Kha's Mock Ciphering Questions

Calculus AB Set 1

LaMAΘ State Convention

Thursday to Saturday

March 26 - 28, 2020

Rules

- Two minutes are allotted for each question.
- All answers must be in exact, simplified form unless otherwise requested.
- Four points are awarded for answering the question correctly within one minute.
- Two points are awarded for answering correctly within two minutes.
- Good luck and have fun!

1. Find $\frac{d^5y}{dx^5}$, given that $y = 2x^5 + 19x^4 + 5x^3 + 7x^2 + 11x + 17$.

2. Evaluate $\lim_{k \to -2} \frac{3k^2-12}{\sqrt{2k+4}}$.

3. Let $a(x) = x^4 + 3x^2 + 6$. Find the equation in slope-intercept form of the tangent line to $a(x)$ when $x = 2$.

4. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. If $h = 3r$ and the radius is increasing at 2 inches per minute when the radius is 6 inches, find the rate of change of volume at that time.

5. The position of a particle at time t is given by $x(t) = x^4 + \frac{8}{3}x^3 + 2x^2 + 8x$. Find the interval(s) where the particle speeds up.

6. Suppose there is a function such that:

$$T(x) = \begin{cases} \frac{\pi}{2} \ln \frac{\pi}{4} \cdot (\log_{\frac{\pi}{4}} x - 1) + \alpha & \text{if } x \leq \frac{\pi}{4} \\ \beta \tan x & \text{if } x > \frac{\pi}{4} \end{cases}$$

α and β are integer constants that make $T(x)$ continuous and differentiable. Find $\alpha + \beta$.

7. The *generalized mean value theorem* states that if f and g are differentiable on $[a, b]$, then there is some $c \in (a, b)$ such that:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If $f(x) = x^2 + 7x + 4$ and $g(x) = x^2 - 3x + 5$, what is the value of c guaranteed by the generalized mean value theorem on $[0, 1]$?

8. Let $f^{-1}(x) = g(x)$ be the inverse function of $f(x)$. Using the information from the table below, find $g'(3)$.

x	f(x)	g(x)	f'(x)
4	3	6	5
5	7	9	2
6	4	8	8
7	2	5	7

9. Suppose $f(x) = x^2 + x$. Approximate the area between $f(x)$ and the x-axis from $x = 0$ to $x = 2$ using a trapezoidal sum with 4 equal subdivisions.

10. Suppose $0 < k < h < a < \pi$. If $\int_a^k f(x)dx = -2$ and $\int_h^k f(x)dx = 9$, find $\int_a^h f(x)dx$.

Answers

1. 240

2. 0

3. $y = 44x - 54$

4. 216π inches^3 per minute

5. $(-2, -1) \cup (-\frac{1}{3}, \infty)$

6. 2

7. $\frac{1}{2}$

8. $\frac{1}{5}$

9. $\frac{19}{4}$

10. -11

Solutions

1. Find $\frac{d^5y}{dx^5}$, given that $y = 2x^5 + 19x^4 + 5x^3 + 7x^2 + 11x + 17$.

Ans: 240

You only have to differentiate $2x^5$ because the other terms will eventually become 0 after iterated differentiation. It will become $2 \cdot 5!$ or 240.

2. Evaluate $\lim\limits_{k \to -2} \frac{3k^2-12}{\sqrt{2k+4}}$.

Ans: 0

Plugging in -2 returns indeterminate form. Apply L'Hospital's rule.

3. Let $a(x) = x^4 + 3x^2 + 6$. Find the equation in slope-intercept form of the tangent line to $a(x)$ when $x = 2$.

Ans: $y = 44x - 54$

Use the point-slope formula where the point $= (2, a\,(2))$ and slope $= a'(2)$.

4. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. If $h = 3r$ and the radius is increasing at 2 inches per minute when the radius is 6 inches, find the rate of change of volume at that time.

Ans: 216π inches^3 per minute

We have to make a choice to use either r or h in our primary function. We do not know h', but we do know r'. Because of this, we put V in terms of r. You get $V = \pi r^3$. From there, differentiate and plug in values.

5. The position of a particle at time t is given by $x(t) = x^4 + \frac{8}{3}x^3 + 2x^2 + 8x$. Find the interval(s) where the particle speeds up.

Ans: $(-2, -1) \cup (-\frac{1}{3}, \infty)$

A particle speeds up when its velocity and acceleration have the same sign. We need to perform the first and second derivative test on $x(t)$.
$x'(t) = 4x^3 + 8x^2 + 4x + 8$
$x'(t) = 4x(x^2 + 1) + 8(x^2 + 1)$
$x'(t) = (x^2 + 1)(4x + 8) = 0$
$x'(t)$ is positive on $(-2, \infty)$ and negative on $(-\infty, -2)$.

$x''(t) = 12x^2 + 16x + 4$
$x''(t) = 4(3x^2 + 4x + 1)$
$x''(t) = 4(3x + 1)(x + 1) = 0$
$x''(t)$ is positive on $(-\infty, -1) \cup (-\frac{1}{3}, \infty)$ and negative on $(-1, -\frac{1}{3})$.
$x'(t)$ and $x''(t)$ share the same sign on $(-2, -1) \cup (-\frac{1}{3}, \infty)$.

6. Suppose there is a function such that:

$$T(x) = \begin{cases} \frac{\pi}{2} \ln \frac{\pi}{4} \cdot (\log_{\frac{\pi}{4}} x - 1) + \alpha & \text{if } x \leq \frac{\pi}{4} \\ \beta \tan x & \text{if } x > \frac{\pi}{4} \end{cases}$$

α and β are integer constants that make $T(x)$ continuous and differentiable. Find $\alpha + \beta$.

Ans: 2

Set $\lim\limits_{x \to \frac{\pi}{4}^-} T(x) = \lim\limits_{x \to \frac{\pi}{4}^+} T(x)$ and $\lim\limits_{x \to \frac{\pi}{4}^-} T'(x) = \lim\limits_{x \to \frac{\pi}{4}^+} T'(x)$.
The first equation results in
$\frac{\pi}{2} \ln \frac{\pi}{4} \cdot (\log_{\frac{\pi}{4}} \frac{\pi}{4} - 1) + \alpha = \beta \tan \frac{\pi}{4}$
$\frac{\pi}{2} \ln \frac{\pi}{4} \cdot (1 - 1) + \alpha = \beta \cdot 1$
$\frac{\pi}{2} \ln \frac{\pi}{4} \cdot (0) + \alpha = \beta$
$\alpha = \beta$

The second equation results in
$\frac{\pi}{2} \ln \frac{\pi}{4} \cdot (\frac{1}{\ln \frac{\pi}{4} \cdot \frac{\pi}{4}}) = \beta \sec^2 \frac{\pi}{4}$
$2 = 2\beta$
$\beta = 1$
$\alpha = \beta = 1$, so $\alpha + \beta = 2$.

7. The *generalized mean value theorem* states that if f and g are differentiable on $[a, b]$, then there is some $c \in (a, b)$ such that:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If $f(x) = x^2 + 7x + 4$ and $g(x) = x^2 - 3x + 5$, what is the value of c guaranteed by the generalized mean value theorem on $[0, 1]$?

Ans: $\frac{1}{2}$

Plug in 0 and 1 into their respective positions in the formula.
$[f(1) - f(0)]g'(c) = [g(1) - g(0)]f'(c)$
$[12 - 4](2c - 3) = [3 - 5](2c + 7)$
$16c - 24 = -4c - 14$
$c = \frac{1}{2}$.

8. Let $f^{-1}(x) = g(x)$ be the inverse function of $f(x)$. Using the information from the table below, find $g'(3)$.

x	f(x)	g(x)	f'(x)
4	3	6	5
5	7	9	2
6	4	8	8
7	2	5	7

Ans: $\frac{1}{5}$

We know that $f(g(x)) = x$. Differentiate this to obtain
$f'(g(x)) \cdot g'(x) = 1$. Plug in 3.
$f'(g(3)) \cdot g'(3) = 1$. Note: $g(3) = 4$ because $f(3) = 4$
$f'(4) \cdot g'(x) = 1$
$5 \cdot g'(3) = 1$
$g'(3) = \frac{1}{5}$

9. Suppose $f(x) = x^2+x$. Approximate the area between $f(x)$ and the x-axis from $x = 0$ to $x = 2$ using a trapezoidal sum with 4 equal subdivisions.

Ans: $\frac{19}{4}$

Each trapezoid has equal height of $\frac{1}{2}$, so we can just calculate $\frac{1}{2} \cdot \frac{1}{2} \cdot (f(0) + 2f(\frac{1}{2}) + 2f(1) + 2f(\frac{3}{2}) + f(2))$, which equals 19/4.

10. Suppose $0 < k < h < a < \pi$. If $\int_a^k f(x)dx = -2$ and $\int_h^k f(x)dx = 9$, find $\int_a^h f(x)dx$.

Ans: -11

Integral properties. Since $0 < k < h < a < \pi$:
$\int_k^h f(x)dx + \int_h^a f(x)dx = \int_k^a f(x)dx$.
$-\int_h^k f(x)dx - \int_a^h f(x)dx = -\int_a^k f(x)dx$
$\int_h^k f(x)dx + \int_a^h f(x)dx = \int_a^k f(x)dx$
$9 + \int_a^h f(x)dx = -2$
$\int_a^h f(x)dx = -11$
You can also solve intuitively with a diagram/graph.