Casey's Theorem

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1 February 2026

In this article we study Casey's Theorem and its applications.

§1 Casey's Theorem

Casey's Theorem is a result on lengths in the configuration of tangent circles. However, this theorem is also applicable to point circles making it a much more powerful result.

Theorem 1.1 (Casey's Theorem)
Given a circle Ω and four circles ω_1, ω_2, ω_3, ω_4 internally tangent to Ω at A, B, C and D such that $ABCD$ is a convex quadrilateral. Let δ_{ij} denote the length of the external common tangent between circles ω_i and ω_j, then

$$\delta_{12} \cdot \delta_{34} + \delta_{23} \cdot \delta_{14} = \delta_{13} \cdot \delta_{24}$$

Infact the converse of this theorem holds too!

Theorem 1.2 (Converse of Casey's Theorem)
Given four circles ω_1, ω_2, ω_3 and ω_4 that satisfy

$$\pm\delta_{12} \cdot \delta_{34} \pm \delta_{23} \cdot \delta_{14} \pm \delta_{13} \cdot \delta_{24} = 0$$

where δ_{ij} is the length of external common tangent of circles ω_i and ω_j then there exists a circle that is tangent to all the four circles.

Since the proof of this result is challenging, we shall omit it here. Instead, let's learn how to use this result on problems.

§2 Point Circles!

Point circles refer to circles with zero radius. They are circles that precisely collapse to a singular point. Such circles are also known as **degenerate circles**. Interestingly, Casey's Theorem is also applicable to point circles! And when we consider all the four circles in Casey's Theorem to be degenerate, the condition boils down to a very famous result in the literature of geometry known as **Ptolemy's Theorem**. In other words, Casey's Theorem can also be understood as a generalisation of Ptolemy's Theorem.

§2.1 Ptolemy's Theorem

Theorem 2.1 (Ptolemy's Theorem)

For four points A, B, C and D in a plane, $ABCD$ is a cyclic quadrilateral if and only if,

$$\overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC} = \overline{AC} \cdot \overline{BD}$$

Notice how this is occurs exactly when the four circles in Casey's Theorem are chosen to be point circles at A, B, C and D. Since we have mentioned Ptolemy's Theorem, it's also worth mentioning the following result.

Theorem 2.2 (Ptolemy's Inequality)

For four points A, B, C and D in a plane, we always have

$$\overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC} \geq \overline{AC} \cdot \overline{BD}$$

Equality holds if and only if $ABCD$ is a cyclic quadrilateral.

§2.2 Examples

Problem 2.3 (IMO Shortlist 1997)

The lengths of the sides of a convex hexagon $ABCDEF$ satisfy $AB = BC$, $CD = DE$, $EF = FA$. Prove that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

Proof. Applying ptolemy's inequality on the quadrilateral $ABCE$, we have that

$$\begin{aligned} \overline{CE} \cdot \overline{AB} + \overline{AE} \cdot \overline{BC} &\geq \overline{AC} \cdot \overline{BE} \\ \implies \frac{\overline{BC}}{\overline{BE}} &\geq \frac{\overline{AC}}{\overline{CE} + \overline{AE}} \end{aligned}$$

Similarly,

$$\frac{\overline{DE}}{\overline{DA}} \geq \frac{\overline{CE}}{\overline{AE} + \overline{AC}} \text{ and } \frac{\overline{FA}}{\overline{FC}} \geq \frac{\overline{AE}}{\overline{AC} + \overline{CE}}$$

However, it is well known that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}$$

Hence,

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}$$

□

§2.3 Exercises

Exercise 2.4 (RMO 2024)**.** Let $ABCD$ be a cyclic quadrilateral such that $AB \parallel CD$. Let O be the circumcenter of $ABCD$ and L be the point on AD such that OL is perpendicular to AD. Prove that

$$OB \cdot (AB + CD) = OL \cdot (AC + BD).$$

Exercise 2.5 (IMO 1995)**.** Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that $AG + GB + GH + DH + HE \geq CF$.

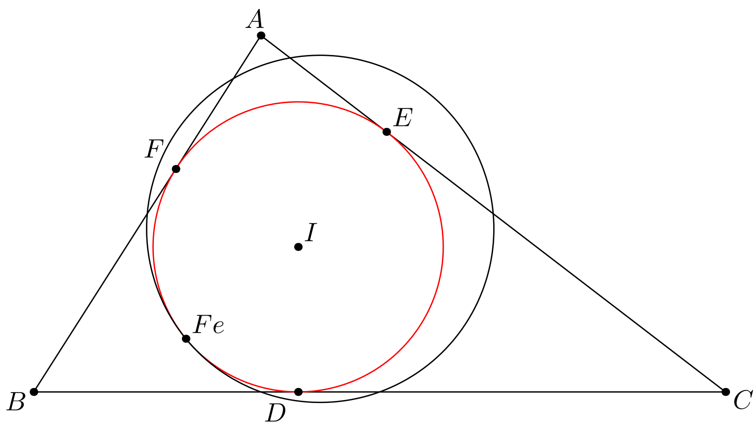
§3 Feuerbach's Theorem

Feuerbach's Theorem is an extremely beautiful and celebrated result in Euclidean Geometry. Here is the statement for the theorem.

Theorem 3.1 (Feuerbach's Theorem)

The **Nine-Point Circle** of a triangle is *internally tangent* to the **Incircle** and *externally tangent* to the three **Excircles**. The points of tangencies are the interior and three exterior **Feuerbach Points**.

It's difficult to prove this theorem with synthetic geometry, however we can easily prove this using Casey's Theorem.



Proof. Apply converse of Casey's Theorem on the incircle $\odot(I)$ and the point circles $\odot(M_A)$, $\odot(M_B)$ and $\odot(M_C)$,

$$\begin{cases} \delta_{DE} \cdot \delta_{FI} = \frac{c}{2} \left| \frac{b-a}{2} \right| \\ \delta_{EF} \cdot \delta_{DI} = \frac{a}{2} \left| \frac{b-c}{2} \right| \\ \delta_{DF} \cdot \delta_{EI} = \frac{b}{2} \left| \frac{c-a}{2} \right| \end{cases}$$

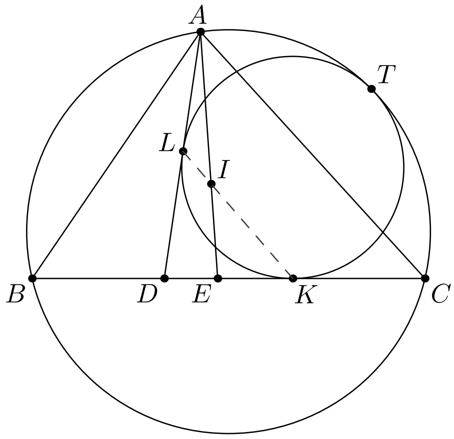
which is easy to show that satisfies

$$\pm\delta_{DE} \cdot \delta_{FI} \pm \delta_{EF} \cdot \delta_{DI} \pm \delta_{DF} \cdot \delta_{EI} = 0$$

Therefore, $\odot(DEF)$ which is the nine point circle is tangent to $\odot(I)$ which is the incircle. Similarly, we can show that the incircle is tangent to all the three excircles. □

§4 Sawayama's Theorem

We can even produce a short proof of **Sawayama's Theorem** using Casey's Theorem.



Proof. We want to show that the points L, I and K are collinear so applying menelaus' theorem on $\triangle ADE$, we get

$$\frac{AL}{LD} \cdot \frac{DK}{KE} \cdot \frac{EI}{IA} = 1$$

However $\overline{DL} = \overline{DK}$, so we just want to show that

$$\frac{AL}{KE} \cdot \frac{EI}{IA} = 1$$

Applying Casey's Theorem on $\odot(TLK)$, and point circles $\odot(A)$, $\odot(B)$ and $\odot(C)$ we get

$$\overline{AL} \cdot \overline{BC} + \overline{CK} \cdot \overline{AB} = \overline{BK} \cdot \overline{AC}$$

this implies that

$$\frac{AL}{KE} = \frac{b+c}{a}$$

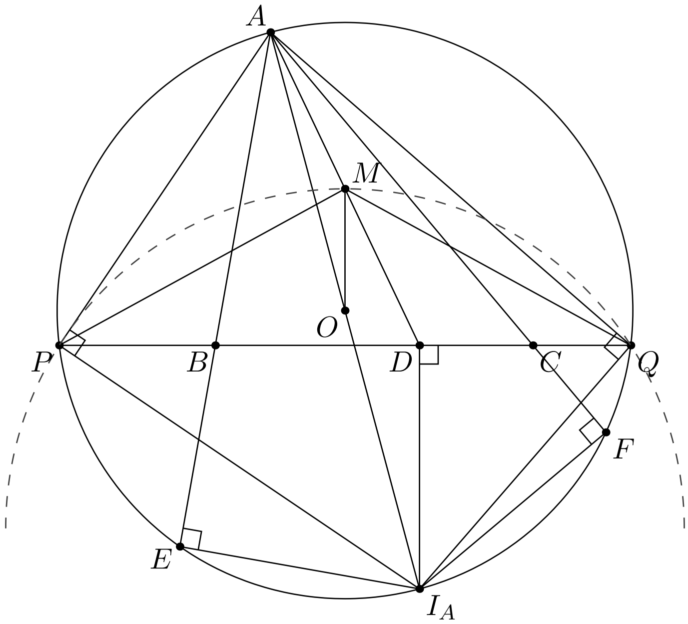
but using angle bisector theorem, we already know that $\frac{AI}{IE} = \frac{b+c}{a} \implies \frac{AL}{KE} \cdot \frac{EI}{IA} = 1$, thus proving the result. $\square$

§5 Examples

Problem 5.1 (IMO Shortlist 2017)

In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

Proof. Let O be the center of $\odot(AEF)$. Since $\odot(AEF)$ passes through the A-excenter and $\angle AEI_A = 90^\circ \implies O$ is the midpoint of $\overline{AI_A}$. Since, $\overline{I_AD} \perp \overline{BC}$ and $\overline{OM} \parallel \overline{I_AD}$



by midpoint theorem $\implies \overline{OM} \perp \overline{BC}$. Since O is the center of $\odot(AEF) \implies \overline{MP} = \overline{MQ}$. Since

$$\begin{aligned}
\overline{PQ}^2 \cdot (\overline{MI_A}^2 - \overline{DI_A}^2) &= \left(\overline{MP} \cdot \overline{QD} + \overline{MQ} \cdot \overline{PD}\right)^2 \\
\iff \overline{PQ}^2 \cdot (\overline{MI_A}^2 - \overline{DI_A}^2) &= \overline{MP}^2 \cdot \overline{PQ}^2 \\
\iff \overline{MI_A}^2 - \overline{DI_A}^2 &= \overline{MP}^2 \\
\iff \overline{MI_A}^2 - \overline{MP}^2 &= \overline{DI_A}^2 \\
\iff \frac{1}{4}\left(2\overline{AI}^2 + 2\overline{DI_A}^2 - \overline{AD}^2\right) - \frac{1}{4}\left(2\overline{PD}^2 + 2\overline{AP}^2 - \overline{AD}^2\right) &= \overline{DI_A}^2 \\
\iff \frac{1}{2}\left(\overline{AI_A}^2 - \overline{PD}^2 - \overline{AP}^2 + \overline{DI_A}^2\right) &= \overline{DI_A}^2
\end{aligned}$$

which is true due to pythagoras' theorem on $\triangle API_A$ and $\triangle PDI_A$. Therefore using the converse of Casey's Theorem on $\odot(DEF)$ and point circles $\odot(M)$, $\odot(P)$ and $\odot(Q)$, we have shown that $\odot(MPQ)$ is tangent to $\odot(DEF)$. $\square$

§6 Practice Problems

Exercise 6.1. Let $ABCD$ be a cyclic quadrilateral. Prove that,

$$\frac{AC}{BD} = \frac{AB \cdot AD + CB \cdot CD}{BA \cdot BC + DA \cdot DC}$$

Exercise 6.2 (USA 1997)**.** Let Q be a quadrilateral whose side lengths are a, b, c, d in that order. Show that the area of Q does not exceed $\frac{ac+bd}{2}$.

Exercise 6.3 (APMO 2014)**.** Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.

Exercise 6.4 (IMO 1997)**.** It is known that $\angle BAC$ is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that $AU = TB + TC$.

Exercise 6.5. Let ABC be a triangle with centroid G, incenter I, incircle ω, and nine-point circle Γ. Let the line IG meet BC at P and let the common tangent ω and Γ meet BC at Q. Prove that the midpoint of BC is also the midpoint of PQ .

Exercise 6.6. Let D, E, F be points on sides BC, CA, AB of triangle ABC respectively such that lines AD, BE, CF concur. Let Ω be the circumcircle of triangle ABC and let ω_A be the circle internally tangent to Ω and tangent to BC at D. Define circle ω_B and ω_C similarly. Show that there exists a circle tangent to circles $\omega_A, \omega_B, \omega_C$ that is also tangent to the incircle of triangle of ABC.

Exercise 6.7 (IMO 2001)**.** Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.