Centroid & Orthocenter

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We will discuss some of the most popular triangle centers and their configurations, investigate their properties and prove results in their configurations.

§1 Centroid

Let's talk about the centroid configuration. A couple of definitions first.

Definition 1.1. In $\triangle ABC$, choose a point D on side $\overline{BC}$. If $\overline{AD}$ bisects $\overline{BC}$, then $\overline{AD}$ is called A-**median** of $\triangle ABC$.

In $\triangle ABC$, there are three medians, emanating from each vertex. It can be shown using **Ceva's Theorem** that all the three medians are concurrent. This point of concurrency is called the **Centroid**.

Definition 1.2. The centroid of a $\triangle ABC$ is the intersection of the three medians of the triangle. This point is usually denoted by G.

A popular naming convention is to let M_A, M_B and M_C be the midpoints of sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$.

Definition 1.3. $\triangle M_AM_BM_C$ is called the **medial triangle**.

There are also names for the sides of the medial triangle.

Definition 1.4. Suppose $\triangle M_AM_BM_C$ is the medial triangle of $\triangle ABC$. Then line M_BM_C is called the A-**midline** of $\triangle ABC$.

Similar to the number of medians, there are three midlines in a triangle. Now let's take a look at a few properties of this configuration.

§1.1 Lots of Parallelograms!

There are three parallelograms in the centroid configuration, one associated with each vertex of the triangle. Namely, $BM_AM_BM_C$, $CM_BM_CM_A$ and $AM_CM_AM_B$ are parallelograms.

Proposition 1.5

The quadrilaterals $BM_AM_BM_C$, $CM_BM_CM_A$ and $AM_CM_AM_B$ are parallelograms.

How do we prove this? This fact follows from **Thales' Proportionality Theorem**. $\triangle AM_CM_B \sim \triangle ABC$ by SAS similarity criterion, and therefore $M_CM_B \parallel BC$. Similarly, we have the other corresponding sides of the medial triangle parallel to the sides of $\triangle ABC$. Since, $2\overline{M_CM_B} = \overline{BC} \implies \overline{M_CM_B} = \overline{BM_A}$ and similarly, $\overline{M_AM_B} = \overline{BM_C}$ which proves the fact that all three quadrilaterals are parallelograms.

§1.2 Areas

Proposition 1.6

The area of the medial triangle is a quarter of the area of the original triangle. That is

$$[\triangle M_AM_BM_C] = \frac{1}{4}\,[\triangle ABC]$$

The key idea to prove this is to observe the fact $\triangle AM_CM_B \cong \triangle M_CBM_A \cong \triangle M_BM_AC \cong M_AM_BM_C$. All of these congruences follow from SSS Congruence Criterion. As a result the areas of these four triangles are the same. Now,

$$\begin{aligned}[\triangle ABC] &= [\triangle AM_CM_B] + [\triangle M_CBM_A] + [\triangle M_BM_AC] + [\triangle M_AM_BM_C] \\ &= 4\,[\triangle M_AM_BM_C]\end{aligned}$$

from which the proposition follows. Another important fact about areas in this configuration is the following:

Proposition 1.7

In $\triangle ABC$, let G be the centroid and $\triangle M_AM_BM_C$ be the medial triangle. Then the areas of the triangles $\triangle BGM_A$, $\triangle M_AGC$, $\triangle CGM_B$, $\triangle M_BGA$, $\triangle AGM_C$ and $\triangle M_CGB$ are all equal.

There are two crucial steps in realizing this fact. The first step is easy to see. The areas of adjacent triangles $\triangle BGM_A$ and $\triangle M_AGC$ are equal. This follows from the fact that the altitudes and bases of both triangles are equal in length. The next step is to show that the areas of $\triangle AM_CM_A$ and $\triangle CM_CM_A$ are equal. This is because lines $\overline{AC} \parallel \overline{M_AM_C}$ so the altitudes of these triangles are the same.

$$\begin{aligned}[\triangle AM_CM_A] &= [\triangle CM_CM_A] \\ [\triangle AGM_C] + [\triangle M_CGM_A] &= [\triangle M_CGM_A] + [\triangle M_AGC] \\ [\triangle AGM_C] &= [\triangle M_AGC]\end{aligned}$$

which implies that all six triangles have equal areas. As a corollary we have the important fact

Corollary 1.8

In $\triangle ABC$, let G be the centroid and $\triangle M_AM_BM_C$ be the medial triangle. Then,

$$[\triangle GBC] = [\triangle GCA] = [\triangle GAB] = \frac{1}{3}\,[\triangle ABC]$$

§1.3 Centroid Division

This is a very important fact in the centroid configuration.

Proposition 1.9

In $\triangle ABC$, let G be the centroid and $\triangle M_A M_B M_C$ be the medial triangle. Then

$$\frac{\overline{AG}}{\overline{GM_A}} = \frac{\overline{BG}}{\overline{GM_B}} = \frac{\overline{CG}}{\overline{GM_C}} = 2$$

There are several proofs to this fact. There are two proofs that seem very interesting and are included here.

The first proof argues with areas. The idea is to consider the triangles $\triangle AM_CG$ and $\triangle AGC$. From the fact proven before,

$$\frac{[\triangle AM_CG]}{[\triangle AGC]} = \frac{1}{2}$$

However, the altitudes of these two triangles are the same. Hence we have $\frac{\overline{M_CG}}{\overline{GC}} = \frac{1}{2}$. This proves the proposition.

The other proof relies on a very clever construction. We construct a point X by extending the line AG such that $\overline{GM_A} = \overline{M_AX}$. Because triangles $\triangle BGM_A \cong \triangle CXM_A$, it follows that $\overline{BG} \parallel \overline{XC}$. This implies that $\triangle AGM_B \sim \triangle AXC$, which implies that G is the midpoint of segment $\overline{AX}$. Hence,

$$\overline{AG} = \overline{GX} = 2\overline{GM_A} \implies \frac{\overline{AG}}{\overline{GM_A}} = 2$$

which proves the fact.

This fact reveals something deeper in this configuration.

Corollary 1.10

There exists a homothetic transformation that maps $\triangle ABC$ to $\triangle M_A M_B M_C$ centered at point G with scaling factor $-\frac{1}{2}$

This immediately follows from the proposition about centroid division. It is not immediately evident how we will use this fact, but we will see how important this is when we introduce the Nine-Point Circle.

§1.4 Medians

A fact worth remembering is the length of the median of a triangle.

Proposition 1.11

In $\triangle ABC$, let G be the centroid and $\triangle M_A M_B M_C$ be the medial triangle.

$$\overline{AM_A} = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$
$$\overline{BM_B} = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$$
$$\overline{CM_C} = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

Let's prove the first one. The result follows from **Law of Cosines** on $\triangle ABM_A$ with angle $\angle ABM_A$ and on $\triangle ABC$ with angle $\angle ABC$.

$$\begin{aligned}\overline{AM_A}^2 &= c^2 + \frac{1}{4}a^2 - ac\cos\angle ABC \\ &= c^2 + \frac{1}{4}a^2 + ac\left(\frac{b^2-a^2-c^2}{2ac}\right) \\ &= \frac{2b^2+2c^2-a^2}{4} \implies \overline{AM_A} = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}\end{aligned}$$

proving the fact. There are a few more interesting facts that show up once we introduce the orthocenter in this diagram. So, let's study the orthocenter configuration first!

§2 Orthocenter

The orthocenter configuration is another popular triangle center configuration. Let's define a few things first.

> **Definition 2.1.** In $\triangle ABC$, choose a point D on side $\overline{BC}$ such that $\overline{AD} \perp \overline{BC}$, then $\overline{AD}$ is called the A-**altitude** of $\triangle ABC$.

In $\triangle ABC$, there are three altitudes emanating from each vertex of the triangle. These altitudes are concurrent and again this can be shown using **Ceva's Theorem**. The point of concurrency is called the **Orthocenter**.

> **Definition 2.2.** The Orthocenter of a $\triangle ABC$ is the intersection of the three altitudes of the triangle. This point is usually denoted by H.

The naming convention for the feet of perpendiculars is to choose D, E, and F or H_A, H_B, and H_C. We will choose the latter, so that our references are more specific.

> **Definition 2.3.** The $\triangle H_AH_BH_C$ is called the **orthic triangle**.

The orthocenter configuration is very rich in properties, as we will see in this section.

§2.1 Lots of Cyclic Quadrilaterals!

> **Proposition 2.4**
> In $\triangle ABC$, let H be the orthocenter and $\triangle H_AH_BH_C$ be the orthic triangle. Then the quadrilaterals BH_CHH_A, CH_AHH_B, AH_BHH_C, ABH_AH_B, BCH_BH_C and CAH_CH_A are all cyclic quadrilaterals.

That is a total of six cyclic quadrilaterals! Let's see how we can prove this fact. By definition of altitudes, we have

$$\begin{aligned}\angle HH_AB = \angle AH_AB = 90^\circ \\ \angle HH_CB = \angle CH_CB = 90^\circ \\ \implies \angle HH_AB + \angle HH_CB = 180^\circ\end{aligned}$$

This shows that the quadrilateral BH_CHH_A is cyclic. Similarly, CH_AHH_B and AH_BHH_C are cyclic quadrilaterals too. Next, observe that

$$\angle AH_AB = 90^\circ$$

$$\begin{aligned} \angle AH_BB &= 90^\circ \\ \implies \angle AH_AB &= \angle AH_BB \end{aligned}$$

which implies that ABH_AH_B is a cyclic quadrilateral. Similarly, BCH_BH_C and CAH_CH_A are cyclic quadrilaterals too, proving the proposition.

Configurations having many cyclic quadrilaterals are a treasure trove for angle chasing. As we shall see, we can show many properties using only angle chasing.

§2.2 Incenter of Orthic Triangle

Proposition 2.5

In $\triangle ABC$, let H be the orthocenter and $\triangle H_AH_BH_C$ be the orthic triangle. Then H is the **Incenter** of the orthic triangle.

We haven't defined an incenter yet, so let's do that first.

> **Definition 2.6.** In $\triangle ABC$, the **Incenter** of $\triangle ABC$ is the point of concurrency of the three internal angle bisectors of $\triangle ABC$. This point is usually denoted by I and is the center of the inscribed circle of $\triangle ABC$, which is called the **Incircle**.

To prove the existence of the Incenter, we can use the **Trigonometric Form** of **Ceva's Theorem**.

Returning to the proposition, if we want to show that H is the incenter of $\triangle H_AH_BH_C$, then we actually want to show that $\overline{AH_A}$ is the angle bisector of angle $\angle H_CH_AH_B$. Let's show this.

$$\begin{aligned} \angle H_CH_AH &= \angle H_CBH \\ &= \angle ABH_B \\ &= \angle AH_AH_B \end{aligned}$$

These angle equalities follow because the quadrilaterals H_CBH_AH and ABH_AH_B are cyclic. Hence, $\overline{AH_A}$ is the angle bisector of $\angle H_CH_AH_B$. Similarly, $\overline{BH_B}$ and $\overline{CH_C}$ are the angle bisectors of $\angle H_AH_BH_C$ and $\angle H_BH_CH_A$ proving that H is indeed the incenter of $\triangle H_AH_BH_C$.

§2.2.1 Exercises

Exercise 2.7. In $\triangle ABC$, let H be the orthocenter and $\triangle H_AH_BH_C$ be the orthic triangle. Reflect point H_A over the line BH to point X. Show that X lies on the line H_BH_C.

Exercise 2.8. In $\triangle ABC$, let H be the orthocenter. Show that the orthocenters of $\triangle BHC$, $\triangle CHA$ and $\triangle AHB$ are points A, B and C, respectively.

For the next two properties that we will investigate, it turns out that reflecting the orthocenter over the sides and midpoints leads to more cyclic quadrilaterals. Of course, there is a motivation to these properties and they all point to a deeper result about **Nine-Point Circles** that we will see in the next section.

§2.3 Reflection of H over $\overline{BC}$

Proposition 2.9

In $\triangle ABC$, let H be the orthocenter. Reflect H over the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ to H_1, H_2 and H_3. Then the three points H_1, H_2 and H_3 lie on the circumcircle of $\triangle ABC$.

So, we want to show that the quadrilateral ABH_1C is cyclic. We could show the same for the other points H_2 and H_3 and we would be done. Due to the definition of point H_1, we have $\angle H_1CB = \angle HCB$ and $\angle H_1BC = \angle HBC$. Therefore,

$$\begin{aligned}\angle H_1CB &= \angle HCB \\ &= \angle HAB \\ &= \angle H_1AB\end{aligned}$$

which implies that ABH_1C is cyclic. To prove the above angle equalities we additionally use the fact that AH_CH_AC is a cyclic quadrilateral.

§2.4 Reflection of H over M_A

Proposition 2.10

In $\triangle ABC$, let H be the orthocenter. Reflect H over the midpoints of sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ to A', B' and C'. Then the three points A', B' and C' lie on the circumcircle of $\triangle ABC$.

Again it suffices to show that $ABA'C$ is cyclic. A similar argument would work for the other points. Because A' is the reflection of H over the midpoint of $\overline{BC}$, $BHCA'$ is a parallelogram. Since $\overline{BH} \perp \overline{AC}$ and $\overline{BH} \parallel \overline{A'C}$, it follows that $\overline{A'C} \perp AC$. Therefore, $\angle ACA' = 90^\circ$. Similarly, $\angle ABA' = 90^\circ$. Therefore $ABA'C$ is cyclic.

§2.5 Centers of Cyclic Quadrilaterals

A few more interesting properties arise when we draw the centers of the six cyclic quadrilaterals in the orthocenter configuration.

Proposition 2.11

In $\triangle ABC$, let H be the orthocenter and $\triangle H_AH_BH_C$ be the orthic triangle. Then the centers of the circles $\odot(BH_CH_BC)$, $\odot(CH_AH_CA)$ and $\odot(AH_BH_AB)$ are the midpoints of the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$.

Proposition 2.12

In $\triangle ABC$, let H be the orthocenter and $\triangle H_AH_BH_C$ be the orthic triangle. Let E_A, E_B and E_C be the midpoints of segments $\overline{AH}$, $\overline{BH}$ and $\overline{CH}$. Then, E_A, E_B and E_C are the centers of the circles $\odot(AH_CHH_B)$, $\odot(BH_CHH_A)$ and $\odot(CH_AHH_B)$.

These results immediately follow from **Thales' Theorem** on diameter of a circle, because $\overline{BC}$ is the diameter of circle $\odot(BH_CH_BC)$ and similarly for the other circles. Similarly, $\overline{AH}$ is the diameter of the circle $\odot(AH_CHH_B)$ and similarly for the other circles and the result follows.

§2.6 Orthogonal Circles

Something even more remarkable is that the six circles that we identified in the orthocenter configuration are pairwise orthogonal to each other.

Proposition 2.13

In $\triangle ABC$, let H be the orthocenter and $\triangle H_A H_B H_C$ be the orthic triangle. Let M_A, M_B and M_C be the midpoints of segments $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$. Let E_A, E_B and E_C be the midpoints of segments $\overline{AH}$, $\overline{BH}$ and $\overline{CH}$. Then,

1. segments $\overline{M_A H_C}$ and $\overline{M_A H_B}$ are tangent to $\odot(AH_C H H_B)$.
2. segments $\overline{E_A H_C}$ and $\overline{E_A H_B}$ are tangent to $\odot(BH_C H_B C)$.

To prove both facts, we need only show that $\angle E_A H_C M_A = 90^\circ$. This follows from angle chasing because

$$\begin{aligned}\angle AH_C E_A &= 90^\circ - \frac{1}{2}\angle AE_A H_C \\ &= 90^\circ - \angle AH_B H_C \\ &= 90^\circ - \angle ABC \\ &= 90^\circ - \frac{1}{2}\angle H_C M_A C \\ &= 90^\circ - \frac{1}{2}(180^\circ - 2\angle M_A H_C C) \\ &= \angle M_A H_C C\end{aligned}$$

This implies $\angle E_A H_C M_A = \angle AH_C C = 90^\circ$ proving the proposition. Similarly, we have $\angle E_A H_B M_A = 90^\circ$ from which both of the statements follow.

Immediate corollaries of this fact are the following:

Corollary 2.14

Pairs of circles $\big(\odot(AH_C H H_B), \odot(BH_C H_B C)\big)$, $\big(\odot(BH_A H H_C), \odot(CH_A H_C A)\big)$, and $\big(\odot(CH_B H H_A), \odot(AH_B H_A B)\big)$ are orthogonal to each other.

Corollary 2.15

Quadrilaterals $E_A H_C M_A H_B$, $E_B H_A M_B H_C$ and $E_C H_B M_C H_A$ are kites.

§2.6.1 Exercises

Exercise 2.16 (2019 RMO P5)**.** In an acute angled triangle ABC, let H be the orthocenter, and let D, E, F be the feet of altitudes from A, B, C to the opposite sides, respectively. Let L, M, N be the midpoints of the segments AH, EF, BC respectively. Let X, Y be the feet of altitudes from L, N on to the line DF respectively. Prove that XM is perpendicular to MY.

§3 Nine-Point Circle

It's time to piece together all the properties in a grand revelation. The **Nine-Point Circle Theorem** is a very celebrated result and we can easily prove this using all the properties described previously. Let's state the theorem first.

Theorem 3.1 (Nine-Point Circle)
In $\triangle ABC$, let $\triangle H_A H_B H_C$ be the orthic triangle, $\triangle M_A M_B M_C$ be the medial triangle. Let H be the orthocenter of $\triangle ABC$ and let E_A, E_B and E_C be the midpoints of segments $\overline{AH}$, $\overline{BH}$ and $\overline{CH}$. Then all the nine points H_A, H_B, H_C, M_A, M_B, M_C, E_A, E_B and E_C lie on a circle.

We use the fact that the reflections of the orthocenter H over the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ and the reflections over the midpoints M_A, M_B and M_C lie on the circle $\odot(ABC)$. Now consider a homothetic transformation centered at point H with factor $\frac{1}{2}$. Under this transformation the circle $\odot(ABC)$ is precisely shrunk to the circle that passes through the nine points described above. This proves the result!

Definition 3.2. The **Nine-Point Center** is the center of the Nine-Point Circle. This point is usually denoted by N_9.

Due to the existence of such a homothetic transformation that maps $\odot(ABC)$ to $\odot(N_9)$, we have the following immediate corollaries:

Corollary 3.3
The point N_9 bisects the segment $\overline{OH}$.

Corollary 3.4
If we pick a point P on the circle $\odot(ABC)$, the Nine-Point Circle $\odot(N_9)$ bisects the segment $\overline{HP}$.

Notice how we have described two homothetic transformations that map $\odot(ABC)$ to $\odot(N_9)$. The first is centered at the centroid G with factor $\frac{-1}{2}$, and the other is centered at the orthocenter H with factor $\frac{1}{2}$. Such a mapping indicates that all four points H, N_9, G and O are collinear!

§3.1 Euler Line

Proposition 3.5
In $\triangle ABC$, let H, N_9, G and O be the orthocenter, nine-point center, centroid and circumcenter of the triangle. Then all four points are collinear and the line that passes through these points is called the **Euler Line**.

This is a very remarkable result, in the sense that these arbitrarily defined four distinct triangle centers are tied up in a deeper result revealing a collinearity between these points. The proof for the existence of collinearity immediately follows from the homothetic transformation. In fact, since we know precisely the scaling factor of the homothetic transformation, we can even comment on the ratios in which these points divide the segment $\overline{OH}$.

Corollary 3.6

The points H, N_9, G and O are collinear in this same order and G divides $\overline{HO}$ in the ratio $2:1$ whereas N_9 bisects the segment $\overline{HO}$.

If we compute the distance $\overline{OH}$, we can easily compute the distances between any of these four triangle centers since we already know the ratios. So, let's take a look at how we can compute the distance.

§3.2 Calculating the distance $\overline{OH}$

There are two popular ways to compute this distance. We will set up a few preliminaries so that we can easily compute this. The following result about distances in the orthocenter configuration is super-handy!

Proposition 3.7

In $\triangle ABC$, let H be the orthocenter and $\triangle H_AH_BH_C$ be the orthic triangle. Let R be the circumradius of $\triangle ABC$. Then

1. $\overline{AH} = 2R\cos A$
2. $\overline{HH_A} = 2R\cos B\cos C$
3. $\overline{AH_A} = 2R\sin B\sin C$
4. $\overline{H_BH_C} = a\cos A$

We will leave the proof of this to the reader, as this is a simple trigonometric exercise. (**Hint.** Use the **Extended Law of Sines**)

§3.2.1 Exercises

Exercise 3.8. Prove the three results,

1. $\overline{AH} = 2R\cos A$
2. $\overline{HH_A} = 2R\cos B\cos C$
3. $\overline{AH_A} = 2R\sin B\sin C$
4. $\overline{H_BH_C} = a\cos A$

Exercise 3.9. Show that $AH = 2OM_A$, where M_A is the midpoint of side $\overline{BC}$.

Exercise 3.10. Prove that, quadrilaterals AE_AM_AO and E_AHM_AO are parallelograms, where E_A is the midpoint of side $\overline{AH}$. Suppose O' is the reflection of O over $\overline{BC}$. Show that $AHO'O$ is a parallelogram too.

§3.2.2 Stewart's Theorem

Another result that is a powerful tool in computing cevian lengths if you know the ratio in which the cevian divides the side of the triangle is **Stewart's Theorem**.

Theorem 3.11 (Stewart's Theorem)
In $\triangle ABC$, let D be a point on side $\overline{BC}$ such that $\overline{BD} = m$ and $\overline{DC} = n$. Then,

$$b^2m + c^2n = a(d^2 + mn)$$

We will leave the proof of this to the reader as well. (**Hint.** Use the **Law of Cosines**)
The first way to compute $\overline{OH}$ uses the **Power of Point Theorem**.

Lemma 3.12 (Distance $\overline{OH}$)
In $\triangle ABC$, let H and O be the orthocenter and the circumcenter. Then,

$$\overline{OH}^2 = R^2\left(1 - 8\cos A\cos B\cos C\right)$$

The proof is straightforward. Suppose H_1 is the reflection of H over $\overline{BC}$. Then $\overline{HH_1}$ $= 4R\cos B\cos C$. By Power of Point Theorem,

$$\begin{aligned}\overline{AH}\cdot\overline{HH_1} &= R^2 - \overline{OH}^2\\ (2R\cos A)\cdot(4R\cos B\cos C) &= R^2 - \overline{OH}^2\\ \implies \overline{OH}^2 &= R^2\left(1 - 8\cos A\cos B\cos C\right)\end{aligned}$$

which proves the result. The second way uses **Stewart's Theorem** to compute $\overline{OG}^2$. Since $\overline{OH}^2 = 9\overline{OG}^2$, we can easily compute $\overline{OH}^2$ if we know $\overline{OG}^2$.

Lemma 3.13 (Distance $\overline{OG}$)
In $\triangle ABC$, let G and O be the centroid and circumcenter. Then,

$$\overline{OG}^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$$

We will leave the proof of this as an exercise as well.

§3.2.3 Exercises

Exercise 3.14. Prove **Stewart's Theorem** using **Law of Cosines**.

Exercise 3.15. Prove the result,

$$\overline{OG}^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$$

(**Hint.** Use **Stewart's Theorem** on $\triangle AOM_A$)

§4 Practice Problems

Exercise 4.1. In $\triangle ABC$, let H and O be the orthocenter and circumcenter. Show that O is the **Isogonal Conjugate** of H with respect to $\triangle ABC$.

Exercise 4.2. In $\triangle ABC$, let H be the orthocenter. Show that the Nine-Point Circles of $\triangle ABC$, $\triangle BHC$, $\triangle CHA$ and $\triangle AHB$ are all the same.

Exercise 4.3. In $\triangle ABC$, let G be the centroid and $\triangle H_A H_B H_C$ be the orthic triangle. Suppose there exists a point L on $\odot(ABC)$ such that $\overline{AL} \parallel \overline{BC}$. Show that the points H_A, G and L are collinear.

Exercise 4.4 (IMO 2010 Shortlist)**.** Let ABC be an acute triangle with D, E, F being the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

Exercise 4.5 (IMO 2013)**.** Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y, and H are collinear.