Coaxial Circles

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In this article, we will discuss about the properties of coaxial circles.

So far, we have studied tools for dealing with pairs of circles that have distinct radical axes. Let us now consider the situation where multiple circles share the same radical axis.

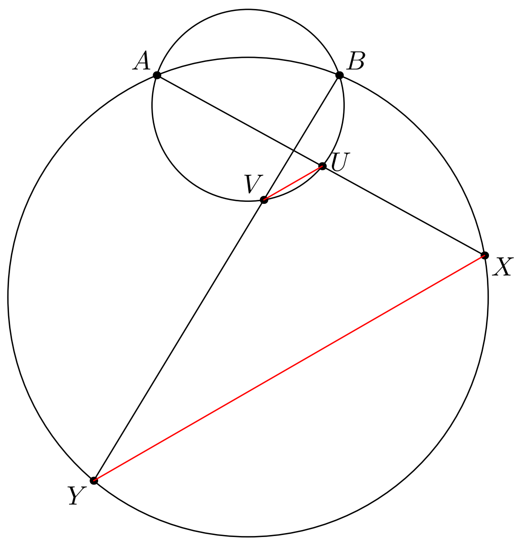
Definition 0.1. A family of circles is called **Coaxial** if they all share the same radical axis.

§1 Reim's Theorem

This particular configuration appears very frequently in geometry problems.

Theorem 1.1 (Reim's Theorem)

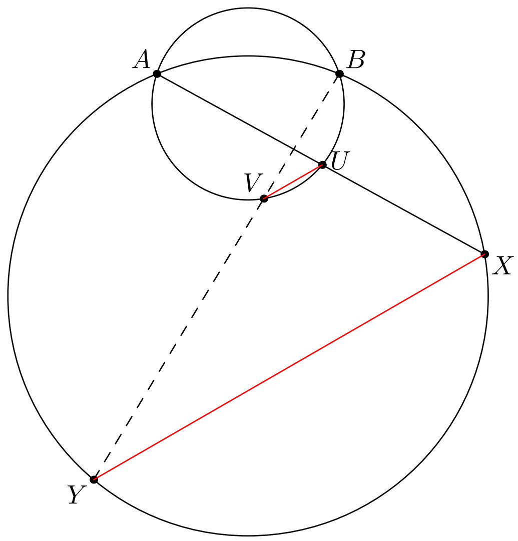
Suppose circles ω_1 and ω_2 intersect in points A and B. Let ℓ_1 and ℓ_2 be two lines through passing through A and B such that they intersect ω_1 in U and V, and ω_2 in X and Y. Then $\overline{UV} \parallel \overline{XY}$.



Furthermore, the converse of this theorem is true too.

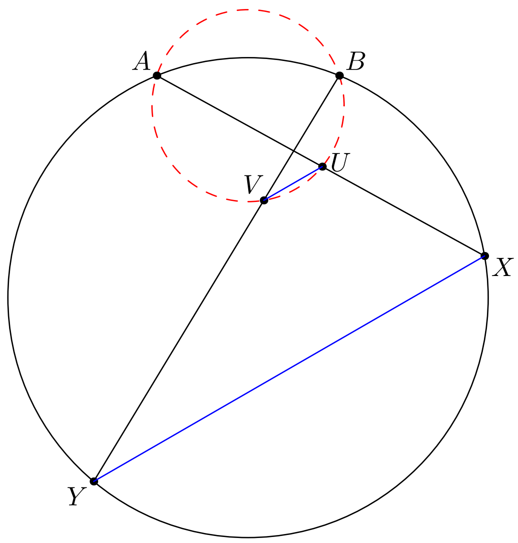
Theorem 1.2 (Converse of Reim's Theorem (Collinearity))

Suppose two circles ω_1 and ω_2 intersect in points A and B. Let ℓ be a line that passes through A and intersects ω_1 and ω_2 at X and U. Suppose Y and V lie on circles ω_1 and ω_2 such that $\overline{UV} \parallel \overline{XY}$. Then, points B, V and Y are collinear.



Theorem 1.3 (Converse of Reim's Theorem (Concyclicity))

Given a circle ω and four points A, B, X and Y on the circle. Choose points U and V on $\overline{AX}$ and $\overline{BY}$ such that $\overline{UV} \parallel \overline{XY}$. Then the points A, B, U and V are concyclic.



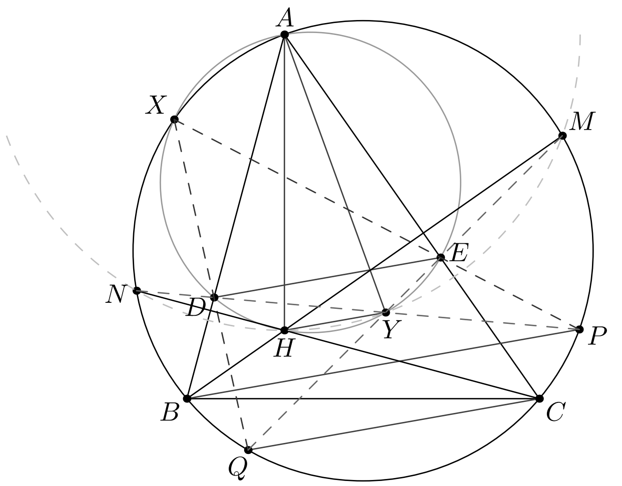
To be precise, both of these results follow from a simple two-step angle chase. Nevertheless, because they appear so frequently in various configurations, it is important to have them at your fingertips to avoid overlooking any pair of parallel lines or cyclic quadrilaterals.

§1.1 Examples

Problem 1.4 (USA TSTST 2019)

Let ABC be an acute triangle with circumcircle Ω and orthocenter H. Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q, respectively. Denote by ω the circumcircle of $\triangle ADE$.

1. Show that lines PE and QD meet on ω.
2. Prove that if ω passes through H, then lines PD and QE meet on ω as well.



Proof. For the first part, suppose $\omega \cap \Omega$ at $X \neq A$. Then applying converse of reim's theorem on $\overline{DE} \parallel \overline{BP}$, we get that PE passes through X. Similarly, QD passes through X proving that PE and QD indeed meet on ω at X.

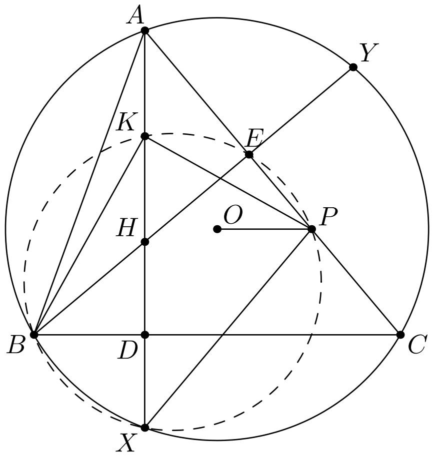
For the second part, suppose Y lies on ω such that $\overline{AH}$ and$\overline{AY}$ are isogonal with respect to $\angle BAC$. Then, $\overline{HY} \parallel \overline{DE}$. Let M and N be the reflections of H over $\overline{AC}$ and $\overline{AB}$. We can show that D lies on $\overline{NY}$, because

$$\begin{aligned}\angle NDH &= 2\angle BDH \\ &= 2\,(180^\circ - \angle ADH) \\ &= 2\angle AYH \\ &= 2\left(90^\circ - \tfrac{1}{2}\angle HAY\right) \\ &= 180^\circ - \angle HDY\end{aligned}$$

Similarly, we can show that E lies on $\overline{MY}$. Since $\overline{DE} \parallel \overline{HY}$, therefore by the converse of reim's theorem, we have $NHYM$ as a cyclic quadrilateral. Again by the converse of reim's theorem applied on circle $\odot(NHYM)$ and $\odot(ABC)$, and pairs of parallel lines $\overline{HY} \parallel \overline{BP}$ and $\overline{HY} \parallel \overline{QC} \implies Y$ lies on $\overline{PD}$ and $\overline{QE}$, as desired. $\square$

Problem 1.5 (Iran 2015)

Let ABC be a triangle with orthocenter H and circumcenter O. Let K be the midpoint of AH. point P lies on AC such that $\angle BKP = 90^\circ$. Prove that $OP \parallel BC$.



Proof. Let D and E be the foot of perpendicular from A and B onto $\overline{BC}$ and $\overline{AC}$, and X and Y be the reflections of H over $\overline{BC}$ and $\overline{AC}$. Since K and E are the midpoints of $\overline{AH}$ and $\overline{HY}$, therefore by the midpoint theorem $\overline{KE} \parallel \overline{AY}$. Applying the converse of reim's theorem, we get that $BKEX$ is a cyclic quadrilateral. Since $\angle BKP = 90^\circ$ and $\angle BEP = 90^\circ \implies P$ lies on the circle $\odot(BKEX)$. Since K is the center of $\odot(AEH)$, we have

$$\angle XAP = \angle KAE = \angle KEA = 180^\circ - \angle KEP = \angle KXP = \angle AXP$$

Therefore, $\triangle PAX$ is isosceles and hence P lies on the perpendicular bisector of $\overline{AX}$. Since, O lies on the perpendicular bisector of $\overline{AX}$ too $\implies \overline{OP} \perp \overline{AX}$. But $\overline{AX} \perp \overline{BC}$ $\implies \overline{OP} \parallel \overline{BC}$. □

§1.2 Exercises

Exercise 1.6 (Iran IMO TST 2008)**.** In the triangle ABC, $\angle B$ is greater than $\angle C$. Suppose T is the midpoint of the arc BAC from the circumcircle of ABC and I is the incenter of ABC. Let E be a point such that $\angle AEI = 90^\circ$ and $AE \parallel BC$. Let $\overline{TE}$ intersect the $\odot(ABC)$ for the second time in P. If $\angle B = \angle IPB$, find the angle $\angle A$.

Exercise 1.7 (IMO 2019)**.** In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$. Prove that points P, Q, P_1 and Q_1 are concyclic.

§2 Practice Problems

Exercise 2.1 (Romania TST 2017)**.** Let $ABCD$ be a trapezium, $AD \parallel BC$, and let E and F be points on the sides AB and CD, respectively. The circumcircle of AEF meets AD again at A_1, and the circumcircle of CEF meets BC again at C_1. Prove that A_1C_1, BD and EF are concurrent.

Exercise 2.2 (IMO Shortlist 2017)**.** Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.