Linearity of Power of a Point

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In this article, we will learn about an advanced technique that exploits the linearity of the power of a point function.

§1 Background

Let's reiterate over the definition of the power of a point function.

> **Definition 1.1.** For a given circle ω centered at O and radius r, and a point P, the **power of a point** is defined as
>
> $$\operatorname{Pow}_{\omega}(P) = \overline{OP}^2 - r^2$$

The big claim here is that the difference of power of a point computed against two circles is *linear*. In mathematics, when we talk about **linear functions**, we mean functions that satisfy the following conditions

$$f(x + y) = f(x) + f(y)$$
$$f(\alpha x) = \alpha f(x)$$

In general, these conditions can be combined and written as

$$f(\alpha x + (1 - \alpha)x) = \alpha f(x) + (1 - \alpha) f(x)$$

§2 Proving Linearity

Theorem 2.1 (Linearity of Power of a Point)
Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(P) = \operatorname{Pow}_{\omega_1}(P) - \operatorname{Pow}_{\omega_2}(P)$$

for two fixed circles ω_1 and ω_2. Then f is linear.

Proof. Suppose the centers of the circles ω_1 and ω_2 are O_1 and O_2, and their radii are r_1 and r_2 respectively. Pick two points A and B and choose point C on $\overline{AB}$ such that

$$\frac{\overline{AC}}{\overline{CB}} = \frac{k}{1 - k}$$

We would like to show that

$$f\left(C\right) = kf\left(A\right) + (1-k)f\left(B\right)$$

Simplifying the left hand side expression

$$\begin{aligned} f\left(C\right) &= \text{Pow}_{\omega_1}\left(C\right) - \text{Pow}_{\omega_2}\left(C\right) \\ &= \left(\overline{O_1C}^2 - r_1^2\right) - \left(\overline{O_2C}^2 - r_2^2\right) \end{aligned}$$

Applying Stewart's Theorem on $\triangle ABO_1$ and $\triangle ABO_2$,

$$\begin{aligned} &= \left(\overline{O_1C}^2 - r_1^2\right) - \left(\overline{O_2C}^2 - r_2^2\right) \\ &= \left(-k(1-k)\overline{AB}^2 + \left((1-k)\overline{O_1B}^2 + k\overline{O_1A}^2\right)\right) \\ &\quad - \left(-k(1-k)\overline{AB}^2 + \left((1-k)\overline{O_2B}^2 + k\overline{O_2A}^2\right)\right) + r_2^2 - r_1^2 \\ &= k\left(\overline{O_1A}^2 - \overline{O_2A}^2\right) + (1-k)\left(\overline{O_1B}^2 - \overline{O_2B}^2\right) + r_2^2 - r_1^2 \\ &= k\left(\overline{O_1A}^2 - r_1^2\right) - k\left(\overline{O_2A}^2 - r_2^2\right) \\ &\quad + (1-k)\left(\overline{O_1B}^2 - r_1^2\right) - (1-k)\left(\overline{O_2B}^2 - r_2^2\right) \\ &= kf(A) + (1-k)f(B) \end{aligned}$$

thus proving the linearity. □

§3 Applications: Radical Axis

Linearity of power of a point turns out to be a good criteria to determine if a point lies on the radical axis of two circles. Since the radical axis is the locus of points having equal power, it essentially means that it's the locus of zeros of f.

Problem 3.1 (USA 2020)

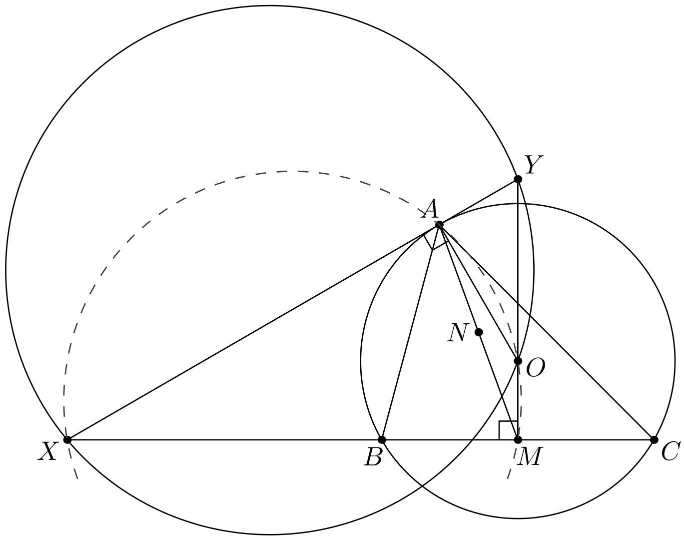
Let O and Γ denote the circumcenter and circumcircle, respectively, of scalene $\triangle ABC$. Furthermore, let M be the midpoint of side $\overline{BC}$. The tangent to Γ at A intersects BC and OM at points X and Y, respectively. If the circumcircle of $\triangle OXY$ intersects Γ at two distinct points P and Q, prove that PQ bisects $\overline{AM}$.

Proof. Suppose N is the midpoint of $\overline{AM}$. We would like to show that N lies on the radical axis of the circles $\odot(ABC)$ and $\odot(OXY)$. Suppose f is defined as

$$f\left(P\right) = \text{Pow}_{\odot(ABC)}\left(P\right) - \text{Pow}_{\odot(OXY)}\left(P\right)$$

Then we would like to show that

$$f\left(N\right) = \frac{1}{2}\left(f(A) + f(M)\right) = 0$$



Keeping in mind that the power of a point function is negative for points inside the circle,

$$
\begin{aligned}
f(A) + f(M) &= (0 - (-\overline{AX} \cdot \overline{AY})) + (-\overline{BM} \cdot \overline{MC} - \overline{OM} \cdot \overline{MY}) \\
&= \overline{AX} \cdot \overline{AY} - \tfrac{1}{4}\overline{BC}^2 - \overline{OM} \cdot \overline{MY} \\
&= \overline{AX} \cdot \overline{AY} + \overline{OM}^2 - \overline{OB}^2 - \overline{OM} \cdot \overline{MY} \\
&= \overline{AX} \cdot \overline{AY} - \overline{OA}^2 - \overline{OM} \cdot \overline{OY} \\
&= \overline{AX} \cdot \overline{AY} + \overline{AY}^2 - \overline{OY}^2 - \overline{OM} \cdot \overline{OY} \\
&= \overline{AY} \cdot \overline{XY} - \overline{OY} \cdot \overline{MY}
\end{aligned}
$$

Since $\angle OAX = \angle OMX = 90^\circ \implies AOMX$ is a cyclic quadrilateral. Therefore,

$$\overline{AY} \cdot \overline{XY} = \overline{YO} \cdot \overline{YM} \implies f(A) + f(M) = 0$$

hence $f(N) = 0$, implying that the midpoint of $\overline{AM}$ lies on the radical axis $\overline{PQ}$. $\square$

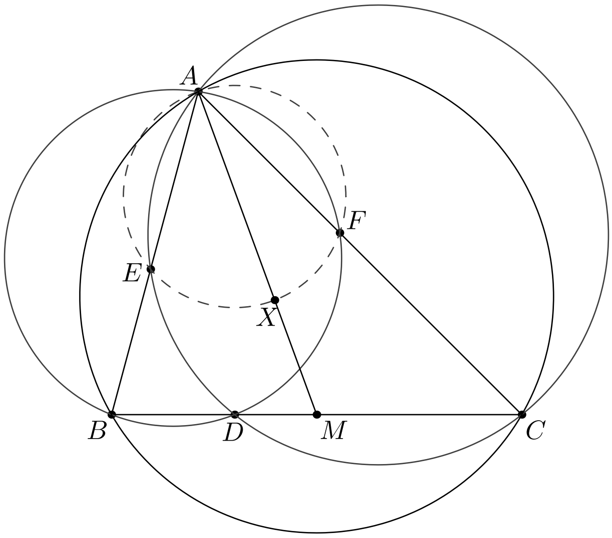
§4 Applications: Identifying Fixed Points

Often we would like to show that a circle passes through a fixed point. There are several ways to prove such results. One of the ways is to pick a fixed point on the secant through the desired fixed point and show that the power of point is constant.

> **Problem 4.1** (ELMO Shortlist 2013)
> In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Proof. If we can show that the power of M with respect to $\odot(AEF)$ is constant, then that would mean $\odot(AEF)$ passes through a fixed point on $\overline{AM}$. Suppose $\overline{AM} \cap \odot(AEF) = X$. Define f as

$$f(P) = \text{Pow}_{\odot(ABC)}(P) - \text{Pow}_{\odot(AEF)}(P)$$



Using the fact that f is linear, we get

$$\begin{aligned} f(M) &= \tfrac{1}{2}\left(f(B) + f(C)\right) \\ &= \tfrac{1}{2}\left(\left(0 - \overline{BE} \cdot \overline{AB}\right) + \left(0 - \overline{CF} \cdot \overline{AC}\right)\right) \\ &= -\tfrac{1}{2}\left(\overline{BE} \cdot \overline{AB} + \overline{CF} \cdot \overline{AC}\right) \\ &= -\tfrac{1}{2}\left(\overline{BD} \cdot \overline{BC} + \overline{CD} \cdot \overline{BC}\right) \\ &= -\tfrac{1}{2}\overline{BC}^2 \end{aligned}$$

Since,

$$\begin{aligned} f(M) &= \operatorname{Pow}_{\odot(ABC)}(M) - \operatorname{Pow}_{\odot(AEF)}(M) \\ &= -\overline{BM} \cdot \overline{MC} - \operatorname{Pow}_{\odot(AEF)}(M) \\ &= -\tfrac{1}{2}\overline{BC}^2 \end{aligned}$$

Therefore,

$$\operatorname{Pow}_{\odot(AEF)} = \frac{1}{4}\overline{BC}^2$$

which is a constant. This shows that the product $\overline{AM} \cdot \overline{MX}$ is a constant and thus $\odot(AEF)$ passes through the fixed point X. □

§5 Applications: Zero Radius Circle

Problem 5.1 (USAMO 2013)

In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.