Coaxial Circles

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In this article, we will discuss about the properties of coaxial circles.

So far, we have studied tools for dealing with pairs of circles that have distinct radical axes. Let us now consider the situation where multiple circles share the same radical axis.

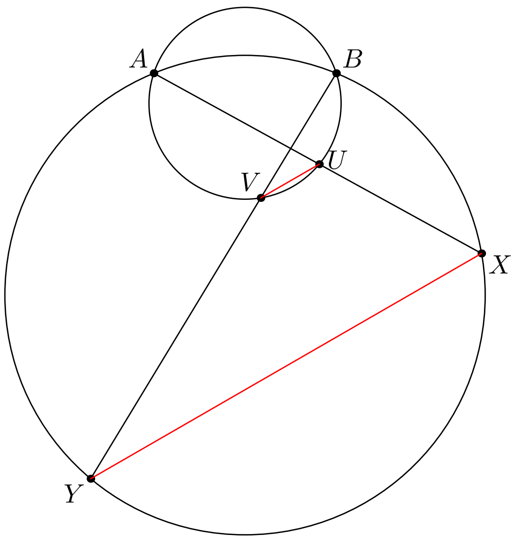
> **Definition 0.1.** A family of circles is called **Coaxial** if they all share the same radical axis.

§1 Reim's Theorem

This particular configuration appears very frequently in geometry problems.

Theorem 1.1 (Reim's Theorem)

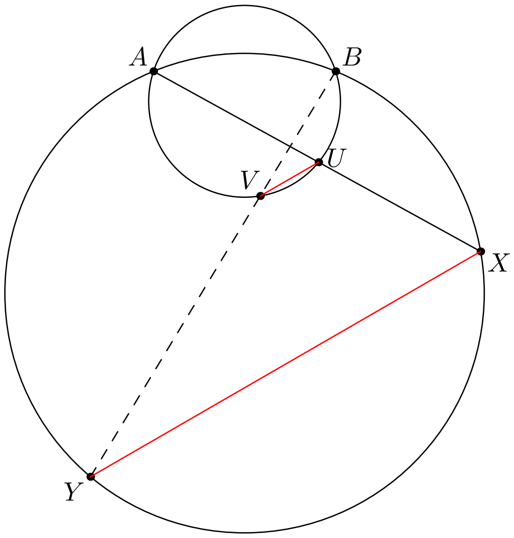
Suppose circles ω_1 and ω_2 intersect in points A and B. Let ℓ_1 and ℓ_2 be two lines through passing through A and B such that they intersect ω_1 in U and V, and ω_2 in X and Y. Then $\overline{UV} \parallel \overline{XY}$.



Furthermore, the converse of this theorem is true too.

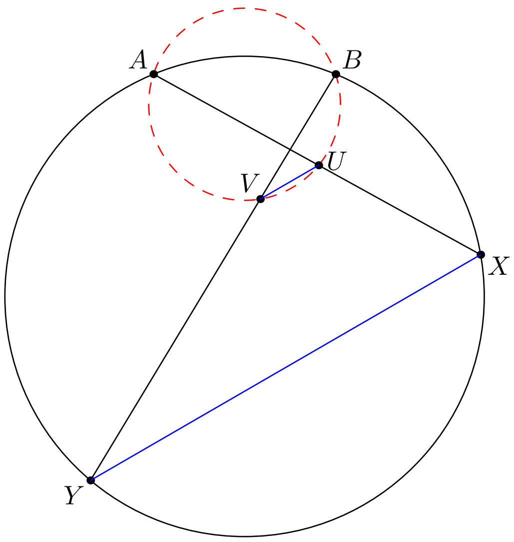
Theorem 1.2 (Converse of Reim's Theorem (Collinearity))

Suppose two circles ω_1 and ω_2 intersect in points A and B. Let ℓ be a line that passes through A and intersects ω_1 and ω_2 at X and U. Suppose Y and V lie on circles ω_1 and ω_2 such that $\overline{UV} \parallel \overline{XY}$. Then, points B, V and Y are collinear.



Theorem 1.3 (Converse of Reim's Theorem (Concyclicity))

Given a circle ω and four points A, B, X and Y on the circle. Choose points U and V on $\overline{AX}$ and $\overline{BY}$ such that $\overline{UV} \parallel \overline{XY}$. Then the points A, B, U and V are concyclic.



To be precise, both of these results follow from a simple two-step angle chase. Nevertheless, because they appear so frequently in various configurations, it is important to have them at your fingertips to avoid overlooking any pair of parallel lines or cyclic quadrilaterals.

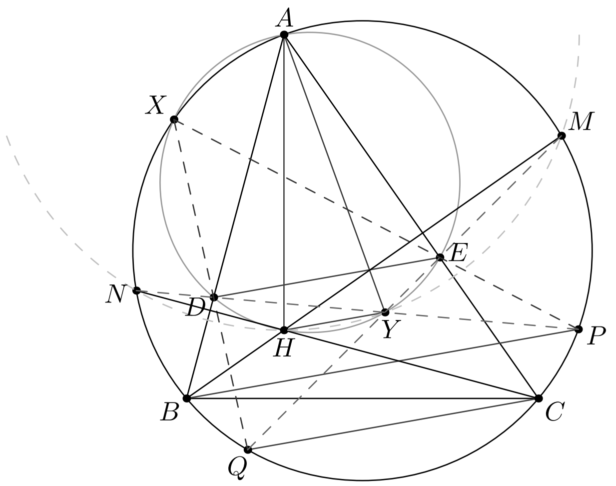
§1.1 Examples

Problem 1.4 (USA TSTST 2019)

Let ABC be an acute triangle with circumcircle Ω and orthocenter H. Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q, respectively. Denote by ω the circumcircle of $\triangle ADE$.

1. Show that lines PE and QD meet on ω.

2. Prove that if ω passes through H, then lines PD and QE meet on ω as well.



Proof. For the first part, suppose $\omega \cap \Omega$ at $X \neq A$. Then applying converse of reim's theorem on $\overline{DE} \parallel \overline{BP}$, we get that PE passes through X. Similarly, QD passes through X proving that PE and QD indeed meet on ω at X.

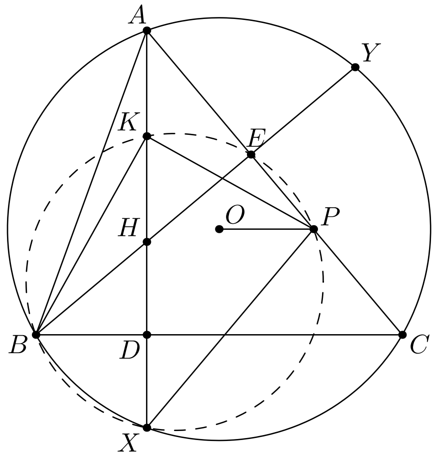
For the second part, suppose Y lies on ω such that $\overline{AH}$ and$\overline{AY}$ are isogonal with respect to $\angle BAC$. Then, $\overline{HY} \parallel \overline{DE}$. Let M and N be the reflections of H over $\overline{AC}$ and $\overline{AB}$. We can show that D lies on $\overline{NY}$, because

$$\begin{aligned}\angle NDH &= 2\angle BDH \\ &= 2\left(180^\circ - \angle ADH\right) \\ &= 2\angle AYH \\ &= 2\left(90^\circ - \tfrac{1}{2}\angle HAY\right) \\ &= 180^\circ - \angle HDY\end{aligned}$$

Similarly, we can show that E lies on $\overline{MY}$. With a simple angle chase, we can show that $NHYM$ is a cyclic quadrilateral. By the converse of reim's theorem applied on circle $\odot(NHYM)$ and $\odot(ABC)$, and pairs of parallel lines $\overline{HY} \parallel \overline{BP}$ and $\overline{HY} \parallel \overline{QC} \implies Y$ lies on $\overline{PD}$ and $\overline{QE}$, as desired. □

Problem 1.5 (Iran 2015)

Let ABC be a triangle with orthocenter H and circumcenter O. Let K be the midpoint of AH. point P lies on AC such that $\angle BKP = 90^\circ$. Prove that $OP \parallel BC$.



Proof. Let D and E be the foot of perpendicular from A and B onto $\overline{BC}$ and $\overline{AC}$, and X and Y be the reflections of H over $\overline{BC}$ and $\overline{AC}$. Since K and E are the midpoints of $\overline{AH}$ and $\overline{HY}$, therefore by the midpoint theorem $\overline{KE} \parallel \overline{AY}$. Applying the converse of reim's theorem, we get that $BKEX$ is a cyclic quadrilateral. Since $\angle BKP = 90^\circ$ and $\angle BEP = 90^\circ \implies P$ lies on the circle $\odot(BKEX)$. Since K is the center of $\odot(AEH)$, we have

$$\angle XAP = \angle KAE = \angle KEA = 180^\circ - \angle KEP = \angle KXP = \angle AXP$$

Therefore, $\triangle PAX$ is isosceles and hence P lies on the perpendicular bisector of $\overline{AX}$. Since, O lies on the perpendicular bisector of $\overline{AX}$ too $\implies \overline{OP} \perp \overline{AX}$. But $\overline{AX} \perp \overline{BC}$ $\implies \overline{OP} \parallel \overline{BC}$. □

§1.2 Exercises

Exercise 1.6 (Iran IMO TST 2008)**.** In the triangle ABC, $\angle B$ is greater than $\angle C$. Suppose T is the midpoint of the arc BAC from the circumcircle of ABC and I is the incenter of ABC. Let E be a point such that $\angle AEI = 90^\circ$ and $AE \parallel BC$. Let $\overline{TE}$ intersect the $\odot(ABC)$ for the second time in P. If $\angle B = \angle IPB$, find the angle $\angle A$.

Exercise 1.7 (IMO 2019)**.** In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$. Prove that points P, Q, P_1 and Q_1 are concyclic.

Exercise 1.8 (APMO 2022)**.** Let ABC be a right triangle with $\angle B = 90^\circ$. Point D lies on the line CB such that B is between D and C. Let E be the midpoint of AD and let F be the second intersection point of the circumcircle of $\triangle ACD$ and the circumcircle of $\triangle BDE$. Prove that as D varies, the line EF passes through a fixed point.

Exercise 1.9 (APMO 2024)**.** Let ABC be an acute triangle. Let D be a point on side AB and E be a point on side AC such that lines BC and DE are parallel. Let X be an interior point of $BCED$. Suppose rays DX and EX meet side BC at points P and Q, respectively, such that both P and Q lie between B and C. Suppose that the circumcircles of triangles BQX and CPX intersect at a point $Y \neq X$. Prove that the points A, X, and Y are collinear.

§2 Forgotten Coaxiality Lemma

Another powerful result to show that a circle is coaxial with other two circles is the following.

Theorem 2.1 (Forgotten Coaxiality Lemma)
Given two circles ω_1 and ω_2 that intersect in A and B. Then the locus of points P such that

$$\frac{\mathrm{Pow}_{\omega_1}(P)}{\mathrm{Pow}_{\omega_2}(P)} = k$$

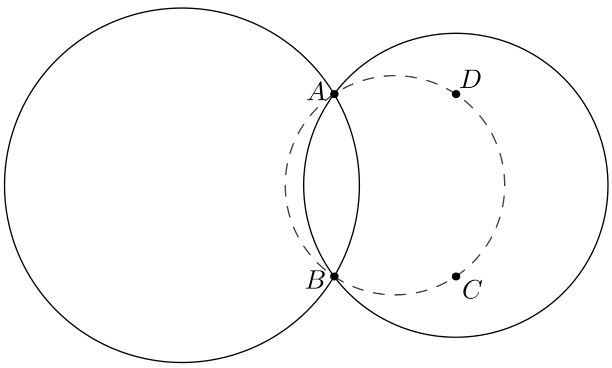
for a real constant k, is a circle that is passes through A and B.

An equivalent way to state the above result is to consider the following situation

Example 2.2
Given two circles ω_1 and ω_2 that intersect in A and B. For some points C and D we would like to show that A, B, C and D are concyclic.

Using the forgotten coaxiality lemma, we can essentially ignore points A and B and the concyclicity would be immediately implied if

$$\frac{\mathrm{Pow}_{\omega_1}(C)}{\mathrm{Pow}_{\omega_2}(C)} = \frac{\mathrm{Pow}_{\omega_1}(D)}{\mathrm{Pow}_{\omega_2}(D)}$$



Proof. Suppose AD intersects ω_1 and ω_2 at D_1 and D_2, and BC intersects ω_1 and ω_2 at C_1 and C_2. Therefore,

$$\frac{\mathrm{Pow}_{\omega_1}(C)}{\mathrm{Pow}_{\omega_2}(C)} = \frac{\overline{CB} \cdot \overline{CC_1}}{\overline{CB} \cdot \overline{CC_2}} = \frac{\overline{CC_1}}{\overline{CC_2}}$$

Similarly,

$$\frac{\mathrm{Pow}_{\omega_1}(D)}{\mathrm{Pow}_{\omega_2}(D)} = \frac{\overline{DA} \cdot \overline{DD_1}}{\overline{DA} \cdot \overline{DD_2}} = \frac{\overline{DD_1}}{\overline{DD_2}}$$

However by reim's theorem, $\overline{C_1D_1} \parallel \overline{CD} \parallel \overline{C_2D_2}$. Hence,

$$\frac{\overline{CC_1}}{\overline{CC_2}} = \frac{\overline{DD_1}}{\overline{DD_2}}$$

which proves that,

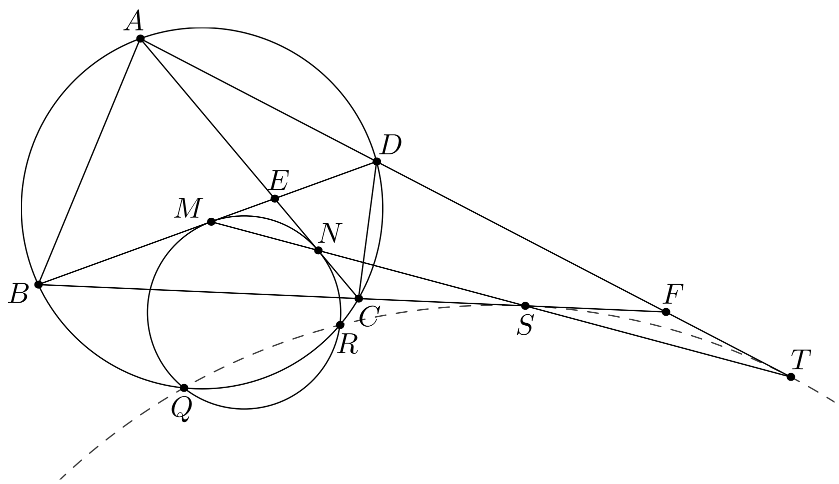
$$\frac{\text{Pow}_{\omega_1}(C)}{\text{Pow}_{\omega_2}(C)} = \frac{\text{Pow}_{\omega_1}(D)}{\text{Pow}_{\omega_2}(D)}$$

□

§2.1 Examples

Problem 2.3 (CGMO 2017)

Let the $ABCD$ be a cyclic quadrilateral with circumcircle ω_1. Lines AC and BD intersect at point E, and lines AD, BC intersect at point F. Circle ω_2 is tangent to segments EB, EC at points M, N respectively, and intersects with circle ω_1 at points Q, R. Lines BC, AD intersect line MN at S, T respectively. Show that Q, R, S, T are concyclic.



Proof. We would like to show that $QRST$ is a cyclic quadrilateral. Since $\overline{QR}$ is the radical axis of $\odot(ABCD)$ and $\odot(MNQR) \implies$ we need to show $QRST$ is coaxial with these two circles. Applying the forgotten coaxiality lemma, we want to show

$$\frac{\text{Pow}_{\omega_1}(S)}{\text{Pow}_{\omega_2}(S)} = \frac{\text{Pow}_{\omega_1}(T)}{\text{Pow}_{\omega_2}(T)}$$
$$\iff \frac{\overline{SB} \cdot \overline{SC}}{\overline{SM} \cdot \overline{SN}} = \frac{\overline{TA} \cdot \overline{TD}}{\overline{TN} \cdot \overline{TM}}$$

Since $\triangle EMN$ is isosceles $\implies \angle CNS = \angle ENM = \angle DMT$. And, $\angle ACB = \angle ADB \implies \angle NCS = \angle MDT$. Therefore by AA similarity criterion, we have $\triangle NCS \sim \triangle MDT$. Similarly, $\triangle BMS \sim \triangle ANT$. These two similar triangles imply

$$\frac{\overline{SC}}{\overline{SN}} = \frac{\overline{TD}}{\overline{TM}} \quad \& \quad \frac{\overline{SB}}{\overline{SM}} = \frac{\overline{TA}}{\overline{TN}}$$

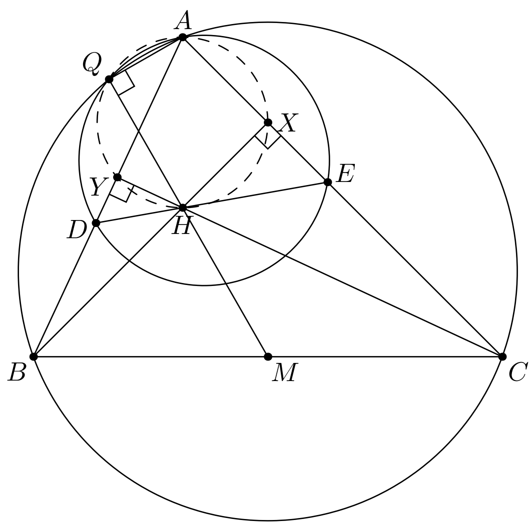
Multiplying these two equalities we get the desired result

$$\frac{\overline{SB} \cdot \overline{SC}}{\overline{SM} \cdot \overline{SN}} = \frac{\overline{TA} \cdot \overline{TD}}{\overline{TN} \cdot \overline{TM}} \implies \frac{\text{Pow}_{\omega_1}(S)}{\text{Pow}_{\omega_2}(S)} = \frac{\text{Pow}_{\omega_1}(T)}{\text{Pow}_{\omega_2}(T)}$$

Therefore, $QRST$ is coaxial with ω_1 and $\omega_2 \implies QRST$ is cyclic. □

Problem 2.4 (IMO Shortlist 2005)

Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.



Proof. It's well known that if the ray MH intersects $\odot(ABC)$ at Q, then $\overline{AQ} \perp \overline{HM}$. This suggests us that Q must lie on $\odot(ADE)$. To show that $ADEQ$ is cyclic, we only need to show

$$\frac{\operatorname{Pow}_{\odot(ABC)}(D)}{\operatorname{Pow}_{\odot(AH)}(D)} = \frac{\operatorname{Pow}_{\odot(ABC)}(E)}{\operatorname{Pow}_{\odot(AH)}(E)}$$

Suppose X and Y are the feet of perpendicular from B and C onto $\overline{AC}$ and $\overline{AB}$. Then,

$$\begin{aligned} \frac{\operatorname{Pow}_{\odot(ABC)}(D)}{\operatorname{Pow}_{\odot(AH)}(D)} &= \frac{\operatorname{Pow}_{\odot(ABC)}(E)}{\operatorname{Pow}_{\odot(AH)}(E)} \\ \iff \frac{\overline{DA} \cdot \overline{DB}}{\overline{DY} \cdot \overline{DA}} &= \frac{\overline{EA} \cdot \overline{EC}}{\overline{EX} \cdot \overline{EA}} \\ \iff \frac{\overline{DB}}{\overline{DY}} &= \frac{\overline{EC}}{\overline{EX}} \end{aligned}$$

Since, $\angle YHB = \angle BAC$ and $\angle YHD = 90^\circ - \angle YDH = \frac{1}{2}\angle BAC$. Therefore, $\overline{DH}$ is the angle bisector of $\angle YHB$ and similarly $\overline{HE}$ is the angle bisector of $\angle CHX$. Using the fact that $BYXC$ is a cyclic quadrilateral, we can write

$$\begin{aligned} \overline{HY} \cdot \overline{HC} &= \overline{HX} \cdot \overline{HB} \\ \implies \frac{\overline{HY}}{\overline{HB}} &= \frac{\overline{HX}}{\overline{HC}} \\ \implies \frac{\overline{DY}}{\overline{DB}} &= \frac{\overline{EX}}{\overline{EC}} \end{aligned}$$

Due to the forgotten coaxiality lemma, we have $AQDE$ is a cyclic quadrilateral $\implies$ $\overline{HM}$ is perpendicular to the radical axis of $\odot(ABC)$ and $\odot(ADE)$. $\square$

§2.2 Exercises

Exercise 2.5. In triangle $\triangle ABC$, let E and F be points on sides AC and AB, respectively, such that $BFEC$ is cyclic. Let lines BE and CF intersect at point P, and M and N be the midpoints of $\overline{BF}$ and $\overline{CE}$, respectively. If U is the foot of the perpendicular from P to BC, and the circumcircles of triangles $\triangle BMU$ and $\triangle CNU$ intersect at second point V different from U, prove that A, P, and V are collinear.

Exercise 2.6 (European Mathematical Cup 2016)**.** Let C_1, C_2 be circles intersecting in X, Y . Let A, D be points on C_1 and B, C on C_2 such that A, X, C are collinear and D, X, B are collinear. The tangent to circle C_1 at D intersects BC and the tangent to C_2 at B in P, R respectively. The tangent to C_2 at C intersects AD and tangent to C_1 at A, in Q, S respectively. Let W be the intersection of AD with the tangent to C_2 at B and Z the intersection of BC with the tangent to C_1 at A. Prove that the circumcircles of triangles YWZ, RSY and PQY have two points in common, or are tangent in the same point.

§3 Practice Problems

Exercise 3.1 (IMO Shortlist 2019)**.** Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

Exercise 3.2 (Iran 2021)**.** Circle ω is inscribed in quadrilateral $ABCD$ and is tangent to segments BC, AD at E, F , respectively.DE intersects ω for the second time at X. if the circumcircle of triangle DFX is tangent to lines AB and CD , prove that quadrilateral $AFXC$ is cyclic.

Exercise 3.3 (Romania TST 2017)**.** Let $ABCD$ be a trapezium, $AD \parallel BC$, and let E and F be points on the sides AB and CD, respectively. The circumcircle of AEF meets AD again at A_1, and the circumcircle of CEF meets BC again at C_1. Prove that A_1C_1, BD and EF are concurrent.

Exercise 3.4 (IMO Shortlist 2017)**.** Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

Exercise 3.5 (IMO Shortlist 2012)**.** Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Exercise 3.6 (IMO Shortlist 2017)**.** A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c and I_d be the incenters of the triangles DAB, ABC, BCD and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90°$.