Lecture Notes (25th Jan, 2026)

Mmukul Khedekar

17 January 2026

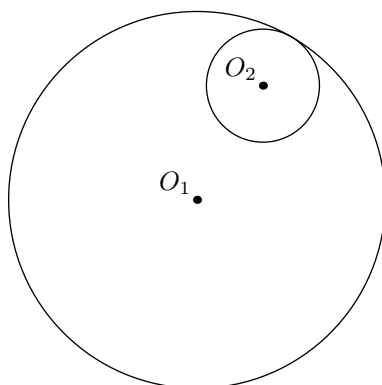In this lecture, we study the properties of tangent circles.

§1 Internally Tangent Circles

We would like to study about configurations that deal with tangent circles. When we say two circles are tangent, it means that these circles only have one point in common. A pair of circles could be either **internally** tangent or **externally** tangent.

> **Definition 1.1.** A pair of circles Γ and ω are
>
> 1. **internally tangent**, if and only if Γ and ω share a single point and the center of the smaller circle is contained inside the larger circle.
>
> 2. **externally tangent**, if and only if Γ and ω share a single point and the center of either of the circles lie outside the other circle.

We would like to focus on pair of circles that are internally tangent for now and study their properties. Let's start with the simplest configuration with two internaly tangent circles.
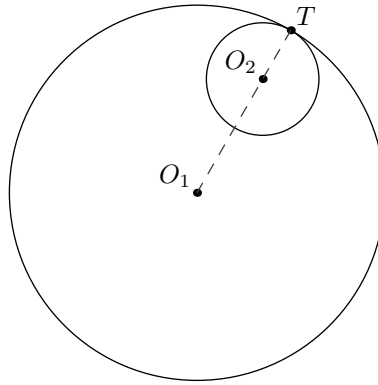

Let Γ and ω be two circles centered at O_1 and O_2, with radius R and r, where $R > r$. Suppose that Γ and ω are internally tangent to each other.

§1.1 Homothetic Mapping

> **Proposition 1.2**
>
> Suppose T is the internal tangency point of Γ and ω, then the points O_1, O_2 and T are collinear.
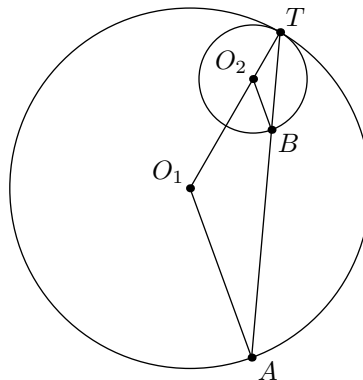
Proof. Consider a homothety at point T that maps ω to Γ. Observe that under this homothetic transformation, we map the point O_2 to O_1. Hence, the points O_1, O_2 and T must be collinear. □

As a consequence of the homothetic mapping, we have the following result

> **Proposition 1.3**
>
> Choose a point B on ω. Suppose TB meets Γ again at A. Then $\overline{O_1A} \parallel \overline{O_2B}$.

Proof. Consider a homothety at point T that maps ω to Γ. Under this homothety, the point B is mapped to A. Hence, $\triangle TO_2B \sim \triangle TO_1A \implies \angle TBO_2 = \angle TAO_1 \implies \overline{O_1A} \parallel \overline{O_2B}$. □

> **Corollary 1.4**
>
> Let T be the point of internal tangency of two circles Γ and ω with radii R and r, where $R > r$. Choose a point B on ω and let TB meet Γ at A. Then
>
> $$\frac{\overline{TB}}{\overline{TA}} = \frac{r}{R}$$

The proof for the above result immediately follows from the fact that $\triangle TO_2B \sim \triangle TO_1A$. Now, we move onto a very celebrated result by Archimedes which appears very frequently in geometry configurations.

§1.2 Archimedes' Lemma

> **Theorem 1.5** (Archimedes' Lemma)
>
> Let Γ and ω be two circles centered at O_1 and O_2. Suppose that these circles are internally tangent at the point T. Let $\overline{XY}$ be the chord of Γ such that $\overline{XY}$ is tangent to ω at point B. Let A be the midpoint of the arc XY that does not contain T. Then
>
> 1. points T, B and A are collinear.
> 2. $\overline{AB} \cdot \overline{AT} = \overline{AX}^2 = \overline{AY}^2$
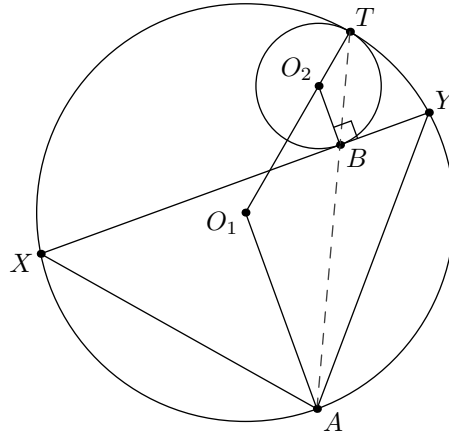
Proof. Consider a homothety at point T that sends ω to Γ. Suppose that this homothety sends point B to A', where A' lies on Γ. Since $\overline{XY}$ is tangent to ω at B, therefore this homothety maps XY to a line ℓ passing through A' that is tangent to Γ. Since ℓ is the image under a homothetic transformation of $XY \implies \overline{XY} \parallel \ell$. Therefore,

$$\angle YXA' = \angle\left(\overline{A'X}, \ell\right) = \angle XYA'$$

This implies that $\triangle XYA'$ is isosceles $\implies$ A' is the midpoint of the arc XY not containing T. Thus $A \equiv A'$, proving that T, B and A are collinear.

For the second part, we shall show that $\overline{AX}$ is tangent to $\odot(TBX)$ at X. This is easy to establish since,

$$\angle XTB = \angle XTA = \angle XYA = \angle AXY = \angle AXB$$

Using the power of a point theorem, we get that

$$\overline{AX}^2 = \overline{AB} \cdot \overline{AT}$$

Since $\overline{AX} = \overline{AY}$, which implies the relation. $\qquad\square$

Exercise 1.6 (Russia 2001)**.** In the above configuration, show that the circumradius of $\triangle ABY$ is a constant that does not depend upon the position of point B.

Let's look at some nice results that revolve around the **Archimedes' Lemma**.

§1.3 Examples

> **Problem 1.7**
>
> Let h be a semicircle with diameter AB. The two circles k_1 and k_2, $k_1 \neq k_2$, touch the segment AB at the points C and D, respectively, and the semicircle h fom the inside at the points E and F, respectively. Prove that the four points C, D, E and F lie on a circle.
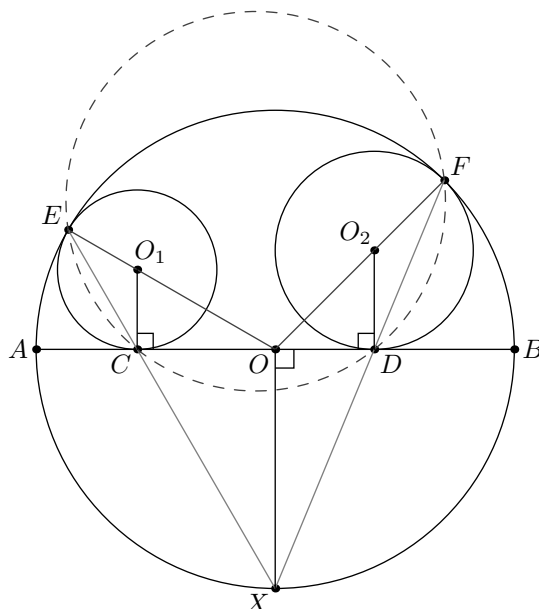
Proof. Suppose X is the midpoint of arc AB not containing E. Then X lies on lines EC and FD by archimedes' lemma. Since,

$$\overline{XC} \cdot \overline{XE} = \overline{AX}^2 = \overline{XD} \cdot \overline{XF}$$

Therefore, by the converse of power of a point theorem $\implies$ points C, D, E and F lie on a circle. $\qquad\qquad\square$

§1.4 Exercises

Exercise 1.8 (INMO 2019)**.** Let AB be the diameter of a circle Γ and let C be a point on Γ different from A and B. Let D be the foot of perpendicular from C on to AB.Let K be a point on the segment CD such that AC is equal to the semi perimeter of ADK.Show that the excircle of ADK opposite A is tangent to Γ.

Exercise 1.9 (RMO 2019)**.** Given a circle τ, let P be a point in its interior, and let l be a line through P. Construct with proof using ruler and compass, all circles which pass through P, are tangent to τ and whose center lies on line l.

Exercise 1.10 (RMO 2017)**.** Let Ω be a circle with a chord AB which is not a diameter. Γ_1 be a circle on one side of AB such that it is tangent to AB at C and internally tangent to Ω at D. Likewise, let Γ_2 be a circle on the other side of AB such that it is tangent to AB at E and internally tangent to Ω at F. Suppose the line DC intersects Ω at $X \neq D$ and the line FE intersects Ω at $Y \neq F$. Prove that XY is a diameter of Ω .

§2 Curvilinear Incircles

Let's move to something more complicated and miraculous.

> **Definition 2.1.** Given $\triangle ABC$ and a point D on $\overline{BC}$, a circle ω is called the **curvilinear incircle** of $\triangle ABC$ if ω is tangent to sides $\overline{AD}$ and $\overline{BC}$, and is internally tangent to $\odot(ABC)$.

Curvilinear incircles are a natural extension of the archimedes' lemma. Essentially, we are choosing another point on the outer circle and adding more tangents.
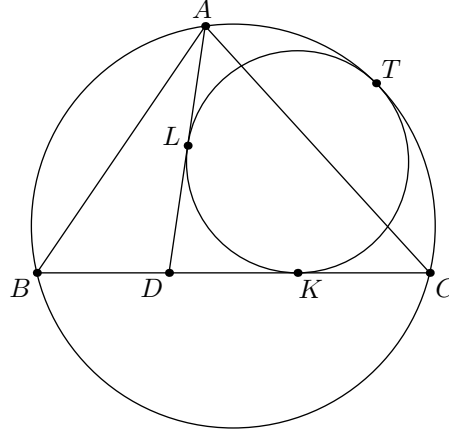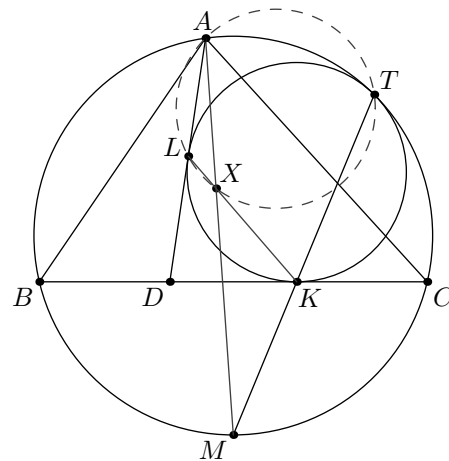

Let's look at some properties of the **curvilinear incircles**.

§2.1 More Circles!

> **Proposition 2.2**
>
> Given $\triangle ABC$ and a point D on the $\overline{BC}$. Suppose ω is the curvilinear incircle of $\triangle ABC$ tangent to $\overline{AD}$ and $\overline{BC}$ at L and K, and tangent to $\odot(ABC)$ at T. Let M be the midpoint of arc BC not containing A. Suppose $\overline{AM}$ intersects $\overline{KL}$ at X. Then the points A, L, X and T are concyclic.


Proof. By archimedes' lemma, we know that M lies on $\overline{KT}$. To show that the points A, L, X and T are concyclic, we just need to angle chase

$$\angle XLT = \angle KLT = \angle CKT = \angle MBT = \angle MAT = \angle XAT$$

where, $\angle CKT = \angle MBT$ holds because, $\overline{TM}$ is the angle bisector of $\angle BTC \implies$ $\triangle TBM \sim \triangle TKC$. Therefore, $\angle XLT = \angle XAT$, which implies that the four points are concyclic. $\qquad \square$

Proposition 2.3

Given $\triangle ABC$ and a point D on the $\overline{BC}$. Suppose ω is the curvilinear incircle of $\triangle ABC$ tangent to $\overline{AD}$ and $\overline{BC}$ at L and K, and tangent to $\odot(ABC)$ at T. Let M be the midpoint of arc BC not containing A. Suppose $\overline{AM}$ intersects $\overline{KL}$ at X. Then, $\overline{MX}$ is tangent to $\odot(XKT)$ at point X.
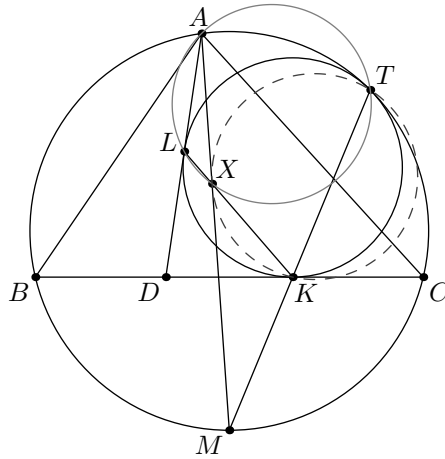
Proof. Effectively, we just want to show that $\angle MXK = \angle MTX$. Fortunately, this is just straightforward angle chasing

$$\angle MTX = \angle MTL - \angle XTL = \angle DLK - \angle XAL = \angle AXL = \angle MKX$$

which proves that $\overline{MX}$ is tangent to $\odot(XKT)$ at X. $\qquad \square$

§2.2 Introducing the Incenter

Proposition 2.4 (Sawayama's Theorem)

Show that the point X is the **Incenter** of $\triangle ABC$.

Proof. Observe that $\overline{MB} = \overline{MC} \implies \overline{AM}$ is the angle bisector of $\angle BAC$. Since,

$$\overline{MX}^2 = \overline{MK} \cdot \overline{MT} = \overline{MB}^2 = \overline{MC}^2$$

Hence by the incenter/excenter lemma, we get that X is the incenter of $\triangle ABC$. $\qquad \square$

It's very surprising how the incenter appears in this configuration. Something even more interesting occurs when we add the other curvilinear incircle of the cevian $\overline{AD}$ to the diagram, which leads to a reknowned result by *Victor Thébault*.

§2.3 Thébault's Theorem

> **Theorem 2.5** (Thébault's Theorem)
>
> Given $\triangle ABC$ and a point D on $\overline{BC}$, let ω_1 and ω_2 be the two curvilinear incircles of $\triangle ABC$ tangent to the cevian $\overline{AD}$. Suppose O_1 and O_2 are the centers of the two curvilinear incircles and I is the incenter of $\triangle ABC$, then points O_1, I and O_2 are collinear.
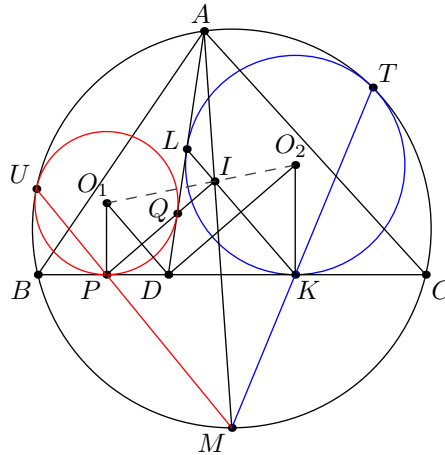
Proof. $\square$

§2.4 Examples

§2.5 Exercises

Exercise 2.6 (IMO Shortlist 1992)**.** Two circles touch externally at a point I. The two circles lie inside a large circle and both touch it. The chord BC of the large circle touches both smaller circles (not at I). The common tangent to the two smaller circles at the point I meets the large circle at a point A, where the points A and I are on the same side of the chord BC. Show that the point I is the incenter of triangle ABC.

§3 Mixtilinear Incircles

> **Problem 3.1** (EGMO 2013)
>
> Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q. Prove that $\angle ACP = \angle QCB$.