DISCLAIMER: *If these strategies don't work it's not my fault! I tried my best, and not every single thing here will work for everyone. You can tell me if something is blatantly wrong, but don't say "Oh I tried checking and I got a 1, so this document deserves a 0/10".*

Note: *If you would like to suggest something, PM me at freeman66 on AoPS*

AIME Strategies

Dylan Yu

0 Table of Contents

1 Abstract

The purpose of this document is to explain some strategies you can employ during the AIME, and maybe it'll help you! If it doesn't, find your own strategy, and if you know any better strategies, let me know!

It should be noted that there has been a "revolution" in problem writing for the AIME, where a lot of the older techniques / types of problems don't show up as much, meaning the problems are a) harder and b) need more creativity. Thus, it is a good idea to expose yourself to all types of problems. Of course, this does not mean these new problems don't rely on the traditional techniques - those are still very important.

2 Preparation

"By failing to prepare, you are preparing to fail." -Benjamin Franklin

Preparing for the AIME is perhaps the greatest deciding factor on how well one does. Sure, some people might have a good testing day or get lucky and have seen one of the problems before. However, there is always a clear difference between the person who put in 1000 hours and the person who only put in 100. Preparing for the AIME is crucial, and to succeed and pass the AIME requires hard work to be put in. Not only must you spend a ton of time preparing, this time must also be used efficiently in order to be well spent.

2.1 Groups of People

Preparing for the AIME depends on what type of person you are and at what stage you are in regards to your math ability. There are typically three types of people: those who just barely qualified for the AIME, those who are aiming to qualify for the USA(J)MO for their first time but isn't sure they can qualify, and those who have already qualified for the USA(J)MO and know that they can for sure qualify. For the first group, don't take the AIME too seriously. If this is only your first time qualifying, then you should congratulate yourself for that success and try your best on the AIME, but don't be sad if you could only solve a couple of the problems. However, you should still practice. Try to be able to consistently solve the first 5 problems of any AIME - those are typically in the range of AMC10/12 as well. On the day of the test, relax and just have fun. The second group is probably the more common group to be in. For the second group, preparing for the AIME is critical. There are some people who skip the second group altogether and go straight to the olympiads as soon as they qualify for the AIME. You want to be those kinds of people. However, many more people find themselves qualifying for AIME year after year, barely missing the USA(J)MO cutoff. This group is probably one of the most stressful to be in, so you want to leave this group as quickly as possible. To do this, you need to start preparing for the AIME the first or second year you qualify. Most people fall into the trap of only trying to improve their AIME score. Do not just do this. Instead, you must also make sure that you will score 135+ on the AMC10/12. Raising your AIME score by 1 or 2 points is incredibly tough, but making sure that you do not silly on the AMC10/12 can bring your index up by 12-18 points. In addition to this, you should spend a year increasing your AIME score to a consistent 10+ on mocks. Only after doing this will you have a chance at qualifying for the USA(J)MO. Lastly, if you are in the third group, do not worry too much. Do a couple AMC10/12's before the AMC10/12 so you don't get a terrible score, and do a couple AIME's before the real contest. Other than that, you should not be stressing out too much, instead focus your efforts on olympiad.

The rest of this section will be focusing on the preparation that the second group should do, as the first and third group do not need to focus too much on their preparation.

2.2 Over the summer and Fall (3-9 months before the AIME)

This time should be spent on improving your math knowledge. You have all the free time in summer and you shouldn't have too much homework in the fall, so make sure to use this time well. Attend camps, read books, attend local lectures, etc. You shouldn't be doing problem sets under time pressure at this point. Instead, try to focus more on theory and learn things that might even be more advanced than what is on the AIME. You should not be doing too many AIME problems instead you should be learning new theorems and their proofs.

2.3 1-3 months before the AIME

The time to transition to this new style of preparation is announced by the beginning of winter break. Start mixing AIME problem sets along with your books and other materials. Try to do one timed AIME every week, and make sure that you understand how to solve all the problems. For each problem that you got wrong, do not just read the solution, and don't even just say that you understood it. Instead, you must understand the solution and be able to write the solution 3 days after reading the solution. This will test if you really understood the solution or if you just read it without thinking. In addition to these AIME problem sets, also mix in computational problems from other contests such as HMMT, PUMAC, CMIMC, ARML, and the like. During this time is the AMC10/12. Two weeks before the AMC10/12, start doing timed AMC10/12 problem sets. Make sure you get a good score on the AMC10/12. Getting a good score practically determines what you can do on the AIME, so you must make sure you get the score that you wished for - preferable 130+.

2.4 1-4 weeks before the AIME

By this point, you should start only doing problems. I recommend doing at least 1 timed AIME practice set once every three days, as well as reading their solutions and rewriting the solutions. This is where every single minute of your day should be spent on doing problems in order to raise your AIME score by as much as possible. The number of practice AIME's that you do during this time period have the greatest effect on your AIME score. In addition to doing past AIME's, I also recommend mocks, which are typically harder than a normal AIME. You can find many high quality ones inside of the Aops Mock Contest Forum on Aops.

2.5 1 week before the AIME

You should try to squeeze in at least one more practice AIME in the last week. You should also start to plan your strategy, ex) What would I go for, a P13 Geo or a P10 Algebra? Planning ahead is crucial because choosing the right problems to do can either give you that one extra point or waste an hour of your time. You should also start to review the problems that you've gotten wrong before and try to resolve them. In addition, review concepts and theorems. If you realize that you have no idea on one of the concepts, then try to cram it in as quickly as possible. The last couple days shouldn't be spent much on math. Instead, relax, maybe review some more theorems, but by this point you should just rest to ensure you have the best possible AIME score.

2.6 Resources

Below are some common resources that I've used for my own AIME preparation. This is by no means an exhaustive list, so if you find a good resource that is not on the list, feel free to use it. Also, by Google searching and chasing threads you can find a lot of good handouts. Study by **individual handouts** (or books if you like those), **not by randomly asking people for resources.**

Volume 1: A lot of people ignore this book as they think it's "too easy" for AIME. However, it's great for teaching you the basics and is sort of like a fundamental

Volume 2: This will teach you much more advanced topics than Volume 1, and it is good for the middle and last 5 AIME problems

Introductory Aops Books: Great for building a foundation - will help you get through the first and middle 5.

This Handout: It has all the AIME problems sorted into categories by subject and then by difficulty.

Intermediate Algebra: Covers pretty much all the AIME algebra topics

Intermediate Counting and Probability: Covers pretty much all the AIME counting and probability topics

BOGTRO's AIME study guide: Very useful study guide for the AIME

Markan's AIME syllabus: Good for review

CMC series: These are a series of mock contests. Their mock AMC's are probably way too hard, but their AIME is fine.

2015 Mock AIME I by djmathman and Binomial-Theorem: Quite a nice, high quality mock AIME

djmathman's AIME practice set: **Link here**

Olympiad Number Theory by Justin Stevens: Probably a bit too advanced, but it covers many topics that would be on the AIME (Many Olympiad NT topics are useful on AIME, for ex.: LTE)

djmathman's 100 Geometry problems handout: Also probably a bit too advanced, but it contains some nice Geometry Problems

Cjquines' Geometry Handout: Most of the theorems on here aren't too useful but reviewing them can't hurt. Link here

3 Time Management

At the AMC we have 3 minutes per problem. At the AIME we have 12. That means the timing strategies need to be different. Indeed, the AMC is so fast-paced that it is reasonable to save time by not reading a problem twice. If you read it, you either solve it or skip it and go on. The student who is not trying to achieve a perfect score can decide in advance not to read those final, highly-difficult problems.

Strategy 1. When deciding between checking your work or doing another problem, and it's crunch time, I recommend using the following algorithm:

1. If less than 15 minutes left
   a. Check your work
2. If less than 30 minutes left
   a. Try the problem for 15 minutes
3. Anything else
   a. Split around ⅔ ratio: in other words, spend two-thirds of the time checking

Note: the ratio depends on preference, and if you get the gut feeling this isn't going to work out, abort and go check your other problems.

4 Problem Management

What problems should we do, and which should we abort? This will tell you what to do!

Strategy 2. When reading through the problems, I recommend using the following algorithm:

1. If solution is immediately found
   a. If solution is a long bash
      i. Do the bash now, put priority on checking (see Checking section)

    b. If solution is short and sweet
        i. (Obviously) Do it now, put checking priority towards end of list
2. If solution is not immediately found but an idea is found
    a. Write down your idea next to the problem, skip
3. If solution is not immediately found with no idea
    a. Skip
4. If step 1 is no longer possible
    a. Proceed to step 2, then step 3, etc.

A few notes: *What if I don't immediately see the solution, but can probably do it?* Duh. do it. *Won't it take a lot of time to put the idea next to the problem?* Then just memorize it.

For the AIME it is not expensive, in relative terms of time, to read all the problems. You can read the problems and choose the most promising ones to start with, knowing that if there is time they can always come back to other problems.

Strategy 3. Streamlined solutions are the best, meaning you just go with the flow, and this reduces the amount of work you actually have to do. This only works if you've seen a problem like this before, because the more you try a certain idea or method the easier it is for you to do it. Another idea is to blindly apply a formula you've learned before. This might allow you to learn something very important regarding the problem.

5 Checking

I've noticed that the accuracy level of students who take the AIME for the first time drops significantly. It seems that they are so used to multiple choice questions that they rely on multiple choices as a confirmation that they are right. So when someone solves a problem, they compare their answer to the given choices and if the answer is on the list they assume that the answer must be correct. Their pattern is broken when there are no choices. So they arrive at an answer and since there is no way to check it against choices, they just submit it. Because of this lack of confirmation, checking their answer in other ways becomes more important.

Strategy 4. When checking the problems, I recommend using the following algorithm:

1. If it was bash
    a. Check longest bash first

i. Methods of checking: check your work, redo the problem, or try to find another way to do the problem

2. If it was not bash

   a. Check longest solution first

As a general rule of thumb, check the longer solutions, because those are more prone to mistakes. That being said, if a #14 seems like a one-liner, you (probably) have an error in your thinking. Reread the problem!

6 Guessing

Guessing at the AMC is very profitable if you can exclude three choices out of the given five. Guessing for the AIME is a waste of time because the answers are integers between 000 and 999. So the probability of a random guess is one in a thousand. Actually, this is not quite right, because the problem writers are human and it is much easier to write a problem with an answer of 10 than one with an answer of 731. But the AIME designers are trying very hard to make answers that are randomly distributed. So the probability of a random guess is not one in a thousand, but it is very close. You can improve your chances by an intelligent guess. For example, you might notice that the answer must be divisible by 10. But guessing is still a waste of time. Thinking about a problem for two minutes in order to increase the probability of a correct guess to one in a 100 means that your expected gain is 1/200 points per minute. Which is usually much less than the gain for checking your answers. You can play the guessing game if you have exhausted your other options.

I remember someone collected (outdated data) that stated the most popular answer choice was 025. This is not likely to be correct, but feel free to put it!

Strategy 5. This is more like “guessing the formula”, but whatever. If you forget part of a formula, try to think of what it basically looked like. Plug in values to make that formula match the data you put in (this works only if you know what goes into the formula and what will come out). Just assume the formula, and if you have time (which you probably won’t) try to prove it.

Strategy 6. If you see a pattern, use **engineer’s induction**! However, note that sometimes the pattern a) grows too big, and it’s hard to find the first few terms, and b) the pattern sometimes isn’t what it seems, so this can be solved by just trying more terms.

Strategy 7. (Engineer's Induction) If a pattern seems reasonable, assume it's true! Sometimes, even if it doesn't seem reasonable, still assume it's true if you are running low on time. Usually, the pattern is right.

Strategy 8. (Collinearity and Concurrency) Draw a few cases you know, and if it works out assume collinear / concurrent. If you running low on time, just assume it without proof, or just try the equilateral case (which usually doesn't work out because everything is nice in an equilateral triangle).

Note: Strategy 8 applies to all areas: if you are running low on time, just look at your diagram and assume a bunch of stuff.

Strategy 9. Draw a VERY DETAILED diagram. This means make lengths reasonable - if the answer is an integer you could very easily guess the answer. This might lead you to solve a #15 before a #3 (most of the time, the AIME writers are smart and add in square roots and fractions, but it couldn't hurt to estimate the answer so you can double check your real answer).

7 Fakesolving

Fakesolving is the idea of getting the correct answer while not necessarily proving or justifying that your answer is correct. It can be used in many cases where the problem statement doesn't specify certain things, and is incredibly useful to get one or two extra points. Obviously, this strategy only works for computational competitions and not olympiads.

Strategy 10. If you have a couple degrees of freedom in a geometry problem, use that to your advantage. Assume the central shape is something like an equilateral triangle or that a point is on top of another point. Doing this can sometimes trivialize the problem.

Strategy 11. Using a compass and a ruler to draw a very accurate diagram, then use a ruler to measure out the length. This can sometimes give you an extra point.

Strategy 12. Using expected value on problems where they ask you to count something can be incredibly useful, especially when paired with symmetry. For example, look at 2017 AIME II Problem 12.

8 Strategies and Techniques for Specific Topics

I wanted to include this, but I wasn't sure where: *look out for the definition to fact problems*. What I mean by this is, if the problem states *"A point is 3 inches away from 3 other points…"*, you immediately know this point is the circumcenter of the other 3 points, and the circumradius has a length of 3 inches! So knowing the definition extremely well is important. For example, when you draw a good figure, and you see a point that looks like the orthocenter, it probably is. Knowing various properties of orthocenters can help prove that.

8.1 Algebra

Variables. Use common notation, like x for distance, t for speed, etc.

Equations. If you get a high-degree polynomial, and it is not symmetric abort. Similarly, a lot of square roots and cube roots could be bad. There is probably a simpler way then, or perhaps $x = 1$ or $x = -1$ will work. If you square an equation, make sure that you check your work, because there is the plus-minus deal to worry about. Basically, check *non-reversible* steps.

Factorizations. The more factorizations you know the better. This is also useful in Number Theory, when you check if a polynomial can be prime. Also, prime factorization is extremely important.

Substitutions. Use substitutions in two ways: a) to reduce a multi-variable equation, or b) to simplify something inside a root. For example, in

$$\sqrt{4n+5},$$

If you make the substitution $n = t^2 - t - 1$,

$$\sqrt{4t^2 - 4t + 1} = 2t - 1,$$

Reducing the square root.

Sequences and Series. Know arithmetic, geometric, arithmetico-geometric (https://artofproblemsolving.com/wiki/index.php?title=Arithmetico-geometric_series) , telescoping, and periodic sequences / series. Also, telescoping series can sometimes come in the form of polynomials (e.g. 2016 AIME I Problem 11: https://artofproblemsolving.com/wiki/index.php/2016_AIME_I_Problems/Problem_11)

Rate Problems. Only equation necessary is $d = rt$. Sometimes, even relativity helps solve the problem (i.e. assume one object isn't moving).

Polynomials. Vieta's kills most. Transformations like $x^n P(z)$ where $z = \frac{1}{x}$ sometimes works, because it reverses all the coefficients.
Logarithms. Knowing all the rules destroys the problems.
Trigonometry. Knowing all the rules really helps. For the problems that require you to sum a bunch of them that form a pattern, use the sum to product / product to sum rules. Also, know that

$$\tan\frac{\pi}{4} = \frac{\tan\theta + \tan(\frac{\pi}{4} - \theta)}{1 - \tan\theta \cdot \tan(\frac{\pi}{4} - \theta)},$$

$$(\tan\theta + 1)(\tan(\frac{\pi}{4} - \theta) + 1) = 2.$$

Inequalities. Rarely shows up, but know that squares are nonnegative. Also, occasionally they will throw a troll out like 2016 AIME II Problem 15 (https://artofproblemsolving.com/wiki/index.php/2016_AIME_II_Problems/Problem_15), where the Cauchy-Schwarz Inequality is actually in the equality case. Sometimes, this helps with maximization / minimization problems, but just know: AM-GM, Cauchy-Schwarz, and Rearrangement and you'll (probably) be fine.
Roots of Unity. This also helps with the *summing a bunch of trig values in a pattern* problems. This usually works for equations of the form $\omega^n - 1$.
Functional Equations. On the AIME, it is likely the only functional equation problems that will show up are the ones where you simply plug in values after another in a pattern. These problems are of the following form:

$$a(f(x)) + b(f(\frac{k}{x})) = g(x),$$

Where a, b, k are constants, and $f(x), g(x)$ are functions. $g(x)$ is given. The solution to this type of problem is:

$$f(x) = \frac{a(g(x)) - b(g(\frac{k}{x}))}{a^2 - b^2}.$$

In addition try to guess the function, which can trivialize the problem sometimes.
Newton Sums. Know them. They're useful sometimes and are not too bad to memorize.
Complex Numbers. This happens more often on geometry problems, but know how to handle them and their basic properties, such as $z\overline{z} = |z|^2$ and other basics.

Lagrange Interpolation. Given a set of points, it can help find the smallest degree polynomial that passes through these points. A precursor to this is if you have a group of points right next to each other, by continuously taking the difference you can find the very next term. So for those questions that have $f(i) = x_i$ for $i = 1, 2, 3, 4, \ldots, 10$, you can find $f(11)$.

The *uvw* Method. By writing things in terms of $u = x + y + z, v = xy + yz + zx, w = xyz,$ you can solve many problems. For example, in the following set of equations,

$$x + y + z = 10,$$
$$x^2 + y^2 + z^2 = 10,$$
$$x^3 + y^3 + z^3,$$

You can derive both v and w (u is given). You can also use the aforementioned Newton's Sums.

8.2 Geometry

Auxiliary Lines. This could be a parallel or perpendicular line, or just connecting two previously unconnected points.

Power of a Point. Know it and know how to use it

Similarity and Congruency. Self-explanatory. This is extremely useful when you have to find the area of two objects that are similar, and instead of finding the area you find two corresponding side lengths and just square it. Works with volumes too, but instead cube it.

Area / Perimeter / Surface Area / Volume Formulas. Memorize them. A weird one is the donut, and that may be useful. You can use calculus to derive them, but that's hard during the AIME. It is important to know a lot for triangles and quadrilaterals, especially cyclic quadrilaterals.

Triangles. Know a bunch of properties of them, including Incenter-Excenter, various concurrencies, and collinear points. In my opinion, most late AIME geometry problems (i.e. #11 - #15) are just JMO problems but with nice numbers. Trig Ceva's, Law of Sines, Law of Cosines, and other trig related formulas are great to use on the AIME - they come in handy a lot. Also, know the Kimberling Centers: https://faculty.evansville.edu/ck6/encyclopedia/ETC.html

Quadrilaterals. Know cyclic and tangential quadrilateral properties. Those are especially useful.

Bashing. Geometry has various bashing techniques: coordinate bashing, complex bashing, and barycentric bashing. The last one is hard, so just know mass points (a kindergarten version of barycentric coordinates). A

note on coordinate bashing: there are many techniques for this, including Shoelace, Pick's Theorem, distance from a point to a line, etc. There is also trig bashing, which involves using angles and trig functions to solve the problem (duh).
Coord Bashing. Use it when given a lot of intersections, perpendicular and parallel lines, or ratios. Also useful when you need to find the area of a figure. Shoelace is your best friend, along with choosing the right origin.
Trig Bashing. Law of sines is incredibly useful and can relate angles to lengths. Use it when you can find a lot of angles and their sine and cosine values.
Complex Bashing. Incredibly useful when you have 60, 90 degree angles. Often used for infinite path walking types of problems.

Note: As said before, doing JMO problems allows you to see the harder AIME questions in a new light and apply some quick techniques to finish off the problem. So do them.

A Short List

1. Projective Geometry
2. Radical Axis
3. Symmedians

Transformations. This includes noticing symmetry, rotations, reflections, translations, etc. An important one is the cut-and-paste method, where you cut up an object and move it around - this preserves area but also simplifies the object.
3D Figures. These problems are hated because you have a 2D space to draw a 3D figure, making it hard to visualize. However, by simply knowing a list of formulas (e.g. for tetrahedrons, look through this: https://en.wikipedia.org/wiki/Tetrahedron), you can bash through the problem without even drawing the figure! Of course, a picture is still better for weird cases. In fact, taking the right cross section usually trivializes these problems.
Pythagorean + De Gua's theorems. Memorize them.
Path over 3-D objects. Learn how to unfold 3-d objects so you can find the shortest path
Heron's Shortest Path Problem Technique. Try to understand this technique and know how to use reflections to get the shortest path.
Ptolemy's Theorem. Appears quite often, knowing the formula is usually enough

Stewart's Theorem. Just memorize the formula to find cevian lengths without having to go through a trig bash

Menelaus/Ceva. Useful for finding ratios of sides and is also useful on olympiads. Use it when you are given a lot of ratios.

Ratio Lemma. Good when you are given ratios of the sine of angles. Can sometimes devolve into a Law of Sines bash, and if you find two it can devolve into length bash.

Pitot's Theorem. Gives you lengths and is sometimes useful.

Moving Points. If SOMEHOW a #15 has this, search it up on AoPS and you will get results for this.

8.3 Combinatorics

Combinatorial Arguments. This includes committee forming and block walking.

Simplification. Try doing a simpler problem by removing a constraint, then adding in the constraint later. For example, if you have something in a circle, do it in a line, then transform it back into a circle.

Combinations and Permutations. Know them. They are the foundation of every AIME combinatorics problem.

Symmetry. (*cough cough one of this year's events with states on AMC.*) If something works the same way as another thing but it's reflected, just multiply by 2 and move on. Pretty self-explanatory, and there are so many examples I can't list them all here.

Stars and Bars. Know it.

The Classic 3. (Casework, Constructive, Complementary Counting) These are basically the ways to count, so this is important.

PIE. Principle of Inclusion Exclusion is very important, especially for case work.

Binary. It's possible to transfer some problems to binary notation, where as long as something is yes or no, on or off, etc. (i.e. two options), you can write it as 1s and 0s.

Invariants. If something doesn't change, take full advantage! This is where JMO techniques help.

Bijection. Be able to see them and create them between problems to make the problem easier to solve.

Events with States. If you are ever doing path walking, or considering things that return to its original position, use this. Take advantage of symmetry to simplify the equations.

Recursion. Know how to form and solve recursions, and when to use recursion and when to use casework.
Expected Value. Linearity of Expectation is super powerful - use it.
Catalan. Know when the catalan numbers appear in which problems and know the formula: $C_n = \frac{1}{n+1}\binom{2n}{n}$. Also be able to form bijections between problems involving catalan.
Fibonacci. Know when fibonacci numbers appear (ex. Stair walking problem, rabbits problem, ...)
Roots of Unity Filter. Read it on BOGTRO's AIME List, it helps for solving the summation of combinations.
Generating Functions. Again, read on BOGTRO. Generating functions can even be used on imaginary numbers.
Geometric Probability. Using this, you can transfer the problem to a geometry problem, you can simply just solve for the ratio of areas or volumes.
Combinatorial Identities. In the 2020 AIME 1, Vandermonde's showed up, so you should know that. Also: Pascal's Identity, the summation of a row in Pascal's triangle, etc. A quick Google search should give you plenty of results. Something important is Generalized Vandermonde's:
https://en.wikipedia.org/wiki/Vandermonde%27s_identity#Generalized_Vandermonde's_identity
Chromatic Polynomials. For some reason, these are super useful. They are good for coloring questions regarding cycles, especially the ones where you can't color two adjacent the same:
https://en.wikipedia.org/wiki/Chromatic_polynomial
(Ex: 2016 AIME II Problem 12 is a one line with chromatic polynomials, but also read the other solutions:
https://artofproblemsolving.com/wiki/index.php/2016_AIME_II_Problems/Problem_12)
Hook Length. From Mathcounts to IMO, these are super useful. It might be a good idea to actually understand how it works, but the formula itself requires no insight: https://en.wikipedia.org/wiki/Hook_length_formula
(Note: It wasn't allowed on the IMO because it trivialized the problem, but on AIME it's fair game.)
Symmetric Groups. Apparently these are useful for those questions where the function maps to itself, and you have to find how many ways

this can be done, but I'm not sure:
https://en.wikipedia.org/wiki/Symmetric_group
(A note on those problems: sometimes, just doing case work on the amount of pairs (x,x) is better)

8.4 Number Theory

Divisibility. This involves taking the prime factorization, or just knowing the rules you learn from doing Number Sense problems (basically fast paced problems like "does 17 divide 51?").
Simon's Favorite Factoring Trick. Useful in many scenarios, one of which is finding the number of solutions to $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$.
Difference of Squares. If you need to find out if a polynomial has integer roots, look at the discriminant, and do difference of squares.
Modular Arithmetic. So many techniques here, I'm not going to list them all - but you have to know them. Know how to do inverses and fractions in modular arithmetic - this might require Legendre's. Also know which mods are good for what, for example 7 and 9 are good for cubes while 8 is good for squares, etc.
Bounding. To show there are no more solutions, just place something between two consecutive integers and say that because of this, there are no more integer solutions.
Fibonacci. This relates to geometry too, especially for pentagons and decagons. Know Binet's theorem, and various properties of Fibonacci numbers (like the summation of all the odd Fibonacci numbers up to a certain point).
Euclidean Algorithm and Fermat's Little Theorem. The phi function is super important. Know these two.
Diophantine Equations. Know Bezout's and know how to find solutions to these equations.
Chicken Mcnuggets. Just memorize it and know when to apply it
Lifting the Exponent. See BOGTRO - it's good for finding how many powers of one prime divide a number.
Cyclotomic Polynomials. These are kind of advanced, but it could still help for the integer polynomial type questions.
Zsigmondy's Theorem. Useful for finding if $p|a^n-b^n$. I'm not sure exactly how it works either, but a quick Google search should give you some good results.

Quadratic Residues. This is actually pretty useful for the ones with weird constraints. Even just trying small residues is a good idea (e.g. 2016 AIME I Problem 12: https://artofproblemsolving.com/wiki/index.php/2016_AIME_I_Problems/Problem_12 (this is not a quadratic residues problem, but it is a "try small residues" problem.)

9 Final Notes / Strategies

Be calm. When in doubt, bash. Go with your gut. Eat a good breakfast. There are two strategies, explained in this video:
https://www.youtube.com/watch?v=x01gndt4qOM
Good luck, have fun, and eat your vegetables! Also, drink a lot of water / boba.

Jeffrey's Advice: Be organized. I recommend having at least 30 sheets of scratch paper and at least 5 pencils, along with your favorite snacks, water, and any necessary medication (inhaler, cough drops, ...). Give each problem its own sheet of paper and write neatly so you can easily check your work. For more difficult problems (particularly those in the last 5), try using 2 or 3 sheets of paper each. Even for the earlier ones, where you might only need ⅓ a sheet of paper per problem, still use an entire sheet, front and back, for that one problem. It makes checking your work and finding your ideas for problems so much easier. If you like, you might also want to leave little notes around your work so that you can easily come back to what you were thinking about for that problem.

10 Sources

1. https://blog.tanyakhovanova.com/2012/02/approaching-the-aime-strategically/
2. BOGTRO's AIME List
3. http://markan.net/aime/syllabus.pdf
4. https://www.youtube.com/watch?v=x01gndt4qOM
5. http://blog.tanyakhovanova.com/2010/01/how-to-boost-your-guessing-accuracy-during-tests/

Properties of 2020

freeman66

February 2, 2020

Abstract

This document provides some properties of 2020. If you have a good one, feel free to let me know! Also, note that the properties in each section go from easy to interesting to overkill. Note that the ones that look weird and/or are overkill are likely from OEIS, so I'm not weird OEIS is.

1 Algebra

1.1 Operations on 2020

2020^2 is 4080400 and 2020^3 is 8242408000. $\sqrt{2020}$ is 44.9444101085 and $\sqrt[3]{2020} = 12.6410686485$. $\ln 2020$ is 7.6108527903953 and $\log_{10} 2020$ is 3.3053513694466. $\sin 2020$ is 0.044061988343923, $\cos 2020$ is -0.99902879897588, and $\tan 2020$ is -0.044104822993183. $\frac{1}{2020}$ has period 4.

1.2 Time

2020 seconds is equal to 33 minutes, 40 seconds. 2020 is a leap year. 2020 has 53 Wednesdays and Thursdays. The previous year this occurs was 1992, and the next is 2048.

1.3 Triangular Numbers

2020 is equal to $\sum_{k=3}^{22} T_k$, where T_k is the kth triangular number. Also, $\mathrm{T}_5 + T_{30} + T_{55} = 2020$.

1.4 Heptagonal Numbers

The alternating sum of the first 40 heptagonal numbers is 2020.

1.5 Sequences

A sequence that starts with 4 then adds 1, then 2, then 3, and so on contains 2020. Also, in the product $f(x) = \Pi_{k=1}^{\infty}(1 + 4 \cdot x^k)$, x^{14} has coefficient 2020. Also, in the expansion of $\frac{\sum_{i=1}^{\infty} \frac{x^{i^2}}{1-x^{i^2}}}{\Pi_{i=1}^{\infty} \frac{x^{i^2}}{1-x^{i^2}}}$, the coefficient of x^{51} is 2020. Also, if $a_0 = 4$, and for $n > 0$, $a_n = a_{n-1} + 2^n - 3$, then $a_{10} = 2020$. In the expansion of $\Pi_{k=1}^{7} \frac{1}{1-x^{2k-1}}$, the coefficient of $x^5 3$ is 2020. Also, in the expansion of $\frac{1}{1-x^3-x^4-x^5-x^6}$, the coefficient of $x^2 8$ is 2020. Also, in the expansion of $\Pi_{k=1}^{\infty} \frac{1}{1+x^k}^{k-1}$, the coefficient of $x^3 5$ is -2020.

1.6 Trigonometry

The generating function of $\frac{\tan x \cdot \sin(\tanh x)}{2}$ has -2020 as the coefficient of x^8.

1.7 Doublets

2020 is the juxtaposition of two identical strings of 20.

1.8 Sperner Systems

The number of monotone Boolean functions of 5 variables with 4 mincuts is 2020.

1.9 Ramanujan's Theta Function

The 52nd term of the expansion of $\frac{1}{f(-x,-x^5)}$ is 2020.

1.10 Hermite Polynomials

The numerator of $H(3, \frac{1}{13})$ is -2020, where H is a Hermite polynomial.

1.11 Pythagorean Approximations

The denominators of the Pythagorean Approximations of 2 to 5 is 2020.

2 Geometry

2.1 Outer Vecten Triangle

The outer vecten triangle of an integer triangle has area 2020. (Note: I need help finding which integer triangle this is!)

2.2 Convolution Triangle of A Fibbonacci-Like Sequence

The convolution triangle of a_n, where $a_n = 2 \cdot (a_{n-1} + a_{n-2})$ has element 2020 in the 40th position.

3 Combinatorics

3.1 Catalan Numbers

32_{10} is 2020_C, where base C is base Catalan Numbers.

3.2 Bernoulli Numbers

B_{2020} has denominator 330, along with $20, 340, 1220, 1420, 2020, 2980, 3340, 3940, 4460, 4540, 4580, 5140, 5660, 5780$.

3.3 Partitions

The number of strict integer partitions of 58 not containing 1 or any part whose prime indices all belong to the partition is 2020. Also, the number of integer partitions of 44 whose augmented differences are weakly decreasing is 2020. Also, the number of overcubic partions of 12 is 2020. Also, the number of partions of 32 such that the sum of the distinct odd parts is greater than $\frac{n}{2}$ is 2020. Also, the number of partitions of 53 into Heegner numbers is 2020. Also, the number of ordered partitions (i.e. compositions) of 24 into 4 relatively prime parts is 2020. Also, the number of compositions of 21 such that no two adjacent parts are equal is 2020.

3.4 Grids

The number of lines going through exactly 8 points in a 4040 grid of points is 2020.

3.5 Tesselations

The number of 1-sided polycairos with 9 cells is 2020.

3.6 Symmetric Group

The number of character table entries of the symmetric group S_{12} greater than 0 is 2020. (Note that the idea of S_n is not completely useless to competition - it describes a set of numbers mapping to themselves, which is an idea featured in the last few combinatorics questions of HMMT!)

3.7 Numbers in a Range

The number of even numbers in the range $10n$ to $10n + 9$ where $n = 40$ is 2020.

3.8 Latin Rectangles

The number of 36 Latin rectangles in which the second row contains 2 cycles with the same order of elements is 2020.

3.9 Graphs

The number of nodes in the 12th level of the Euclid-Mullin Graph starting with 1 is 2020.

3.10 Conway's Game of Life

The number of active (ON, black) cells in 39th stage of growth of two-dimensional cellular automaton defined by "Rule 118", based on the 5-celled von Neumann neighborhood is 2020.

3.11 Fredholm-Rueppel Inverse Triangle

In the Fredholm-Rueppel Inverse Triangle, the 42nd and 43rd row sums are both -2020.

4 Number Theory

4.1 Prime Factorization

The prime factorization of 2020 is

$$2^2 \cdot 5 \cdot 101.$$

From here we can deduce 2020 is composite and even.

4.2 Number of Primes

The exist 3 distinct prime factors: $2, 5, 101$.

4.3 Primes

Prime numbers close to 2020 include: 2011, 2017, 2027, 2029. The 2020th prime is 17573. Also, the sum of the primitive roots of the 52nd prime (239) is 2020. Also, the sequence in which any two consecutive digits in the sequence sum up to a prime (starting with 1) contains 2020. Also, the 15th prime multiplied by the 14th prime is 1 more than 2020 (15th prime is 47 and 14th prime is 43). Also, $6 \cdot 2020^2 - 1$ and $6 \cdot 2020^2 + 1$ are twin primes. Also, in the set of the number of positive integers z such that $\pi(x^3 + y^3) = \pi(z^3)$ for some $0 < x \leq y \leq z$, 2020 is included. Also, the number of primes under 26^3 is 2020. Also, $k = 2020$ is the smallest integer k such that $\frac{n!-k}{n}$ is prime, where $n = 20$. Also, for the set of numbers n such that $n^{1024} + (n+1)^{1024}$, 2020 is included in that set. 202099 is prime. Also, the average of the 2020th prime number and the 2021st prime number is a perfect cube. Also, the 2020th prime minus 2020 is a brilliant number (15553).

4.4 Number of Divisors

There are 12 divisors: $1, 2, 4, 5, 10, 20, 101, 202, 404, 505, 1010, 2020$.

4.5 Sum of Divisors

The sum of the divisors of n is 4284. The sum of the proper divisors (i.e. not including 2020) is 2264.

4.6 Product of Divisors

The product of the divisors of 2020 is 2020^6.

4.7 Abundance

2020 is an abundant number with abundance 244. Also, $2020, 2024, 2026$ are an abundant triple (all are abundant).

4.8 Bases

2020 in binary is 11111100100_2. 2020 in hexadecimal is $7E4_{16}$. 2020 is a palindrome in base 21: $4C4_{21}$. It is a pernicuous number because it has 7 ones in binary (pernicuous means it has an odd number of ones in binary). 2020_4 is a self-descriptive number, where a self-descriptive number is an integer m that in a given base b is b digits long in which each digit d at position n (the most significant digit being at position 0 and the least significant at position b - 1) counts how many instances of digit n are in m.

4.9 Digits

2020 has 4 digits. The sum of digits of 2020 is 4, which makes it a Harshad Number (divisible by the sum of its digits).

4.10 Squares and Sum of Squares

2020 has 2 representations as a sum of 2 squares: $2020 = 16^2 + 42^2 = 24^2 + 38^2$. Also, note that $2020 = 17^2 + 19^2 + 23^2 + 29^2$, where these are the 7th, 8th, 9th, and 10th prime numbers respectively. The smallest square less than 2020 is 1936. It can be written as the sum of 10 consecutive even squares: $4^2 + 6^2 + \cdots + 20^2 + 22^2 = 2020$. Also, $45, 2020)$ is a lattice point of $y = x^2 - 5$.

4.11 Factorial

$2020! = 2^{2013}3^{1005}5^{503}7^{334}\cdots$.

4.12 Exponentiation

$2020^3|k^{2020^2}-1$ for $k=3,9,11$. 2020 is equal to $2^{11}-28$, and 2020 divides 91^4-1 (note that this also bleeds into divisibility). Also, $2020^3|3^{2020^2}-1$. Also, $2020^3|11^{2020^2}-1$. Also, $\lfloor 21^{\frac{5}{2}} \rfloor = 2020$. $2020, 2020^2, 2020^3$ all only use even digits.

4.13 Modulos

$2020 \pmod m$ is $0,1,0,0,4,4,4,4$ for $m=1,2,3,4,5,6,7,8,9$ respectively. Also, $1616^k \equiv 1616 \pmod{2020}$ for all $k \geq 1$.

4.14 Fibbonacci

The 8-step Fibbonacci sequence, $a_n = \sum_{k=1}^{7} a_{n-k}$, has 19th term 2020. Also, in the 3-step Fibbonacci sequence (a.k.a Tribonacci), $a_n = a_{n-1}+a_{n-2}+a_{n-3}$, a_{2020} is a prime.

4.15 Roman Numerals

2020 is MMXX in Roman Numerals.

4.16 Totient Function

The number of positive integers less than 2020 relatively prime to 2020 is $\phi 2020 = 800$. The set of record values for $\sigma(m)+\phi m)$ includes 2020, where record values are when the sum $\sigma(m)+\phi m)$ is greater than any previous values.

4.17 Palindrome / Reverse Numbers

Adding 2020 to its palindrome 202 gives you the palindrome 2222. It is a plaindrome (non-decreasing) in base 11 and nialpdrome (nonincreasing) in base 14. Also, the divisors of 2020 reversed (i.e. 12 goes to 21) multiplied is 2020.

4.18 Euler Transform

The Euler Transform of the powers of 5 includes the term 2020.

4.19 Niven Number

2020 is a Super Niven Number, which means it is divisible by the sum of any subset of its nonzero digits.

4.20 Undulating Number

2020 is a undulating numbers, which it has the form $ABABA\ldots$.

4.21 Gapful Number

2020 is a gapful number because it is divisible by its first and last digit.

4.22 Zumkeller Number

2020 is a Zumkeller Number because its divisors can be partitioned into two sets with the same sum 2142.

4.23 Iban Number

2020 is an iban number because when written as two thousand twenty it does not contain an "i".

4.24 Self-Sliding Number

2020 is a self-sliding number because it can each digit d by d places either to the left or the right.

4.25 Collatz Conjecture

It takes the number 2020 exactly 64 steps to reach 1.

4.26 Abelian Squares

2020 is an abelian square, where an abelian square is a string of length 2n where the last n symbols form a permutation of the first n symbols

4.27 Continued Fractions

The set of the denominators of the continued fractions of $\sqrt{102}$ and $\sqrt{918}$ includes 2020.

4.28 Other / Weird Representations

2020 can be written as $n \cdot (5n+1)$, where $n = 20$. Also, 2020 can be represented as $4n \cdot (4n^2+1)$, where $n = 5$. Also, 2020 can be written as $\frac{10n^2+4n+(1-(-1)^n)}{8}$, where $n = 40$. Also, 2020 is expressible as $x^4 + y^2$, and $x^2 + 24y$ is an integer (in this case $x = 4, y = 42$). Also, 2020 is expressible as $\frac{n^3+9n^2+26n}{6}$, where $n = 20$. Also, 2020 can be written as $81n^2 - n$, where $n = 5$. Also, 2020 can be written as $25n^2 - 5$, where $n = 9$. Also, π, e, ϕ have the same digit in the 2020th decimal place. Also, 2020 can be written as $\lfloor \frac{8^n}{7^n} \rfloor$, where $n = 57$. The string 4,4 occurs in 2020 but not 2019 when using base 8. The string 8,4 occurs in 2020 but not 2019 when written in base 9.

Unorthodox Problems

freeman66

Computational Problems

Problem C1. Compute $0.1 + 0.02 + 0.003 + \dots$.
Problem C2. 1st grader Nairit knows only the number 1. This means he can only repeatedly write down 1s, creating numbers such as $1, 11, 111, \cdots$; somehow, he manages to write down the smallest number of this form divisible by 2019. Let x be the number of 1s he wrote down. Find the remainder of x when divided by 1000.
Problem C3. Let a number be imajinary if it can be expressed as aj, where a is a real number, and $j = |\sqrt{2i}|$, where $i = \sqrt{-1}$. Let a number be khanplex if it can expressed as $-aj + bk$, where a, b are real number constants and $k = a + bi$ is complex. If a number x is imajinary and a number y is khanplex, $xy + x + y = zj$, and there exists only one value of $z = \dfrac{m}{n}$, find $m + n$.
Problem C4. There are 108 distinct heptominos, which are figures made out of seven unit squares that are connected to at least one other unit square by an edge. How many ways are there to arrange / connect these 108 heptominos such that they form a square?
Problem C5. There exists a natural number n whose square and cube contain exactly one of each of the numerals from 0 to 9. Find n.
Problem C6. In a group of males, there are exactly 2 father-son relationships, 2 uncle-nephew relationships, a grandfather-grandson relationship, and one elder and younger brother. What is the least number of people in this group? For example: If there were exactly 1 father and 1 son, it would be 2, because we would just need a father and a son.
Problem C7. If

$$\begin{aligned}
&x^2 + y^2 + z^2 + t^2 = 50,\\
&y^2 + t^2 - x^2 - z^2 = 24,\\
&xz = yt,\\
&x + z + t = y,
\end{aligned}$$

Find x, y, z, t where $x, y, z, t \in \mathbb{R}$.

Proof Problems

Problem P1. Each point A in the plane is assigned a real number $f(A)$. Given $f(M) = f(A) + f(B) + f(C)$, where M is the centroid of $\triangle ABC$, prove that $f(A) = 0$ for all points A.

Problem P2. One glass contains 5 spoons of milk, and the other glass contains 5 spoons of tea. A spoon of milk was taken from the second glass and put in the first, then mixed thoroughly. Next, a spoon of tea (with milk) was poured back into the second glass. Is there more milk in the first glass or more tea in the second glass? Will the answer change after 10 such transfusions?

Problem P3. Given

$$a_n = an + b_{n=1}^{\infty},$$

$$a_n \cap F_n = \emptyset,$$

Find the minimum value of a and b, where F_n is the Fibonacci sequence.

Problem P4. Prove the following quantity diverges:

$$\frac{2}{3 - \frac{2}{3 - \frac{2}{\cdots}}}.$$

Problem P5. Compute

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{\ldots}}}.$$

Problem P6. For what value of x is the following maximum?

$$x^{x^{x^{\cdots}}}$$

Problem P7. Find n-digit positive integers A, B, C where

$$A + B = C.$$

MAO Diophantine Equations

Dylan Yu

October 20, 2020

1 Definitions

Here we introduce some important notation and ideas that we will use throughout the handout.

Definition 1 (Diophantine Equation)
A ***diophantine equation*** is an equation that can be solved over the integers.

For example, $a + b = 32$, where a, b are integers, is a diophantine equation.

Definition 2 ($\mathbb{Z}$)
If $a \in \mathbb{Z}$, then a is an integer.

Furthermore, $\mathbb{Z}^-$ is the set of negative integers, $\mathbb{Z}^+$ is the set of positive integers, $\mathbb{Z}^{0+}$ is the set of nonnegative integers, and $\mathbb{Z}^{0-}$ is the set of nonpositive integers.

2 Modular Arithmetic

When we say "$a \equiv b \pmod{m}$" (this is read as "a is congruent to b mod m"), we mean that when we add or subtract a with some integer number of m's, we will get b. For example, $27 \equiv 2 \pmod{5}$ because if we subtract 5 5's from 27, we get 2. We can also say that $a \equiv b \pmod{m}$ if $a \div m$ and $b \div m$ have the same remainder. Now let's note a few important properties we will use in solving diophantines:

1. **Parity**. Taking odd numbers in mod 2 are always 1, and even numbers are always 0.

2. **Checking Squares**. In mod 3, squares are either 0 or 1. In mod 4, squares are also either 0 or 1.

3. **Checking Cubes**. In mod 4, cubes are either 0, 1, or 3.

There are more properties, but they are easily derived (just check all the possibilities).

Example 3 (Folklore)
Prove that if $x \in \mathbb{Z}$, $x^2 \equiv 3 \pmod{4}$ has no solutions.

Solution. Note that x is either $0, 1, 2$, or 3 in mod 4. Let's make a chart:

$x \pmod 4$	$x^2 \pmod 4$
0	0
1	1
2	0
3	1

Thus, in mod 4, squares are either 0 or 1 mod 4. This means x^2 can never be 3 mod 4. □

Example 4 (Balkan MO)
Prove that the equation $x^5 - y^2 = 4$ has no solutions over the integers.

Solution. Note that x^5 is either $-1, 0$, or 1 mod 11 and y^2 is either $0, 1, 3, 4, 5$, or 9 mod 11. Thus, if we have the equation

$$x^5 - y^2 = 4 \pmod{11},$$

we realize that regardless of what we choose for the pair of mods from the list above, it will always never equal 4 (if you don't believe me, try it out!). Thus, there are no solutions. □

Remark 5
Mod 11 is a strange thing to do, but with practice it becomes more natural. This is why practice is necessary – it allows you to more accurately pinpoint which mod to apply.

3 Factoring

Sometimes we can just factor the equation. However, it is usually extremely disguised, so **if you see a strangely arranged equation with many terms, try factoring!**

Theorem 6 (SFFT)
Simon's Favoring Factoring Trick, abbreviated SFFT, states that $xy + ax + by + ab = (x + b)(y + a)$.

This isn't very special, but sometimes it is disguised.

Example 7
Find all integral solutions to $xy - x + y = 0$.

Solution. Note that this is equivalent to $x(y - 1) + y = 0$. If we subtract 1 from both sides, we get $x(y - 1) + y - 1 = -1$, so

$$(x + 1)(y - 1) = -1,$$

implying we have $x + 1 = 1$ and $y - 1 = -1$ or $x + 1 = -1$ or $y - 1 = 1$. Thus, the solutions for (x, y) are $\boxed{(0, 0)}$ or $\boxed{(-2, 2)}$. □

Example 8 (Titu)
Find all integral solutions to the equation

$$(x^2+1)(y^2+1)+2(x-y)(1-xy)=4(1+xy).$$

Solution. Let's expand (almost) everything:

$$x^2y^2+x^2+y^2+1+2(x-y)(1-xy)=4+4xy,$$

$$\begin{aligned} x^2y^2+x^2+y^2+1+2(x-y)(1-xy)&=4+4xy, \\ x^2y^2-2xy+1+x^2+y^2-2xy-2(x-y)(xy-1)&=4, \\ (xy-1)^2+(x-y)^2-2(x-y)(xy-1)&=4, \\ (xy-1-(x-y))^2&=4, \end{aligned}$$

implying $xy-x+y-1=2$ or -2. Note that $xy-x+y-1=(x+1)(y-1)$, which gives us solutions of $\boxed{(-3,2),(-2,3),(0,-1),(1,0)}$. □

Here is an important theorem to keep in mind while solving:

Theorem 9
Let x,y be positive integers and let $n=p_1^{e_1}p_2^{e_2}\dots p_k^{e_k}$ (in other words, its prime factorization). Then the equation

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$$

has $(2e_1+1)(2e_2+1)\dots(2e_k+1)$ solutions.

Knowing key factorizations is important. For example,

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

can help you solve problems of this nature quickly.

4 Inequalities

Sometimes, to show there are finite (or no) possibilities, we can use an inequality to bound the equation.

Theorem 10 (Trivial Inequality)
Squares are always greater than or equal to 0, i.e. $x^2\geq 0$ for all real x.

Example 11
Find all pairs (x,y) of integers such that

$$x^3+y^3=(x+y)^2.$$

Solution. Factoring the LHS (left hand side), we get

$$(x+y)(x^2-xy+y^2)=(x+y)^2,$$

so if $x+y\neq 0$, then

$$\begin{aligned}x^2-xy+y^2&=x+y,\\ x^2-xy+y^2-(x+y)&=0,\\ 2x^2-2xy+2y^2-2x-2y&=0,\\ x^2-2xy+y^2+x^2-2x+y^2-2y&=0,\\ (x-y)^2+(x-1)^2+(y-1)^2&=2,\end{aligned}$$

and by the Trivial Inequality, two of these squares are equal to 1 and one of them is equal to 0. We can easily solve for the solutions then: $\boxed{(0,1),(1,0),(1,2),(2,1),(2,2)}$. However, we said this is what happens if $x+y\neq 0$. That means when $x+y=0$, we can have the solutions $\boxed{(k,-k)}$, and they all suffice. □

Another strategy is assuming $x\geq y\geq z$ without loss of generality (abbreviated WLOG). This sometimes holds if the equation is symmetric.

Example 12 (UK MO)
Find all triples (x,y,z) of positive integers such that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)=2.$$

Solution. WLOG let $x\geq y\geq z$. This means that $\frac{1}{x}\leq\frac{1}{y}\leq\frac{1}{z}$, so

$$1+\frac{1}{x}\leq 1+\frac{1}{y}\leq 1+\frac{1}{z}.$$

Thus,

$$2=\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)\leq\left(1+\frac{1}{z}\right)^3,$$

implying

$$1+\frac{1}{z}\geq\sqrt[3]{2},$$

and solving this inequality gives us $z\leq 3$. Thus, we just test the possibilities where $z=1,2,$ or 3, giving us $\boxed{(7,6,2),(9,5,2),(15,4,2),(8,3,3),(5,4,3)}$ in any order. □

Remark 13
Notice how we assumed $x\geq y\geq z$, but then we have to convert back to the original problem. This meant that if we considered something different, like $y\geq z\geq x$, it would be the exact same problem, except the variables would be moved around. That's why we put "in any order" in the last sentence.

5 Problems

5.1 Modular Arithmetic

Problem 14
Prove that the equation

$$(x+1)^2 + (x+2)^2 + \ldots + (x+2001)^2 = y^2$$

is not solvable.

Problem 15 (Russian MO)
Find all pairs (p, q) of prime numbers such that

$$p^3 - q^5 = (p+q)^2.$$

Problem 16 (IMO 1982/4)
Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers x, y, then it has at least three such solutions. Prove that the equation has no integer solution when $n = 2891$.

Problem 17 (IMO 1990/3)
Determine all integers $n \geq 1$ such that $\frac{2^n+1}{n^2}$ is an integer.

5.2 Factoring

Problem 18
Let p, q be primes. Solve, in positive integers, the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{pq}.$$

Problem 19 (Indian MO)
Determine all nonnegative integral pairs (x, y) for which

$$(xy - 7)^2 = x^2 + y^2.$$

Problem 20 (Polish MO)
Solve the following equation in integers x, y:

$$x^2(y-1) + y^2(x-1) = 1.$$

Problem 21 (Romanian MO)
Find all pairs (x, y) of integers such that

$$x^6 + 3x^3 + 1 = y^4.$$

5.3 Inequalities

Problem 22 (Romanian MO)
Solve the following equation in positive integers x, y, z:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}.$$

Problem 23 (Romanian MO)
Determine all triples (x, y, z) of positive integers such that

$$(x+y)^2 + 3x + y + 1 = z^2.$$

Problem 24 (Australian MO)
Determine all pairs (x, y) of integers that satisfy the equation

$$(x+1)^4 - (x-1)^4 = y^3.$$

Problem 25 (Russian MO)
Find all integer solutions to the equation

$$(x^2 - y^2)^2 = 1 + 16y.$$

MAO Invariants

Dylan Yu

November 23, 2020

1 Introduction

1.1 Definitions

Invariant
An ***invariant*** is a property or quantity that does not change under certain operations

Monovariant
A ***semi-invariant*** or ***monovariant*** is a quantity that always increases or always decreases after the corresponding operation.

1.2 More Exposition

Classical examples of invariants are parity or algebraic expressions such as sums or products. Finding an invariant is a common idea in problems asking to prove that something cannot be achieved. Monovariants are also very efficient in showing that the corresponding process must stop after finitely many moves.

2 Classics

I'll skip over the "find the invariant and win" questions, since those just involve ***algebraic manipulation***. In other words, I'm skipping over to the main course.[1] In these problems we do one of three things (or a combination of them):

1. use algorithms, or
2. use modular arithmetic, or
3. use AM-GM.

Note AM-GM is for bounding.

Example 2.1 (ISL 1989)
A natural number is written in each square of an $m \times n$ chessboard. The allowed move is to add an integer k to each of two adjacent numbers in such a way that nonnegative numbers are obtained (two squares are adjacent if they share a common side). Find a

[1] Try the problems in the problem set if you would like to see examples of these.

necessary and sufficient condition for it to be possible for all the numbers to be zero after finitely many operations.

The following solution is by Pranav Sriram:

Solution. Note that in each move, we are adding the same number to 2 squares, one of which is white and one of which is black (if the chessboard is colored alternately black and white). If S_b and S_w denote the sum of numbers on black and white squares respectively, then $S_b - S_w$ is an invariant. Thus if all numbers are 0 at the end, $S_b - S_w = 0$ at the end and hence $S_b - S_w = 0$ in the beginning as well. Thus, this condition is necessary; now we prove that it is sufficient.

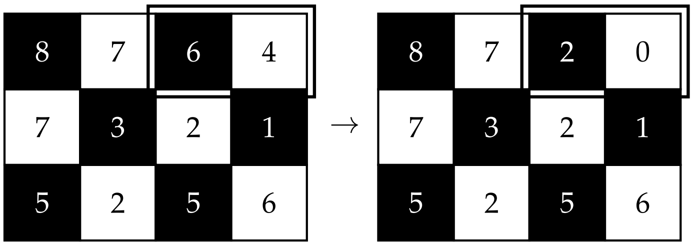


Figure 1: A move on the $m \times n$ board

Suppose a, b, c are numbers in cells A, B, C respectively, where A, B, C are cells such that A and C are both adjacent to B. If $a \le b$, we can add $(-a)$ to both a and b, turning a to 0. If $a \ge b$, then add $a - b$ to b and c. Then b becomes a, and now we can add $-a$ to both of them, making them 0. Thus we have an algorithm for reducing a positive integer to 0. Apply this in each row, making all but the last 2 entries 0. Now all columns have only zeroes except the last two. Now apply the algorithm starting from the top of these columns, until only two adjacent nonzero numbers remain. These two numbers must be equal since $S_b = S_w$. Thus we can reduce them to 0 as well. □

This implies an important heuristic: **use an invariant to show a condition is necessary, and use an algorithm to show it's sufficient**.

Example 2.2 (ELMO 1999)
Jimmy moves around on the lattice point. From points (x, y) he may move to any of the points $(y, x), (3x, -2y), (-2x, 3y), (x + 1, y + 4)$ and $(x - 1, y - 4)$ show that if he starts at $(0, 1)$ he can never get to $(0, 0)$.

Solution. Let us take mod 5 of $x + y$. Note that since

$$3x - 2y \equiv 3(x + y) \pmod 5,$$

$$-2x + 3y \equiv 3(x + y) \pmod 5,$$

$$x + 1 + y + 4 \equiv x + y \pmod 5,$$

$$x - 1 + y - 4 \equiv x + y \pmod 5,$$

the sum of the two coordinates is either constant or multiplied by 3. Thus, $(0, 0)$ cannot be achieved. □

Always try invariants like:

1. sums
2. products
3. AM/GM/HM

This helps motivate what modulo is necessary.

Example 2.3 (IMO Shortlist 2014)
The number 1 is written on each of 2^n sheets of paper. Each minute we are allowed to choose two distinct sheets, erase the two numbers a and b appearing on them and writing the number $a+b$ instead on both sheets. Prove that after $n2^{n-1}$ minutes the sum of the numbers on all sheets is at least 4^n.

Solution. Consider the product P of the number on the sheets. Say we choose a, b and replace them by $a+b, a+b$. The quotient between the product of all numbers after the operation and the one before the operation is $\frac{(a+b)^2}{ab}$, which by AM-GM is greater than or equal to 4. Thus, the product is at least $4^{n \cdot 2^{n-1}}$, and by AM-GM again, we get that the sum S is

$$\left(\frac{S}{2^n}\right)^{2^n} \geq P,$$

implying the desired result $S \geq 4^n$. □

Example 2.4
The numbers $1, 2, \ldots, 2008$ are written on a blackboard. Every second, Jimmy erases four numbers of the form $a, b, c, a+b+c$, and replaces them with the numbers $a+b, b+c, c+a$. Prove that this can continue for at most 10 minutes.

Solution. Note that $a+b+c+(a+b+c) = (a+b)+(b+c)+(c+a)$. Thus, for every operation he does, the sum is constant, but the number of numbers decreases by 1. Even more important,

$$a^2+b^2+c^2+(a+b+c)^2 = (a+b)^2+(b+c)^2+(c+a)^2,$$

implying the sum of squares is also invariant. Let $x_1, x_2, \ldots, x_n$ be on the blackboard. Then by Cauchy-Schwarz,

$$n(x_1^2+x_2^2\ldots+x_n^2) \geq (x_1+x_2+\ldots+x_n)^2.$$

Taking into account our two invariants, we obtain

$$n \geq \frac{(1+2+\ldots+2008)^2}{1^2+2^2+\ldots+2008^2} = 1506 + \frac{502}{1339},$$

implying this can take place at most $2008-1506 = 502$ times, which is less than 600 seconds, or 10 minutes. □

Example 2.5 (Saint Petersburg 2013)
There are 100 numbers from the interval $(0,1)$ on the board. Every minute we can replace two numbers a, b on the board with the roots of $x^2 - ax + b = 0$ (if it has two real roots). Prove that this process must stop at some moment.

Solution. Suppose this process is endless. There exist real number $N < 1$ such that all of 100 initial numbers are smaller than N. Since if $a, b < N$, we get

$$\frac{a + \sqrt{a^2 - 4b}}{2} < N.$$

So, all numbers on the board will always smaller than N. Denote by S and P the sum and product respectively of all numbers on the board. Let S_0 and P_0 be that of the initial 100 numbers. Each move gives us $S \to S - b$ and $P \to \frac{P}{a} > \frac{P}{N}$. After M moves, we get that $S < S_0$ and $P > \frac{P_0}{N^M}$. By AM-GM, we get $S \geq 100 \sqrt[100]{P}$. This gives contradiction for sufficiently large M, implying the desired result. $\square$

Fact 2.6. When sums and products are used in invariant/monovariant questions, it is a good idea to use AM-GM.

Now for a monovariant:

Example 2.7 (IMO Shortlist 2012)
Several positive integers are written in a row. Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

Solution. Clearly the maximum number on the board does not change, say it was M initially. Let $a_1, a_2, \ldots, a_n$ be the numbers on the board. I claim the quantity

$$S = a_1 + 2a_2 + \ldots + ka_k + \ldots + na_n$$

always increases. Say that at some moment $x > y$ with x to the left of y and x in position i. The difference between the new value of S and the old one is either

$$i(y + 1) + (i + 1)x - (ix + (i + 1)y) = x - y + i \geq 1$$

or

$$i(x - 1) + (i + 1)x - (ix + (i + 1)y) = -i + (i + 1)(x - y) \geq -i + i + 1 = 1,$$

proving the claim and the desired result. $\square$

3 Brutal Examples

Example 3.1 (Tournament of Towns 2016)
On a blackboard several polynomials of degree 37 are written, each of them having leading coefficient equal to 1 and all coefficients nonnegative.
It is allowed to erase any pair of polynomials f, g and replace it by another pair of polynomials f_1, g_1 of degree 37 with leading coefficients to 1 such that either $f_1 + g_1 = f + g$ or $f_1 g_1 = fg$.
Can we reach a blackboard on which all polynomials have 37 distinct positive roots?

Solution. Let's suppose that

$$(X^{37}+a_1X^{36}+\ldots+a-37)(X^{37}+b_1X^{36}+\ldots+b_{37})$$
$$=(X^{37}+c_1X^{36}+\ldots+c-37)(X^{37}+d_1X^{36}+\ldots+d_{37}).$$

Looking at the coefficient of X^{36+37} we obtain

$$a_1+b_1=c_1+d_1.$$

This also holds if

$$(X^{37}+a_1X^{36}+\ldots+a-37)+(X^{37}+b_1X^{36}+\ldots+b_{37})$$
$$=(X^{37}+c_1X^{36}+\ldots+c-37)+(X^{37}+d_1X^{36}+\ldots+d_{37}).$$

Thus, the sum of coefficients of X^{36} is invariant. Since initially all coefficients are nonnegative, at each step the sum of coefficients of X^{36} stays nonnegative. This implies that at least one polynomial, say $P(X)=X^{37}+a_1X^{36}+\ldots+a-37$, has $a_1\geq 0$ at every step. If $x_1,x_2,\ldots,x_{37}$ are the complex roots of P, then

$$x_1+x_2+\ldots+x_{37}=-a_1\leq 0.$$

Thus, the polynomial cannot have 37 positive roots, which implies the desired result. □

Fact 3.2. If we have more than one option as to what to turn the objects into (e.g. Tournament of Towns 2016), it is often good to find a way to find an invariant that works for all options.

The following is not necessarily hard, but realizing the weighting is nontrivial.

Example 3.3

The first quadrant is divided into unit squares. We are allowed to perform the following move: if the square (x,y) has a token, while $(x,y+1),(x+1,y)$ are empty, then we take the token on (x,y) and put a token on each of the other two squares. Initially, we have tokens on $(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)$. Can we clear these six squares by a sequence of moves?

Solution. Consider the sum of $\frac{1}{2^{x+y}}$ over all pairs (x,y) for which there is a token at (x,y). Thus,

$$\frac{1}{2^{x+(y+1)}}+\frac{1}{2^{(x+1)+y}}=\frac{1}{2^{x+y}},$$

implying the sum is invariant. Initially, the sum is

$$\frac{1}{2^2}+\frac{2}{2^3}+\frac{3}{2^4}=\frac{11}{16}.$$

Suppose that at some moment the six initial squares have no token. Then the sum is at most

$$\sum_{x,y\geq 1}\frac{1}{2^{x+y}}-\frac{11}{16}=1-\frac{11}{16}=\frac{5}{16},$$

which is a contradiction, proving the desired result. □

Fact 3.4. Sometimes ***weighting*** is important for invariant questions, especially grid problems.

In general, when there's symmetry, attempt to weight the objects.

Example 3.5 (Russia 2014)
The polynomials $X^3 - 3X^2 + 5$ and $X^2 - 4X$ are written on the blackboard. If the polynomials $f(X)$ and $g(x)$ are written on the blackboard, we are allowed to write down the polynomials $f(X) \pm g(X), f(X) \cdot g(X), f(g(X))$ and $c \cdot f(x)$, where c is an arbitrary real constant. Can we write a nonzero polynomial of form $X^n - 1$ after a finite number of steps?

Solution. Let $f(X) = a_0 + a_1 X + \ldots + a_n X^n$, then its derivative is

$$f'(x) = a_1 + 2a_2 X + \ldots + na_n X^{n-1}.$$

This satisfies all conditions, because

$$\begin{aligned}(f \pm g)' &= f' \pm g',\\ (cf)' &= cf',\\ (f \cdot g)' &= f' \cdot g + g' \cdot f,\\ (f \circ g)' &= (f' \circ g) \cdot g'.\end{aligned}$$

Thus, if f' and g' have a common root z, then z is a common root of $(f \pm g)', (cf)', (f \cdot g)'$ and $(f \circ g)'$. The derivatives of the initial polynomials are $3X^2 - 6X$ and $2X - 4$, and 2 is a common root. However, $(X^n - 1)' = nX^{n-1}$ does not have the root $X = 2$, implying we can never get $X^n - 1$. □

Example 3.6 (RMM Shortlist 2016)
Start with any finite list of distinct positive integers. We may replace any pair $n, n+1$ (not necessarily adjacent in the list) by the single integer $n-2$, now allowing negatives and repeats in the list. We may also replace any pair $n, n+4$ by $n-1$. We may repeat these operations as many times as we wish. What is the most negative integer which can appear in a list?

Solution. Let's look for an invariant of the form $\sum_{n \in \mathbb{L}} x^n$, where $\mathbb{L}$ is a subset of $\mathbb{Z}$. To have an invariant, we want

$$\begin{aligned}x^n + x^{n+1} &= x^{n-2},\\ x^n + x^{n+4} &= x^{n-1},\end{aligned}$$

for all n. This reduces to

$$\begin{aligned}x^2 + x^3 &= 1,\\ x^5 + x &= 1,\end{aligned}$$

which is easily solvable since they are secretly the same equation, because

$$x^5 + x - 1 = (x^3 + x^2 - 1)(x^2 - x + 1).$$

Thus, we choose x such that $x^3 + x^2 = 1$ and get $\sum_{n \in \mathbb{L}} x^n$ is constan.. Thus,

$$\sum_{n \in \mathbb{L}} x^n \leq \sum_{n \geq 1} x^n = \frac{x}{1-x} = x^{-4}.$$

This must be true for all steps, and since $0 < x < 1$, we know that $n > -4$. Working backwards from -3, we get eventually get $1, 2, 3, 4, 5$ implying $\boxed{-3}$ works. □

> *Remark 3.7.* The invariant in this problem is similar to the one in Conway's soldiers. The motivation behind this is ***recursion***, then transfer it to a ***characteristic polynomial***. Note that this is again a weighting problem.

Fact 3.8. For most invariant/monovariant questions, it is pretty easy to identify if the answer is yes or no (otherwise it wouldn't be a invariant/monovariant question!). The hard part is **proving** why your claim is true.

4 Problems

Let's eat a three course meal.

4.1 Appetizer

Problem 1. The cells of a 7×7 board are chess-painted (alternating colors) so that the corners are black. One is allowed to repaint any two adjacent cells to the opposite color. Is it possible to repaint the entire board white using such operations?

Problem 2. The numbers $1, 2, \ldots, 20$ are written on the board. One is allowed to erase any two numbers a and b and instead write the number $a + b - 1$. What number can remain on the board after 19 such operations?

Problem 3. Given a 1000-digit number with no zeroes, prove that from this number you can delete several (or none) last digits so that the resulting number is not a natural power less than 500 (a^1 is not considered a power).

Problem 4. The numbers 1 through 1000 are written on the board. One is allowed to erase any two numbers and and write the numbers ab and $a^2 + b^2$ instead. Is it possible with such operations to ensure that among the numbers written on the board, there are 700 at least that are the same?

Problem 5. Initially we have the numbers $\frac{49}{1}, \frac{49}{2}, \ldots, \frac{49}{97}$ on a board. A move consists in replacing two numbers, say a and b, with $2ab - a - b + 1$. After a series of moves, there is only one number left on the board. Find it!

4.2 Entree

Problem 6 (Russia 2008). A natural number is written on the blackboard. Whenever a number x is written, one can write either the number $2x + 1$ or $\frac{x}{x+2}$. At some point the number 2008 appears on the blackboard. Show that it was there from the beginning.

Problem 7 (Saint Petersburg 2020). The points $(1, 1)$, $(2, 3)$, $(4, 5)$ and $(999, 111)$ are marked in the coordinate system. If points (a, b) are marked then (b, a) and $(a - b, a + b)$ can be marked. If points (a, b) and (c, d) are marked then so can be $(ad + bc, 4ac - 4bd)$.

Can we, after some finite number of these steps, mark a point belonging to the line $y = 2x$?

Problem 8 (Tuymaada Junior 2018). The numbers $1, 2, 3, \ldots, 1024$ are written on a blackboard. They are divided into pairs. Then each pair is wiped off the board and non-negative difference of its numbers is written on the board instead. 512 numbers obtained in this way are divided into pairs and so on. One number remains on the blackboard after ten such operations. Determine all its possible values.

4.3 Dessert

Full yet?

Problem 9. The numbers $1, 2, \ldots, n$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers x and y, erases them, and writes the number $2x + 2y$ on the board. This continues until only one number remains. Prove that this number is at least $\frac{4}{9}n^3$.

Problem 10. Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x + y)^4$ on the blackboard. Show that after $n - 1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.

Problem 11 (USAMO 2019/5). Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

Problem 12 (Russia 2017). Initially a positive integer n is on the blackboard. Every minute we are allowed to take a number a on the blackboard, erase it and write instead all divisors of a except for a. After some time there are n^2 numbers on the blackboard. For which n is this possible?

Problem 13 (Iran RMM TST 2020). A 9×9 table is filled with zeroes. At every step we can either take a row, add 1 to every cell and shift it one unit to the right (the rightmost number in that row ends up in the leftmost position of the row) or take a column, subtract 1 from every number on that column and shift it one cell down (with the same convention as for rows). Can the table with the top right -1 and bottom left $+1$ and all other cells zero be reached?

MAΘ Diagram Perturbation

Dylan Yu

January 10, 2021

1 Introduction

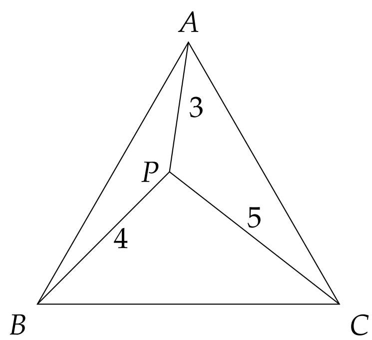
1.1 What is diagram perturbation?

The idea of ***diagram perturbation*** is to manipulate a diagram in a geometry problem in some way so that the result becomes easier to find. Let's try a few classic examples.

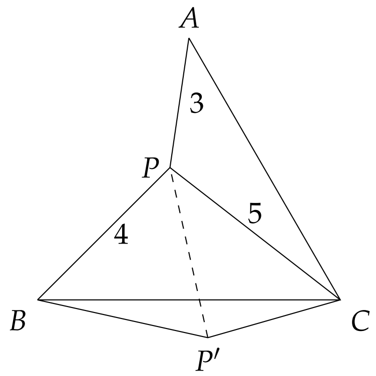
1.2 Rotations

Example 1.1
If a point P lies in an equilateral triangle ABC such that $AP = 3, BP = 4, CP = 5$, find the area of $\triangle ABC$.

Solution. Let's draw a figure:



Note that $3, 4, 5$ are special numbers, because they form the side lengths of a right triangle. Now we're going to rotate $\triangle APB$ around B such that A goes to C:



Let's angle chase. Note that $\angle ABP = \angle CBP'$ and $\angle APB + \angle CBP = 60^\circ$. Thus, $\angle PBP' = 60^\circ$, and combined with the fact $BP = BP' = 4$, we must have that $\triangle BPP'$ is an equilateral triangle. Furthermore, $CP' = 3$, implying $\triangle PP'C$ is a $3-4-5$ right triangle. Thus,

$$\angle APB = \angle CP'B = \angle BP'P + \angle CP'P = 60^\circ + 90^\circ = 150^\circ,$$

and we can use Law of Cosines on $\triangle APB$ to get

$$AB^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos 150^\circ = 25 + 12\sqrt{3},$$

implying

$$[ABC] = \frac{AB^2\sqrt{3}}{4} = \boxed{\frac{25\sqrt{3}+36}{4}}.$$

□

We used a few strategies here:

- rotations (***cutting and pasting***),
- angle chasing, and
- trigonometry.

We will focus on rotations and similar ideas in this handout – in other words, rotations are an example of diagram perturbation.

> *Remark 1.2.* You will see a common pattern throughout problems of this type:
> - do something smart with the figure (i.e. diagram perturbation),
> - find some angles, and
> - apply length bashing techniques (e.g. trigonometry) to finish the problem.

Let's try one more example.

1.3 Reflections

I don't remember exactly how this was stated, but it's a problem any math enthusiast has heard of.

Example 1.3 (Folklore)
A man is at point A and wants to go to point B. However, he must first go to a river to get water, which is effectively a straight line. Note that points A and B are on the same side of the river. If he must go from A to a point on the river then to B, what is the path he should take?

Solution. Let's reflect B across the river line to get B'. Then if the point on the river he goes to is P, we are trying to maximize $AP + BP$, but $BP = B'P$ since it is a reflection, so

$$AP + BP = AP + B'P.$$

But the shortest distance from A to B' in general is just the line segment AB', and in this case, P would be the intersection of AB' and the river line. Thus, we reflect $B'P$ back across the line, and this is the path we should take. □

Convince yourself this is true. With rotations and reflections explained, let's move on to some harder ideas.

1.4 Isn't this just transformations?

You might be asking yourself, "I've known about rotations, reflections, translations, and dilations since 6th grade. What's different here?"

This is a good question. There actually is **no difference**. However, I've chosen to call this *diagram perturbation* as opposed to *transformations* because we aren't just reflecting the whole object. Finding what pieces to perturb and what auxillary lines to draw is wildly harder than moving all the pieces. We're going to focus on drawing extra lines here.

2 Parallelograms

Parallelograms are effectively just reflecting a triangle across one of its midpoints. Let's take advantage of the angles formed.

Example 2.1
Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$ in triangle ABC. Prove $MN = \frac{1}{2}BC$ without using similar triangles.

Solution. Let L be the midpoint of BC. Then $MN \parallel LC$ and $NC \parallel ML$, implying $MNCL$ is a parallelogram. Thus, $MN = LC = LB$, and we're done. □

Example 2.2
Let M be the midpoint of $\overline{BC}$ in a triangle ABC. Given that $AM = 2$, $AB = 3$, $AC = 4$, find the area of ABC.

Solution. Reflect A across M to get A'. Then AA' and BC bisect each other, implying $ACA'B$ is a parallelogram. Furthermore, $AM = MA' = 2$, so $BA' = A'A = 4$, which means $\triangle AA'B$ is isosceles. We can easily find

$$[AA'B] = \frac{3\sqrt{55}}{4},$$

but we know that $[AA'B] = [AMB] + [MBA']$ and furthermore

$$[AMB] = [CMA] = [A'MC] = [BMA'],$$

implying

$$[ABC] = [AA'B] = \boxed{\frac{3\sqrt{55}}{4}}.$$

□

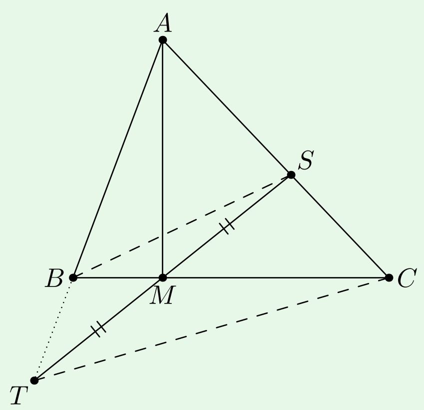
Example 2.3
A triangle ABC has medians of lengths m_a, m_b, m_c. Find the ratio of the area of the triangle formed by these medians to the area of triangle ABC.

Solution. Let l be the line through A parallel to BC, and let D, E, F be the midpoints of BC, CA, AB respectively. Furthermore, let A' be a point on l such that $AA' = EF$. We can easily prove through parallelograms that $\triangle A'CF$ is a triangle formed by the medians of $\triangle ABC$ (prove this yourself!). Thus, if we let S be one of the four equal areas formed by parallelogram $AA'EF$ and its diagonals, we have that $[AA'EF] = 4S$, and $[A'CF] = 6S$. Furthermore, since $[AEF] = 2S$, we have that $[ABC] = 8S$. Thus, the answer is

$$\frac{[AC'F]}{[ABC]} = \frac{6S}{8S} = \boxed{\frac{3}{4}}.$$

□

Example 2.4 (NIMO 8.8)
The diagonals of convex quadrilateral $BSCT$ meet at the midpoint M of $\overline{ST}$. Lines BT and SC meet at A, and $AB = 91$, $BC = 98$, $CA = 105$. Given that $\overline{AM} \perp \overline{BC}$, find the positive difference between the areas of $\triangle SMC$ and $\triangle BMT$.



Solution. Let's get rid of B and C first. Set $\beta = \angle BAM$, $\gamma = \angle CAM$ and note that $\sin\beta = \frac{5}{13}$ and $\sin\gamma = \frac{3}{5}$. Compute $AM = 84$. Now, let A_1 be the reflection of A over M.

We can compute

$$\begin{aligned}[AST] &= [AA_1T] \\ &= \frac{AM^2 \sin\beta \sin\gamma}{2\sin(\beta+\gamma)} \\ &= 7^2 \cdot 288 \cdot \frac{\frac{5}{13}\cdot\frac{3}{5}}{\frac{5}{13}\cdot\frac{4}{5}+\frac{12}{13}\cdot\frac{3}{5}} \\ &= 7^2 \cdot 288 \cdot \frac{15}{56} \\ &= 3780.\end{aligned}$$

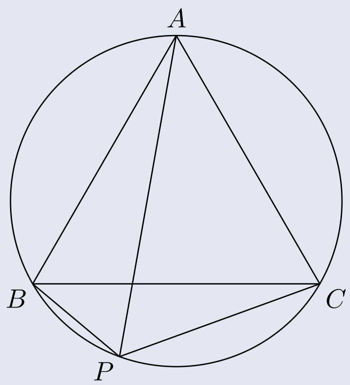
In that case, the desired quantity is $[ABC] - [AST] = 84 \cdot 7^2 - 3780 = \boxed{336}$. □

3 Equilateral Triangles

3.1 Same Point on the Same Side

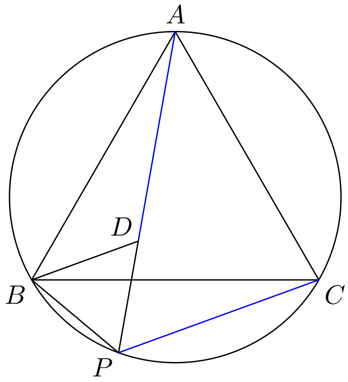
The idea is to split up a segment into two parts, or you can also think of it as adding two segments and seeing if that new segment can be found in the figure.

Theorem 3.1 (van Schooten's Theorem)
Let P be a point on the minor arc BC of equilateral triangle ABC. Then $PA = PB + PC$.



There is a quick solution using Ptolemy's theorem by applying it to quadrilateral $ABPC$. We'll try to prove this theorem without using Ptolemy's.

Proof. To prove that PA is the sum of PB and PC, let's try to split up PA into two segments. One will have length PB, and the other should have length PC. We pick the point D on segment PA such that $PD = PB$.



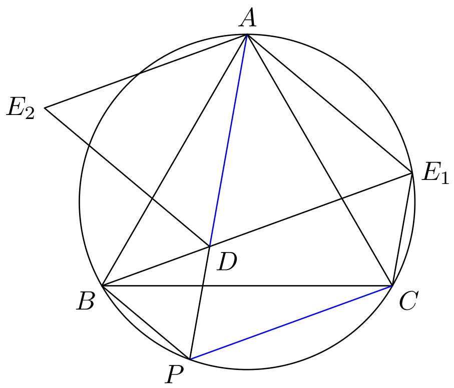
From construction, we have $PD = PB$, so to finish, we need to prove $AD = PC$. Let's look at what we can get from $PD = PB$. If we draw BD, it doesn't just look like triangle BDP is isosceles, but it looks like it's equilateral too.

Angle chasing, we get

$$\angle BPD = \angle BPA = \angle BCA = 60^\circ.$$

Since $\triangle BDP$ is isosceles, the remaining two angles are 60°, making it equilateral.

We can show that $AD = PC$ by constructing a similar equilateral triangle. Let E be the point such that $\triangle ADE$ is equilateral. But there's a problem: there are *two* possible choices of E on opposite sides of AD. Let's draw both and see what happens.



Surprisingly, it looks like E_1 lies on the circumcircle. It even looks like B, D, and E_1 are collinear! It also seems that this line is parallel to PC, which would make DE_1CP is a parallelogram. In fact, if it was a parallelogram, we'd be done.

We will try to prove these:

1. If DE_1CP is a parallelogram, then $AD = PC$.
2. DE_1CP is a parallelogram.

For the first one, if it was a parallelogram, then $PC = DE_1$ because they're opposite sides. But $DE_1 = AD$ because $\triangle ADE_1$ is equilateral, so that finishes the proof. The second one's a bit harder, and needs some more steps. Try to show them individually:

- *B, D and E_1 are collinear*: $\angle ADE_1 = 60^\circ = \angle BDP$.
- *E_1 lies on the circumcircle*: $\angle AE_1B = \angle AE_1D = 60^\circ = \angle ACB$.
- *DE_1 and PC are parallel*: $\angle ADE_1 = 60^\circ = \angle ABC = \angle APC$.

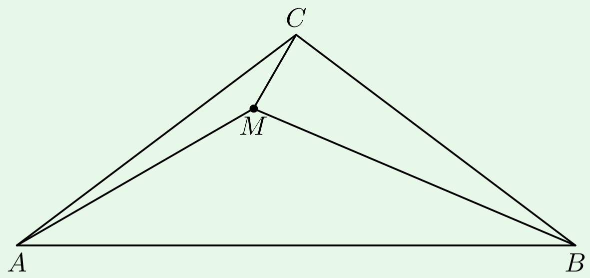
- *DP and E_1C are parallel*: $\angle BE_1C = \angle BAC = \angle PAE_1 = 180^\circ - \angle PCE_1$.

This gives us the desired result. □

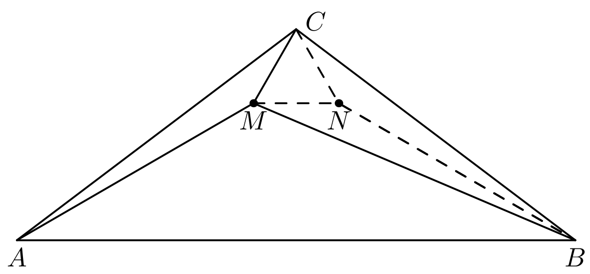
3.2 Reflections

Let's just dive into an example:

Example 3.2 (AIME I 2003/10)
Triangle ABC is isosceles with $AC = BC$ and $\angle ACB = 106^\circ$. Point M is in the interior of the triangle so that $\angle MAC = 7^\circ$ and $\angle MCA = 23^\circ$. Find the number of degrees in $\angle CMB$.



Solution. Let's reflect M across the perpendicular from C to AB to get N:



Then obviously $\angle CBN = 7^\circ$ and $\angle BCN = 23^\circ$. Thus,

$$\angle MCN = 106^\circ - 2 \cdot 23^\circ = 60^\circ.$$

Furthermore, since $\triangle AMC$ and $\triangle BNC$ are congruent by ASA, we must have that

$$CM = CN.$$

Hence $\triangle CMN$ is an equilateral triangle, so $\angle CNM = 60^\circ$. Thus

$$\angle MNB = 360^\circ - \angle CNM - \angle CNB = 360^\circ - 60^\circ - 150^\circ = 150^\circ.$$

We now see that $\triangle MNB$ and $\triangle CNB$ are congruent. Therefore, $CB = MB$, so $\angle CMB = \angle MCB = \boxed{83^\circ}$. □

4 Translations

We've seen how rotation and reflection are useful. What about translation? In this case, we're just going to move **one point** and see how everything else follows. Here's a problem I came up with to demonstrate this:

Example 4.1
An equilateral triangle ACK is located inside a regular decagon $ABCDEFGHIJ$. If the area of the decagon is 2020, find the area of $HIJAK$.

Solution. Let O be the center of this decagon. Note that O lies on the line BG, as does K. Note that $AH \parallel BG$, so O to AH is the same distance as K to AH. Thus, by same base-same height, we must have $[AHO] = [AHK]$. Thus,

$$[HIJAK] = [HIJA] + [AHK] = [HIJA] + [AHO] = [HIJAO],$$

which is equivalent to three-tenths of the decagon (because $[HIJAO] = [HIO] + [IJO] + [JAO]$, and each of these are equal isosceles triangles, and the total area of the decagon is 10 of these equal isosceles triangles). Thus, the answer is

$$\frac{3}{10} \cdot 2020 = \boxed{606}.$$

□

The idea was to take advatange of **same base-same height** (if the heights and bases are equal in length for two triangles, their areas are the same). This is the basis of moving a point, i.e. translation.

5 An IMO Shortlist Teaser

This is going to use the trick of creating a segment out of the sum of two segments.

⚠ We are **not** going to solve this problem! This is just a hint as to how to solve it. A solution is given here if you'd like to see the rest – we are just going to examine the construction. The other part, Ceva, is a part left to the reader to prove.

Example 5.1 (ISL 2000/G3)
ABC is an acute-angled triangle with orthocenter H and circumcenter O. Show that there are points D, E, F on BC, CA, AB respectively such that $OD + DH = OE + EH = OF + FH$ and AD, BE, CF are concurrent.

The constraint $OD + DH = OE + EH = OF + FH$ is extremely weird. It is well known that H and O share some nice connections. How could this help us?

In a triangle ABC, if we reflect the orthocenter H across BC, we get a point H_A that lies on the circumcircle. Thus, $HD = H_AD$. But wait a minute! What if we connect O to H_A? What if we let D be the intersection of OH_A and BC?

We realize that $R = OD + DH_A = OD + DH$! This tells us that our weird constraint is actually just saying that their sum is equal to R. From here we can apply the other constraint and use Ceva to solve the problem.

6 Strategies

- **Symmetry**: take advantage of this. In particular, you can create symmetry by applying transformations.

- **Angle chasing**: use cyclic quadrilaterals and similarity to get some angles. Transformations can also help.
- **Auxillary lines**: draw lines, because they help you find out what exactly you're missing.
  - **Parallograms**: construct them when dealing with midpoints or, more obviously, parallel lines.
  - **Equilateral triangles**: if there is one in the figure, refer back to the bullet point above about symmetry. If there isn't, try applying transformations to find a hidden one.
- **Same point on the same side**: make the sum of two segments into a segment. If $AX + AY$ appears on one side, construct a point Y' on ray AX such that $XY' = AY$. Then $AX + AY = AY'$, and AY' hopefully makes an isosceles triangle, parallelogram, isosceles trapezoid, or cyclic quadrilateral. Note that you can try constructing on ray AY instead. Try to construct in the opposite direction.
- **Same point on opposite sides**: make the difference of two segments into a segment.
- **Transformations**: obviously, these are useful. But how?
  - **Rotations**: cut and paste a bit of the figure and attach it elsewhere. Usually, you want to attach it so that two sides line up because they have the same length.
  - **Reflections**: reflect isosceles figures.
  - **Translations**: try moving one point and see what happens. We take advantage of same base-same height here.
- **Length bashing**: this is mostly just used for answer extraction. However, sometimes bashing out that two lengths are the same is a good indication something interesting is occurring.

7 Problems

7.1 Classics

These are examples of Langley's problems that might serve better as brainteasers. Here is a generalized way to solve them.

1. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20°$. Points D and E are on AC and AB respectively such that $\angle CBD = 40°$ and $\angle BCE = 50°$. Determine $\angle CED$.
2. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20°$. Points D and E are on AC and AB respectively such that $\angle CBD = 50°$ and $\angle BCE = 60°$. Determine $\angle CED$.
3. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20°$. Points D and E are on AC and AB respectively such that $\angle CBD = 60°$ and $\angle BCE = 70°$. Determine $\angle CED$.
4. In convex quadrilateral $ABCD$, $\angle ABD = 12°$, $\angle ACD = 24°$, $\angle DBC = 36°$, and $\angle BCA = 48°$. Determine $\angle ADC$.

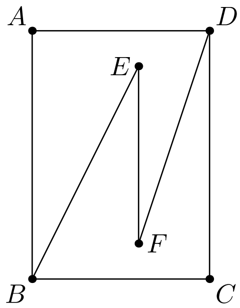
5. In convex quadrilateral $ABCD$, $\angle ABD = 38^\circ$, $\angle ACD = 48^\circ$, $\angle DBC = 46^\circ$, and $\angle BCA = 22^\circ$. Determine $\angle ADC$.

Have these shown up in contest? Yep!

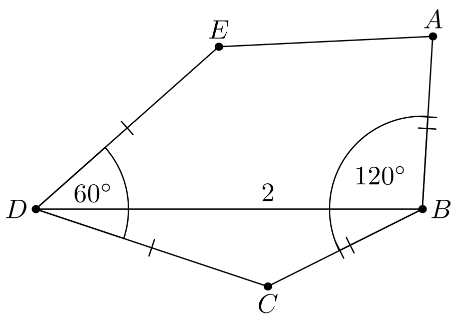
Problem 1 (AMC 10B 2008/24). In convex quadrilateral $ABCD$, $AB = BC = CD$, $\angle ABC = 70^\circ$, and $\angle BCD = 170^\circ$. Determine $\angle DAB$.

7.2 Parallelograms

Problem 2 (AIME 2011/2). In rectangle $ABCD$, $AB = 12$ and $BC = 10$. Points E and F lie inside rectangle $ABCD$ so that $BE = 9$,$DF = 8$,$\overline{BE} \parallel \overline{DF}$,$\overline{EF} \parallel \overline{AB}$, and line BE intersects segment $\overline{AD}$. The length EF can be expressed in the form $m\sqrt{n} - p$, where m,n, and p are positive integers and n is not divisible by the square of any prime. Find $m + n + p$.

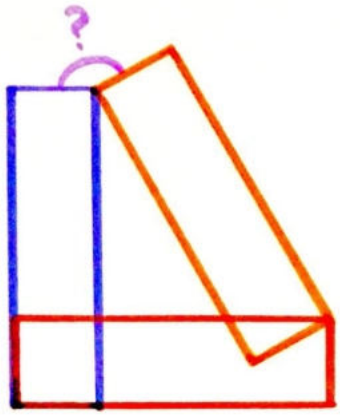


Problem 3. Let $ABCDE$ be a convex pentagon with $AB = BC$ and $CD = DE$. If $\angle ABC = 2\angle CDE = 120^\circ$ and $BD = 2$, find the area of $ABCDE$.



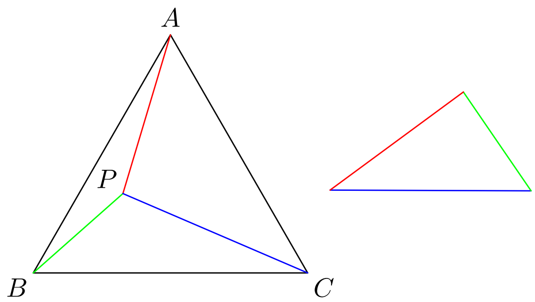
7.3 Equilateral Triangles

Problem 4 (Catriona Shearer). Find the angle labeled by the question mark.

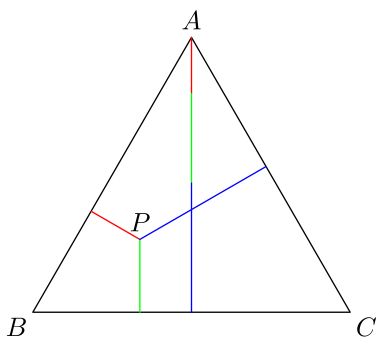


Remark 7.1. For the above problem, rotations would also work.

Problem 5 (Pompeiu's Theorem). Let P be a point *not* on the circumcircle of an equilateral triangle ABC. Then there exists a triangle with side lengths PA, PB, and PC.



Problem 6 (Viviani's Theorem). Let P be a point inside equilateral triangle ABC. Then the sum of the distances from P to the sides of the triangle is equal to the length of its altitude.

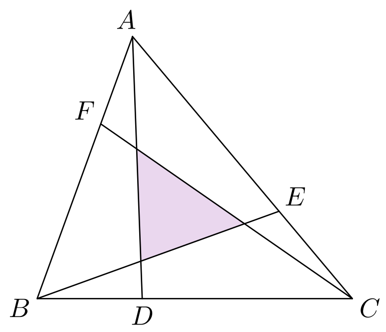


The following is not an equilateral triangle problem, but it is still has a similar idea nonetheless:

Problem 7 (One-Seventh Area Triangle). In triangle ABC, points D, E, and F lie on sides BC, CA, and AB respectively, such that

$$\frac{CD}{BD} = \frac{AE}{CE} = \frac{BF}{AF} = 2.$$

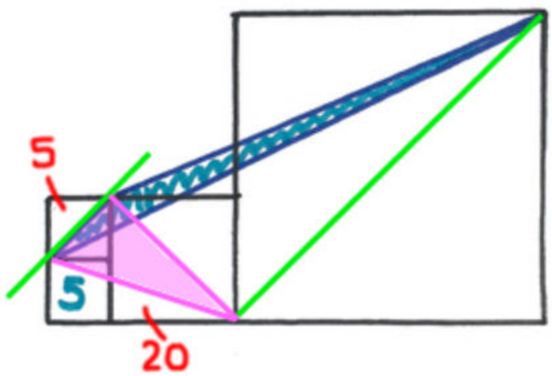
Then the area of the inner triangle formed by the lines AD, BE, and CF is one-seventh the area of ABC.



> *Remark 7.2.* A generalization of the above is Routh's theorem.

7.4 Translations

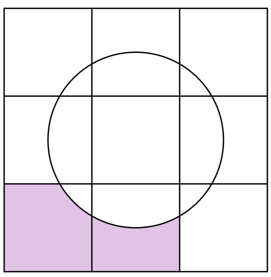
Problem 8 (Catriona Shearer). In this figure there are four squares. The area of the two little squares is 5 and the area of the middle square is 20. What is the area of the blue triangle? (Note that the figure hints at the answer.)



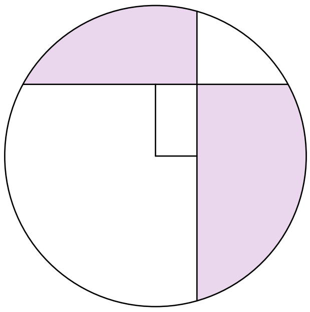
7.5 Miscellaneous

This is just a few problems that I thought were interesting. Have fun!

Problem 9. A circle with radius 1 is drawn centered on a 3×3 grid of unit squares. Find the area inside the lower-left and bottom squares but outside the circle.



Problem 10 (UKMT 2014). A circle with area 2500 is divided by two perpendicular chords into four regions. The two regions next to the region with the circle's center, shaded in the figure, have combined area 1000. The center of the circle and the intersection of the chords form opposite corners of a rectangle, whose sides are parallel to the chords. What is the area of this rectangle?



Christmas Problems

Dylan Yu

December 25, 2020

Problem 1 (239 2009 J8). Each of the 11 girls wants to mail each of the other a gift for Christmas. The packages contain no more than two gifts. If they have enough time, what is the smallest possible number of packages that they have to send?

Problem 2 (ToT 2014/6). During Christmas party Santa handed out to the children 47 chocolates and 74 marmalades. Each girl got 1 more chocolate than each boy but each boy got 1 more marmalade than each girl. What was the number of the children?

Problem 3 (Slovenia TST 2018/1). Let n be a positive integer. On the table, we have n^2 ornaments in n different colours, not necessarily n of each colour. Prove that we can hang the ornaments on n Christmas trees in such a way that there are exactly n ornaments on each tree and the ornaments on every tree are of at most 2 different colours.

Problem 4 (Chistmas Assessment Exercise). The elves in Santa's factory have received, between them, exactly 2^n Christmas cards. After Christmas they seek to dispose of them at the Lapland recycling centre. Santa is a harsh taskmaster, and will only allow one elf to take time off work to visit the dump. The elves can pass bundles of cards between themselves as follows: if elf X has x cards and elf Y has y cards, and $x \geq y$, then X may pass y cards to Y, so that X has $x - y$ cards and Y has $2y$ cards. Can the cunning elves arrange their exchanges so that they can recycle all the cards?

> *Remark 0.1.* More of these problems can be found here.

Problem 5 (Estonia TST 2020/2). The city has 2020 inhabitants. Everyone is happy before the New Year; but if before on holiday, the resident will not receive a greeting card from another resident, he will be sad. Unfortunately, there is only one postal company in the city, which provides only one service: each a resident can name two other citizens before the New Year, and during the holiday the postal company will send a postcard on behalf of the sender one of them of your choice. The postal company is known to choose recipients in such a way that as many citizens as possible become sad. Find the smallest possible number of townspeople who can become sad.

Problem 6 (Italian MO 2004/1). Observing the temperatures recorded in Cesenatico during the December and January, Stefano noticed an interesting coincidence: in each day of this period, the low temperature is equal to the sum of the low temperatures the preceeding day and the succeeding day. Given that the low temperatures in December 3 and January 31 were 5°C and 2°C respectively, find the low temperature in December 25.

Problem 7 (Geoff Smith). The Elves have a huge supply of wrapped presents in identical cubical boxes, with plenty wrapped in red paper, and plenty more wrapped in blue paper. To prepare for the big night, Santa orders that 12 boxes be placed on the ground in a row. Then 11 boxes are placed on top of these, so that the rightmost bottom box has no box on top of it. Then 10 boxes are to be placed on top of these, so that the last box on the right in the second row has no box on top of it. This continues until the last row has just one box in it, and it is on the far left. Thus 78 boxes are stacked in a triangular shape.

Santa is very superstitious, and insists that the stacking obeys *Lapland Rules*.

(a) In each of the 12 columns, the colours of the top box and the bottom box must be the same.

(b) If a box has a blue box underneath and a blue box to the left, then it must be a red box.

(c) If a box has a red box underneath and a red box to the left, then it must be a red box.

(d) If a box has different coloured boxes underneath and to the left, then it must be a blue box.

How many different stackings are there which obey the Lapland rules?

Problem 8 (Geoff Smith). **Follow up**: How many such completed stackings are possible, given that Santa insists that each column should have the same colour top present as its bottom present?

Problem 9 (ToT 2015/7). Santa Clause has k candies from each of n types of candies. He puts them randomly in k bags with and equal number of candies in each bag and gives the bags to k children, one to each. The children organise trade exchanges as follows: If one of them doesn't have candies of type i but has some of type j and the other has some of type i but none of type j then they may exchange one candy of these types among themselves. Prove that they can organise a sequence of exchanges which will eventually lead each of them having candies of all types.

Problem 10 (expii). The Christmas Tree at Rockefeller Center in New York City is an annual tradition. The 2016 tree is 94 feet high and 56 feet wide. Conveniently, tree weight does not scale with the volume of the cone determined by the tree, because the internal branching pattern makes the tree sparser inside than at the fringes. Evergreen trees grow this way so they have a large surface area to capture sunlight while the interior can support the tree structurally.

To see how quickly a branching structure can expand, consider a growth process which starts with a single branch, forking into two sub-branches after a foot of growth, and where every sub-branch forks into two more sub-branches after a foot of growth. After how many feet of growth would there be 1 million sub-branches?

x^y Problems

Dylan Yu

January 31, 2021

Remark 1. These are problems of the form x^y, i.e. variables to the power of variables.

Problem 1. Find all positive integer solutions $x^y + y^x = (x-y)^{x+y}$.

Problem 2 (Estonia TST 2005/3). Find all pairs (x, y) of positive integers satisfying the equation $(x+y)^x = x^y$.

Problem 3 (APMC Team 1999/1). Find all pairs (x, y) of positive integers such that $x^{x+y} = y^{y-x}$.

Problem 4 (Delta Polish Magazine). Find all rational numbers $x, y > 1$ satisfying

$$x^y = xy.$$

Problem 5 (M.M. Circles). Prove that if $x > 1, y > 1$, and $x^y + y^x = x^x + y^y$, then $x = y$.

Problem 6 (Kazakhstan MO Grade 11 2000/4). Find all triples of natural numbers (x, y, z) that satisfy the condition $(x+1)^{y+1} + 1 = (x+2)^{z+1}$.

Problem 7. Let $0 < x, y < 1$. Prove $x^y + y^x > 1$.

Problem 8 (IMAC Arhimede 2009/5). Find all natural numbers x and y such that $x^y - y^x = 1$.

Problem 9 (Romania MO Grade 10 2009/1).

(a) Show that two real numbers $x, y > 1$ chosen so that $x^y = y^x$, are equal or there exists a positive real number $m \neq 1$ such that $x = m^{\frac{1}{m-1}}$ and $y = m^{\frac{m}{m-1}}$.

(b) Solve the following equation in $(1, \infty)^2$: $x^y + x^{x^{y-1}} = y^x + y^{y^{x-1}}$.

Problem 10 (USAMTS 5/1/32). Find all pairs of rational numbers (a, b) such that $0 < a < b$ and $a^a = b^b$.

Integration

Dennis Chen and Dylan Yu

2021

Just like in differentiation, where we want to reconcile two different definitions – the approximation-based definition and the repeated differentiation-based definition – we want to take the two different definitions of the integral and understand why they're really the same: the area under a curve definition and the antiderivative definition. We unify the two definitions with the **Fundamental Theorem of Calculus**.

§ 1 The Integral

Because integration can be somewhat hard to grasp at the beginning, we spend a lot of time bashing you over the head with the "area under a curve" and antiderivative concepts. If you find yourself going "I understand all of this, and this is going very slowly," feel free to skim/skip the section **until you get to the Fundamental Theorem of Calculus**.

§ 1.1 Area Under a Curve and Some Exposition

We crudely define the integral as the area under a curve.[1]

The purpose of the antiderivative is to answer the question: if $F'(x) = f(x)$, and we are given f, what is F? You'll soon see that a function has infinitely many antiderivatives, and all differ only by a constant. Furthermore, just like the derivative of f may not exist, the antiderivative can also be non-existent.

Definition 1 (Integral) For a function f that is continuous over $[a, b]$, we define the **integral** $\int\limits_a^b f(x)dx$ as the area under f in the range $[a, b]$.

If f goes below the y axis, then the area counts as negative.

Definition 2 (Indefinite Integral) Let $f(x)$ be a function that has an antiderivative $F(x)$. The **indefinite integral** of $f(x)$ is denoted by and equal to

$$\int f(x)\,dx = F(x) + C,$$

where C is a constant.

Definition 3 (Definite Integral) Let $f(x)$ be a function that has an antiderivative $F(x)$. The **definite integral** of $f(x)$ from $x = a$ to $x = b$ is denoted by and equal to

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\big|_a^b .$$

[1] If you want a non-terrible definition, you'll have to wait until real analysis.

We use the area under a curve definition to extract a couple of (probably obvious) facts about integrals.

Theorem 1 (Integral Properties) For a function f that is continuous over $[a, b]$, where all variables in the integrals are assumed to be in the range $[a, b]$, the following are true:

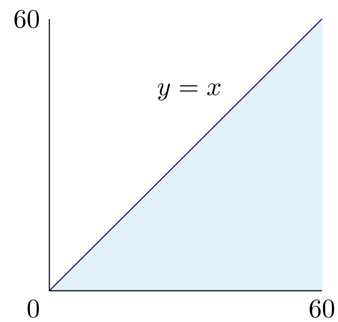
- Joining integrals: $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$[a]
- Linearity of integrals: $\int_a^b (pf(x) + qg(x))d(x) = p\int_a^b f(x)dx + q\int_a^b g(x)dx.$

[a] Actually, provided that the function is continuous over the right interval, c does not have to be between a and b.

This limited understanding of integration is enough to do the most basic of mechanics problems, so we do a couple of them as examples.

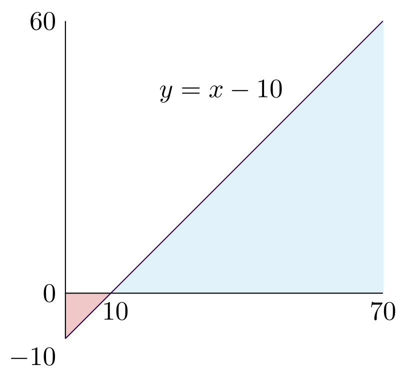
Example 1 A car is accelerating at $1m/s^2$ (meter per second squared). If the car starts at a speed of 0 meters, find the distance the car has traveled after it reaches a speed of $60m/s$.

Solution: Note that the condition is just code for "how far does the car travel after 60 seconds?" So we just need to integrate $\int_0^{60} xdx$. Note that this is an isosceles right triangle with side length 60, so the area is just $\frac{60^2}{2} = 1800$. Thus the car has traveled 1800 meters.



Example 2 (Negative Area) A car is going in reverse at $-10m/s$ and accelerating at $1m/s^2$. How far is the car from its original position after 70 seconds?

Solution: Note that the red area below the y axis represents the car going backwards, so we subtract it from the overall integral. Thus $\int_0^{70} xdx = \frac{60^2}{2} - \frac{10^2}{2} = 1750$, and the car is 1750 meters away from its original position.



If the concept of negative area is confusing, just use the linearity of integrals to shift the function up by a constant, and adjust for it by subtracting the area of a rectangle at the end.[2]

There are dozens of these types of practice problems in any standard high-school calculus textbook, particularly because it is very easy to draw lines on a graph and ask for the area under said lines. If you want practice on these problems, you can probably get away with doing the same thing yourself and finding the area – this is just simple geometry.[3]

§ 1.2 Antiderivatives

Because the derivative is a function, it has an inverse. We examine the inverse, known as the antiderivative.

Definition 4 (Antiderivative) The antiderivative of a function $f(x)$ is the function F that satisfies $F'(x) = f(x)$.

The antiderivative is not unique, and the possible antiderivatives differ by a constant.

Theorem 2 (Constant perturbance) If $F'(x) = f(x)$ and $g(x) = F(x) + C$ for any constant C, then $g'(x) = F(x)$.

Proof: We use the fact that derivatives are additive. Note that

$$g'(x) = (F(x) + C)' = F'(x),$$

as desired. ■

Now how do we go about finding the antiderivative? The Power Rule makes it very easy to find the antiderivatives for polynomials:

Exercise 1 (Antiderivative with Power Rule) Prove that the antiderivative of x^n is $nx^{n+1} + C$ for all $x \neq -1$.

Exercise 2 (Followup to Power Rule) Why is $x = -1$ the exception, and what is the antiderivative of $\frac{1}{x}$?

In theory, we could take the Maclaurin Series of any function and then find the antiderivative of each term. But this is often more trouble than it's worth, so here's a list of antiderivatives you should know.

[2] After you think of this, the concept is probably going to be less confusing anyways...

[3] I do not think spamming these sorts of exercises are the right thing to do anyway, because you either spend a lot of time missing the point or you've already gotten the point anyways. The only real consideration you have to make is towards negative areas, and shifting the function by a constant explains the motivation behind this very well.

Theorem 3 (Common Antiderivatives) We might mention a few of these later, so take them for granted for now:

- $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$
- $\int \sin(x)\,dx = -\cos(x) + C$
- $\int \cos(x)\,dx = \sin(x) + C$
- $\int \sec^2 x\,dx = \tan(x) + C$
- $\int e^x\,dx = e^x + C$
- $\int \sinh(x)\,dx = \cosh(x) + C$
- $\int \cosh(x)\,dx = \sinh(x) + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int \frac{1}{1-x^2}\,dx = \arcsin(x) + C$
- $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$
- $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C$

You can easily show most of these by differentiating the right-hand side. Here's a heuristic you should be aware of: **whenever you see something of the form $1 \pm x^2$ in the denominator of a function**, think trig substitutions.

Exercise 3 (Antiderivative of $e^{f(x)}$) Show that the antiderivative of $e^{f(x)}$ is $\frac{e^{f(x)}}{f'(x)}$.

§ 1.3 Putting it Together

Now that we've presented the two definitions of the integral, we'll follow up by linking them together with the Fundamental Theorem of Calculus.

Theorem 4 (The Fundamental Theorem of Calculus) Let f be a continuous function f on $[a, b]$ and let

$$F(x) = \int_a^x f(x)dx$$

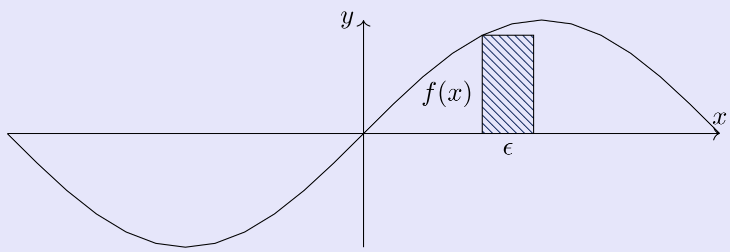
for all $x \in (a, b)$, where the integral function denotes the area under the curve. Then

1. For all $x \in (a, b)$, $F'(x) = f(x)$.
2. If f and g are continuous on $[a, b]$ and $g'(x) = f(x)$, then $\int_a^b f(x)dx = g(b) - g(a)$. (In other words, $g(x) = F(x) + C$ for some constant C.)

Proof: The first part follows from the limit definition of the derivative. I'll explicitly state the key idea: **we're adding a rectangle of width ϵ and height $f(x)$.** Note that

$$F'(x) = \lim_{\epsilon \to 0} \frac{F(x+\epsilon) - F(x)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\int\limits_{x}^{x+\epsilon} f(x)dx}{\epsilon} = \frac{\epsilon f(x)}{\epsilon} = f(x).$$

The second part follows from the first. Note that if $F'(x) = f(x)$, then by properties of the derivative, the only functions g that satisfy $g'(x) = f(x)$ are of the form $F'(x) + C$ for any constant C. Thus $g(b) - g(a) = F(b) - F(a) = \int\limits_{a}^{b} f(x)dx$.



■

The two parts are respectively called the **second** and **first** fundamental theorems of calculus, respectively.[4]

§ 1.3.1 Indefinite Integrals

Now we really connect the idea of integrals and anti-derivatives with the **indefinite integral**, which is a representation of an integral as an anti-derivative.

Definition 5 (Indefinite Integral) Given $F'(x) = f(x)$, we say that

$$\int f(x)dx = F(x) + C.$$

In general, we will choose to express the constant term of $F(x)$ as 0, since it is neater and $+C$ doesn't really care what the constant term is anyways.

§ 2 Evaluating the Integral

Because integrals are really just the reverse of derivatives, it follows that the two fundamental rules of differentiation hold in some for integration: the **chain rule** and **product rule**. Their counterparts in integration are u-substitution (and the use of u-substitution is followed by something still called the chain rule) and integration by parts.

[4] I don't know why the theorems are ordered like this either and I think it makes sense to present the second before the first, but if anyone knows why, please shoot me an email at proofprogram@gmail.com or message me on Discord at shootinglucky#4186 so I can replace this footnote with something more useful.

§ 2.1 u-substitution

The chain rule is really just saying that $\frac{d}{dx} = \frac{d}{du} \cdot \frac{du}{dx}$. When we do a u-substitution, what we're really trying to do is **get rid of unruly expressions in** x and turn them into managable expressions in terms of u.

Theorem 5 (Chain Rule) If u is a function about x, $g(u) = f(x)$, and $f(x)$ is an integratable function, then $\int f(x)dx = \int \frac{g(u)}{u'}du$.

This is implied by the chain rule for derivatives.

Example 3 Find $\int 8x(2x^2+1)dx$.

Solution: Of course we can just integrate with linearity, but for the sake of this example we will avoid doing that and use u-substitution instead.

The "unruly term" here is $2x^2+1$. Note that the derivative of this term is $4x$, so substituting $u = 2x^2+1$ gives

$$\int 8x(2x^2+1)dx = \int \frac{8xu}{u'}du = \int \frac{8xu}{4x}du = \int 2udu = u^2.$$

If we're dealing with definite integrals, then in our u-substitution, **we must make sure to change the bounds according to u**. Let's reuse the previous example.

Example 4 (Changing Bounds) Find $\int\limits_{-1}^{3} 8x(2x^2+1)dx$.

Solution: The bounds have to be expressed in terms of u. Since $x = -1$ and $x = 3$ are the bounds, the new bounds are $u = 2(-1)^2+1 = 3$ and $u = 2(3)^2+1 = 19$. Thus the integral is equivalent to

$$\int_3^{19} 2udu = 19^2 - 3^2 = 352.$$

Exercise 4 This is seemingly suspicious because this implies that $\int\limits_{-a}^{a} 8x(2x^2+1)dx = \int\limits_{2a^2+1}^{2a^2+1} u^2dx = 0$. In other words, $-a$ and a in this case would be indistinguishable as bounds. So why is this correct?

Example 5 Find $\int x(2x^3+1)^2$.

Remark: One might initially believe that that u-sub does not work here: allowing $u = 2x^3+1$ gives us

$$\int \frac{1}{6x}u^2du,$$

and expressing x in terms of u is not even worth the effort.

Solution: With a little bit of ingenuity, u-substitution will work. If you let $u = x^2$ instead, then you get

$$\int \frac{(2u\sqrt{u}+1)^2}{2}du.$$

At this point, crudely expanding will just work: you get

$$\int \frac{4u^3+4u\sqrt{u}+1}{4}du = \frac{u^4}{4} + \frac{8u^2\sqrt{u}}{5} + u + C = \frac{x^8}{2} + \frac{4x^5}{5} + \frac{x^2}{2} + C.$$

Indeed, there is no good reason to use integration by parts here – we only presented it as an example that could be verified through other approaches.

Notice that in these types of integrals, we want to **contrive** the u-sub into working: Take a value of u that might seem inconvenient at first to force the x to disappear.

Here is a very tricky and non-obvious application of the chain rule: the integral of $\sec x$.

Example 6 (sec x) Find $\int \sec x dx$.

Solution: Multiply by $\frac{\sec x+\tan x}{\sec x+\tan x}$ and make the integral

$$\int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}.$$

Set $u = \sec x + \tan x$, and then $u' = \sec x \tan x + \sec^2 x$. Now the integral is equivalent to

$$\int \frac{\frac{\sec^2 x+\sec x \tan x}{\sec x+\tan x}}{u'} du = \int \frac{1}{\sec x + \tan x} du = \int \frac{1}{u} du =$$
$$\ln|u| + C = \ln|\sec x + \tan x| + C.$$

Exercise 5 (Nearly arctan) Find $\int \frac{1}{\sqrt{1+x^2}}$.

Example 7 (Reverse u-sub) Find $\int_{-1}^{8} \sqrt{1+\sqrt{1+x}}\, dx$.

Solution: Let $x = u^2 - 1$ and $dx = 2u\, du$. Then we have

$$\int_0^3 \sqrt{1+u} \cdot 2u\, du,$$

and we can set $u = t^2 - 1$ and $du = 2t\, dt$ to get

$$\int_1^2 t \cdot 2(t^2-1) \cdot 2t\, dt = \left(\frac{4}{5}t^5 - \frac{4}{3}t^3\right)\Bigg|_1^2 = \boxed{\frac{232}{15}}.$$

Remark: In this case, we let x be the variable separated from the rest of the equation rather than u, and then we substituted something in for u to finish the problem. Furthermore, notice how I **changed the domain** when I used u-sub on definite integrals. You should always change to the domain of the new variable.

§ 2.2 Even/Odd Identities

Theorem 6 (Integrating an Odd Function over Symmetric Bounds) Let f be odd and a be a real number. Then

$$\int_{-a}^{a} f(x)\, dx = 0.$$

Theorem 7 (Integrating an Odd Function over Symmetric Bounds) Let f be even and a be a real number. Then

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx.$$

The idea is simple, so let's just leave an exercise:

Exercise 6 Evaluate $\int_{-1}^{1} \frac{\arctan(x^7)}{\cos(x)+\sec(x)}\, dx$.

§ 2.3 Integration by Parts

We just presented the inverse of the chain rule; we now show the inverse of the product rule, otherwise known as integration by parts.

Theorem 8 (Integration by Parts) Given functions u, v,

$$\int uv' dx = uv - \int vu' dx.$$

Remark: Typically, we write $u' = du, v' = dv$, so the most common way it is written is

$$\int udv = uv - \int vdu.$$

Note we are using this to find the integral of $\int udv$, not $\int uv$.

This just follows by the product rule for derivatives; differentiating gives an obviously true statement.

Proof: Note that $(uv)' = uv' + u'v$, so integrating gives

$$uv = \int uv' dx + \int u'v dx.$$

Now rearranging yields

$$\int uv' dx = uv - \int vu' dx.$$

■

We are not going to be including the dx at the end of the integral if it is clear what we are integrating. This is to make reading the problems and their solutions easier.

Example 8 Find $\int x \sin x dx$.

Solution: We integrate by parts. Let $x = u$ and $v' = \sin x$. Then

$$x \sin x = -x \cos x + \cos x + C.$$

Remark: Not that unlike the product rule, integration by parts **should not be used symmetrically**; that is, the choice of which function to make u and which function to make v' makes a major difference between how feasible a calculation is.

Remark: When integrating by parts, never forget to add $+C$ at the end!

There are some times when the parts are non-obvious or it's hard to see that it's integration by parts at all, as the next example will show.

Example 9 Find $\int \ln x dx$.

Solution: Our parts are $u = \ln x$ and $dv = 1$. Then

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} = x \ln x - x + C.$$

What's the motivation here? Once you're told this is an "integration by parts" problem, it isn't hard to see why we set $u = \ln x$ and $dv = 1$ – we want the $\ln x$ to disappear from the integral, and $du = \frac{1}{x}$ is a lot easier to handle. But why would we try integration by parts to begin with?

In a nutshell, it's because **it's the easiest approach that could work**. We can't "just integrate this function" because it's an elementary function – there are no tricks here. The Chain Rule actually *does* work in this case – though it is a quite complicated argument, and you can't even get away from integration by parts in the end.[5] But the integration by parts solution is so simple that it is feasible to discover, even if it isn't the most natural – just by the "infinite monkey theorem[6]."

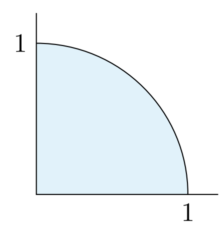
In this case, it's actually not impossible to think of – this is the first thing an experienced contestant will think of after deciding to try integration by parts.

§ 2.4 Trigonometric Substitutions

We begin with a geometric problem.

Example 10 Find $\int_0^1 \sqrt{1-x^2}dx$.

Solution: Geometrically, note that this is a quarter circle, as shown below. Thus the area is $\frac{\pi}{4}$.



Algebraically, let $x = \sin\theta$. Note that $dx = \cos\theta d\theta$, so the integral is

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta =$$

$$\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}(\sin^{-1} x + x\sqrt{1-x^2})\Bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}(\frac{\pi}{2} - 0) = \frac{\pi}{4}.$$

Let's generalize the above problem.

Exercise 7 Find $\int \sqrt{1-x^2}dx$.

What of the reciprocal of this function? You may know from differentiation that the derivative of arcsin is $\frac{1}{1-x^2}$ – now, let's use integration to confirm this.

---

[5] Set $u = \ln x$ and note

$$\int \ln x dx = \int ue^u du.$$

Then use integration by parts to get

$$\int ue^u du = ue^u - \int e^u = ue^u - e^u + C.$$

Now substitute $u = \ln x$ to get $x\ln x - x + C$ as before.

[6] Math contestants aren't monkeys, and a one-line solution is significantly shorter than Hamlet – so it is *actually feasible*, not just *theoretically* so.

Example 11 (arcsin) Find $\int \frac{1}{\sqrt{1-x^2}} dx$.

Solution: Let $x = \sin\theta$; then note $dx = \cos\theta d\theta$, and we have $\int \frac{1}{\cos\theta} \cos\theta d\theta = \theta + C$. Note that $\theta = \arcsin x$, so the integral evaluates to $\arcsin x + C$, as desired.

In general, expressions of the form $x^2 \pm 1$ strongly suggest trigonometric substitutions. **This is not just something you'll see in calculus** – in fact, many algebra problems (particularly at college contests, which are run for high schoolers despite their name) involve these sorts of substitutions.

§ 2.5 Partial Fraction Decomposition

The reader's ability to do partial fraction decomposition is assumed. The appendix contains a short explanation of it.

How would you integrate something like

$$\int \frac{1}{x^2 - 3x + 2} dx?$$

At first glance, there's no obvious antiderivative for the denominator; fractions generally tend to complicate things. You could express it as $\int \frac{4}{(2x-3)^2-1}$ and use a trigonometric substitution, but that would also be pretty annoying. This also wouldn't be sustainable; what about something like $\frac{1}{x^3-3x^2+2x} dx$? There is a better solution, however.

Solution: Note that

$$\int \frac{1}{x^2 - 3x + 2} dx = \int \left(\frac{1}{x-1} - \frac{1}{x-2} \right) dx.$$

Both of these are easily integratable; the answer is $\ln|x-1| - \ln|x-2| + C$.

Such an approach is generalizable. For instance, if you were given some fraction equivalent to, say, $\frac{1}{x-1} + \frac{2}{x-3} - \frac{3}{x}$, it would be much easier to integrate this than $\frac{6x^2-17x+9}{x^3-4x^2+3x}$. However, not all rational functions have a convenient partial fraction decomposition; for instance, $\frac{3}{x^3-1} = \frac{1}{x-1} - \frac{x+2}{x^2+x+1}$ has a disgusting integral.

§ 3 Applications of the Integral

§ 3.1 Average of a Function

Finding the average of a function is quite easy; the procedure is just integrating and dividing by the length of the segment you're integrating on.

Definition 6 (Average of a Function) The average of $f(x)$ in the range $[a, b]$ is

$$\frac{\int_a^b f(x)dx}{b-a}.$$

Drawing an analogy to a discrete average, the intuitive reason this works is because the "sum" of the different values it can take is expressed by the integral, and the "number" of values taken is $b - a$.

We present a simple example.

Example 12 Say x is a randomly chosen number from 0 to 1. What is the expected value of $x(1-x)$?

Solution: Note that the desired value is

$$\frac{\int_0^1 x(1-x)dx}{1} = \int_0^1 (x - x^2)dx = (\frac{1^2}{2} - \frac{1^3}{3}) - (\frac{0^2}{2} - \frac{0^3}{3}) = \frac{1}{6}.$$

Finally, we present an example of how the Fundamental Theorem of Calculus is tightly intertwined with our intuition of averages.

Example 13 (Average Velocity) Say a car is traveling on a road. At time a it is at location $f(a)$ and at time b it is at location $f(b)$. What is its average speed between the interval $[a, b]$?

Solution: Our intuition says that the answer is $\frac{f(b)-f(a)}{b-a}$, and we want to see if integrating gives the same result. Without loss of generality, assume $a < b$.

By the way the problem is worded, we assume that f is continuous. (This is how cars in real life work, after all!) For the sake of convenience, let us also assume that f is differentiable. Now note that we really want to integrate the *speed*, or f', to find the average value it takes. Note that the requested value is

$$\frac{\int_a^b f'(x)dx}{b-a} = \frac{(f(b)+C)-(f(a)+C)}{b-a} = \frac{f(b)-f(a)}{b-a},$$

as expected.

§ 3.2 Area Between Two Curves

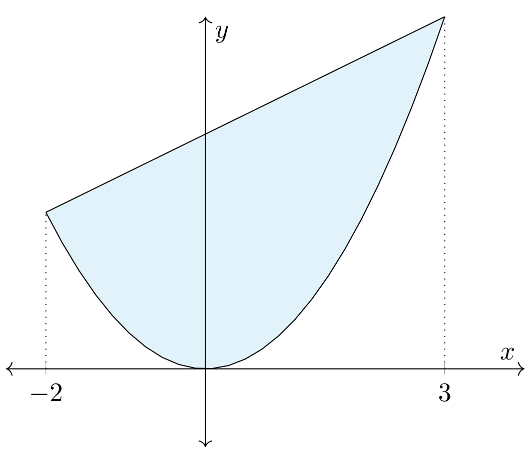
Let's begin with a simple example, and afterwards, let's generalize for any two curves.

Example 14 Find the area enclosed by $f(x) = x^2$ and $g(x) = x + 6$.

Solution: Note that the height between the two curves at x is $x + 6 - x^2$ – since we want to find the total area between them, we integrate this height. The area is

$$\int_{-2}^{3} (x + 6 - x^2)dx = \left(-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right)\Bigg|_{-2}^{3} = \frac{125}{6},$$

where we get the bounds of integration from the intersection points of x^2 and $x+6$, or the zeroes of $x+6-x^2$.



The procedure for finding the area between two curves is quite simple.

Theorem 9 (Area Between Two Curves) Say integratable functions $f(x), g(x)$ intersect at $x = a$ and $x = b$, and furthermore, a is the leftmost intersection point and b is the rightmost intersection point. Then the area bounded by $f(x)$ and $g(x)$ is

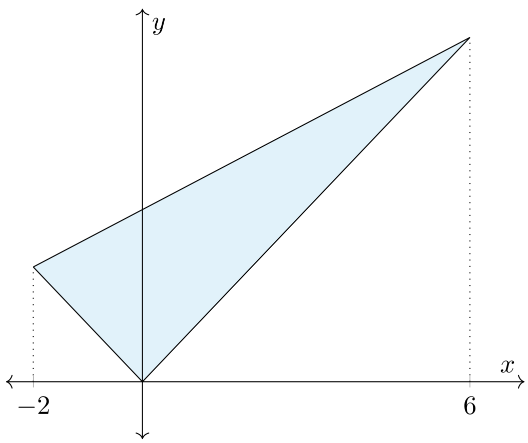
$$\int_a^b |f(x) - g(x)|\, dx.$$

Let's do a harder example.

Example 15 Find the area between $f(x) = |x|$ and $f(x) = \frac{1}{2}x + 3$.

Solution: Note that the intersection points are $(-2, 2)$ and $(6, 6)$, and further note that $|x| \leq \frac{1}{2}x + 3$ in this range, so the area is

$$\int_{-2}^{6} \left(\frac{1}{2}x + 3 - |x|\right) dx = \left(\frac{6^2}{4} - \frac{(-2)^2}{4} + 8 \cdot 3\right) - \left(\frac{6^2}{2} + \frac{(-2)^2}{2}\right) = 12.$$



§ 3.3 Applied Example: 3D Geometry

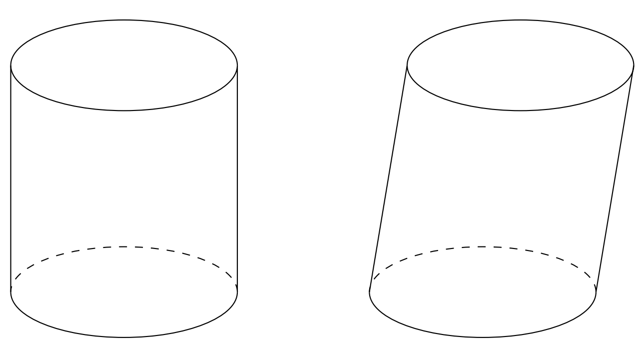
Before we discuss volume, we should first define it. **This definition is not intended to be rigorous**; instead, it is meant to highlight the connection between integration and volumes.

Definition 7 (Volume) The volume of a solid is the integral of the areas of all cross-sections made with planes parallel to a certain reference plane.

This basically means you sum up all of the cross-sections, similar to finding the area of a triangle. Usually our reference plane will be the base.

Cavalieri's Principle is an immediate consequence of this definition.

Fact 1 (Cavalieri's Principle) If in two solids of equal altitudes, the planes parallel to and at the same distance from their respective bases always create cross-sections with equal area, then the two solids have the same volume.



As examples, we prove the volume formulas for cylinders/prisms and cones/pyramids, leaving the volume of the sphere as an exercise.

Example 16 The volume of a prism/cylinder with a base of area B and a height of h is Bh.

The proof follows obviously by the definition.

Proof: Let the reference plane be one of the bases. Then note that the cross-section always has area B over a height of h. Let k be the distance of the cross-section from the base. Then the volume is $\int_0^h Bkdk = Bh$. ■

If you've ever taken geometry class, you might wonder why the cone and pyramid volume formulae have a coefficient of $\frac{1}{3}$. Perhaps you have may be suspecting by now that the reason is that $\int x^2 dx = \frac{x^3}{3} + C$; we demonstrate how this produces the coefficient of $\frac{1}{3}$.[7]

Proof: Let the reference plane be the plane through the apex parallel to the base and let k be the distance of the cross-section from the reference plane. (The cross-section lies on the same side of the reference plane as the base.)

Then by similarity, the volume is $\int_0^k B\frac{k^2}{h^2}dk = \frac{B}{h^2}\int_0^k k^2 dk = \frac{B}{h^2} \cdot \frac{h^3}{3} = \frac{Bh}{3}$. ■

To finish off, prove the volume of a sphere yourself.

Theorem 10 (Volume of a Sphere) The volume of a sphere with radius r is $\frac{4\pi r^3}{3}$.

Surface area is actually surprisingly tricky to define; however, we just need a couple of properties to give an idea of how to work with them.

Fact 2 (Additivity) The surface area of an object is the sum of the surface area of its parts.

Fact 3 (Surface Area of Flat Shapes) The surface area of a flat shape is the same as the area of the flat shape.

Fact 4 (Straight Lines) If part of the surface consists of lines, then the surface area of that part can be found by integrating the lengths of the lines.

As an example, consider the side of a cylinder or the lines joining the apex of a cone to the circumference of its base.

Fact 5 (Curves) If part of the surface consists of curves, then the surface area of that part can be found by integrating the lengths of the curves.

Before we move onto the surface area of a sphere, convince yourself that surface areas can be found through integration as a natural consequence of these properties.

Theorem 11 (Surface Area of a Sphere) The surface area of a sphere with radius r is $4\pi r^2$.

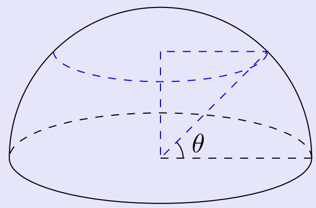
[7] Note that the coefficient of x^3 is $\frac{1}{3}$ in the antiderivative.

Proof: We instead prove that the surface area of a hemisphere, not counting the base, is $2\pi r^2$.

We integrate about the arc of the circumference. Let θ be the angle a point on the cross-section forms with the radius containing the foot from the point onto the base. Then we integrate about $t = r\theta$. Note integrating the circumferences gives

$$\int_0^{\frac{\pi r}{2}} 2\pi r \cos \frac{t}{r} dt = 2\pi r^2.$$

Multiplying by 2 implies that the surface area of the sphere is $4\pi r^2$.



■

Exercise 8 If we try to integrate about the height, as in

$$\int_0^r 2\pi\sqrt{r^2 - k^2} dk = 2\pi \int_0^r \sqrt{r^2 - k^2} dk,$$

we end up getting that the surface area of a hemisphere is $\frac{\pi r^2}{2}$. Why is this wrong?

§ 3.3.1 Rotating About An Axis: aka. Disks, Washers, and Shells

For those of you taking or have taken high-school calculus, you will know this as the disk and washer method, as well as the shell method. Despite the specific nomenclature of the disk and washer method, the underlying principles are quite intuitive.

Theorem 12 (Disk Method) To find the volume of layered disks perpendicular to an axis ℓ (these disks will usually be generated through rotation on the coordinate axis), evaluate

$$\int_a^b R(x) dx,$$

where $R(x)$ is the radius of the disk x above an arbitrary base.

This is (informally) apparent from Cavalieri's principle.[8]

Example 17 (Gabriel's Horn) The graph of $xy = 1$ where $x \geq 1$ is rotated about the x axis. Find the volume of the resultant solid, if it is finite at all.

Solution: Note that this is an improper integral; we are asked to evaluate

$$\lim_{b\to\infty} \pi \int_1^b \left(\frac{1}{x}\right)^2 dx = \lim_{b\to\infty} \pi \left(-\frac{1}{b} - \left(-\frac{1}{1}\right)\right) = \pi.$$

[8]We do not include the actual proof here, because that is out of the scope of this handout.

Exercise 9 (Extension of Gabriel's Horn) What if our problem with Gabriel's Horn had no domain restrictions; that is, what would the volume have been had we rotated the entirety of $xy = 1$ around the x axis? Would it still have been finite?

The disk method extends pretty naturally to the washer method. The question asked is "what happens to the volume if I cut circles of varying radiuses out of each layer?" and the answer is, "the cutout can also be integrated; evaluate the volume of the part you cut out."

Theorem 13 (Washer Method) Say we are finding the volume of the rotation of a curve around a line ℓ. We integrate about a plane perpendicular to the axis ℓ; if the maximum distance from a point to the axis is $R(x)$ and the minimum distance from a point to the axis is $r(x)$ on the plane x **directed** units away from an arbitrary base, then the volume of the curve is

$$\int_a^b (R(x)^2 - r(x)^2)dx,$$

provided that **every value between the minimum and maximum on each plane is somehow covered by the rotation.**[a]

In terms of the coordinate plane, you want the integral

$$\int_a^b (R(x)^2 - r(x)^2)dx,$$ [b]

where $y = R(x)$ is the outer radius (larger value) and $y = r(x)$ is the inner radius (smaller value) at x.

[a] In high-school calculus classes, this will usually be the case. High-school calculus contests such as Integration Bee will probably not ask you to know this.

[b] I am aware the formula is the same. No, it is not a coincidence or a mistake.

> **Remark:** This can easily be generalized to weirder curves with several "cutouts;" however, the computation becomes worse.

A variant of the dish and washer method is the **shell** method. The primary difference is that instead of integrating about the axis of revolution, we integrate instead about a line **perpendicular** to the axis of revolution.

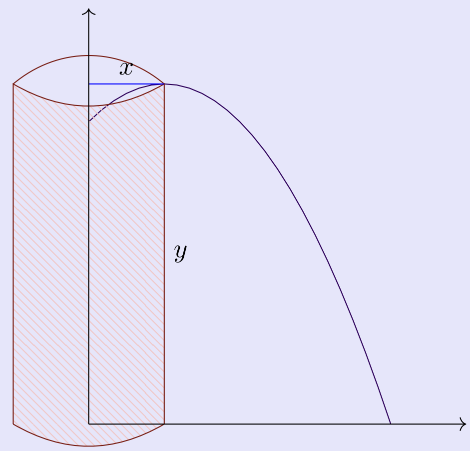
Theorem 14 (Shell Method) Say we are rotating some function $y = f(x)$ with bounds (a, b) around the **y**-axis. Then the volume of the resultant solid is

$$2\pi \int_a^b xydx.$$ [a]

[a] Another way to write the sum is $2\pi \int_a^b xf(x)dx$.

Proof: We could probably do this just with pure integral manipulation.[a] However, to do this would basically be missing the entire picture; as those of you more familiar with contest math might already have noticed, 2π strongly suggests the circumference of a circle.

It turns out that $2\pi xy$ actually represents the lateral surface area of a cylinder. I will let the following diagram do the rest of the talking for me.



Since the sum of these lateral surface areas is equivalent to the volume, we are done. ■

[a]For anyone interested, the integral manipulation follows:

We present a somewhat funny solution to Gabriel's Horn with this.

Solution (Gabriel's Horn): Note that by symmetry, this is equivalent to rotating $y \geq 1$ around the y axis, or $0 \leq x \leq 1$ around the y axis. Shift the function down by 1, since the range $0 \leq y \leq 1$ is not going to be rotated around the axis. Now by the Shell Method, this is just

$$2\pi \int_0^1 x\left(\frac{1}{x} - 1\right) dx = \pi.$$

Hilarious.

So what would we do if we encountered an axis of integration that was not an axis? All we have to do is transform it.

Example 18 Consider the curve bound by $y = -x^2 + 4$ and $y = 0$. Find the volume of the curve when rotated around $x = -1$.

Solution: We shift the axis of rotation to $x = 0$, attaining $y = -(x-1)^2 + 4$ as the new equation of our curve. Clearly, $|\sqrt{y}+1| > |\sqrt{y}-1|$ for positive y, so we can actually disregard the second quarant altogether.

Now we just use the Shell Method; the answer is

$$2\pi \int_0^3 x(-(x-1)^2 + 4)dx = \frac{45}{2}.$$

To finish off this section, we end with a quite insane example.

Example 19 The curve $xy = 1$, defined for $1 \leq x \leq a$ for positive $a > 1$, is rotated around the line $y = x$. Find, in terms of a, the volume of the resultant solid.

Solution: Read the appendix for an explanation of the techniques used here.

Rotate the axes by 45° and note that $x = x' \cos 45° - y' \sin 45°$ and $y = x' \sin 45° + y \cos 45°$, where x' and y' are the new coordinates. Then

$$\frac{1}{2}(x' - y')(x' + y') = 1,$$

or

$$x'^2 - y'^2 = 2,$$

where the domain is $x' \geq \sqrt{2}$ and the range is $y' \leq 0$. Now we need to interpret $x \leq a$; note that $x' = x \cos 45° + y \sin 45° = \frac{x+y}{\sqrt{2}}$. By AM-GM, the larger x is (or the further x is from y), the larger $x + y$ becomes. Thus, $x' \leq \frac{a+\frac{1}{a}}{\sqrt{2}} = \frac{a^2+1}{a\sqrt{2}}$.

Now we can just use the washer method; the integral is

$$\pi \int_{\sqrt{2}}^{\frac{a^2+1}{a\sqrt{2}}} (-\sqrt{x^2 - 2})^2 dx = \pi \int_{\sqrt{2}}^{\frac{a^2+1}{a\sqrt{2}}} (x^2 - 2) dx = \pi \left(\frac{x^3}{3} - 2x \right) \bigg|_{\sqrt{2}}^{\frac{a^2+1}{a\sqrt{2}}} = \frac{(a-1)^4(a^2+4a+1)}{6\sqrt{2}a^3}\pi.$$

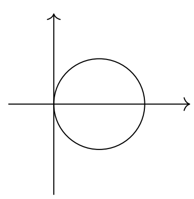
§ 3.4 Polar Functions

To determine a point in $\mathbb{R}^2$, you need two pieces of information. However, these two pieces of information need not directly be the x and y coordinates (rectangular form). They can be in another form: polar.

In polar coordinates, a point is defined with a distance from the origin and an angle.

Definition 8 (Polar Coordinates) The point (r, θ) in polar coordinates is the point $(r, 0)$ (rectangular coordinates) rotated by θ counterclockwise around the origin. In other words, it is equivalent to $(r \cos \theta, r \sin \theta)$ in rectangular coordinates.

Curves can be defined in the xy plane, like $x^2 + y^2 = 1$. However, we can also define the radius as a function of θ. Consider $r = 1$ as a very simple example: this is a circle because the output is the same for all θ. Another example is $r = \cos \theta$; this turns out to be the following graph.



There are plenty of good resources for learning polar coordinates out there, but this handout is not meant to act as an introduction to them. If you don't understand them, I would skip this section entirely.

Let's get to the area of a polar curve.

Theorem 15 (Area of a Polar Curve) The area of the polar curve

$$r = f(\theta)$$

on the interval (α, β) is

$$\int_{\alpha}^{\beta} \frac{1}{2} \theta r^2 d\theta.$$

The proof is fundamentally identical to the rectangular version, even if it looks different.

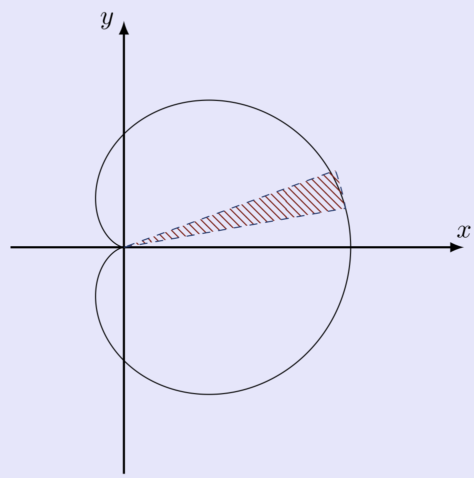
Proof: Say $F(\theta) = \int_{\alpha}^{\theta} \frac{1}{2}\gamma r^2 d\gamma$. (We use γ just because we're using θ as the input of the function; it is identical to the rest of the integrals.)

Note that

$$F'(\theta) = \lim_{\epsilon \to 0} \frac{F(\theta+\epsilon) - F(\theta)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\int_{\theta}^{\theta+\epsilon} f(\gamma) d\gamma}{\epsilon} = \frac{\frac{\epsilon}{2\pi} \cdot \pi f(\theta)^2}{\epsilon} = \frac{f(\theta)^2}{2}.$$

Note that if $F'(\theta) = f(\theta)$, then by the Fundamental Theorem of Calculus,

$$F(\beta) - F(\alpha) = \int_{\alpha}^{\beta} \theta r^2 d\theta.$$



■

Here are some motivating remarks. This is all based on the same idea as Riemann Sums for rectangles: where we added rectangles before, we are adding circular arcs now. If you wanted to be a little less rigorous, you could say that you're "adding" a lot of small circular arcs together and skip the Fundamental Theorem of Calculus stuff altogether. [9]

Why is r^2 squared? Think in terms of dimensional analysis: if the radius is doubled, it would quadruple the area, which is why we need the square. And why is there a factor of $\frac{1}{2}$? Because with a circular arc of radius r and angle ϵ, the area is $\frac{\epsilon}{2\pi}(\pi r^2) = \frac{1}{2}r^2$.

§ 3.5 Length of a Curve

There are three ways to describe a curve: through rectangular, parametric, and polar coordinates.[10] Each of them has their own length formula. It's easy to memorize them, but the important thing is understanding how they are all the same.

We begin with parametric curves and show that the other two are equivalent.

Theorem 16 (Arc Length with Parametric Curves) The arc length of a parametric function from a to b is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

[9] That isn't to say this proof is rigorous at all, because we haven't been taking things from an analysis perspective whatsoever.

[10] Strictly speaking, polar curves are a subset of parametric curves.

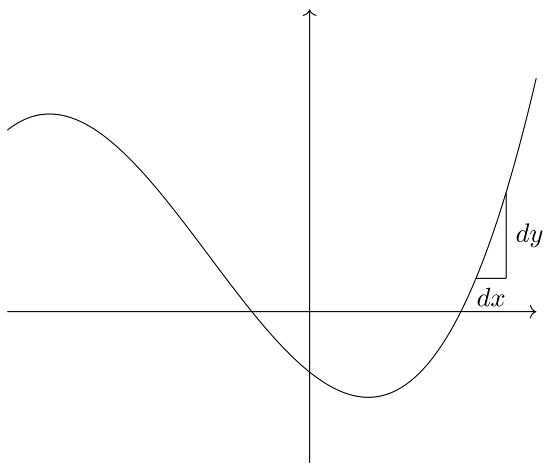
Here is a short explanation of why this *should* be true. Let me emphasize that **this is not a proof**. Note that the distance between $(x(t), y(t))$ and $(x(t+\epsilon), y(t+\epsilon))$ for small ϵ is

$$\sqrt{(dx)^2 + (dy)^2}$$

by the Pythagorean Theorem, and note this is equal to

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

as desired.



Theorem 17 (Arc Length in Rectangular Form) The arc length of a rectangular function from a to b is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Notice that this just follows from the parametric curve

$$\begin{aligned} x &= t \\ y &= f(t). \end{aligned}$$

Theorem 18 (Arc Length in Polar Form) The arc length of a polar function $r = f(\theta)$ from α to β is

$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Proof. This is just a standard parametric substitution plus some differentiation.

Note $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$, so by the Product Rule,

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta = \frac{dr}{d\theta}\cos\theta - r\sin\theta,$$

and similarly,

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta.$$

Now note

$$\begin{aligned}
\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2} &= \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2+\left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2} \\
&= \sqrt{\left(\frac{dr}{d\theta}\right)^2 (cos^2\theta+\sin^2\theta) + r\frac{dr}{d\theta}(-2\sin\theta\cos\theta+2\sin\theta\cos\theta)+r^2(\sin^2\theta+\cos^2\theta)} \\
&= \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}.
\end{aligned}$$

So the initial parametric integral simplifies to

$$\int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

□

§ 3.6 Bound Tricks

It is often hard to bound an integral. Here are a few tricks:

- If $1+a^x$ is in the denominator with bounds $-b$ to b, then apply $u=-x$.
- If the integrand is in the form $\frac{f(x-c)}{f(x-c)+f(-x+d)}$ with bounds from a to b, and $a+b=c+d$, then apply $u=a+b-x$. This is usually seen in the form $u=\frac{pi}{2}-x$ for trigonometric functions over $\left(0,\frac{\pi}{2}\right)$ or $u=1-x$ over $(0,1)$.
- If x^2+a^2 is in the denominator with bounds 0 to ∞ or 1 to a (which is common with $\ln x$ and $\arctan x$), apply $u=\frac{a}{x}$.

Exercise 10 Evaluate $\int_{-\infty}^{\infty}\frac{1}{(1+x^2)(1+e^x)}\,dx$.

§ 3.7 Weierstrass Substitution

The **Weierstrass substitution** is simply $t=\tan\left(\frac{x}{2}\right)$. Under this substitution, we have

$$\sin x=\frac{2t}{1+t^2}\qquad \cos x=\frac{1-t^2}{1+t^2}\qquad dx=\frac{2}{1+t^2}\,dx.$$

his allows us to turn trigonometric expressions into rational functions. Using $t=\tanh\left(\frac{x}{2}\right)$, we can similarly derive

$$\sinh x=\frac{2t}{1-t^2}\qquad \cosh(x)=\frac{1+t^2}{1-t^2}\qquad dx=\frac{2}{1-t^2}\,dx.$$

Exercise 11 Evaluate $\int_0^{\frac{\pi}{2}}\frac{1}{3+\cos x}\,dx$.

§ 4 List of Integration Techniques

Your toolbox, with a few additions not covered in this handout.

- Substitutions
  - u-Substitution
  - Trigonometric Substitution
  - Weierstrass Substitution
  - Bounding Tricks
  - Inversion (Reverse u-Substitution)
- Partial Fraction Decomposition
- Integration by Parts
  - $u = e^x$
  - $u' = 1$ (i.e. $u = x$)
  - Tabular Integration (i.e. making a chart)
- Parametrization
- Feynman's Technique (Differentiation Under the Integral)

§ 5 Problems

Minimum is [32✐]. Problems with the ♦ symbol are required.

> "Oh, I *will* find it. You may pretend he is not here, but I will find him, though I dig forever!"
>
> The Count of Monte Cristo

[2✐] **Problem 1** (SMT 2018/4) Compute

$$\int_0^4 \frac{dx}{\sqrt{|x-2|}}.$$

[2✐] **Problem 2** (Putnam Calculus Problems 2016-I/1) Show that for all $x > 1$,

$$\int_1^x e^{-t^2} dt < \frac{1}{2e}.$$

[2♦] **Problem 3** Find $\int f(x)f'(x)dx$.

[3♦] **Problem 4** Find $\int e^x \sin x dx$.

[3✐] **Problem 5** (SMT 2018/3) Find the value of a such that

$$\int_1^a \left(3x^2 - 6x + 3\right) dx = 27.$$

[3✐] **Problem 6** (Vishal Muthuvel) Find the average value of the function $f(x) = \cos(2x) - \sin(\frac{x}{2})$ on the interval $[\frac{-\pi}{2}, \pi]$.

[4✐] **Problem 7** Evaluate

$$\int \frac{1}{x^2 + c} dx.$$

[4✐] **Problem 8** (SMT 2019/3) Compute $\int_0^{\pi/4} \cos x - 2 \sin x \sin 2x dx$.

[4♦] **Problem 9 (AMC 12A 2016/23)** Three numbers in the interval $[0, 1]$ are chosen independently and at random. What is the probability that the chosen numbers are the side lengths of a triangle with positive area?

[6✐] **Problem 10** (MIT OCW) Evaluate the following limits:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\sqrt{1 + \frac{2i}{n}}\right] \frac{2}{n}$$

$$\lim_{h\to\infty} \frac{1}{h} \int_2^{2+h} \sin(x^2) dx.$$

[6 ✎] **Problem 11** (Joe Foster) Find
$$\int \frac{1}{\sqrt{8x - x^2}} dx.$$

[6 ✎] **Problem 12** (SMT Calculus 2019/10) Evaluate $\int_0^2 \frac{\ln(1+x)}{x^2-x+1}\, dx$.

[6 ✎] **Problem 13** (David Altizio) Evaluate
$$\int \frac{2x}{x^4+1}, dx.$$

[9 ✎] **Problem 14** Evaluate
$$\int \frac{dx}{e^x - 1}.$$

[9 ◇] **Problem 15 (AMC 10A 2015/25)** Let S be a square of side length 1. Two points are chosen independently at random on the sides of S. The probability that the straight-line distance between the points is at least $\frac{1}{2}$ is $\frac{a-b\pi}{c}$, where a, b, and c are positive integers and $\gcd(a, b, c) = 1$. What is $a + b + c$?

[9 ✎] **Problem 16** (SMT 2019/7) Turn the graph of $y = \frac{1}{x}$ by $45°$ counter-clockwise and consider the bowl-like top part of the curve (the part above $y = 0$). We let a 2D fluid accummulate in this 2D bowl until the maximum depth of the fluid is $\frac{2\sqrt{2}}{3}$. What's the area of the fluid used?[11]

[9 ✎] **Problem 17** (MIT Integration Bee 2006) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos(x)+\sin(x)}{9+16\sin(2x)}\, dx$.

[13 ✎] **Problem 18** (AIME I 2015/15) A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge of one of the circular faces of the cylinder so that $\widehat{AB}$ on that face measures $120°$. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. The area of one of these unpainted faces is $a \cdot \pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.

[11] Reading Section B of the Appendix may help, but is strictly unnecessary.

§ A Partial Fraction Decomposition

The contents of this section can be found on the appendix of AQU-Telescoping, but they are also reproduced here for your convenience.

Definition 9 (Partial Fraction Decomposition) The partial fraction decomposition of a fraction

$$\frac{f(x)}{(x-r_1)^{c_1}(x-r_2)^{c_2}\cdots(x-r_n)^{c_n}}$$

is of the form

$$\sum_{i=1}\frac{f_i(x)}{(x-r_i)^{c_i}} = \frac{f_1(x)}{(x-r_1)^{c_1}} + \frac{f_2(x)}{(x-r_2)^{c_2}} + \cdots + \frac{f_n(x)}{(x-r_n)^{c_n}}.$$

The traditional way to solve this is by setting a system of equations. We take the well-known $\frac{1}{x} - \frac{1}{x+1}$ partial fraction decomposition as an example.

Example 20 Find the partial fraction decomposition of $\frac{1}{x(x+1)}$.

Solution: Note that the partial fraction decomposition is of the form

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}.$$ [12]

Now we multiply out the fraction to get

$$1 = A(x+1) + Bx.$$

This implies that $A+B=0$ and that $A=1$. Thus $B=-1$, and the PFD is

$$\frac{1}{x} - \frac{1}{x+1}.$$

We proceed with a harder example.

Example 21 (Three Terms) Find the partial fraction decomposition of $\frac{1}{n(n+1)(n+2)}$.

Solution: Let the decomposition be $\frac{1}{n(n+1)(n+2)} = \frac{I}{n} + \frac{J}{n+1} + \frac{K}{n+2}$. This implies that

$$I(n+1)(n+2) + J(n)(n+2) + K(n)(n+1) = 1.$$

We see that the following system of equations results from coefficient matching:

$$n^2(I+J+K) = 0 \implies I+J+K=0$$

$$n(3I+2J+K) = 0 \implies 3I+2J+K=0$$

$$2I = 1.$$

Solving gives us $I = 1/2, J = -1, K = 1/2$, which means our partial fraction decomposition is $\frac{1/2}{n} - \frac{1}{n+1} + \frac{1/2}{n+2}$.

In calculus classes, the forbidden values method is taught, though usually with little explanation given as to why it should be true or why it should work. My goal is to give the reader a feeling for why this should be true, give concrete examples as to when it works, and show when it doesn't.

[12] We can use A and B as constants because the degree of x and $x+1$ is one less than the degree of $x(x+1)$.

Theorem 19 (Forbidden Values) Given some partial fraction decomposition

$$\frac{f(x)}{g_1(x)g_2(x)\cdots g_n(x)} = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)} + \cdots + \frac{f_n(x)}{g_n(x)},$$

the functions $f_1, f_2, \ldots, f_n$ can be determined via substituting the roots of $g_i(x)$ for all $1 \le i \le n$ into

$$f(x) = f_1(x)g_2(x)\cdots g_n(x) + \cdots + f_n(x)g_1(x)\cdots g_{n-1}(x).$$

The natural instinct to have after using this 'trick' a couple times is, "Wait, why does this actually work?" and it is very easy to just relegate this to a trick that you don't think about. However, there is a well-founded reason that this works, and it is rooted in polynomial and root analysis. We remind the reader of the following theorem from **AQU-Factorize**.

Theorem 20 (Infinite Roots) If a degree n polynomial has more than n roots, it must have infinite roots.

Now the proof follows naturally.

Proof of Forbidden Values: Note that as

$$\frac{f(x)}{g_1(x)g_2(x)\cdots g_n(x)} = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)} + \cdots + \frac{f_n(x)}{g_n(x)}$$

has infinite roots, so must

$$f(x) = f_1(x)g_2(x)\cdots g_n(x) + \cdots + f_n(x)g_1(x)\cdots g_{n-1}(x).$$

This implies that

$$f(x) - (f_1(x)g_2(x)\cdots g_n(x) + \cdots + f_n(x)g_1(x)\cdots g_{n-1}(x)) = 0$$

for all values of x, even the roots of $g_i(x)$. ■

Forbidden values can instantly tell us how a fraction decomposes. We present the following corollary that eliminates most algebraic manipulation from PFD, aside from the initial factorization of the denominator.

Theorem 21 (Only Linear g_i Work) Given some partial fraction decomposition

$$\frac{f(x)}{g_1(x)g_2(x)\cdots g_n(x)} = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)} + \cdots + \frac{f_n(x)}{g_n(x)},$$

where g_i is of the form $(x - r_i)^{c_i}$,

$$f_i(r_i) = \frac{f(r_i)}{\prod\limits_{1 \le k \le n, k \ne i} g_i(r_i)},$$

where r_i is the root of g_i.

In the case where all g_i are linear, f_i are all constant, so

$$f_i = \frac{f(r_i)}{\prod\limits_{1 \le k \le n, k \ne i} g_i(r_i)}.$$

Keep in mind that this only explicitly describes the values achieved from forbidden values; you should almost always clear the fractions and substitute the forbidden values yourself instead of trying to recall the exact formula. The corollary does little good especially when some g_i is not linear. If no g_i are linear, the forbidden values method is completely useless, since no f_i will be linear. In fact, forbidden values works exactly when g_i is linear, even if other g are not.

Let me state it more explicitly: **substituting forbidden values only works for linear g_i.**

Example 22 Find the partial fraction decomposition of $\frac{1}{x(x+1)}$.

Solution: With our new forbidden values method in hand, we set

$$\frac{1}{x(x+1)} = \frac{f_1}{x} + \frac{f_2}{x+1}$$

and multiply out to get

$$1 = f_1 \cdot (x+1) + f_2 \cdot (x).$$

Since x and $x+1$ are both linear, f_1 and f_2 are both constants. Substituting $x = 0$ gives $f_1 = 1$ and $x = -1$ gives $f_2 = -1$, so our PFD is

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

Here's a partial forbidden values solution of a PFD; one of the terms in the denominator is linear while the other is not.

Example 23 Find the partial fraction decomposition of $\frac{2x^2-4x+1}{(x-1)^2(x-2)}$.

Solution: Note that the fraction decomposes into the form

$$\frac{2x^2-4x+1}{(x-1)^2(x-2)} = \frac{f_1(x)}{(x-1)^2} + \frac{f_2(x)}{x-2},$$

which is equivalent to

$$2x^2 - 4x + 1 = f_1(x)(x-2) + f_2(x)(x-1)^2.$$

Plugging in $x = 1$ yields $-1 = -f_1(1)$ or $1 = f_1(1)$, and $x = 2$ yields $1 = f_2(2)$. Degree analysis only tells us that f_2 is constant, or $f_2(x) = 1$. Now we have

$$2x^2 - 4x + 1 = f_1(x)(x-2) + (x-1)^2.$$

Now we directly solve for f_1. Note that the equation is equivalent to

$$x^2 - 2x = f_1(x)(x-2)$$

$$f_1(x) = x.$$

Thus, the PFD is

$$\frac{2x^2-4x+1}{(x-1)^2(x-2)} = \frac{x}{(x-1)^2} + \frac{1}{x-2}.$$

Here is a crown example of the forbidden values method.

Example 24 (AMC 10A 2019/24) Let p, q, and r be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. It is given that there exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \notin \{p, q, r\}$. What is $\frac{1}{A} + \frac{1}{B} + \frac{1}{C}$?

Solution: This is the same as solving for A, B, C such that

$$1 = A(s-q)(s-r) + B(s-r)(s-p) + C(s-p)(s-q).$$

Substitute the forbidden values of $s = p, q, r$ to get

$$\begin{aligned} 1 &= A(p-q)(p-r) \\ 1 &= B(q-r)(q-p) \\ 1 &= C(r-p)(r-q), \end{aligned}$$

which we get from substituting $s = p, q, r$, respectively. This then implies

$$\begin{aligned} \frac{1}{A} &= (p-q)(q-r) \\ \frac{1}{B} &= (q-r)(q-p) \\ \frac{1}{C} &= (r-p)(r-q). \end{aligned}$$

At this point we can just finish with Vieta's Formulas. Note that

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = (p-q)(p-r) + (q-r)(q-p) + (r-p)(r-q) =$$
$$p^2 - pq - pr + qr + q^2 - qr - qp + rp + r^2 - rp - rq + pq =$$
$$(p+q+r)^2 - 3(pq+qr+rp) = 22^2 - 3 \cdot 80 = 244.$$

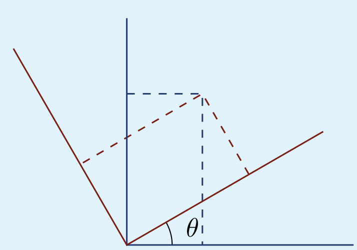
Exercise 12 (Adapted from AoPS Calculus 5.35) Find the partial fraction decomposition of

$$\frac{3}{x^3-1}.$$

§ B Rotation of Coordinates

This is just for fun for anyone who really wants to understand the last example presented.

Theorem 22 (Rotation of Axes) If the coordinate axes x, y are rotated by θ to produce a new coordinate system x', y', then a point (x, y) in the old coordinate system becomes $(x', y') = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$.



Understanding this theorem requires a good bit of experience with polar coordinates. We express the coordinates of the point in polar form; if $(x, y) = (r, \alpha)$, then by definition, $(x', y') = (r, \alpha - \theta)$. Now by the sine and cosine addition formulae,

$$x' = r\cos(\alpha - \theta) = r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = x\cos\theta + y\sin\theta$$

$$y' = r\sin(\alpha - \theta) = r\sin\alpha\cos\theta - r\cos\alpha\sin\theta = y\cos\theta - x\sin\theta.$$

That's all fine and dandy, but this is the perfect excuse to show you a little bit of linear algebra. Note that the above system of equations can perfectly be expressed in the following equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Why do we express the rotation matrix like this? Because what we're doing is really equivalent to **rotating the point** by $-\theta$ around the origin. Thus, commonly, the rotation matrix is defined as following.

Definition 10 (Rotation Matrix) The two-dimensional rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is used to rotate some two-dimensional vector $\vec{v}$ by θ, and the rotation is done by

$$R\vec{v} = \vec{v_1},$$

where $\vec{v_1}$ is the resultant vector.

Keep in mind this definition follows from the use of polar coordinates and the sine and cosine addition formulae. With the rotation matrix, we can easily write the converse of our original transformation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos -\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Example 25 Evaluate

$$\begin{bmatrix} \sqrt{6}+\sqrt{2} & -\sqrt{6}+\sqrt{2} \\ \sqrt{6}-\sqrt{2} & \sqrt{6}+\sqrt{2} \end{bmatrix}^6.$$

Solution: The first instinct of someone who sees this problem might be to diagonalize; however, the expressions $\sqrt{6} \pm \sqrt{2}$ remind of us $\cos 15°$ and $\sin 15°$. Indeed, note that if our matrix is A,

$$A = 4R_{15°},$$

where R_θ is defined as the rotation matrix of θ. Thus,

$$A^6 = 4^6 R_{15°}^6,$$

and

$$R_{15°}^6 = R_{90°}{}^{13} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

so the answer is

$$\begin{bmatrix} 0 & -4^6 \\ 4^6 & 0 \end{bmatrix}.$$

Now we condense the exposition above into a proof of our initial theorem.

Proof: This is equivalent to rotating the point by $-\theta$ around the origin; thus, we multiply

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}.$$

■

[13] This is because six rotations of 15° are equivalent to a rotation of 90°.

I cannot stress this enough: **matrices are convenient representations of linear mappings**, not random arrays of numbers; hopefully this has shown you one of the fantastic uses of matrices and you don't write them off as "arcane magic we learned about in high school that one time." There's a reason I included the last example in Rotating About An Axis, and it wasn't just to show how transformations could make Disks, Washers, and Shells usable.

Contest Rules

for CMIMC, HMMT, and PUMaC

Dylan Yu

February 24, 2021

A basic rundown of the rules. The official lists can be found here for CMIMC, here for HMMT, and here for PUMaC. The rules marked with ☣ are related to online testing.

1 CMIMC

- TCS Round
  - Proof
  - 90 minutes
  - 3 problems
  - Done with team
- Team Round
  - Short answer
  - 60 minutes
  - 15 problems
- Individual Rounds
  - Short answer
  - 60 minutes
  - 8 problems
  - Three rounds: algebra & NT, combinatorics & CS, and geometry
  - Two divisions: 1 (harder) and 2 (easier)
- Submitting
  - You are given some time to submit at the end
  - ☣ For TCS submit a PDF to a Google form (i.e. solutions are required)
  - ☣ For Team and Individual submit on the website
- Proctoring
  - ☣ Stay muted
  - ☣ Turn your camera on
  - ☣ No computational tools (e.g. calculators)

2 HMMT

- Guts Round
  - Short answer
  - 80 minutes
  - 36 problems, given in sets of 3 in **November** and 4 in **February** (after completing a set you turn it in to receive the next one)
  - Done with team
- Team Round
  - Short answer
  - 60 minutes
  - 10 problems
  - **November** is *short-answer*
  - **Feburary** is *proof*
- Individual Round
  - Short answer
  - 50 minutes
  - 10 problems
  - **November** has two rounds: general and theme
  - **February** has three rounds: algebra & NT, geometry, and combinatorics
- Submitting
  - Everything must be handwritten (in particular, do not type or LaTeX)
  - Use website to submit everything
  - Don't forget to save your answers; otherwise the website won't record them
  - Scratch work must be uploaded (for each question submit an image/PDF of your answer and work for that question)[1]
- Proctoring
  - You must join a Zoom meeting for all rounds and events
  - Cameras must be on and do **not** use virtual backgrounds

3 PUMaC

- Power Round
  - Proof
  - 1 week
  - # of problems varies; includes several parts
  - Done with team

[1] Download something like TinyScanner or CamScanner to easily take pictures with.

  - You can use question a to prove question b if $a < b$, but (to my knowledge) not vice versa

- Individual Round
  - Short answer
  - 60 minutes
  - 8 problems
  - Four rounds: algebra, combinatorics, geometry, and NT (each contestant chooses 2)
  - Two divisions: A (harder) and B (easier)
- Team Round
  - Short answer
  - 30 minutes (sometimes varies)
  - # of problems varies
  - There is a "game" component in which teams have the opportunity to increase their score (see previous tests)
- Individual Finals
  - Proof
  - 90 minutes
  - 3 problems
  - Only top 10 individuals in each test and each division take this test
- Live Round
  - Literally just a random set of problems with everyone doing the test at the same time (similar to Guts)
  - Done with team

> *Remark 1.* I have no idea how submissions/proctoring will be done at PUMaC; someone please let me know.