A Beautiful Journey Through Olympiad Geometry

Stefan Lozanovski

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Contents

Introduction

This book is aimed at anyone who wishes to prepare for the geometry part of the mathematics competitions and Olympiads around the world. No previous knowledge of geometry is needed. Even though I am a fan of non-linear storytelling, this book progresses in a linear way, so everything that you need to know at a certain point will have been already visited before. We will start our journey with the most basic topics and gradually progress towards the more advanced ones. The level ranges from junior competitions in your local area, through senior national Olympiads around the world, to the most prestigious International Mathematical Olympiad.

The word "Beautiful" in the book's title means that we will explore only synthetic approaches and proofs, which I find elegant and beautiful. We will not see any analytic approaches, such as Cartesian or barycentric coordinates, nor we will do complex number or trigonometry bashing.

Structure

This book is structured in two parts. The first one provides an introduction to concepts and theorems. For the purpose of applying these concepts and theorems to geometry problems, a number of useful properties and examples with solutions are offered. At the end of each chapter, a selection of unsolved problems is provided as an exercise and a challenge for the reader to test their skills in relation to the chapter topics. This part can be roughly divided in two parts: Junior (the first 10 chapters) and Senior (the other 12 chapters). The second part of this book contains mixed problems, mostly from competitions and Olympiads from all around the world.

Acknowledgments

I would like to thank my primary school math teacher Ms. Vesna Todorovikj for her dedication in training me and my friend Bojan Joveski for the national math competitions. She introduced me to problem solving and thinking logically, in general. I'll never forget the handwritten collection of geometry problems that she gave us, which made me start loving geometry.

I would also like to thank my high school math Olympiad mentor, Mr. Özgür Kırçak. He boosted my Olympiad spirit during the many Saturdays in "Olympiad Room" while eating burek, drinking tea and solving Olympiad problems. Under his guidance, I started preparing geometry worksheets and teaching the younger Olympiad students. Those worksheets are the foundation of this book.

Finally, I would like to thank all of my students for working through the geometry worksheets, shaping the Olympiad geometry curriculum together with me and giving honest feedback about the lessons and about me as a teacher. Their enthusiasm for geometry and thirst for more knowledge were a great inspiration for me to write this book.

Support & Feedback

This book is part of my project for sharing knowledge with the whole world. If you are satisfied with the book contents, please support the project by donating at olympiadgeometry.com.

Tell me what you think about the book and help me make this Journey even more beautiful. Write a general comment about the book, suggest a topic you'd like to see covered in a future version or report a mistake at the same web site.

You can also follow us on Facebook (facebook.com/olympiadgeometry) for the latest news and updates. Please leave you honest review there.

The Author

Notations

Since the math notations slightly differ in various regions of the world, here is a quick summary of the ones we are going to use throughout our journey.

Notation	Explanation
$\angle ABC$	angle ABC; or measurement of said angle
$\overline{AB}$	length of the line segment AB
$\overrightarrow{AB}$	vector AB
$\widehat{AB}$	arc AB
$(ABCD)$	circumcircle of the cyclic polygon $ABCD$
$\equiv$	coincide Example: if $A - B - C$ are collinear, then $AB \equiv AC$.
$\cap$	intersection
$\perp$	perpendicular
$\parallel$	parallel
$P_{\triangle ABC}$	area of the triangle ABC
P_{ABCD}	area of the polygon $ABCD$
$d(P,\, AB)$	distance from the point P to the line AB
$\angle(p, q)$	angle between the lines p and q
$\alpha, \beta, \gamma, \ldots$	unless otherwise noted, the angles at the vertices $A, B, C, \ldots$ in a polygon $ABC\ldots$; or measurements of said angles
a, b, c	unless otherwise noted, the sides opposite the vertices A, B, C in a triangle ABC; or lengths of said sides
$\iff$	if and only if (shortened iff) Example: $p \iff q$ means "if p then q AND if q then p".
$\therefore$	therefore
$\because$	because
LHS \ RHS	The left-hand side \ the right-hand side of an equation
WLOG	Without loss of generality
$\blacksquare$	Q.E.D. (initialism of the Latin phrase "quod erat demonstrandum", meaning "which is what had to be proved".)

Part I

Lessons

Chapter 1

Congruence of Triangles

Two triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are said to be congruent when their corresponding sides and corresponding angles are equal.

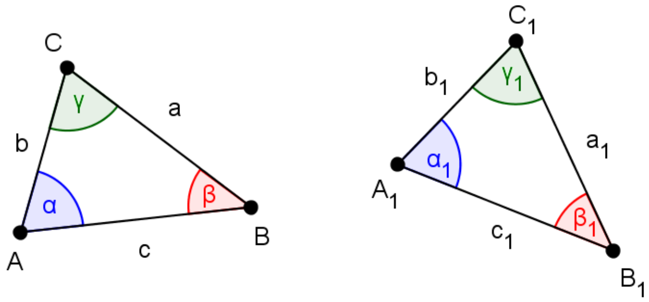


Figure 1.1: Congruent triangles.

$$\triangle ABC \cong \triangle A_1B_1C_1 \iff a = a_1, b = b_1, c = c_1, \alpha = \alpha_1, \beta = \beta_1, \gamma = \gamma_1$$

However, in most of the problems, the equality of all these six pairs of elements will not be given, so we will need to use some criteria for congruence. With these criteria, we will prove the congruence of two triangles only by using the equality of three pairs of corresponding elements.

Criterion SSS (side-side-side) If three pairs of corresponding sides are equal, then the triangles are congruent.

Criterion SAS (side-angle-side) If two pairs of corresponding sides and the angles between them are equal, then the triangles are congruent.

Criterion ASA (angle-side-angle) If two pairs of corresponding angles and the sides formed by the common rays of these angles are equal, then the triangles are congruent.

These criteria are part of our axioms, so we will not prove them. However, in Figure 1.2, you can see that we can construct exactly one triangle given the corresponding set of elements for each criteria. We can also see why there can not exist an ASS congruence criterion.

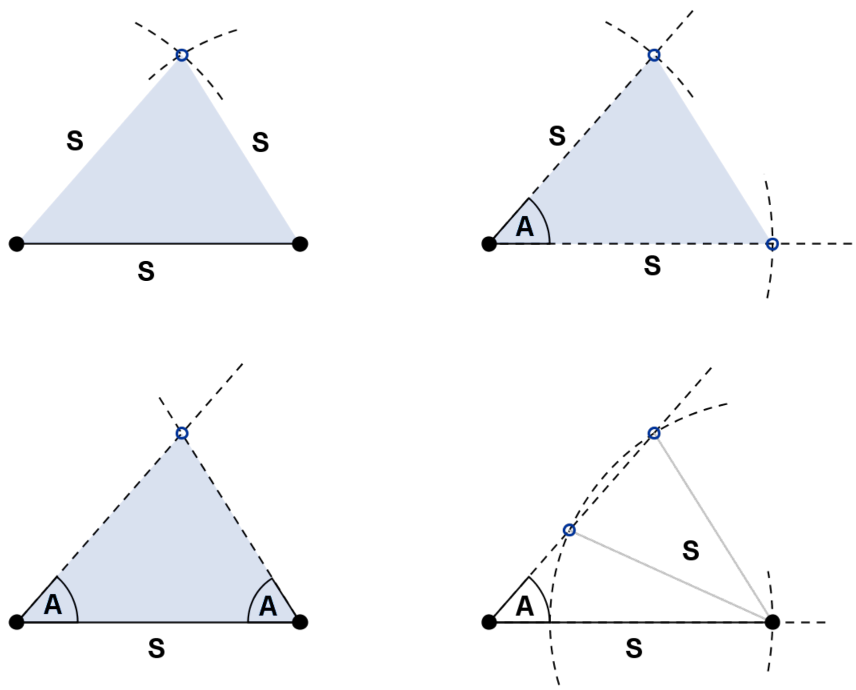


Figure 1.2: Criteria for congruence of triangles.

Chapter 2

Angles of a Transversal

When two lines p and q are intersected by a third line t, we get eight angles. The line t is called a *transversal*. The pairs of angles, depending on their position relative to the transversal and the two given lines are called:

corresponding angles if they lie on the same side of the transversal and one of them is in the interior of the lines p and q, while the other one is in the exterior (e.g. α_1 and α_2);

alternate angles if they lie on different side of the transversal and both of them are either in the interior or in the exterior of the lines p and q (e.g. β_1 and β_2); or

opposite[1]**angles** if they lie on the same side of the transversal and both of them are either in the interior or in the exterior of the lines p and q (e.g. γ_1 and γ_2).

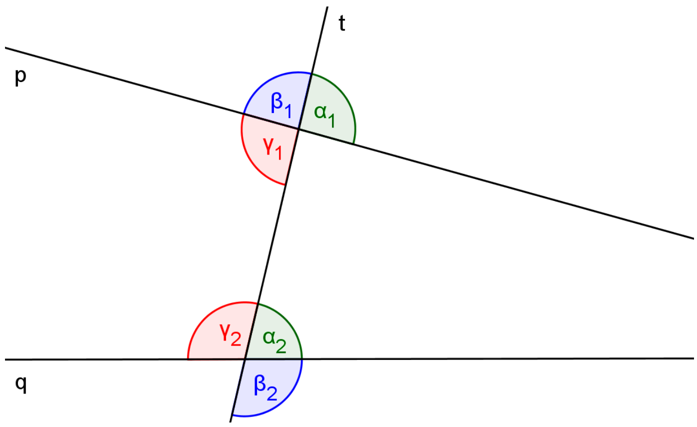


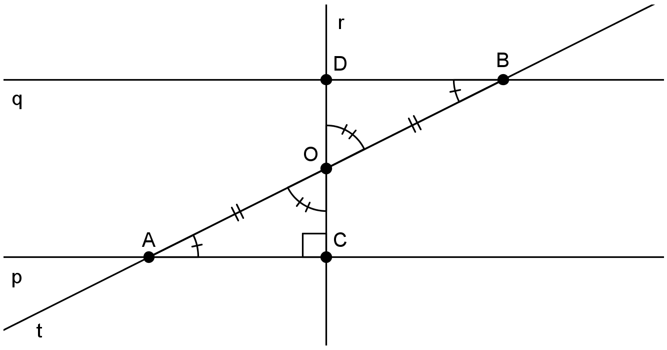
Figure 2.1: Angles of a transversal.

[1] In some resources, the interior opposite angles are called *consecutive interior angles*, but there is no name for the exterior opposite angles, which have the same property. Since in some languages these angles are called opposite, in this book we'll call them that in English, too, even though I haven't seen this terminology used in other resources in English.

Property 2.1. If the lines p and q are parallel, then the corresponding angles are equal, the alternate angles are equal and the opposite angles are supplementary. The converse is also true.

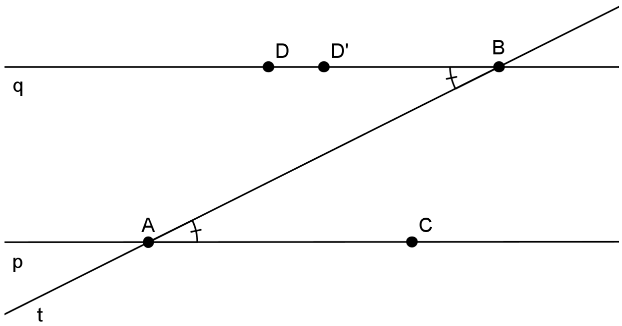
$$p \parallel q \iff \alpha_1 = \alpha_2,\ \beta_1 = \beta_2,\ \gamma_1 + \gamma_2 = 180^\circ$$

Proof. Let the transversal t intersect p and q at A and B, respectively and let O be the midpoint of the line segment AB, i.e. $\overline{AO} = \overline{BO}$. Let r be a line



through O that is perpendicular to p. Let $r \cap p = C$ and $r \cap q = D$. Then $\angle OCA = 90^\circ$. Let's prove one of the directions, i.e. let $\angle OAC = \angle OBD$. The angles $\angle AOC$ and $\angle BOD$ are vertical angles and therefore equal. So, by the criterion ASA, $\triangle AOC \cong \triangle BOD$. Therefore, their corresponding elements are equal, i.e. $\angle ODB = \angle OCA = 90^\circ$. So, $r \perp q$. Therefore, $p \parallel q$. $\square$

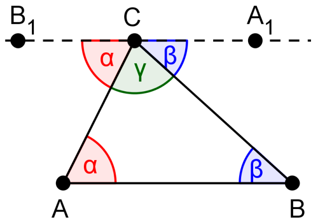
Now, let's prove the other direction. Let $p \parallel q$. Let t be a transversal, such



that $t \cap p = A$ and $t \cap q = B$. Let $C \in p$ and $D \in q$, such that C and D are on different sides of t. We want to prove that $\angle BAC = \angle ABD$. Let D' be a point such that $\angle BAC = \angle ABD'$. By the direction we just proved, $AC \parallel BD'$. Since B lies on both BD and BD' and $BD' \parallel AC \parallel BD$, then $BD \equiv BD'$ and consequently, $\angle ABD \equiv \angle ABD'$. Therefore, $\angle BAC = \angle ABD$.

Remark. The other angles with vertices at A and B are either vertical to (and therefore equal) or form a linear pair (and therefore supplementary) with the angles $\angle BAC$ and $\angle ABD$, so it is easy to prove the rest. ■

Example 2.1 (Sum of angles in a triangle)**.** Prove that the sum of the interior angles in a triangle is 180 degrees.

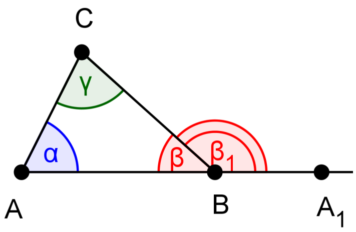


Proof. Let ABC be a triangle. Let's draw a line B_1A_1 which passes through C and is parallel to AB. Then, by Property 2.1, we have:

$$\begin{aligned}
&\angle B_1CA = \angle CAB = \alpha \quad \text{(alternate interior angles; transversal } AC\text{)}\\
&\angle A_1CB = \angle CBA = \beta \quad \text{(alternate interior angles; transversal } BC\text{)}\\
&\angle ACB = \gamma\\
\therefore\ &\angle B_1CA + \angle A_1CB + \angle ACB = \alpha + \beta + \gamma\\
&\angle B_1CA_1 = \alpha + \beta + \gamma\\
&180^\circ = \alpha + \beta + \gamma
\end{aligned}$$

■

Example 2.2. Prove that an exterior angle equals the sum of the two non-adjacent interior angles.



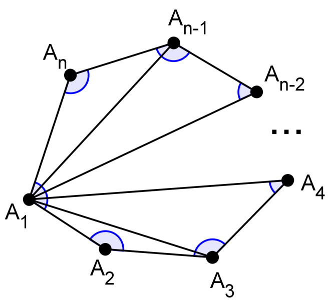
Proof. Let ABC be a triangle and let A_1 be a point on the extension of AB.

$$\begin{aligned}
&\angle A_1BC + \angle ABC = 180^\circ \quad \text{(linear pair)}\\
&\angle ABC + \angle BCA + \angle CAB = 180^\circ \quad \text{(Sum of angles in a triangle)}\\
\therefore\ &\angle A_1BC = 180^\circ - \angle ABC = \angle BCA + \angle CAB
\end{aligned}$$

■

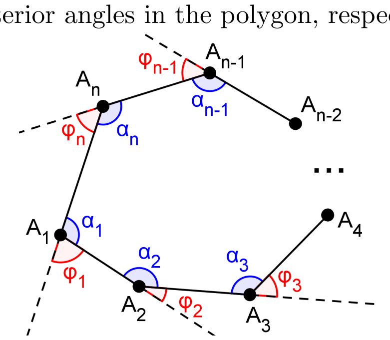
Example 2.3. Find the sum of the interior angles in an n-gon.

Proof. Let $A_1A_2A_3\ldots A_n$ be a polygon with n sides. If we draw the diagonals from A_1 to all the other $(n-3)$ vertices, we get $(n-2)$ distinct triangles. By Example 2.1, the sum of all the interior angles in these triangles is $(n-2)\cdot 180^\circ$. Note that these angles actually form all the interior angles in the n-gon. So, the sum of the interior angles in an n-gon is $(n-2)\cdot 180^\circ$. ■



Example 2.4. Find the sum of the exterior angles in an n-gon.

Proof. Let $A_1A_2A_3\dots A_n$ be a polygon with n sides. Let α_i and φ_i $(i = 1, 2, \dots, n)$ be the interior and exterior angles in the polygon, respectively.



Since each exterior and its corresponding interior angle form a linear pair, we have $\alpha_i + \beta_i = 180°,\ i = 1, 2, \dots, n$. If we sum these equations, we get

$$\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \varphi_i = n \cdot 180°.$$

From Example 2.3, we know that

$$\sum_{i=1}^{n} \alpha_i = (n-2) \cdot 180°.$$

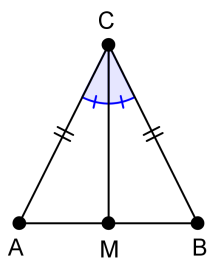
In order to find the sum of the exterior angles, we need to subtract the two previous equations.

$$\sum_{i=1}^{n} \varphi_i = \big(n - (n-2)\big) \cdot 180° = 2 \cdot 180° = 360°.$$

So, the sum of the exterior angles in any polygon does not depend on the number of sides n and is always $360°$. ∎

Example 2.5 (Isosceles Triangle)**.** In $\triangle ABC$, two of the sides are equal, i.e. $\overline{CA} = \overline{CB}$. Prove that $\angle CAB = \angle CBA$.

Proof. Let the angle bisector of $\angle BCA$ intersect the side AB at M. Then, $\angle ACM = \angle BCM$. Combining with $\overline{CA} = \overline{CB}$ and CM-common side, by SAS, we get that $\triangle ACM \cong \triangle BCM$. Therefore, their corresponding angles are equal, i.e.



$$\angle CAB \equiv \angle CAM = \angle CBM \equiv \angle CBA.$$

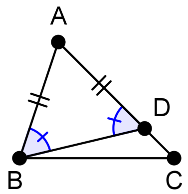
Additionally, as a consequence of the congruence, we can also get two other things: $\overline{AM} = \overline{MB}$ and $\angle AMC = \angle BMC$, which means that $CM \perp AB$. Therefore, as a conclusion, the angle bisector, the median and the altitude from the vertex C in an isosceles triangle coincide with the side bisector of AB. ∎

Remark. The converse is also true (if $\angle CAB = \angle CBA$, then $\overline{CA} = \overline{CB}$). Can you prove it by yourself?

Example 2.6 (Equilateral triangle)**.** In $\triangle ABC$, all three sides are equal. Prove that all the angles are equal to $60°$.

Proof. Combining Example 2.1 and Example 2.5, we directly get the desired result. ■

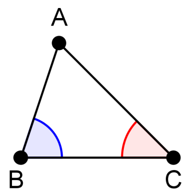
Example 2.7. In any triangle, a greater side subtends a greater angle.



Proof. In $\triangle ABC$, let $\overline{AC} > \overline{AB}$. Then we can choose a point D on the side AC, such that $\overline{AD} = \overline{AB}$. Since $\triangle ABD$ is isosceles, we have $\angle ABD = \angle ADB$.

$$\angle ABC > \angle ABD = \angle ADB \overset{2.2}{=} \angle DBC + \angle DCB > \angle DCB \equiv \angle ACB \qquad ■$$

Example 2.8. In any triangle, a greater angle is subtended by a greater side.



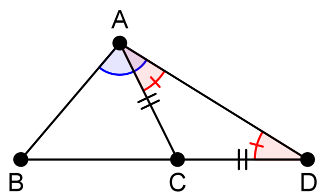
Proof. In $\triangle ABC$, let $\angle ABC > \angle ACB$. We want to prove that $\overline{AC} > \overline{AB}$. Let's assume the opposite, i.e. $\overline{AC} \leq \overline{AB}$.

i) If $\overline{AC} = \overline{AB}$, then by Example 2.5, $\angle ABC = \angle ACB$, which is not true.

ii) If $\overline{AC} < \overline{AB}$, then by Example 2.7, $\angle ABC < \angle ACB$, which is not true.

Therefore, our assumption is wrong, so $\overline{AC} > \overline{AB}$. ■

Example 2.9 (Triangle Inequality)**.** In any triangle, the sum of the lengths of any two sides is greater than the length of the third side.



Proof. In $\triangle ABC$, let D be a point on the extension of the side BC beyond C, such that $\overline{CD} = \overline{CA}$. Then, $\triangle CAD$ is isosceles, so $\angle CAD = \angle CDA$. Now, in $\triangle BAD$ we have

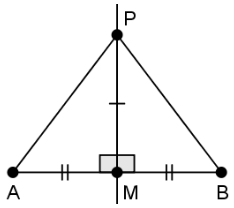
$$\angle BAD = \angle BAC + \angle CAD > \angle CAD = \angle CDA \equiv \angle BDA,$$

which by Example 2.8 means that $\overline{BD} > \overline{AB}$. Therefore,

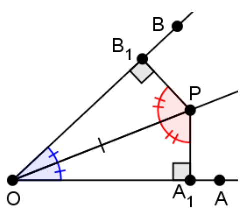
$$\overline{BC} + \overline{CA} = \overline{BC} + \overline{CD} = \overline{BD} > \overline{AB} \qquad ■$$

Example 2.10. Any point P that lies on the side bisector of a line segment AB is equidistant from the endpoints.

Proof. Let p and M be the side bisector and the midpoint of AB, respectively. Therefore, $M \in p$. As $p \perp AB$, we have $\angle PMA = \angle PMB = 90°$. Combining with $\overline{MA} = \overline{MB}$ and MP - common side, we get $\triangle PMA \cong \triangle PMB$ (by the SAS criterion). Therefore, $\overline{PA} = \overline{PB}$, i.e. P is equidistant from the endpoints. ∎



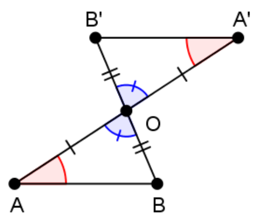
Example 2.11. Any point P that lies on the angle bisector of an angle $\angle AOB$ is equidistant from the rays.



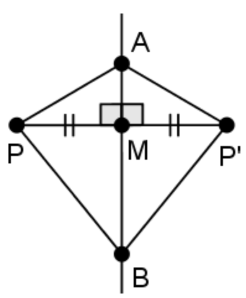
Proof. Let p be the angle bisector of $\angle AOB$ and let A_1 and B_1 be the feet of the perpendiculars from P to OA and OB, respectively. Therefore, $\angle POA_1 = \angle POB_1 = \frac{\alpha}{2}$ and $\angle OPA_1 = 90° - \frac{\alpha}{2} = \angle OPB_1$. Since OP is a common side, we get $\triangle OPA_1 \cong \triangle OPB_1$ (by the ASA criterion). Therefore, $\overline{PA_1} = \overline{PB_1}$, i.e. P is equidistant from the rays of $\angle AOB$. ∎

Example 2.12. Let A' and B' be the reflections of the points A and B, respectively, with respect to the point O. Prove that $\overline{AB} = \overline{A'B'}$ and $AB \parallel A'B'$.

Proof. Since $\overline{OA} = \overline{OA'}$, $\overline{OB} = \overline{OB'}$ and $\angle AOB = \angle A'OB'$ as vertical angles, by the SAS criterion we have that $\triangle OAB \cong \triangle OA'B'$. Therefore, $\overline{AB} = \overline{A'B'}$ and $\angle OAB = \angle OA'B'$ which implies that $AB \parallel A'B'$ because the alternate angles of the transversal AA' and the lines AB and $A'B'$ are equal. ∎



Example 2.13. Let P' be the reflection of the point P with respect to the line AB. Prove that $\triangle PAB \cong \triangle P'AB$.



Proof. Let $M \cap AB = PP'$. Then, $\overline{PM} = \overline{MP'}$ and $PM \perp AB$. Since $\overline{PM} = \overline{MP'}$, $\angle PMA = \angle P'MA = 90°$ and AM is a common side, by the SAS criterion we get that $\triangle PMA \cong \triangle P'MA$ and therefore $\overline{PA} = \overline{P'A}$. Similarly, $\triangle PMB \cong \triangle P'MB$ and therefore $\overline{PB} = \overline{P'B}$. Finally, by the SSS criterion we get that $\triangle PAB \cong \triangle P'AB$. ∎

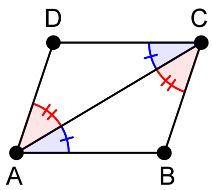
The quadrilaterals, depending on the number of parallel opposite sides, are divided in 2 categories:

trapezoid [2] with at least 1 pair of parallel opposite sides

parallelogram with 2 pairs of parallel opposite sides

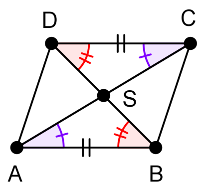
In the next 2 examples, we will present properties of the parallelograms and later we will present criteria for parallelograms, i.e. we will show 3 different ways how to prove that a quadrilateral is a parallelogram (apart from the obvious way, by definition, by proving that both pairs of opposite sides are parallel).

Example 2.14. Let $ABCD$ be a parallelogram. Prove that its opposite sides are of equal length.



Proof. Let's draw the diagonal AC. Since $AB \parallel CD$, by Property 2.1, $\angle CAB = \angle ACD$. Similarly, since $BC \parallel AD$, $\angle ACB = \angle CAD$. Therefore, since AC is a common side for the triangles $\triangle ABC$ and $\triangle CDA$, by the ASA criterion, $\triangle ABC \cong \triangle CDA$. Therefore, their corresponding elements, are equal, i.e. $\overline{AB} = \overline{CD}$ and $\overline{BC} = \overline{DA}$. ∎

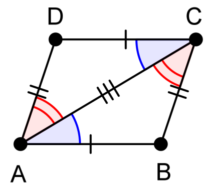
Example 2.15. Let $ABCD$ be a parallelogram. Prove that its diagonals bisect at their intersection point.



Proof. Let the intersection of the diagonals AC and BD be S. Because $AB \parallel CD$, from Property 2.1 we get that $\angle SAB = \angle SCD$. Similarly, $\angle SBA = \angle SDC$. Also, from Example 2.14 we know that $\overline{AB} = \overline{CD}$, so by combining these three facts, by the ASA criterion we get that $\triangle SAB \cong \triangle SCD$. Therefore, $\overline{SA} = \overline{SC}$ and $\overline{SB} = \overline{SD}$, i.e. the diagonals bisect at their intersection point. ∎

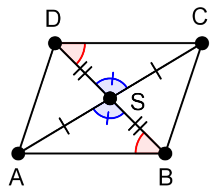
[2] American English (in British English, a quadrilateral with 1 pair of parallel opposite sides is called a "trapezium", while the term "trapezoid" refers to a quadrilateral with no parallel opposite sides). In this book, we will use the American English terminology.

Example 2.16. In the quadrilateral $ABCD$, the opposite sides are of equal length. Prove that $ABCD$ is a parallelogram.



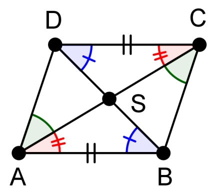
Proof. Let's draw the diagonal AC. Since $\overline{AB} = \overline{CD}$, $\overline{BC} = \overline{DA}$ and AC is a common side, by the SSS criterion we get that $\triangle ABC \cong \triangle CDA$. Therefore $\angle BAC = \angle DCA$, which by Property 2.1 implies that $AB \parallel CD$. Similarly, $\angle BCA = \angle DAC$ and therefore $BC \parallel AD$. Hence, $ABCD$ is a parallelogram. ■

Example 2.17. In the quadrilateral $ABCD$, the intersection point of the diagonals bisects them. Prove that $ABCD$ is a parallelogram.



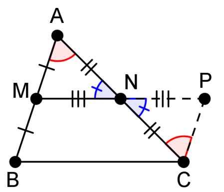
Proof. Let the intersection of the diagonals AC and BD be S. Then, from the condition, we have that $\overline{AS} = \overline{SC}$ and $\overline{BS} = \overline{SD}$. Let's take a look at $\triangle ABS$ and $\triangle CDS$. We have $\overline{AS} = \overline{CS}$, $\angle ASB = \angle CSD$ as vertical angles and $\overline{BS} = \overline{DS}$. So, by the SAS criterion, $\triangle ABS \cong \triangle CDS$. Therefore, the corresponding elements are equal, i.e. $\angle ABS = \angle CDS$. Since these angles are alternate angles of the transversal BD and the lines AB and CD, we have that $AB \parallel CD$. Similarly, $\triangle BCS \cong \triangle DAS$ and $\angle BCS = \angle DAS$. Therefore, $BC \parallel DA$. ■

Example 2.18. In the quadrilateral $ABCD$, $\overline{AB} = \overline{CD}$ and $AB \parallel CD$. Prove that $ABCD$ is a parallelogram.



Proof. Let the intersection of the diagonals AC and BD be S. Since $AB \parallel CD$, the alternate angles of the transversal BD are equal, i.e. $\angle ABS = \angle CDS$. Similarly, $\angle BAS = \angle DCS$. Combining with the fact that $\overline{AB} = \overline{CD}$, by the ASA criterion, we get that $\triangle ABS \cong \triangle CDS$. Therefore, as the corresponding elements are equal, $\overline{AS} = \overline{CS}$ and $\overline{BS} = \overline{DS}$. Combining with the fact that $\angle ASD = \angle CSB$ as vertical angles, by the SAS criterion we get that $\triangle ASD \cong \triangle CSB$. Therefore, $\angle DAS = \angle BCS$, so $DA \parallel BC$. ■

Example 2.19 (Midsegment Theorem)**.** In a triangle, the segment joining the midpoints of any two sides is parallel to the third side and half its length.



Proof. In $\triangle ABC$, let M and N be the midpoints of the sides AB and AC, respectively. Let P be a point on the ray MN beyond N, such that $\overline{MN} = \overline{NP}$. Since $\angle MNA = \angle PNC$ as vertical angles, by SAS we have that $\triangle AMN \cong \triangle CPN$. Therefore, $\overline{AM} = \overline{CP}$ and $\angle MAN = \angle PCN$ which means that $AM \parallel CP$. Now, we have $\overline{BM} = \overline{AM} = \overline{CP}$ and $BM \equiv AM \parallel CP$. By Example 2.18, since the opposite sides in the quadrilateral $MBCP$ are of equal length and parallel, it must be a parallelogram. Therefore,

$$MN \equiv MP \parallel BC$$

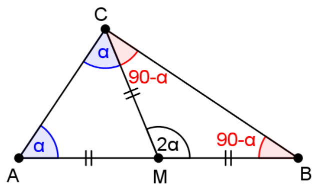
and because of Example 2.14,

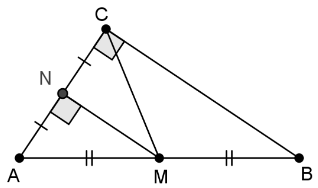
$$\overline{MN} = \frac{1}{2}\overline{MP} = \frac{1}{2}\overline{BC}. \qquad \blacksquare$$

Example 2.20. Let M be the midpoint of the side AB in the triangle ABC. Prove that $\angle ACB = 90°$ if and only if $\overline{MA} = \overline{MB} = \overline{MC}$.

Proof. Let $\overline{MA} = \overline{MB} = \overline{MC}$.
Let $\angle BAC = \alpha$. Since $\triangle MAC$ is isosceles, $\angle MCA = \angle MAC \equiv \angle BAC = \alpha$. As an exterior angle of $\triangle MAC$, $\angle BMC = \angle MAC + \angle MCA = 2\alpha$. Now, since $\triangle MBC$ is isosceles, $\angle MCB = \frac{1}{2} \cdot (180° - \angle BMC) = 90° - \alpha$. Finally, $\angle ACB = \angle ACM + \angle MCB = 90°$. $\square$

Now, let's prove the other direction. Let $\angle ACB = 90°$.
Let N be the midpoint of AC. Then, MN is a midsegment in $\triangle ABC$ and therefore $MN \parallel BC$. Since $AC \perp BC$, we get $AC \perp MN$, i.e. MN is altitude in $\triangle MAC$. Since MN is both median and altitude in $\triangle MAC$, then $\triangle MAC$ is isosceles. Therefore, $\overline{MA} = \overline{MC}$. Since M is the midpoint of AB, we get $\overline{MA} = \overline{MB} = \overline{MC}$. $\blacksquare$



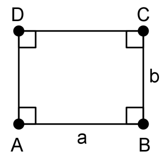


Related problems: 1, 2, 3, 4, 5, 6, 7, 8, 11, 13, 14, 16, 22, 23, 24, 26, 28 and 29.

Chapter 3

Area of Plane Figures

Rectangle

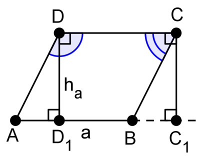


The area of a rectangle $ABCD$ is defined as the product of the length $a = \overline{AB} = \overline{CD}$ and the width $b = \overline{BC} = \overline{AD}$ of the rectangle.

$$P_{ABCD} = a \cdot b$$

Using this fact, we will derive the formulae for the area of other plane figures.

Parallelogram



Let $ABCD$ be a parallelogram. WLOG, let $\angle ABC > 90°$. Let C_1 and D_1 be the feet of the perpendiculars from C and D, respectively, to the line AB. Since $AD \parallel BC$, by Property 2.1, $\gamma = 180° - \delta$.

$$\angle BCC_1 = 90° - \gamma = \delta - 90° = \angle ADD_1$$

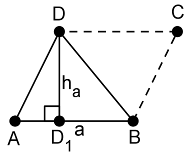
Additionally, $\overline{CC_1} = d(AB, CD) = \overline{DD_1}$ and $\angle CC_1B = 90° = \angle DD_1A$. Therefore, by the ASA criterion, $\triangle BCC_1 \cong \triangle ADD_1$. So $P_{\triangle BCC_1} = P_{\triangle ADD_1}$.

$$P_{ABCD} = P_{\triangle ADD_1} + P_{DD_1BC} = P_{\triangle BCC_1} + P_{DD_1BC} = P_{DD_1C_1C}$$

Since DD_1C_1C is a rectangle with length $\overline{CD} = \overline{AB} = a$ and width $\overline{CC_1} = h_a$, we get

$$P_{ABCD} = a \cdot h_a$$

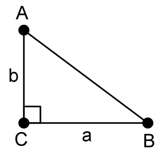
Triangle



Let $ABCD$ be a parallelogram. The diagonal BD divides the parallelogram in two triangles $\triangle ABD$ and $\triangle BCD$. By Example 2.14, the opposite sides of the parallelogram are equal, i.e. $\overline{AB} = \overline{CD}$ and $\overline{BC} = \overline{DA}$. Therefore, since $\angle BAD = 180^\circ - \angle ADC = \angle DCB$, by the SAS criterion, $\triangle BAD \cong \triangle DCB$. Since congruent triangles have equal areas, then the area of each of the triangles is half the area of the parallelogram, i.e.

$$P_{\triangle ABD} = \frac{a \cdot h_a}{2}$$

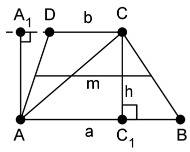
Right Triangle



In right triangle, the altitude opposite of the side a is in fact the side b, so

$$P_{\triangle ABC} = \frac{a \cdot b}{2}$$

Trapezoid



Let $ABCD$ be a trapezoid, such that $AB \parallel CD$. Let A_1 and C_1 be the feet of the altitudes from A and C to the lines CD and AB, respectively.

$$P_{ABCD} = P_{\triangle ABC} + P_{\triangle CDA} = \frac{\overline{AB} \cdot \overline{CC_1}}{2} + \frac{\overline{CD} \cdot \overline{AA_1}}{2}$$

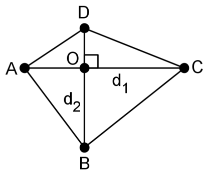
Let $h = d(AB, CD)$, $a = \overline{AB}$ and $b = \overline{CD}$. Then $\overline{AA_1} = \overline{CC_1} = h$. Therefore,

$$P_{ABCD} = \frac{a+b}{2} \cdot h$$

Since the midsegment in $ABCD$, m, is the sum of the midsegments in $\triangle ABC$ and $\triangle CDA$, the area of the trapezoid is sometimes expressed as

$$P_{ABCD} = m \cdot h$$

Quadrilateral with perpendicular diagonals



Let $ABCD$ be a quadrilateral with perpendicular diagonals. Let $AC \cap BD = O$. Then the triangles $\triangle ABO$, $\triangle BCO$, $\triangle CDO$ and $\triangle DAO$ are right triangles. Therefore,

$$\begin{aligned} P_{ABCD} &= P_{\triangle ABO} + P_{\triangle BCO} + P_{\triangle CDO} + P_{\triangle DAO} = \\ &= \frac{\overline{AO} \cdot \overline{BO}}{2} + \frac{\overline{BO} \cdot \overline{CO}}{2} + \frac{\overline{CO} \cdot \overline{DO}}{2} + \frac{\overline{DO} \cdot \overline{AO}}{2} = \\ &= \frac{(\overline{AO} + \overline{CO}) \cdot (\overline{BO} + \overline{DO})}{2} = \frac{\overline{AC} \cdot \overline{BD}}{2} \end{aligned}$$

Let the diagonals AC and BD be d_1 and d_2, respectively. Then,

$$P_{ABCD} = \frac{d_1 \cdot d_2}{2}$$

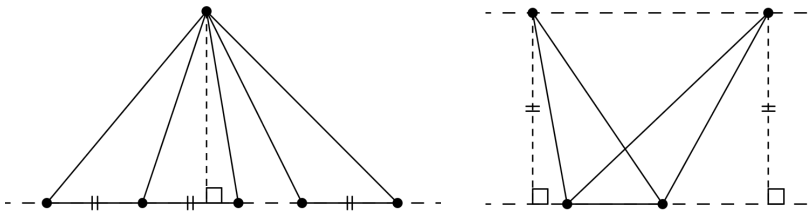
3.1 Area of Triangles

We will now show some properties that are often used in geometry problems.

Property 3.1.

(a) Two triangles that have base sides of equal length and a common altitude, have equal areas.

(b) Two triangles that have a common base side and altitudes of equal length, have equal areas.



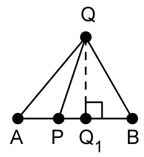
(a) Common altitude (b) Common base

Figure 3.1: Triangles with equal area

Proof. Follows directly by the formula for area of triangle $P_{\triangle ABC} = \dfrac{a \cdot h_a}{2}$. ■

Property 3.2. Let $A - P - B$ be collinear points in that order and let Q be a point that is not collinear with them. Then

$$\frac{P_{\triangle APQ}}{P_{\triangle BPQ}} = \frac{\overline{AP}}{\overline{PB}}.$$



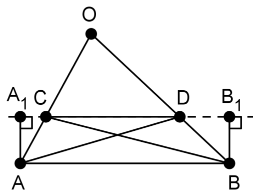
Proof. Let Q_1 be the foot of the perpendicular from Q to AB. Then,

$$\frac{P_{\triangle APQ}}{P_{\triangle BPQ}} = \frac{\frac{\overline{AP} \cdot \overline{QQ_1}}{2}}{\frac{\overline{PB} \cdot \overline{QQ_1}}{2}} = \frac{\overline{AP}}{\overline{PB}}.$$ ■

We will use the proof of the following well-known theorem to present how these properties can be used.

Example 3.1 (Thales' Proportionality Theorem)**.** Let OAB be a triangle and let CD be a line that intersects its sides OA and OB at C and D, respectively. Prove that

$$AB \parallel CD \iff \frac{\overline{OC}}{\overline{CA}} = \frac{\overline{OD}}{\overline{DB}}$$



Proof. Let A_1 and B_1 be the feet of the perpendiculars from A and B, respectively, to the line CD. Then,

$$\begin{aligned} & AB \parallel CD \\ \iff\; & \overline{AA_1} = \overline{BB_1} \\ \overset{\text{Property 3.1}}{\iff}\; & P_{\triangle CDA} = P_{\triangle CDB} \\ \iff\; & \frac{P_{\triangle OCD}}{P_{\triangle CDA}} = \frac{P_{\triangle OCD}}{P_{\triangle CDB}} \\ \overset{\text{Property 3.2}}{\iff}\; & \frac{\overline{OC}}{\overline{CA}} = \frac{\overline{OD}}{\overline{DB}} \end{aligned}$$ ■

Related problems: 10, 12, 15, 17 and 27.

Chapter 4

Similarity of Triangles

Two triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are said to be similar when their corresponding angles are equal and their corresponding sides are proportional.

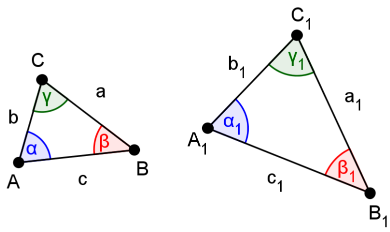


Figure 4.1: Similar triangles.

$$\triangle ABC \sim \triangle A_1B_1C_1 \iff \alpha = \alpha_1, \beta = \beta_1, \gamma = \gamma_1, \frac{a_1}{a} = \frac{b_1}{b} = \frac{c_1}{c} = k$$

The positive real number k is called the *ratio of similarity*. If it is greater than 1, then $\triangle A_1B_1C_1$ is proportionally greater than $\triangle ABC$. If it is less than 1, then $\triangle A_1B_1C_1$ is proportionally smaller than $\triangle ABC$. If it is equal to 1, then $\triangle ABC$ and $\triangle A_1B_1C_1$ are congruent.

This ratio doesn't apply only for the lengths of the sides, but also for the lengths of other corresponding elements (for example, the length of an altitude, a median, etc). So, for the ratio of the areas of two similar triangles, we get:

$$\frac{P_1}{P} = \frac{\frac{a_1 \cdot h_{a_1}}{2}}{\frac{a \cdot h_a}{2}} = \frac{a_1}{a} \cdot \frac{h_{a_1}}{h_a} = k \cdot k = k^2 \quad \text{or} \quad k = \sqrt{\frac{P_1}{P}}.$$

There are also criteria for similarity of triangles.

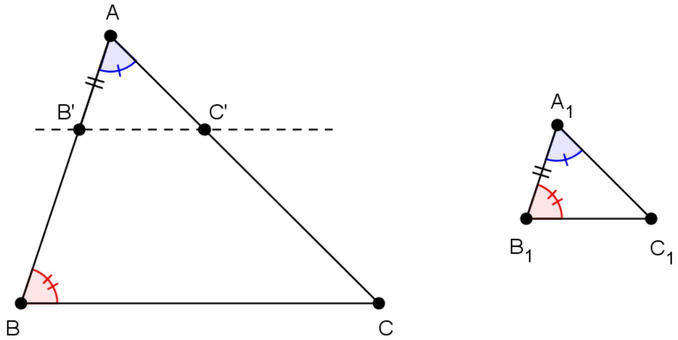
Criterion AA (angle-angle) If two pairs of corresponding angles are equal, then the triangles are similar.

Criterion SSS (side-side-side) If three pairs of corresponding sides are proportional, then the triangles are similar.

Criterion SAS (side-angle-side) If two pairs of corresponding sides are proportional and the angles between them are equal, then the triangles are similar.

We will now present the proofs of these criteria, for the sake of completeness. Although they use only the things that we learned until now, if you are a beginner, you may want to skip them (page 22) since the main point is to know how to use them. But if you are skeptical and don't believe that the criteria for similarity are really true, here are the proofs :)

Proof (AA). Let $\triangle ABC$ and $\triangle A_1B_1C_1$ be two trianges with $\alpha = \alpha_1$ and $\beta = \beta_1$. By Example 2.1, $\gamma = \gamma_1$, too. WLOG, let $\overline{A_1B_1} < \overline{AB}$. Then, we



can construct a point $B' \in AB$, such that $\overline{AB'} = \overline{A_1B_1}$. The parallel line to BC through B' intersects AC at C'. Then, by Property 2.1, $\angle AB'C' = \angle ABC$. So, by the ASA criterion for congruent triangles, we have $\triangle AB'C' \cong \triangle A_1B_1C_1$.

Since $BC \parallel B'C'$, by Thales' Proportionality Theorem, we have

$$\frac{\overline{AB'}}{\overline{B'B}} = \frac{\overline{AC'}}{\overline{C'C}}.$$

$$\frac{\overline{B'B}}{\overline{AB'}} = \frac{\overline{C'C}}{\overline{AC'}}$$

$$\frac{\overline{B'B}}{\overline{AB'}} + 1 = \frac{\overline{C'C}}{\overline{AC'}} + 1$$

$$\frac{\overline{B'B} + \overline{AB'}}{\overline{AB'}} = \frac{\overline{C'C} + \overline{AC'}}{\overline{AC'}}$$

$$\frac{\overline{AB}}{\overline{AB'}} = \frac{\overline{AC}}{\overline{AC'}}$$

$$\frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}}$$

Now, by substituting the correpsonding sides from the congruence we just proved, we get

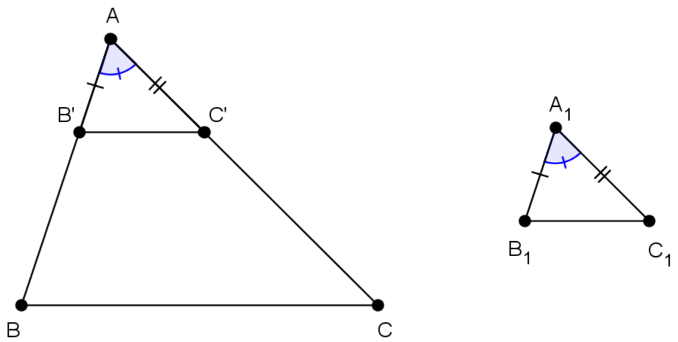
$$\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{A_1C_1}}{\overline{AC}}.$$

Similarly, by constructing a point $A'' \in BA$ and then a line $A''C''$ that is parallel to AC, we can get that

$$\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{B_1C_1}}{\overline{BC}}.$$

Therefore, all the three corresponding angles are equal and the three corresponding pairs of sides are proportional, so $\triangle ABC \sim \triangle A_1B_1C_1$. ■

Proof (SAS). Let $\triangle ABC$ and $\triangle A_1B_1C_1$ be two triangles with $\alpha = \alpha_1$ and $\dfrac{\overline{A_1B_1}}{\overline{AB}} = \dfrac{\overline{A_1C_1}}{\overline{AC}} = k$.



WLOG, let $k < 1$. Then, we can construct points $B' \in AB$ and $C' \in AC$, such that $\overline{AB'} = \overline{A_1B_1}$ and $\overline{AC'} = \overline{A_1C_1}$. By substituting the line segments with equal lengths, we get

$$k = \frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}}.$$

Similarly as in the previous proof, by algebraic transformations (taking the reciprocal value, subtracting 1 on both sides, and taking the reciprocal value once again), we get

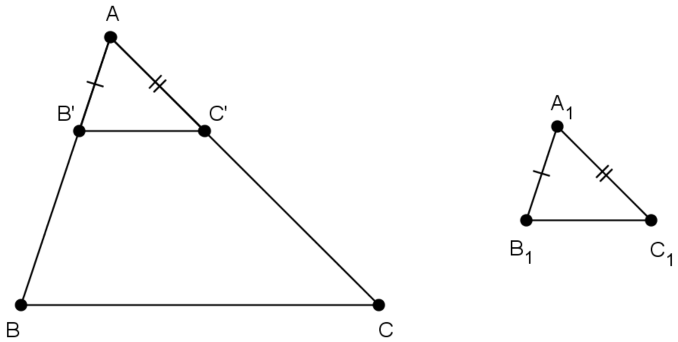
$$\frac{\overline{AB'}}{\overline{B'B}} = \frac{\overline{AC'}}{\overline{C'C}},$$

which by Thales' Proportionality Theorem means that $B'C' \parallel BC$. Therefore, by Property 2.1, we get that $\angle AB'C' = \angle ABC$ and $\angle AC'B' = \angle ACB$.

By the SAS criterion for congruence, we get $\triangle AB'C' \cong \triangle A_1B_1C_1$. Therefore, $\angle AB'C' = \angle A_1B_1C_1$ and $\angle AC'B' = \angle A_1C_1B_1$. By combining this with the previous result, we get that $\beta = \beta_1$ and $\gamma = \gamma_1$.

In conclusion, all the angles in the triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are equal, so by the criterion AA that we previously proved, we get that $\triangle ABC \sim \triangle A_1B_1C_1$. ∎

Proof (SSS). Let $\triangle ABC$ and $\triangle A_1B_1C_1$ be two triangles with $\dfrac{\overline{A_1B_1}}{\overline{AB}} = \dfrac{\overline{A_1C_1}}{\overline{AC}} = \dfrac{\overline{B_1C_1}}{\overline{BC}} = k$.



WLOG, let $k < 1$. Then, we can construct points $B' \in AB$ and $C' \in AC$, such that $\overline{AB'} = \overline{A_1B_1}$ and $\overline{AC'} = \overline{A_1C_1}$. Therefore, we have

$$k = \frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}},$$

which, as in the previous proof, by algebraic transformations (taking the reciprocal value, subtracting 1 on both sides, and taking the reciprocal value once again), becomes

$$\frac{\overline{AB'}}{\overline{B'B}} = \frac{\overline{AC'}}{\overline{C'C}}.$$

Therefore, by Thales' Proportionality Theorem, $B'C' \parallel BC$, so by Property 2.1, $\angle AB'C' = \angle ABC$ and $\angle AC'B' = \angle ACB$. By the AA criterion that we earlier proved, we get that $\triangle AB'C' \sim \triangle ABC$ and therefore

$$\frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}} = \frac{\overline{B'C'}}{\overline{BC}}.$$

By substituting the line segments with equal length that we constructed, we get

$$\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{A_1C_1}}{\overline{AC}} = \frac{\overline{B'C'}}{\overline{BC}}.$$

Combining this with the condition, we can conclude that

$$\frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{B_1C_1}}{\overline{BC}}, \text{ i.e. } \overline{B'C'} = \overline{B_1C_1}.$$

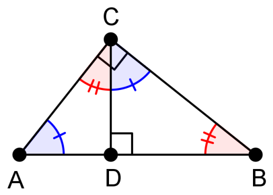
Now, by the SSS criterion for congruence, we get that $\triangle AB'C' \cong \triangle A_1B_1C_1$ and therefore $\angle B'AC' = \angle B_1A_1C_1$. But $\angle B'AC' \equiv \angle BAC$, so $\angle B_1A_1C_1 = \angle BAC$. Combining this with the condition, by the SAS criterion for similarity that we earlier proved, we get that $\triangle A_1B_1C_1 \sim \triangle ABC$. ∎

Example 4.1 (Euclid's laws)**.** In a right triangle ABC, with the right angle at C, let D be the foot of the perpendicular from C to AB. Prove that:

$$\overline{CD}^2 = \overline{AD} \cdot \overline{DB}$$

$$\overline{AC}^2 = \overline{AD} \cdot \overline{AB}$$

$$\overline{BC}^2 = \overline{BD} \cdot \overline{BA}.$$



Proof. Let $\angle CAB = \alpha$ and $\angle CBA = \beta$. Since $\angle ACB = 90^\circ$ and we know that all the angles in a triangle add up to 180°, then $\alpha + \beta = 90^\circ$. Now looking at the triangles ACD and BCD, and remembering again the sum of angles in a triangle, we get that $\angle ACD = 180^\circ - 90^\circ - \alpha = \beta$ and $\angle BCD = 180^\circ - 90^\circ - \beta = \alpha$.

$$\triangle ADC \sim \triangle CDB \text{ (by the criterion AA)}$$

$$\therefore \frac{\overline{AD}}{\overline{DC}} = \frac{\overline{CD}}{\overline{DB}}, \text{ i.e. } \overline{CD}^2 = \overline{AD} \cdot \overline{DB}$$

$$\triangle ACD \sim \triangle ABC \text{ (by the criterion AA)}$$

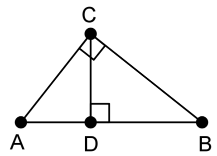
$$\therefore \frac{\overline{AC}}{\overline{AD}} = \frac{\overline{AB}}{\overline{AC}}, \text{ i.e. } \overline{AC}^2 = \overline{AD} \cdot \overline{AB}$$

$$\triangle BCD \sim \triangle BAC \text{ (by the criterion AA)}$$

$$\therefore \frac{\overline{BC}}{\overline{BD}} = \frac{\overline{BA}}{\overline{BC}}, \text{ i.e. } \overline{BC}^2 = \overline{BD} \cdot \overline{BA}$$

■

Example 4.2 (Pythagorean Theorem)**.** Prove that the square of the hypothenuse in a right triangle is equal to the sum of the squares of the legs.



Proof. Let ABC be a right triangle with right angle at C and let CD be an altitude in that triangle. From Example 4.1, we know that $\overline{AC}^2 = \overline{AD} \cdot \overline{AB}$ and $\overline{BC}^2 = \overline{BD} \cdot \overline{BA}$. By adding these equations, we get

$$\overline{AC}^2 + \overline{BC}^2 = \overline{AB} \cdot (\overline{AD} + \overline{BD}) = \overline{AB}^2$$

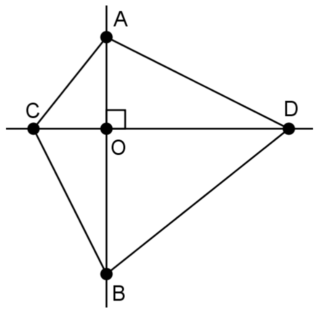
■

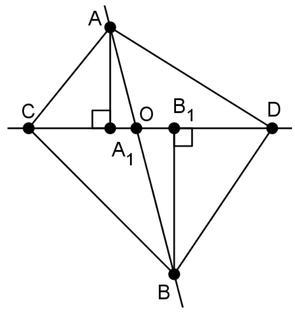
Example 4.3. Let AB and CD be two intersecting lines. Then,

$$AB \perp CD \iff \overline{CA}^2 - \overline{CB}^2 = \overline{DA}^2 - \overline{DB}^2.$$

Proof. Let $AB \cap CD = O$. We will firstly prove the first direction, so let $AB \perp CD$. Then, the triangles $\triangle ACO$, $\triangle BCO$, $\triangle ADO$ and $\triangle BDO$ are right triangles, so by the Pythagorean Theorem, we get

$$\overline{CA}^2 - \overline{CB}^2 = (\overline{OC}^2 + \overline{OA}^2) - (\overline{OC}^2 + \overline{OB}^2) = \overline{OA}^2 - \overline{OB}^2 =$$
$$= (\overline{OD}^2 + \overline{OA}^2) - (\overline{OD}^2 + \overline{OB}^2) = \overline{DA}^2 - \overline{DB}^2 \quad \square$$





Now, let's prove the other direction. Let $\overline{CA}^2 - \overline{CB}^2 = \overline{DA}^2 - \overline{DB}^2$. We will discuss the case where O is between A and B and between C and D. Let the feet of the perpendiculars from A and B to CD be A_1 and B_1, respectively. Then, triangles $\triangle CAA_1$, $\triangle CBB_1$, $\triangle DAA_1$ and $\triangle DBB_1$ are right triangles, so by using the Pythagorean Theorem and substituting in the condition, we get

$$(\overline{CA_1}^2 + \overline{AA_1}^2) - (\overline{CB_1}^2 + \overline{BB_1}^2) = (\overline{DA_1}^2 + \overline{AA_1}^2) - (\overline{DB_1}^2 + \overline{BB_1}^2)$$

After canceling on both sides, we get

$$\overline{CA_1}^2 - \overline{CB_1}^2 = \overline{DA_1}^2 - \overline{DB_1}^2$$
$$\overline{CA_1}^2 - \overline{DA_1}^2 = \overline{CB_1}^2 - \overline{DB_1}^2$$

Using the formula for difference of squares, we get

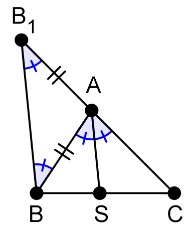
$$(\overline{CA_1} - \overline{DA_1}) \cdot (\overline{CA_1} + \overline{DA_1}) = (\overline{CB_1} - \overline{DB_1}) \cdot (\overline{CB_1} + \overline{DB_1})$$
$$(\overline{CA_1} - \overline{DA_1}) \cdot \overline{CD} = (\overline{CB_1} - \overline{DB_1}) \cdot \overline{CD}$$
$$\overline{CA_1} - \overline{CB_1} = \overline{DA_1} - \overline{DB_1}$$
$$-\overline{A_1B_1} = \overline{A_1B_1}$$
$$0 = 2 \cdot \overline{A_1B_1}$$
$$A_1 \equiv B_1$$

Therefore, the perpendiculars to CD from A and B pass through a common point on CD, so they must be the same line, i.e. $AB \perp CD$.

In the cases where O is not between A and B or between C and D, the proof follows exactly the same steps. There might be a different operation when dealing with the line segments (addition or subtraction) depending on the configuration, but the result will always be the same. ■

Example 4.4 (Angle Bisector Theorem)**.** The angle bisector in a triangle divides the opposite side in segments proportional to the other two sides of the triangle.

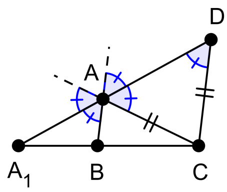
Proof. Here is an idea how to prove this theorem. Let ABC be a triangle and let S be a point on BC, such that AS is an angle bisector in $\triangle ABC$. We need to prove that $\frac{\overline{BS}}{\overline{SC}} = \frac{\overline{AB}}{\overline{AC}}$. If we rearrange this equality, we get that we need to prove that $\frac{\overline{CS}}{\overline{SB}} = \frac{\overline{CA}}{\overline{AB}}$. This resembles the Thales' Proportionality Theorem, with the exception that the points C, A and B are not collinear. So if we take a point B_1 on the extension of CA, such that $\overline{AB_1} = \overline{AB}$, then we will only need to prove that SA is parallel to BB_1.



Let $B_1 \in CA$, such that $\overline{AB_1} = \overline{AB}$. The triangle $\triangle ABB_1$ is isosceles, so $\angle ABB_1 = \angle AB_1B = \varphi$. The angle $\angle BAC$ is exterior angle of $\triangle ABB_1$, so $\angle BAC = \angle ABB_1 + \angle AB_1B = 2\varphi$. Since AS is an angle bisector, $\angle BAS = \frac{1}{2}\angle BAC = \varphi$. So, $\angle BAS = \angle ABB_1$, which means that $SA \parallel BB_1$. By the Thales' Proportionality Theorem, we get that $\frac{\overline{CS}}{\overline{SB}} = \frac{\overline{CA}}{\overline{AB_1}}$. By substituting $\overline{AB}$ for $\overline{AB_1}$, we get $\frac{\overline{CS}}{\overline{SB}} = \frac{\overline{CA}}{\overline{AB}}$. ■

Example 4.5 (External Angle Bisector Theorem)**.** Let the bisector of the exterior angle at vertex A in $\triangle ABC$ intersect the line BC at A_1. Prove that

$$\frac{\overline{BA_1}}{\overline{A_1C}} = \frac{\overline{AB}}{\overline{AC}}.$$



Proof. WLOG, let $\overline{AB} < \overline{AC}$, i.e. $\overline{A_1B} < \overline{A_1C}$. Let D be a point on the line AA_1, such that $AB \parallel CD$. Then, by Property 2.1, $\angle A_1AB = \angle A_1DC$, so $\triangle A_1AB \sim \triangle A_1DC$ and therefore

$$\frac{\overline{A_1B}}{\overline{A_1C}} = \frac{\overline{AB}}{\overline{DC}}. \tag{*}$$

Let α' be the external angle at the vertex A in $\triangle ABC$. Then, as vertical angles,

$$\angle DAC = \frac{\alpha'}{2} = \angle A_1AB = \angle A_1DC \equiv \angle ADC,$$

so $\triangle ADC$ is isosceles, i.e. $\overline{AC} = \overline{DC}$. By substituting in (*), we get the desired ratio. ■

Related problems: 25 and 30 and 32.

Chapter 5

Circles

A circle is a set of point equidistant from one previously chosen point, called the center. The distance from the center to the circle is called the radius of the circle. We will usually notate a circle with center O and radius r as $\omega(O, r)$.

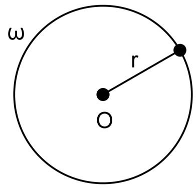
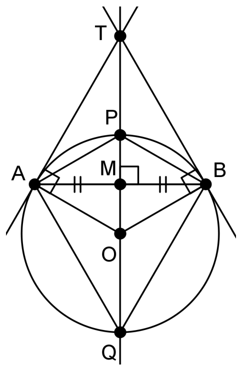


Figure 5.1: Circle ω with center O and radius r.

5.1 Symmetry in a Circle



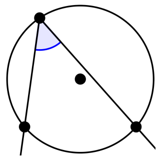
Let AB be a chord in a circle. If we connect the points A and B with the center O, we get an isosceles triangle ABO. If M is the midpoint of AB, then $\triangle AMO \cong \triangle BMO$ (by SSS) and therefore $\angle AMO = \angle BMO$, i.e. $OM \perp AB$. Also, $\angle AOM = \angle BOM$, so if we denote by P and Q the intersections of OM with the circle, we get that $\triangle OAP \cong \triangle OBP$ (by SAS) which yields $\overline{AP} = \overline{BP}$ and consequently $\widehat{AP} = \widehat{BP}$. Similarly, $\triangle AOQ \cong \triangle BOQ$ (by SAS) and $\widehat{AQ} = \widehat{BQ}$. Looking from a different perspective, this all means that the center of the circle O and the midpoints of the minor and major arc $\widehat{AB}$, P and Q, all lie on the perpendicular bisector of the chord AB. Hence, the center of any circle can be found as the intersection of the perpendicular bisectors of any two chords.

Moreover, let T be the intersection of the tangents at A and B. By the Pythagorean Theorem, $\overline{TA}^2 = \overline{TO}^2 - \overline{OA}^2 = \overline{TO}^2 - \overline{OB}^2 = \overline{TB}^2$, i.e. $\overline{TA} = \overline{TB}$. So the tangent segments from a point to the circle are equal. Now, $\triangle OAT \cong \triangle OBT$ (by SSS), so $\angle TOA = \angle TOB$, which combined with the previous findings, means that T also lies on the perpendicular bisector of the chord AB.

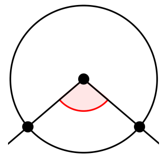
5.2 Angles in a Circle

An inscribed angle is an angle whose vertex lies on a circle and its rays intersect that circle.

A central angle is an angle whose vertex is the center of the circle and its rays intersect that circle.

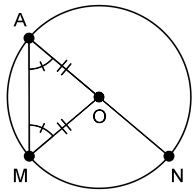


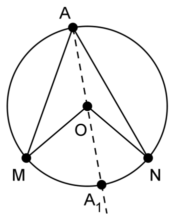
(a) Inscribed angle

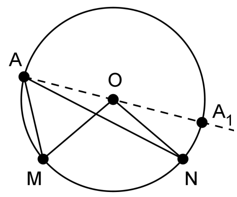


(b) Central angle

Now, let's take a look at the relation between an inscribed angle and a central angle that subtend the same arc. Let $\angle MAN$ and $\angle MON$ be an inscribed and the central angle that subtend the arc $\widehat{MN}$, respectively. The center O can be in three positions relative to $\angle MAN$.







i) O lies on one of the rays of $\angle MAN$, WLOG let O lie on the ray AN.

$$\overline{OA} = r = \overline{OM}$$

$$\therefore \triangle OAM \text{ is isosceles.}$$

$$\therefore \angle OAM = \angle OMA$$

$$\therefore \angle MON = \angle OAM + \angle OMA = 2 \cdot \angle OAM \equiv 2 \cdot \angle MAN$$

ii) O is in the interior of $\angle MAN$.
Let A_1 be the second intersection of AO with the circle.

$$\angle MOA_1 = 2 \cdot \angle MAA_1 \text{ (from case i)}$$

$$\angle A_1ON = 2 \cdot \angle A_1AN \text{ (from case i)}$$

$$\therefore \angle MON = \angle MOA_1 + \angle A_1ON = 2 \cdot \angle MAA_1 + 2 \cdot \angle A_1AN = 2 \cdot \angle MAN$$

iii) O is in the exterior of $\angle MAN$, WLOG O is closer to the ray AN.
Let A_1 be the second intersection of AO with the circle.

$$\angle MOA_1 = 2 \cdot \angle MAA_1 \text{ (from case i)}$$

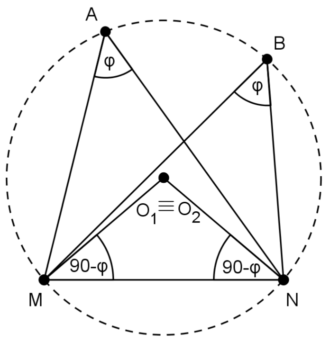
$$\angle NOA_1 = 2 \cdot \angle NAA_1 \text{ (from case i)}$$

$$\therefore \angle MON = \angle MOA_1 - \angle NOA_1 = 2 \cdot \angle MAA_1 - 2 \cdot \angle NAA_1 = 2 \cdot \angle MAN$$

Therefore, any inscribed angle is half the central angle that subtends the same arc. It also implies that all the inscribed angles that subtend the same arc are equal.

The converse is also true. The proof is "less attractive", but it will be presented for the sake of completeness :) We will prove that if two angles $\angle MAN$ and $\angle MBN$ are equal (and their vertices A and B lie on the same side of the line MN), then their vertices, A and B, and the intersection points of their corresponding rays, M and N, are concyclic.

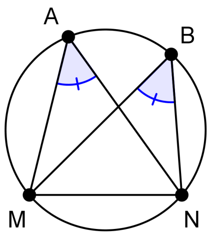
Let $\omega_1(O_1, r_1)$ be the circumcircle of $\triangle MAN$. Let $\varphi = \angle MAN = \angle MBN$. Therefore, $\angle MO_1N = 2 \cdot \angle MAN = 2\varphi$. Since $\triangle MO_1N$ is isosceles (because $\overline{O_1M} = r_1 = \overline{O_1N}$), $\angle O_1MN = \angle O_1NM = 90° - \varphi$. Similiarly, if $\omega_2(O_2, r_2)$ is the circumcirle of $\triangle MBN$, then $\angle O_2MN = \angle O_2NM = 90° - \varphi$. Therefore, by the ASA criterion, $\triangle MO_1N \cong \triangle MO_2N$. Since A and B, and consequently O_1 and O_2 lie on the same side of MN, we get that $O_1 \equiv O_2$. Therefore, $r_1 = \overline{O_1M} = \overline{O_2M} = r_2$, so $\omega_1 \equiv \omega_2$, i.e. the points M, A, B and N lie on a single circle.



In conclusion, we get two important properties of the angles in a circle:

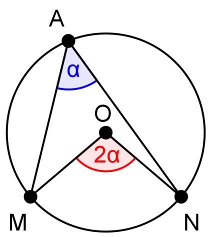
Property 5.1. Inscribed angles that subtend the same arc are equal. The converse is also true, i.e. if two angles are equal, then their vertices and the intersection points of their corresponding rays are concyclic.

$$M, A, B, N \in \omega \text{ (in that order)} \iff \angle MAN = \angle MBN \tag{5.1}$$



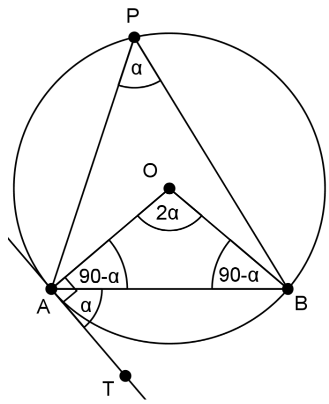
Property 5.2. The central angle is twice an inscribed angle that subtends the same arc.

$$M, A, N \in \omega(O, r) \implies \angle MON = 2 \cdot \angle MAN \tag{5.2}$$



Finally, let's investigate the angle between a tangent and a chord through the tangent point.

Let AB be a chord in $\omega(O, r)$ and let TA be a tangent to ω at A. Let $\angle BAT = \alpha$. Since TA is a tangent, then it must be perpendicular to OA, i.e. $\angle OAT = 90°$.



$$\therefore \angle OAB = \angle OAT - \angle BAT = 90° - \alpha$$

$$\overline{OA} = r = \overline{OB}$$

$$\therefore \triangle OAB \text{ is isosceles}$$

$$\therefore \angle OAB = \angle OBA = 90° - \alpha$$

$$\angle AOB = 180° - 2(90° - \alpha) = 180° - 180° + 2\alpha = 2\alpha$$

Let $\angle APB$ be any inscribed angle over the arc $\widehat{AB}$. Then,

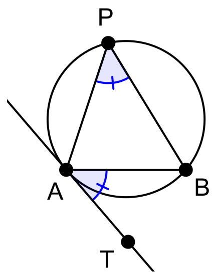
$$\angle APB \overset{(5.2)}{=} \frac{1}{2} \cdot \angle AOB = \frac{1}{2} \cdot 2\alpha = \alpha.$$

In conclusion, we get the following property:

Property 5.3. The angle between a tangent and a chord is equal to any inscribed angle that subtends that chord.

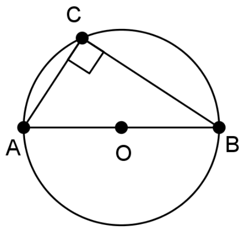
$$\angle TAB = \angle APB \tag{5.3}$$

The converse is also true, i.e. if an angle between a chord and a line through one of the endpoints of the chord is equal to an inscribed angle that subtends that chord, then that line must be tangent to the circle.



We will now see a few useful consequences of the relation between an inscribed and a central angle.

Example 5.1 (Thales' Theorem)**.** Every inscribed angle that subtends a diameter is a right angle.

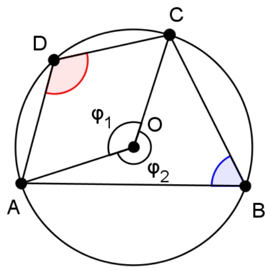


Proof. Let AB be a diameter in a circle with center O, and let C be another point on the circle.

$$\angle ACB \overset{(5.2)}{=} \frac{1}{2} \cdot \angle AOB = \frac{1}{2} \cdot 180^\circ = 90^\circ \qquad \blacksquare$$

Remark. Moreover, we can see that inscribed angles that subtend an arc greater than half the circumference are obtuse and inscribed angles that subtend an arc smaller than half the circumference are acute.

Example 5.2. The opposite angles of a cyclic quadrialateral are supplementary.



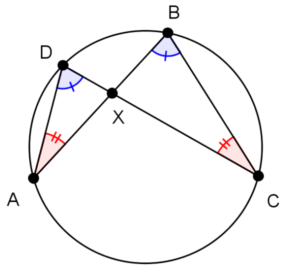
Proof. Let $ABCD$ be a cyclic quadrilateral and let its circumcircle be centered at O. Let φ_1 and φ_2 be the central angles that subtend the arcs $\widehat{ADC}$ and $\widehat{ABC}$, respectively.

$$\angle ABC \overset{(5.2)}{=} \frac{1}{2}\varphi_1 \text{ (over the arc } \widehat{ADC})$$

$$\angle ADC \overset{(5.2)}{=} \frac{1}{2}\varphi_2 \text{ (over the arc } \widehat{ABC})$$

$$\therefore \angle ABC + \angle ADC = \frac{1}{2} \cdot (\varphi_1 + \varphi_2) = \frac{1}{2} \cdot 360^\circ = 180^\circ \qquad \blacksquare$$

Example 5.3 (Intersecting Chords Theorem)**.** Let AB and CD be two line segments that intersect at X. Then the quadrilateral $ACBD$ is cyclic if and only if $\overline{AX} \cdot \overline{XB} = \overline{CX} \cdot \overline{XD}$.



Proof. Let's notice that

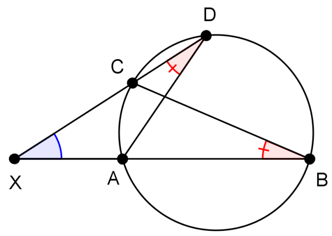
$$\angle AXD = \angle CXB. \tag{*}$$

Then,

$$\begin{aligned} &ACBD \text{ is cyclic} \\ \overset{(5.1)}{\iff}\ &\angle ADC = \angle ABC \quad \text{and} \quad \angle DAB = \angle DCB \\ \iff\ &\angle ADX = \angle XBC \quad \text{and} \quad \angle DAX = \angle XCB \\ \iff\ &\triangle ADX \sim \triangle CBX \\ \overset{(*)}{\iff}\ &\frac{\overline{AX}}{\overline{XD}} = \frac{\overline{CX}}{\overline{XB}} \\ \iff\ &\overline{AX} \cdot \overline{XB} = \overline{CX} \cdot \overline{XD} \end{aligned}$$

■

Example 5.4 (Intersecting Secants Theorem)**.** Let AB and CD be two lines that intersect at X, such that $X - A - B$ and $X - C - D$. Then the quadrilateral $ABDC$ is cyclic if and only if $\overline{XA} \cdot \overline{XB} = \overline{XC} \cdot \overline{XD}$.



Proof. Let's notice that

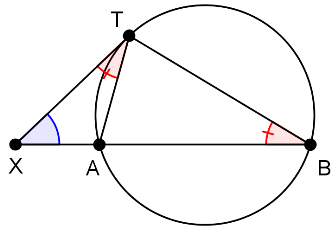
$$\angle CXB \equiv \angle AXD. \tag{*}$$

Then,

$$\begin{aligned} &ABDC \text{ is cyclic} \\ \overset{(5.1)}{\iff}\ &\angle ABC = \angle ADC \\ \iff\ &\angle XBC = \angle ADX \\ \overset{(*)}{\iff}\ &\triangle XBC \sim \triangle XDA \\ \overset{(*)}{\iff}\ &\frac{\overline{XB}}{\overline{XC}} = \frac{\overline{XD}}{\overline{XA}} \\ \iff\ &\overline{XA} \cdot \overline{XB} = \overline{XC} \cdot \overline{XD} \end{aligned}$$

■

Example 5.5 (Secant-Tangent Theorem)**.** Let ABT be a triangle and let X be a point on AB, such that $X - A - B$. Then XT is tangent to the circumcircle of $\triangle ABT$ if and only if $\overline{XT}^2 = \overline{XA} \cdot \overline{XB}$.



Proof. Let's notice that

$$\angle TXA \equiv \angle BXT. \tag{*}$$

Then,

$$\begin{aligned} & XT \text{ is tangent to } (ABT) \\ \overset{(5.3)}{\iff} \ & \angle ATX = \angle TBA \\ \iff \ & \angle ATX = \angle TBX \\ \overset{(*)}{\iff} \ & \triangle XTA \sim \triangle XBT \\ \overset{(*)}{\iff} \ & \frac{\overline{XT}}{\overline{XA}} = \frac{\overline{XB}}{\overline{XT}} \\ \iff \ & \overline{XT}^2 = \overline{XA} \cdot \overline{XB} \end{aligned}$$

■

Related problems: 34, 35, 36, 37, 38, 39, 40, 45, 46, 47, 48, 50, 52, 53, 54, 55, 56, 57, 58, 60, 64, 69, 70, 72, 75, 90, 97, 101, 105, 111 and 112.

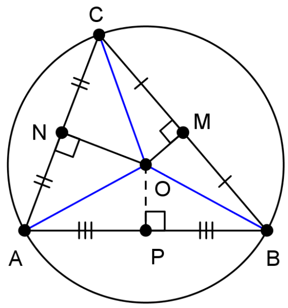
Chapter 6

A Few Important Centers in a Triangle

Property 6.1 (Circumcenter)**.** The three *side bisectors* of a triangle are concurrent. The point of concurrence is the center of a circle that passes through all three vertices of the triangle.

The point of concurrence is called the *circumcenter* of the triangle. The circle that is circumscribed around the triangle is called the *circumcircle* of the triangle.

Proof. Let M, N and P be the midpoints of the sides BC, CA and AB, respectively. Let O be the intersection of the side bisectors of BC and CA. Then $OM \perp BC$ and $ON \perp CA$.



Let's take a look at the triangles $\triangle OMB$ and $\triangle OMC$. They have a common side OM, $\angle OMB = 90° = \angle OMC$ and $\overline{MB} = \overline{MC}$, so by SAS, they are congruent. Therefore, their corresponding sides are equal, i.e. $\overline{OB} = \overline{OC}$. Similarly, $\triangle ONC \cong \triangle ONA$, so $\overline{OC} = \overline{OA}$.

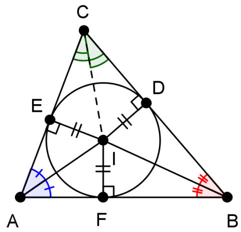
Therefore, $\overline{OA} = \overline{OB}$, so $\triangle OAB$ is isosceles. Therefore, since P is the midpoint of AB, by Example 2.5, we get that OP is the side bisector of AB.

Since $\overline{OA} = \overline{OB} = \overline{OC}$, then O is the center of a circle that passes through the vertices of $\triangle ABC$. ■

Property 6.2 (Incenter)**.** The three *angle bisectors* of a triangle are concurrent. The point of concurrence is the center of a circle that is tangent to all three sides of the triangle.

The point of concurrence is called the *incenter* of the triangle. The circle that is inscribed inside the triangle is called the *incircle* of the triangle.

Proof. Let I be the intersection of the angle bisectors of $\angle CAB$ and $\angle ABC$. Let D, E and F be the feet of the perpendiculars from I to the sides BC, CA and AB, respectively.



Let's take a look at the triangles $\triangle AIE$ and $\triangle AIF$. They are right triangles and $\angle IAE = \frac{\alpha}{2} = \angle IAF$, so they are similar. But they have a common corresponding side AI, so their ratio of similarity is 1, i.e. they are congruent. Therefore, $\overline{IE} = \overline{IF}$. Similarly, $\triangle BIF \cong \triangle BID$, so $\overline{IF} = \overline{ID}$.

Therefore, $\overline{IE} = \overline{ID}$. The triangles $\triangle CIE$ and $\triangle CID$ are right triangles, so by the Pythagorean Theorem, we get

$$\overline{CE}^2 = \overline{IC}^2 - \overline{IE}^2 = \overline{IC}^2 - \overline{ID}^2 = \overline{CD}^2, \text{ i.e. } \overline{CE} = \overline{CD}.$$

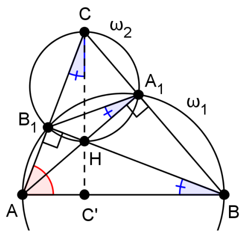
So, by SSS, $\triangle CIE \cong \triangle CID$ and therefore $\angle ICE = \angle ICD$, i.e. CI is the angle bisector of $\angle ECD \equiv \angle ACB$.

Since $\overline{ID} = \overline{IE} = \overline{IF}$, $ID \perp BC$, $IE \perp CA$ and $IF \perp AB$, then I is the center of a circle that is tangent to the sides of $\triangle ABC$. ■

Property 6.3 (Orthocenter)**.** The three *altitudes* of a triangle are concurrent.

The point of concurrence is called the *orthocenter* of the triangle.

Proof. Let the altitudes AA_1 and BB_1 intersect at H.



Since $AA_1 \perp BC$ and $BB_1 \perp AC$, then $\angle AA_1B = 90^\circ = \angle AB_1B$. Therefore, ABA_1B_1 is a cyclic quadrilateral. Let (ABA_1B_1) be ω_1. Also,

$$\angle CB_1H + \angle CA_1H \equiv \angle CB_1B + \angle CA_1A = 90^\circ + 90^\circ = 180^\circ,$$

so CB_1HA_1 is a cyclic quadrilateral. Let (CB_1HA_1) be ω_2. Let $CH \cap AB = C'$. Then,

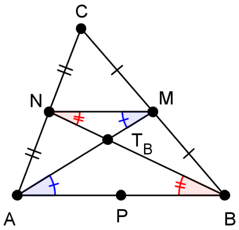
$$\angle ACC' \equiv \angle B_1CH \overset{\omega_2}{=} \angle B_1A_1H \equiv \angle B_1A_1A \overset{\omega_1}{=} \angle B_1BA \overset{\triangle ABB_1}{=} 90^\circ - \alpha$$

Finally, from Sum of angles in a triangle $\triangle ACC'$, we get that $\angle AC'C = 90^\circ$, i.e. $CH \perp AB$. ■

Property 6.4 (Centroid)**.** The three *medians* of a triangle are concurrent. The point of concurrence divides the medians in ratio $2:1$.

The point of concurrence is called the *centroid* of the triangle.

Proof. Let M, N and P be the midpoints of the sides BC, CA and AB, respectively. Then, MN is a midsegment in $\triangle ABC$, so $MN \parallel AB$ and $\overline{AB} = 2 \cdot \overline{MN}$.



Let the B-median intersect the A-median at a point T_B. Then, by Property 2.1, $\angle T_B AB = \angle T_B MN$ and $\angle T_B BA = \angle T_B NM$, so $\triangle T_B AB \sim \triangle T_B MN$ and therefore

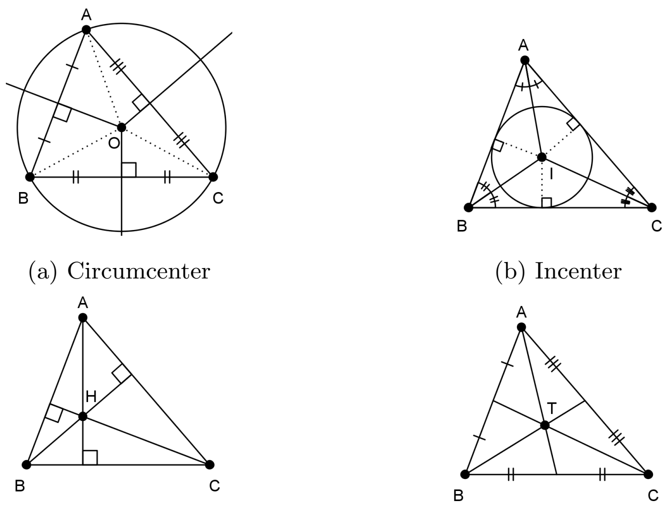
$$\frac{\overline{AT_B}}{\overline{T_B M}} = \frac{\overline{AB}}{\overline{MN}} = 2.$$

Similarly, if the C-median intersect the A-median at T_C, we can get

$$\frac{\overline{AT_C}}{\overline{T_C M}} = 2.$$

So $T_B \equiv T_C \equiv T$, i.e. the B-median and the C-median intersect the A-median at the same point T. Additionally,

$$\frac{\overline{AT}}{\overline{TM}} = \frac{\overline{BT}}{\overline{TN}} = \frac{\overline{CT}}{\overline{TP}} = 2. \qquad \blacksquare$$



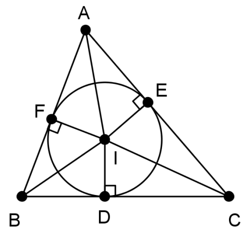
(a) Circumcenter (b) Incenter

(c) Orthocenter (d) Centroid

Figure 6.1: The four most important centers of a triangle ABC.

Property 6.5. Let s and r be the semiperimeter and the radius of the incircle, respectively, in a triangle $\triangle ABC$. Then,

$$P_{\triangle ABC} = r \cdot s.$$



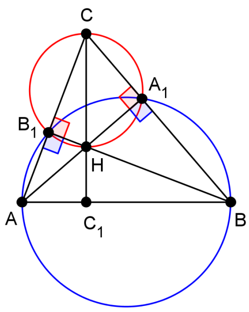
Proof. Let D, E and F be the tangent points of the incircle with the sides BC, CA and AB, respectively. Let I be the incenter of $\triangle ABC$. Then $ID \perp BC$, $IE \perp CA$ and $IF \perp AB$.

$$\begin{aligned} P_{\triangle ABC} &= P_{\triangle BCI} + P_{\triangle CAI} + P_{\triangle ABI} = \\ &= \frac{\overline{BC} \cdot \overline{ID}}{2} + \frac{\overline{CA} \cdot \overline{IE}}{2} + \frac{\overline{AB} \cdot \overline{IF}}{2} = \\ &= \frac{a \cdot r}{2} + \frac{b \cdot r}{2} + \frac{c \cdot r}{2} = \\ &= r \cdot \frac{a+b+c}{2} = \\ &= r \cdot s \end{aligned}$$

■

Property 6.6. Let AA_1, BB_1 and CC_1 be the altitudes in a $\triangle ABC$ and let H be its orthocenter. Then,

- ABA_1B_1, BCB_1C_1 and CAC_1A_1 are cyclic quadrilaterals
- AB_1HC_1, BC_1HA_1 and CA_1HB_1 are cyclic quadrilaterals



Proof. Since $AA_1 \perp BC$ and $BB_1 \perp AC$, then $\angle AA_1B = 90^\circ = \angle AB_1B$. Therefore, ABA_1B_1 is a cyclic quadrilateral. Similarly, BCB_1C_1 and CAC_1A_1 are cyclic quadrilaterals. □

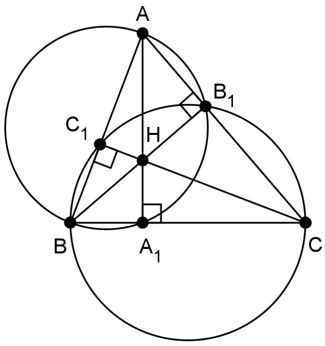
On the other hand,

$$\angle CB_1H + \angle CA_1H \equiv \angle CB_1B + \angle CA_1A = 90^\circ + 90^\circ = 180^\circ,$$

so CA_1HB_1 is a cyclic quadrilateral. Similarly, AB_1HC_1 and BC_1HA_1 are cyclic quadrilaterals. ■

Property 6.7. Let AA_1, BB_1 and CC_1 be the altitudes in a $\triangle ABC$ and let H be its orthocenter. Then,

$$\overline{AH} \cdot \overline{HA_1} = \overline{BH} \cdot \overline{HB_1} = \overline{CH} \cdot \overline{HC_1}.$$



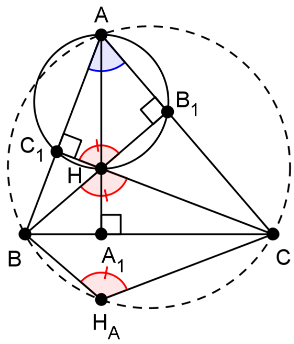
Proof. From Property 6.6, we know that ABA_1B_1 is a cyclic quadrilateral. Since the altitudes AA_1 and BB_1 intersect at the orthocenter H, by the Intersecting Chords Theorem, we get

$$\overline{AH} \cdot \overline{HA_1} = \overline{BH} \cdot \overline{HB_1}.$$

Similarly, from the cyclic quadrilateral BCB_1C_1, we get

$$\overline{BH} \cdot \overline{HB_1} = \overline{CH} \cdot \overline{HC_1}.$$ ■

Property 6.8. The reflections of the orthocenter with respect to the sides of a triangle lie on the circumcircle of the triangle.

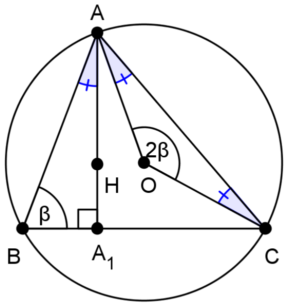


Proof. Let B_1 and C_1 be the feet of the altitudes from B and C, respectively, in $\triangle ABC$. Let H be its orthocenter. From Property 6.6, we know that AB_1HC_1 is a cyclic quadrilateral and therefore $\angle B_1HC_1 = 180^\circ - \alpha$. As vertical angles, $\angle BHC = \angle B_1HC_1$. Let H_A be the reflection of H with respect to the side BC. By symmetry, $\angle BH_AC = \angle BHC$. Therefore, $\angle BH_AC = 180^\circ - \alpha$. Finally,

$$\angle CAB + \angle BH_AC = \alpha + 180^\circ - \alpha = 180^\circ,$$

so $H_A \in (ABC)$. Similarly, $H_B, H_C \in (ABC)$. ■

Property 6.9. The orthocenter and the circumcenter in a triangle are isogonal conjugates[1].



Proof. WLOG, $\overline{AB} < \overline{AC}$. Let O and H be the circumcenter and the orthocenter, respectively, in $\triangle ABC$. We need to prove that $\angle HAB = \angle OAC$.

Let A_1 be the foot of the altitude from A to BC. Then, from $\triangle ABA_1$, we get

$$\angle HAB \equiv \angle A_1AB = 90^\circ - \angle ABA_1 = 90^\circ - \beta. \tag{1}$$

Since $\angle ABC$ and $\angle AOC$ are inscibed and central angle, respectively, over $\widehat{AC}$ in (ABC), we have

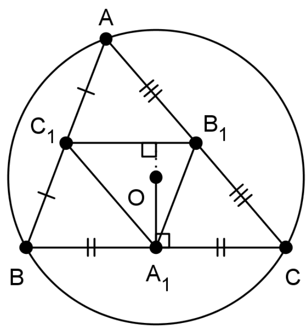
$$\angle AOC = 2 \cdot \angle ABC = 2\beta.$$

Since $\overline{OA} = R = \overline{OC}$, from sum of the angles in the isosceles $\triangle AOC$, we have

$$\angle OAC = \frac{180^\circ - \angle AOC}{2} = \frac{180^\circ - 2\beta}{2} = 90^\circ - \beta. \tag{2}$$

From (1) and (2), we get that $\angle HAB = \angle OAC$. Similarly, $\angle HBC = \angle OBA$ and $\angle HCA = \angle OCB$, so H and O are isogonal conjugates in $\triangle ABC$. ■

Property 6.10. The circumcenter of a triangle is the orthocenter of its medial triangle[2].

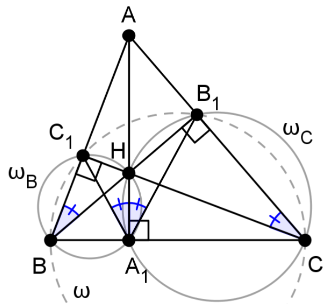


Proof. Let A_1, B_1 and C_1 be the midpoints of the sides BC, CA and AB in $\triangle ABC$, respectively. Let O be the circumcenter of $\triangle ABC$. Since A_1 is the midpoint of the chord BC in (ABC), $OA_1 \perp BC$. Since B_1C_1 is the midsegment in $\triangle ABC$, $B_1C_1 \parallel BC$. Therefore, $OA_1 \perp B_1C_1$, i.e. A_1O is an altitude in $\triangle A_1B_1C_1$. Similarly, B_1O and C_1O are also altitudes in $\triangle A_1B_1C_1$, so O is the orthocenter of $\triangle A_1B_1C_1$. ■

[1] Two points P and P^* are called isogonal conjugates if XP^* is the reflection of XP across the angle bisector of the angle at the vertex X in a triangle $\triangle ABC$, where X is any of the vertices A, B or C. In other words, the lines XP and XP^* make equal angles with the sides of the triangle that contain X, e.g. $\angle PAB = \angle P^*AC$.

[2] The medial triangle is the triangle with vertices the midpoints of a triangle.

Property 6.11. The orthocenter of a triangle is the incenter of its orthic triangle[3].



Proof. Let AA_1, BB_1 and CC_1 be the altitudes in a $\triangle ABC$. Let H be the orthocenter of $\triangle ABC$. We want to prove that A_1H is the angle bisector of $\angle C_1A_1B_1$, i.e. $\angle C_1A_1H = \angle HA_1B_1$. From Property 6.6, we know that BC_1HA_1, CA_1HB_1 and BCB_1C_1 are cyclic quadrilaterals. Let's call them ω_B, ω_C and ω, respectively. Then,

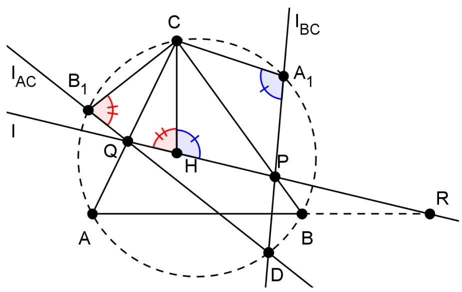
$$\angle C_1A_1H \overset{\omega_B}{=} \angle C_1BH \equiv \angle C_1BB_1 \overset{\omega}{=} \angle C_1CB_1 \equiv \angle HCB_1 \overset{\omega_C}{=} \angle HA_1B_1.$$

Similarly, B_1H and C_1H are angle bisectors of $\angle A_1B_1C_1$ and $\angle B_1C_1A_1$, so H is the incenter of $\triangle A_1B_1C_1$. ■

Property 6.12. Let l be any line through the orthocenter of $\triangle ABC$. Prove that the reflections of the line l with respect to the lines AB, BC and CA are concurrent at the circumcircle of $\triangle ABC$.

Proof. Let the line l intersect the lines BC, CA and AB at P, Q and R, respectively. We will examine the case when H is inside $\triangle ABC$ (the other cases should be similar). Since one of these points will be on the extension of a side and two of these points will be on the sides of the triangle, WLOG, let R be on the extension of the side AB.

Let A_1, B_1 and C_1 be the reflections of the orthocenter with respect to the sides BC, CA and AB, respectively. From Property 6.8, we know that $A_1, B_1, C_1 \in (ABC)$. Therefore, l_{BC}, the reflection of the line l with respect to the line BC, will contain A_1 (and similarly for the other lines). Let D be the intersection of the lines l_{BC} and l_{AC}. We want to prove that $D \in (ABC)$.



$$\angle DA_1C + \angle CB_1D \equiv \angle PA_1C + \angle CB_1Q = \angle PHC + \angle CHQ = 180^\circ$$

$$\therefore D \in (A_1CB_1) \equiv (ABC)$$

Similarly, we can prove that the intersection of the lines l_{AB} and l_{BC} lies on (ABC). ■

Related problems: See problems 9, 19, 20, 21, 61, 62 and 63.

[3] The orthic triangle is the triangle with vertices the feet of the altitudes of a triangle.

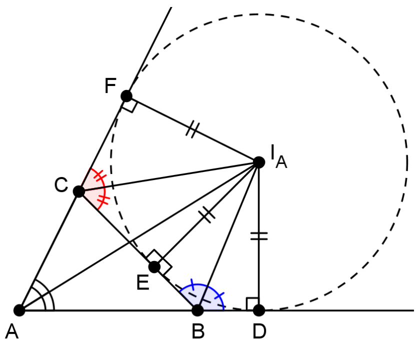
Chapter 7

Excircles

Property 7.1 (Excenter)**.** The external bisectors of two of the angles and the internal angle bisector of the third angle in a triangle are concurrent. The point of concurrence is the center of a circle that is externally tangent to one of the sides and the extensions of the other two sides of a triangle.

The point of concurrence is called an *excenter* of the triangle. The circle that is exscribed outside the triangle is called an *excircle* of the triangle. There are three excircles for each triangle.

Proof. Let I_A be the intersection of the external angle bisectors at B and C, in a triangle $\triangle ABC$. Let D, E, and F be the feet of the perpendiculars from I_A to the lines AB, BC and AC, respectively. The triangles $\triangle BDI_A$ and $\triangle BEI_A$



are similar because they have two equal angles. Moreover, they have a common corresponding side, so they are congruent. Therefore, $\overline{I_AD} = \overline{I_AE}$. Similarly, $\overline{I_AF} = \overline{I_AE}$. The triangles $\triangle I_ADA$ and $\triangle I_AFA$ are right triangles with two equal corresponding sides, so by the Pythagorean Theorem, the third sides are also equal. By the criterion SSS, these triangles are congruent. Therefore, their corresponding angles $\angle I_AAD$ and $\angle I_AAF$ are equal, so AI_A is angle bisector of $\angle BAC$.

Since $\overline{I_AD} = \overline{I_AE} = \overline{I_AF}$, $I_AD \perp AB$, $I_AE \perp BC$ and $I_AF \perp CA$, then I is the center of a circle that is tangent to the lines AB, BC and CA. ■

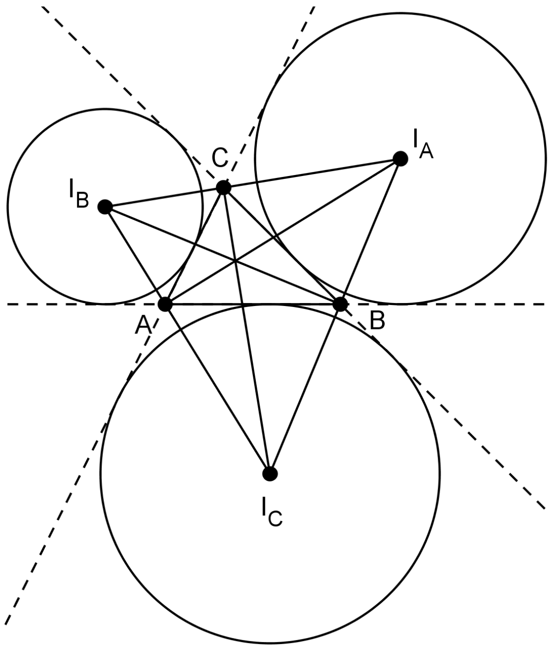
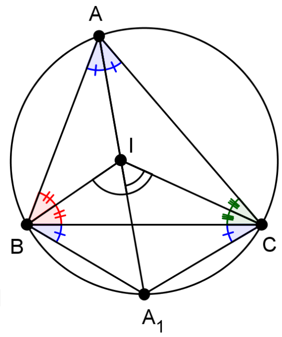


Figure 7.1: The three excircles of a triangle ABC.

Example 7.1. Let I be the incenter of $\triangle ABC$. Let A_1 be the second intersection of the angle bisector of $\angle BAC$ with the circumcircle of $\triangle ABC$. Prove that

$$\overline{A_1B} = \overline{A_1I} = \overline{A_1C}.$$

Proof.



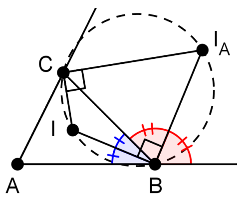
$$\angle A_1BI = \angle A_1BC + \angle CBI = \angle A_1AC + \frac{\beta}{2} = \frac{\alpha}{2} + \frac{\beta}{2}$$

$$\angle A_1IB = \angle IAB + \angle IBA = \frac{\alpha}{2} + \frac{\beta}{2}$$

$$\therefore \triangle A_1BI \text{ is isosceles, i.e. } \overline{A_1B} = \overline{A_1I}$$

Similarly, $\overline{A_1C} = \overline{A_1I}$. ∎

Example 7.2. Let I and I_A be the incenter and A-excenter in $\triangle ABC$, respectively. Prove that the quadrilateral IBI_AC is cyclic.

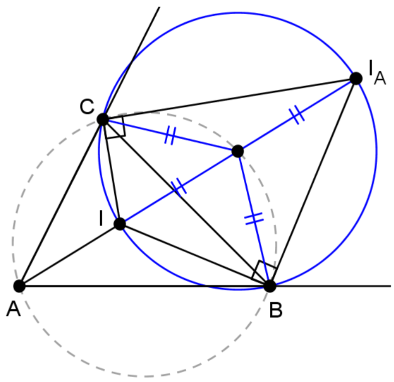


Proof. Since I and I_A are incenter and excenter, BI and BI_A are internal and external angle bisectors.

$$\therefore \angle IBI_A = \angle IBC + \angle CBI_A = \frac{\beta}{2} + \frac{180^\circ - \beta}{2} = 90^\circ$$

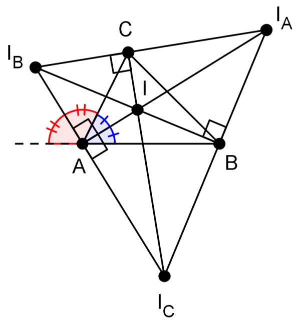
Similarly, $\angle ICI_A = 90^\circ$.
Therefore, $\angle IBI_A + \angle ICI_A = 180^\circ$, so IBI_AC is cyclic. ∎

With these two examples, we actually proved that the circle with diameter II_A passes through B and C and its center is the intersection of AI with the circumcircle of $\triangle ABC$.



Example 7.3. Let I_A, I_B and I_C be the excenters opposite of A, B, and C in $\triangle ABC$, respectively. Prove that the incenter of $\triangle ABC$ is the orthocenter of $\triangle I_A I_B I_C$.

Proof. Let I be the incenter of $\triangle ABC$. Since I and I_B are incenter and excenter, AI and AI_B are internal and external angle bisectors.



$$\therefore \angle IAI_B = \angle IAC + \angle CAI_B = \frac{\alpha}{2} + \frac{180^\circ - \alpha}{2} = 90^\circ$$

Similarly, $\angle IAI_C = 90^\circ$. Therefore, since $\angle IAI_B + \angle IAI_C = 180^\circ$, $A \in I_B I_C$ and $I_A A \equiv IA \perp I_B I_C$, so $I_A A$ is an altitude in $\triangle I_A I_B I_C$. Similarly, $I_B B$ and $I_C C$ are altitudes, too, so I is the orthocenter of $\triangle I_A I_B I_C$. ∎

Example 7.4. Let I and I_A be the incenter and the A-excenter in $\triangle ABC$. Prove that

$$\overline{AI} \cdot \overline{AI_A} = \overline{AB} \cdot \overline{AC}.$$

Proof 1. Let's look at the triangles $\triangle AIC$ and $\triangle ABI_A$.

$$\angle CAI = \frac{\alpha}{2} = \angle I_A AB \tag{1}$$

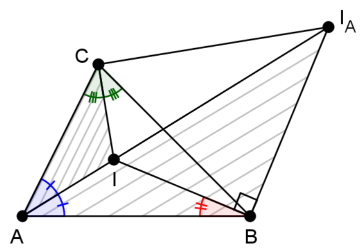
$$\angle AIC = 180^\circ - (\angle IAC + \angle ICA) = 180^\circ - \left(\frac{\alpha + \gamma}{2}\right) = 90^\circ + \frac{\beta}{2}$$

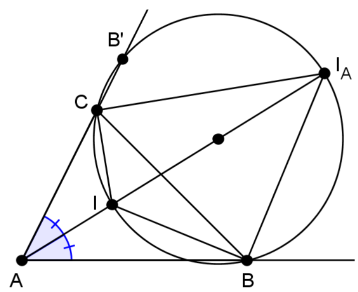
$$\angle ABI_A = \angle ABI + \angle IBI_A = \frac{\beta}{2} + 90^\circ$$

$$\therefore \angle AIC = \angle ABI_A \tag{2}$$

From (1) and (2), we can conclude that $\triangle AIC \sim \triangle ABI_A$.

$$\therefore \frac{\overline{AI}}{\overline{AC}} = \frac{\overline{AB}}{\overline{AI_A}}, \text{ i.e. } \overline{AI} \cdot \overline{AI_A} = \overline{AB} \cdot \overline{AC}$$ ∎





Proof 2. Recall from Example 7.2 that IBI_AC is a cyclic quadrilateral and that the center of this circle lies on AI. Notice that the line AC is a reflection of the line AB with respect to the angle bisector of $\angle BAC$, AI. Let the second intersection of (IBI_AC) with AC be B'. By symmetry, $\overline{AB} = \overline{AB'}$. Now, by the Intersecting Secants Theorem for the point A, we have

$$\overline{AI} \cdot \overline{AI_A} = \overline{AB'} \cdot \overline{AC} = \overline{AB} \cdot \overline{AC}$$ ∎

Related problems: 51, 65, 68, 85, 92 and 107.

Chapter 8

Collinearity

Three points are *collinear* if they lie on a single line. We will now present a few approaches that will help us prove that three points are collinear when solving geometry problems.

8.1 Manual Approach

There are three most common angle chasing ways to prove that three points A, B and C are collinear.

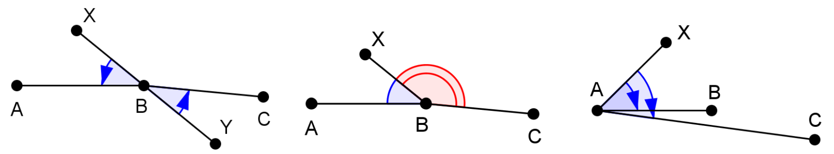


Figure 8.1: Three collinearity configurations

In the first configuration, we will need two extra points that are already collinear with our "middle" point B. Let those points be X and Y. If $\angle XBA = \angle YBC$, then the points A, B and C are collinear.

In the second configuration, we will need one extra point X that doesn't lie on the supposed line $A - B - C$. If $\angle ABX + \angle XBC = 180°$, then the points A, B and C are collinear.

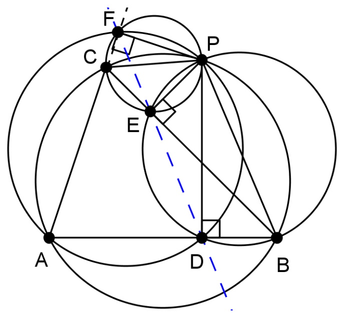
In the third configuration, we will also need one extra point X that doesn't lie on the supposed line $A - B - C$. If $\angle XAB = \angle XAC$, then the points A, B and C are collinear.

In the proof of the following theorem, we will demonstrate all three approaches.

Example 8.1 (Simson Line Theorem)**.** Let P be a point on the circumcircle ω of a triangle ABC. If D, E and F are the feet of the perpendiculars from P to the lines AB, BC and CA, prove that the points D, E and F are collinear.

Proof. WLOG let P be on the arc $\overset{\frown}{BC}$ that doesn't contain A.

$$\angle PDB = 90^\circ = \angle PEB$$
$$\therefore PEDB \text{ is cyclic} \tag{1}$$
$$\angle CEP + \angle CFP = 180^\circ$$
$$\therefore CEPF \text{ is cyclic} \tag{2}$$
$$\angle ADP + \angle AFP = 180^\circ$$
$$\therefore ADPF \text{ is cyclic} \tag{3}$$



We will now finish the proof in three different ways, demonstrating all of the approaches mentioned before.
I way: We will prove that $\angle CEF = \angle BED$.

$$\angle CEF \overset{(2)}{=} \angle CPF \overset{\triangle CFP}{=} 90^\circ - \angle FCP$$

$$\angle BED \overset{(1)}{=} \angle BPD \overset{\triangle BDP}{=} 90^\circ - \angle DBP$$

$$\angle FCP = 180^\circ - \angle ACP \overset{\omega}{=} \angle ABP \equiv \angle DBP$$

II way: We will prove that $\angle FEP + \angle PED = 180^\circ$.

$$\angle FEP \overset{(2)}{=} \angle FCP = 180^\circ - \angle PCA$$

$$\angle PED \overset{(1)}{=} 180 - \angle PBD \equiv 180^\circ - \angle PBA$$

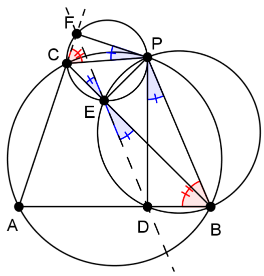
$$\angle FEP + \angle PED = 360^\circ - (\angle PCA + \angle PBA) \overset{\omega}{=} 360^\circ - 180^\circ = 180^\circ$$

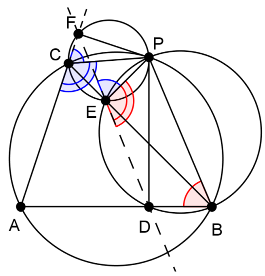
III way: We will prove that $\angle PFE = \angle PFD$.

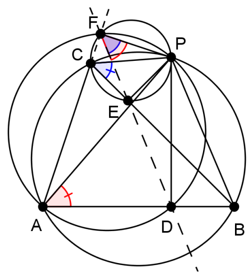
$$\angle PFE \overset{(2)}{=} \angle PCE \equiv \angle PCB$$

$$\angle PFD \overset{(3)}{=} \angle PAD \equiv \angle PAB$$

$$\angle PCB \overset{\omega}{=} \angle PAB$$ ■

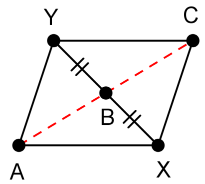






8.2 Parallelogram Trick

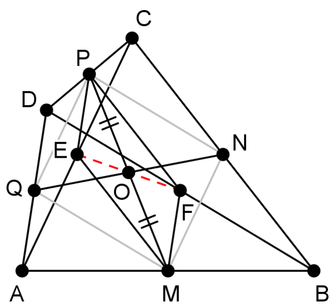
Sometimes (although much more rarely), we can use the following approach to prove that three points A, B and C are collinear. If we know that the "middle" point B is the midpoint of some line segment XY, then by showing that $AXCY$ is a parallelogram, we will prove that A, B and C are collinear. This is because we know that the diagonals in a parallelogram bisect at the intersection point, so if B is the midpoint of the diagonal XY, then it must also be the midpoint of the other diagonal AC, i.e. it must lie on AC.



We will now solve one problem as an example of how this approach can be used.

Example 8.2. Prove that in any convex quadrilateral $ABCD$ the midpoints of its diagonals and the point which is the intersection of the lines through the midpoints of the opposite sides are collinear.

Proof. Let E and F be the midpoints of the diagonals AC and BD, respectively. Let M, N, P and Q be the midpoints of the sides AB, BC, CD and DA, respectively, and let O be the intersection of MP and NQ. We need to prove that E, O and F are collinear.



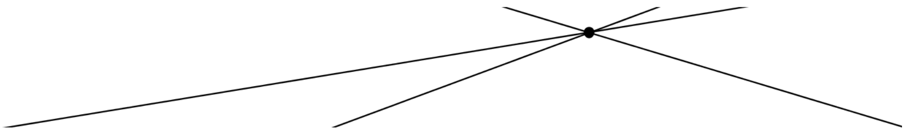
Firstly, let's take a look at the quadrilateral $MNPQ$. MN is midsegment in $\triangle ABC$. Therefore, $MN \parallel AC$ and $\overline{MN} = \frac{\overline{AC}}{2}$. Similarly, PQ is midsegment in $\triangle DAC$, so $PQ \parallel AC$ and $\overline{PQ} = \frac{\overline{AC}}{2}$. Therefore, $MN \parallel PQ$ and $\overline{MN} = \overline{PQ}$. Thus, by Example 2.18, $MNPQ$ is a parallelogram. Since we know that the diagonals in a parallelogram bisect at the intersection point and O is the intersection of the diagonals MP and NQ, we get that O is the midpoint of MP.

Now, since we want to prove that E, O and F are collinear and we know that O is the midpoint of MP, it is enough to prove that $EMFP$ is a parallelogram. Notice that ME is midsegment in $\triangle ABC$. Therefore, $ME \parallel BC$ and $\overline{ME} = \frac{\overline{BC}}{2}$. Similarly, FP is midsegment in $\triangle BCD$, so $FP \parallel BC$ and $\overline{FP} = \frac{\overline{BC}}{2}$. Therefore, $ME \parallel FP$ and $\overline{ME} = \overline{FP}$. Thus, by Example 2.18, $EMFP$ is a parallelogram. ∎

Related problems: (Collinearity) 41, 42, 43, 44 and 99.

Chapter 9

Concurrence

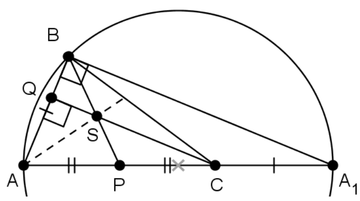


Three lines are *concurrent* if they pass through a common point. We will firstly present a few approaches to proving concurrence by using things we have already visited during our journey. Then, we will learn a new theorem related to concurrence.

9.1 Manual Approach

The most basic way to prove that three lines are concurrent is to take the intersection of two of them and then somehow prove that the third line passes through this intersection.

Example 9.1. Let C be a point on the diameter AA_1 in a circle ω. Let B be a point on ω, such that $\overline{AB} = \overline{CA_1}$. Prove that in $\triangle ABC$, the internal angle bisector at the vertex A, the median from the vertex B and the altitude from the vertex C are concurrent.



Proof. We will take the intersection of the median and the altitude and we will prove that the angle bisector passes through this point. Let P be the midpoint of AC and let Q be the foot of the altitude from the vertex C in $\triangle ABC$. Let $S = BP \cap CQ$. We need to prove that AS bisects the angle $\angle CAB$.

$$CQ \perp AB \quad (\because CQ \text{ is altitude in } \triangle ABC)$$

$$A_1B \perp AB \quad (\because AA_1 \text{ is diameter})$$

$$\therefore CQ \parallel A_1B \text{ , i.e. } CS \parallel A_1B$$

$$\therefore \frac{\overline{PC}}{\overline{CA_1}} = \frac{\overline{PS}}{\overline{SB}} \text{ (by Thales' Proportionality Theorem)}$$

Substituting $\overline{PC} = \overline{AP}$ and $\overline{CA_1} = \overline{AB}$, we get

$$\frac{\overline{AP}}{\overline{AB}} = \frac{\overline{PS}}{\overline{SB}},$$

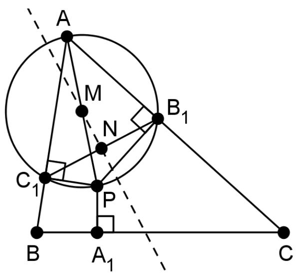
which by the Angle Bisector Theorem means that AS is the angle bisector of $\angle PAB$. Since $\angle PAB \equiv \angle CAB$, we get that AS bisects $\angle CAB$. ■

Remark. This approach can, in fact, be used not only for proving concurrent lines, but also for proving that any three objects (lines or circles or any combination of those) pass through a common point. For example, if we need to prove that three circles pass through a point, we will take the intersection of two of the circles and then prove that this intersection lies on the third circle. Otherwise, if we need to prove that two lines intersect on a circle, we can either take the intersection of the lines and prove that this intersection lies on the circle, or we can take the intersection of one of the lines and the circle and prove that this intersection lies on the other line.

9.2 Special Lines

Another way to prove that three lines are concurrent is by proving that they are "special lines" (such as side bisectors, angle bisectors, altitudes, ...) in a triangle in the figure. This is because we already know that these special lines concur at one of the important centers that we mentioned in chapter 6.

Example 9.2. Let P be an arbitrary point inside the triangle ABC. Let A_1, B_1 and C_1 be the feet of the perpendiculars from P to BC, CA and AB, respectively. Prove that the lines that pass through the midpoints of PA and B_1C_1, PB and C_1A_1, and PC and A_1B_1 are concurrent.



Proof. We will prove that these lines are in fact side bisectors in $\triangle A_1B_1C_1$, so they will concur at the circumcircle of $\triangle A_1B_1C_1$. Let M and N be the midpoints of PA and B_1C_1, respectively.

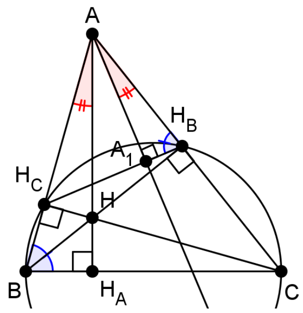
$$\angle PC_1A + \angle PB_1A = 180^\circ \quad (\because PC_1 \perp AB,\ PB_1 \perp AC)$$

Therefore, AC_1PB_1 is cyclic. Since $\angle AC_1P = 90^\circ$, AP is a diameter in (AC_1PB_1), so M is its center. Since N is the midpoint of the chord B_1C_1 and M is the center, MN is the side bisector of B_1C_1. Similarly, the other two lines are also side bisectors in $\triangle A_1B_1C_1$, so they are concurrent. ■

9.3 Special Point

If the lines in question are not "special lines", there is another way that the important centers can help us–by proving somehow that the lines pass through a "special point", i.e. an important center in a triangle in the figure.

Example 9.3 (Macedonia MO 2015)**.** Let AH_A, BH_B and CH_C be altitudes in $\triangle ABC$. Let p_A, p_B and p_C be the perpendicular lines from vertices A, B and C to H_BH_C, H_CH_A and H_AH_B, respectively. Prove that p_A, p_B and p_C are concurrent.



Proof. We will prove that the lines pass through the circumcenter of $\triangle ABC$. Let $A_1 = p_A \cap H_BH_C$.

$$BCH_BH_C \text{ is cyclic } (\because \angle BH_BC = 90^\circ = \angle BH_CC)$$

$$\therefore \angle AH_BA_1 \equiv \angle AH_BH_C = \angle H_CBC \equiv \angle ABC = \beta$$

$$\angle CAA_1 \equiv \angle H_BAA_1 = 90^\circ - \angle AH_BA_1 = 90^\circ - \beta \quad (\because AA_1 \perp H_BH_C)$$

$$\angle BAH_A = 90^\circ - \angle ABH_A = 90^\circ - \beta$$

In conclusion, $\angle CAA_1 = \angle BAH_A$, so AH_A and $AA_1 \equiv p_A$ are symmetric with respect to the angle bisector of $\angle BAC$. Since the orthocenter lies on the altitude AH_A, its isogonal conjugate, the circumcenter (Property 6.9), must lie on p_A. Similarly, the circumcenter of $\triangle ABC$ lies on p_B and p_C, so the lines p_A, p_B and p_C are concurrent. ∎

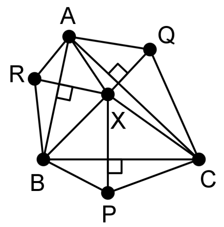
9.4 Concurrent Perpendiculars

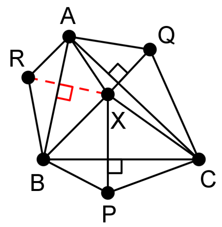
Property 9.1 (Carnot's Extended Theorem)**.** Let P, Q and R be points in the plane of triangle ABC. Then, the lines ℓ_P, ℓ_Q and ℓ_R, which are the perpendiculars from P, Q and R to BC, CA and AB, respectively, are concurrent if and only if

$$\overline{PB}^2 - \overline{PC}^2 + \overline{QC}^2 - \overline{QA}^2 + \overline{RA}^2 - \overline{RB}^2 = 0$$

Proof. Let's prove the first direction, i.e. let ℓ_P, ℓ_Q and ℓ_R be concurrent and let the point of concurrence be X. By the perpendicularity condition in Example 4.3, we get that $XP \perp BC \iff \overline{PB}^2 - \overline{PC}^2 = \overline{XB}^2 - \overline{XC}^2$. If we substitute this for all three perpendiculars, we get

$$\begin{aligned} &\overline{PB}^2 - \overline{PC}^2 + \overline{QC}^2 - \overline{QA}^2 + \overline{RA}^2 - \overline{RB}^2 = \\ = \ &\overline{XB}^2 - \overline{XC}^2 + \overline{XC}^2 - \overline{XA}^2 + \overline{XA}^2 - \overline{XB}^2 = 0 \quad \square \end{aligned}$$





Now, let's prove the other direction, i.e. let $\overline{PB}^2 - \overline{PC}^2 + \overline{QC}^2 - \overline{QA}^2 + \overline{RA}^2 - \overline{RB}^2 = 0$. Let $\ell_P \cap \ell_Q = X$. In order for the three perpendiculars to be concurrent, we need to prove that $X \in \ell_R$.

By using the same perpendicularity condition as before for the perpendiculars ℓ_P and ℓ_Q and substituting in the given condition, we get

$$\overline{XB}^2 - \overline{XC}^2 + \overline{XC}^2 - \overline{XA}^2 + \overline{RA}^2 - \overline{RB}^2 = 0$$

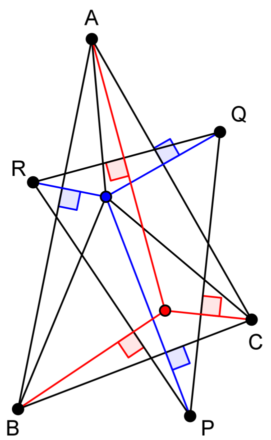
$$\overline{XB}^2 - \overline{XA}^2 = \overline{RB}^2 - \overline{RA}^2$$

$$\therefore XR \perp AB \quad \text{(by Example 4.3)}$$

$$\therefore X \in \ell_R$$

■

Property 9.2. Let P, Q and R be points in the plane of triangle ABC. Then, the perpendiculars from P, Q and R to BC, CA and AB, respectively, are concurrent if and only if the perpendiculars from C, A and B to PQ, QR and RP, respectively, are concurrent.



Proof. By using Carnot's Extended Theorem, rearranging the terms and using Carnot's Extended Theorem again, we get

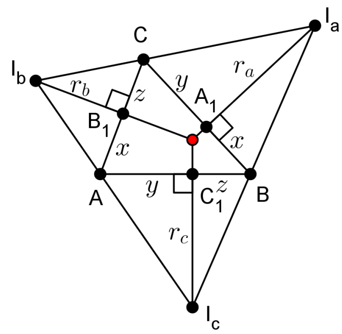
$$\begin{aligned} &LHS \iff \\ &\iff \overline{PB}^2 - \overline{PC}^2 + \overline{QC}^2 - \overline{QA}^2 + \overline{RA}^2 - \overline{RB}^2 = 0 \\ &\iff 0 = \overline{CP}^2 - \overline{CQ}^2 + \overline{AQ}^2 - \overline{AR}^2 + \overline{BR}^2 - \overline{BP}^2 \\ &\iff RHS \end{aligned}$$

■

In the following problem, we will present a solution with each of the aforementioned properties.

Example 9.4 (Serbia 2017, Drzavno IIIA)**.** Let I_a, I_b and I_c be the excenters of triangle ABC opposite the vertices A, B and C, respectively. Let A_1, B_1 and C_1 be the tangent points of the $A-$, $B-$ and $C-$excircle with the sides BC, CA and AB, respectively. Prove that the lines I_aA_1, I_bB_1 and I_cC_1 are concurrent.

Proof 1. By Carnot's Extended Theorem the three perpendiculars are concurrent if and only if $\overline{I_aB}^2 - \overline{I_aC}^2 + \overline{I_bC}^2 - \overline{I_bA}^2 + \overline{I_cA}^2 - \overline{I_cB}^2 = 0$

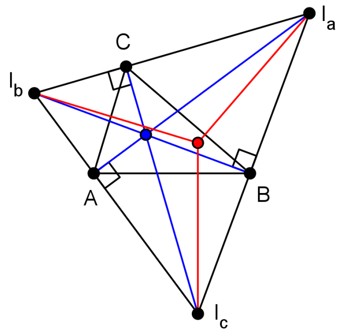


As we will shortly see (in Example 10.3.2), the tangent points of the excircles have such property that $\overline{BA_1} = s - c = \overline{AB_1}$, where s is the semiperimeter of $\triangle ABC$. Let $x = s - c$, $y = s - b$ and $z = s - a$ and let r_a, r_b and r_c be the radii of the $A-$, $B-$ and $C-$excircle, respectively. By using the Pythagorean Theorem six times, the above statement is equivalent to

$$r_a^2 + x^2 - r_a^2 - y^2 + r_b^2 + z^2 - r_b^2 - x^2 + r_c^2 + y^2 - r_c^2 - z^2 = 0$$

which is true because everything on the left-hand side cancels out. ■

Proof 2. By Property 9.2, the perpendiculars from I_a, I_b and I_c to BC, CA and AB, respectively, are concurrent if and only if the perpendiculars from C, A and B to I_aI_b, I_bI_c and I_cI_a, respectively, are concurrent. We are going to prove the latter.



Let's recall, from Example 7.3, that A, B and C are the feet of the altitudes in $\triangle I_aI_bI_c$. Thus, the perpendiculars from C, A and B to I_aI_b, I_bI_c and I_cI_a are in fact the altitudes in $\triangle I_aI_bI_c$, so they concur at its orthocenter. ■

Related problems: (Concurrence) 20 and 77.

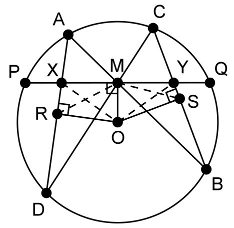
Chapter 10

A Few Useful Lemmas

10.1 Butterfly Theorem

Example 10.1.1 (Butterfly Theorem)**.** Let M be the midpoint of a chord PQ of a circle ω, through which two other chords AB and CD are drawn. Let $AD \cap PQ = X$ and $BC \cap PQ = Y$. Prove that M is also the midpoint of XY.

Proof. Let O be the center of ω. Since M is the midpoint of PQ, $OM \perp PQ$. Thus, in order to show that $\overline{XM} = \overline{MY}$, we need to prove that $\angle MOX = \angle MOY$.



Let R and S be the feet of the perpendiculars from O to AD and BC, respectively. Therefore, $\overline{AR} = \overline{RD}$ and $\overline{BS} = \overline{SC}$.

$$\angle DAM \equiv \angle DAB \overset{\omega}{=} \angle DCB \equiv \angle MCB \quad \text{and} \quad \angle AMD = \angle CMB,$$

$$\therefore \triangle AMD \sim \triangle CMB$$

$$\therefore \frac{\overline{AD}}{\overline{AM}} = \frac{\overline{CB}}{\overline{CM}}$$

$$\therefore \frac{\overline{AR}}{\overline{AM}} = \frac{\overline{CS}}{\overline{CM}} \quad (\because \frac{\overline{AD}}{\overline{AR}} = 2 = \frac{\overline{CB}}{\overline{CS}})$$

$$\therefore \triangle AMR \sim \triangle CMS \quad (\because \angle RAM \equiv \angle DAB \overset{\omega}{=} \angle DCB \equiv \angle MCS)$$

$$\therefore \angle MRA = \angle MSC \tag{*}$$

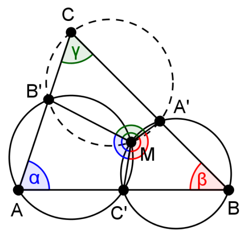
Since $OM \perp PQ$ and $OR \perp AD$, $\angle ORX + \angle OMX \equiv \angle ORA + \angle OMP = 180°$. Therefore, $OMXR$ is cyclic. Similarly, $OMYS$ is cyclic. Therefore,

$$\angle MOX = \angle MRX \equiv \angle MRA \overset{(*)}{=} \angle MSC \equiv \angle MSY = \angle MOY \qquad \blacksquare$$

Related problem: 78.

10.2 Miquel's Theorem

Example 10.2.1. Let ABC be a triangle, with arbitrary points A', B' and C' on sides BC, CA and AB, respectively (or their extensions). The circumcircles of $\triangle AB'C'$, $\triangle A'BC'$ and $\triangle A'B'C$ intersect in a single point, called the Miquel point.



Proof. Let's assume that the points A', B' and C' are on the sides (not on the extensions). In the other cases, the proof follows a similar structure.

Let $M = (AB'C') \cap (A'BC')$. We will prove that M lies on $(A'B'C)$, too. Since $AB'MC'$ and $BC'MA'$ are cyclic, we have

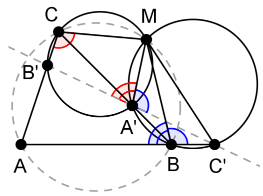
$$\angle B'MC' = 180^\circ - \alpha \quad \text{and} \quad \angle C'MA' = 180^\circ - \beta.$$

$$\therefore \angle A'MB' = 360^\circ - (\angle B'MC' + \angle C'MA') = \alpha + \beta$$

$$\therefore \angle A'CB' + \angle A'MB' = \gamma + \alpha + \beta = 180^\circ$$

$$\therefore M \in (A'B'C)$$ ■

Example 10.2.2. Let ABC be a triangle, with arbitrary points A', B' and C' on sides BC, CA and AB, respectively (or their extensions). The Miquel point lies on the circumcircle of $\triangle ABC$ if and only if the points A', B' and C' are collinear.



Proof. We will see the configuration when one of the points is on the extension of the sides, WLOG let it be C'. The other case, where all three points are on the extensions of the sides follows a similar structure.

Let M be the Miquel point of $\triangle ABC$. Then $MA'BC'$ and $MCB'A'$ are cyclic quadrilaterals.

$$\angle MA'C' = \angle MBC' = 180^\circ - \angle MBA$$

$$\angle MA'B' = 180^\circ - \angle MCB' \equiv 180^\circ - \angle MCA$$

$$\therefore \angle MA'C' + \angle MA'B' = 360^\circ - (\angle MBA + \angle MCA)$$

The points $C' - A' - B'$ are collinear iff the left-hand side is 180°. The right-hand side is 180° iff $ABMC$ is a cyclic quadrilateral, i.e. $M \in (ABC)$. ■

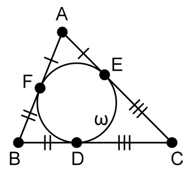
Related problems: 44, 86, 109, 113 and 149.

10.3 Tangent Segments

From section 5.1, we know that the tangent segments from a point to the circle are of equal length. We will now present some useful properties based on this fact.

Example 10.3.1. Let ω be the incircle in $\triangle ABC$. Let D be the tangent point of ω with the side BC. Prove that $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD}$.

Proof. Let E and F be the tangent points of ω with the sides CA and AB, respectively. Then, as tangent segments from A, B and C to ω, we get

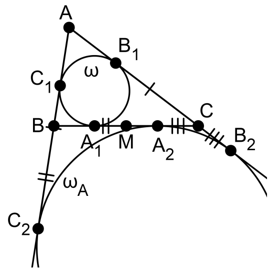


$$\overline{AF} = \overline{AE}\,, \quad \overline{BF} = \overline{BD} \quad \text{and} \quad \overline{CD} = \overline{CE}.$$

$$\therefore \overline{AB}+\overline{CD} = \overline{AF}+\overline{FB}+\overline{CD} = \overline{AE}+\overline{EC}+\overline{BD} = \overline{AC}+\overline{BD} \ \blacksquare$$

Example 10.3.2. Let ω and ω_A be the incircle and the A-excircle in $\triangle ABC$. Let A_1, B_1 and C_1 be the tangent points of ω with the sides BC, CA and AB, respectively. Let A_2, B_2 and C_2 be the tangent points of ω_A with the lines BC, CA and AB. Prove that:

- $\overline{AB} + \overline{BA_2} = \overline{AC} + \overline{CA_2}$;
- $\overline{BA_2} = \overline{CA_1}$, i.e. $\overline{A_1M} = \overline{MA_2}$, where M is the midpoint of BC;



Proof. As tangent segments from the points A, B and C to ω_A, we get

$$\overline{AB_2} = \overline{AC_2}\,, \quad \overline{BA_2} = \overline{BC_2} \quad \text{and} \quad \overline{CA_2} = \overline{CB_2}.$$

$$\therefore \overline{AB} + \overline{BA_2} = \overline{AB} + \overline{BC_2} = \overline{AC_2} = \overline{AB_2} = \overline{AC} + \overline{CB_2} = \overline{AC} + \overline{CA_2} \quad \square$$

Since the sum of both sides equals the whole perimeter of $\triangle ABC$, then each side is equal to its semiperimeter s.

$$\therefore \overline{BA_2} = s - \overline{AB}$$

From Example 10.3.1, we have

$$\overline{AC} + \overline{BA_1} = \overline{AB} + \overline{CA_1}.$$

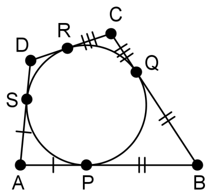
Again, the sum of both sides equals the whole perimeter of $\triangle ABC$, so each side is equal to its semiperimeter s.

$$\therefore \overline{CA_1} = s - \overline{AB}$$

Thus, we can conclude that $\overline{BA_2} = \overline{CA_1}$. Since $\overline{BM} = \overline{CM}$, then we also have $\overline{A_1M} = \overline{MA_2}$. $\blacksquare$

Example 10.3.3 (Tangential quadrilateral)**.** Let $ABCD$ be a quadrilateral such that there exists an incircle that is tangent to its sides. Prove that the sums of the opposite sides are equal, i.e.

$$\overline{AB} + \overline{CD} = \overline{BC} + \overline{AD}.$$



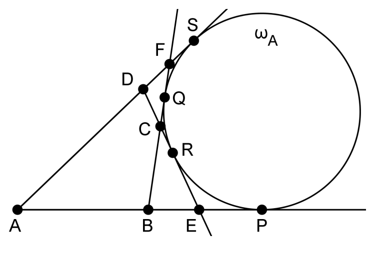
Proof. Let P, Q, R and S be the tangent points of the incircle with the sides AB, BC, CD and DA, respectively. Then, as tangent segments,

$$\overline{AP} = \overline{AS}\,, \quad \overline{BP} = \overline{BQ}\,, \quad \overline{CQ} = \overline{CR} \quad \text{and} \quad \overline{DR} = \overline{DS}.$$

$$\therefore \overline{AB} + \overline{CD} = \overline{AP} + \overline{PB} + \overline{CR} + \overline{RD} = \overline{AS} + \overline{BQ} + \overline{CQ} + \overline{DS} = \overline{BC} + \overline{AD} \quad \blacksquare$$

Example 10.3.4 (Ex-tangential quadrilateral)**.** Let $ABCD$ be a quadrilateral such that there exists an excircle ω_A that is tangent to the rays AB (beyond B) and AD (beyond D) and is also tangent to the lines BC and CD. Let E and F be the intersections of the opposite sides. Prove that

$$\overline{AB} + \overline{BC} = \overline{AD} + \overline{DC}$$
$$\overline{EA} + \overline{EC} = \overline{FA} + \overline{FC}$$
$$\overline{EB} + \overline{ED} = \overline{FB} + \overline{FD}.$$

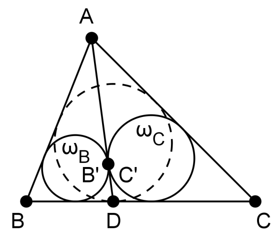


Proof. Let P, Q, R and S be the tangent points of the excircle with the lines AB, BC, CD and DA, respectively. Then, as tangent segments,

$$\overline{AP} = \overline{AS}\,,\ \overline{BP} = \overline{BQ}\,,\ \overline{CQ} = \overline{CR}\,,\ \overline{DR} = \overline{DS}\,,\ \overline{EP} = \overline{ER} \quad \text{and} \quad \overline{FQ} = \overline{FS}.$$

$$\begin{aligned}
\overline{AB} + \overline{BC} &= \overline{AP} - \overline{BP} + \overline{BQ} - \overline{CQ} = \overline{AP} - \overline{CQ} = \\
&= \overline{AS} - \overline{CR} = \overline{AS} - \overline{DS} + \overline{DR} - \overline{CR} = \overline{AD} + \overline{DC} \\
\overline{EA} + \overline{EC} &= \overline{AP} - \overline{EP} + \overline{ER} + \overline{CR} = \overline{AP} + \overline{CR} = \\
&= \overline{AS} + \overline{CQ} = \overline{AS} - \overline{FS} + \overline{FQ} + \overline{CQ} = \overline{FA} + \overline{FC} \\
\overline{EB} + \overline{ED} &= \overline{BP} - \overline{EP} + \overline{ER} + \overline{DR} = \overline{BP} + \overline{DR} = \\
&= \overline{BQ} + \overline{DS} = \overline{BQ} + \overline{FQ} + \overline{DS} - \overline{FS} = \overline{FB} + \overline{FD}
\end{aligned} \quad \blacksquare$$

Example 10.3.5. Let ABC be a triangle, and let D be the point where the incircle touches the side BC. Let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Prove that ω_B and ω_C are tangent to each other.



Proof. Let B' and C' be the tangent points of ω_B and ω_C, respectively, to the side AD. We need to prove that $B' \equiv C'$.
Using Example 10.3.1 on $\triangle ABD$ and $\triangle ACD$, we get

$$\overline{AB} + \overline{DB'} = \overline{BD} + \overline{AB'} \quad \text{and} \quad \overline{CD} + \overline{AC'} = \overline{AC} + \overline{DC'}$$

By adding these two equations side by side, we get

$$\overline{AB} + \overline{CD} + \overline{AC'} + \overline{DB'} = \overline{AC} + \overline{BD} + \overline{AB'} + \overline{DC'}$$

From, Example 10.3.1, we know that $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD}$, so

$$\overline{AC'} + \overline{DB'} = \overline{AB'} + \overline{DC'}.$$

By adding $\overline{AB'} + \overline{AC'}$ on both sides, we get:

$$2 \cdot \overline{AC'} + \overline{AD} = 2 \cdot \overline{AB'} + \overline{AD}$$

$$\therefore \overline{AC'} = \overline{AB'}, \text{ i.e. } B' \equiv C' \qquad \blacksquare$$

Related problems: 91 and 123.

10.4 Euler Line

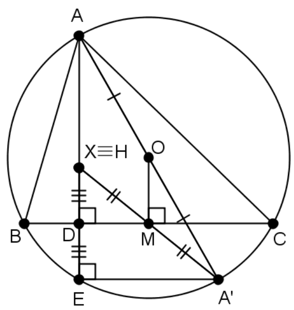
Example 10.4.1. Let M be the midpoint of the side BC in $\triangle ABC$. Let A' be the antipode of A on the circumcircle, i.e. the point on the circumcircle such that AA' is a diameter. Finally, let H be the orthocenter of $\triangle ABC$. Prove that the points $H - M - A'$ are collinear.

Proof. Let D be the feet of the altitude from A to BC. Let $A'M \cap AD = X$. We want to prove that $X \equiv H$.

The lines OM and AX are parallel because they are both perpendicular to BC. Since O is the midpoint of AA', OM is midsegment in $\triangle AXA'$. Therefore, $\overline{XM} = \overline{MA'}$.

Let AD intersect the circumcircle of $\triangle ABC$ again at E. Then, $\angle AEA' = 90°$ as an inscribed angle over the diameter, which means that $AE \perp EA'$. Since $AE \equiv AD \perp BC$, we have $EA' \parallel BC$, i.e. $EA' \parallel DM$.

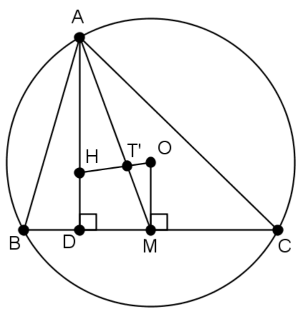
Since $\overline{XM} = \overline{MA'}$ and $DM \parallel EA'$, DM is midsegment in $\triangle XEA'$, so $\overline{XD} = \overline{DE}$. We know (from Property 6.8) that the reflections of the orthocenter lie on the circumcircle, so $X \equiv H$. ■



Example 10.4.2. Let M be the midpoint of the side BC in $\triangle ABC$. Let H and O be the orthocenter and circumcenter of $\triangle ABC$, respectively. Prove that $\overline{AH} = 2 \cdot \overline{OM}$.

Proof. Using the same notations as in Example 10.4.1 and continuing from there, we got that OM is midsegment in $\triangle AHA'$. So $\overline{AH} = 2 \cdot \overline{OM}$. ■

Example 10.4.3 (Euler Line)**.** Let H, T and O be the orthocenter, centroid and circumcenter in $\triangle ABC$, respectively. Prove that the points $H - T - O$ are collinear and $\overline{HT} = 2 \cdot \overline{TO}$.



Proof. Let M be the midpoint of BC. Let $T' = AM \cap HO$. We will prove that $T' \equiv T$. The lines AH and OM are parallel because they are both perpendicular to BC. Therefore, $\triangle AHT' \sim \triangle MOT'$ and

$$\therefore \frac{\overline{AH}}{\overline{MO}} = \frac{\overline{AT'}}{\overline{MT'}} = \frac{\overline{HT'}}{\overline{OT'}}$$

Combining with $\overline{AH} = 2 \cdot \overline{OM}$ (from Example 10.4.2), we get that the ratio of similarity is 2.

$$\therefore \frac{\overline{AT'}}{\overline{T'M}} = 2 : 1,$$

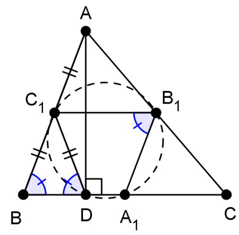
which means that T' is the centroid of the triangle, i.e. $T' \equiv T$. So the points $H - T - O$ are collinear. This line is known as the *Euler line* of $\triangle ABC$.

From the same similarity, we also get that $\overline{HT} = 2 \cdot \overline{TO}$. ■

Related problems: 18, 31, 49, 74, 83, 87, 99, 100 and 103.

10.5 Nine Point Circle

Example 10.5.1. Let A_1, B_1 and C_1 be the midpoints of the sides BC, CA and AB in $\triangle ABC$, respectively. Let D be the foot of the altitude from A to BC. Prove that D lies on the circumcircle of $\triangle A_1B_1C_1$.

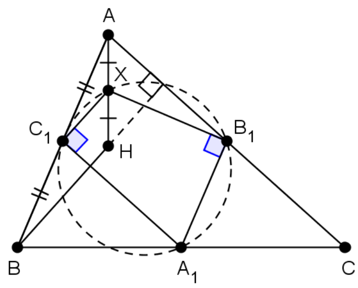


Proof. $\triangle ABD$ is a right triangle and C_1 is the midpoint of the hypotenuse, so $\overline{C_1D} = \overline{C_1B}$. Therefore, $\angle C_1DB = \angle C_1BD = \beta$. B_1C_1 is a midsegment in $\triangle ABC$, so $B_1C_1 \parallel BC$. Similarly, $A_1B_1 \parallel AB$. Therefore, $\angle C_1B_1A_1 = \beta$.

$$\angle C_1B_1A_1 + \angle C_1DA_1 = \beta + (180^\circ - \beta) = 180^\circ$$

$$\therefore D \in (A_1B_1C_1)$$ ■

Example 10.5.2. Let A_1, B_1 and C_1 be the midpoints of the sides BC, CA and AB in $\triangle ABC$, respectively. Let H be the orthocenter in $\triangle ABC$. Let X be the midpoint of AH. Prove that X lies on the circumcircle of $\triangle A_1B_1C_1$.



Proof.

$$C_1X \parallel BH \quad (\because C_1X \text{ is midsegment in } \triangle ABH)$$

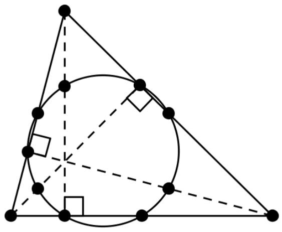
$$BH \perp CA$$

$$CA \parallel C_1A_1 \quad (\because C_1A_1 \text{ is midsegment in } \triangle ABC)$$

$$\therefore C_1X \perp C_1A_1, \text{ i.e. } \angle XC_1A_1 = 90^\circ$$

Similarly, $\angle XB_1A_1 = 90^\circ$. Therefore, $\angle XC_1A_1 + \angle XB_1A_1 = 180^\circ$, so X lies on the circumcircle of $A_1B_1C_1$. ■

With these two examples, we proved that the midpoints of the sides, the feet of the altitudes and the midpoints of the line segments from each vertex to the orthocenter (totally nine points) all lie on one circle. This circle is called the *nine point circle* of the triangle.

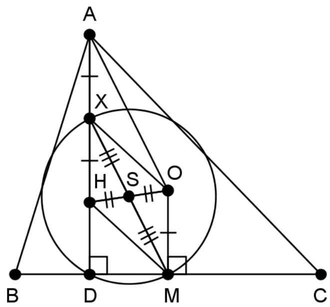


Now, let's try to find the center and the radius of this circle.

Let M be the midpoint of the side BC in $\triangle ABC$. Let H and O be the orthocenter and circumcenter of $\triangle ABC$, respectively. Let X be the midpoint of AH and let D be the foot of the altitude from A. As we know from Example 10.4.2, $\overline{AH} = 2 \cdot \overline{OM}$, so

$$\overline{XH} = \frac{1}{2} \cdot \overline{AH} = \overline{OM}.$$

Also, $XH \parallel OM$ because they are both perpendicular to BC. So $XHMO$ is a parallelogram, which means that the intersection point of its diagonals, let it be S, is their midpoint, i.e. $\overline{HS} = \overline{SO}$ and $\overline{XS} = \overline{SM}$.



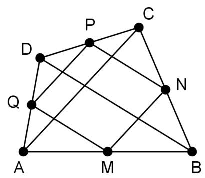
Since D, M and X lie on the nine point circle of $\triangle ABC$ and since $\angle XDM = 90^\circ$, the center of the nine point circle must be on the midpoint of XM, i.e. the point S and SX is a radius in that circle. Also, since SX is midsegment in $\triangle HOA$, $\overline{SX} = \frac{1}{2} \cdot \overline{OA} = \frac{1}{2} \cdot R$. In conclusion,

Property 10.5.1. The center of the nine point circle lies on the Euler line, more specifically it is the midpoint of OH. The radius of the nine point circle is one half of the radius of the circumcircle of $\triangle ABC$.

Related problem: 67.

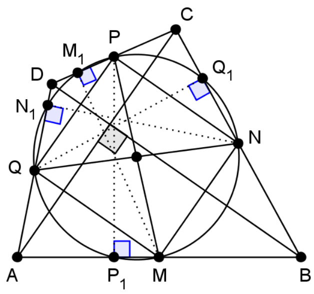
10.6 Eight Point Circle

Example 10.6.1. Let $ABCD$ be a convex quadrilateral. Let M, N, P and Q be the midpoints of the sides AB, BC, CD and DA, respectively. Prove that $MNPQ$ is a parallelogram.



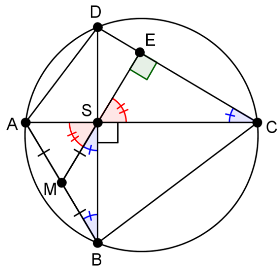
Proof. MN is midsegment in $\triangle ABC$. Therefore, $MN \parallel AC$. Similarly, PQ is midsegment in $\triangle DAC$, so $PQ \parallel AC$. Therefore, $MN \parallel PQ$. Similarly, $MQ \parallel NP$. Therefore, $MNPQ$ is a parallelogram. ■

Example 10.6.2 (Eight Point Circle)**.** Let $ABCD$ be a convex quadrilateral with perpendicular diagonals. Let M, N, P and Q be the midpoints of the sides AB, BC, CD and DA, respectively. Let M_1, N_1, P_1 and Q_1 be the feet of the perpendiculars from M, N, P and Q, respectively, to the opposite sides in the quadrilateral. Prove that the points M, N, P, Q, M_1, N_1, P_1 and Q_1 all lie on a single circle.



Proof. Combining the proof of Example 10.6.1 with $AC \perp BD$, we get that $MNPQ$ is a rectangle, i.e. a quadrilateral inscribed in a circle where the diagonals MP and NQ are diameters. From the definition of M_1, $MM_1 \perp CD$, i.e. $\angle MM_1P = 90°$, so $M_1 \in (MNPQ)$. Similarly, N_1, P_1 and Q_1 also lie on $(MNPQ)$. ■

Example 10.6.3 (Brahmagupta Theorem)**.** Let $ABCD$ be a cyclic quadrilateral with perpendicular diagonals that intersect at S. Let M be the midpoint of the side AB. Prove that $MS \perp CD$.



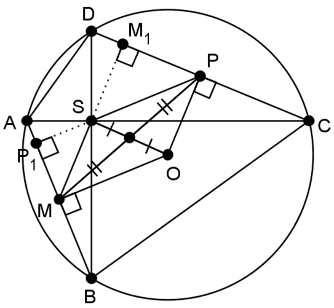
Proof. Let $MS \cap CD = E$. M is the midpoint of the hypotenuse in the right $\triangle ABS$, so $\overline{MB} = \overline{MS}$, i.e. $\angle MBS = \angle MSB$.

$$\angle SCE \equiv \angle ACD = \angle ABD \equiv \angle MBS = \angle MSB$$

$$\angle ESC + \angle SCE = \angle MSA + \angle MSB = \angle ASB = 90^\circ$$

$$\angle SEC = 180^\circ - (\angle ESC + \angle SCE) = 90^\circ, \text{ i.e. } MS \perp CD$$ ∎

Example 10.6.4. Let $ABCD$ be a cyclic quadrilateral with perpendicular diagonals that intersect at S. Let O be the center of $(ABCD)$. Prove that the eight point circle of $ABCD$ is centered at the midpoint of OS.



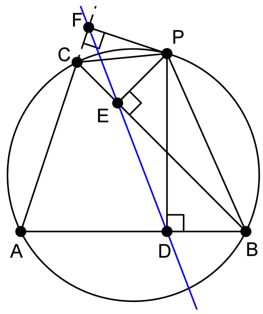
Proof. Let M, N, P and Q be the midpoints of the sides AB, BC, CD and DA, respectively. Let M_1, N_1, P_1 and Q_1 be the feet of the perpendiculars from M, N, P and Q, respectively, to the opposite sides in the quadrilateral. From Example 10.6.2, we know that MP is a diameter of the eight point circle, so its center is the midpoint of MP. We need to prove that the midpoint of MP coincides with the midpoint of OS. We will prove that $MOPS$ is a parallelogram.

Since M is a midpoint of the chord AB and O is the center of $(ABCD)$, we get that $OM \perp AB$. From Example 10.6.3, we know that the lines MM_1, NN_1, PP_1 and QQ_1 pass through S, so $PS \equiv PP_1 \perp AB$. Therefore, $OM \parallel PS$. Similarly, $OP \parallel MS$. Therefore, $MOPS$ is a parallelogram. ∎

Related problems: 99 and 106.

10.7 Simson Line Theorem

Example 10.7.1 (Simson Line Theorem)**.** Let P be a point on the circumcircle ω of a triangle ABC. If D, E and F are the feet of the perpendiculars from P to the lines AB, BC and CA, prove that the points D, E and F are collinear.

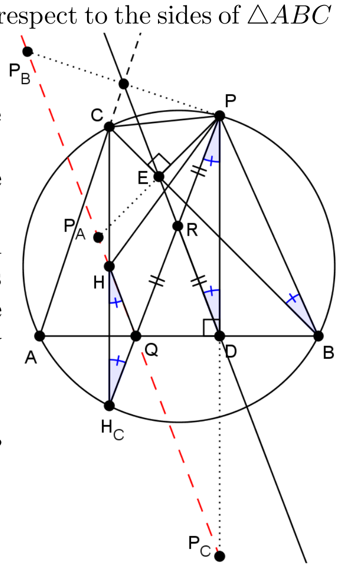


Proof. In Example 8.1, we already gave 3 different proofs of this theorem. ■

Now, we are going to present one property of the Simson Line.

Example 10.7.2. Let P be a point on the circumcircle ω of $\triangle ABC$ and let H be its orthocenter. Prove that the reflections of P with respect to the sides of $\triangle ABC$ are collinear with H.

Proof. From Example 8.1, we know that the feet of perpendiculars from P to the sides of $\triangle ABC$ lie on the P-Simson line of $\triangle ABC$. Then, by Thales' Proportionality Theorem, the reflections of P with respect to the sides of $\triangle ABC$ will also be collinear. We just need to prove that H lies on that line. Since the distance from a point to the foot of the perpendicular to a line is half the distance from the point to its reflection with respect to the line, we need to prove that the P-Simson line bisects the line segment PH.



WLOG let P be on the arc $\overset{\frown}{BC}$ that doesn't contain A. Let D and E be the feet of the perpendiculars from P to AB and BC, respectively. Let H_C be the reflection of H with respect to the side AB. By Property 6.8, we know that $H_C \in \omega$.

$DP \parallel H_CH \quad (\because DP \perp AB \land H_CH \perp AB)$ (1)

Let $PH_C \cap AB = Q$. Then, $\angle EDP \overset{(PEDB)}{=} \angle EBP \equiv \angle CBP \overset{\omega}{=} \angle CH_CP \equiv \angle HH_CQ = \angle H_CHQ$. (2)

Because of (1), we get that $ED \parallel HQ$. (3)

Let $ED \cap PH_C = R$.

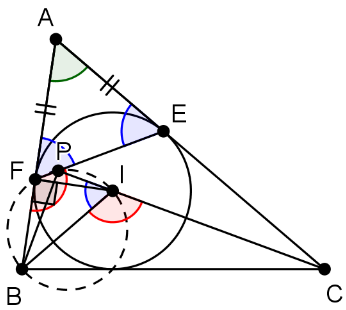
Then, $\angle RPD \overset{(1)}{=} \angle RH_CC \equiv \angle PH_CC \overset{(2)}{=} \angle EDP \equiv \angle RDP$. Since $\triangle PDQ$ is right triangle, we can also get $\angle RQD = \angle RDQ$. Therefore, $\overline{RP} = \overline{RD} = \overline{RQ}$, i.e. ED bisects PQ. Combining with (3), we get that the P-Simson line ED bisects the line segment PH. ■

Related problems: 66, 73 and 94.

10.8 In-Touch Chord

Example 10.8.1. Let I be the incenter of $\triangle ABC$ and let E and F be the tangent points of the incircle with the sides AC and AB, respectively. Let $CI \cap EF = P$. Then, $BP \perp PC$.

Proof. Because I is the incenter and F is the tangent point of AB and the incircle, we have $\angle BFI = 90^\circ$. We want to prove that $\angle BPC \equiv \angle BPI = 90^\circ$, so we need to prove that the quadrilateral $BFPI$ is cyclic.
From $\triangle BIC$, we get $\angle BIC = 180^\circ - (\frac{\alpha+\beta}{2}) = 90^\circ + \frac{\alpha}{2}$. Therefore,



$$\angle BIP = 90^\circ - \frac{\alpha}{2}.$$

Because AE and AF are tangent to the incircle, as tangent segments, they are of equal length and therefore $\triangle AEF$ is isosceles.

$$\therefore \angle AFE = \angle AEF = 90^\circ - \frac{\alpha}{2}$$

$$\therefore \angle BFP = 180^\circ - \angle AFE = 90^\circ + \frac{\alpha}{2}.$$

Finally, $\angle BIP + \angle BFP = 180^\circ$ and thus $BFPI$ is cyclic.

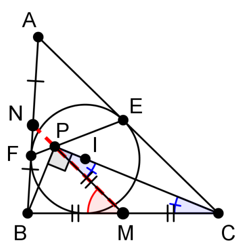
$$\therefore \angle BPC \equiv \angle BPI = \angle BFI = 90^\circ \qquad \blacksquare$$

Example 10.8.2. The A-intouch chord, B-midsegment and $\angle C$-bisector are concurrent.

Proof. Let I be the incenter of $\triangle ABC$ and let E and F be the tangent points of the incircle with the sides AC and AB, respectively. Let $CI \cap EF = P$. Let M and N be the midpoints of BC and BA, respectively. We will prove that $P \in MN$.

From Example 10.8.1 we know that $\triangle BPC$ is right-angled. Since M is the midpoint of its hypotenuse, we get that $\overline{MB} = \overline{MP} = \overline{MC}$. Therefore, as an exterior angle in $\triangle MCP$

$$\angle BMP = 2\angle MCP \equiv 2\angle BCI = 2 \cdot \frac{\gamma}{2} = \gamma$$



Also, MN is midsegment in $\triangle ABC$, so $MN \parallel AC$.

$$\therefore \angle BMN = \angle BCA = \gamma.$$

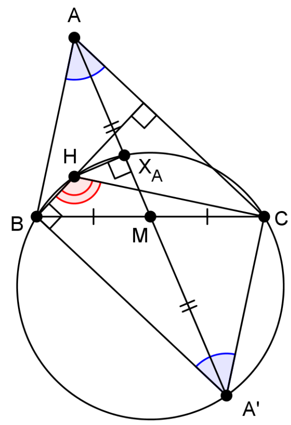
Finally, $\angle BMP = \angle BMN$, so $M - P - N$ are collinear, i.e. $P \in MN$. $\blacksquare$

Related problems: 79, 81, 88, 93, 102 and 143.

10.9 HM Point

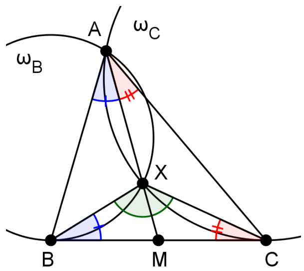
This section is about a set of points that have many properties, but still have no official name; On the Internet, they are known as the "HM points" (there are 3 in every triangle). In a triangle ABC, the $A-HM$ point, denoted by X_A, is the foot of the perpendicular from the orthocenter H to the median AM.

Example 10.9.1. Let ABC be a triangle with orthocenter H. Prove that the point X_A lies on the circumcircle of $\triangle BHC$.



Proof. Let M be the midpoint of the side BC and let $A' \in AM$, such that $\overline{AM} = \overline{MA'}$. Then $ABA'C$ is a parallelogram.
Since the opposite angles in a parallelogram are equal, we have $\angle BA'C = \angle BAC = \alpha$. We know that $\angle BHC = 180° - \alpha$. Therefore, $\angle BA'C + \angle BHC = 180°$, i.e. $A' \in (BHC)$.
Since $BH \perp AC$ and $AC \parallel BA'$, we get that $BH \perp BA'$ and therefore $\angle HBA' + \angle HX_AA' = 180°$, i.e. $X_A \in (HBA'C)$. ■

Example 10.9.2. Let ABC be a triangle and let ω_B be the circle that passes through A and B and is tangent to the line BC. Similarly, let ω_C be the circle that passes through A and C and is tangent to the line BC. Prove that the second intersection of ω_B and ω_C is X_A.



Proof. Let X be the second intersection of ω_B and ω_C. We will prove that $X \equiv X_A$ by proving that X lies on the $A-$median and on the circumcircle of $\triangle BHC$, where H is the orthocenter of $\triangle ABC$.

Let $AX \cap BC = M$. From Secant-Tangent Theorem we get that $\overline{MB}^2 = \overline{MX} \cdot \overline{MA} = \overline{MC}^2$, so M is the midpoint of BC, i.e. X lies on the $A-$median.

Since BM is tangent to ABX, we get that $\angle MBX = \angle BAX$. Similarly, $\angle MCX = \angle CAX$. Therefore, from $\triangle BXC$,

$$\angle BXC = 180° - (\angle XBC + \angle XCB) = 180° - (\angle BAX + \angle CAX) = 180° - \alpha.$$

We also know that $\angle BHC = 180° - \alpha$, so $X \in (BHC)$. ■

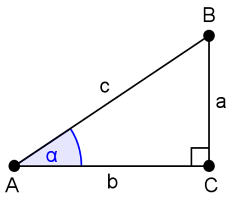
You can find many more properties of the HM point, solved example problems and unsolved exercises in [1]. Some of these are more advanced, so you may want to finish the remaining chapters in this book before trying them.

Related problem: 110.

Chapter 11

Basic Trigonometry

Trigonometric Functions in Right Triangle



Let ABC be a right triangle ($\gamma = 90°$). Then, we define the sine and cosine functions as follows:

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

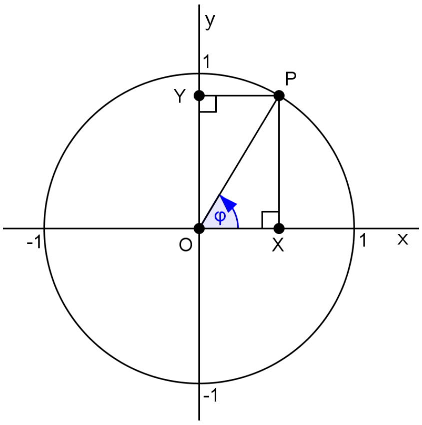
$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

Although they may sound scary, they are nothing more than ratios of sides :)

The Unit Circle

We defined the basic trigonometric functions for angles $0° < \varphi < 90°$. Let's try to extend them for all values of φ.

Let's take a look at the unit circle. That is a circle which is centered at the origin $O\,(0, 0)$ of the coordinate plane and has a radius of length 1. Let's represent any angle φ with a point P on the unit circle, such that the angle starting from the positive x-axis and going in the counter-clockwise direction to the line OP is equal to φ.



Let $0° < \varphi < 90°$. Let P be a point on the unit circle that represents the angle φ. Let X and Y be the feet of the perpendicular from P to the x- and y-axis, respectively. Then the triangle $\triangle OPX$ is a right triangle with hypotenuse $\overline{OP} = 1$, so by the definitions above, we get that $\cos\varphi = \overline{OX}$ and $\sin\varphi = \overline{PX} = \overline{OY}$. That's right, the cosine and sine values are in fact the x- and y-component of the point P in the coordinate system.

So why not extend this defintion for all possible values of φ? Those are, in fact, the actual definitions for the cosine and sine functions. The cosine is the x-component and the sine is the y-component of the point P representing the angle φ. For example, $\cos(120°) = -\frac{1}{2}$ and $\sin(90°) = 1$. As we can see, the ranges of both the cosine and sine functions are $[-1, 1]$.

Using the unit circle and the definition above, very simply, using congruence of triangles, or the Pythagorean Theorem, we can prove various properties, like:

$$\sin(180° - \alpha) = \sin\alpha \qquad \cos(180° - \alpha) = -\cos\alpha$$

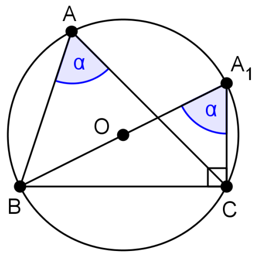
$$\sin(90° + \alpha) = \cos\alpha \qquad \cos(90° + \alpha) = -\sin\alpha$$

$$\cos^2\alpha + \sin^2\alpha = 1$$

Property 11.1 (Law of Sines)**.** In a triangle $\triangle ABC$ with circumradius R,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R.$$

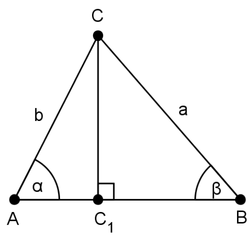
Proof. Let ω be the circumcirle of $\triangle ABC$ and let O be its center. Let A_1 be the second intersection of BO and ω. Then $\angle BA_1C = \angle BAC = \alpha$. On the other hand, $\angle A_1CB = 90°$ as an inscribed angle over the diameter, so $\triangle A_1BC$ is a right triangle. By definition,



$$\sin\alpha = \frac{\overline{BC}}{\overline{A_1B}} = \frac{a}{2R}, \text{ i.e. } \frac{a}{\sin\alpha} = 2R \qquad \blacksquare$$

Property 11.2 (Law of Cosines)**.** In a triangle $\triangle ABC$, for any side

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$



Proof. Let C_1 be the feet of the altitude from C to AB. Let's investigate the case when C_1 is between A and B. From the two right triangles $\triangle ACC_1$ and $\triangle BCC_1$ we get $\overline{AC_1} = b\cos\alpha$ and $\overline{BC_1} = a\cos\beta$. Since $\overline{AB} = \overline{AC_1} + \overline{BC_1}$, we get

$$c = a\cos\beta + b\cos\alpha.$$

We get exactly the same result even when C_1 is not between A and B because of the property $\cos(180° - \alpha) = -\cos\alpha$. Multiplying the last equation by c on both sides, we get

$$c^2 = ac\cos\beta + bc\cos\alpha.$$

Similarly, we can get

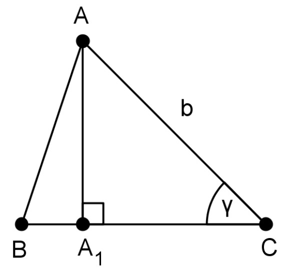
$$ab\cos\gamma + ac\cos\beta = a^2$$

$$ab\cos\gamma + bc\cos\alpha = b^2$$

By adding the last three equations side by side, we get the desired result. $\blacksquare$

Property 11.3. The area of a triangle $\triangle ABC$ can be expressed as

$$P_{\triangle ABC} = \frac{1}{2}ab\sin\gamma.$$



Proof. Let A_1 be the feet of the altitude from A to BC. Then $\triangle CAA_1$ is a right triangle, so we have

$$\sin\gamma = \frac{\overline{AA_1}}{\overline{AC}}, \text{ i.e. } \overline{AA_1} = b\sin\gamma.$$

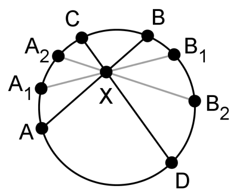
Then, the area of $\triangle ABC$ is

$$P_{\triangle ABC} = \frac{1}{2}\cdot\overline{BC}\cdot\overline{AA_1} = \frac{1}{2}ab\sin\gamma. \qquad \blacksquare$$

Related problems: 33 and 133.

Chapter 12

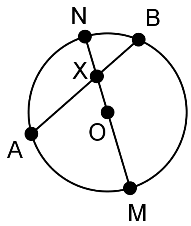
Power of a Point



Let AB and CD be two intersecting chords in a circle and let their intersection be X. By the Intersecting Chords Theorem, $\overline{AX} \cdot \overline{XB} = \overline{CX} \cdot \overline{XD}$. This means that for a fixed point X in the fixed circle $\omega(O,\, r)$, the product $\overline{AX} \cdot \overline{XB}$ will be constant and will not depend on the choice of the chord A_iB_i which passes through X, i.e.

$$\overline{AX} \cdot \overline{XB} = \overline{A_1X} \cdot \overline{XB_1} = \overline{A_2X} \cdot \overline{XB_2} = const.$$

So, this product must depend on the position of X (relative to ω) and on ω itself.



Well, let's try to express this product as a function of the known elements, i.e. the radius of the circle and the distance from the center of the circle to the point X. Let's draw a diameter through X (in order to include the center in all of this) and let M and N be its endpoints. Then, as previously proved,

$$\overline{AX} \cdot \overline{XB} = \overline{MX} \cdot \overline{XN} = (\overline{MO} + \overline{OX})(\overline{ON} - \overline{OX}) =$$
$$= (r + \overline{OX})(r - \overline{OX}) = r^2 - \overline{OX}^2.$$

This is, in fact, the absolute value (since X is inside the circle) of what we will call the power of X with respect to ω.

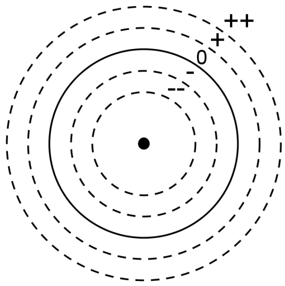
We will define the *power of the point* X with respect to the circle $\omega(O,\, r)$ as a real number which reflects the relative distance of the point X to the circle ω:

$$pow(X,\, \omega) = \overline{OX}^2 - r^2.$$

Consequently, we can conclude the following property:

Property 12.1. Points that are on equal distances from the center have equal powers with respect to the circle.

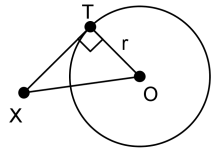
By the definition, it also means that the points inside the circle (for which $0 \le \overline{OX} < r$) will have negative power, the points on the circle (for which $\overline{OX} = r$) will have zero power and the points outside the circle (for which $\overline{OX} > r$) will have positive power with respect to the circle.



Property 12.2. If a point X is outside the circle $\omega(O, r)$, then the power of the point equals the square of the length of the tangent segment from X to the tangent point T.

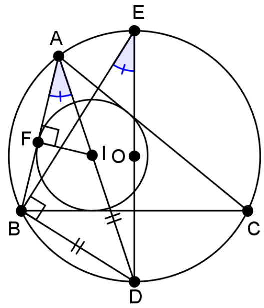
Proof. By the definition of power of a point and using the Pythagorean Theorem:

$$pow(X, \omega) = \overline{OX}^2 - r^2 = \overline{OX}^2 - \overline{OT}^2 = \overline{XT}^2 \quad \blacksquare$$



Example 12.1 (Euler's Theorem in Geometry)**.** Let O and I be the circumcenter and incenter of $\triangle ABC$, respectively. Let R and r be the circumradius and inradius, respectively. Prove that

$$\overline{OI}^2 = R(R - 2r)$$



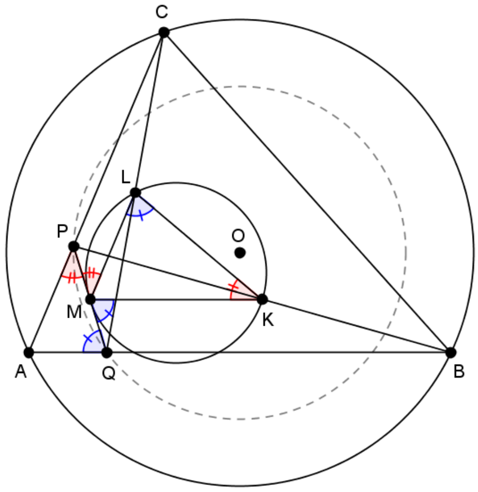
Proof. Let D be the second intersection of AI with the circumcircle ω of $\triangle ABC$. We need to prove that $2Rr = R^2 - \overline{OI}^2 = |pow(I, \omega)| = \overline{AI} \cdot \overline{ID}$. By Example 7.1, we know that $\overline{DI} = \overline{DB}$. So, we need to find similar triangles that contain the sides AI, BD, r and $2R$. Let E be the diametrically opposite point of D on ω; Thus $\overline{ED} = 2R$ and $\angle EBD = 90°$. Let F be tangent point of the incircle with the side AB; Thus $\overline{IF} = r$ and $\angle IFA = 90°$. Since $\angle FAI \equiv \angle BAD = \angle BED$, the right triangles $\triangle AIF$ and $\triangle EDB$ are similar. Therefore,

$$\frac{\overline{AI}}{\overline{IF}} = \frac{\overline{ED}}{\overline{DB}} \tag{*}$$

$$\overline{AI} \cdot \overline{ID} = \overline{AI} \cdot \overline{DB} \overset{(*)}{=} \overline{ED} \cdot \overline{IF} = 2Rr \quad \blacksquare$$

Remark. From this theorem, we can derive the *Euler inequality*. Since $\overline{OI}^2$ is non-negative and R is always positive, we can conclude that $R - 2r$ is non-negative, i.e. $R \geq 2r$. Equality holds iff $O \equiv I$, i.e. when $\triangle ABC$ is equilateral.

Example 12.2 (IMO 2009/2)**.** Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $\overline{OP} = \overline{OQ}$.



Proof. We need to prove that P and Q are on the same distance from the circumcenter O, so by Property 12.1, we need to prove that their power with respect to the circumcircle (ABC) are equal, i.e. we need to prove that

$$\overline{AP} \cdot \overline{PC} = \overline{AQ} \cdot \overline{QB}.$$

Since MK and ML are midsegments in $\triangle QBP$ and $\triangle PCQ$, we have

$$MK \parallel QB \quad \text{and} \quad ML \parallel PC \tag{1}$$

$$\overline{MK} = \frac{1}{2} \cdot \overline{QB} \quad \text{and} \quad \overline{ML} = \frac{1}{2} \cdot \overline{PC} \tag{2}$$

Since Γ is tangent to PQ at M, and using (1), we get

$$\angle KLM = \angle KMQ = \angle MQA \equiv \angle PQA$$

$$\angle LKM = \angle LMP = \angle MPA \equiv \angle QPA$$

Therefore, $\triangle APQ \sim \triangle MKL$.

$$\therefore \frac{\overline{AP}}{\overline{AQ}} = \frac{\overline{MK}}{\overline{ML}} \overset{(2)}{=} \frac{\overline{QB}}{\overline{PC}}, \text{ i.e. } \overline{AP} \cdot \overline{PC} = \overline{AQ} \cdot \overline{QB} \qquad \blacksquare$$

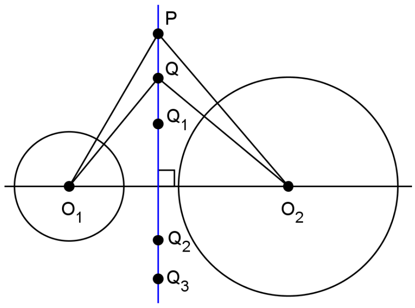
Before continuing, let's mention that sometimes it's useful to consider a point as a degenerate circle, i.e. a circle with radius zero. Then, the power of a point P with respect to the degenerate circle $\omega(A, 0)$ is

$$pow(P, \omega) = \overline{AP}^2 - r^2 = \overline{AP}^2$$

If it is unclear to you what this means, wait just a bit; We will use this in Example 12.3.

12.1 Radical axis

We learned about the power of a point with respect to a circle. Now, let's find the locus of the points that have equal power with respect to two given circles $\omega_1(O_1, r_1)$ and $\omega_2(O_2, r_2)$. Let P be a point that satisfies this condition. Then,



by the definition of power of a point,

$$\overline{PO_1}^2 - r_1^2 = \overline{PO_2}^2 - r_2^2.$$

Let Q be another point that satisfies the condition. Similarly, we have

$$\overline{QO_1}^2 - r_1^2 = \overline{QO_2}^2 - r_2^2.$$

From the two equations above, we get:

$$\overline{PO_1}^2 - \overline{PO_2}^2 = r_1^2 - r_2^2 = \overline{QO_1}^2 - \overline{QO_2}^2,$$

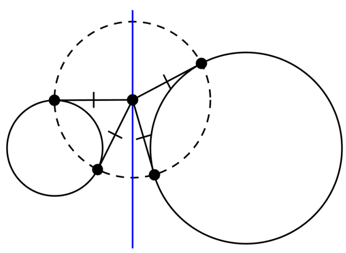
which, by Example 4.3 means that $PQ \perp O_1O_2$. But this will also be true for all other points $Q_1, Q_2, \ldots$ that have same power with respect to both circles, i.e. $PQ_1 \perp O_1O_2$, $PQ_2 \perp O_1O_2$, $\ldots$, which means that the set of all such points is a straight line perpendicular to O_1O_2.

The radical axis of two circles is a line that is the locus of all points that have equal powers with respect to both circles.

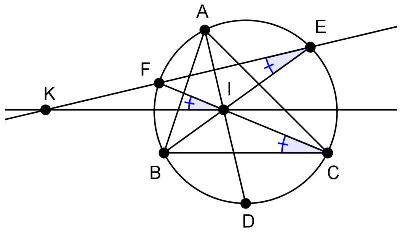
Property 12.3. The radical axis of two circles is perpendicular to the line connecting the centers of the circles.

As a consequence, the radical axis of two intersecting circles will be the line that passes through their intersection points, because those points have zero power with respect to both circles. The radical axis of two tangent circles will be their common tangent through their tangent point, because it is perpendicular to the line connecting the centers and because the tangent point has zero power with respect to both circles.

Let's recall that if a point is outside the circle, the power of the point with respect to the circle equals the square of the length of the tangent segment from the point to the circle. Hence, the tangent segments from such point Q_i to both circles are of equal length, which means that each point on the radical axis is a center of a circle that intersects both given circles orthogonally.



Example 12.3 (BMO 2015)**.** Let $\triangle ABC$ be a scalene triangle with incentre I and circumcircle ω. Lines AI, BI and CI intersect ω for the second time at points D, E and F, respectively. The parallel lines from I to the sides BC, AC and AB intersect EF, DF and DE at points K, L and M, respectively. Prove that the points K, L and M are collinear.



Proof. Because of the parallel lines KI and BC,

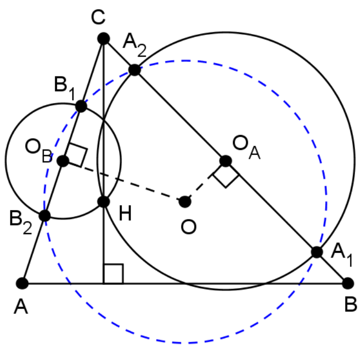
$$\angle KIF = \angle BCI \equiv \angle BCF \stackrel{\omega}{=} \angle BEF \equiv \angle IEK$$

In addition to this, $\angle IKE \equiv \angle IKF$ is a common angle for the triangles $\triangle KIF$ and $\triangle KEI$, so the triangles are similar and therefore

$$\frac{\overline{KI}}{\overline{KF}} = \frac{\overline{KE}}{\overline{KI}}, \text{ i.e. } \overline{KI}^2 = \overline{KF} \cdot \overline{KE}$$

The left hand side is the power of the point K with respect to the degenerate circle I and the right hand side is the power of the point K with respect to the circle (EFD). Therefore, K lies on the radical axis r of I and (EFD). Similarly, L and M also lie on r, so they are collinear. ■

Example 12.4 (IMO 2008/1)**.** Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2. Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.



Proof. A precise drawing may give us a hint about this problem. If we draw the figure correctly, we will see that the second intersection of the circles Γ_A and Γ_B lies on the altitude CH. So, we firstly need to prove that this is indeed true, i.e. we need to prove that CH is the radical axis of Γ_A and Γ_B and then use this fact to solve the problem.

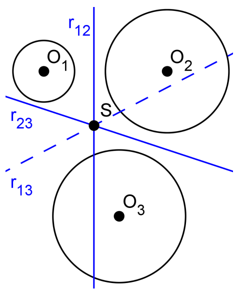
Let O_A and O_B be the centers of Γ_A and Γ_B (i.e. the midpoints of BC and CA), respectively. Since O_AO_B is a midsegment in $\triangle ABC$, $O_AO_B \parallel AB$. Since CH is altitude in $\triangle ABC$, $AB \perp CH$. Therefore, $O_AO_B \perp CH$. By Property 12.3 and recalling that $H \in \Gamma_A$ and $H \in \Gamma_B$, we can conclude that CH is the radical axis of Γ_A and Γ_B.

Now, using the fact that any point on the radical axis has equal power with respect to both circles, we get that

$$\overline{CA_1} \cdot \overline{CA_2} = \overline{CB_1} \cdot \overline{CB_2}.$$

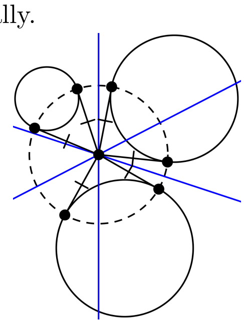
Therefore, $A_1A_2B_1B_2$ is a cyclic quadrilateral. The center of $(A_1A_2B_1B_2)$ can be found as the intersection of the side bisectors of A_1A_2 and B_1B_2. But, since the midpoint of BC coincides with the midpoint of A_1A_2 and the midpoint of CA coincides with the midpoint of B_1B_2, then the side bisectors of A_1A_2 and B_1B_2 intersect at the circumcenter O of $\triangle ABC$. Similarly, $(B_1B_2C_1C_2)$ is a circle centered at O. Therefore, the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic. ∎

12.2 Radical center



Let's find the locus of the points that have equal power with respect to three circles ω_1, ω_2 and ω_3, whose centers are not collinear. By definition, the set of points that satisfy $pow(X, \omega_1) = pow(X, \omega_2)$ is the radical axis of ω_1 and ω_2 and the set of points that satisfy $pow(X, \omega_2) = pow(X, \omega_3)$ is the radical axis of ω_2 and ω_3. These axes are not parallel (because the centers of the circles are not collinear), so let their intersection be S. By transitivity, we get that for this point S, the following is true $pow(S, \omega_1) = pow(S, \omega_3)$, which means that the radical axis of ω_1 and ω_3 also passes through S. So, S is the only point that has equal power to all three circles and it is called the *radical center* of the three circles.

Note that if the radical center lies outside of all three circles, then the tangent segments from it to all three circles will be of equal length. So, the radical center is the center of the unique circle (called the *radical circle*) that intersects the three given circles orthogonally.



Geometric construction of radical axis

Now, let's see how we can geometrically construct the radical axis of two non-concentric circles.

i) $\omega_1 \cap \omega_2 = \{A, B\}$
The points A and B lie on both circles, so they both have zero power to both circles. Since we know that the radical axis is a line, we can construct it by drawing the line through the points A and B.

ii) $\omega_1 \cap \omega_2 = \{T\}$
As discussed in the previous case, the point T lies on the radical axis. Since we proved that the radical axis is a line perpendicular to the line joining the centers, we can construct it easily as the common tangent of the circles through T.

iii) $\omega_1 \cap \omega_2 = \emptyset$
Let's draw another circle ω_3 that intersects both ω_1 and ω_2. Let's construct the radical axes of ω_1 and ω_3, $r_{1,3}$, and ω_2 and ω_3, $r_{2,3}$. The intersection of $r_{1,3}$ and $r_{2,3}$, S, is the radical center of the three circles, so it must lie on the radical axis $r_{1,2}$ that we are trying to construct. Now, we can continue in two different ways: we can either do the same thing with another circle that intersect ω_1 and ω_2, thus finding another point that lies on $r_{1,2}$; or we can use the fact the radical axis $r_{1,2}$ is perpendicular to the line joining the centers, O_1O_2.

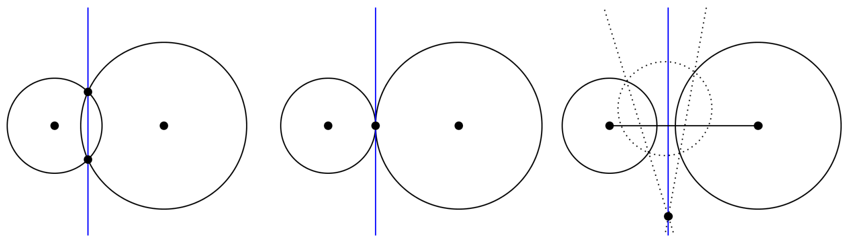


Figure 12.1: Radical axis of two circles

Related problems: 44, 82, 84 and 119.

Chapter 13

Collinearity II

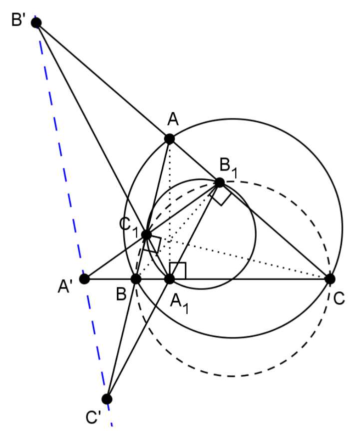
13.1 Radical Axis

In some problems, we can prove that three points are collinear if we show that they all lie on the Radical axis of some two circles.

Example 13.1 (Orthic axis)**.** Let AA_1, BB_1 and CC_1 be the altitudes in $\triangle ABC$. Let A' be the intersection of the lines BC and B_1C_1 and similarly define the points B' and C'. Prove that A', B' and C' lie on a line.

Proof. Since $\angle BB_1C = 90° = \angle BC_1C$, the quadrilateral BCB_1C_1 is cyclic. Therefore, by the intersecting secant theorem, we have

$$\overline{A'B} \cdot \overline{A'C} = \overline{A'C_1} \cdot \overline{A'B_1}.$$

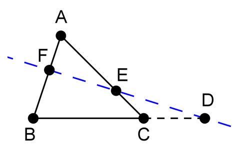


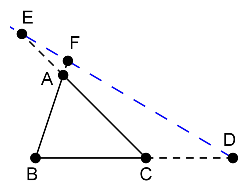
The left-hand side is in fact the power of the point A' to the circle (ABC) and the right-hand side is the power of the point A' to the circle $(A_1B_1C_1)$. Since it has same power with respect to both circles, then it must lie on the radical axis of those circle. Similarly, B' and C' also lie on the radical axis of (ABC) and $(A_1B_1C_1)$, so A', B' and C' are collinear. ∎

13.2 Menelaus' Theorem

Example 13.2 (Menelaus' Theorem)**.** Let ABC be a triangle. Let D, E and F be points on the lines BC, CA and AB, respectively, such that odd number of them (one or three) are on the extensions of the sides. The points D, E and F are collinear if and only if

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$





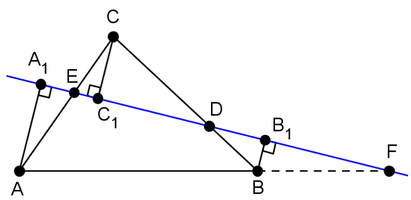
Remark. An easy way to remember how to write these ratios is the following. If we have a triangle $\triangle XYZ$ and the points $M \in XY$, $N \in YZ$ and $P \in ZX$ lie on its sides, then we will write the ratios as follows:
Firstly we are going to write its sides in a cyclic manner, like this

$$\frac{\overline{X\quad}}{\overline{\quad Y}} \cdot \frac{\overline{Y\quad}}{\overline{\quad Z}} \cdot \frac{\overline{Z\quad}}{\overline{\quad X}}$$

and then we will just add each point in the numerator and denominator in the fraction of the corresponding side, like this

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YN}}{\overline{NZ}} \cdot \frac{\overline{ZP}}{\overline{PX}}$$

Proof. Let D, E and F be collinear and let the line defined by them be p. Let A_1, B_1 and C_1 be the feet of the perpendiculars from A, B and C, respectively, to the line p.



$$\triangle AA_1F \sim \triangle BB_1F \quad (\because \angle AA_1F = 90^\circ = \angle BB_1F, \angle AFA_1 \equiv \angle BFB_1)$$

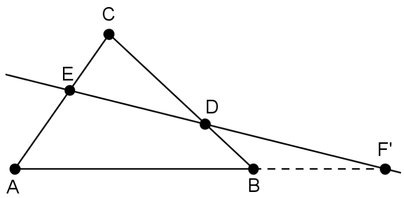
$$\therefore \frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AA_1}}{\overline{BB_1}}$$

$$\text{Similarly, } \frac{\overline{BD}}{\overline{DC}} = \frac{\overline{BB_1}}{\overline{CC_1}} \text{ and } \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{CC_1}}{\overline{AA_1}}.$$

$$\therefore \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{AA_1}}{\overline{BB_1}} \cdot \frac{\overline{BB_1}}{\overline{CC_1}} \cdot \frac{\overline{CC_1}}{\overline{AA_1}} = 1 \quad \square$$

Now, let's prove the other direction. Let

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$



We will discuss the case when two of the points are on the sides of the triangle and one on the extension of the other side. The other case, when all three points are on the extensions of the sides is analogous. WLOG, let D and E be on the sides BC and CA, respectively and F be on the extension of the side AB. We should prove that the points D, E and F are collinear. Let the line DE intersect the line AB at F' (note that F' cannot lie between A and B). Because the points D, E and F' are collinear, we can use the direction of the Menelaus' Theorem that we just proved. So,

$$\frac{\overline{AF'}}{\overline{F'B}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Combining with the given condition, we get

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AF'}}{\overline{F'B}}.$$

Since $\overline{AF} - \overline{FB} = \overline{AB} = \overline{AF'} - \overline{F'B}$, by subtracting 1 from both sides in the above equation, we get

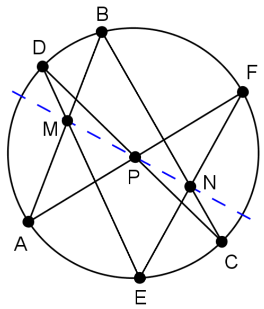
$$\frac{\overline{AB}}{\overline{FB}} = \frac{\overline{AB}}{\overline{F'B}}.$$

We conclude that $\overline{FB} = \overline{F'B}$. Because both F and F' are on the extension of the side AB, we get that $F \equiv F'$, i.e. the points D, E and F are collinear. ■

We will see how this theorem can be used in both directions, while proving the next theorem.

13.3 Pascal's Theorem

Example 13.3 (Pascal's theorem)**.** Let A, B, C, D, E and F be points on a circle (not necessarily in cyclic order). Let $M = AB \cap DE$, $N = BC \cap EF$ and $P = CD \cap FA$. Then M, N and P are collinear.

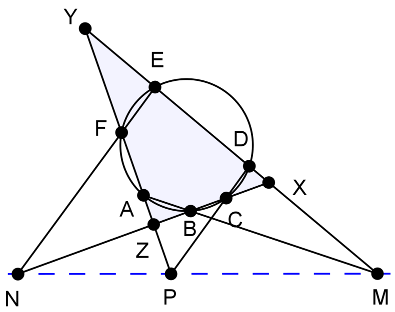


$$ABCDEFA$$

$$M - N - P$$

Remark. An easy way to remember these intersections is the following: Take two consecutive letters for a line, skip one letter, and take two more letters for the second line. Their intersection is the first of the three collinear points. Then shift to the right and repeat two times.

Proof. Let $X = BC \cap DE$, $Y = DE \cap FA$ and $Z = FA \cap BC$.



If we use Menelaus' Theorem three times on $\triangle XYZ$, firstly with the collinear points $A - B - M$, then with the collinear points $P - C - D$ and finally with the collinear points $F - N - E$, we get:

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YA}}{\overline{AZ}} \cdot \frac{\overline{ZB}}{\overline{BX}} = 1$$

$$\frac{\overline{XD}}{\overline{DY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZC}}{\overline{CX}} = 1$$

$$\frac{\overline{XE}}{\overline{EY}} \cdot \frac{\overline{YF}}{\overline{FZ}} \cdot \frac{\overline{ZN}}{\overline{NX}} = 1$$

By multiplying these 3 equations and reordering the members, we get:

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZN}}{\overline{NX}} \cdot \frac{(\overline{YA} \cdot \overline{YF}) \cdot (\overline{ZB} \cdot \overline{ZC}) \cdot (\overline{XD} \cdot \overline{XE})}{(\overline{AZ} \cdot \overline{FZ}) \cdot (\overline{BX} \cdot \overline{CX}) \cdot (\overline{DY} \cdot \overline{EY})} = 1.$$

From the Intersecting Secants Theorem for the points X, Y and Z we get:

$$\overline{XD} \cdot \overline{XE} = \overline{XC} \cdot \overline{XB}$$

$$\overline{YF} \cdot \overline{YA} = \overline{YE} \cdot \overline{YD}$$

$$\overline{ZB} \cdot \overline{ZC} = \overline{ZA} \cdot \overline{ZF}$$

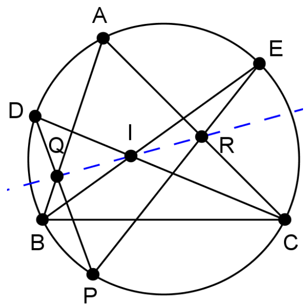
From the previous four equations, we get:

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZN}}{\overline{NX}} = 1,$$

which by Menelaus' Theorem means that M, N and P are collinear. ■

Here is an example to show how the Pascal's Theorem can be used in a problem.

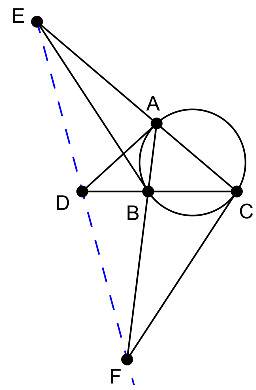
Example 13.4. Let D and E be the midpoints of the minor arcs $\widehat{AB}$ and $\widehat{AC}$ on the circumcircle of $\triangle ABC$, respectively. Let P be on the minor arc $\widehat{BC}$, $Q = PD \cap AB$ and $R = PE \cap AC$. Prove that the line QR passes through the incenter I of $\triangle ABC$.



Proof. Since D is the midpoint of the arc $\widehat{AB}$, CD is the angle bisector of $\angle BCA$. Similarly, BE is the angle bisector of $\angle ABC$. Therefore, $CD \cap BE = I$. Now, we apply Pascal's Theorem to the points C, D, P, E, B and A and we get that the points $CD \cap EB = I$, $DP \cap BA = Q$ and $PE \cap AC = R$ are collinear. ■

We remark that there are limiting cases of Pascal's Theorem. For example, we may move A to approach B. In the limit, A and B will coincide and the line AB will become the tangent line at B. Here is an example to show how this works.

Example 13.5. Let ω be the circumcircle of $\triangle ABC$. Let the tangent lines to ω at A, B and C intersect the lines BC, CA and AB at points D, E and F, respectively. Prove that the points D, E and F are collinear.

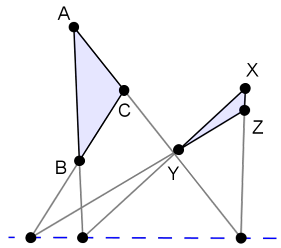


Proof. Let's apply the Pascal's Theorem to the points A, A, B, B, C and C. We get that the points $AA \cap BC = D$, $AB \cap CC = F$ and $BB \cap CA = E$ are collinear. ■

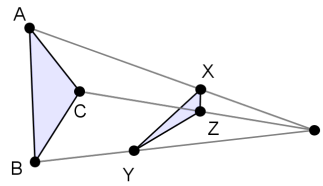
13.4 Desargues' Theorem

Two triangles $\triangle ABC$ and $\triangle XYZ$ are *perspective from a line* if the points $AB \cap XY$, $BC \cap YZ$ and $CA \cap ZX$ are collinear.

Two triangles $\triangle ABC$ and $\triangle XYZ$ are *perspective from a point* if the lines AX, BY and CZ are concurrent.

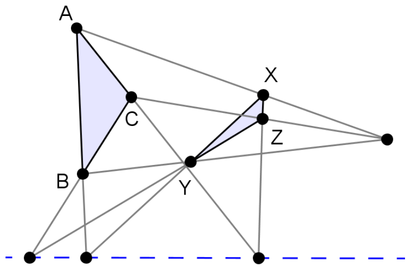


(a) Perspective from a line.



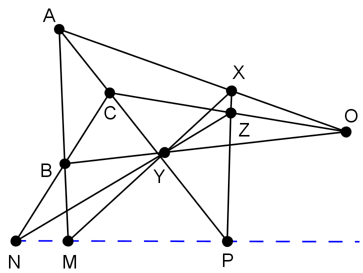
(b) Perspective from a point.

Example 13.6 (Desargues' Theorem)**.** Two triangles are perspective from a line if and only if they are perspective from a point.



Proof. Let $\triangle ABC$ and $\triangle XYZ$ be perspective from a point, i.e. AX, BY and CZ are concurrent, and let the point of concurrence be O. Let $AB \cap XY = M$, $BC \cap YZ = N$ and $CA \cap ZX = P$.

We firstly apply the Menelaus' Theorem to $\triangle OAB$ and the points $M-Y-X$, then to $\triangle OBC$ and $N-Z-Y$, and finally to $\triangle OCA$ and $P-X-Z$:



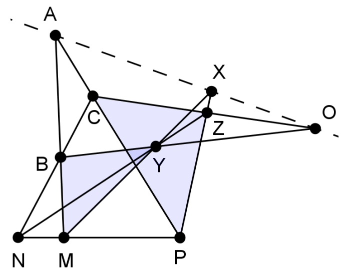
$$\frac{\overline{OX}}{\overline{XA}} \cdot \frac{\overline{AM}}{\overline{MB}} \cdot \frac{\overline{BY}}{\overline{YO}} = 1$$
$$\frac{\overline{OY}}{\overline{YB}} \cdot \frac{\overline{BN}}{\overline{NC}} \cdot \frac{\overline{CZ}}{\overline{ZO}} = 1$$
$$\frac{\overline{OZ}}{\overline{ZC}} \cdot \frac{\overline{CP}}{\overline{PA}} \cdot \frac{\overline{AX}}{\overline{XO}} = 1$$

By multiplying these three equations, we get:

$$\frac{\overline{AM}}{\overline{MB}} \cdot \frac{\overline{BN}}{\overline{NC}} \cdot \frac{\overline{CP}}{\overline{PA}} = 1,$$

which by the Menelaus' Theorem for $\triangle ABC$, means that the points M, N and P are collinear, i.e. $\triangle ABC$ and $\triangle XYZ$ are perspective from a line. $\square$

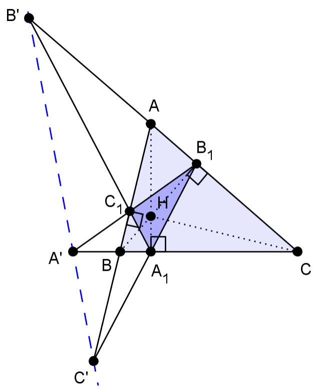
Now, let's prove the other direction.
Let $\triangle ABC$ and $\triangle XYZ$ be perspective from a line, i.e. the points $M = AB \cap XY$, $N = BC \cap YZ$ and $P = CA \cap ZX$ are collinear. Let $O = BY \cap CZ$. We should prove that AX also passes through O.



Let's take a look at $\triangle PCZ$ and $\triangle MBY$. The lines PM, CB and ZY are concurrent at N, so the triangles are perspective from a point. By the direction of the Desargues' Theorem that we just proved, it follows that the triangles must be perspective from a line, i.e. the points $PC \cap MB = A$, $CZ \cap BY = O$ and $ZP \cap YM = X$ are collinear. With this, we proved that AX passes through O, so $\triangle ABC$ and $\triangle XYZ$ are perspective from a point. ■

Let's see it in action. We will give an alternate proof to Example 13.1:

Example 13.7 (Orthic axis)**.** Let AA_1, BB_1 and CC_1 be the altitudes in $\triangle ABC$. Let A' be the intersection of the lines BC and B_1C_1 and similarly define the points B' and C'. Prove that A', B' and C' lie on a line.



Proof. Since AA_1, BB_1 and CC_1 are concurrent at the orthocenter of $\triangle ABC$, the triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are perspective from a point. Then, by the Deargues' Theorem, they are also perspective from a line, i.e. the points $AB \cap A_1B_1 = C'$, $BC \cap B_1C_1 = A'$ and $CA \cap C_1A_1 = B'$ are collinear. ■

Remark. This proof can be used for more generalized problem, where AA_1, BB_1 and CC_1 are any cevians in $\triangle ABC$ that are concurrent.

We will end this chapter here, but we must mention that collinearity plays an important role in the chapter Homothety, so we will continue this theme later in our journey.

Related problems: (Menelaus' Theorem) 108, 114, 115 and 134.

Chapter 14

Concurrence II

14.1 Radical Center

Recall section 12.2, where we saw that the pairwise radical axes of three circles concur at the radical center. This is another approach of proving concurrence in geometry problems.

Example 14.1 (IMO 1995/1)**.** Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN and XY are concurrent.

Proof. We will prove that these three lines are radical axis of three circles. Let the circle with diameter AC be ω_1 and the circle with diameter BD be ω_2.

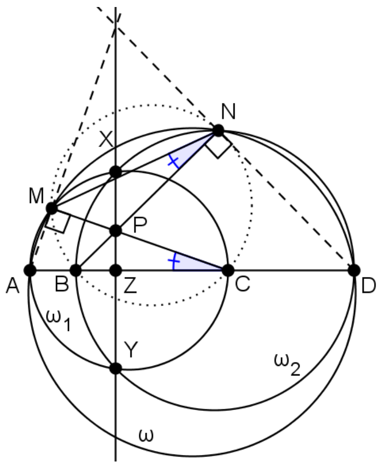
$$\overline{PM}\cdot\overline{PC} \overset{\omega_1}{=} \overline{PX}\cdot\overline{PY} \overset{\omega_2}{=} \overline{PB}\cdot\overline{PN}$$

$$\therefore BCNM \text{ is cyclic} \qquad (*)$$

Since AC and BD are diameters of ω_1 and ω_2, then $\angle AMC = 90^\circ = \angle BND$.

$$\begin{aligned}\angle MND &= \angle MNB + \angle BND \overset{(*)}{=} \\ &= \angle MCB + 90^\circ \equiv \\ &\equiv \angle MCA + 90^\circ \overset{\triangle AMC}{=} \\ &= 90^\circ - \angle MAC + 90^\circ \equiv \\ &\equiv 180^\circ - \angle MAD\end{aligned}$$

$$\therefore MADN \text{ is cyclic}$$

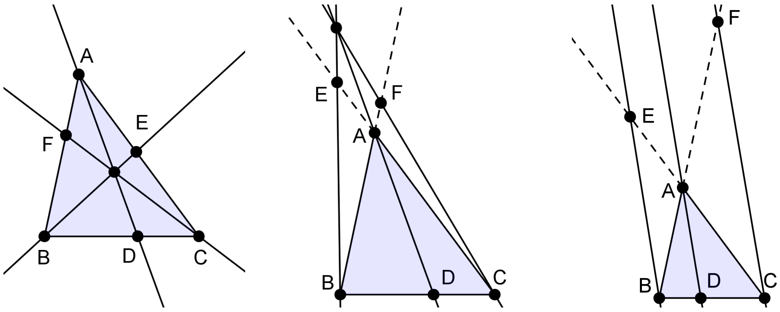


Now, we have three circles: $(MAYCX)$, $(MADN)$ and $(NXBYD)$. Their pairwise radical axes are MA, DN and XY, so they are concurrent at the radical center of these three circles. ■

14.2 Ceva's Theorem

Example 14.2 (Ceva's Theorem)**.** Let ABC be a triangle. Let D, E and F be points on the lines BC, CA and AB, respectively, such that even number of them (zero or two) are on the extensions of the sides. The lines AD, BE and CF are concurrent or parallel if and only if

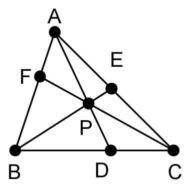
$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$



Remark. We write the ratio in exactly the same manner as we showed in Menelaus' Theorem.

Proof. Let the lines AD, BE and CF be concurrent at P. Assume that the point P is inside the triangle ABC. (When P is outside, the proof is similar)

$$\frac{P_{\triangle CAF}}{P_{\triangle CFB}} = \frac{\overline{AF}}{\overline{FB}}$$
$$\frac{P_{\triangle PAF}}{P_{\triangle PFB}} = \frac{\overline{AF}}{\overline{FB}}$$



$$\therefore \frac{P_{\triangle CAF} - P_{\triangle PAF}}{P_{\triangle CFB} - P_{\triangle PFB}} = \frac{\overline{AF}}{\overline{FB}}$$

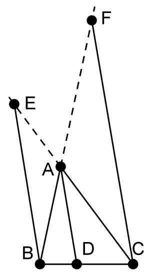
$$\text{i.e. } \frac{P_{\triangle CAP}}{P_{\triangle BCP}} = \frac{\overline{AF}}{\overline{FB}}.$$

$$\text{Similarly, } \frac{P_{\triangle ABP}}{P_{\triangle CAP}} = \frac{\overline{BD}}{\overline{DC}} \text{ and } \frac{P_{\triangle BCP}}{P_{\triangle ABP}} = \frac{\overline{CE}}{\overline{EA}}.$$

$$\therefore \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1. \quad \square$$

Let the lines AD, BE and CF be parallel. Exactly one of the points must be on the side of the triangle, WLOG let that point be D (the other two points are on the extensions of the sides). By Thales' Proportionality Theorem, we get

$$\begin{aligned} \frac{\overline{AF}}{\overline{FB}} &= \frac{\overline{DC}}{\overline{CB}} \quad (\because DA \parallel CF) \\ \frac{\overline{CE}}{\overline{EA}} &= \frac{\overline{CB}}{\overline{BD}} \quad (\because DA \parallel BE) \\ \therefore \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} &= \frac{\overline{DC}}{\overline{CB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CB}}{\overline{BD}} = 1. \quad \square \end{aligned}$$

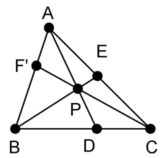


Now, let's prove the other direction. Let

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$$

and let even number of the points be on the extensions of the sides. Let the intersection of the lines AD and BE be P. In the case when there is no intersection, i.e. when $AD \parallel BE$, it can be easily proven that AD, BE and CF are parallel.

Let CP intersect AB at F'. Similarly as in the proof of Menelaus' Theorem, we are using the direction of Ceva's Theorem that we just proved (for AD, BE and CF' which do concur) and we get:



$$\frac{\overline{AF'}}{\overline{F'B}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Combining with the condition, we get:

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AF'}}{\overline{F'B}}.$$

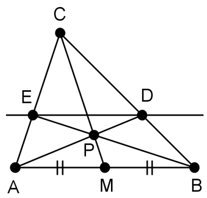
Keeping in mind that $\overline{AF} + \overline{FB} = \overline{AB} = \overline{AF'} + \overline{F'B}$, by adding 1 to both sides, we get:

$$\frac{\overline{AB}}{\overline{FB}} = \frac{\overline{AB}}{\overline{F'B}},$$

which means $\overline{FB} = \overline{F'B}$, i.e. $F \equiv F'$. We should note that in the last part, we assumed that F' is between A and B. That is a safe assumption because there are either zero or two points on the extensions of the sides; If there are zero, then D and E are on the sides BC and CA, so F' must also lie on the side AB; If there are two points on the extensions, then WLOG let them be D and E and F' will again lie on the side AB. ■

In the next few examples, we will show how we can use Ceva's Theorem in both directions.

Example 14.3. In $\triangle ABC$, let M be the midpoint of the side AB. Let P be an arbitrary point on the segment CM ($P \neq C$, $P \neq M$). Let $AP \cap BC = D$ and $BP \cap AC = E$. Prove that $ED \parallel AB$.



Proof. The lines AD, BE and CM are concurrent, so we can use Ceva's Theorem:

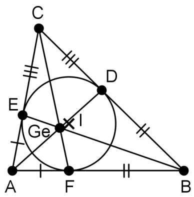
$$\frac{\overline{AM}}{\overline{MB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Since M is the midpoint of AB, $\overline{AM} = \overline{MB}$, so by canceling and then rearranging, we get:

$$\frac{\overline{CE}}{\overline{EA}} = \frac{\overline{CD}}{\overline{DB}},$$

which by Thales' Proportionality Theorem means that $ED \parallel AB$. ∎

Example 14.4 (Gergonne Point)**.** Let D, E and F be the tangent points of the incircle of $\triangle ABC$ with the sides BC, CA and AB, respectively. Prove that AD, BE and CF are concurrent.



Proof. $\overline{AF} = \overline{AE} = x$ as tangent segments from the point A to the incircle. Similarly, $\overline{BF} = \overline{BD} = y$ and $\overline{CD} = \overline{CE} = z$.

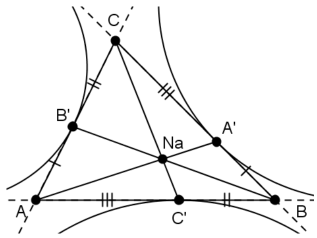
$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x} = 1,$$

so by Ceva's Theorem, AD, BE and CF are concurrent. ∎

Remark. This point of concurrence is known as the *Gergonne Point* of the triangle ABC.

Example 14.5 (Nagel Point)**.** Let A', B' and C' be the tangent points of the A-excircle, B-excircle and C-excircle with the sides BC, CA and AB in the $\triangle ABC$, respectively. Prove that AA', BB' and CC' are concurrent.

Proof. From Example 10.3.2, we know that $\overline{AB} + \overline{BA'} = \overline{AC} + \overline{CA'}$. We see that LHS and RHS add up to the perimeter of $\triangle ABC$, so each of them is equal



to the semiperimeter s. Therefore, $\overline{BA'} = s - c$ and $\overline{CA'} = s - b$. Similarly, $\overline{CB'} = s - a$, $\overline{AB'} = s - c$, $\overline{AC'} = s - b$ and $\overline{BC'} = s - a$.

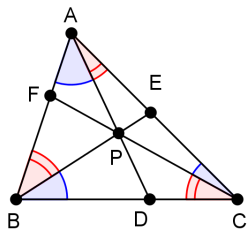
$$\frac{\overline{AC'}}{\overline{C'B}} \cdot \frac{\overline{BA'}}{\overline{A'C}} \cdot \frac{\overline{CB'}}{\overline{B'A}} = \frac{s-b}{s-a} \cdot \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} = 1,$$

so by Ceva's Theorem, AA', BB' and CC' are concurrent. ■

Remark. This point of concurrence is known as the *Nagel Point* of the triangle ABC.

Example 14.6 (Trigonometric Ceva's Theorem)**.** Given a triangle ABC and points D, E and F that lie on the lines BC, CA and AB, respectively; the lines AD, BE and CF are concurrent or parallel if and only if

$$\frac{\sin\angle BAD}{\sin\angle CAD} \cdot \frac{\sin\angle CBE}{\sin\angle ABE} \cdot \frac{\sin\angle ACF}{\sin\angle BCF} = 1.$$



Proof. By using the Law of Sines in $\triangle ABD$, we get

$$\frac{\overline{BD}}{\sin\angle BAD} = \frac{\overline{AB}}{\sin\angle BDA}, \text{ i.e. } \sin\angle BAD = \frac{\overline{BD} \cdot \sin\angle BDA}{\overline{AB}}.$$

Similarly, for $\triangle ACD$, we get

$$\sin\angle CAD = \frac{\overline{CD} \cdot \sin\angle CDA}{\overline{AC}}.$$

Since $D \in BC$, then the angles $\angle BDA$ and $\angle CDA$ are always equal or supplementary. Therefore, $\sin\angle BDA = \sin\angle CDA$. By dividing the previous equations, we get

$$\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{\overline{BD}}{\overline{CD}} \cdot \frac{\overline{AC}}{\overline{AB}}.$$

Analogously, we can get similar equations for the cevians BE and CF. By multiplying these three equations, we get

$$\frac{\sin\angle BAD}{\sin\angle CAD} \cdot \frac{\sin\angle CBE}{\sin\angle ABE} \cdot \frac{\sin\angle ACF}{\sin\angle BCF} = \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}},$$

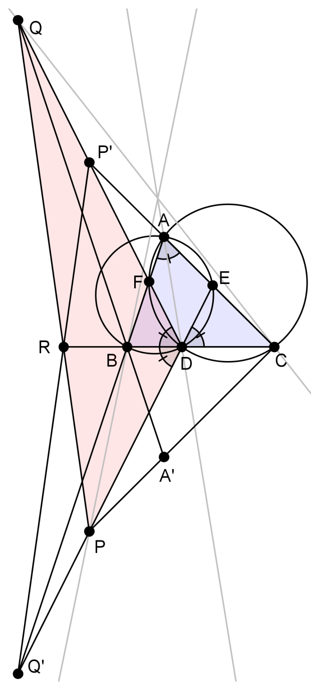
so by Ceva's Theorem we are done. ■

14.3 Desargues' Theorem

Here is an example that shows how we can use Desargues' Theorem when we need to prove concurrence.

Example 14.7 (RMM 2016)**.** Let ABC be a triangle and let D be a point on the segment BC, $D \neq B$ and $D \neq C$. The circle (ABD) meets the segment AC again at an interior point E. The circle (ACD) meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).

Proof. Let σ denote reflection in the line BC. Since $\angle BDF = \angle BAC = \angle CDE$ (because of the cyclic quadrilaterals $ABDE$ and $ACDF$), the lines DE and DF are images of one another under σ, so the lines AC and DF meet at $P' \equiv \sigma(P)$, and the lines AB and DE meet at $Q' \equiv \sigma(Q)$. Consequently, the lines PQ and $P'Q' \equiv \sigma(PQ)$ meet at some point R on the line BC. Since the points $Q' = AB \cap DP$, $R = BC \cap PQ$ and $P' = CA \cap QD$ are collinear, the triangles $\triangle ABC$ and $\triangle DPQ$ are perspective from a a line. Therefore, by Desargues' Theorem, they are also perspective from a point, i.e. the lines AD, BP and CQ are concurrent. ■

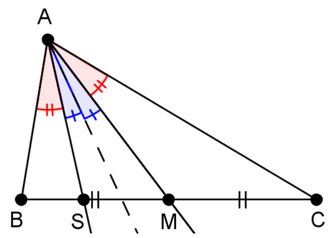


Related problems: (Concurrence) 77, 124 and 129. (Ceva's Theorem) 116.

Chapter 15

Symmedian

Symmedian is the reflection of the median across the corresponding angle bisector.



We will now see and prove a few properties of the symmedians.

Property 15.1. The symmedian AS divides the opposite side in the ratio of the square of the sides, i.e.

$$\frac{\overline{BS}}{\overline{CS}} = \left(\frac{\overline{AB}}{\overline{AC}}\right)^2.$$

Proof. Since the symmedian is the reflection of the median with respect to the angle bisector, we have $\angle BAS = \angle CAM$ and $\angle BAM = \angle CAS$.

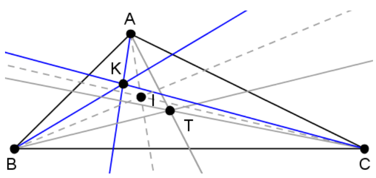
$$\frac{\overline{BS}}{\overline{MC}} = \frac{P_{\triangle BAS}}{P_{\triangle MAC}} = \frac{\overline{BA} \cdot \overline{AS}}{\overline{AM} \cdot \overline{AC}}$$

$$\frac{\overline{BM}}{\overline{SC}} = \frac{P_{\triangle BMA}}{P_{\triangle CSA}} = \frac{\overline{BA} \cdot \overline{AM}}{\overline{AS} \cdot \overline{AC}}$$

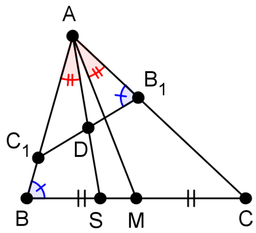
By multiplying these equalities, we are done. ∎

Property 15.2 (Lemoine Point)**.** The three symmedians in a triangle are concurrent.

Proof. Using Property 15.1, by Ceva's Theorem, it immediately follows that the three symmedians in a triangle are concurrent. This point of concurrence is called the *Lemoine Point* of the triangle. ∎



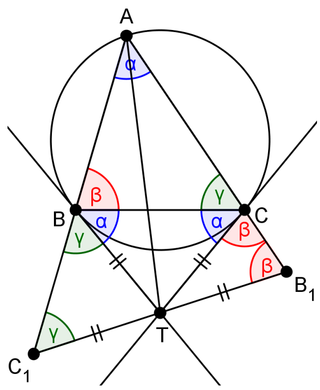
Property 15.3. A symmedian drawn from a vertex of a triangle bisects the antiparallels to the opposite side (with respect to the adjacent sides).



Proof. Let AS and AM be the symmedian and the median from the vertex A in $\triangle ABC$, respectively. Let $B_1 \in AC$ and $C_1 \in AB$, such that B_1C_1 is antiparallel to BC with respect to the lines AB and AC, i.e. $\angle AB_1C_1 = \angle ABC$. Therefore, $\triangle ABC \sim \triangle AB_1C_1$. Let $AS \cap B_1C_1 = D$. By the definition of symmedian, $\angle BAS = \angle CAM$, which means that the similarity "maps" AM in $\triangle ABC$ to $AS \equiv AD$ in $\triangle AB_1C_1$. Therefore, AD is median in $\triangle AB_1C_1$, i.e. the symmedian AS bisects B_1C_1 which is antiparallel to the opposite side BC. ∎

Property 15.4. Given a triangle ABC and its circumcircle, let the intersection of the tangents at the points B and C intersect at T. Then, AT is a symmedian in $\triangle ABC$.

Proof. Since the angle between a tangent and a chord is equal to any inscribed angle that subtends the same chord, $\angle CBT = \angle CAB = \alpha$ and $\angle BCT = \angle BAC = \alpha$, so $\triangle BCT$ is isosceles and therefore $\overline{TB} = \overline{TC}$. Let $B_1 \in AC$ and $C_1 \in AB$, such that B_1C_1 is an antiparallel line to BC (with respect to the lines AB and AC) that passes through T. Then, $\angle AB_1C_1 = \angle ABC = \beta$. Now, $\angle TCB_1 = 180^\circ - (\angle ACB + \angle BCT) = 180 - (\gamma + \alpha) = \beta$ and $\angle TB_1C \equiv \angle C_1B_1A = \beta$, so $\triangle TCB_1$ is isosceles, i.e. $\overline{TC} = \overline{TB_1}$. Similarly, $\overline{TB} = \overline{TC_1}$.
In conclusion, $\overline{TC_1} = \overline{TB} = \overline{TC} = \overline{TB_1}$, so T is the midpoint of B_1C_1. By Property 15.3, it follows that AT is the symmedian from the vertex A in $\triangle ABC$. ∎



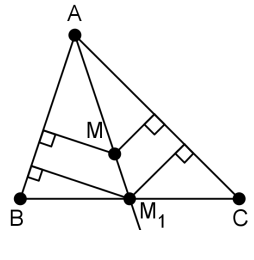
Remark. The previous example shows that the Lemoine point of a triangle is the Gergonne point of the tangential triangle.

Property 15.5. The A-symmedian is the locus of the points P such that

$$\frac{d(P,\, AB)}{d(P,\, AC)} = \frac{\overline{AB}}{\overline{AC}}.$$

Proof. We will firstly prove that the median is the locus of the points M such that

$$\frac{d(M,\, AB)}{d(M,\, AC)} = \frac{\overline{AC}}{\overline{AB}}.$$



Let M be a point in the interior of $\angle BAC$. Let AM meet BC at M_1. By similarity of triangles, we get that

$$\frac{d(M,\, AB)}{d(M_1,\, AB)} = \frac{\overline{AM}}{\overline{AM_1}} = \frac{d(M,\, AC)}{d(M_1,\, AC)}.$$

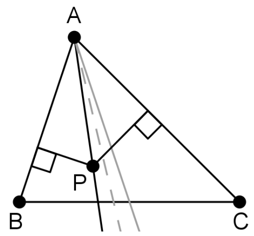
By rearranging, we get

$$\frac{d(M_1,\, AB)}{d(M_1,\, AC)} = \frac{d(M,\, AB)}{d(M,\, AC)} = \frac{\overline{AC}}{\overline{AB}}$$

$$\iff d(M_1,\, AB) \cdot \overline{AB} = d(M_1,\, AC) \cdot \overline{AC}$$

$$\iff P_{\triangle ABM_1} = P_{\triangle ACM_1}$$

$$\iff \overline{BM_1} = \overline{M_1C} \quad \square$$



Since the symmedian is the reflection of the median with respect to the angle bisector, by symmetry we have that it is the locus of the points P such that

$$\frac{d(P,\, AB)}{d(P,\, AC)} = \frac{d(M,\, AC)}{d(M,\, AB)} = \frac{\overline{AB}}{\overline{AC}} \qquad \blacksquare$$

Now, we will present a problem with many different solutions to see how we can use these properties in an Olympiad problem.

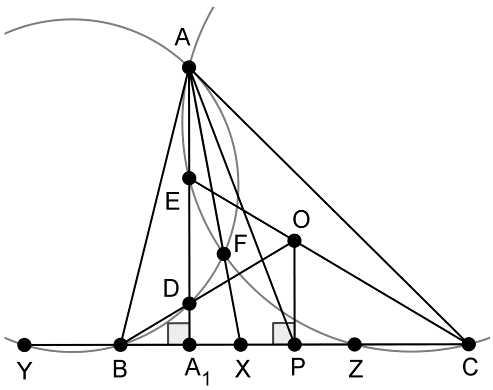
Example 15.1 (Macedonia MO 2017, Stefan Lozanovski)**.** Let O be the circumcenter of the acute triangle ABC ($\overline{AB} < \overline{AC}$). Let A_1 and P be the feet of the perpendiculars from A and O to BC, respectively. The lines BO and CO intersect AA_1 in D and E, respectively. Let F be the second intersection point of (ABD) and (ACE). Prove that the angle bisector od $\angle FAP$ passes through the incenter of $\triangle ABC$.

Proof. We need to prove that $\angle BAF = \angle CAP$. Since OP is perpendicular to BC and O is the circumcenter, then P is the midpoint of BC. Since AP is the median from A, we need to prove that AF is the symmedian from A.

Proof 1. We will use Property 15.1 to prove that AF is symmedian. Let the line AF intersect the side BC at X and let the circumcircles of $\triangle ABD$ and $\triangle ACE$ meet the line BC again at Y and Z, respectively. Then, by the Intersecting Secants Theorem, we have

$$\overline{XB} \cdot \overline{XY} = \overline{XF} \cdot \overline{XA} = \overline{XZ} \cdot \overline{XC}$$

$$\frac{\overline{XB}}{\overline{XC}} = \frac{\overline{XZ}}{\overline{XY}} = \frac{\overline{XB}+\overline{XZ}}{\overline{XC}+\overline{XY}} = \frac{\overline{BZ}}{\overline{CY}} \tag{1}$$



Now, let's use the fact that the point E is defined as the intersection of the altitude and the circumradius.

$$\angle ACE \equiv \angle ACO = \frac{1}{2}(180^\circ - \angle AOC) = \frac{1}{2}(180^\circ - 2\angle ABC) =$$
$$= 90^\circ - \angle ABC \equiv 90^\circ - \angle ABA_1 = \angle BAA_1 \equiv \angle BAE$$

Therefore, BA is tangent to (ACE). Similarly, CA is tangent to (ABD). Now, by the Secant-Tangent Theorem, we have

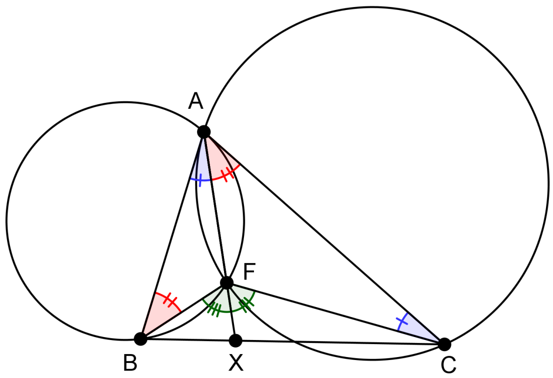
$$\overline{BA}^2 = \overline{BZ} \cdot \overline{BC}$$
$$\overline{CA}^2 = \overline{CB} \cdot \overline{CY}$$

By dividing these equations and using (1), we get

$$\frac{\overline{BA}^2}{\overline{CA}^2} = \frac{\overline{BZ}}{\overline{CY}} = \frac{\overline{XB}}{\overline{XC}}$$ ■

Proof 2. As in Proof 1, we will use Property 15.1 to prove that AF is symmedian. In Proof 1, we also proved that BA is tangent to $(ACE) \equiv (ACF)$. Therefore, $\angle BAF = \angle ACF$. Similarly, CA is tangent to ABF and therefore $\angle CAF = \angle ABF$. Thus, by the criterion AA, $\triangle BAF \sim \triangle ACF$ which gives

$$\frac{\overline{BF}}{\overline{CF}} = \frac{\frac{\overline{BF}}{\overline{AF}}}{\frac{\overline{CF}}{\overline{AF}}} = \frac{\frac{\overline{BA}}{\overline{AC}}}{\frac{\overline{AC}}{\overline{AB}}} = \frac{\overline{AB}^2}{\overline{AC}^2}$$



Also, $\angle BFX = 180° - \angle BFA = 180° - \angle AFC = \angle CFX$, so FX is an angle bisector in $\triangle BFC$ and

$$\frac{\overline{BF}}{\overline{CF}} = \frac{\overline{BX}}{\overline{CX}}$$

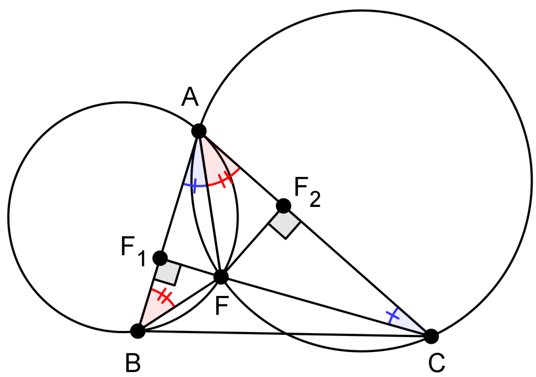
Finally, we get that

$$\frac{\overline{BX}}{\overline{CX}} = \frac{\overline{AB}^2}{\overline{AC}^2} \qquad \blacksquare$$

Proof 3. Same as in Proof 2, we get that $\triangle BAF \sim \triangle ACF$. Let F_1 and F_2 be the feet of the perpendiculars from F to AB and AC, respectively. Then, from the similarity, we get

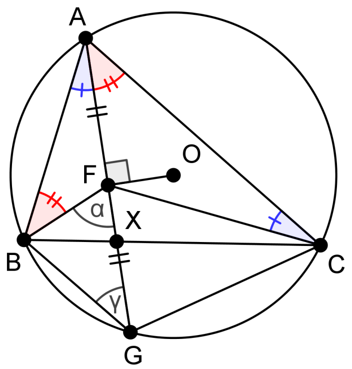
$$\frac{\overline{FF_1}}{\overline{FF_2}} = \frac{\overline{AB}}{\overline{AC}}$$

which, by Property 15.5, means that F lies on the $A-$symmedian. $\blacksquare$



Proof 4. Same as in Proof 2, we get that $\triangle BAF \sim \triangle ACF$ and therefore

$$\frac{\overline{BA}}{\overline{BF}} = \frac{\overline{AC}}{\overline{AF}} \tag{1}$$



Let AX intersect (ABC) again at G. Then,

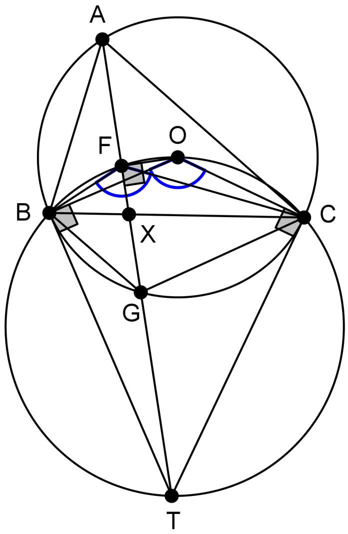
$$\angle BFG = 180^\circ - \angle BFA = \angle FBA + \angle FAB = \angle FAC + \angle FAB = \alpha$$

$$\angle BGF \equiv \angle BGA = \angle BCA = \gamma$$

$$\therefore \triangle ABC \sim \triangle FBG$$

$$\therefore \frac{\overline{AB}}{\overline{FB}} = \frac{\overline{AC}}{\overline{FG}} \tag{2}$$

From (1) and (2), we get that $\overline{AF} = \overline{FG}$, i.e. F is the midpoint of the chord AG. Since O is the circumcenter, we get $OF \perp AG$, i.e. $\angle OFG = 90^\circ$ (3)



Now, let's draw the tangents to the circumcircle at B and C, and let them intersect at T. The quadrilateral $OBTC$ is cyclic with diameter OT. Earlier in this solution, we proved that $\angle BFG = \alpha$. Similarly, $\angle CFG = \alpha$.

$$\therefore \angle BFC = \angle BFG + \angle CFG = 2\alpha = \angle BOC$$

$$\therefore F \in (BOCT)$$

$$\therefore \angle OFT = 90^\circ$$

$$\therefore \angle OFT \overset{(3)}{=} \angle OFG$$

$$T \in FG \equiv AF$$

Finally, since AF passes through the intersection of the tangents at B and C, by Property 15.4, AF is symmedian. ■

Related problems: 89, 96, 98, 121, 125 and 126.

Chapter 16

Homothety

Definition and properties

A homothety with center O and ratio k is a function that sends every point on the plane P to a point P' such that

$$\overrightarrow{OP'} = k \cdot \overrightarrow{OP}.$$

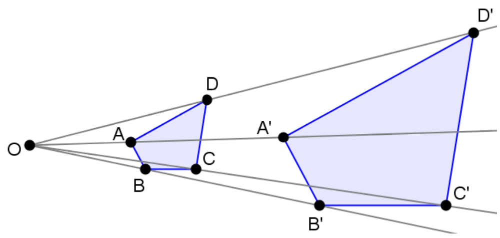


Figure 16.1: A homothety with center O and ratio $k = 2.5$

From this definition, we can directly conclude the following properties:

Property 16.1. The image point, the original point and the center of the homothety are collinear.

Property 16.2. A homothety always sends a figure to a similar figure, such that the corresponding sides are parallel.

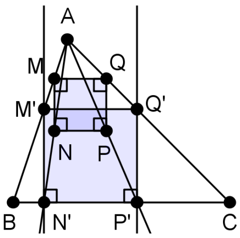
If $k > 0$, then the image and the original will be on the same side of the center; If $k < 0$, the image and the original will be on different sides of the center, i.e. the center will be between them. If $|k| > 1$, then the homothety is a magnification; If $|k| < 1$, then it is a reduction.

We will use the notation $\mathcal{X}_{O,\,k} : P \to P'$ to denote that P' is the image of P under the homothety centered at O with ratio k.

Getting started

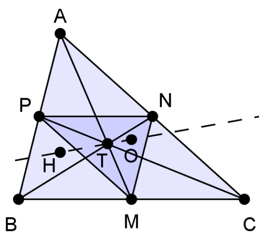
As an exercise, let's try to construct a square that is "inscribed" in a $\triangle ABC$, such that one vertex lies on the side AB, one on the side AC and two adjacent vertices of the square lie on the side BC. Firstly try by yourself and then see the solution presented below.

Construct any square $MNPQ$ such that $M \in AB$ and $Q \in AC$ and $MQ \parallel NP \parallel BC$. We will now define a homothety centered at A that will send $MNPQ$ to the desired square. Let $AN \cap BC = N'$. We define the ratio of the homothety $k = \dfrac{\overline{AN'}}{\overline{AN}}$, so that $\mathcal{X} : N \to N'$. Since $NP \parallel BC$ and $N' \in BC$, the image of P will be a point P' on BC. Also the center of the homothety (A), the original (P) and the image (P') must be collinear, so $P' = AP \cap BC$.



Now, let's find M'. $M'N'$ should be parallel to MN, but also $A - M - M'$ should be collinear, so M' is the intersection of the perpendicular to BC through N' and the line $AM \equiv AB$. We can find Q' similarly to M'. The resulting quadrilateral $M'N'P'Q'$ is similar to its original $MNPQ$, so it must be a square. It is also "inscribed" in $\triangle ABC$ per the given conditions, so we are done.

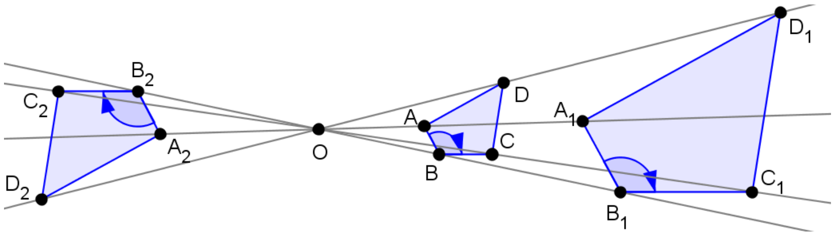
As another exercise, we will give an alternate proof of Example 10.4.3, where we proved that in any triangle ABC, the orthocenter H, the centroid T and the circumcenter O are collinear and that $\overline{HT} = 2 \cdot \overline{TO}$.



Let M, N and P be the midpoints of the sides BC, CA and AB, respectively. Remember that the medians AM, BN and CP intersect at the centroid T and moreover, it divides them in ratio 2:1. Therefore, $\mathcal{X}_{T,-1/2} : \triangle ABC \to \triangle MNP$. Also, $\mathcal{X} : H_{ABC} \to H_{MNP}$, so the points $H_{ABC} - T_{ABC} - H_{MNP}$ are collinear. From Property 6.10, we know that $O_{ABC} \equiv H_{MNP}$, so $H_{ABC} - T_{ABC} - O_{ABC}$ are collinear. Because $|k| = \frac{1}{2}$, we can also conclude that $\overline{HT} = 2 \cdot \overline{TO}$.

16.1 Homothetic center of circles

Homothetic centers may be external ($k > 0$) or internal ($k < 0$). If the center is internal, the two geometric figures are scaled, 180°-rotated and translated images of one another. Otherwise, if the center is external, the two figures are scaled and translated similar to one another. Sometimes, the external and internal homothetic centers (centers of similitude) are called *exsimilicenter* and *insimilicenter*, respectively.



Circles are geometrically similar to one another and "rotation invariant". Hence, a pair of circles has both types of homothetic centers, internal and external (unless the centers coincide or the radii are equal; we will discuss these special cases later). These two homothetic centers lie on the line joining the centers of the two given circles.

How can we find those homothetic centers? Let's draw two parallel diameters A_1B_1 and A_2B_2, one for each circle. These make the same angle with the line connecting the centers. The lines A_1A_2, and B_1B_2, intersect each other and the line connecting the centers at the external homothetic center. Conversely, the lines A_1B_2 and B_1A_2 intersects each other and the line connecting the centers at the internal homothetic center. As a limiting case of this construction, a line tangent to both circles passes through one of the homothetic centers, as it forms right angles with both the corresponding diameters, which are thus parallel. The common external tangents pass through the external homothetic center, while the common internal tangents pass through the internal homothetic center. If the circles have the same radius (but different centers), they have no external homothetic center. If the circles have the same center, they have only one homothetic center and that is the common center of the circles.

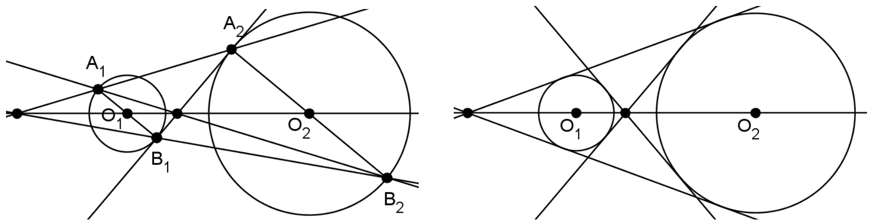
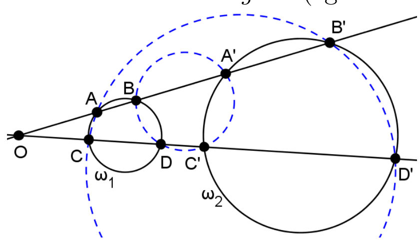


Figure 16.2: Internal and external homothetic center of two circles.

A line through a homothetic center that intersects the circles, will intersect each circle at two places. Of these four points, any two are said to be *homologous* if radii drawn to them make the same angle with the line connecting the centers (eg. A and A'). Out of these four, any two that lie on different circles and are not homologous are said to be *antihomologous* (eg. A and B').



We will now prove that any two pairs of antihomologous points (defined by lines through the same homothetic center) are concyclic. Let O be a homothetic center of ω_1 and ω_2. Let a line through O intersect ω_1 at A and B and ω_2 at A' and B' (such that A and A' are closer to O than B and B', respectively). Then,

$$\overline{OA} \cdot \overline{OB'} = \overline{OA} \cdot (k \cdot \overline{OB}) = k \cdot \overline{OA} \cdot \overline{OB}.$$

If we similarly define points C, D, C' and D' for a different line through O, we have

$$\overline{OC} \cdot \overline{OD'} = \overline{OC} \cdot (k \cdot \overline{OD}) = k \cdot \overline{OC} \cdot \overline{OD}.$$

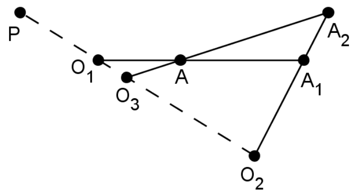
From the intersecting secants theorem for ω_1, we have $\overline{OA} \cdot \overline{OB} = \overline{OC} \cdot \overline{OD}$, so combining the previous equations, we get $\overline{OA} \cdot \overline{OB'} = \overline{OC} \cdot \overline{OD'}$ which means that the points A, B', C and D' (which are two pairs of antihomologous points) are concyclic. Just as a trivia for our more curious readers, we will mention that because of this concyclicity, the intersection of the lines AC and $B'D'$ lies on the radical axis of the circles ω_1 and ω_2. Can you see why?

16.2 Composition of homotheties

Composition of two homotheties, $\mathcal{X}_1$ with center O_1 and ratio k_1, and $\mathcal{X}_2$ with center O_2 and ratio k_2 is a homothety $\mathcal{X}_3$ (unless $k_1 \cdot k_2 = 1$).

Property 16.3. Let $\mathcal{X}_3$ be the composition of homotheties $\mathcal{X}_2 \circ \mathcal{X}_1$. Then, the center of $\mathcal{X}_3$ lies on the line O_1O_2 and the ratio of $\mathcal{X}_3$ is $k_1 \cdot k_2$.

Proof. Let A be a point. Let A_1 and A_2 be points such that $\mathcal{X}_1 : A \to A_1$ and $\mathcal{X}_2 : A_1 \to A_2$. In other words, $\mathcal{X}_3 : A \to A_2$ because $\mathcal{X}_3(A) = \mathcal{X}_2 \circ \mathcal{X}_1(A) = \mathcal{X}_2(\mathcal{X}_1(A)) = \mathcal{X}_2(A_1) = A_2$.



Let's prove that the center of $\mathcal{X}_3$, O_3, lies on O_1O_2. Let $\mathcal{X}_2 : O_1 \to P$. The point P doesn't have a special meaning, but it will help us prove our claim. We will also use the fact that the center of homothety is fixed under a homothety, i.e. $\mathcal{X}_1 : O_1 \to O_1$.

$$\mathcal{X}_3(O_1) = \mathcal{X}_2(\mathcal{X}_1(O_1)) = \mathcal{X}_2(O_1) = P$$

From this equation, we have $\mathcal{X}_3 : O_1 \to P$ and $\mathcal{X}_2 : O_1 \to P$. Using Property 16.1, which says that the center of homothety, the original, and the image are collinear, we have $O_3 - O_1 - P$ and $O_2 - O_1 - P$, i.e. all four points are collinear, so the center of $\mathcal{X}_3$, which is O_3, lies on O_1O_2.

We will now find the ratio of $\mathcal{X}_3$. Let's get back to the points A, A_1 and A_2 that we defined earlier. By definition of A_1, we get $\overline{O_1A_1} = k_1 \cdot \overline{O_1A}$. By definition of A_2, we get $\overline{O_2A_2} = k_2 \cdot \overline{O_2A_1}$. Because $\mathcal{X}_3 : A \to A_2$, we have $\overline{O_3A_2} = k_3 \cdot \overline{O_3A}$. Keep in mind the fact that we just proved, that O_3 is collinear with O_1 and O_2. Because of this collinearity, we can apply Menelaus' Theorem to $\triangle AA_1A_2$ and the points $O_2 - O_3 - O_1$

$$\frac{\overline{AO_1}}{\overline{O_1A_1}} \cdot \frac{\overline{A_1O_2}}{\overline{O_2A_2}} \cdot \frac{\overline{A_2O_3}}{\overline{O_3A}} = 1$$

$$\frac{1}{k_1} \cdot \frac{1}{k_2} \cdot \frac{k_3}{1} = 1$$

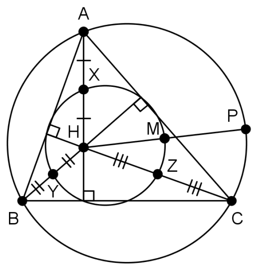
Finally, $k_3 = k_1 \cdot k_2$. ■

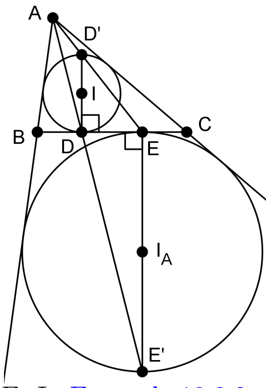
Related problems: (Homothety) 105, 120, 128, 141 and 150.

16.3 Useful Lemmas

Example 16.1. Prove that the nine point circle bisects any line segment from the orthocenter to the circumcircle.

Proof. Let ABC be a triangle with orthocenter H. Let X, Y and Z be the midpoints of AH, BH and CH, respectively. Then, there is a homothety centered at H, with ratio 2, that sends $\triangle XYZ$ to $\triangle ABC$. Since the circumcirle of $\triangle XYZ$ is the nine point circle of $\triangle ABC$, this homothety sends the nine point circle of $\triangle ABC$ to its circumcirle. Let P be any point on the circumcircle of $\triangle ABC$. Let the nine point circle intersect HP at M. Then, $\mathcal{X}_{H,2} : M \to P$. Therefore, $\overline{HP} = 2 \cdot \overline{HM}$, i.e. $\overline{HM} = \overline{MP}$. ■





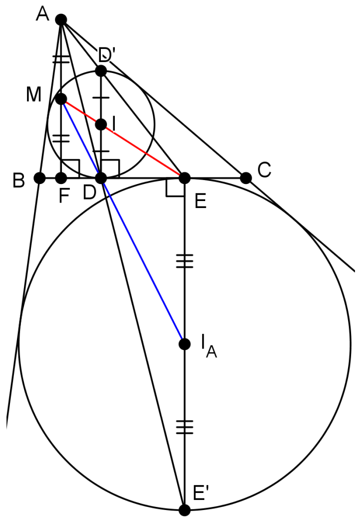
Example 16.2 (Diameter of the incircle)**.** Let the incircle of $\triangle ABC$ touch the side BC at D and let DD' be a diameter of the incircle. Let $AD' \cap BC = E$. Prove that $\overline{BD} = \overline{EC}$.

Proof. Consider the homothety with center A that sends the incircle to the A-excircle. The diameter DD' of the incircle must be mapped to the diameter of the excircle that is perpendicular to BC. It follows that D' must get mapped to the point of tangency between the excircle and BC. Since the image of D' must lie on the line AD', it must be E. That is, the excircle is tangent to BC at E. In Example 10.3.2, we already proved that the tangent points of the incircle and the excircle to BC are equidistant from the midpoint, so $\overline{BD} = \overline{EC}$. ■

Remark (1). Similarly, because of the same homothety, if EE' is diameter of the excircle, then $A - D - E'$ are collinear.

Remark (2). Notice that, as a consequence to these collinearities, the line joining the incenter and the midpoint of BC is parallel to the line AE, while the line joining the $A-$excenter and the midpoint of BC is parallel to the line AD.

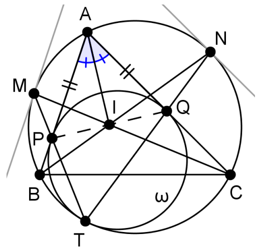
Example 16.3 (Midpoint of the altitude)**.** Let ABC be a triangle and let D and E be the tangent points of the side BC with the incircle (centered at I) and $A-$excircle (centered at I_A), respectively. If M is the midpoint of the altitude AF, prove that $M = EI \cap DI_A$.



Proof. From Example 16.2, we know that $A - D' - E$ are collinear. Since $AF \parallel D'D$, the homothety centered at E that takes DD' to FA also takes the midpoint I of DD' to the midpoint M of FA and therefore $E - I - M$ are collinear.
Similarly, $A - D - E'$ are collinear, so the homothety centered at D (with negative coefficient) that takes EE' to FA also takes I_A to M and therefore $I_A - D - M$ are collinear. ■

Example 16.4. Let ABC be a triangle. A circle ω is internally tangent to the circumcircle of $\triangle ABC$ and also to the sides AB and AC at P and Q, respectively. Prove that the midpoint of PQ is the incenter of $\triangle ABC$.

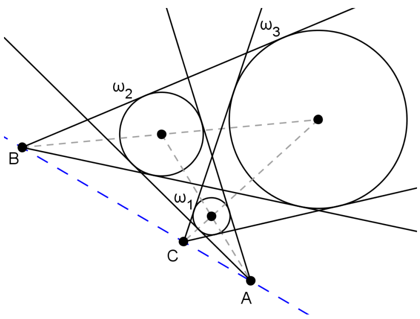
Proof. Let the tangent point of (ABC) and ω be T. Let $\mathcal{X}$ be a homothety that sends ω to (ABC), i.e. $\mathcal{X}_T : \omega \to (ABC)$. Let $TP \cap (ABC) = M$. Since AB is tangent to ω at P, the parallel line to AB through M should be tangent to (ABC) at M. This means that M must be the midpoint of the arc $\widehat{AB}$ in (ABC). So CM is an angle bisector in $\triangle ABC$, i.e. $C - I - M$ are collinear, where I is the incenter in $\triangle ABC$. Let $TQ \cap (ABC) = N$. Similarly, $B - I - N$ are collinear. By applying Pascal's Theorem to the points T, M, C, A, B and N, we get that the points $P - I - Q$ are collinear. Also, $\overline{AP} = \overline{AQ}$ as tangent segment, and AI is the angle bisector of $\angle BAC \equiv \angle PAQ$, so I must be the midpoint of PQ. ∎



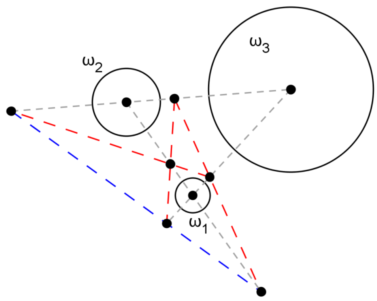
Example 16.5 (Monge's Theorem)**.** The exsimilicenters of three circles are collinear.

Proof. Let ω_1, ω_2 and ω_3 be three circles. Let A, B and C be the intersections of the external tangents of ω_1 and ω_2; ω_2 and ω_3; and ω_1 and ω_3, respectively. One of the two homotheties that sends ω_1 to ω_2 is centered at A and has a coefficient $k_1 > 0$, i.e. $\mathcal{X}_{A,\,k_1} : \omega_1 \to \omega_2$. Similarly, $\mathcal{X}_{B,\,k_2} : \omega_2 \to \omega_3$, where $k_2 > 0$. Therefore, the composition homothety $\mathcal{X}_{comp} = \mathcal{X}_B \circ \mathcal{X}_A$ sends ω_1 to ω_3. By the properties of composition of homotheties, we know that the center of $\mathcal{X}_{comp}$ lies on the line AB and the coefficient is positive (as it is equal to $k_1 \cdot k_2$). But the center of the homothety that sends ω_1 to ω_3 with positive coefficient is found as the intersection of the common external tangents, so it is C. In conclusion, $C \in AB$, i.e. the points A, B and C are collinear. ∎

Remark. Can you prove this using Menelaus' Theorem? What about using Desargues' Theorem?



(a) Monge's Theorem

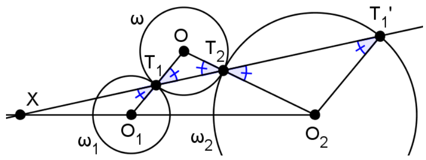


(b) Monge-d'Alembert Theorem

Example 16.6 (Monge-d'Alembert Theorem)**.** Given three circles, the insimilicenters of any two pairs of circles and the exsimilicenter of the third one are collinear.

Proof. The proof is analogous to the proof of Monge's Theorem. ∎

Example 16.7. Let ω be a circle that is tangent to ω_1 and ω_2 at T_1 and T_2, respectively. Prove that the line T_1T_2 passes through one of the homothetic centers of ω_1 and ω_2.



Proof 1. We will discuss the case where ω is externally tangent to both ω_1 and ω_2. The other cases should be analogous. Let O, O_1 and O_2 be the centers of ω, ω_1 and ω_2, respectively.

$$\angle OT_1T_2 = \angle OT_2T_1 \quad (\because \overline{OT_1} = \overline{OT_2})$$

Let $X = T_1T_2 \cap O_1O_2$. Because of the tangency of ω and ω_1, we know that $T_1 \in OO_1$, so

$$\angle OT_1T_2 = \angle O_1T_1X.$$

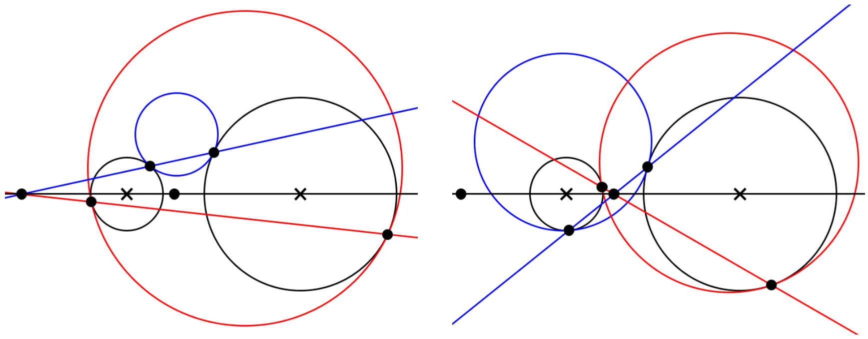
Let $T_1' \in T_1T_2 \cap \omega_2$. From the tangency of ω and ω_2, we know that $T_2 \in OO_2$, so

$$\angle OT_2T_1 = \angle O_2T_2T_1' = \angle O_2T_1'T_2 \equiv \angle O_2T_1'X.$$

Combining the three equations, we get $\angle O_1T_1X = \angle O_2T_1'X$, so X is a homothetic center of ω_1 and ω_2. ■

Proof 2. Using the same case and the same notations as in the previous proof, the tangent point T_1 is the insimilicenter of the tangent circles ω and ω_1. Similarly, T_2 is the insimilicenter of ω and ω_2. Therefore, by Monge-d'Alembert Theorem, T_1T_2 passes through the exsimilicenter of ω_1 and ω_2. ■

Remark. The line T_1T_2 passes through the external homothetic center of ω_1 and ω_2 when ω is either internally or externally tangent to both ω_1 and ω_2. Otherwise, when ω is internally tangent to one of ω_1 and ω_2 and externally tangent to the other one, then the line T_1T_2 passes through the internal homothetic center of ω_1 and ω_2.



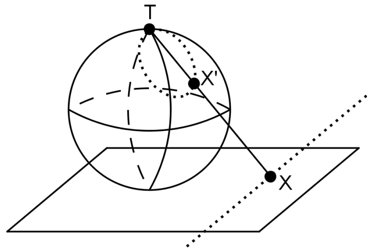
Related problems: (Lemmas) 117, 136, 139, 142, 145 and 148.

Chapter 17

Inversion

Inversion, like homothety, is a function that sends a point to another point. However, before continuing, let's firstly introduce the term "extended plane", because inversion is defined there.

The *extended plane* is a set of all the points in a plane together with one special point that we will call *the point at infinity* (P_∞). We also imagine that all the lines pass through this point. To make this a little bit clearer, let's imagine a sphere sitting on a horizontal plane. Let the top-most point of the sphere be T. Then, for every point X on the plane, the line TX will intersect the sphere at a unique point X'. If we move a point X on a line, the image points X' on the sphere will make a circle. However, as we go towards infinity on *any* line on the plane, the image circle will pass through the *same* image, i.e. the point T. Thus, in our scenario, it is OK to imagine that all the lines pass through the same point at infinity.



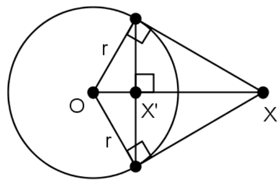
Now, back to the formal definition of inversion. Inversion with center O and radius r is a function on the extended plane that sends a point X to a point X' on the ray OX, such that $\overline{OX} \cdot \overline{OX'} = r^2$. If X is the center of inversion, then it is sent to the point at infinity and vice versa. The circle with center O and radius r is called the *circle of inversion*. From the definition, we can easily check that $(X')' = X$.

$$\mathcal{J}_{O,r} : X \leftrightarrow \begin{cases} X' \in OX, \overline{OX} \cdot \overline{OX'} = r^2 & O \neq X \neq P_\infty \\ P_\infty & X \equiv O \\ O & X \equiv P_\infty \end{cases}$$

A point

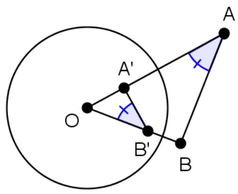
It is easy to see that if a point lies on the circle of inversion, since $\overline{OX} = r$, the image will be the same point, i.e. $X' \equiv X$. If a point is inside the circle of inversion, then its image will be outside and vice versa. How can we construct those images?

Well, if the point X is outside the circle of inversion, we draw the tangents from X to the circle of inversion. The image X' is found as the intersection of the line connecting the tangent points and the line OX. It can be easily checked, by similarity of triangles, that the equation in the definition is satisfied. Since $(X')' = X$, it can be easily figured out what the image is if X is inside the circle of inversion.



Two points

Let A' and B' be the images of the points A and B under inversion with center O and radius r.



Then, $\overline{OA} \cdot \overline{OA'} = r^2 = \overline{OB} \cdot \overline{OB'}$.

$$\therefore \frac{\overline{OA}}{\overline{OB}} = \frac{\overline{OB'}}{\overline{OA'}}$$

Keeping in mind that $O - A - A'$ and $O - B - B'$ are collinear, i.e. $\angle AOB = \angle A'OB'$, we get that $\triangle OAB \sim \triangle OB'A'$. From this similarity, we conclude two important properties that will be further used when solving problems:

Property 17.1. Let A' and B' be the images of the points A and B under inversion with center O and radius r. Then,

$$\angle OB'A' = \angle OAB \tag{17.1}$$

Property 17.2. Let A' and B' be the images of the points A and B under inversion with center O and radius r. Then,

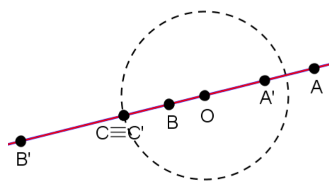
$$\overline{A'B'} = \overline{AB} \cdot \frac{r^2}{\overline{OA} \cdot \overline{OB}} \tag{17.2}$$

Proof. From the similarity $\triangle OAB \sim \triangle OB'A'$, we get

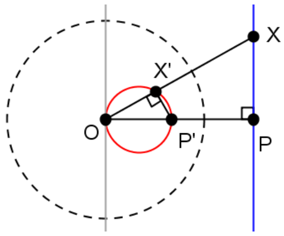
$$\frac{\overline{A'B'}}{\overline{AB}} = \frac{\overline{OA'}}{\overline{OB}} = \frac{\overline{OA'} \cdot \overline{OA}}{\overline{OB} \cdot \overline{OA}} = \frac{r^2}{\overline{OA} \cdot \overline{OB}} \qquad \blacksquare$$

A line

A line that passes through the center is sent to itself. (Each point is not sent to itself, obviously, but the line as a figure is sent to itself.)



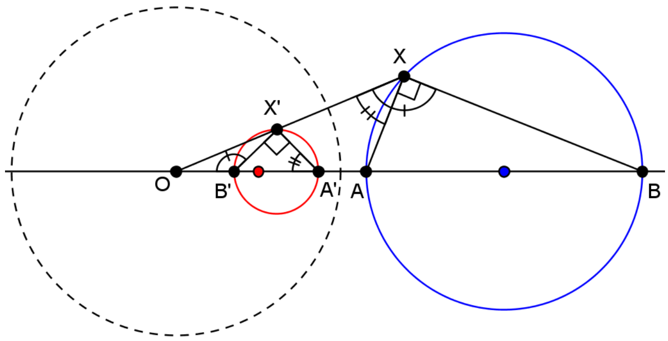
Let's see what happens when a line doesn't pass through the center. Let P be the foot of the perpendicular from the center O to the line, with inverse P', and let X be any point on the line, with inverse X'. Then, by Property 17.1, $\angle OX'P' = \angle OPX = 90°$. Therefore, X' lies on a circle with diameter OP'. In conclusion, a line that doesn't pass through the center is sent to a circle through the center. Moreover, this circle is tangent to the line through O parallel to the original line.



A circle

From the previous case, it's obvious that a circle that passes through the center is sent to a line.

Let's see what happens when a circle doesn't pass through the center. Let A and B be the points on the original circle that are closest and furthest, respectively, to the center of inversion. Then AB is diameter of the original circle. Let A' and B' be the images of A and B, respectively.

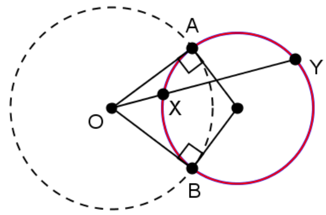


Let X be another point on the original circle. Then $\angle OXB - \angle OXA = \angle AXB = 90°$. By Property 17.1, we get

$$\begin{aligned} \angle OB'X' - \angle OA'X' &= 90° \quad [\angle OB'X' \text{ is an exterior angle for } \triangle A'B'X'] \\ \angle B'A'X' + \angle A'X'B' - \angle OA'X' &= 90° \quad [\angle B'A'X' \equiv \angle OA'X'] \\ \angle A'X'B' &= 90° \end{aligned}$$

So, X' lies on a circle with diameter $A'B'$. In conclusion, a circle that doesn't pass through the center of inversion is sent to a circle. Moreover, the center of the original circle and the center of the image circle are collinear with the center of inversion. However, the center of the original cicle *is not sent* to the center of the image circle.

In the case when the original circle is outside the circle of inversion, the image circle is inside and vice versa. When the original circle intersects the circle of inversion, it shares two common points with the image circle (the ones that are on the circle of inversion). So, is it possible, under any conditions, that a circle is sent to itself? Let's assume it is and see if we can understand the conditions when that happens. Let ω be a circle that intersects the circle of inversion at A and B. Let X be any other point on ω and let Y be the second intersection of OX and ω. Since we assumed that ω is sent to itself under the inversion, Y must be the image of X. Therefore, $\overline{OX} \cdot \overline{OY} = r^2$. But A lies on the circle of inversion, so $\overline{OA} = r$. Therefore, $\overline{OX} \cdot \overline{OY} = \overline{OA}^2$, so by the Secant-Tangent Theorem, we get that OA is tangent to ω, i.e. the circle of inversion and ω are orthogonal. In conclusion, a circle orthogonal to the circle of inversion is sent to itself.

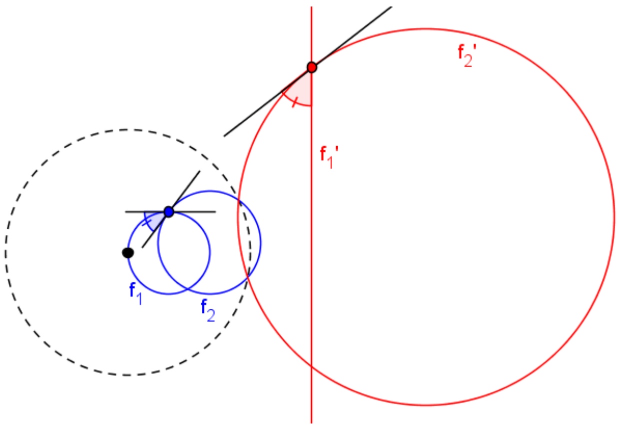


Property 17.3. Let f_1 and f_2 be two figures (line or circle). Let $\mathcal{J} : f_1 \leftrightarrow f_1'$ and $\mathcal{J} : f_2 \leftrightarrow f_2'$. Then, inversion preserves angles between figures[1], i.e.

$$\angle(f_1, f_2) = \angle(f_1', f_2')$$

As a consequence, f_1 is tangent to f_2 if and only if f_1' is tangent to f_2'.

Proof. This can be proven using Property 17.1 in the intersection point. ■

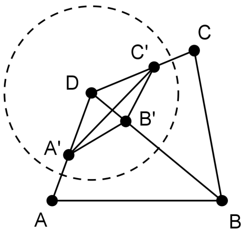


Summary

1. A point on the circle of inversion is sent to itself.
2. A line passing through the center is sent to itself.
3. A line not passing through the center is sent to a circle through the center.
4. A circle passing through the center is sent to a line.
5. A circle not passing through the center is sent to a circle.
    5.1. A circle orthogonal to the circle of inversion is sent to itself.

[1]The angle between two circles is defined as the angle between the tangents to the circles at a point of intersection. Analogously, the angle between a circle and a line is defined as the angle between the line and the tangent to the circle at a point of intersection with the line.

Example 17.1 (Ptolemy's Theorem)**.** Let $ABCD$ be a quadrilateral. Prove that $\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} \geq \overline{AC} \cdot \overline{BD}$ and that equality holds iff $ABCD$ is a cyclic quadrilateral.



Proof. Let's invert with center D and any radius r. Let the images of A, B and C be A', B' and C', respectively. By the Triangle Inequality for $\triangle A'B'C'$, we have $\overline{A'B'} + \overline{B'C'} \geq \overline{A'C'}$. By Property 17.2, this is equivalent to

$$\overline{AB} \cdot \frac{r^2}{\overline{DA} \cdot \overline{DB}} + \overline{BC} \cdot \frac{r^2}{\overline{DB} \cdot \overline{DC}} \geq \overline{AC} \cdot \frac{r^2}{\overline{DA} \cdot \overline{DC}}.$$

Multiplying by $\overline{DA} \cdot \overline{DB} \cdot \overline{DC}$ and dividing by r^2 on both sides, we get:

$$\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} \geq \overline{AC} \cdot \overline{BD}.$$

Equality holds iff the points A', B' and C' are collinear, i.e. iff the points A, B and C are concyclic with the center of inversion D. ■

Example 17.2 (IMO 1996/2)**.** Let P be a point inside triangle ABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Let D and E be the incenters of $\triangle APB$ and $\triangle APC$, respectively. Show that AP, BD and CE are concurrent.

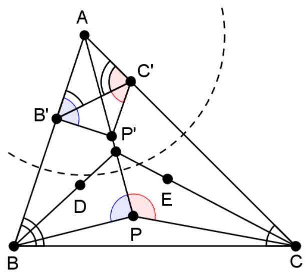
Proof. Here, we see that there are many angles in the condition that are in the form $\angle AXY$ with fixed A. That's why we will try to invert through A.

By Property 17.1, the condition now becomes

$$\angle AB'P' - \angle AB'C' = \angle AC'P' - \angle AC'B'$$

$$\angle P'B'C' = \angle P'C'B'$$

$$\therefore \overline{P'B'} = \overline{P'C'} \qquad (*)$$



If we want to prove that the angle bisectors BD and CE intersect the line segment AP at the same point, then by the Angle Bisector Theorem, we need to prove that

$$\frac{\overline{AB}}{\overline{BP}} = \frac{\overline{AC}}{\overline{CP}}$$

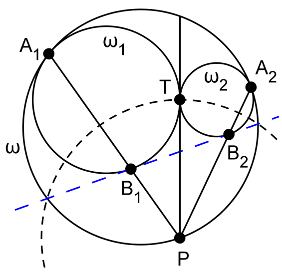
From the similarity of the triangles $\triangle APB \sim \triangle AB'P'$ and $\triangle ACP \sim \triangle AP'C'$ we get

$$\frac{\overline{AB}}{\overline{BP}} = \frac{\overline{AP'}}{\overline{P'B'}} \overset{(*)}{=} \frac{\overline{AP'}}{\overline{P'C'}} = \frac{\overline{AC}}{\overline{CP}}$$

■

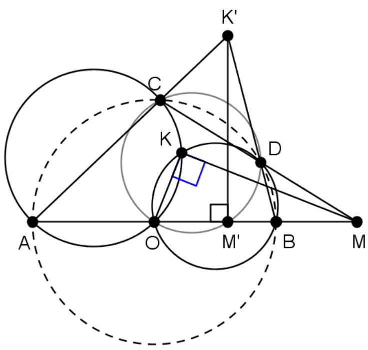
Example 17.3. Two circles ω_1 and ω_2 touch each other externally at T. They also touch a circle ω internally at A_1 and A_2, respectively. Let P be one point of intersection of ω with the common tangent to ω_1 and ω_2 at T. The line PA_1 meets ω_1 again at B_1 and the line PA_2 meets ω_2 again at B_2. Prove that B_1B_2 is a common tangent to ω_1 and ω_2.

Proof. In many problems, we tend to invert through a "busy" point, i.e. a point through which many lines or circles pass. Consider the inversion $\mathcal{J}(P, \overline{PT})$. In this way, since PT is tangent to ω_1, i.e. ω_1 is orthogonal to the circle of inversion, $\mathcal{J} : \omega_1 \leftrightarrow \omega_1$. Let's see what is A_1 sent to. The image of A_1 must be on PA_1. Also, since A_1 lies on ω_1, then the image of A_1 must lie on the image of ω_1. Therefore, $\mathcal{J} : A_1 \leftrightarrow B_1$. Similarly, $\mathcal{J} : \omega_2 \leftrightarrow \omega_2$ and $\mathcal{J} : A_2 \leftrightarrow B_2$. Now, the circle ω passes through the center of inversion P, so it will be sent to a line. The points A_1 and A_2 lie on this circle, so their images will lie on the image line. Therefore, $\mathcal{J} : \omega \leftrightarrow B_1B_2$. Finally, since ω is tangent to ω_1 and ω_2, then by Property 17.3, its image will be tangent to their images, i.e. B_1B_2 will be tangent to ω_1 and ω_2. ■



Example 17.4. A semicircle with diameter AB is centered at O. A line intersects the semicircle at C and D and the line AB at M, such that $\overline{MB} < \overline{MA}$ and $\overline{MD} < \overline{MC}$. Let K be the second point of intersection of the circumcircles of $\triangle AOC$ and $\triangle BOD$. Prove that $\angle MKO = 90°$.

Proof. The point O is one of the busy points in this diagram and also the angle $\angle MKO$ which is of interest for us is in the form $\angle OXY$, so it is wise to try to invert through O. Consider the inversion $\mathcal{J}(O, \overline{OA})$. By Property 17.1, we need to prove that $\angle OM'K' = 90°$. The points A, B, C and D are sent to themselves since they lie on the circle of inversion. The circles (OAC) and (OBD) pass through the center of inversion, so they are sent to the lines AC and BD, respectively. Since K lies on these circles, then K' must lie on their images, i.e. $AC \cap BD = K'$. The line AB passes through the center of inversion, so it is sent to itself. The line CD doesn't pass through the center, so it is sent to the circle (OCD). Since M lies on the lines AB and CD, its image M' will lie on their images, i.e. $M' = AB \cap (OCD)$. Now, since AB is the diameter of $(ABCD)$, C and D are feet of the altitudes in $\triangle ABK'$ and O is a midpoint in the same triangle. So, (OCD) is the nine point circle of $\triangle ABK'$. Since M' lies on AB and the nine point circle, then it must be the feet of the altitude from K' to AB and therefore $\angle OM'K' = 90°$. ■

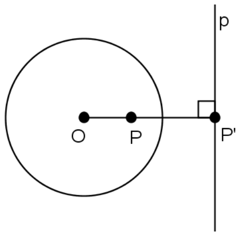


Related problems: (Inversion) 59, 71 and 132. (Ptolemy's Theorem) 57 and 122.

Chapter 18

Pole & Polar

Let the image of the point P under inversion with respect to the circle with center O and radius r be P'. The *polar* of P is the line p perpendicular to the line OP at P'. In this case, the point P is called the *pole* of p.



We will now present some properties that will be useful when solving problems.

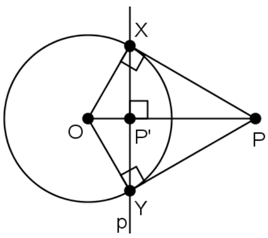
Property 18.1. If P is outside the circle ω, and X and Y are points on ω, such that PX and PY are tangents, then the polar p of P is the line XY.

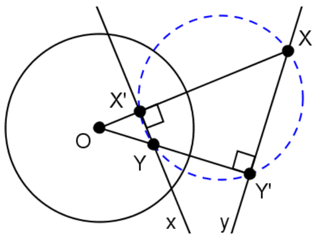
Proof. Recall that the image point P' can be found as the intersection of XY and OP, i.e. $P' \in XY$. By symmetry, $XY \perp OP$. Therefore, by the definition of polar, $p \equiv XY$. ■

Property 18.2 (La Hire's Theorem)**.** Let x and y be the polars of X and Y, respectively. Then, $X \in y \iff Y \in x$.

Proof. Let X' and Y' be the images of X and Y under the inversion. Then $\overline{OX} \cdot \overline{OX'} = r^2 = \overline{OY} \cdot \overline{OY'}$, which means that the points X, Y, X' and Y' are concyclic. Therefore,

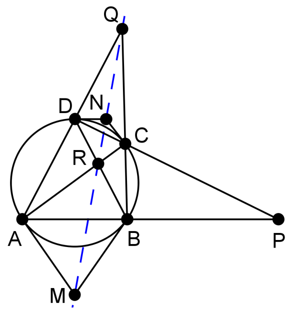
$$X \in y \iff \angle XY'Y = 90^\circ \iff \angle XX'Y = 90^\circ \iff Y \in x$$ ■

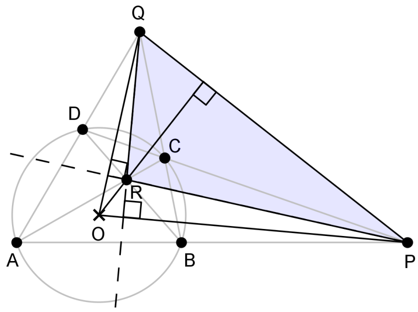




Property 18.3 (Brocard's Theorem)**.** Let $ABCD$ be a cyclic quadrilateral centered at O. Let $AB \cap CD = P$, $BC \cap AD = Q$ and $AC \cap BD = R$. Then, the polar of P is QR. Moreover, the triangle $\triangle PQR$ is autopolar and O is the orthocenter of $\triangle PQR$.

Proof. Let the intersection of the tangents at A and B be M. Then, $AB \equiv m$. Let the intersection of the tangent at C and D be N. Then, $CD \equiv n$. By La Hire's Theorem, since $P \in m$ and $P \in n$, then $M \in p$ and $N \in p$, i.e. $MN \equiv p$. By applying Pascal's Theorem on the points $AACBBD$, we get that $M-R-Q$ are collinear. By applying Pascal's Theorem on the points $CCADDB$, we get that the points $N-R-Q$ are collinear. Therefore $QR \equiv MN \equiv p$.

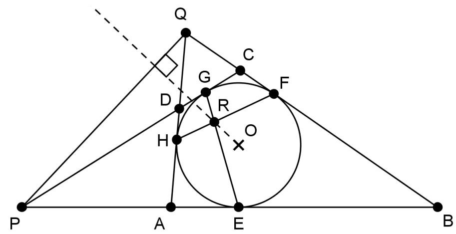




Similarly, we can get that $PR \equiv q$. Also, since R lies on the polars of P and Q, then the polar of R, $r \equiv PQ$. Therefore, the triangle $\triangle PQR$ is autopolar, i.e. the polar of each of the vertices is the opposite side. So, by the definition of polar, it also follows that O is the orthocenter of $\triangle PQR$. ■

We will now solve a few examples to see how these properties of polars can be used in problems. In these examples, we will use lowercase letters to denote the polars of the points in uppercase (e.g. p is the polar of P).

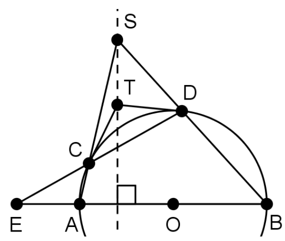
Example 18.1. The quadrilateral $ABCD$ has an inscribed circle ω which is tangent to the sides AB, BC, CD and DA at E, F, G and H, respectively. Let $AB \cap CD = P$, $AD \cap BC = Q$ and $EG \cap FH = R$. If O is the center of ω, then prove that $OR \perp PQ$.



Proof. By Property 18.1, we get $p \equiv EG$ and $q \equiv FH$. Since $R \in p$ and $R \in q$, by La Hire's Theorem, we get $P \in r$ and $Q \in r$, i.e. $r \equiv PQ$. By the definition of polar, $OR \perp r$, i.e. $OR \perp PQ$. ■

Example 18.2. Let AB be a diameter of a semicircle. C and D are two points on the semicircle such that $\overset{\frown}{AC} < \overset{\frown}{AD}$. The tangents to the semicircle at C and D meet at T. If $S = AC \cap BD$, prove that $ST \perp AB$.

Proof. Let $E = CD \cap AB$. By Property 18.1, $t \equiv CD$. Since $E \in t$, by La Hire's Theorem, $T \in e$. On the other hand, by Property 18.3, $S \in e$. Therefore, $ST \equiv e$, so by the definition of polar $OE \perp ST$, i.e. $AB \perp ST$. ■

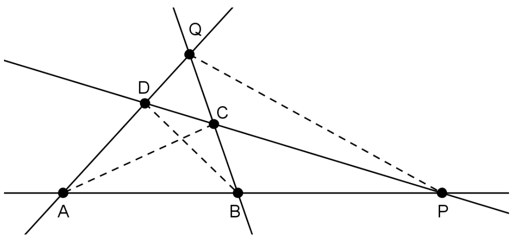


Related problems: 95 and 104.

Chapter 19

Complete quadrilateral

A complete quadrilateral is a system of four lines (no three of which pass through the same point) and the six points of intersection of these lines.



Among the six points of a complete quadrilateral there are three pairs of points that are not already connected by lines. The line segments connecting these pairs are called *diagonals* of the complete quadrilateral.

In all of the following properties, let $ABCD$ be a quadrilateral such that the *rays* AB and DC intersect at P and the *rays* BC and AD intersect at Q.

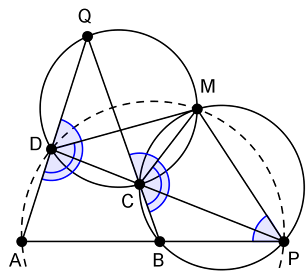
By taking any three of the four lines of a complete quadrilateral, we can get four triangles. For the complete quadrilateral $ABCDPQ$, those triangles are $\triangle ABQ$, $\triangle BCP$, $\triangle CDQ$ and $\triangle DAP$.

Property 19.1 (Miquel Point)**.** The circumcircles of the four triangles mentioned above pass through a common point, called the Miquel point of the quadrilateral.

Proof 1. Observe that this is a different wording of Example 10.2.2 in the direction when it is given that the points are collinear, which we already proved. ∎

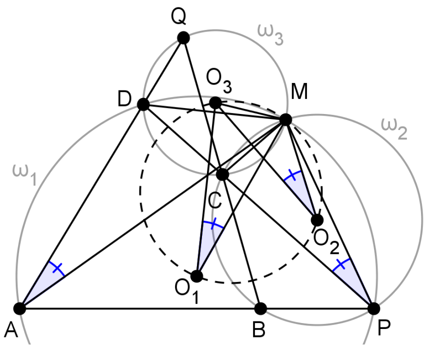
Proof 2. Let M be the second intersection of (BCP) and (CDQ). Then,

$$\begin{aligned}\angle APM \equiv \angle BPM = 180^\circ - \angle BCM &= \\ = \angle QCM = \angle QDM &= \\ = 180^\circ - \angle ADM\end{aligned}$$



Therefore, $ADMP$ is cyclic, i.e. $M \in (DAP)$. In exactly the same manner, we can prove that $M \in (ABQ)$. ∎

Property 19.2. The circumcenters of the four triangles mentioned above, and the Miquel point are concyclic.



Proof. Let M be the Miquel point of $ABCDPQ$. Let the circles $(DAPM)$, $(BCMP)$, $(CDQM)$ and $(ABMQ)$ be ω_1, ω_2, ω_3 and ω_4, respectively and let O_i be the center of ω_i. We will firstly prove that $O_1O_2MO_3$ is cyclic.

MD is the radical axis of ω_1 and ω_3, so O_1O_3 is the bisector of MD and therefore the angle bisector of $\angle MO_1D$. Similarly, O_2O_3 is the bisector of MC and the angle bisector of $\angle MO_2C$.

$$\angle MO_1O_3 = \frac{\angle MO_1D}{2} \overset{\omega_1}{\equiv} \angle MAD \overset{\omega_1}{\equiv} \angle MPD \equiv \angle MPC \overset{\omega_2}{\equiv} \frac{\angle MO_2C}{2} = \angle MO_2O_3$$

Therefore, the quadrilateral $O_1O_2MO_3$ is cyclic. Similarly, $O_2MO_3O_4$ is cyclic.

■

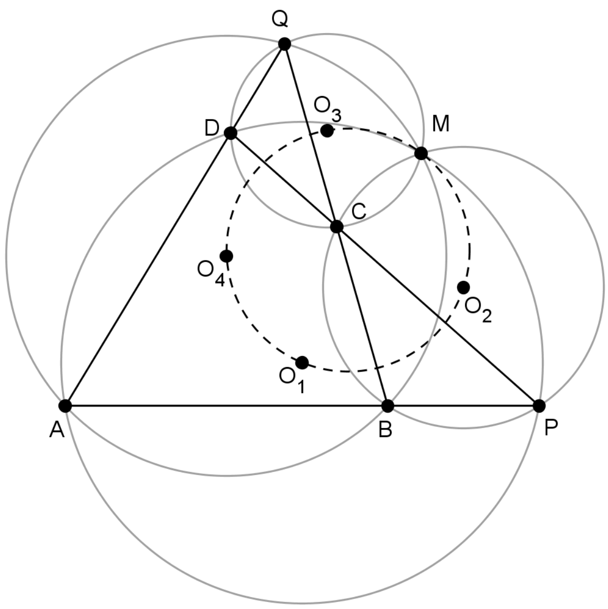
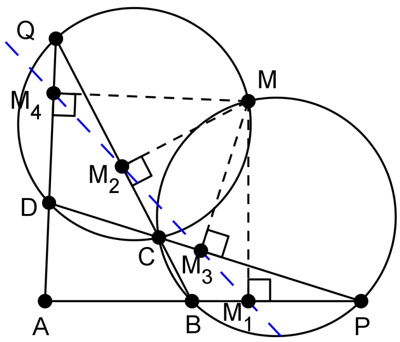


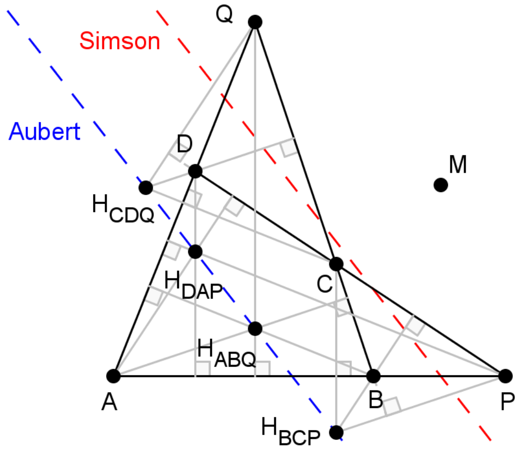
Figure 19.1: The circumcircles of $\triangle ABQ$, $\triangle BCP$, $\triangle CDQ$ and $\triangle DAP$ pass through the Miquel point M. Their circumcenters and M are concyclic.

Property 19.3 (Simson's Line)**.** The feet of the perpendiculars from the Miquel point to the sides of the complete quadrilateral lie on a line, called the *Simson's line* of the complete quadrilateral.



Proof. Let the feet of the perpendiculars from M to AB, BC, CD and DA be M_1, M_2, M_3 and M_4. Using the Simson Line Theorem for $\triangle PBC$ and the point M which lies on its circumcircle, we get that M_1, M_2 and M_3 are collinear. Similarly, by using the Simson Line Theorem for $\triangle CQD$ and the point M, we get that the points M_2, M_3 and M_4 are collinear. ■

Property 19.4 (Aubert's Line)**.** The orthocenters of the four triangles mentioned above lie on a line, called the *Aubert's line*, which is parallel to the Simson's line of the complete quadrilateral.



Proof. By Example 10.7.2, we know that the Simson line from M bisects the line segment MH, where H is the orthocenter of the triangle. Therefore, the homothety $\mathcal{X}_{M,2}$ sends the Simson line of a triangle, to a line through its orthocenter which is parallel to the Simson line. Since in Property 19.3, we proved that the Simson lines of all four triangles coincide, we get that the orthocenters of all four triangles lie on a line parallel to Simson's line. ■

Property 19.5. The three circles with diameters the diagonals of the complete quadrilateral have a common chord.

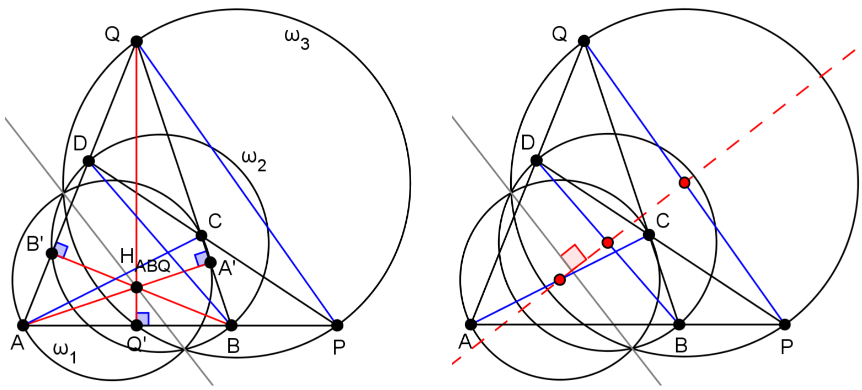
Proof. We will prove that the common chord lies on Aubert's line. Let ω_1, ω_2, and ω_3 be the circles with diameters the diagonals AC, BD and PQ, respectively. Let $\omega_1 \cap BQ = A'$, $\omega_2 \cap AQ = B'$ and $\omega_3 \cap AB = Q'$. Since the inscribed angles over the diameter are right angles, we get that $AA' \perp BQ$, $BB' \perp AQ$ and $QQ' \perp AB$. Therefore, AA', BB' and QQ' pass through the orthocenter of $\triangle ABQ$, H_{ABQ}. From Property 6.7, we know that

$$\overline{AH_{ABQ}} \cdot \overline{H_{ABQ}A'} = \overline{BH_{ABQ}} \cdot \overline{H_{ABQ}B'} = \overline{QH_{ABQ}} \cdot \overline{H_{ABQ}Q'},$$

which is equivalent to

$$pow(H_{ABQ}, \omega_1) = pow(H_{ABQ}, \omega_2) = pow(H_{ABQ}, \omega_3).$$

Therefore, H_{ABQ} has equal powers to all three circles. Similarly, we can get that the other three orthocenters also have equal powers to the three circles. Therefore, there isn't a single radical center of the three circles, but all the points on the line containing the orthocenters have equal powers to all three circles. Therefore, Aubert's line is the common chord of the circles with diameters the diagonals of the complete quadrilateral. ∎



Property 19.6 (Gauss' Line)**.** The midpoints of the diagonals of the complete quadrilateral lie on a line, called the *Gauss' line*, which is perpendicular to Simson's and Aubert's line.

Proof. Since the circles ω_1, ω_2, and ω_3 defined in Property 19.5 have a common chord and their centers are the midpoints of the diagonals of the complete quadrilateral, we can conclude that the midpoints of the diagonals are collinear. We also know that the line joining the centers of two circles is perpendicular to their common chord (Property 12.3), thus Gauss's Line is perpendicular to Aubert's Line. ∎

19.1 Cyclic Quadrilateral

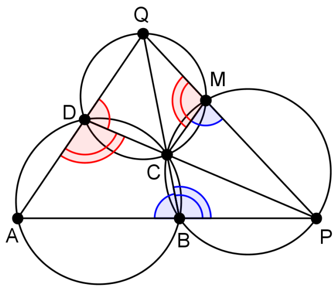
Property 19.7. The Miquel point of $ABCD$ lies on the line PQ if and only if $ABCD$ is cyclic.

Proof. Let M be the Miquel point of the quadrilateral $ABCD$. Then $MCBP$ and $MQDC$ are cyclic quadrilaterals.

$$\angle PMC = 180° - \angle PBC = \angle ABC$$

$$\angle QMC = 180° - \angle QDC = \angle ADC$$

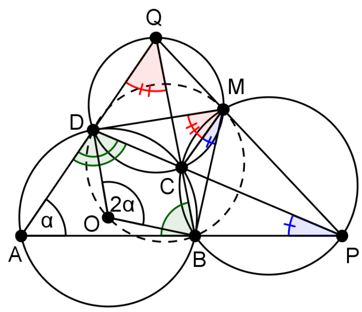
$$\therefore \angle PMC + \angle QMC = \angle ABC + \angle ADC$$



The Miquel point M lies on PQ iff the left-hand side is 180°. The right-hand side is 180° iff $ABCD$ is a cyclic quadrilateral. ■

Property 19.8. Let $ABCD$ be a cyclic quadrilateral, inscribed in a circle ω centered at O. Let the intersection of the diagonals AC and BD be R. Let M be the Miquel point of $ABCD$. Then,

- The point M lies on the circumcircles of $\triangle AOC$ and $\triangle BOD$
- The point M is the image of the point R under the inversion with respect to ω
- The point M lies on the line OR
- The line $O-R-M$ bisects $\angle AMC$ and $\angle BMD$
- The line $O-R-M$ is perpendicular to PQ
- PQ is the polar of R



Proof.

$$\angle BMC \overset{(MCBP)}{=} \angle BPC \equiv \angle APD \overset{\triangle ADP}{=} 180° - \alpha - \delta$$

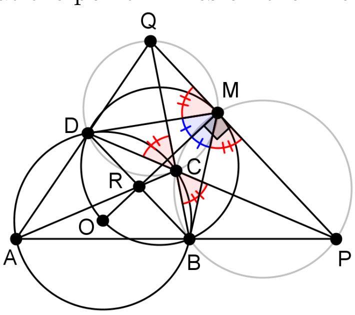
$$\angle CMD \overset{(MCDQ)}{=} \angle CQD \equiv \angle BQA \overset{\triangle ABQ}{=} 180° - \alpha - \beta$$

$$\therefore \angle BMD = \angle BMC + \angle CMD = 360° - (\beta + \delta) - 2\alpha \; = 180° - 2\alpha$$

$$\therefore \angle BOD + \angle BMD = 2\alpha + 180° - 2\alpha = 180°$$

So, $M \in (BOD)$. In a similar way, we can prove that $M \in (AOC)$.

Now, let's consider the inversion with respect to ω. The points A, B, C and D are sent to themselves. Therefore, the line AC is sent to the circle (OAC) and the line BD is sent to the circle (OBD). The lines AC and BD intersect at R, so their images will intersect at the image of R. Since we proved that M lies on (OAC) and (OBD), we can conclude that M is the image of R, i.e. $R' \equiv M$. Since the center of inversion, the original and the image are collinear, we can also conclude that the point M lies on the line OR.



From the cyclic quadrilateral $OBMD$, since $\overline{OB} = \overline{OD}$, we get that

$$\angle OMB = \angle OMD.$$

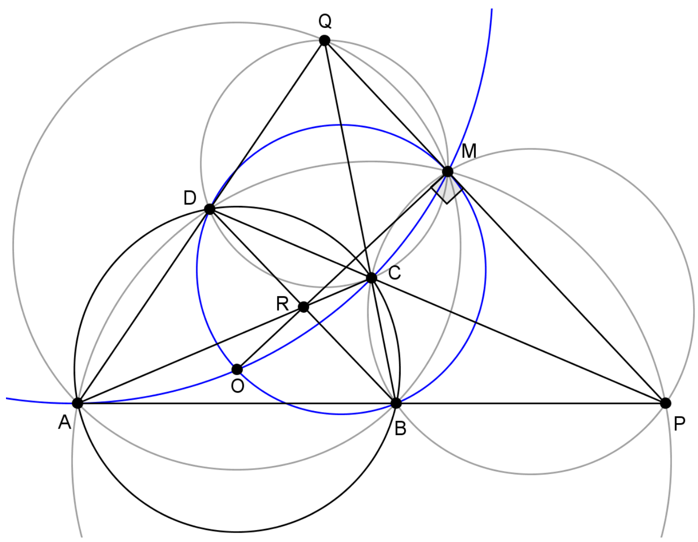
Therefore, OM bisects $\angle BMD$. Similarly, we get that OM bisects $\angle AMC$. On the other hand,

$$\angle BMP = \angle BCP = \angle DCQ = \angle DMQ.$$

Adding these last two equations side by side, we get that

$$\angle OMP = \angle OMB + \angle BMP = \angle OMD + \angle DMQ = \angle OMQ.$$

Since $ABCD$ is cyclic, by Property 19.7, we know that $M \in PQ$. Therefore, $OM \perp PQ$. By the definition of polar, since $R' \equiv M$ and $OM \perp PQ$, we get that PQ is the polar of R. ■



Property 19.9. Let $ABCD$ be a cyclic quadrilateral that is inscribed in a circle ω centered at O. Let the intersection of the diagonals AC and BD be R. Let M be the Miquel point of $ABCD$. Let P' and Q' be the images of P and Q, respectively, under inversion with respect to ω. Then,

- The points $P - R - Q'$ and $Q - R - P'$ are collinear.
- The quadrilaterals $ABRQ'$, $BCRP'$, $CDQ'R$ and $DAP'R$ are cyclic.
- The quadrilaterals $ABP'O$, $BCQ'O$, $CDOP'$ and $DAOQ'$ are cyclic.
- The quadrilaterals $AP'CQ$, $BP'DQ$, $AQ'CP$ and $BQ'DP$ are cyclic.

Proof. Let's recall Brocard's Theorem, where we proved that the triangle $\triangle PQR$ is autopolar and that O is its orthocenter.

The point P', by definition, must lie on the line OP. On the other hand, it must lie on the polar of P, which is QR. Therefore, $P' = OP \cap QR$. Similarly, $Q' = OQ \cap PR$.

We know, from Property 19.8, that $\mathcal{J}_\omega : R \leftrightarrow M$. We also know, from Property 19.1, that the circumcircle of $\triangle ABQ$ passes through M. Therefore, the image of the circle $(ABMQ)$ under inversion with respect to ω is the circle $(ABRQ')$, i.e. the quadrilateral $ABRQ'$ is cyclic. Similarly, the quadrilaterals $BCRP'$, $CDQ'R$ and $DAP'R$ are cyclic, too.

Since AB is a line that doesn't pass through O and $A, B \in \omega$, we get $\mathcal{J}_\omega : AB \leftrightarrow (ABO)$. Since $P \in AB$, we get that $P' \in (ABO)$, i.e. $ABP'O$ is cyclic. Similarly, $BCQ'O$, $CDOP'$ and $DAOQ'$ are cyclic, too.

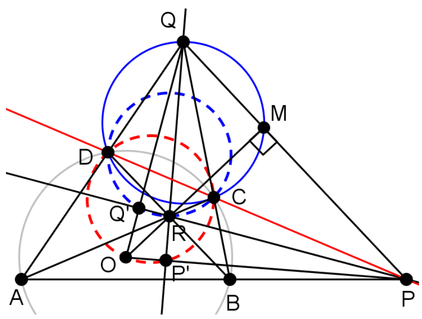
Earlier, in Property 19.8, we proved that M lies on the circumcircle of $\triangle AOC$ and also, that $O - R - M$ are collinear. Therefore, by the Intersecting Chords Theorem we get

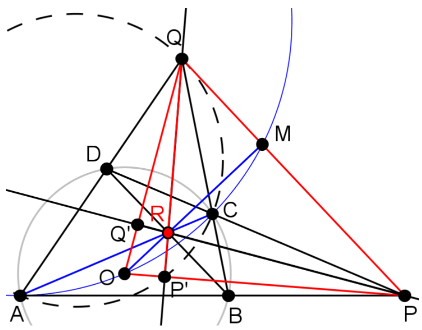
$$\overline{AR} \cdot \overline{RC} = \overline{OR} \cdot \overline{RM}. \tag{1}$$

O is the orthocenter of PQR, so R is the orthocenter of OPQ. Therefore, by Property 6.7, we get

$$\overline{OR} \cdot \overline{RM} = \overline{QR} \cdot \overline{RP'}. \tag{2}$$

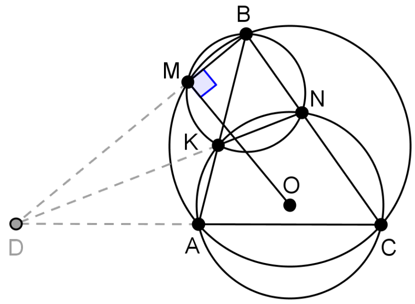
By combining (1) and (2), by the converse of the Intersecting Chords Theorem, we get that $AP'CQ$ is cyclic. Similarly, $BP'DQ$, $AQ'CP$ and $BQ'DP$ are cyclic, too. ■





It is very important to learn to recognize these configurations because they show up in many olympiad problems. However, the configuration is not always complete, so sometimes you have to draw additional points, lines or circles in order to come to these "well-known" configurations.

Example 19.1 (IMO 1985/5)**.** A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.



Proof. Let $AC \cap KN = D$. Let's take a look at the complete quadrilateral $ACNKBD$. The triangles $\triangle ACB$ and $\triangle KNB$ are two of the four triangles formed by the lines of the complete quadrilateral, so their circumcircles intersect at the Miquel Point of the complete quadrilateral, i.e. M is the Miquel point of $ACNKBD$. Since $ACNK$ is cyclic, by Property 19.7, $M \in BD$. Finally, by Property 19.8, we get that $OM \perp BD$, i.e. $\angle OMB = 90°$. ■

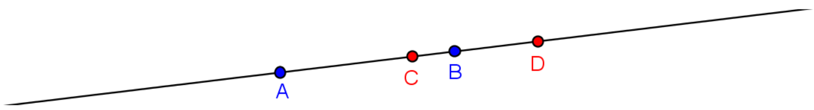
Related problems: 127 and 130.

Chapter 20

Harmonic Ratio

If A, B, C and D are collinear points, then their *cross-ratio* is defined as:

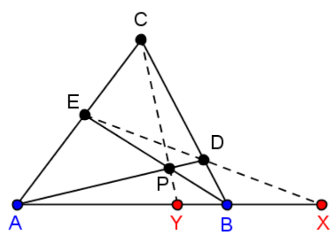
$$(A, B; C, D) = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}.$$



If $(A, B; C, D) = 1$ and the order of the points on the line is such that the line segments AB and CD partially overlap (e.g. $A - C - B - D$), then the ratio is called *harmonic ratio* and the four-tuple $(ACBD)$ is called a *harmonic division*, or simply *harmonic*. The points C and D are *harmonic conjugates* with respect to the points A and B and vice versa.

Notice, by the definition, that if $(ACBD)$ is a harmonic, then $(DBCA)$ is also harmonic.

Property 20.1. Let X be a point on the extension of the side AB in $\triangle ABC$. A line which passes through X meets the sides BC and CA at points D and E, respectively. Let P be the intersection of AD and BE. The line CP meets AB at Y. Then, X and Y are harmonic conjugates with respect to the points A and B.



Proof. We need to prove that $\frac{\overline{AX}}{\overline{BX}} = \frac{\overline{AY}}{\overline{BY}}$. By using Menelaus Theorem for $\triangle ABC$ and the collinear points $D - E - X$ and Ceva's Theorem for $\triangle ABC$

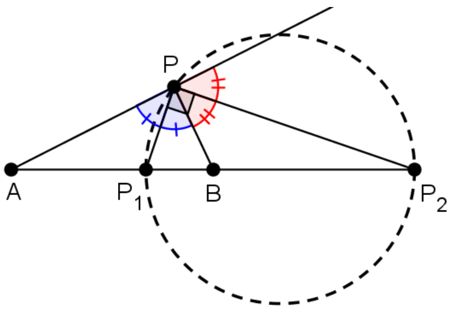
and the concurrent cevians AD, BE and CY, we get:

$$\frac{\overline{AX}}{\overline{XB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1 = \frac{\overline{AY}}{\overline{YB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}}.$$

By canceling, we get the needed ratio. ∎

Property 20.2. Given two points A and B, find the locus of the points P such that

$$\frac{\overline{AP}}{\overline{PB}} = \lambda, \ \lambda > 0.$$



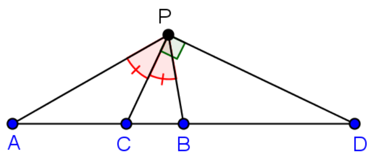
Proof. If $\lambda = 1$, then the locus of the points P is the side bisector of AB. Let's investigate the case when $\lambda \neq 1$. WLOG, let $\lambda > 1$. Obviously, there is a point P_1 between A and B (in this case, closer to B) that satisfies the condition. Also, there is another point, P_2, on the extension of the line (in this case, beyond B), that also satisfies the condition. Note that we know how to construct the latter using Property 20.1. Now let P be a point that doesn't lie on the line AB, but satisfies the condition. Then,

$$\frac{\overline{AP}}{\overline{PB}} = \lambda = \frac{\overline{AP_1}}{\overline{P_1B}} = \frac{\overline{AP_2}}{\overline{P_2B}},$$

so by the Angle Bisector Theorem, we get that PP_1 is the internal angle bisector of $\angle APB$. By the External Angle Bisector Theorem, we get that PP_2 is the external angle bisector of $\angle APB$. Therefore $PP_1 \perp PP_2$, because $\angle P_1PP_2 = \frac{1}{2} \cdot 180° = 90°$, so by Thales' Theorem P lies on the circle with diameter P_1P_2. ∎

Property 20.3. Let A, C, B and D be four collinear points lying on a line l in this order. Let P be a point not lying on l. Then, if any two of the following propositions are true, then the third is also true:

1. The division $(ACBD)$ is harmonic.
2. PC is the angle bisector of $\angle APB$.
3. $PC \perp PD$.



Proof. This is a direct consequence of the result of Property 20.2. See its proof for details. ∎

Property 20.4. Let A, C, B and D be four collinear points lying on a line in this order. Then, the division $(ACBD)$ is harmonic if and only if C is the image of D under inversion with respect to the circle with diameter AB.

Proof. Let O be the midpoint of AB and $r = \frac{1}{2}\overline{AB}$.

$$(ACBD) \text{ is harmonic}$$

$$\iff \frac{\overline{CA}}{\overline{CB}} = \frac{\overline{DA}}{\overline{DB}}$$

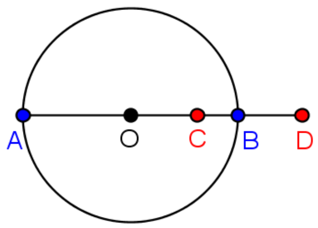
$$\iff \frac{r + \overline{OC}}{r - \overline{OC}} = \frac{\overline{DO} + r}{\overline{DO} - r}$$

$$\iff (r + \overline{OC}) \cdot (\overline{DO} - r) = (\overline{DO} + r) \cdot (r - \overline{OC})$$

$$\iff r \cdot \overline{OD} - r^2 + \overline{OC} \cdot \overline{OD} - r \cdot \overline{OC} = r \cdot \overline{OD} - \overline{OC} \cdot \overline{OD} + r^2 - r \cdot \overline{OC}$$

$$\iff \overline{OC} \cdot \overline{OD} = r^2$$

$$\iff \mathcal{J}_{O,\,r} : C \leftrightarrow D \qquad \blacksquare$$



Property 20.5. Let A, C, B and D be four collinear points lying on a line in this order. Let O be the midpoint of AB. Then, the division $(ACBD)$ is harmonic if and only if $\overline{DA} \cdot \overline{DB} = \overline{DC} \cdot \overline{DO}$.

Proof. Let $r = \frac{1}{2}\overline{AB}$.

$$\overline{DA} \cdot \overline{DB} = \overline{DC} \cdot \overline{DO}$$

$$\iff (\overline{OD} + \overline{r}) \cdot (\overline{OD} - \overline{r}) = (\overline{OD} - \overline{OC}) \cdot \overline{OD}$$

$$\iff \overline{OD}^2 - r^2 = \overline{OD}^2 - \overline{OC} \cdot \overline{OD}$$

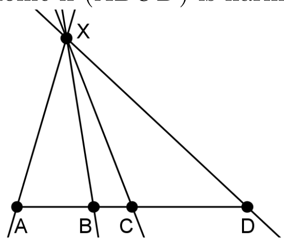
$$\iff r^2 = \overline{OC} \cdot \overline{OD}$$

$$\iff \mathcal{J}_{O,\,r} : C \leftrightarrow D$$

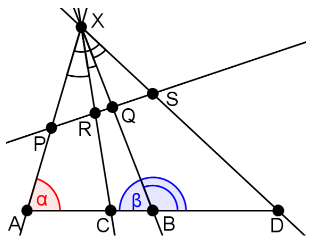
$$\overset{\textit{Property } 20.4}{\iff} (ACBD) \text{ is harmonic} \qquad \blacksquare$$

20.1 Harmonic Pencil

Four points A, B, C and D, are given on a line l in this order. If X is a point not lying on l, then the pencil $X(ABCD)$, which consists of the four lines XA, XB, XC and XD, is harmonic if $(ABCD)$ is harmonic.



Property 20.6. If any pencil $X(ABCD)$ is intersected with another line at points P, Q, R and S, then $(A, B; C, D) = (P, Q; R, S)$. As a consequence, if a harmonic pencil is intersected with a line, the intersection points form a harmonic division.



Proof. WLOG let A, C, B and D (and P, R, Q and S) be collinear in this order. Let $\angle XAC = \alpha$ and $\angle XBC = \beta$. By using the Law of Sines in the triangles $\triangle CXA$, $\triangle CXB$, $\triangle DXA$ and $\triangle DXB$, we get:

$$\frac{\overline{CA}}{\sin(\angle CXA)} = \frac{\overline{CX}}{\sin(\alpha)}$$

$$\frac{\overline{CB}}{\sin(\angle CXB)} = \frac{\overline{CX}}{\sin(\beta)}$$

$$\frac{\overline{DA}}{\sin(\angle DXA)} = \frac{\overline{DX}}{\sin(\alpha)}$$

$$\frac{\overline{DB}}{\sin(\angle DXB)} = \frac{\overline{DX}}{\sin(180^\circ - \beta)}.$$

By rearranging and using that $\sin(\beta) = \sin(180^\circ - \beta)$, we get

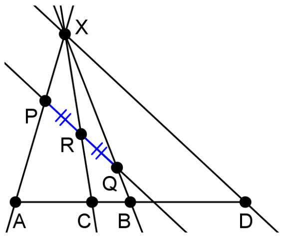
$$(A, B; C, D) = \frac{\sin(\angle CXA)}{\sin(\angle CXB)} : \frac{\sin(\angle DXA)}{\sin(\angle DXB)} \tag{20.1}$$

Since $\angle CXA \equiv \angle RXP$, $\angle CXB \equiv \angle RXQ$, $\angle DXA \equiv \angle SXP$ and $\angle DXB \equiv \angle SXQ$, it follows that $(A, B; C, D) = (P, Q; R, S)$. ■

Remark. Since the cross ratio is not dependend on the line intersecting the pencil, we can define the cross ratio of a pencil $X(ABCD)$ to be

$$(XA, XB; XC, XD) = (A, B; C, D).$$

Property 20.7. Given a pencil $X(ABCD)$ and a line parallel to XD that intersect XA, XB and XC at points P, Q and R, respectively, then $X(ABCD)$ is a harmonic pencil if and only if $\overline{PR} = \overline{RQ}$.



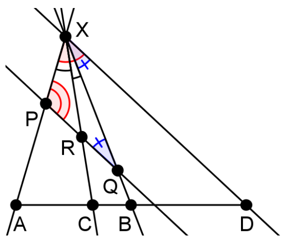
Proof 1. We will firstly give a not so Euclidean proof :)
Since $XD \parallel PQ$, $XD \cap PQ = P_\infty$. Then, by Property 20.6

$$1 = (A, B; C, D) = (P, Q; R, P_\infty) = \frac{\overline{RP}}{\overline{RQ}} : \frac{\overline{PP_\infty}}{\overline{QP_\infty}}$$

Since P_∞ is the point at infinity, then we can take $\overline{PP_\infty} = \overline{QP_\infty}$, giving us $\overline{PR} = \overline{RQ}$. ■

Proof 2. For the more skeptical readers, here is a valid Euclidean proof. Let A, C, B and D be collinear in this order. From Equation 20.1, we know that

$$(A, B; C, D) = \frac{\sin(\angle CXA)}{\sin(\angle CXB)} : \frac{\sin(\angle DXA)}{\sin(\angle DXB)} \tag{1}$$



By using the Law of Sines in the triangles $\triangle PRX$ and $\triangle QRX$, we get

$$\frac{\overline{PR}}{\sin(\angle RXP)} = \frac{\overline{XR}}{\sin(\angle XPR)} \quad \text{and} \quad \frac{\overline{QR}}{\sin(\angle RXQ)} = \frac{\overline{XR}}{\sin(\angle XQR)},$$

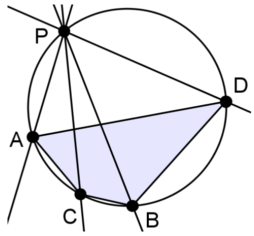
$$\text{i.e. } \frac{\overline{PR}}{\overline{QR}} = 1 \iff \frac{\sin(\angle RXP)}{\sin(\angle RXQ)} : \frac{\sin(\angle XPR)}{\sin(\angle XQR)} = 1. \tag{2}$$

We have $\angle CXA \equiv \angle RXP$ and $\angle CXB \equiv \angle RXQ$. Since $XD \parallel PQ$, we also have $\angle DXA \equiv \angle DXP = 180^\circ - \angle XPR$ and $\angle DXB \equiv \angle DXQ = \angle XQR$. Combining with (1) and (2), we get that

$$\overline{PR} = \overline{QR} \iff (A, B; C, D) = 1$$ ■

20.2 Harmonic Quadrilateral

Let $ABCD$ be a cyclic quadrilateral and P be a point on the circle. Then, $ABCD$ is called *harmonic quadrilateral* if the pencil $P(ABCD)$ is harmonic, i.e. if $(PA, PB; PC, PD) = 1$ and AB and CD intersect inside the circle (the order of the points on the circle is $A - C - B - D - A$, in any direction).



Property 20.8. Let A, C, B and D be points on a circle in this order. Let P be any point on that circle. Then the cross ratio $(PA, PB; PC, PD)$ does not depend on P.

Proof. Let the radius of the circle be R. By the Sine Law for $\triangle CPA$, we get

$$\frac{\overline{CA}}{\sin(\angle CPA)} = 2R.$$

We can get similar equations for the triangles $\triangle CPB$, $\triangle DPA$ and $\triangle DPB$. Therefore, by the defintion of a cross ratio of a pencil and by Equation 20.1, we get

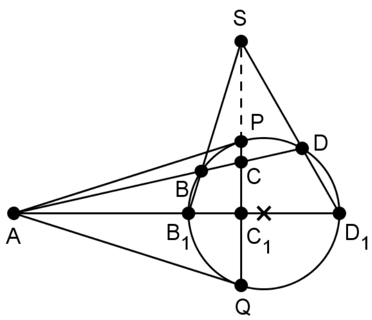
$$(PA, PB; PC, PD) = \frac{\sin(\angle CPA)}{\sin(\angle CPB)} : \frac{\sin(\angle DPA)}{\sin(\angle DPB)} = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}$$

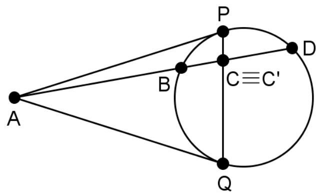
which doesn't depend on the point P.

Remark. As a consequence, for a harmonic quadrilateral $ACBD$, the products of the opposite sides are equal, i.e. $\overline{AC} \cdot \overline{BD} = \overline{BC} \cdot \overline{AD}$. ∎

Property 20.9. Let A be a point outside of a circle ω. A line l which passes through A, meets ω at points B and D. C is a point on the line segment BD. Prove that the division $(ABCD)$ is harmonic if and only if C lies on the polar of A.

Proof 1. Let P and Q be points on ω such that AP and AQ are tangents. Then, PQ is the polar of A. Let's prove the first direction. Let $C = BD \cap PQ$. We need to prove that $(ABCD)$ is harmonic. Let the secant through A that passes through the center of ω intersect ω at B_1 and D_1 and the line PQ at C_1, such that the points $A - B_1 - C_1 - D_1$ are collinear in that order. Then C_1 is the image of A under inversion with respect to ω. By Property 20.4, we know that $(AB_1C_1D_1)$ is harmonic. Let $B_1B \cap D_1D = S$. By Property 18.3, $S \in a \equiv PQ$. Therefore, B_1B, C_1C and D_1D concur at S. Since $(AB_1C_1D_1)$ is harmonic, then the pencil $S(AB_1C_1D_1)$ is harmonic. When it is intersected by another line, by Property 20.6, the intersection points form a harmonic division, i.e. $(ABCD)$ is harmonic. □





For proving the other direction, let C' be a point on the segment BC such that $(ABC'D)$ is harmonic. We need to prove that $C' \in PQ$. From above, we know that $(ABCD)$ is harmonic, where $C = BD \cap PQ$. Since three of the points in the cross ratio coincide and the cross ratio is equal, then the fourth point must also coincide. We will prove this property once here, but remember it because it is often used in problems.

$$\frac{\overline{BA}}{\overline{BC'}} : \frac{\overline{DA}}{\overline{DC'}} = 1 = \frac{\overline{BA}}{\overline{BC}} : \frac{\overline{DA}}{\overline{DC}}$$

$$\frac{\overline{DC'}}{\overline{BC'}} = \frac{\overline{DC}}{\overline{BC}}$$

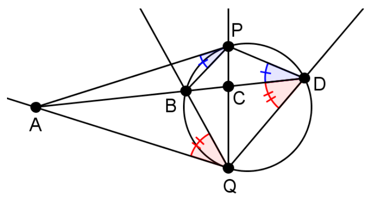
By adding 1 on both sides, we get

$$\frac{\overline{DB}}{\overline{BC'}} = \frac{\overline{DB}}{\overline{BC}}$$

$$\overline{BC'} = \overline{BC}$$

Since we know that both C and C' are between B and D, we get that $C' \equiv C$, i.e. $C' \in PQ$. ■

Proof 2. Again, let P and Q be points on ω such that AP and AQ are tangents and let $C = BD \cap PQ$. We will give an alternate proof of the first direction, i.e. that $(ABCD)$ is harmonic.



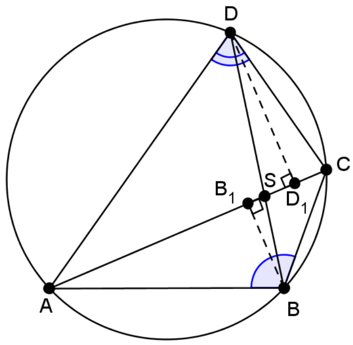
It's obvious that $\triangle ABP \sim \triangle APD$ and $\triangle ABQ \sim \triangle AQD$. Since $\overline{AP} = \overline{AQ}$:

$$\frac{\overline{BP}}{\overline{PD}} = \frac{\overline{AB}}{\overline{AP}} = \frac{\overline{AB}}{\overline{AQ}} = \frac{\overline{BQ}}{\overline{QD}}$$

So, $\overline{BP} \cdot \overline{QD} = \overline{PD} \cdot \overline{BQ}$, which by Property 20.8, means that $QBPD$ is a harmonic quadrilateral. Then, the pencil $Q(QBPD)$ is a harmonic pencil. By Property 20.6, we know that if we intersect it by the line AB, then the intersection points $QQ \cap AB = A$, $QB \cap AB = B$, $QP \cap AB = C$ and $QD \cap AB = D$ will form a harmonic division, i.e. $(ABCD)$ is a harmonic.

The other direction is the same as in the previous proof. ■

Property 20.10. Given a cyclic quadrilateral $ABCD$, let S be the intersection of the diagonals AC and BD. Then, $ABCD$ is a harmonic quadrilateral if and only if AS is a symmedian in $\triangle ABD$.



Proof. Firstly, let's investigate something that is true in any cyclic quadrilateral. Let B_1 and D_1 be the feet of the perpendiculars from B and D, respectively, to AC. Then, $\triangle BB_1S \sim \triangle DD_1S$. Also, $\sin\beta = \sin(180° - \delta) = \sin\delta$.

$$\frac{\overline{BA}\cdot\overline{BC}}{\overline{DA}\cdot\overline{DC}} = \frac{\frac{1}{2}\,\overline{BA}\,\overline{BC}\sin\beta}{\frac{1}{2}\,\overline{DA}\,\overline{DC}\sin\delta} = \frac{P_{\triangle ABC}}{P_{\triangle ADC}} = \frac{\frac{1}{2}\,\overline{AC}\,\overline{BB_1}}{\frac{1}{2}\,\overline{AC}\,\overline{DD_1}} = \frac{\overline{BB_1}}{\overline{DD_1}} = \frac{\overline{BS}}{\overline{SD}} \qquad (*)$$

Then,

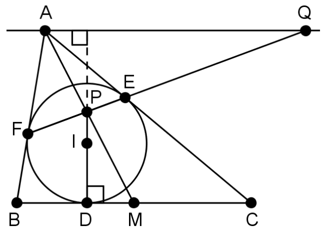
$$\frac{\overline{BS}}{\overline{SD}} = \left(\frac{\overline{AB}}{\overline{AD}}\right)^2 \overset{(*)}{\iff} \frac{\overline{BA}\cdot\overline{BC}}{\overline{DA}\cdot\overline{DC}} = \left(\frac{\overline{AB}}{\overline{AD}}\right)^2 \iff \frac{\overline{BC}}{\overline{CD}} = \frac{\overline{AB}}{\overline{AD}}$$

By Property 15.1, the left-hand side is true iff AS is a symmedian in $\triangle ABD$. By Property 20.8, the right-hand side is true iff $ABCD$ is a harmonic quadrilateral. ∎

20.3 Useful Lemmas

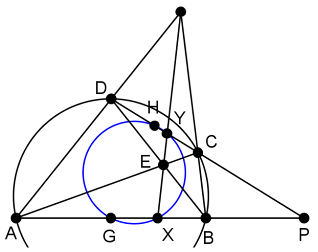
Example 20.1. In $\triangle ABC$, the incircle ω centered at I touches the sides BC, CA and AB at D, E and F, respectively. Let $DI \cap EF = P$ and let $AP \cap BC = M$. Prove that $\overline{BM} = \overline{MC}$.

Proof. We need to prove that $\overline{BM} = \overline{MC}$, so our main idea, by Property 20.7, is to prove that the pencil $A(BMCQ)$ is harmonic, where AQ is some line parallel to BC. Let Q be a point such that $AQ \parallel BC$ and Q lies on the line EF. We will use polars, so let x denote the polar of a point X with respect to ω. AF and AE are tangents to ω, so by Property 18.1, $EF \equiv a$. $P \in a$, so by La Hire's Theorem, $A \in p$. Also, since $IP \perp AQ$ (because $ID \perp BC$ and $BC \parallel AQ$), $AQ \equiv p$. Since $Q \in a$ and $Q \in p$, then by La Hire's Theorem, $AP \equiv q$.



Since $P \in q$, by Property 20.9, the division $(QEPF)$ is harmonic. Then, the pencil $A(QEPF)$ is harmonic. By Property 20.7, by intersecting the harmonic pencil $A(QEPF)$ with the line BC which is parallel to AQ, we get that $(P_\infty CMB)$ is harmonic, i.e. $\overline{CM} = \overline{MB}$. ■

Example 20.2. Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. The line EF intersects AB and CD at X and Y, respectively. Prove that $GXYH$ is a cyclic quadrilateral.



Proof. Let $AB \cap CD = P$. In $\triangle ABF$, the cevians AC, BD and FX are concurrent, so by Property 20.1, we get that $(AXBP)$ is a harmonic. Since $\overline{GA} = \overline{GB}$, by Property 20.5, we get that

$$\overline{PA} \cdot \overline{PB} = \overline{PX} \cdot \overline{PG}. \tag{1}$$

Since $(AXBP)$ is harmonic, then $F(AXBP)$ is a harmonic pencil. If we intersect it with the line CD, by Property 20.6 , the intersection points $(DYCP)$ form a harmonic division. Again, since $\overline{DH} = \overline{HC}$, by Property 20.5, we get that

$$\overline{PC} \cdot \overline{PD} = \overline{PY} \cdot \overline{PH}. \tag{2}$$

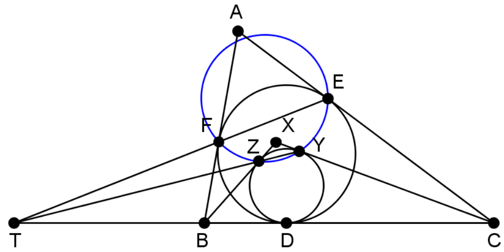
Since $ABCD$ is cyclic, by the Intersecting Secants Theorem, we have

$$\overline{PX} \cdot \overline{PG} \stackrel{(1)}{=} \overline{PA} \cdot \overline{PB} = \overline{PC} \cdot \overline{PD} \stackrel{(2)}{=} \overline{PY} \cdot \overline{PH}.$$

Therefore, by the converse of the Intersecting Secants Theorem, $GXYH$ is a cyclic quadrilateral. ■

Now, let's solve some problems.

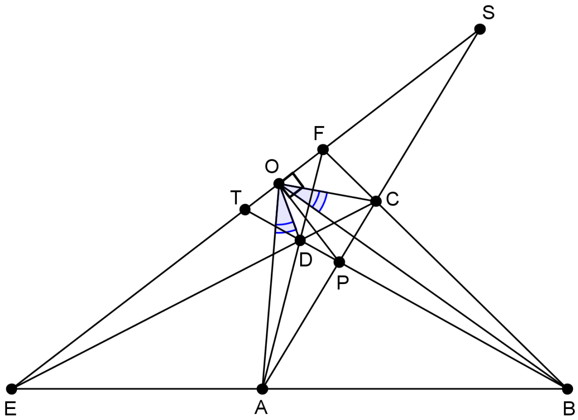
Example 20.3 (IMO Shortlist 1995/G3)**.** The incircle of $\triangle ABC$ touches the sides BC, CA and AB at D, E and F, respectively. X is a point inside $\triangle ABC$ such that the incircle of $\triangle XBC$ touches BC at D and touches CX and XB at Y and Z, respectively. Show that E, F, Z and Y are concyclic.



Proof. Let EF intersect BC at T_1. Since AD, BE and CF are concurrent at the Gergonne Point of $\triangle ABC$, by Property 20.1, we get that (T_1BDC) is a harmonic. Similarly, if $YZ \cap BC = T_2$, then (T_2BDC) is a harmonic. Since three of the points are fixed, then the fourth one must also be fixed, i.e. $T_1 \equiv T_2 \equiv T$.

Now, by the Secant-Tangent Theorem for the circle (DEF) and then for the circle (DYZ), we get $\overline{TF} \cdot \overline{TE} = \overline{TD}^2 = \overline{TZ} \cdot \overline{TY}$, which by the converse of the Intersecting Secants Theorem means that E, F, Z and Y are concyclic. ∎

Example 20.4 (China TST 2002)**.** Let E and F be the intersections of opposite sides of a convex quadrilateral $ABCD$. The two diagonals meet at P. Let O be the foot of the perpendicular from P to EF. Show that $\angle BOC = \angle AOD$.

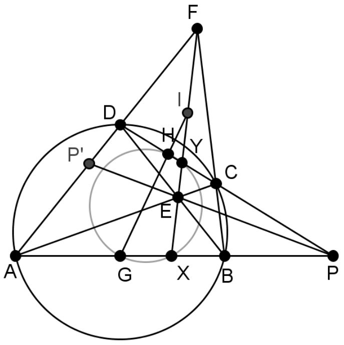


Proof. Let $E = BA \cap CD$ and $F = BC \cap AD$. Also, let $AC \cap EF = S$ and $BD \cap EF = T$. Since the cevians EC, FA and BT in $\triangle EFB$ are concurrent (at D), by Property 20.1, we get that the division $(ETFS)$ is harmonic. Therefore, the pencil $B(ETFS)$ is a harmonic pencil. If we intersect it with the line AC, by Property 20.6, the intersection points also form a harmonic division, i.e. $(APCS)$ is harmonic. Since $OP \perp OS$, by Property 20.3, $\angle AOP = \angle POC$.

On the other hand, since $(APCS)$ is harmonic, the pencil $E(APCS)$ is harmonic, so by intersecting it with the line BD, we get that $(BPDT)$ is harmonic. Again, since $OP \perp OT$, we get $\angle BOP = \angle POD$.

Finally, $\angle BOC = \angle POC - \angle POB = \angle AOP - \angle POD = \angle AOD$. ∎

Example 20.5 (IMO Shortlist 2009/G4)**.** Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.



Proof 1. Let the line EF intersect the lines AB, CD and GH at X, Y and I, respectively. By Property 19.6, the midpoints of the diagonals of the complete quadrilateral $FDECAB$ are collinear, so I is the midpoint of EF. Let $AB \cap CD = P$ and $PE \cap AD = P'$. In $\triangle ADP$, the cevians AC, DB and PP' are concurrent, so by Property 20.1, $(FDP'A)$ is a harmonic. Therefore, $P(FDP'A)$ is a harmonic pencil. If we intersect it with the line FE, by Property 20.6, the intersection points will form a harmonic division, i.e. $(FYEX)$ is a harmonic. By Property 20.4, $\mathcal{J}_{I,\overline{IE}} : X \leftrightarrow Y$, i.e.

$$\overline{IE}^2 = \overline{IX} \cdot \overline{IY}. \tag{1}$$

From Example 20.2, we know that $GXYH$ is a cyclic quadrilateral, so

$$\overline{IX} \cdot \overline{IY} = \overline{IH} \cdot \overline{IG}. \tag{2}$$

By combining (1) and (2), by the converse of the Secant-Tangent Theorem, we get that $IE \equiv FE$ is tangent to (EHG). ■

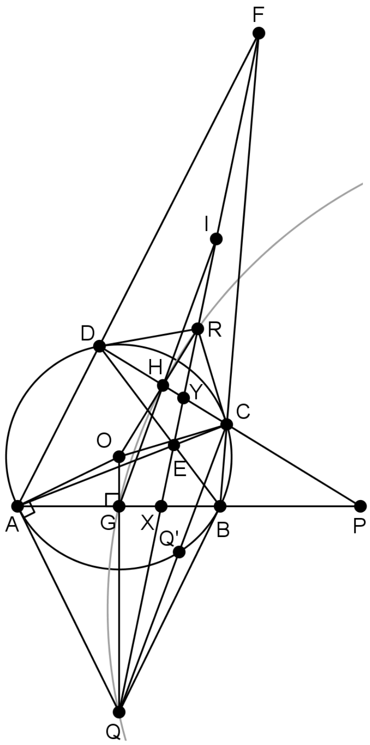
Proof 2. Let $AB \cap CD = P$ and $GH \cap FE = I$. Let $\omega \equiv (ABCD)$ and let O be its center. Let the tangents to ω at A and B intersects at Q. Let the tangents to ω at C and D intersect at R.

Since G is the midpoint of the AB, $G \in OQ$ and $OG \perp AB$. By the Euclid's laws for $\triangle OAQ$, we get $\overline{OA}^2 = \overline{OG} \cdot \overline{OQ}$. Similarly, $\overline{OC}^2 = \overline{OH} \cdot \overline{OR}$. Since $\overline{OA} = \overline{OC}$ as radii in ω, we have $\overline{OG} \cdot \overline{OQ} = \overline{OH} \cdot \overline{OR}$, so by the converse of the Intersecting Secants Theorem $GQRH$ is a cyclic quadrilateral. Therefore, for the secants through the point I, by the Intersecting Secants Theorem, we get

$$\overline{IG} \cdot \overline{IH} = \overline{IQ} \cdot \overline{IR} \tag{1}$$

By Property 18.1, AB is the polar of Q, i.e. $AB \equiv q$. Since $P \in q$, then by La Hire's Theorem, $Q \in p$. On the other hand, from Brocard's Theorem we know that $FE \equiv p$. Therefore, $Q \in FE$. Similarly, $R \in FE$.

Since QA and QB are tangents to (ABC), then by Property 15.4, CQ is a symmedian in $\triangle ABC$. If $Q' = CQ \cap (ABC)$, then by Property 20.10, $AQ'BC$



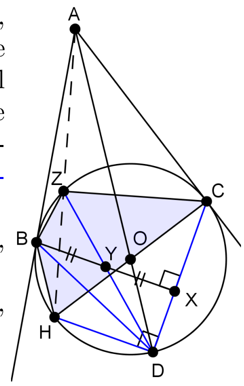
is a harmonic quadrilateral. Therefore, $C(A, B; Q', C)$ is a harmonic pencil. By intersecting it with the line FE, by Property 20.6, we get that the intersection points form a harmonic division, i.e. $(E, F; Q, R)$ is harmonic. By Property 19.6, the midpoints of the diagonals of the complete quadrilateral $FDECAB$ are collinear, so I is the midpoint of EF. Therefore, by Property 20.4, $\mathcal{J}_{I,\overline{IE}} : Q \leftrightarrow R$, i.e.

$$\overline{IE}^2 = \overline{IQ} \cdot \overline{IR}. \tag{2}$$

By combining (1) and (2), by the converse of the Secant-Tangent Theorem, we get that $IE \equiv FE$ is tangent to (EHG). ■

Example 20.6 (BMO Shortlist 2007, Cosmin Pohoata)**.** Let ω be a circle centered at O and A be a point outside it. Denote by B and C the points where the tangents from A with respect to ω meet the circle. Let D be the point on ω, for which O lies on the line segment AD. Let X be the foot of the perpendicular from B to CD, Y be the midpoint of the line segment BX and Z be the second intersection of DY with ω. Prove that $ZA \perp ZC$.

Proof. Let $CO \cap \omega = H$. Then, by Thales' Theorem, $DH \perp DC$. Since $XB \perp DC$, we get $DH \parallel XB$. Since $\overline{XY} = \overline{YB}$, by Property 20.7, we get that the pencil (DX, DY, DB, DH) harmonic. Therefore, by definition, the cyclic quadrilateral formed by the intersections of the pencil with ω, $CZBH$, is a harmonic quadrilateral. By Property 20.10, HZ is symmedian in $\triangle HBC$.
Since BA and CA are tangents to ω, then by Property 15.4, HA is a symmedian in $\triangle HBC$.
Finally, $HA \equiv HZ$, i.e. $H - Z - A$ are collinear. Therefore,



$$\angle AZC = 180^\circ - \angle CZH = 180^\circ - 90^\circ = 90^\circ,$$

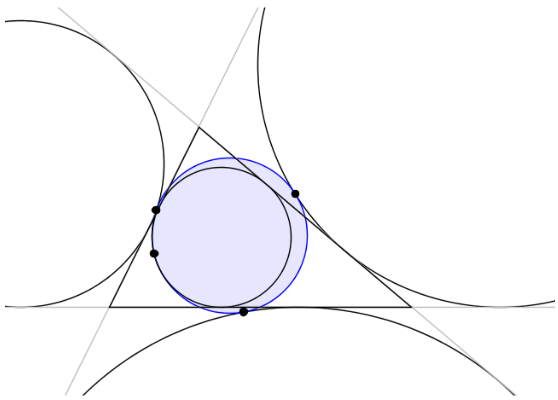
$$\text{i.e. } ZA \perp ZC$$ ■

Related problems: 131, 140 and 147.

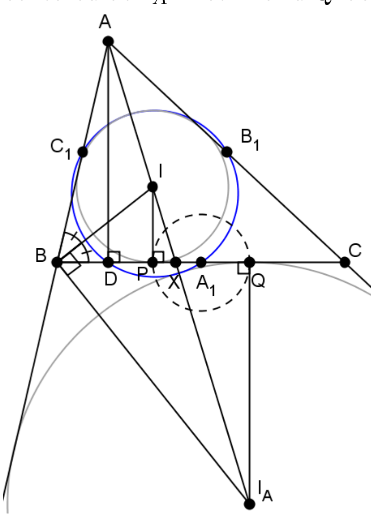
Chapter 21

Feuerbach's Theorem

Example 21.1 (Feuerbach's Theorem)**.** The nine point circle of a triangle is internally tangent to its incircle and externally tangent to its three excircles.



Proof. Let A_1, B_1 and C_1 be the midpoints of BC, CA and AB, respectively. Let ω be the incircle of the triangle, centered at I. Let ω_A be the A-excircle, centered at I_A. Let P and Q be the tangent points of ω and ω_A, respectively to BC. From Example 10.3.2, we know that $\overline{A_1P} = \overline{A_1Q}$. Let $\mathcal{J}$ be the inversion with center A_1 and radius $\overline{A_1P}$. Since A_1P is tangent to ω, $\mathcal{J} : \omega \leftrightarrow \omega$. Similarly $\mathcal{J} : \omega_A \leftrightarrow \omega_A$. Since the nine point circle, C_9, passes through the center of inversion A_1, it will be sent to a line, i.e. $\mathcal{J} : C_9 \leftrightarrow l_9$. Because we want to prove that the nine point circle is tangent to the incircle and A-excircle, we will prove that its image, l_9, is tangent to their images, ω and ω_A.

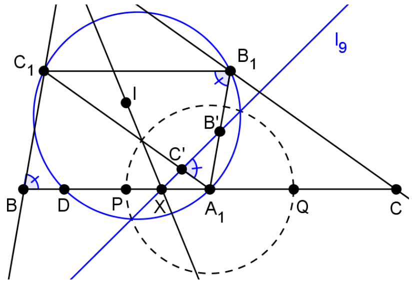


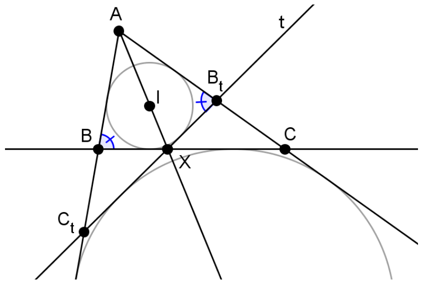
Let $AI \cap BC = X$. Since BI is the internal angle bisector of $\angle ABC$ and $BI \perp BI_A$, by Property 20.3, the division $(AIXI_A)$ is harmonic. Let D be the foot of the altitude from A to BC. Then $AD \parallel IP \parallel I_AQ$ (because they are all perpendicular to BC). So, by Thales'

Proportionality Theorem, the division $(DPXQ)$ is also harmonic. By Property 20.4, since PQ is the diameter of the circle of inversion, $\mathcal{J} : D \leftrightarrow X$. Since $D \in C_9$, then $X \in l_9$. Let $\mathcal{J} : B_1 \leftrightarrow B'$ and $\mathcal{J} : C_1 \leftrightarrow C'$. Then by Property 17.1

$$\angle A_1C'B' = \angle A_1B_1C_1 = \angle ABC. \tag{1}$$

Also, since $B_1, C_1 \in C_9$, then $B', C' \in l_9$.





Since X is the intersection of the line connecting the centers of ω and ω_A, II_A, and one of the common internal tangents, BC, then the other common internal tangent, t, must also pass through X. We want to prove that $l_9 \equiv t$. Let $t \cap AB = C_t$ and $t \cap AC = B_t$. By symmetry with respect to the line AI,

$$\angle AB_tC_t = \angle ABC. \tag{2}$$

From (1), we know that $\angle(A_1C_1, l_9) = \angle ABC$. From (2), we know that $\angle(AC, t) = \angle ABC$. Since $A_1C_1 \parallel AC$, it means that $l_9 \parallel t$. But we already know that $X \in l_9$ and $X \in t$, so $l_9 \equiv t$.

Therefore, the nine point circle is tangent to the incircle and the A-excircle. Similarly, it is tangent to the other two excircles. ■

Remark. The tangent point of the incircle and the nine point circle is called the *Feuerbach point* of the triangle.

Chapter 22

Apollonius' Problem

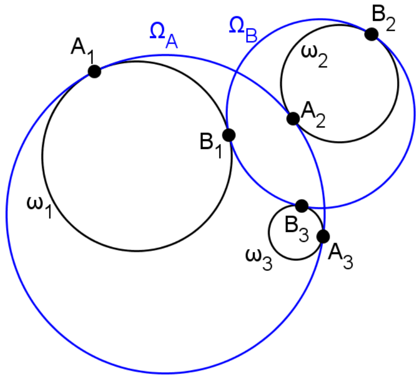
This topic is not directly related to Olympiad geometry problems, but it is a nice collection of properties that we already visited during our journey combined in a beautiful result.

Example 22.1 (Apollonius' problem)**.** Construct circles that are tangent to three given circles in a plane, ω_1, ω_2 and ω_3.

We will examine the case where the three circles are in general position, i.e. none of them intersect and all of them have different radii.

Firstly, let's find the number of solution circles that are tangent to all three circles. The solution circles can be tangent either internally or externally to any of the three circles. So the number of solution circles is $2^3 = 8$.

We will explain Gergonne's approach to solving this problem. It considers the solution circles in pairs such that if one of the solution circles is internally tangent to a given circle, then the other solution circle is externally tangent to that circle and vice versa. For example, if a solution circle is internally tangent to ω_1 and ω_3, but externally tangent to ω_2, then the paired solution circle is externally tangent to ω_1 and ω_3, but internally tangent to ω_2.



Let Ω_A and Ω_B be a pair of solution circles. Let Ω_A be tangent to ω_1, ω_2 and ω_3 at A_1, A_2 and A_3. Let Ω_B be tangent to ω_1, ω_2 and ω_3 at B_1, B_2 and B_3.

So, we somehow need to find these 6 tangent points. Then, the circumcircles of $\triangle A_1A_2A_3$ and $\triangle B_1B_2B_3$ would be the solution circles. Gergonne's approach was to construct a line l_1 such that A_1 and B_1 must always lie on it. Then, A_1

and B_1 could be obtained as the intersection points of l_1 and ω_1. Similarly, by finding lines l_2 and l_3 that contained A_2 and B_2, and A_3 and B_3, respectively, we would find all 6 tangent points.

Let's recall, from section 12.2, that the radical center of three circles is the center of the unique circle (called the radical circle) that intersects the three given circles orthogonally. Let R be the radical center of ω_1, ω_2 and ω_3. Now, consider the inversion $\mathcal{J}$ with the radical circle as the circle of inversion. Since the radical circle is orthogonal to the three given circles, each of them will be sent to itself. Since the solution circles are tangent to the three given circles, their images need to be tangent to the images of the given circles (which happen to be the three circles themselves), so the solution circles will be sent one into the other, i.e.

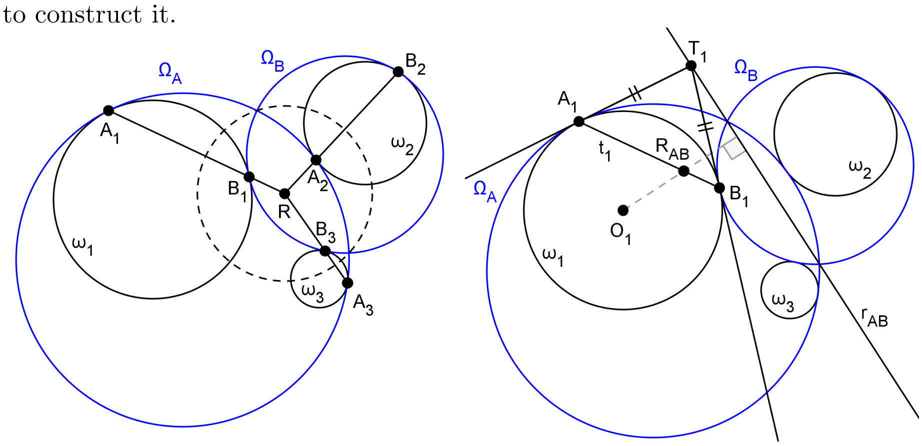
$$\mathcal{J} : \omega_1 \leftrightarrow \omega_1 \quad \mathcal{J} : \omega_2 \leftrightarrow \omega_2 \quad \mathcal{J} : \omega_3 \leftrightarrow \omega_3$$

$$\mathcal{J} : \Omega_A \leftrightarrow \Omega_B$$

Therefore, the tangent point $A_1 = \omega_1 \cap \Omega_A$ will be sent to a point

$$A_1' = \omega_1' \cap \Omega_A' = \omega_1 \cap \Omega_B = B_1, \text{ i.e. } \mathcal{J} : A_1 \leftrightarrow B_1.$$

Since the center of inversion, the original point and the image point are collinear, we get that the radical center lies on the line A_1B_1, i.e. $R \in l_1$. Thus, we found one point on the line l_1. Now we need to find another one in order to be able to construct it.

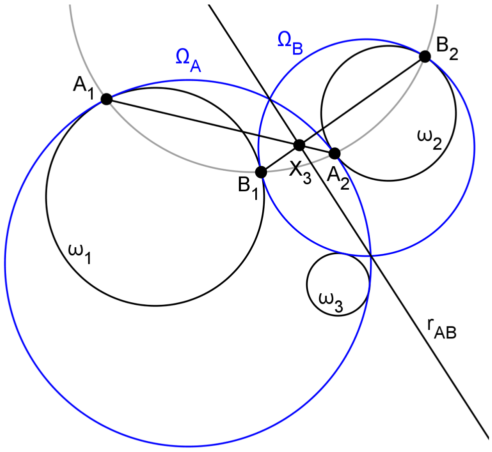


(a) $R \in l_1$ (b) $R_{AB} \in l_1$

Let the tangents to ω_1 at A_1 and B_1 intersect at T_1. Then, $\overline{T_1A_1} = \overline{T_1B_1}$ and also, by Property 18.1, A_1B_1 is the polar of T_1 with respect to ω_1, i.e. $A_1B_1 \equiv t_1$. Notice that T_1A_1 and T_1B_1 are also tangents to the circles Ω_A and Ω_B. By Property 12.2 and since $\overline{T_1A_1} = \overline{T_1B_1}$, the power of the point T_1 with respect to Ω_A and Ω_B is equal, which means that T_1 lies on their radical axis r_{AB}. By La Hire's Theorem, the pole of r_{AB} with respect to ω_1 lies on t_1. So, here is our second point on the line $t_1 \equiv A_1B_1 \equiv l_1$.

But in order to construct the pole of r_{AB} with respect to ω_1, we firstly need to construct r_{AB}. How do we constuct the radical axis of two circles Ω_A and Ω_B if we don't have them yet? Well, we will find points that should lie on the radical axis and then we will construct the radical axis as the line through those points.

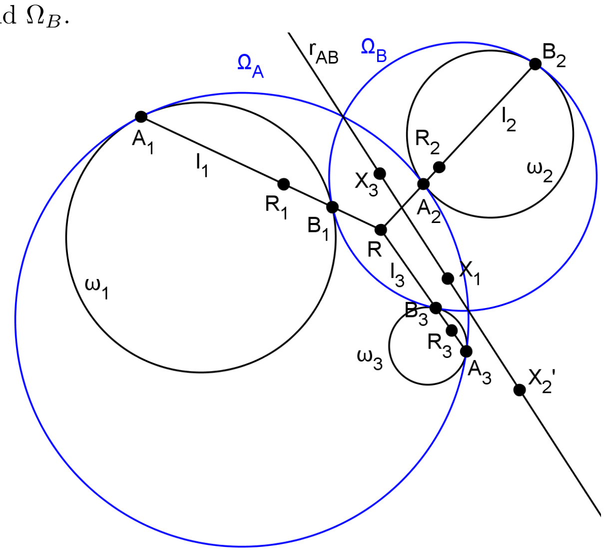
Let's recall, from Example 16.7 that if a circle is tangent to two other circles, then the line through the tangent points passes through one of the homothetic centers of the two circles. Let X_3 be a homothetic center of ω_1 and ω_2, which can be constructed as the intersection of their common tangents. Then $X_3 \in A_1A_2$ and $X_3 \in B_1B_2$.



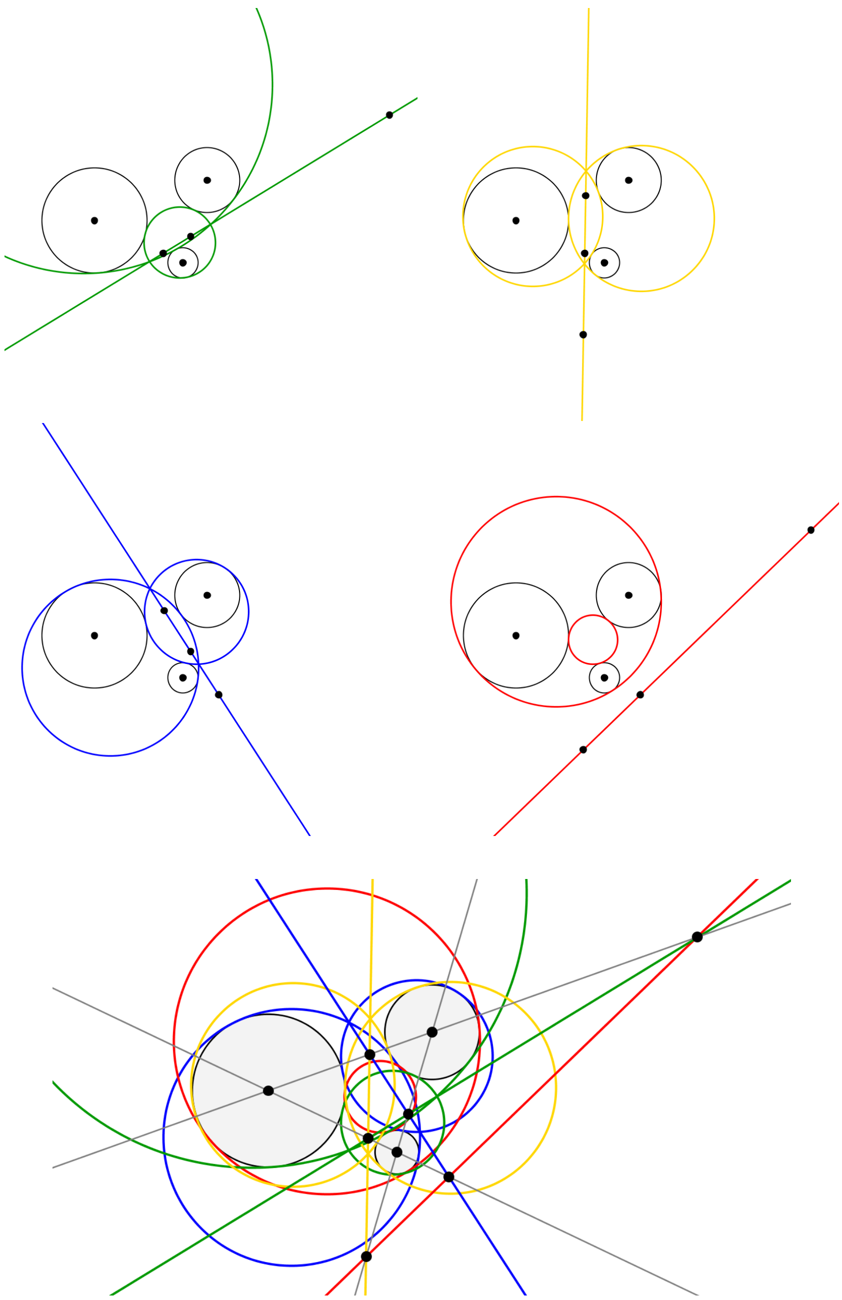
Recall also, from section 16.1, the definition and the properties of antihomologous points. This means that in our case, since Ω_A is tangent to ω_1 and ω_2, A_1 and A_2 are a pair of antihomologous points. Similarly, B_1 and B_2 are antihomologous points. Because, we know that two pairs of antihomologous points are concyclic, then:

$$\overline{X_3A_1} \cdot \overline{X_3A_2} = \overline{X_3B_1} \cdot \overline{X_3B_2}$$

But since $A_1, A_2 \in \Omega_A$ and $B_1, B_2 \in \Omega_B$, this means that the power of the point X_3 with respect to Ω_A and Ω_B is the same, i.e. X_3 lies on their radical axis r_{AB}. Similarly, the homothetic centers X_1 and X_2 (which can be constructed as the intersection of the common tangents of ω_2 and ω_3, and ω_1 and ω_3, respectively) also lie r_{AB}. Thus, we know how to construct the radical axis r_{AB} of the solution circles Ω_A and Ω_B.



In summary, the desired line l_1 is defined by two points: the radical center R of the three given circles and the pole with respect to ω_1 of the line connecting the homothetic centers. Depending on whether we choose all three external homothetic centers (1 possibility), or we choose one external and the other two internal homothetic centers (3 possibilities), we have 4 ways of defining "the line connecting the homothetic centers"[1]. Each of these 4 lines generates a different pair of solution circles, so that's how we can get all 8 solution circles.



[1]Recall Monge-d'Alembert Theorem, page 99

Part II

Mixed Problems

Problem 1. Let C be a point on the line segment AB. Let D be a point that doesn't lie on the line AB. Let M and N be points on the angle bisectors of $\angle ACD$ and $\angle BCD$, respectively, such that $MN \parallel AB$. Prove that the line CD bisects MN.

Problem 2. Let ABC be a right triangle with $\angle BCA = 90^\circ$ and $\overline{CA} < \overline{CB}$. Let $D \in BC$, such that $\overline{DA} = \overline{DB}$ and $E \in AB$ such that $\overline{CA} = \overline{CE}$. Prove that $AD \perp CE$.

Problem 3. Let ABC be a triangle and let M be a point on the ray AB beyond B, such that $\overline{BM} = \overline{BC}$. Prove that MC is parallel to the angle bisector of $\angle ABC$.

Problem 4. Let ABC be an isosceles triangle ($\overline{CA} = \overline{CB}$). Let S_{XY} denote the side bisector of a segment XY. Let $S_{CA} \cap CB = P$ and $S_{CB} \cap CA = Q$. Prove that $PQ \parallel AB$.

Problem 5. Let M and N be midpoints of the sides CA and CB, respectively, in a triangle ABC. The angle bisector of $\angle BAC$ intersects the line MN at D. Prove that $\angle ADC = 90^\circ$.

Problem 6. Let ABC be an equilateral triangle. Let $D \in AB$ and $E \in BC$, such that $\overline{AD} = \overline{BE}$. Let $AE \cap CD = F$. Find $\angle CFE$.

Problem 7. Let BD be a median in $\triangle ABC$. The points E and F divide the median BD in three equal parts, such that $\overline{BE} = \overline{EF} = \overline{FD}$. If $\overline{AB} = 1$ and $\overline{AF} = \overline{AD}$, find the length of the line segment CE.

Problem 8. Let I be the incenter of $\triangle ABC$. Let l be a line through I, parallel to AB, that intersects the sides CA and CB at M and N, respectively. Prove that

$$\overline{AM} + \overline{BN} = \overline{MN}.$$

Problem 9 (Serbia 2017, Opstinsko IIB)**.** The diagonals of a convex quadrilateral $ABCD$ intersect at O. Prove that the circumcenters of the triangles ABO, BCO, CDO and DAO are vertices of a parallelogram.

Problem 10. Let $ABCD$ be a convex quadrilateral with area 1. Let A_1 be a point on the ray AB beyond B, such that $\overline{AB} = \overline{BA_1}$. Similarly define the points B_1, C_1 and D_1. Prove that the area of $A_1B_1C_1D_1$ is 5.

Problem 11. Let $ABCD$ be a convex quadrilateral with right angle at the vertex C. Let $P \in CD$, such that $\angle APD = \angle BPC$ and $\angle BAP = \angle ABC$. Prove that

$$\overline{BC} = \frac{\overline{AP} + \overline{BP}}{2}.$$

Problem 12. Let $ABCD$ be a convex quadrilateral with area 3. The points M and N divide the line segment AB in three equal parts, such that $\overline{AM} = \overline{MN} = \overline{NB}$. The points P and Q divide the line segment CD in three equal parts, such that $\overline{CP} = \overline{PQ} = \overline{QD}$. Prove that the area of $MNPQ$ is 1.

Problem 13. Let $ABCD$ be a convex quadrilateral ($\overline{AB} > \overline{CD}, \overline{AD} > \overline{BC}$). Let $AB \cap DC = P$ and $AD \cap BC = Q$. If $\overline{AP} = \overline{AQ}$ and $\overline{AB} = \overline{AD}$, prove that AC is the angle bisector of $\angle BAD$.

Problem 14. Let $ABCD$ be a convex quadrilateral such that the side bisector of BC passes through the midpoint of AD and $\overline{AC} = \overline{BD}$. Prove that $\overline{AB} = \overline{CD}$.

Problem 15. Let $ABCD$ be a trapezoid ($AB \parallel CD$). Let its diagonals AC and BD intersect at P. Let the areas of the triangle $\triangle ABP$ and $\triangle CDP$ be m and n, respectively. Prove that $P_{ABCD} = (\sqrt{m} + \sqrt{n})^2$.

Problem 16. Let $ABCD$ be a parallelogram. Let M and N be the midpoints of the sides BC and DA, respectively. Prove that the lines AM and CN divide the diagonal BD in three equal parts.

Problem 17. Let $ABCD$ be a parallelogram with area 1. Let M be the midpoint of the side AD. Let $BM \cap AC = P$. Find the area of $MPCD$.

Problem 18. Let H and O be the orthocenter and circumcenter in a triangle ABC, respectively. Prove that $\overline{AH} = \overline{AO}$ if and only if $\angle BAC = 60°$.

Problem 19 (Serbia 2017, Opstinsko IIA)**.** Let T be the centroid of a triangle ABC and let t be a line that passes through T, such that A and B are on one side of t and C is on the other side. Let A', B' and C' be the orthogonal projections of A, B and C, respectively, to the line t. Prove that $\overline{AA'} + \overline{BB'} = \overline{CC'}$.

Problem 20 (Serbia 2014, Okruzno IB)**.** Let $ABCDEF$ be a convex hexagon with $\overline{AB} = \overline{AF}$, $\overline{BC} = \overline{CD}$ and $\overline{DE} = \overline{EF}$. Prove that the angle bisectors of $\angle BAF$, $\angle BCD$ and $\angle DEF$ are concurrent.

Problem 21 (Serbia 2014, Okruzno IB)**.** Let ABC be a triangle with $\angle B > \angle C$. The angle bisector of $\angle A$ intersects BC at D. The perpendicular from B to AD intersects the circumcircle of $\triangle ABD$ again at E. Prove that the circumcenter of $\triangle ABC$ lies on the line AE.

Problem 22. In the triangle ABC, let A_1 be the midpoint of BC and let B_1 and C_1 be the feet of the altitudes from the vertices B and C, respectively. Prove that the triangle $A_1B_1C_1$ is equilateral if and only if $\angle BAC = 60°$.

Problem 23 (Serbia 2014, Opstinski IA)**.** Let $ABCD$ be a quadrilateral such that $\angle BCA + \angle CAD = 180°$ and $\overline{AB} = \overline{AD} + \overline{BC}$. Prove that

$$\angle BAC + \angle ACD = \angle CDA$$

Problem 24 (Serbia 2016, Okruzno IA)**.** Let $ABCD$ be a convex quadrilateral with $\overline{AD} = \overline{BC}$ and $\angle A + \angle B = 120°$. Let E be the midpoint of the side CD and let F and G be the midpoints of the diagonals AC and BD, respectively. Prove that EFG is an equilateral triangle.

Problem 25. Let ABC be a right triangle ($\angle BCA = 90°$). Let CD be the altitude from the vertex C. Prove that the distances from the point D to the legs of the triangle are proportional to the lengths of the legs.

Problem 26. Let M and N be midpoints of the sides AB and AC, respectively, in a triangle ABC. Let P and Q be points outside the triangle, such that $PM \perp AB, \overline{PM} = \frac{1}{2}\overline{AB}$ and $QN \perp AC, \overline{QN} = \frac{1}{2}\overline{AC}$. If L is the midpoint of BC, prove that $\overline{LP} = \overline{LQ}$ and $\angle PLQ = 90°$.

Problem 27. Let P be a point on the side AB in $\triangle ABC$, such that $\overline{AP} = 3 \cdot \overline{PB}$. Let $Q \in AC$, such that $\overline{AQ} = 4 \cdot \overline{QC}$. Prove that BQ bisects the line segment CP.

Problem 28. Let ABC be a right triangle ($\angle BCA = 90°$). Let AD and BE be angle bisectors ($D \in BC, E \in CA$). Let N and M be the feet of the perpendiculars from D and E, respectively, to the hypotenuse AB. Prove that $\angle MCN = 45°$.

Problem 29 (Serbia 2018, Drzavno VI)**.** Let ABC be an acute triangle and let AX and AY be rays, such that the angles $\angle XAB$ and $\angle YAC$ have no common interior point with $\triangle ABC$ and $\angle XAB = \angle YAC < 90°$. Let B' and C' be feet of the perpendiculars from B and C to AX and AY, respectively. If M is the midpoint of BC, prove that $\overline{MB'} = \overline{MC'}$.

Problem 30. In the triangle ABC, let BE and CF be perpendiculars to the angle bisector AD. Prove that $\overline{AE} \cdot \overline{DF} = \overline{AF} \cdot \overline{DE}$.

Problem 31 (Romania JBMO TST 2016)**.** Let ABC be an acute triangle where $\angle BAC = 60°$. Prove that if the Euler's line of $\triangle ABC$ intersects AB and AC at D and E, respectively, then $\triangle ADE$ is equilateral.

Problem 32. Let $\triangle ABC$ be a right triangle ($\gamma = 90°$). The angle bisector of $\angle ABC$ intersects AC at D. If $\overline{AD} = 5$ and $\overline{CD} = 3$, find $\overline{AB}$.

Problem 33 (Stefan Lozanovski)**.** In the triangle ABC, $\gamma = 60°$. Let O be the circumcenter of $\triangle ABC$. AO intersects BC at M and BO intersects AC at N. Prove that $\overline{AN} = \overline{BM}$.

Problem 34 (Serbia 2018, Opstinsko IIIA)**.** Let I be the incenter of a triangle ABC ($\overline{AB} < \overline{AC}$). The line AI intersects the circumcircle of ABC again at D. The circumcircle of CDI intersects BI again at K. Prove that $\overline{BK} = \overline{CK}$.

Problem 35. Let ABC be an isosceles triangle, such that $\overline{AC} = \overline{BC}$. Let P be a point on the side AC. The tangent to (ABP) at the point P intersects (BCP) at D. Prove that $CD \parallel AB$.

Problem 36. The angle bisectors of the adjacent angles in a quadrilateral $ABCD$ intersect at the points E, F, G and H. Prove that $EFGH$ is cyclic.

Problem 37 (JBMO Shortlist 2015)**.** Let t be the tangent at the vertex C to the circumcircle of triangle ABC. A line p parallel to t intersects BC and AC at points D and E, respectively. Prove that the points A, B, D and E are concyclic.

Problem 38. Two circles intersect at A and B. One of their common tangents touches the circles at P and Q. Let A' be the reflection of A across the line PQ. Prove that $A'PBQ$ is a cyclic quadrilateral.

Problem 39. Let ABC be an acute triangle. Let E and F be the feet of the altitudes in $\triangle ABC$ from B and C, respectively, and let M be the midpoint of BC. Prove that ME and MF are tangents to (AEF).

Problem 40. Two circles intersect at A and B. One of their common tangents touches the circles at P and Q. Prove that the line AB bisects the line segment PQ.

Problem 41. Let $ABCD$ be a cyclic quadrilateral and let S be the intersection of its diagonals ($\angle ASB < 90°$). If H is the orthocenter of $\triangle ABS$ and O is the circumcenter of $\triangle CDS$, prove that the points H, S and O are collinear.

Problem 42. Let $ABCD$ be a cyclic quadrilateral. The rays AB and DC intersect at P and the rays AD and BC intersect at Q. The circumcircles of $\triangle BCP$ and $\triangle CDQ$ intersect at R. Prove that the points P, Q and R are collinear.

Problem 43. The diagonals of a cyclic quadrilateral $ABCD$ intersect at S. The circumcircle of $\triangle ABS$ intersects line BC at M, and the circumcircle of $\triangle ADS$ intersects line CD at N. Prove that S, M and N are collinear.

Problem 44. Let D, E and F be points on the sides BC, CA and AB, respectively, such that $BCEF$ is a cyclic quadrilateral. Let P be the second intersection of the circumcircles of $\triangle BDF$ and $\triangle CDE$. Prove that A, D and P are collinear.

Problem 45. Two circles are tangent to each other internally at a point T. Let the chord AB of the larger circle be tangent to the smaller circle at a point P. Prove that TP is the internal angle bisector of $\angle ATB$.

Problem 46. Let AD be an altitude in triangle ABC. Let E and F be the feet of the perpendiculars from D to the sides AB and AC, respectively. Prove that the quadrilateral $BCFE$ is cyclic.

Problem 47. In a triangle ABC let AD be an angle bisector ($D \in BC$). Let E and F be points on the interior segments AC and AB, respectively, such that $\angle BFD = \angle BDA$ and $\angle CED = \angle CDA$. Prove that EF is parallel to BC.

Problem 48 (Iran MO (3rd Round), 2017)**.** Let ABC be a triangle. Suppose that X and Y are points in the plane such that BX and CY are tangent to the circumcircle of $\triangle ABC$, $\overline{AB} = \overline{BX}$, $\overline{AC} = \overline{CY}$ and X, Y and A are in the same side of BC. If I be the incenter of $\triangle ABC$ prove that $\angle BAC + \angle XIY = 180°$.

Problem 49 (JBMO Shortlist 2012)**.** Let ABC be an acute-angled triangle with circumcircle ω, and let O and H be the triangle's circumcenter and orthocenter, respectively. Let also A' be the point where the angle bisector of $\angle BAC$ meets ω. If $\overline{A'H} = \overline{AH}$, then find the measure of $\angle BAC$.

Problem 50 (EGMO 2012)**.** Let ABC be a triangle with circumcenter O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. Let K be the circumcenter of triangle AFE. Prove that the lines DK and BC are perpendicular.

Problem 51 (EGMO 2015)**.** Let $\triangle ABC$ be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle $\triangle ADE$ again at F. If $\angle ADF = 45°$, show that CF is tangent to ω.

Problem 52. Let $ABCD$ be a parallelogram with $\overline{AC} > \overline{BD}$. The circumcircle of $\triangle BCD$ intersects AC again at P. Prove that BD is a common tangent for the circumcircles of $\triangle ABP$ and $\triangle ADP$.

Problem 53 (Bosnia and Herzegovina TST 2013)**.** Triangle ABC is right angled at C. Lines AM and BN are internal angle bisectors. AM and BN intersect the altitude CD at points P and Q, respectively. Prove that the line which passes through the midpoints of the segments QN and PM is parallel to AB.

Problem 54. Let P be a point outside a circle ω. Let A and B be points on ω, such that PA and PB are tangents to ω. On the minor arc $\widehat{AB}$ lies an arbitrary point C. Let D, E and F be the feet of the perpendiculars from C to AB, PA and PB, respectively. Prove that $\overline{CD}^2 = \overline{CE} \cdot \overline{CF}$.

Problem 55 (Serbia 2017, Okruzno IVA)**.** Let PA and PB be the tangents from P to a circle ω $(A, B \in \omega)$. Let Q be a point on the line PA, such that A is between P and Q and $\overline{PA} = \overline{AQ}$ and let C be a point on the line segment AB. The circumcircle of $\triangle PBC$ intersects ω again at D. Prove that $\angle PBD = \angle QCA$.

Problem 56. Let C be a point on a semicircle with diameter AB and let D be the midpoint of arc AC. Let E be the projection of D onto the line BC and F the intersection of line AE with the semicircle. Prove that BF bisects the line segment DE.

Problem 57. Let ABC be an equilateral triangle. Let S be a point on the arc $\widehat{AB}$ of (ABC) that doesn't contain C. Prove that $\overline{SA} + \overline{SB} = \overline{SC}$.

Problem 58 (JBMO Shortlist 2015)**.** The point P is outside the circle Ω. Two tangent lines, passing through P touch Ω at points A and B. The median AM in the triangle ABP intersects Ω at C and PC intersects Ω again at D. Prove that $AD \parallel BP$.

Problem 59. Let k_1 and k_2 be two circles intersecting at A and B. Let t_1 and t_2 be the tangents to k_1 and k_2 at point A and let $t_1 \cap k_2 = \{A, C\}$, $t_2 \cap k_1 = \{A, D\}$. If E is a point on the ray AB, such that $\overline{AE} = 2 \cdot \overline{AB}$, prove that $ACED$ is cyclic.

Problem 60. In a triangle ABC, let $M \in BC$, $N \in AC$, $K = AM \cap BN$, such that the circumcircles of $\triangle AKN$ and $\triangle BKM$ intersect at the orthocenter H of $\triangle ABC$. Prove that $\overline{AM} = \overline{BN}$.

Problem 61. Let A_1, B_1 and C_1 be the second intersections of the angle bisectors of $\triangle ABC$ with its circumcircle. Prove that the incenter of $\triangle ABC$ is the orthocenter of $\triangle A_1B_1C_1$.

Problem 62. Let A_1, B_1 and C_1 be the second intersections of the altitudes of $\triangle ABC$ with its circumcircle. Prove that the orthocenter of $\triangle ABC$ is the incenter of $\triangle A_1B_1C_1$.

Problem 63 (JBMO Shortlist 2015)**.** Let ω be a circle with center O and let A and B be two points on ω that are not diametrically opposite. The bisector of $\angle ABO$ intersects ω again at C, the circumcircle of $\triangle AOB$ at K and the circumcircle of $\triangle AOC$ at L. Prove that K is the circumcenter of $\triangle AOC$ and L is the incenter of $\triangle AOB$.

Problem 64. Let I be the incenter and AB the shortest side of the triangle ABC. The circle centered at I passing through C intersects the ray AB in P and the ray BA in Q. Prove that the circumcircles of $\triangle CAQ$ and $\triangle CBP$ intersect at the angle bisector of $\angle ACB$.

Problem 65 (Romania JBMO TST 2016)**.** Let O be the circumcenter of a triangle ABC. Let D, E and F be the tangent points of the $A-$excircle with the lines BC, CA and AB, respectively. If the $A-$excircle has radius equal to the circumradius of $\triangle ABC$, prove that $OD \perp EF$.

Problem 66 (Macedonia MO 2018)**.** Given is an acute $\triangle ABC$ with orthocenter H. The point H' is symmetric to H over the side AB. Let N be the intersection point of HH' and AB. The circle passing through A, N and H' intersects AC for the second time in M, and the circle passing through B, N and H' intersects BC for the second time in P. Prove that M, N and P are collinear.

Problem 67. In a triangle ABC ($\overline{AB} \neq \overline{AC}$), let the incircle centered at I touch the sides BC, CA and AB at points D, E and F, respectively. Let Y and Z be the intersections of the line through A parallel to BC with the lines DF and DE, respectively. Let M and N be midpoints of DY and DZ. Prove that the quadrilateral $IMAN$ is cyclic.

Problem 68 (APMO 2018)**.** Let H be the orthocenter of the triangle ABC. Let M and N be the midpoints of the sides AB and AC, respectively. Assume that H lies inside the quadrilateral $BMNC$ and that the circumcircles of triangles BMH and CNH are tangent to each other. The line through H parallel to BC intersects the circumcircles of the triangles BMH and CNH in the points K and L, respectively. Let F be the intersection point of MK and NL and let J be the incenter of triangle MHN. Prove that $\overline{FJ} = \overline{FA}$.

Problem 69 (St. Petersburg City MO 1996)**.** Let BD be the angle bisector of angle ABC in $\triangle ABC$ with D on the side AC. The circumcircle of $\triangle BDC$ meets AB at E, while the circumcircle of $\triangle ABD$ meets BC at F. Prove that $\overline{AE} = \overline{CF}$.

Problem 70. Let BB' and CC' be altitudes in the acute-angled triangle ABC. Let M and N be points on the line segments BB' and CC', respectively, such that $\angle AMC = 90° = \angle ANB$. Prove that $\overline{AM} = \overline{AN}$.

Problem 71 (Serbia 2016, Drzavno)**.** In $\triangle ABC$, the angle bisector of $\angle BAC$ intersects BC at D. Let M be the midpoint of BD. Let k be a circle through A that is tangent to BC at D and let the second intersections of k with the lines AM and AC be P and Q, respectively. Prove that the points B, P and Q are collinear.

Problem 72 (USAMO 1990)**.** An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CE and its extension at points M and N, and the circle with diameter AC intersects altitude BD and its extension at points P and Q. Prove that the points M, N, P and Q lie on a common circle.

Problem 73 (USAMO 2010)**.** Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R and S the feet of the perpendiculars from Y onto lines AX, BX, AZ and BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Problem 74 (JBMO Shortlist 2014)**.** Let ABC be a triangle such that $\overline{AB} \neq \overline{AC}$. Let M be the midpoint of BC and H be the orthocenter of $\triangle ABC$. Let D be the midpoint of AH and O the circumcenter of triangle HBC. Prove that $DAMO$ is a parallelogram.

Problem 75 (EGMO 2012)**.** Let ABC be an acute-angled triangle with circumcircle Γ and orthocenter H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM and BC are concurrent.

Problem 76 (EGMO 2016)**.** Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and BD intersect at X.Let C_1, D_1 and M be the midpoints of segments CX, DX and CD, respectively. Lines AD_1 and BC_1 intersect at Y, and line MY intersects diagonals AC and BD at different points E and F, respectively. Prove that line XY is tangent to the circle through E, F and X.

Problem 77 (USAMO 1997)**.** Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA and AB, respectively. Show that the lines through A, B and C perpendicular to EF, FD and DE, respectively, are concurrent.

Problem 78 (IMO Shortlist 1996, G3)**.** Let O be the circumcenter and H the orthocenter of an acute-angled triangle ABC such that $\overline{CB} > \overline{CA}$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\angle FHP = \angle BAC$.

Problem 79 (AIME 2011, modified)**.** In a triangle ABC, let P and Q be the feet of the perpendiculars from C to the angle bisectors of $\angle ABC$ and $\angle CAB$, respectively. Prove that $\overline{PQ}$ is equal to the length of the tangent segment from C to the incircle of $\triangle ABC$.

Problem 80 (India MO 2010)**.** Let ABC be a triangle with circumcircle Γ. Let M be a point in the interior of $\triangle ABC$ which is also on the bisector of $\angle BAC$. Let AM, BM and CM meet Γ in A_1, B_1 and C_1, respectively. Let P be the point of intersection of A_1C_1 with AB and Q be the point of intersection of A_1B_1 with AC. Prove that $PQ \parallel BC$.

Problem 81. Let M be the midpoint of the side BC in $\triangle ABC$. Let E and F be the tangent points of the incircle and the sides CA and AB, respectively. Let the angle bisectors of $\angle B$ and $\angle C$ intersect the line EF at X and Y, respectively. Prove that $\triangle MXY$ is equilateral if and only if $\angle A = 60°$.

Problem 82. On the sides AB and AC of a triangle ABC are given points P and Q, respectively, such that $PQ \parallel BC$. Prove that the circles with diameters BQ and CP intersect on the line through A that is perpendicular to BC.

Problem 83 (APMO 2013)**.** Let ABC be an acute triangle with altitudes AD, BE, and CF and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC and OE dissect the triangle ABC into three pairs of triangles that have equal areas.

Problem 84. Given a semicircle with diameter AB, let C and D be points on the semicircle, such that D is between A and C. Let P be the intersection of AD and BC. Prove that the value of $\overline{AP} \cdot \overline{AD} + \overline{BP} \cdot \overline{BC}$ doesn't depend on the choice of the points C and D.

Problem 85 (All-Russian MO 2005, Round 4)**.** Let I be an incenter of ABC ($\overline{AB} < \overline{BC}$). Let M be the midpoint of AC and N be the midpoint of the arc $\widehat{ABC}$. Prove that $\angle IMA = \angle INB$.

Problem 86 (IMO 2013/4)**.** Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of $\triangle BWN$, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle $\triangle CWM$, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Problem 87 (EGMO 2017)**.** Let ABC be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid G and the circumcenter O of ABC in its sides BC, CA, AB are denoted by G_1, G_2, G_3 and O_1, O_2, O_3, respectively. Show that the circumcircles of triangles G_1G_2C, G_1G_3B, G_2G_3A, O_1O_2C, O_1O_3B, O_2O_3A and ABC have a common point.

Problem 88 (Serbia 2016, Opstinsko IIA)**.** The incircle of ABC ($\overline{AB} < \overline{AC}$) touches the sides BC, CA and AB at D, E and F, respectively. The angle bisector of $\angle BAC$ intersects the lines DE and DF at M and N, respectively. Let K be the foot of the altitude from A to BC. Prove that D is the incenter of $\triangle MNK$.

Problem 89 (Poland MO 2000)**.** Let a triangle ABC satisfy $\overline{AC} = \overline{BC}$. Let P be a point inside the triangle ABC such that $\angle PAB = \angle PBC$. Denote by M the midpoint of the segment AB. Show that $\angle APM + \angle BPC = 180°$.

Problem 90 (Macedonia MO 2015)**.** Let k_1 and k_2 be two circles that intersect at points A and B. A line through B intersects k_1 and k_2 at C and D, respectively, such that C doesn't lie inside of k_2 and D doesn't lie inside of k_1. Let M be the intersection point of the tangent lines to k_1 and k_2 that pass through C and D, respectively. Let P be the intersection of the lines AM and CD. The tangent line to k_1 passing through B intersects AD in point L. The tangent line to k_2 passing through B intersects AC in point K. Let KP intersect MD at N and LP intersect MC at Q. Prove that $MNPQ$ is a parallelogram.

Problem 91 (Canada MO 2012)**.** Let $ABCD$ be a convex quadrilateral such that $\overline{AC} + \overline{AD} = \overline{BC} + \overline{BD}$ and let P be the point of intersection of AC and BD. Prove that the internal angle bisectors of $\angle ACB$, $\angle ADB$ and $\angle APB$ meet at a common point.

Problem 92 (IMO 2012/1)**.** Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.
(The excircle of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 93. Let ABC be a triangle, and let the tangent points of the incircle with the sides BC, CA, AB be D, E, F, respectively. Let P, Q, R be the midpoints of BC, CA, AB, respectively. Let $PR \cap DE = K$ and $PQ \cap DF = L$. Prove that $\dfrac{\overline{BI}}{\overline{CI}} = \dfrac{\overline{KE}}{\overline{LF}}$.

Problem 94. Let $ABCD$ be a cyclic quadrilateral. Prove that the intersection of the $A-$Simson line of $\triangle BCD$ with the $B-$Simson line of $\triangle ACD$ is collinear with C and the orthocenter of $\triangle ABD$.

Problem 95 (China MO 1997)**.** Let $ABCD$ be a cyclic quadrilateral. Let $AB \cap CD = P$ and $AD \cap BC = Q$. Let the tangents from Q meet the circumcircle of $ABCD$ at E and F. Prove that P, E and F are collinear.

Problem 96 (Serbia 2016, Drzavno)**.** Let $\triangle ABC$ be an acute-angled triangle with $\overline{AB} < \overline{AC}$. Let D be the midpoint of BC and let p be the reflection of the line AD with respect to the angle bisector of $\angle BAC$. If P is the foot of the perpendicular from C to the line p, prove that $\angle APD = \angle BAC$.

Problem 97 (IMO 2014/4)**.** Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumcircle of $\triangle ABC$.

Problem 98 (Vietnam TST 2001)**.** Two circles intersect at A and B and a common tangent intersects the circles at P and Q. Let the tangents at P and Q to the circumcircle of $\triangle APQ$ intersect at S and let H be the reflection of B across the line PQ. Prove that the points A, S and H are collinear.

Problem 99 (JBMO 2013, Stefan Lozanovski)**.** Let ABC be an acute-angled triangle and let O be the center of its circumcircle ω. Let D be a point on the line segment BC such that $\angle BAD = \angle CAO$. Let E be the second point of intersection of ω and the line AD. If M, N and P are the midpoints of the line segments BE, OD and AC, respectively, show that the points M, N and P are collinear.

Problem 100 (JBMO 2014)**.** Consider an acute triangle ABC of area S. Let $CD \perp AB$ ($D \in AB$), $DM \perp AC$ ($M \in AC$) and $DN \perp BC$ ($N \in BC$). Denote by H_1 and H_2 the orthocenters of the triangles MNC and MND, respectively. Find the area of the quadrilateral AH_1BH_2 in terms of S.

Problem 101 (JBMO 2015)**.** Let ABC be an acute triangle. The lines ℓ_1 and ℓ_2 are perpendicular to AB at the points A and B, respectively. The perpendicular lines from the midpoint M of AB to the lines AC and BC intersect ℓ_1 and ℓ_2 at the points E and F, respectively. If D is the intersection point of the lines EF and MC, prove that $\angle ADB = \angle EMF$.

Problem 102 (JBMO 2016)**.** A trapezoid $ABCD$ ($AB \parallel CD, \overline{AB} > \overline{CD}$) is circumscribed about a circle. The incircle of triangle ABC touches the lines AB and AC at the points M and N, respectively. Prove that the incenter of the trapezoid $ABCD$ lies on the line MN.

Problem 103 (JBMO 2017)**.** Let ABC be an acute triangle such that $\overline{AB} \neq \overline{AC}$, with circumcircle Γ and circumcenter O. Let M be the midpoint of BC and D be a point on Γ such that $AD \perp BC$. Let T be a point such that $BDCT$ is a parallelogram and Q a point on the same side of BC as A such that $\angle BQM = \angle BCA$ and $\angle CQM = \angle CBA$. Let the line AO intersect Γ at $E \neq A$ and let the circumcircle of $\triangle ETQ$ intersect Γ at point $X \neq E$. Prove that the points A, M and X are collinear.

Problem 104 (EMC 2012, Senior)**.** Let ABC be an acute triangle with orthocenter H. AH and CH intersect BC and AB in points A_1 and C_1, respectively. BH and A_1C_1 meet at point D. Let P be the midpoint of the segment BH. Let D' be the reflection of the point D with respect to AC. Prove that the quadrilateral $APCD'$ is cyclic.

Problem 105 (EGMO 2018)**.** Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q. Prove that $\angle ABP = \angle QBC$.

Problem 106 (BMO 2010)**.** Let ABC be an acute triangle with orthocentre H, and let M be the midpoint of AC. The point C_1 on AB is such that CC_1 is an altitude of the triangle ABC. Let H_1 be the reflection of H in AB. The orthogonal projections of C_1 onto the lines AH_1, AC and BC are P, Q and R, respectively. Let M_1 be the point such that the circumcentre of triangle PQR is the midpoint of the segment MM_1. Prove that M_1 lies on the segment BH_1.

Problem 107 (Macedonia MO 2009, corrected)**.** Let I be the incenter of $\triangle ABC$. Points K and L are the intersection points of the circumcircles of $\triangle BIC$ and $\triangle AIC$ with the bisectors of $\angle BAC$ and $\angle ABC$, respectively ($K, L \neq I$). Let P be the midpoint of the segment KL. Let M be the reflection of I with respect to P and N be the reflection of I with respect to C. Prove that the points K, L, M and N lie on the same circle.

Problem 108 (Macedonia MO 2016, modified)**.** Let K be the midpoint of a given segment AB. Let C be a point that doesnt lie on the line AB. Let N be the intersection of AC and the line passing through B and the midpoint of CK. Let U be the intersection point of AB and the line passing through C and $L$the midpoint of BN. Prove that the ratio of the areas of $\triangle CNL$ and $\triangle BUL$ does not depend on the choice of the point C.

Problem 109 (APMO 2015)**.** Let ABC be a triangle, and let D be a point on the side BC. A line through D intersects side AB at X and ray AC at Y. The circumcircle of $\triangle BXD$ intersects the circumcircle ω of $\triangle ABC$ again at point Z distinct from point B. The lines ZD and ZY intersect ω again at V and W respectively. Prove that $\overline{AB} = \overline{VW}$.

Problem 110 (Rioplatense MO 2013, Level 3)**.** Let ABC be an acute-angled scalene triangle, with centroid G and orthocenter H. The circle with diameter AH cuts the circumcircle of BHC at A', distinct from H. Analogously define B' and C'. Prove that A', B', C' and G are concyclic.

Problem 111 (RMM 2018)**.** Let $ABCD$ be a cyclic quadrilateral an let P be a point on the side AB. The diagonal AC meets the segment DP at Q. The line through P parallel to CD meets the extension of the side CB beyond B at K. The line through Q parallel to BD meets the extension of the side CB beyond B at L. Prove that the circumcircles of triangles BKP and CLQ are tangent.

Problem 112 (JBMO Shortlist 2015)**.** Let ABC be an acute triangle with $\overline{AB} \neq \overline{AC}$. The incircle ω of the triangle touches the sides BC, CA and AB at points D, E and F, respectively. The perpendicular line erected at C onto BC meets EF at M and similarly, the perpendicular line erected at B onto BC meets EF at N. The line DM meets ω again at P and the line DN meets ω again at Q. Prove that $\overline{DP} = \overline{DQ}$.

Problem 113 (MEMO 2016, Team)**.** Let ABC be an acute triangle, $\overline{AB} \neq \overline{AC}$, and let O be its circumcenter. Line AO meets the circumcircle of $\triangle ABC$ again in D, and the line BC in E. The circumcircle of $\triangle CDE$ meets the line CA again in P. The lines PE and AB intersect in Q. Line passing through O parallel to the line PE intersects the $A-$altitude of $\triangle ABC$ in F. Prove that $\overline{FP} = \overline{FQ}$.

Problem 114 (Hong Kong TST 2003)**.** In the triangle ABC, the point M is the midpoint of AC and D is a point on AB. BM and CD meet at O, with $\overline{AB} = \overline{CO}$. Prove that AB is perpendicular to BC if and only if $ADOM$ is a cyclic quadrilateral.

Problem 115 (Stefan Lozanovski)**.** Let D be a point on the side AB in $\triangle ABC$. Let F be a point on CD such that $\overline{AB} = \overline{CF}$. The circumcircle of $\triangle BDF$ intersects BC again at E. Assume that A, F and E are collinear. If $\angle ACB = \gamma$, find the measurement of $\angle ADC$.

Problem 116 (Stefan Lozanovski)**.** Let AA' be a median in the triangle ABC. Let D be a point on AA' and let the intersection of BD and AC be E. The circumcircle of $\triangle BCE$ intersects AB again at F. If C, D and F are collinear, prove that $\triangle ABC$ is isosceles.

Problem 117 (EGMO 2016)**.** Two circles ω_1 and ω_2 with equal radii intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

Problem 118 (APMO 2016)**.** We say that a triangle ABC is *great* if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC. Prove that triangle ABC is great if and only if $\angle A = 90°$ and $\overline{AB} = \overline{AC}$.

Problem 119 (Serbia MO 2017)**.** Let $ABCD$ be a convex cyclic quadrilateral. The lines AD and BC intersect at E. Let M and N be points on the sides AD and BC, respectively, such that $\dfrac{\overline{AM}}{\overline{MD}} = \dfrac{\overline{BN}}{\overline{NC}}$. The circumcircles of $\triangle EMN$ and $ABCD$ intersect at X and Y. Prove that the lines AB, CD and XY are concurrent or all parallel.

Problem 120 (Russia 2003)**.** Let ABC be a triangle with $AB \neq AC$. Point E is such that $\overline{AE} = \overline{BE}$ and $BE \perp BC$. Point F is such that $\overline{AF} = \overline{CF}$ and $CF \perp BC$. Let D be the point on line BC such that AD is tangent to the circumcircle of $\triangle ABC$. Prove that D, E and F are collinear.

Problem 121 (BMO 2009)**.** Let MN be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. The lines BN and CM meet at point P. The circumcircles of $\triangle BMP$ and $\triangle CNP$ meet at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$.

Problem 122 (JBMO 2002)**.** ABC is an isosceles triangle ($\overline{CA} = \overline{CB}$). Let P be a point on the arc $\widehat{AB}$ on (ABC) that doesn't contain C. Let D be the foot of the perpendicular from C to PB. Show that $\overline{PA} + \overline{PB} = 2 \cdot \overline{PD}$.

Problem 123 (RMM 2015/4)**.** Let ABC be a triangle, and let D be the point where the incircle touches the side BC. Let I_B and I_C be the incentres of the triangles $\triangle ABD$ and $\triangle ACD$, respectively. Prove that the circumcentre of $\triangle AI_BI_C$ lies on the angle bisector of $\angle BAC$.

Problem 124 (Stefan Lozanovski)**.** Let S be a point on AC, such that BS is an angle bisector in the triangle ABC. Let O_1 and O_2 be the circumcenters of $\triangle ABS$ and $\triangle BSC$, respectively. The median AM in $\triangle ABC$ intersects BS at X. Prove that the lines AB, O_1O_2 and CX are concurrent.

Problem 125 (USA JMO 2014)**.** Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides BC, CA, AB and let E, F be the tangent points of γ with CA, AB, respectively. Let U, V be the intersections of line EF with lines MN, MP, respectively, and let X be the midpoint of arc BAC of Γ. Prove that XI bisects UV.

Problem 126 (BMO 2017)**.** Consider an acute-angled triangle ABC with $\overline{AB} < \overline{AC}$ and let ω be its circumscribed circle. Let t_B and t_C be the tangents to the circle ω at points B and C, respectively, and let L be their intersection. The line through B parallel to AC intersects t_C at D. The line through C parallel to AB intersects t_B at E. The circumcircle of the triangle BDC intersects AC in T, where T is located between A and C. The circumcircle of the triangle BEC intersects the line AB in S, where B is located between S and A. Prove that ST, AL, and BC are concurrent.

Problem 127 (China MO 1992). A convex quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC, BD of $ABCD$ meet at P. Circumcircles of $\triangle ABP$ and $\triangle CDP$ meet at P and Q (O, P and Q are pairwise distinct). Show that $\angle OQP = 90°$.

Problem 128 (IMO 1983/2). Let A be one of the two points of intersection of the circles ω_1 and ω_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches ω_1 at P_1 and ω_2 at P_2, while the other touches ω_1 at Q_1 and ω_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

Problem 129 (Macedonia MO 2008). ABC is an acute-angled triangle ($\overline{AB} \neq \overline{BC}$). Let AV and AD be the angle bisector and the altitude from vertex A, respectively. The circumcircle of $\triangle AVD$ intersects CA and AB in points E and F, respectively. Prove that AD, BE and CF are concurrent.

Problem 130 (Russia MO 1999). A circle through vertices A and B of triangle ABC meets the side BC again at D. A circle through B and C meets the side AB at E and the first circle again at F. Prove that if the points A, E, D and C lie on a circle with center O then $\angle BFO = 90°$.

Problem 131 (BMO 2018). A quadrilateral $ABCD$ is inscribed in a circle k where $\overline{AB} > \overline{CD}$ and AB is not parallel to CD. Point M is the intersection of diagonals AC and BD, and the perpendicular from M to AB intersects the segment AB at a point E.If EM bisects the angle CED prove that AB is diameter of k.

Problem 132 (Israel MO 1995). Let ω be a semicircle with diameter PQ. A circle k is tangent internally to ω and to the segment PQ at C. Let AB be the tangent to k perpendicular to PQ, with A on ω and B on the segment CQ. Show that AC bisects $\angle PAB$.

Problem 133 (IMO 2007/4). In triangle ABC, the bisector of $\angle BCA$ intersects the circumcircle of $\triangle ABC$ again at R, the perpendicular bisector of BC at P and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles $\triangle RPK$ and $\triangle RQL$ have the same area.

Problem 134 (IMO 2003/4). Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of the perpendiculars from D to the lines BC, CA and AB, respectively. Show that $\overline{PQ} = \overline{QR}$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

Problem 135 (IMO Shortlist 2012/G2). Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F and G are concyclic.

Problem 136 (MEMO 2014, Team). Let the incircle k of the triangle ABC touch its side BC at D. Let the line AD intersect k at $L \neq D$ and denote the excentre of ABC opposite to A by K. Let M and N be the midpoints of BC and KM, respectively. Prove that the points B, C, N, and L are concyclic.

Problem 137 (Serbia MO 2017)**.** Let k be the circumcircle of $\triangle ABC$ and let k_a be its $A-$excircle. Let the two common tangents of k and k_a intersect BC at P and Q. Prove that $\angle PAB = \angle CAQ$.

Problem 138 (Serbia MO 2018)**.** Let $\triangle ABC$ be a triangle with incenter I. Points P and Q are chosen on segments BI and CI such that $2\angle PAQ = \angle BAC$. If D is the tangent point of the incircle with BC, prove that $\angle PDQ = 90°$.

Problem 139 (IMO Shortlist 2002/G7)**.** The incircle of a triangle ABC touches its side BC at K. Let M be the midpoint of the altitude AD of triangle ABC. The line MK meets the incircle of triangle ABC at a point N (apart from K). Show that the circumcircle of triangle BNC is tangent to the incircle of triangle ABC at the point N.

Problem 140. The incircle of $\triangle ABC$ touches BC, CA and AB at D, E and F, respectively. The A-excircle touches BC, CA and AB at D_1, E_1 and F_1, respectively. Let $K = FD \cap E_1D_1$. Prove that $AK \perp BC$.

Problem 141 (IMO Shortlist 2007/G3)**.** The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that the line CD separates the points P and Q and $\angle AQD = \angle CQB$. Prove that $\angle BQP = \angle DAQ$.

Problem 142 (Vietnam TST 2003)**.** Given a triangle ABC. Let O be the circumcenter of this triangle ABC. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has center I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent. (When the point O concides with I, we consider the line OI as an arbitrary line passing through O.)

Problem 143 (IMO Shortlist 2004/G7)**.** For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

Problem 144 (Peru TST for IberoAmerican MO 2014)**.** The incircle of $\triangle ABC$, centered at I, touches AC and AB at E and F, respectively. Let H be the foot of the altitude from A and let $R = CI \cap AH$ and $Q = BI \cap AH$. Prove that the midpoint of AH lies on the radical axis of (REC) and (QFB).

Problem 145 (Serbia MO 2016)**.** Let ABC be a triangle and I its incenter. Let M be the midpoint of BC and D the tangent point of the incircle and BC. Prove that the perpendiculars from M, D and A to AI, IM and BC, respectively are concurrent.

Problem 146 (Iran TST 2009)**.** In triangle ABC, D, E and F are the points of tangency of the incircle (centered at I) to BC, CA and AB respectively. Let M be the foot of the perpendicular from D to EF. P is on DM such that $\overline{DP} = \overline{MP}$. If H is the orthocenter of $\triangle BIC$, prove that PH bisects EF.

Problem 147 (Turkey MO 2015)**.** In a cyclic quadrilateral $ABCD$ whose largest interior angle is D, lines BC and AD intersect at point E, while lines AB and CD intersect at point F. A point P is taken in the interior of quadrilateral $ABCD$ for which $\angle EPD = \angle FPD = \angle BAD$. O is the circumcenter of quadrilateral $ABCD$. Line FO intersects the lines AD, EP, BC at X, Q, Y, respectively. If $\angle DQX = \angle CQY$, show that $\angle AEB = 90°$.

Problem 148 (USA TST 2015)**.** Let ABC be a triangle ($\overline{AB} < \overline{AC}$) with incenter I whose incircle is tangent to BC, CA, AB at D, E, F, respectively. Denote by M the midpoint of BC. Let Q be a point on the incircle such that $\angle AQD = 90°$. Let P be the point inside the triangle on line AI for which $\overline{MD} = \overline{MP}$. Prove that $\angle PQE = 90°$.

Problem 149 (Serbia MO 2016)**.** Let ABC be a triangle and O be its circumcenter. A line tangent to the circumcircle of the triangle BOC intersects sides AB at D and AC at E. Let A' be the image of A with respect to the line DE. Prove that the circumcircle of $\triangle A'DE$ is tangent to the circumcircle of $\triangle ABC$.

Problem 150 (IMO 2008/6)**.** Let $ABCD$ be a convex quadrilateral ($\overline{BA} \neq \overline{BC}$). Denote the incircles of triangles $\triangle ABC$ and $\triangle ADC$ by ω_1 and ω_2, respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

In case you solved all the problems from a previous version, here is a list of the new problems added in each of the later versions:

Problems added in **v1.1**:
1, 2, 3, 4, 5, 6, 9, 11, 13, 14, 19, 20, 21, 22, 23, 24, 25, 26, 28, 30, 34, 55, 68, 79, 81, 83, 85, 88, 91, 92, 93, 102, 109, 111, 118, 119, 125, 126, 131, 137, 138, 143, 147 and 148.

Problems added in **v1.2**:
18, 29, 31, 36, 37, 39, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 58, 60, 63, 64, 65, 66, 67, 70, 73, 74, 75, 76, 77, 78, 82, 84, 87, 94, 101, 103, 105, 108, 110, 112, 113, 117, 136, 139, 140, 142, 144 and 146.

Appendix A

Contests Abbreviations

Here is a list of all the abbreviated mathematical contests mentioned in this book.

Abbreviation	Full Name
MO	Mathematical Olympiad
JMO	Junior Mathematical Olympiad
IMO	International Mathematical Olympiad
TST	Team Selection Test (unless otherwise noted, for the IMO team)
BMO	Balkan Mathematical Olympiad
JBMO	Junior Balkan Mathematical Olympiad
APMO	Asian Pacific Mathematics Olympiad
EGMO	European Girls' Mathematical Olympiad
MEMO	Middle European Mathematical Olympiad
RMM	Romanian Master of Mathematics
EMC	European Mathematical Cup
AIME	American Invitational Mathematics Examination

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