q, as $\mathbf{Per}_P^{\mathcal{C}}(X) \cap \mathcal{T}$ and $\mathbf{Per}_Q^{\mathcal{C}}(X) \cap \mathcal{T}$ both lie on TX.

The fact that Z lies on $\mathcal{T}$ can also be proven by noting that

$$P(A,Z;B,C) = (AP \cap BC, X_{P,A}; B, C) \stackrel{P_A}{=} (A,X;B,C)_{\mathcal{C}}$$

which suffices by symmetry. □

Let's now establish some shorthand.

Definition 1.8. $\mathcal{D}_{P,Q}$ now refers to conic $(ABCPQ)$.

Definition 1.9. Let T be the fourth intersection of $\mathcal{C}$ and $\mathcal{D}$. Then define $\mathbf{Li}_{P,Q}^{\mathcal{C}}(X) = \mathbf{Li}_{\mathcal{D}_{P,Q}}^{\mathcal{C}}(X) = \mathcal{D} \cap TX$.

By Proposition 1.7, we then get that $\mathbf{Li}_{\mathcal{D}}^{\mathcal{C}}(X) \in \mathbf{Per}_D^{\mathcal{C}}(X)$ for any $D \in \mathcal{D}$.

(Li is just the name of one of the based and cool and awesome authors).

> **Remark 1.10.** Here's a bad way to remember that $\mathbf{Li}_{\mathcal{D}}^{\mathcal{C}}$ lies on $\mathcal{D}$. The *Li*ttle Dipper is a constellation, so the point lies on the dipper or lower conic, wihch is $\mathcal{D}$.

> **Remark 1.11.** This gives us another lens to look at the fact that $\mathbf{Li}_{\mathcal{D}}^{\mathcal{C}}(X)$ lies on TX – this simply follows because $\mathbf{Per}_T^{\mathcal{C}}(X) = TX$!

Proposition 1.12

Take $\triangle ABC$ with circumconics $\mathcal{C}, \mathcal{D}$.

For a fixed $X \in \mathcal{C}$, the map $\mathbf{Per}_{\bullet}^{\mathcal{C}}(X)\colon \mathcal{D} \to \mathbf{Li}_{\mathcal{D}}^{\mathcal{C}}(X)P$ is projective.

Proof. Follows since $\mathbf{Per}^{\mathcal{C}}_{P}(X) = P\mathbf{Li}^{\mathcal{C}}_{\mathcal{D}}(X)$. □

Proposition 1.13

Take $\triangle ABC$ with circumconic $\mathcal{C}$ and another point X. Let ℓ be an arbitrary line.

Then the envelope $\{\mathbf{Per}^{\mathcal{C}}_{P}(X) \mid P \in \ell\}$ inscribes a inconic $\mathcal{B}^{\mathcal{C}}_{\ell}(X)$ of $\triangle ABC$, and the map $\mathbf{Per}^{\mathcal{C}}_{\bullet}(X) \colon \ell \to \mathcal{B}^{\mathcal{C}}_{\mathcal{D}}(X)$ is projective.

Proof. The original document notes this as "trivial". Anyways, the map $P \to X_{P,A}$ has no fixed points and is projective, so this result follows by what is called Steinr inconic. (Reference can be found at https://artofproblemsolving.com/community/c6h1884540p12835147).

The fact that the map is projective is immediate. □

§1.2 Isoconjugations

Let's introduce a new concept now. You may already know the following:

Definition 1.14. An **isoconjugation** φ wrt to $\triangle ABC$ and a point $P = (p : q : r)$ maps points $U = (u : v : w)$ to $(pvw : qwu : ruv)$ where no point lies on the side of a triangle.

See https://bernard-gibert.fr/gloss/isoconjugation.html for a reference.

We use $\mathcal{L}^{\varphi}$ as shorthand for the image. The isoconjugation as we approach A is BC and so on.

Theorem 1.15 (Isoconjugations preserve harmonics)

An isoconjugation maps a line $\mathcal{L}$ not through the vertices of a triangle to a conic $\varphi(\mathcal{L})$ such that

$$(A, B; C, D)_{\mathcal{L}} = (\varphi(A), \varphi(B); \varphi(C), \varphi(D))_{\varphi(\mathcal{L})}$$

Proof. The conic part follows through bary.

Similarly, it turns out that for some pencil of conics, $\varphi(A)$ lies on the polar of A wrt to all conics in this family. Taking the conic which is tangent to $\mathcal{L}$ at l then gives that

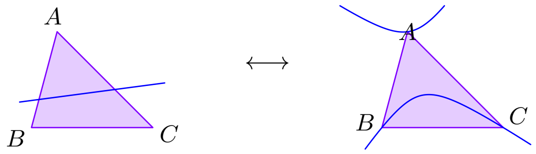
$$(A, B; C, D)_{\mathcal{L}} = L(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (\varphi(A), \varphi(B); \varphi(C), \varphi(D))_{\varphi(\mathcal{L})}$$

which suffices. □

Example 1.16 (Examples of isoconjugations)

Isogonal conjugation and isometric conjugations come to mind. In particular, isogonal conjugation $(\bullet)^*$ maps the line at infinity $\mathcal{L}_\infty$ to the circumcircle.

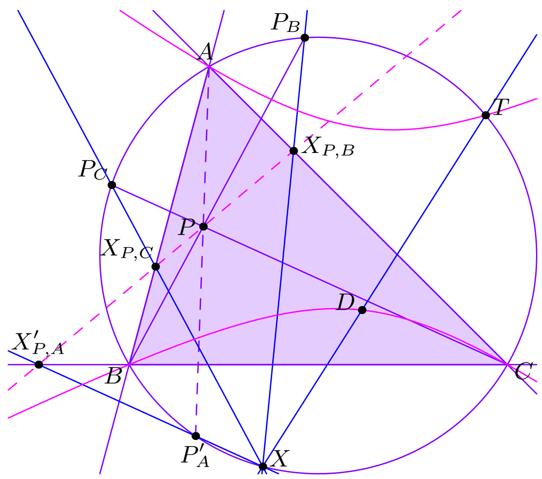
The corresponding family of conics for isogonal conjugation are those through I, I_A, I_B, I_C where I is the incenter and I_A, I_B, I_C are the excenters.



Here's why isoconjugations are related. We can actually use permutation lines to find the isoconjugation of a circumconic:

Proposition 1.17

Take a $\triangle ABC$ with isoconjugation φ and circumconics $\mathcal{C}, \mathcal{D}$, and a X on $\mathcal{C}$. Let $D = \mathbf{Li}^{\mathcal{C}}_{\mathcal{D}}(X)$ and let $P = D^\varphi$. We then have that $\mathcal{D}^\varphi$ is in fact just the same as $\mathbf{Per}^{\mathcal{C}}_P(X)$.



Proof. By definition, we get that P lies on $\mathcal{D}^\varphi$ so $\mathcal{D}^\varphi$ is some line through P. It remains to show that it is in fact the permutation line.

Let $X'_{P,A} = \mathcal{D}^\varphi \cap BC$ and $P'_A = XD \cap \mathcal{C}$. Then it remains to show that P'_A lies on AP.

Let T be the fourth intersection point of the two circumconics.

Now, we have that

$$
\begin{aligned}
(B,C;X'_{P,A},AP'_A\cap BC) &\stackrel{P'_A}{=} (B,C;X,A)_{\mathcal{C}} = T(B,C;P^{\varphi},A) = (B,C;D,A)_{\mathcal{D}} \\
&\stackrel{\varphi}{=} (\mathcal{D}^{\varphi}\cap AC,\mathcal{D}^{\varphi}\cap AB;P,X'_{P,A}) \stackrel{A}{=} (C,B;AP\cap BC,X'_{P,A})
\end{aligned}
$$

which finishes. $\square$

Take $\mathcal{D}=\mathcal{D}_{P,\varphi(P)},\mathcal{D}=P\varphi(P)$ gives a corollary.

Corollary 1.18

If we take $\mathcal{D}=\mathcal{D}_{P,\varphi(P)}$, then for $L=\mathbf{Li}^{\mathcal{C}}_{P,\varphi(P)}(X)=\mathbf{Li}^{\mathcal{C}}_{\mathcal{D}}(X)$ we get that $P\varphi(P)=\mathbf{Per}^{\mathcal{C}}_{L^{\varphi}}(X)$.

Here's a theorem that holds when we have two isoconjugations.

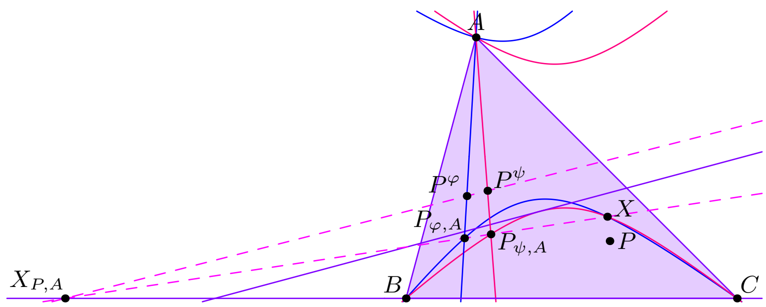
Theorem 1.19 (Fundamental Theorem of Permutation Lines)

For a $\triangle ABC$, if we have two isoconjugations φ,ψ, then for a line $\mathcal{L}$, let X be the fourth intersection of $\mathcal{L}^{\varphi}$ and $\mathcal{L}^{\psi}$. Let P be a point in the plane.

Define $P_{\varphi,A}=A\varphi(P)\cap\mathcal{L}^{\varphi}$ and define $P_{\psi,A}$ similarly.

Then it follows that $X,P_{\varphi,A},P_{\psi,A}$ are collinear. Furthermore,

$$\varphi(P)\psi(P)=\mathbf{Per}^{\mathcal{L}^{\varphi}}_{\varphi(P)}(X)=\mathbf{Per}^{\mathcal{L}^{\psi}}_{\psi(P)}(X).$$



Proof. Note that

$$A(A,P_{\varphi,A};B,C)_{\mathcal{L}^{\varphi}} \stackrel{\varphi}{=} A(\mathcal{L}\cap BC,P;B,C) \stackrel{\psi}{=} A(A,P_{\psi,A};B,C)_{\mathcal{L}^{\psi}}.$$

It thus follows that $X(A,P_{\varphi,A};B,C)=X(A,P_{\psi,A};B,C)$, giving the concurrency.

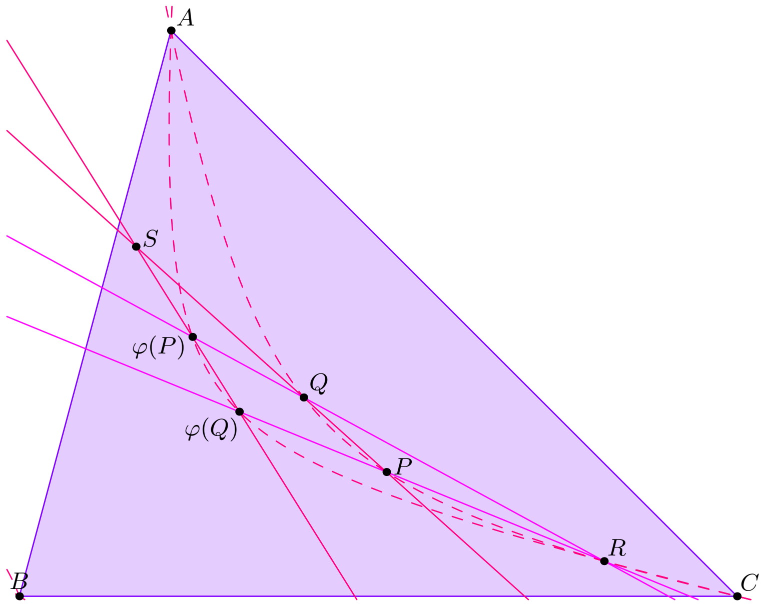
Now, let $XP_{\varphi,A}P_{\psi,A}$ intersect BC at $X_{P,A}$. Define $X_{P,B},X_{P,C}$ simlarly. The second claim now just follows as

$$X_{P,A}\varphi(P)=\mathbf{Per}^{\mathcal{L}^{\varphi}}_{\varphi(P)}(X)=X_{P,A}X_{P,B}X_{P,C}=\mathbf{Per}^{\mathcal{L}^{\psi}}_{\psi(P)}(X)=X_{P,A}\psi(P)$$

as both permutation lines pass through $X_{P,A},X_{P,B},X_{P,C}$. $\square$

Example 1.20 (Isoconjugates Lead to More Isoconjugates)

For $\triangle ABC$, let φ be a isoconjugation and let P and Q be points. If $P\varphi(Q) \cap \varphi(P)Q = R$, $PQ \cap \varphi(P)\varphi(Q) = S$, then R and S are φ-conjugates.



Proof. Let $\mathcal{L}$ be an arbitrary line.

Define an isoconjugation ψ that maps P to Q and let $\mathcal{L}^\varphi$ intersect $\mathcal{L}^\psi$ at X as a fourth intersection.

Then, by Theorem 1.19 it follows that

$$\mathbf{Per}_P^{\mathcal{L}^\psi}(X) = P\varphi(Q) = \mathbf{Per}_{\varphi(Q)}^{\mathcal{L}^\varphi}(X), \ \mathbf{Per}_Q^{\mathcal{L}^\psi}(X) = Q\varphi(P) = \mathbf{Per}_{\varphi(P)}^{\mathcal{L}^\varphi}(X)$$

As such, it follows that $R = \mathbf{Per}_P^{\mathcal{L}^\psi}(X) \cap \mathbf{Per}_Q^{\mathcal{L}^\psi}(X) = \mathbf{Li}_{P,Q}^{\mathcal{L}^\psi}(X)$. By the same logic, $R = \mathbf{Li}_{\varphi(P),\varphi(Q)}^{\mathcal{L}^\varphi}(X)$. It then follows that $\varphi(R)$ lies on both PQ and $\varphi(P)\varphi(Q)$ and is thus S. □

Remark 1.21. This is also a standard DDIT result – Note that we get an involution of $(AP, A\varphi(P)), (AQ, A\varphi(Q)), (AR, AS)$, so $A\varphi(R) = AS$. Doing for B and C as well gives the result.

Example 1.22

For $\triangle ABC$, define the **orthocorrespondent** P^o of a point P wrt to $\triangle ABC$ to be the trilinear polar of its orthotransversal.

Show that for P on the Euler line, its isogonal conjugate P^* and orthocorrespondent P^o are collinear with the orthocenter.

Proof. Take $\mathcal{L}$ to be the Euler line. We can check that H is fixed by the orthocorrespondent, it follows that $H \in \mathcal{L}^* \cap \mathcal{L}^o$ and that the orthocorrespondent is an isoconjugate.

As such, it follows by Theorem 1.19 that $AP^* \cap \mathcal{L}^*$, $AP^o \cap \mathcal{L}^o$, and H are collinear, which finishes. $\square$

Remark 1.23. Here's a sketch of an alternative proof of the $H, P^o, \psi(P)$ concurrence.

We can check that the eight fixed points of the two isoconjugations $(\bullet)^*, (\bullet)^o$ (which here are the orthocenter, incenter, and the versions of these with negative coordinates in bary) lie on the same conic. This can be proven through considering conics of the form $C_1x^2 + C_2y^2 + C_3z^2 = 0$.

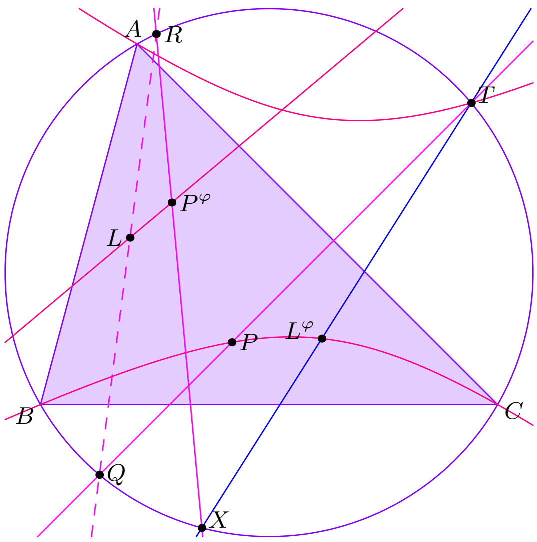
Let H' the polar of the Euler line wrt to this the conic. Then the pole of P wrt to this conic is P^*H' and P^oH', so it follows that $\varphi(P)\psi(P)$ passes through H.

We can check that H' lies on the isogonal conjugate of the Euler line because polar reciprocation maps P to $\varphi(P)H'$ which must be unique.

By symmetry it follows then that $H' = H$.

Proposition 1.24

For $\triangle ABC$ with isoconjugation φ, let $\mathcal{C}$ and $\mathcal{D}$ be circumconics and let T be their fourth intersection point. Then for points $P \in \mathcal{D}$ and $X \in \mathcal{C}$, let that $Q = TP \cap \mathcal{C}, R = XP^\varphi \cap \mathcal{C}, L = \mathbf{Li}_{\mathcal{D}}^{\mathcal{C}}(X)^\varphi$ are collinear.



Proof. Note that the map $Q \to P \to P^\varphi \to R$ is projective, so it remains to check three cases of P to show equivalence with the projective map through L.

When $P = A$, we get that $Q = A$ and that $P^\varphi = (\mathcal{D}^\varphi \cap BC)$. As such, it follows that $R = X(\mathcal{D}^\varphi \cap BC) \cap \mathcal{C}$. Then, by Proposition 1.17 we get that $\mathcal{D}^\varphi = \mathbf{Per}_L^{\mathcal{C}}(X)$, so L lies on QR. The other two cases of $P = B, P = C$, are symmetric so we are done. $\square$

Remark 1.25. In other words, if we have any $L, P \in \mathcal{D}$, and $L^\varphi, P^\varphi \in \mathcal{D}^\varphi$, then T is fixed under $T \to Q \to R \to X \mapsto T$ by projecting through $P, L, P^\varphi, R^\varphi$.

§1.3 The circumcircle

Most geo problems don't involve things that aren't circles, so let's reduce to that case. Take Ω as the circumcircle in the following sections.

Example 1.26 (Circumcircle Permutation Lines)
When $\mathcal{C}$ is the circumcircle Ω, a bunch of things occur. Let O, H, K be the circumcenter, orthocenter, and symmedian. Then

a. $\mathbf{Per}^\Omega_O(X)$ is the orthotransversal of X.

b. $\mathbf{Per}^\Omega_H(X)$ is the Steiner line of X.

c. $\mathbf{Per}^\Omega_K(X)$ is the trilinear polar of X.

Proof. The first one follows as if A' is the antipode of A, then $\angle AXA' = 90°$.

The second one follows as if H_A, X_A are the reflections of H, X over BC, then the lines BC, H_AX, HX_A concur on BC.

For the third one, note that it remains to show that the polar of $AX \cap BC, XD, BC$ concur.

This then follows as

$$(A, D; B, C) \stackrel{X}{=} (AX \cap BC, XD \cap BC; B, C) = -1.$$

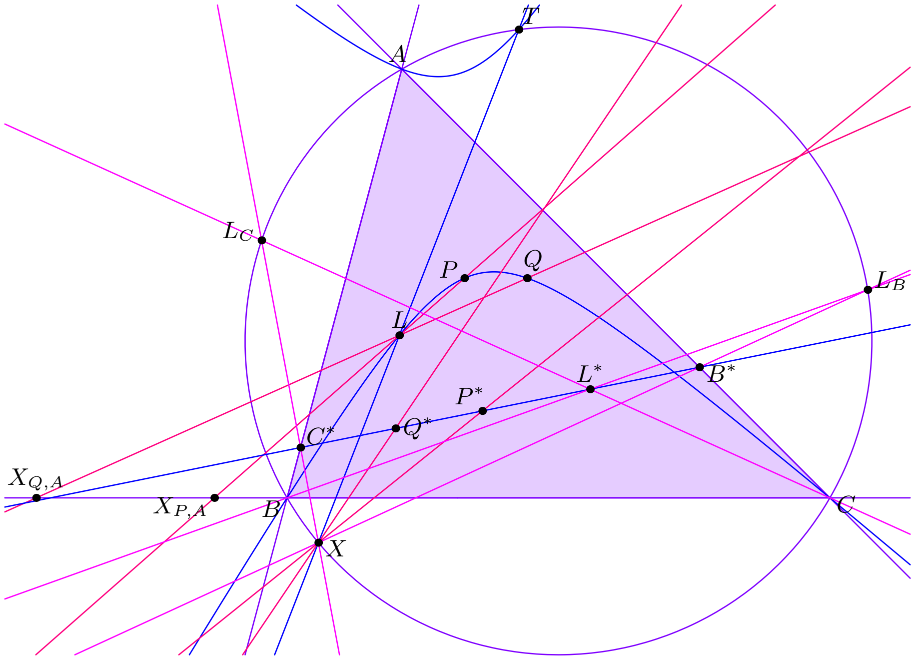
which finishes. □

When $\mathcal{C}$ is the circumcircle, we will omit the symbol to get $\mathbf{Per}_P(X)$ and $\mathbf{Li}_P(X)$.

Proposition 1.27
For $\triangle ABC$ with point X on Ω and arbitrary P and Q with isogonal conjugates P^* and Q^*, we have that

$$\measuredangle(\mathbf{Per}_P(X), \mathbf{Per}_Q(X)) = \measuredangle Q^*XP^*$$

Notably, when $Q \in BC$ this becomes $\measuredangle(\mathbf{Per}_P(X), BC) = AXP^*$.



Proof. Let $\mathcal{D} = \mathcal{D}_{P,Q}$ and let $\ell = \mathcal{D}^* = P^*Q^*$.

As such, by Proposition 1.12 it follows that $m\colon \mathbf{Per}_P(X) \to P \to P^* \to XP^*$ is projective.

We only need to consider the cases where $P, Q \in \{A, B, C\}$. After which, it follows that m shares 3 points with a rotation plus translation which means it is a rotation and translation.

Let $L = \mathbf{Li}_{\mathcal{D}}(X)$ and let $B^* = \ell \cap CA$, and $C^* = \ell \cap BA$. It remains to show by symmetry $\measuredangle BLC = \measuredangle C^*XB^*$.

By Proposition 1.24 with $P = B$ and $P = C$, it follows that $L_B = XB^* \cap BL^*$ and $L_C = XC^* \cap CL^*$ are on the circumcircle.

We now angle chase to get

$$\begin{aligned} \measuredangle BLC &= \measuredangle LBC + \measuredangle BCL = \measuredangle ABL^* + \measuredangle L^*CA \\ &= \measuredangle AL_CL_B + \measuredangle L_CL_BA = \measuredangle L_CAL_B = \measuredangle C^*XB^*. \end{aligned}$$

□

Remark 1.28. The specialized case when $Q \in BC$ has a specialized proof.

Let $P_A = AP \cap \Omega$, $P_A^* = AP^* \cap \Omega$, $D = AP \cap BC$, $X_{P,A} = XP_A \cap BC$. Note that $P_AP_A^* \parallel BC$.

As such, since $\measuredangle X_{P,A}P_AD = \measuredangle XP_AA = \measuredangle XP_A'D$ and $\measuredangle DX_{P,A}P_A = \measuredangle XP_AP_A' = \measuredangle XAP_A'$, it follows that $\triangle X_{P,A}P_AD \stackrel{-}{\sim} \triangle AP_A^*X$.

Define E on $P_AX_{P,A}$ such that $DE \parallel PX_{P,A}$. Then it is a well known lemma (see second spoiler at https://artofproblemsolving.com/community/c284651h1512821) that

$$\frac{AP^*}{P^*P_A^*} = \frac{PD}{DP_A} = \frac{X_{P,A}E}{EP_A}$$

It thus follows that $\triangle X_{P,A}ED \stackrel{-}{\sim} \triangle AP^*X$, hence it follows that $\measuredangle AXP^* = \measuredangle EDX_{P,A} = \measuredangle(\mathbf{Per}_P(X), BC)$ as desired.

Example 1.29

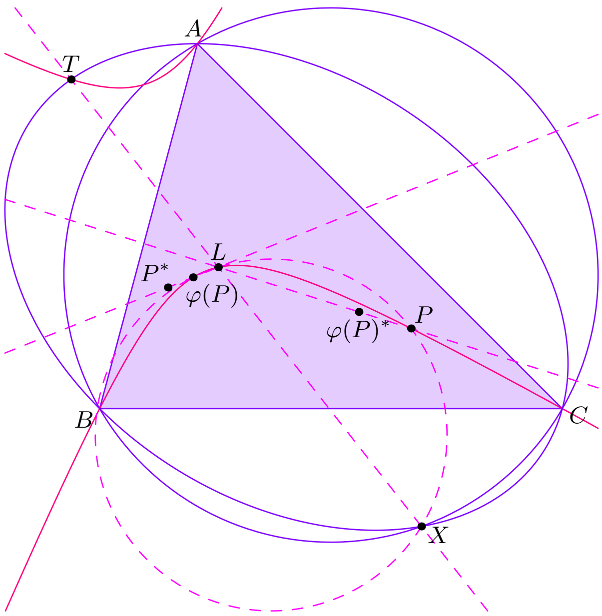
When φ is the isotomic conjugate, the center is the centroid G and $\varphi(G) = G$. When φ is the isogonal conjugate, then the center is the circumcenter O and $\varphi(O) = H$.

We now set $\mathcal{L}$ as the line at infinity. Let's see how Zhihuan Zhang deals with circles now!

Theorem 1.30

Take an isoconjugation φ on triangle $\triangle ABC$ and let $\mathcal{L}^\varphi$ intersect $\Omega = \mathcal{L}^*$ at X as its fourth intersection. Let P be an arbitrary point, and let $(ABCP\varphi(P))$ intersect $\mathcal{L}^\varphi$ at T as its fourth intersection.

Then $TX, P\varphi(P)^*, P^*\varphi(P), (P\varphi(P)X), (ABCP\varphi(P))$ all concur at a point $L = \mathbf{Li}^{\mathcal{L}^\varphi}_{P,\varphi(P)}(X)$.



Proof. By Theorem 1.19 and Proposition 1.7, it follows that

$$TX \cap (ABCP\varphi(P)) = L = \mathbf{Per}_P^{\mathcal{L}^\varphi}(X) \cap \mathbf{Per}_{\varphi(P)}^{\mathcal{L}^\varphi}(X) = P\varphi(P)^* \cap P^*\varphi(P).$$

Now by Proposition 1.27 it follows that

$$\measuredangle PL\varphi(P) = \measuredangle(P\varphi(P)^*, \varphi(P)P^*) = \measuredangle(\mathbf{Per}_{\varphi(P)^*}(X), \mathbf{Per}_{P^*}(X)) = \measuredangle PX\varphi(P)$$

which implies the result. □

We will use this variant of the above theorem.

Corollary 1.31 (Cyclic Variant)
Take an isoconjugation φ on triangle $\triangle ABC$ and let $\mathcal{L}^\varphi$ intersect Ω at X as its fourth point. Let P be an arbitrary point.

Let $L = P^*\varphi(P) \cap P\varphi(P)^*$. Then quadrilateral $P\varphi(P)XL$ is cyclic.

By taking φ as isoconjugation in a limiting sense.

Corollary 1.32
Let P be an arbitrary point, and let $(ABCPP^*)$ intersect (ABC) at T as its fourth intersection point and let X be a point on (ABC). Let TX intersect $(ABCPP^*)$ at L.

Then quadrilateral PP^*XL is cyclic.

Let's also generalize the fact about $\mathbf{Per}_K(X)$ now.

Proposition 1.33
Let $\mathcal{C}$ be a circumconic of $\triangle ABC$, and let P be the perspective center of $\triangle ABC$ and the triangle formed by the polars of AB, BC, CA.

Then $\mathbf{Per}^{\mathcal{C}}_P(X)$ is the trilinear polar of X.

Proof. Let $\triangle A_1B_1C_1$ be the Cevian triangle of P.

Let AP intersect $\mathcal{C}$ at P_A again. Note that quadrilateral ABP_AC is harmonic.

As such, it follows that

$$(B, C; AX \cap BC, \mathbf{Per}^{\mathcal{C}}_P(X) \cap BC) \overset{X}{=} (B, C; A, P_A)_{\mathcal{C}} = -1.$$

It then follows that $\mathbf{Per}^{\mathcal{C}}_P(X), BC, B_1C_1$ concur.

As such, by symmetry the result follows. □

Remark 1.34. This also just gets nuked by a homography.

Example 1.35

We can take $\mathcal{C}$ as the steiner circumellipse (isotomic conjugate of the line at infinity) and P be the centroid.

Likewise, we can take $\mathcal{C}$ as the orthocorrespondent of the line at infinity and P as the orthocenter.

§2 More on Isoconjugates

§2.1 Applications of the Fundamental Theorem

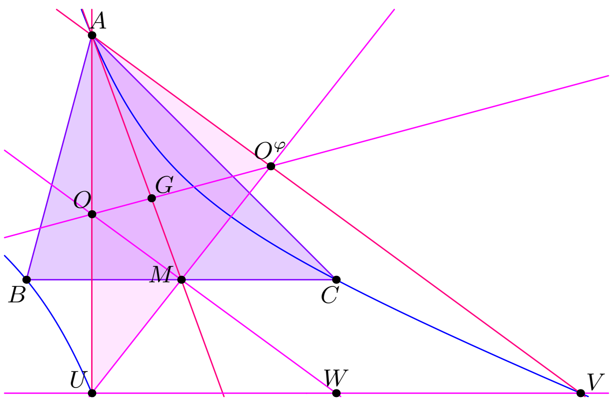
At this point, you may have realized something quirky regarding applying Theorem 1.19 on arbitrary isoconjugations.

We basically don't know much regarding how the operation itself acts. If I told you that φ exchanged N and O, that doesn't immediately give what $\mathcal{L}^{\varphi}$ is, or what else φ exchanges. With only one pair, there's at most two applications possible for the fundamental theorem.

As such, here we derive more lemmas on isoconjugation, to strenghten the fundamental theorem.

Proposition 2.1 (Anticomplement = Isoconjugation Center)
Let φ be an isoconjugation on $\triangle ABC$. Let $\mathcal{L}^{\varphi}$ be the image of the line at infinity. Then a point O is the center of $\mathcal{L}^{\varphi}$ iff $\varphi(O)$ is the anticomplement of O.

(The anticomplement is defined by dilating by -2 through G).



Proof. First suppose O is the center. Now, let AO and $A\varphi(O)$ intersect $\mathcal{L}^{\varphi}$ at U and V respectively.

We then get that $\varphi(U) = \infty_{AV}$ and $\varphi(V) = \infty_{AU}$.

As such, by Example 1.20, the conjugate of $UV \cap \infty_{AV}\infty_{AU}$ is $U\infty_{AU} \cap V\infty_{AV} = A$, so it must be BC.

As such, the midpoint W of UV, M of BC, and O all lie on the polar of ∞_{BC} and thus are concurrent.

Since O is the center, it follows that it is the midpoint of AU. Now, the reflection $\varphi(O)'$ of U across M lies on $AV = A\varphi(O)$. Then, note that G is the centroid of $\triangle AU\varphi(O)'$, so

$\varphi(O)'$ is also the anticomplement of O wrt to G.

By symmetry it follows that $\varphi(O)'$ lies on $B\varphi(O), C\varphi(O)$, so so $\varphi(O)' = \varphi(O)$.

Now, suppose that $\varphi(O)$ is the anticomplement of O. Let O be the center of the unique circumconic $\mathcal{C}$. There exists a map φ' such that $\mathcal{L}^{\varphi'} = \mathcal{C}$. We know then that $\varphi'(O)$ is the anticomplement of O by the converse. It then follows that $\varphi(O) = \varphi'(O)$ and thus $\varphi = \varphi'$. □

Corollary 2.2

Let X_{110} be the focus of the Kiepert Parabola / the antisteiner of the Euler line. Then if an isoconjugation φ exchanges N and O, then $\varphi(X_{110}) \in \mathcal{L}$.

Proof. Let X be the fourth intersection of Ω and $\mathcal{L}^\varphi$. It then follows by considering $(\bullet)' \circ \varphi$ and Theorem 1.19 that

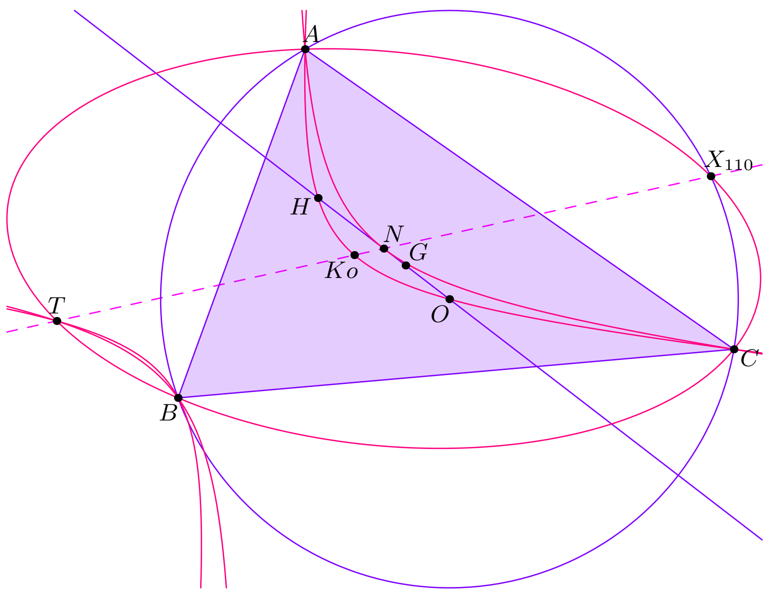
$$\mathbf{Per}_N^{\mathcal{L}^\varphi}(X) = \mathbf{Per}_H(X) = NH.$$

It thus follows that $X = X_{110}$ is the antisteiner point of the Euler line. □

Proposition 2.3

$N, Ko = X_{54}, X_{110}$ are collinear.

(Ko is the isogonal conjugate of N here.)



Proof. Define φ as above. By Proposition 2.1 it follows that N is the center of $\mathcal{L}^\varphi$ and that $X_{110} \in \mathcal{L}^\varphi$. It follows that the reflection T of X_{110} across N lies on $\mathcal{L}^\varphi$.

Let $\mathcal{C} = (ABCNT)$. It then follows by the above that $\mathbf{Li}_{\mathcal{C}}^{\mathcal{L}^\varphi}(X_{110}) = N$, so $\mathbf{Per}_N^{\mathcal{L}^\varphi}(X_{110}) = \overline{NH}$ is also the tangent N to $\mathcal{C}$.

If we take the DIT of conics through A, B, C, T, then we get an involution ψ on intersections with OH.

Then since $\mathcal{L}^\varphi$ is symmetric about N and $(ABCNT)$ is tangent, it follows that this involution is in fact reflection.

As such, T lies on the Jerabek Hyperbola $\mathcal{J} = (ABCOH)$. Note that Ko is the isogonal conjugate of N, so $Ko \in \mathcal{J}$ as well.

It thus follows by Theorem 1.19 and the map $(\bullet)' \circ \varphi$ again that

$$\mathbf{Per}_O^{\mathcal{L}^\varphi}(X_{110}) = \mathbf{Per}_{Ko}(X_{110}) = OKo$$

which implies $\mathbf{Li}_{\mathcal{J}}^{\mathcal{L}^\varphi}(X_{110}) = Ko$, and thus Ko lies on $X_{110}T$ which implies the result. $\square$

Proposition 2.4

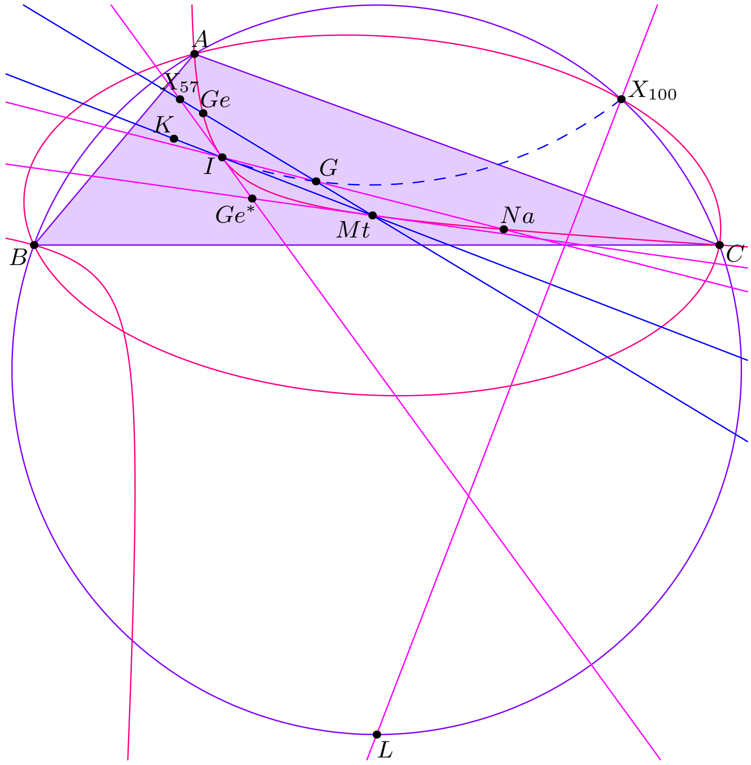
Suppose that an isoconjugation φ exchanges the Gergonne point $Ge = X_7$ and the Mittenpunkt X_9.

Then

(a) It also exchanges I and G.

(b) $X_{100} \in \mathcal{L}^\varphi$ (X_{100} is the anticomplement of Feuerbach point.)

(c) It also exchanges the Nagel point $Na = X_8$ and X_{57}. (The isotomic and isogonal conjugates of X_7 and X_9 respectively.)



Proof. Note that X_{57}, Mt, Ge, and G lie on a line. (Reference: https://mathworld.wolfram.com/IsogonalMittenpunkt.html).

Let X be the fourth intersection of $\mathcal{L}^\varphi$ and Ω. Let $\mathcal{H}$ be the Feuerbach hyperbola, which Ge, I, Na, Mt lie on. Note that $\varphi(\mathcal{H}) = GeMt$.

Applying the fundamental theorem to $(\bullet)^* \circ \varphi$ gives that

$$\mathbf{Per}^{\mathcal{L}^\varphi}_{Ge}(X) = \mathbf{Per}_{X_{57}}(X) = GeX_{57} = GeMt, \mathbf{Per}^{\mathcal{L}^\varphi}_{Mt}(X) = \mathbf{Per}_{Ge^*}(X) = Ge^*Mt.$$

As such, it follows that $\mathbf{Li}^{\mathcal{L}^\varphi}_{\mathcal{H}}(X) = \mathbf{Per}^{\mathcal{L}^\varphi}_{Ge}(X) \cap \mathbf{Per}^{\mathcal{L}^\varphi}_{Mt}(X) = Mt$.

Since G lies on $GeMt$, it follows that $\varphi(G)$ lies on $\mathcal{H}$. Then, by the fundamental theorem it follows that

$$\mathbf{Per}^{\mathcal{L}^\varphi}_{\varphi(G)}(X) = \mathbf{Per}_K(X) = KIMt$$

where K is the symmedian point due to being the trilinear polar. As such, it follows that $\varphi(G) \in KMt$. Notice that $I \in KMt$, so $\varphi(G) = I$ as it can not be Mt.

We now show that $X = X_{100}$.

Once again using the fundamental theorem, we get

$$\mathbf{Per}^{\mathcal{L}^\varphi}_{G}(X) = \mathbf{Per}_I(X) = GI.$$

Let L be the arc midpoint of BC and let I_A be the A-excenter. Then since LX_{100} is the Euler line of $\triangle BI_AC$, it follows that GI, LX_{100}, BC concur so $X = X_{100}$ as a result.

Note that $\mathcal{H}$ is the isogonal conjugate of IX_{57} so IX_{57} is tangent to the hyperbola. Also ntoe that I, G, Na are collinear. As such, it follows that

$$(Ge, Mt; G, \varphi(Na)) \stackrel{\varphi}{=} (Ge, Mt; Na, I)_{\mathcal{H}} \stackrel{I}{=} (Ge, Mt; G, X_{57})$$

which implies the third result. □

Definition 2.5. Take triangle $\triangle ABC$ and points D, E in the same plane. Let $\triangle A_1B_1C_1$ be the Cevian triangle wrt to D. Let $A_2 = AE \cap B_1C_1$ and so forth. Then by Cevian's nest, A_1A_2, B_1B_2, C_1C_2 concur at some point X, which is the **crosspoint** of D and E.

The crosspoint is also the pole of DE wrt to $(ABCDE)$.

Corollary 2.6

Ge^* is the cross point of I and Mt.

Proof. Ge^* lies on IK by taking an isogonal conjugation, which is the tangent to $\mathcal{H}$ at I.

Similarly, since $\mathbf{Li}_{\mathcal{H}}^{\mathcal{L}^{\varphi}}(X) = Mt$ and $Ge^*Mt = \mathbf{Per}_{Mt}^{\mathcal{L}^{\varphi}}(X)$, it follows that Ge^* lies on the tangent at Mt to $\mathcal{H}$.

This implies that Ge^* is the polar of IMt wrt to $(ABCIMt)$ which finishes. □

Corollary 2.7

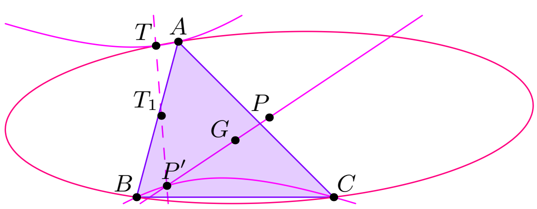
IK is tangent to (IGX_{100}).

Proof. Follows by Corollary 1.31 with P approaching I. □

§2.2 Centers of Conics

Proposition 2.8 (Generalized Poncelet Point)

Take $\triangle ABC$ with circumconic $\mathcal{C}$. Let P be the center of $\mathcal{C}$. Let P' be the anticomplement of P, and let T be a point on $\mathcal{C}$. Then the midpoint T_1 of TP' is the center of $\mathcal{H} = (ABCTP')$.



Proof. Take the isoconjugation φ such that $\mathcal{L}^{\varphi} = \mathcal{C}$.

Then $\varphi(\mathcal{H}) = P\varphi(T)$. Let PT^{φ} intersect $\mathcal{C}$ at K_1, K_2. Then $K_1^{\varphi}, K_2^{\varphi}$ are the intersections of $\mathcal{H}$ with infinity.

We then have that

$$-1 = (P, T^{\varphi}; K_1, K_2) \overset{\varphi}{=} (P', T; K_1^{\varphi}, K_2^{\varphi}) = (T, P'; K_1^{\varphi}, K_2^{\varphi})$$

which implies that the center of $\mathcal{H}$ lies on TP', so it must be the midpoint.

It follows that P' lies on $\mathcal{H}$. □

Proposition 2.9

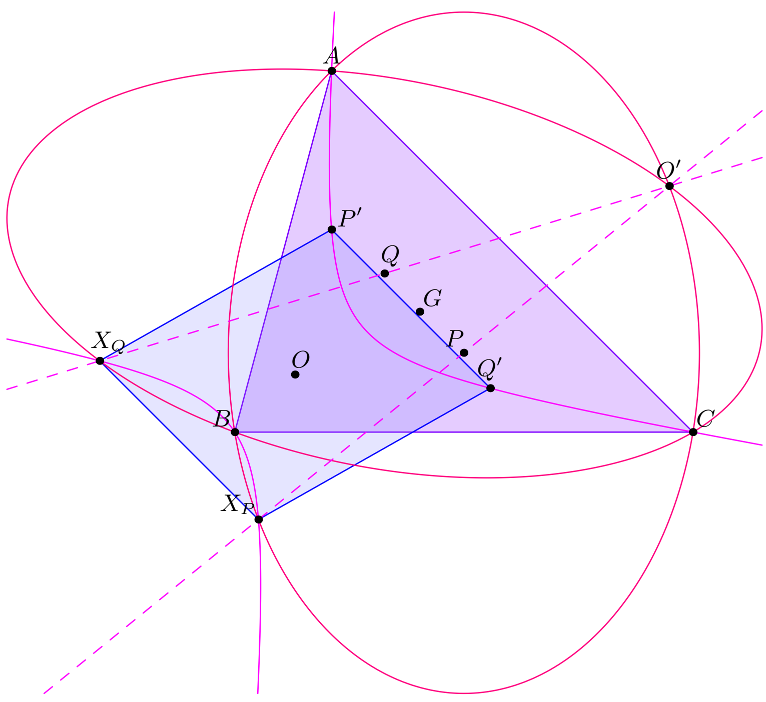
Let φ, ψ be isoconjugations such that P, Q are the centers of $\mathcal{L}^{\varphi}, \mathcal{L}^{\psi}$. Let P', Q' be the anticomplements of P, Q respectively.

Let $\mathcal{C} = (ABCP'Q')$, and let the fourth intersections of $\mathcal{C}$ with $\mathcal{L}^{\varphi}, \mathcal{L}^{\psi}$ be X_P, X_Q respectively.

Then $X_P P, X_Q Q$ concur at the fourth intersection of $\mathcal{L}^{\varphi}$ and $\mathcal{L}^{\psi}$.

Proof. By the previous proposition we get that the midpoint of $X_P P'$, $X_Q Q'$ are the same and the center of $\mathcal{C}$, call it O.

Let G be the centroid and let O' be the anticomplement of O. We can check that O' is the reflection of X_P over P, and the same for X_Q over Q. □



Taking φ as taking an isogonal conjugation, we get the following variant.

Proposition 2.10

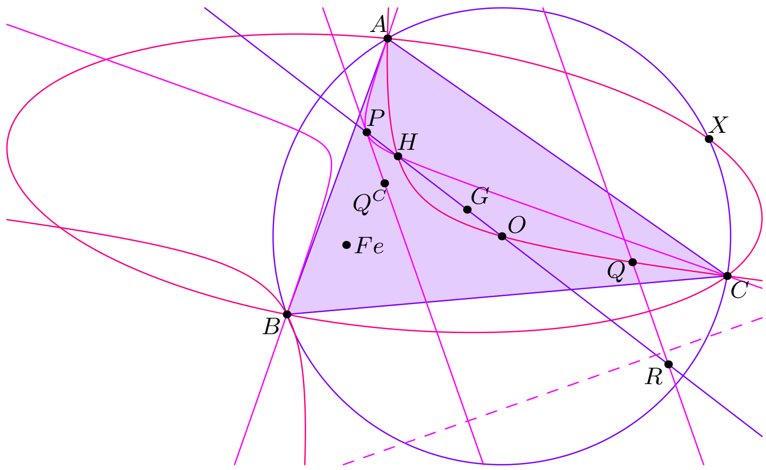
Let φ be an isoconjugation such that P is the center of $\mathcal{L}^\varphi$. Let P' be the anticomplements of P respectively.

Let $\mathcal{C} = (ABCP'H)$, and let the fourth intersections of $\mathcal{C}$ with (ABC) be T.

The antipode of T with respect to (ABC) lies on $\mathcal{L}^\varphi$, which is also the anticomplement of the Poncelet point of $\mathcal{C}$.

Proposition 2.11 (幾何大俠斬 TS 題)

Let P be a point on the Euler line of $\triangle ABC$, and let Q, R be the isogonal conjugate and anticomplement of P respectively. Show that QR is perpendicular to the orthotransversal of P with respect to $\triangle ABC$.



Proof. It's equivalent to show that the tangent at P to $\mathcal{H} = (ABCPH)$ is parallel to QR by Proposition 4.1.

Let Q^C be the complement of Q wrt to $\triangle ABC$. Define the isoconjugation φ which maps Q^C to Q. Then φ has center Q^C.

Let X be the fourth intersection of (ABC) and $\mathcal{L}^\varphi$.

Now note that

$$\mathbf{Per}_P(X) = \mathbf{Per}_{Q^C}^{\mathcal{L}^\varphi} = PQ^C$$

follows by Theorem 1.19. Let $L = \mathbf{Li}_{\mathcal{H}}(X)$. Since $PQ^C \parallel QR$, it remains to show that $\mathbf{Per}_P(X) = PL$ is tangent to $\mathcal{H}$,or that $P = L$.

Then since $\mathbf{Per}_H(X) = HL$, if we show that $X = X_{110}$ then HL is the Euler line and $L = P$ as desired.

As such, by Proposition 2.10 it follows that X is the anticomplement of the Feuerbach point or $X = X_{110}$. $\square$

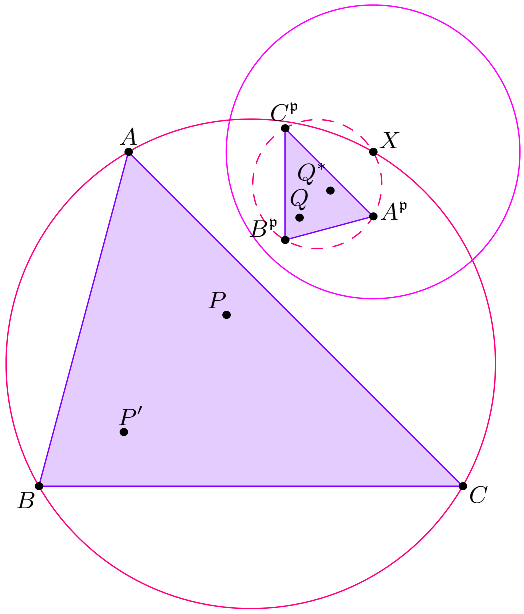
§3 Poles and Polars

Let's talk about poles and polars. We use $\mathfrak{p}_\Gamma$ to refer to the operation of taking a polar wrt to Γ.

Theorem 3.1

Let X be a point on the circumcircle Ω of $\triangle ABC$. Let Γ be a circle with center X. Let (X) denote the polar wrt to Γ. Define $A^\mathfrak{p} = \mathfrak{p}(BC)$ and so forth cyclically. Let P, P^* be a point and its isogonal conjugate. Let Q, Q^* be $\mathfrak{p}(\mathbf{Per}_P(X)), \mathfrak{p}(\mathbf{Per}_{P^*}(X))$ respectively.

Then it follows that $X \in (A^\mathfrak{p} B^\mathfrak{p} C^\mathfrak{p})$ and that $\triangle ABC \cup P \cup P^* \overset{-}{\sim} \triangle A^\mathfrak{p} B^\mathfrak{p} C^\mathfrak{p} \cup Q^* \cup Q$.



Proof. For the first part, note that

$$\measuredangle B^\mathfrak{p} X C^\mathfrak{p} = \measuredangle(\mathfrak{p}(B^\mathfrak{p}), \mathfrak{p}(C^\mathfrak{p})) = \measuredangle CAB = \measuredangle CXB = \measuredangle(\mathfrak{p}(C), \mathfrak{p}(B)) = \measuredangle B^\mathfrak{p} A^\mathfrak{p} C^\mathfrak{p}.$$

For the second part, the similarity of $\triangle ABC, \triangle A^\mathfrak{p} B^\mathfrak{p} C^\mathfrak{p}$ follows from the above.

Then note that

$$\measuredangle Q A^\mathfrak{p} B^\mathfrak{p} = \measuredangle(X\mathfrak{p}(QA^\mathfrak{p}), X(A^\mathfrak{p} B^\mathfrak{p})) = \measuredangle(\mathbf{Per}_P(X) \cap BC) XC = \measuredangle PAC = -\measuredangle P^\star AB$$

which follows by Proposition 1.27. □

The same proposition remains true similarly on

Proposition 3.2

Let X be a point on the steiner circumellipse $\mathcal{S}$ of $\triangle ABC$. Let Γ be a circle with center X. Let (X) denote the polar wrt to Γ. Define $A^{\mathfrak{p}} = \mathfrak{p}(BC)$ and so forth cyclically. Let P, P^* be a point and its isotomic conjugate. Let Q, Q^* be $\mathfrak{p}(\mathbf{Per}_P^{\mathcal{S}}(X)), \mathfrak{p}(\mathbf{Per}_{P'}^{\mathcal{S}}(X))$ respectively.

Then it follows that X is on the Steiner circumellipse $\mathcal{S}^{\mathfrak{p}}$ of $\triangle A^{\mathfrak{p}}B^{\mathfrak{p}}C^{\mathfrak{p}}$. Similarly, $\triangle ABC \cup P \cup P' \mathrel{\bar{\sim}} \triangle A^{\mathfrak{p}}B^{\mathfrak{p}}C^{\mathfrak{p}} \cup Q' \cup Q$.

Proof. Note that the tangent at A to $\mathcal{S}$ is tangent to the steiner circumellipse.

Harmonic chase to get

$$\begin{aligned} A(A, B, C, X)_{\mathcal{S}} &= (\infty_{BC}, B; C, AX \cap BC) \\ &\stackrel{\mathfrak{p}}{=} (A^{\mathfrak{p}}x, A^{\mathfrak{p}}C^{\mathfrak{p}}; A^{\mathfrak{p}}B^{\mathfrak{p}}; A^{\mathfrak{p}}\infty_{B^{\mathfrak{p}}C^{\mathfrak{p}}}) = A^{\mathfrak{p}}(A^{\mathfrak{p}}, B^{\mathfrak{p}}; C^{\mathfrak{p}}, X)_{\mathcal{S}^{\mathfrak{p}}}. \end{aligned}$$

Doing the same for B, C implies that X lies on the steiner inellipse.

Similarly, we can harmonic chase to get that

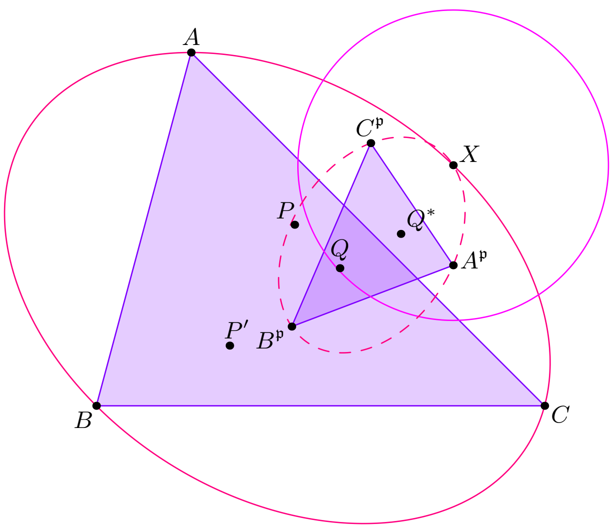
$$A^{\mathfrak{p}}(B^{\mathfrak{p}}, C^{\mathfrak{p}}; A^{\mathfrak{p}}, Q)_{\mathcal{S}^{\mathfrak{p}}} \stackrel{\mathfrak{p}}{=} (C, B; A, \mathbf{Per}_P^{\mathcal{S}}(X)) \stackrel{X}{=} (C, B; A, AP \cap \mathcal{S})_{\mathcal{S}}$$

Then since $(C, B; A, AP \cap \mathcal{S})_{\mathcal{S}} \stackrel{\infty_{BC}}{=} (B, C; A, AP' \cap \mathcal{S})_{\mathcal{S}}$ it follows by symmetry that

$$A^{\mathfrak{p}}(B^{\mathfrak{p}}, C^{\mathfrak{p}}; A^{\mathfrak{p}}, Q)_{\mathcal{S}^{\mathfrak{p}}} = A^{\mathfrak{p}}(C^{\mathfrak{p}}, B^{\mathfrak{p}}; A^{\mathfrak{p}}, Q')_{\mathcal{S}^{\mathfrak{p}}}$$

as desired.

Likewise, this implies the desired similarity. □



Theorem 3.3

Continue from the above theorem. Define $\mathbf{Li}^{\mathcal{S}}_{P,P'}(X) = L, \mathbf{Li}^{\mathcal{S}^{\mathfrak{p}}}_{Q,Q'}(X) = L^{\mathfrak{p}}$. Define $L', L^{\mathfrak{p}'}$ as their isotomic conjugates wrt to $\triangle ABC, \triangle A^{\mathfrak{p}}B^{\mathfrak{p}}C^{\mathfrak{p}}$. Then it follows that

$$\frac{PL'}{P'L'} = \frac{Q'L^{\mathfrak{p}'}}{QL^{\mathfrak{p}'}}$$

Proof. Let $\mathcal{H} = (ABCPP')$ and define $\mathcal{H}^{\mathfrak{p}}$ similarly. Have $'$ mean isotomic conjugation here.

Then we have that

$$\begin{aligned}\frac{PL'}{P'L'} &= (P, P'; L', \infty_{PP'})_{\mathcal{H}'} \overset{'}{=} (\mathbf{Per}^{\mathcal{S}}_{P}(X), \mathbf{Per}^{\mathcal{S}}_{P'}(X); \mathbf{Per}^{\mathcal{S}}_{L'}(X), \mathbf{Per}^{\mathcal{S}}_{\infty_{PP'}}(X))_{L_{\bullet}} \\ &\overset{\mathfrak{p}}{=} (Q, Q', L^{\mathfrak{p}}, \infty'_{QQ'})_{\mathcal{H}^{\mathfrak{p}}} = (Q', Q; L^{\mathfrak{p}'}, \infty_{QQ'})_{\mathcal{H}^{\mathfrak{p}'}} = \frac{Q'L^{\mathfrak{p}'}}{QL^{\mathfrak{p}'}}\end{aligned}$$

as desired. □

§4 Orthotransversals

From before we know that the orthotransversal of X on Ω is its O-permutation line. However, that fact is rather narrow. Let's generalize it some.

Let $\mathcal{O}(X)$ denote the orthotransversal.

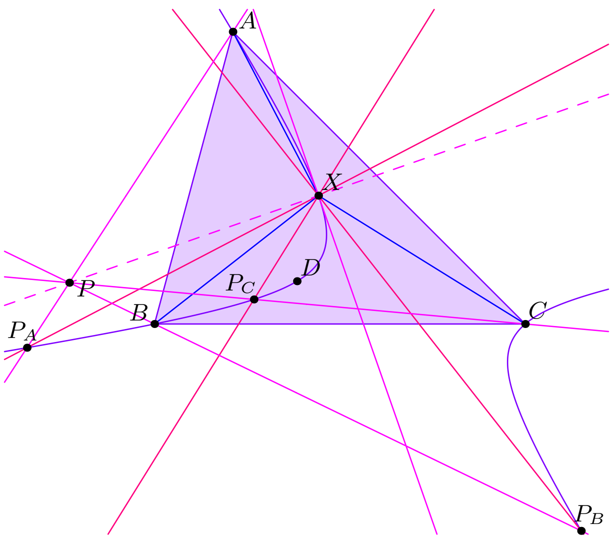
Proposition 4.1

Let X be any point, and let $\mathcal{D} = (ABCHX)$ be the circumrectangular hyperbola through X with respect to $\triangle ABC$. Let P be the point at infinity perpendicular to the tangent at X to $\mathcal{D}$. Then $\mathbf{Per}_P^{\mathcal{D}}(X) = \mathcal{O}(X)$.

Proof. We mainly want to show that if $P_A = AP \cap \mathcal{D}$, then $\angle APP_A = 90°$. Thi follows because the map $A \mapsto P_A$ is an involution and by considering when A is at infinity. It follows that $X_{P,A} \in \mathcal{O}(X)$ by definition. □

Proposition 4.2

Let X be an arbitrary point and let $\mathcal{D}$ be an arbitrary circumconic through A, B, C, X. Then there exists a point P such that $\mathbf{Per}_P^{\mathcal{D}}(X) = \mathcal{O}(X)$, and PX is perpendicular to the tangent at X.



Proof. The map $Y \to f(Y)$ on $\mathcal{D}$ such that about $\angle YXf(Y) = 90°$ is still an involution, and thus $Yf(Y)$ goes through some fixed point P. Taking $Y = A, B, C$ gives $\mathbf{Per}_P^{\mathcal{D}}(X) = \mathcal{O}(X)$, and taking Y approaching X gives the other result. □

Proposition 4.3

Given a point $P \in \mathcal{O}(X)$, there likewise exists a circumconic $\mathcal{D}$ through A, B, C, X such that $\mathbf{Per}_P^{\mathcal{D}}(X) = \mathcal{O}(X)$.

Proof. Take the circumconic through P_A to get that $P = AP_A \cap \mathcal{O}(X)$ then works. □

Proposition 4.4

Given a point X on ω and an isoconjugation φ such that $\varphi(X) \in \mathcal{L}$, it follows that $\mathcal{O}(X) = \mathbf{Per}_H^{\mathcal{L}^\varphi}(X)$, and notably $\varphi(H) \in \mathcal{O}(X)$.

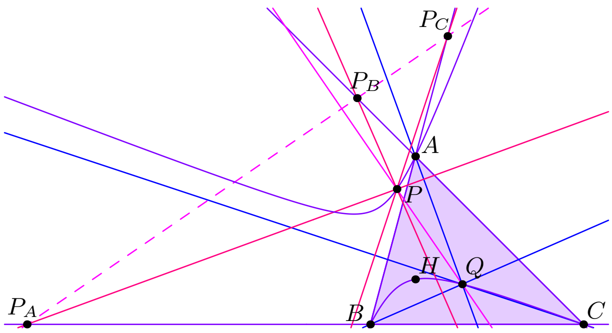
Proof. By Theorem 1.19, we get that

$$\mathbf{Per}_{\varphi(H)}^{\mathcal{L}^\varphi} = \mathbf{Per}_O(X) = \mathcal{O}(X).$$

which finishes. □

Proposition 4.5 (Generalized Orthotransversal)

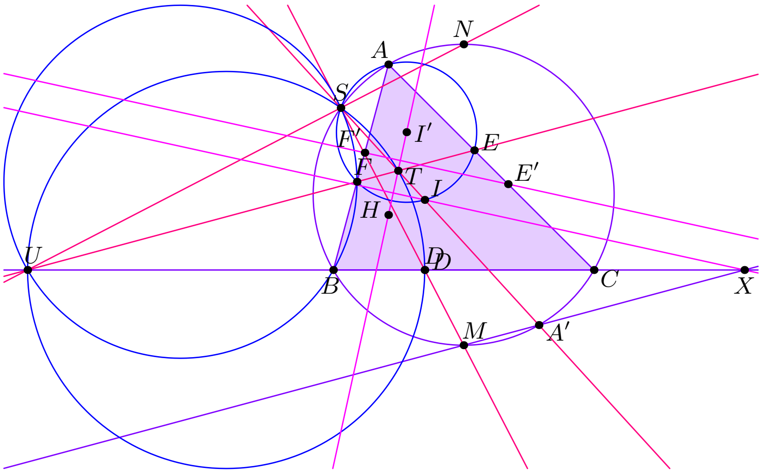
Let $PQABC$ be a circumrectangular hyperbola. Let P_A be a point on BC such that $P_AP \perp AQ$. Define P_B, P_C similarly. Show that P_A, P_B, P_C are collinear and that $P_AP_BP_C \perp PQ$.



Proof. MMP $P = A, B, C, Q$ twice to get both results. □

§5 Heartside / 心世界

Problem 1. Let $\triangle ABC$ have extouch points D', E', F' respectively. Let A' be the A antipode. Let M be the midpoint of arc BC, and let $X = MA' \cap BC$. Show that $XI \parallel E'F'$.



Proof. Let H be the orthocenter of $\triangle ABC$, and let $\triangle DEF$ be the intouch triangle.

Let I' be the reflection of I over EF. Note that I' is the orthocenter of IX. Now, we have that $\mathbf{Per}_I(A') = IX$, so by Proposition 1.27 it follows that $\measuredangle(IX, BC) = AA'I$.

Let $S = (AI) \cap (ABC)$ be the Sharkey-devil point. Let $U = EF \cap BC$, then it follows that $SUFB$ is cyclic as S is the Miquel point of $ABCEFU$. As such,

$$\measuredangle USB = \measuredangle UFB = \measuredangle EFA = \measuredangle CBN = \measuredangle NSB$$

and U lies on SN as well.

Now, since $\measuredangle ASI = 90°$, it follows that $\measuredangle(\mathbf{Per}_I(S), BC) = 90°$ so $\mathbf{Per}_I(S) = ID$.

Let T be the foot from D to EF. Then $\measuredangle UTD = \measuredangle USD = 90°$ so S is the Miquel point of $ABCTDU$ as well.

As such, it follows that the Steiner line of S is HTI'. Then, the $E'F'$ is parallel to the Newton-Gauss line of $BFEC$, which is perpendicular by Gauss-Bodenmiller to HTI'.

As such, it follows that $E'F' \perp HTI'$.

We can now angle chase to get

$$\begin{aligned}
\measuredangle(XI, E'F') &= \measuredangle(XI, BC) + \measuredangle(BC, AI) + \measuredangle(AI, E'F') \\
&= \measuredangle AA'I + \measuredangle(AH + EF) + \measuredangle(EF, TI') \\
&= \measuredangle AA'I + \measuredangle(EF, AA') + \measuredangle(IA', EF) = 0
\end{aligned}$$

as desired. $\square$

Problem 2 (幾何毒書會 X_n 馬拉松 P15)**.** Let $\mathcal{H}$ be the Feuerbach hyperbola and let Sc be the intersection of OH with it. Show that the isogonal conjugate of Sc' is X_{65}, the orthocenter of the intouch triangle.

Proof. We can find that X_{65} lies on $X_{55}X_{56}OI$ and $GeNa$. The latter relation follows because X_{65} is the cross point of H, I and $(H, I; Ge, Na) = -1$.

Then, note that $\mathbf{Li}_{\mathcal{H}}(X_{110}) = Sc$.

Define $\psi = (\bullet)^* \circ \varphi \circ (\bullet)^*$ where φ is isotomic conjugation.

Now we claim that $\psi(X_{110})$ is on the line at infinity $\mathcal{L}$. Note that ψ maps $[x : y : z]$ to $\left[\frac{a^4}{x} : \frac{b^4}{y} : \frac{c^4}{z}\right]$ As such, since $X_{110} = \left[\frac{a^2}{b^2-c^2} ::\right]$, we get that $\psi(X_{110}) = [a^2(b^2 - c^2) ::]$ which has barycentric coordinates summing to 0 as desired.

As such, by Theorem 1.19

$$\mathbf{Per}_{Ge}(X_{110}) = \mathbf{Per}^{\mathcal{L}^\psi}_{X_{56}}(X_{110}) = GeX_{56}, \mathbf{Per}_{Na}(X_{110}) = \mathbf{Per}^{\mathcal{L}^\psi}_{X_{55}}(X_{110}) = NaX_{55}$$

Ergo, it follows that $Sc = GeX_{56} \cap NaX_{55}$, so by Example 1.20 $Sc^* = GeNa \cap X_{55}X_{56}$ as desired. □

Problem 3 (幾何毒書會 X_n 馬拉松 P2)**.** Show that $O = X_3, H = X_4, X_{69}, X_{99}$ are concyclic.

Proof. Note that X_{99} is the trilinear polar of $X_2X_6 = GK$. Since X_{69} is the Symmedian of the anticomplementary triangle, orthocenter, it lies on GK. Since $X_{69} = H^*$, $X_{69} \in (ABCOH)$.

As such,

$$\mathbf{Li}_{O,H}(X_{99}) = (ABCOH) \cap \mathbf{Per}_K(X_{99}) = GK \cap (ABCOH) = X_{69}.$$

By Theorem 1.30 we are done. □

Problem 4. Show that $OH = X_3X_4 \cap X_{69}X_{99}$ is the reflection of $H = X_3$ over $G = X_2$.

Proof. Continue from above. Let X_{74} be the fourth intersection of $\mathcal{H} = (ABCOH)$ with (ABC) which is also the Euler infinity point's isogonal conjugate.

Then it follows that since $\mathbf{Li}_{\mathcal{H}}(X_{99}) = X_{69}$, X_{74} lies on $X_{69}X_{99}$.

As such, it follows that

$$\begin{aligned}\measuredangle GX_{99}X_{69} &= \measuredangle AX_{99}X_{74} + \measuredangle GX_{99}A = \measuredangle(OH, BC) + \measuredangle(BC, \mathbf{Per}_K(X_{99}) \\ &= \measuredangle(OG, BC) + \measuredangle(BC, GX_{69}) = \measuredangle(OG, GX_{69})\end{aligned}$$

so OGH is tangent at G to $(GX_{69}X_{99})$.

Let $T = X_{69}X_{99} \cap OGH$.

As such, by Problem 3 it follows that $TG \cdot TG = TO \cdot TH$ which implies the result. $\square$

Problem 5 (幾何毒書會 X_n 馬拉松 P12)**.** Let $\triangle ABC$ have incenter I, circumcenter O, gergonne point Ge, X_{104} as the isogonal conjugate of OI, X_{999} as the midpoint of I, X_{57}. Show that Ge, X_{104}, X_{999} are collinear.

Proof. Let $\mathcal{H}$ be the Feuerbach hyperbola. Let $X_{104}Ge$ intersect $\triangle ABC$ again at X. We then have that $Ge = \mathbf{Li}_{\mathcal{H}}(X)$.

Now, we can angle chase

$$\measuredangle GeXI = \measuredangle AXGe + \measuredangle AXI = \measuredangle(OI, BC) + \measuredangle(BC, \mathbf{Per}_I(X)) = \measuredangle OIGe$$

so OI is tangent to $(XGeI)$.

On the other hand, X_{56}, Ge, X_{21} lie on a line. As such,

$$\begin{aligned}\measuredangle X_{65}X_{56}Ge &= \measuredangle(OI, \mathbf{Per}_{X_{21}}(X)) = \measuredangle(OI, BC) + \measuredangle(BC, \mathbf{Per}_{X_{21}}(X)) \\ &= \measuredangle AXGe + \measuredangle X_{65}XA = \measuredangle X_{65}XGe\end{aligned}$$

so $X_{56}GeX_{21}X$ is cyclic.

As such, since by magic $-1 = (I, X_{57}, X_{65}, X_{56})$, it follows that X_{999} lies on the radical axis of $(X_{56}GeX_{21}X), (XGeI)$ which finishes. $\square$

Problem 6 (幾何毒書會 X_n 馬拉松 P13)**.** Show that X_7, X_8, X_{21}, X_{99} are concyclic.

Proof. Note that X_{99} is the fourth intersection of $\mathcal{L}^{\varphi}$ and $\mathcal{L}^*$ where φ is isotomic conjugation. Note that X_7, X_8 are also isotomic conjugates of each other.

Then it follows by Theorem 1.30 that X_7, X_8, X_{88}, and $\mathbf{Li}^{\mathcal{L}^{\varphi}}_{X_7,X_8}(X_{99})$ are concyclic. Then we have that

$$\mathbf{Li}^{\mathcal{L}^{\varphi}}_{X_7,X_8}(X_{99}) = X_7(X_8)^{\varphi} \cap X_8(X_7)^{\varphi} = X_7X_{56} \cap X_8X_{55} = X_{21}$$

as desired. $\square$

Problem 7. Show that $X_{21}, X_{55}, X_{56}, X_{110}$ are concyclic.

Proof. Follow from Problem 2.

Define $\psi = (\bullet)^* \circ \varphi \circ (\bullet)^*$ where φ is isotomic conjugation as before. Then $\psi(X_{55}) = X_{56}$ and $\mathbf{Li}^{\mathcal{H}}_{(}X_{110}) = GeX_{56} \cap NaX_{55} = Sc$, so the result follows by Theorem 1.30. $\square$

Problem 8. Let K_θ be the point on the Kiepert hyperbola with angle θ. Then $G, X_{110}, K_\theta^*, K_{-\theta}^*$ are concyclic.

Proof. Let $\mathcal{H}$ be the Kiepert hyperbola. Note that the steiner line of X_{110} is HG, so $\mathbf{Per}_H(X_{110}) = HG$ and thus $\mathbf{Li}_{\mathcal{H}}(X_{110}) = G$. It's well known that $K, K_\alpha, K_{-\alpha}$ are collinear. Similarly, taking the isogonal conjugate of $\mathcal{H}$ gives that $K, K_\alpha^*, K_{-\alpha}^*$ are collinear.

As such, by Example 1.20 it follows that $G, K_\alpha, K_{-\alpha}^*$ are collinear.

As such, $\mathbf{Per}_{K_\theta}(X_{110}) = GK_{-\theta}^*$ and similar.

Thus, we can angle chase with Proposition 1.27 to get

$$\measuredangle K_\theta^* G K_{-\theta}^* = \angle(\mathbf{Per}_{K_{-\theta}}(X_{110}), \mathbf{Per}_{K_\theta}(X_{110})) = \measuredangle K_\theta^* X_{110} K_{-\theta}^*.$$

□

> **Remark 5.9.** There's also a bary sol in the original handout considering $K_\theta^* \to K_{-\theta}^*$.

Problem 10. $N, Ko = X_{54}, X_{110}$ are collinear.

Proof. Let $\mathcal{H}$ be the Jerabek hyperbola. Note that $\mathbf{Li}_{N,H}(X_{110}) = N, \mathbf{Li}_{\mathcal{H}}(X_{110}) = O$.

It's known that the orthotransversal of N is perpendicular to OKo, so OKo is parallel to the tangent at N to $(ABCNH)$ which is $\mathbf{Per}_N(X_{110})$.

As such, we can angle chase with Proposition 1.27 to get

$$\measuredangle AX_{110}N = \measuredangle(BC, \mathbf{Per}_{Ko}((X_{110})) = \measuredangle(BC, OKo) = \measuredangle(BC, \mathbf{Per}_N(X_{110}) = \measuredangle AX_{110}Ko$$

as desired. □

Problem 11. Define K as the symmedian point and let G_H be the centroid of the intouch triangle. Show that KG is tangent to $(KX_{69}X_{110})$.

Proof. Let $\mathcal{H}$ be the Jerabek hyperbola of $\triangle ABC$¿

Consider $\psi = (\bullet)' \circ (\bullet)^o \circ (\bullet)^*$ where $'$ denotes isotomic conjugation and o indicates the orthocorrespondent. Then we can check that $\psi(K) = X_{69})$.

We can check that the trilinear polar of X_{110}' is the Euler line, so $(X_{110}')^o$ lies on the Steiner circumellipse so $\psi(X_{110})$ is at infinity.

As such, by Theorem 1.19 we get

$$\mathbf{Per}_G(X_{110}) = \mathbf{Per}_{X_{69}}^{\mathcal{L}^\psi}(X_{110}) = GKX_{69}$$

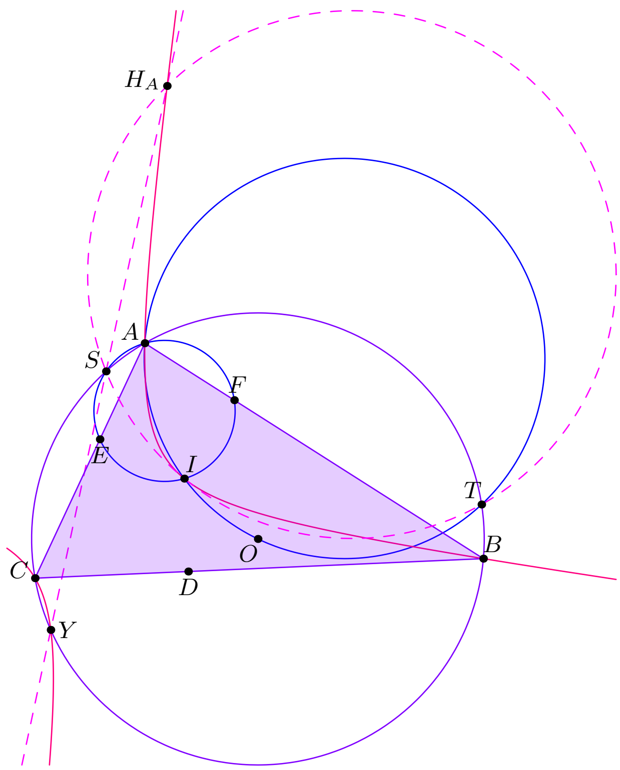
and thus $\mathbf{Li}_{\mathcal{H}}^{\mathcal{L}^{\psi}}(X_{110}) = K$.

We can check that G_H is the crosspoint of the orthocenter H and K, so it follows by Proposition 1.27 that

$$\begin{aligned}\measuredangle X_{69}X_{110}K &= \measuredangle(\mathbf{Per}_G(X_{110}), \mathbf{Per}_{X_{69}^*}(X_{110})) \\ &= \measuredangle(\mathbf{Per}_{X_{69}}^{\mathcal{L}^{\psi}}(X_{110}), \mathbf{Per}_{K}^{\mathcal{L}^{\psi}}(X_{110})) = \measuredangle X_{69}KG_H.\end{aligned}$$

as desired. □

Problem 12. Let $\triangle ABC$ have incenter I, circumcenter O, and intouch triangle $\triangle DEF$. Let H_A be the orthocenter of $\triangle BIC$. Let $S = (AEF) \cap (ABC)$ and let $T = (AIO) \cap (ABC)$. Show that T, H_A, I, S are concyclic.



Proof. Let $\mathcal{H} = (ABCHIH_A)$ be the Feuerbach hyperbola of $\triangle ABC$.

Then it follows that $\mathbf{Per}_I(S) = ID = IH_A$ which implies that $\mathbf{Li}_{\mathcal{H}}(S) = H_A$.

Let Y be the isogonal conjugate of ∞_{OI}, which is also the fourth intersection of (ABC) and $\mathcal{H}$. It then follows that $Y \in SH_A$.

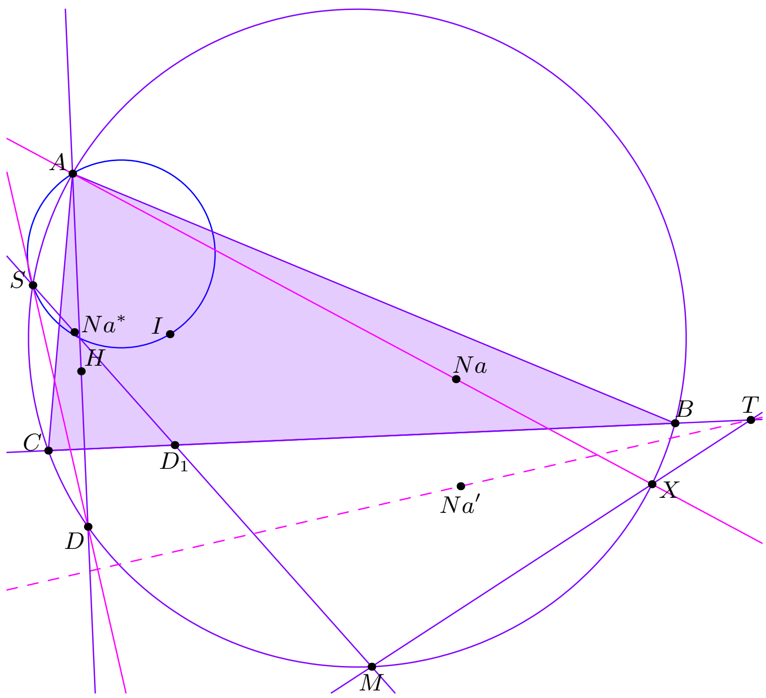
As such, it follows that

$$\measuredangle H_AST = \measuredangle YST = \measuredangle YAT = \measuredangle YAO + \measuredangle OAT = \measuredangle(H_AI, OI) + \measuredangle(OIT) = \measuredangle H_AIT$$

as desired. □

Problem 13 (GAMO P3)**.** Let $\triangle ABC$ have incenter, orthocenter, and Nagel point I, H, Na respectively. Let $D = AH \cap (ABC)$, let $S = (AI) \cap (ABC)$. Let Na' be the reflection of Na over BC. Let M be midpoint of arc BC and let $ANa \cap (ABC) = X$.

Show that the perpendicular from Na' to SD, BC, and MX concur.



Proof. Define $T = MX \cap BC$. It remains to show that $Na'T \perp SD$, which since $AD \perp BC$ is the same as $\measuredangle(BC, TNa') = \measuredangle ADS$.

Now, notice that

$$\measuredangle(BC, TNa') = \measuredangle(NaT, BC) = \measuredangle(\mathbf{Per}_{Na}(M), BC) = \measuredangle AMNa^*$$

Let D_1 be the foot from I to BC. Since Na^* is the exsimilicenter of I and O, it follows that it lies on D_1M and thus on SM.

As such, $\measuredangle(BC, TNa') = \measuredangle AMNa^* = \measuredangle AMS = \measuredangle ADS$ which finishes. $\square$