IMO 2024 Shortlist

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18 December 2025

This article contains my solutions to the IMO 2024 Shortlist.

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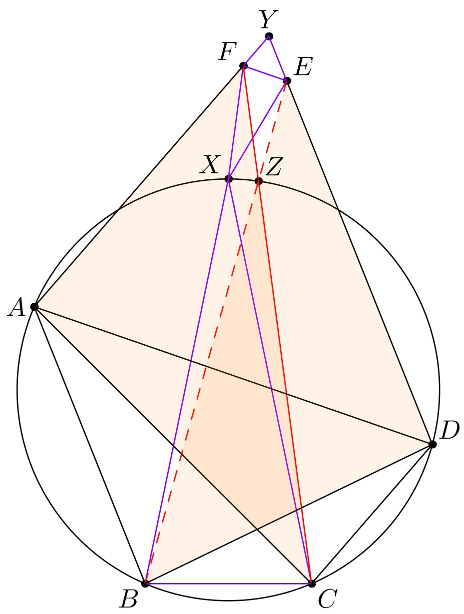
§1 IMO Shortlist 2024 G1

Problem. Let $ABCD$ be a cyclic quadrilateral such that $AC < BD < AD$ and $\angle DBA < 90^\circ$. Point E lies on the line through D parallel to AB such that E and C lie on opposite sides of line AD, and $AC = DE$. Point F lies on the line through A parallel to CD such that F and C lie on opposite sides of line AD, and $BD = AF$. Prove that the perpendicular bisectors of segments BC and EF intersect on the circumcircle of $ABCD$.

Proposed by Mykhailo Shtandenko, Ukraine



Proof. We shall introduce a few points in the diagram. Let $AF \cap DE = Y$, $FC \cap \odot (ABCD) = Z$ and X be a point on $(ABCD)$ such that $BX = CX$.

Claim 1.1 — $\triangle\ EDB$ is congruent to $\triangle\ CAF$.

Proof. Since, we are given that $ED = CA$ and $DB = AF$. It suffices to show that $\angle EDB = \angle CAF$. We use the face that $AF \parallel CD$ and $DE \parallel BA$.

$$\begin{aligned}\angle EDB &= 180^\circ - \angle ABD \\ &= 180^\circ - \angle ACD \\ &= \angle CAF\end{aligned}$$

which proves the congruency using SAS congruency criterion. □

Claim 1.2 — Z lies on the line segment BE.

Proof. Observe that,

$$\begin{aligned}\angle ABZ &= \angle ACZ \\ &= \angle ACF \\ &= \angle DEB \\ &= \angle ABE\end{aligned}$$

which forces the collinearity. □

Claim 1.3 — $\triangle\ FXC$ is congruent to $\triangle\ EXB$.

Proof. Since, $\triangle\ EDB \cong \triangle\ CAF \implies FC = BE$. From the definition of point X, we have $BX = CX$. We can prove this claim using SAS congruency criterion, if we can show that $\angle XBE = \angle XCF$. Fortunately, this is a trivial angle chase.

$$\begin{aligned}\angle XBE &= \angle XBZ \\ &= \angle XCZ \\ &= \angle XCF\end{aligned}$$

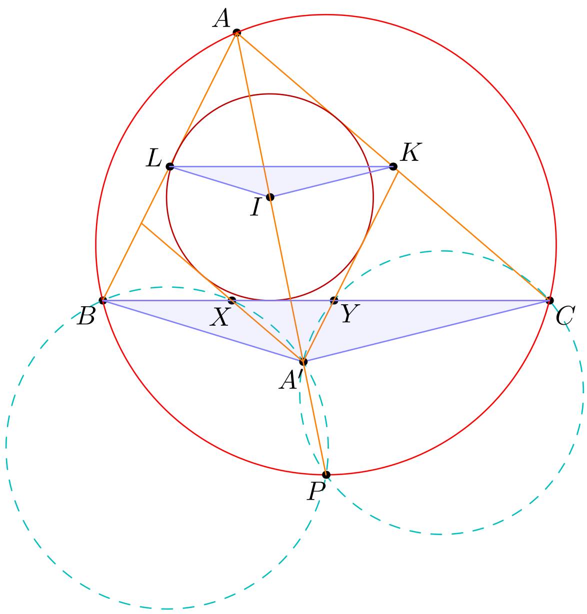
which proves the congruency. □

Since, $\triangle\ XBE \cong \triangle\ XCF \implies FX = XE \implies X$ lies on the perpendicular bisector of line segment FE and by definition of point X, X is the intersection of the perpendicular bisector of line segment BC and $\odot(ABCD)$, which proves that the perpendicular bisectors of line segments BC and EF intersect on the circumcircle of $ABCD$. □

§2 IMO Shortlist 2024 G2

Problem. Let ABC be a triangle with $AB < AC < BC$. Let the incenter and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively. Prove that $\angle KIL + \angle YPX = 180°$.

Proposed by Dominik Burek, Poland



Proof. Let A' be the reflection of point A over point I.

Claim 2.1 — $A'X \parallel AC$ and $A'Y \parallel AB$.

Proof. Construct point E on line segment AC such that ω touches AC at E. Reflect the point E over I to E'. By SAS congruency criterion $\triangle AIE \cong A'IE'$. Since, $\angle IE'A' = 90° \implies A'E'$ is tangent to ω and $A'E' \parallel AC$. However, X lies on the line parallel to AC and tangent to $\omega \implies A'X \parallel AC$, and similarly $A'Y \parallel AB$ which proves the claim. ☐

Claim 2.2 — $BXA'P$ and $CYA'P$ are cyclic quadrilaterals.

Proof. Just angle chasing.

$$\angle CXA' = \angle BCA = \angle BPA = \angle BPA'$$

which proves that $BXA'P$ is cyclic. Similarly, we can show that $CYA'P$ is cyclic. ☐

Claim 2.3 — $\triangle KIL$ is homothetic to $\triangle CA'B$ from point A.

Proof. Since lines BL, $A'I$ and CK are concurrent at point A and $AB = 2AL$, $A'A = 2AI$ and $AC = 2AK \implies \triangle KIL \mapsto \triangle CA'B$ under a homothetic transformation with scaling factor $= 2$. ☐

Finally combining all the information from the proved claims,

$$\begin{aligned}\angle KIL + \angle XPY &= \angle BA'C + \angle XPY \\ &= \angle BA'C + \angle XPA' + \angle YPA' \\ &= \angle BA'C + \angle XBA' + \angle YCA' \\ &= \angle BA'C + \angle CBA' + \angle BCA' \\ &= 180^\circ\end{aligned}$$

□

§3 IMO Shortlist 2024 G3

Problem. Let $ABCDE$ be a convex pentagon and let M be the midpoint of AB. Suppose that segment AB is tangent to the circumcircle of triangle CME at M and that D lies on the circumircles of AME and BMC. Lines AD and ME interesect at K, and lines BD and MC intersect at L. Points P and Q lie on line EC so that $\angle PDC = \angle EDQ = \angle ADB$.

Prove that lines KP, LQ, and MD are concurrent.