Kha’s Mock Ciphering Questions

Calculus BC Set 1
LaMAΘ State Convention
Thursday to Saturday
March 26 - 28, 2020

Rules

- Two minutes are allotted for each question.
- All answers must be in exact, simplified form unless otherwise requested.
- Four points are awarded for answering the question correctly within one minute.
- Two points are awarded for answering correctly within two minutes.
- Good luck and have fun!

1. Determine the length of $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$.
(Note: knowing $\displaystyle\int (\sec x)\, dx = \ln|\tan x + \sec x| + c$ may help you)

2. Evaluate $\displaystyle\int e^{-x}(x^2 + 2x + 3)\,dx$.

3. Evaluate $\int e^{\theta}(cos\theta)\, d\theta$.

4. For what value of P is the population growing the fastest if $\frac{dP}{dt} = \frac{P}{5}(1 - \frac{P}{12})$?

5. Consider a function $f(x)$ such that $f(0) = 1$ and $f'(x) = x - y$. Using Euler's method with step size 0.1, find the resulting approximation of $f(2)$.

6. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{2}{1+x^2}\, dx.$

7. Evaluate $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \dfrac{1}{4 \cdot 5} + \cdots$

8. Determine if the following series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3+1}.$$

9. For the following power series, determine the interval of convergence:

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}.$$

10. Consider a function $f(x)$ such that $f(0) = 1$ and $f'(x) = 2$. Using Euler's method with step size 1, find the resulting approximation of $f(34)$.

Answers

1. $\ln(1+\sqrt{2})$
2. $-e^{-x}(x^2+4x+7)$
3. $\frac{1}{2}(e^\theta \cos\theta + e^\theta \sin\theta) + C$
4. 6
5. 0.82
6. 2π
7. 1
8. Absolute Convergence
9. $(-1, 1]$
10. 69

Solutions

1. Determine the length of $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$. (Note: knowing $\int (\sec x)\, dx = \ln|\tan x + \sec x| + c$ may help you)

Ans: $\ln(1 + \sqrt{2})$

Arc Length: $\int_a^b \sqrt{1 + (\frac{dy}{dx})^2}$
$\frac{dy}{dx} = \frac{1}{\cos x} \cdot -\sin x = -\frac{\sin x}{\cos x} = -\tan x$
$\int_0^{\frac{\pi}{4}} \sqrt{1 + (-\tan x)^2}$
$\int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x} = \int_0^{\frac{\pi}{4}} \sec x$
Using the given formula:
$[\ln|\tan x + \sec x|]_0^{\frac{\pi}{4}}$
$\ln|\tan \frac{\pi}{4} + \sec \frac{\pi}{4}| - \ln|\tan 0 + \sec 0|$
$\ln(1 + \sqrt{2})$

2. Evaluate $\int e^{-x}(x^2 + 2x + 3)\, dx$.

Ans: $-e^{-x}(x^2 + 4x + 7)$

$+/-$	$\frac{dy}{dx}$	$\int$
$+$	$x^2 + 2x + 3$	e^{-x}
$-$	$2x + 2$	$-e^{-x}$
$+$	2	e^{-x}
$-$	0	$-e^{-x}$

We set up a Hindu table as shown above, resulting in:
$-e^{-x}(x^2 + 2x + 3) - e^{-x}(2x + 2) - e^{-x}(2)$
$= -e^{-x}(x^2 + 4x + 7)$

3. Evaluate $\int e^{\theta}(cos\theta)\, d\theta$.

Ans: $\frac{1}{2}(e^{\theta}\cos\theta + e^{\theta}\sin\theta) + C$

Using two integration by parts:

$u = \cos\theta \qquad dv = e^{\theta}\, d\theta$

$du = -\sin\theta \qquad v = e^{\theta}$

$$\int e^{\theta}\cos\theta\, d\theta = e^{\theta}\cos\theta + \int e^{\theta}\sin\theta\, d\theta$$

$u = \sin\theta \qquad dv = e^{\theta}\, d\theta$

$du = \cos\theta \qquad v = e^{\theta}$

$$\int e^{\theta}\cos\theta\, d\theta = e^{\theta}\cos\theta + e^{\theta}\sin\theta - \int e^{\theta}\cos\theta\, d\theta$$

$$2\int e^{\theta}\cos\theta\, d\theta = e^{\theta}\cos\theta + e^{\theta}\sin\theta$$

$\frac{1}{2}(e^{\theta}\cos\theta + e^{\theta}\sin\theta) + C$

Note: It doesn't matter what you pick as u and dv as long as you remain consistent with that choice.

4. For what value of P is the population growing the fastest if $\frac{dP}{dt} = \frac{P}{5}(1 - \frac{P}{12})$?

Ans: 6

P grows fastest at half the carrying capacity.

Logistic differential equations are in the form $\frac{dP}{dt} = kP(1 - \frac{P}{L})$ where L is the carrying capacity.

L is 12 in the problem. Half of 12 is 6.

5. Consider a function $f(x)$ such that $f(0) = 1$ and $f'(x) = x - y$. Using Euler's method with step size 0.1, find the resulting approximation of $f(2)$.

Ans: 0.82

x	y	$\frac{dy}{dx}$
0	1	$0 - 1 = -1$
0.1	$1 + (0.1)(-1) = 0.9$	$0.1 - 0.9 = -0.8$
0.2	$0.9 + (0.1)(-0.8) = 0.82$	

Make an $x/y/\frac{dy}{dx}$ chart like above

6. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{2}{1+x^2}\,dx$.

Ans: 2π

$$\int_{-\infty}^{\infty} \frac{2}{1+x^2}\,dx = 2\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$
$$2[\arctan x]_{-\infty}^{\infty}$$
$$2(\arctan(\infty) - \arctan(-\infty))$$
$$2(\tfrac{\pi}{2} - (-\tfrac{\pi}{2})) = 2\pi$$

7. Evaluate $\displaystyle\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$

Ans: 1

We can rewrite the series as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n)(n+1)}$

We recognize this as a partial fraction $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

We recognize the result as a telescoping series:

$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots$

$1 + (-\frac{1}{2} + \frac{1}{2}) + (-\frac{1}{3} + \frac{1}{3}) + (-\frac{1}{4} + \frac{1}{4}) + \cdots$

$1 + 0 + 0 + 0 + 0 + \cdots = 1$

8. Determine if the following series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3+1}.$$

Ans: Absolute Convergence

$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3+1}$ is convergent by the Alternating Series Test since both $b_n = \frac{1}{n^3+1}$ is decreasing and $\lim_{n\to\infty} b_n = 0$.

$\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^{n+1}}{n^3+1}\right| = \sum_{n=2}^{\infty} \frac{1}{n^3+1}$ converges by direct comparison with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3}$.

Therefore, the original series is absolutely convergent.

(Note: Actually, you only have to test the absolute value case because a series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ is convergent.)

9. For the following power series, determine the interval of convergence:

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}.$$

Ans: $(-1, 1]$

Using ratio test will give you the interval $-1 < x < 1$.

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Therefore, the interval of convergence is $-1 < x \le 1$.

10. Consider a function $f(x)$ such that $f(0) = 1$ and $f'(x) = 2$. Using Euler's method with step size 1, find the resulting approximation of $f(34)$.

Ans: 69

x	$f(x)$	$\frac{dy}{dx}$
0	1	2
1	$1 + (1)(2) = 3$	2
2	$3 + (1)(2) = 5$	2
3	$3 + (1)(2) = 7$	2
$\vdots$	$\vdots$	$\vdots$

Make an $x/f(x)/\frac{dy}{dx}$ chart like above.
Notice the pattern: $(0, 1), (1, 3), (2, 5), (3, 7)$.
With each step size of 1, the y increases by 2.
Using this linear relationship, we can conclude that $(x, f(x))$ is simply $(x, 2x + 1)$. Plug in 34 into $f(x) = 2x + 1$ to obtain 69.