2019 SCHOLARSHIP TEST

1. If the diameter of a right cylindrical can with circular bases is increased by 25%, by what percent should the height be increased in order to double the volume of the original can?

$V = \pi r^2 h$

$V_1 = \pi\left(\frac{d_1}{2}\right)^2 h$

$2V_1 = \pi\left(\frac{1.25d}{2}\right)^2 kh$

$2V_1 = \pi\left(\frac{5}{4}\right)^2\left(\frac{d}{2}\right)^2 kh$

$2V_1 = V_1\left(\frac{5}{4}\right)^2 k$

$k = \frac{2}{\left(\frac{5}{4}\right)^2}$

$k = \frac{32}{25}$

$\frac{7}{25} = \boxed{28\%}$

2. Let $\downarrow n \downarrow$ be the largest prime number less than n and let $\uparrow n \uparrow$ be the smallest prime number greater than n. Find the value of $41+(\downarrow 35 \downarrow)-(\uparrow 53 \uparrow)+(\uparrow\downarrow 35 \downarrow\uparrow)$

11 13 17 19
23 29
31 37
41 43 47
53 59

$41 + (31) - (59) + (37)$

$\boxed{50}$

3. Find the exact value of the sixth term of the sequence $\frac{\sqrt{3}}{4}, \sin\left(\frac{\pi}{3}\right), \cot\left(\frac{\pi}{6}\right)$.......

$\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}, \sqrt{3}, 2\sqrt{3}, 4\sqrt{3}, \boxed{8\sqrt{3}}$

4. Find the value of $9-3+1-\frac{1}{3}+\cdots$ (answer as a fraction)

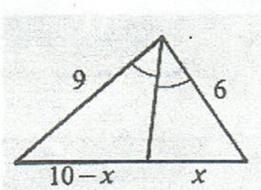
$\frac{9}{1-\left(-\frac{1}{3}\right)} = \frac{9}{\frac{4}{3}} = 9\cdot\frac{3}{4}$

$\frac{27}{4}$

$\frac{9}{1-\left(-\frac{1}{3}\right)} = \frac{9}{\frac{4}{3}} \, 9\cdot\frac{3}{4} = \boxed{\frac{27}{4}}$

$\frac{9}{1-\left(\frac{1}{9}\right)} + \frac{-3}{1-\left(\frac{1}{9}\right)} = 9\cdot\frac{9}{8} - 3\cdot\frac{9}{8} \; \boxed{\frac{54}{8}}$

5. Find the value of x in the triangle below



$\frac{9}{10-x} = \frac{6}{x}$

$9x = 60 - 6x$

$15x = 60$

$\boxed{x = 4}$

6. Solve for a if $\dfrac{1}{\log_{1024} a}+\dfrac{1}{\log_{729} a}-\dfrac{1}{\log_{36} a}=4$ $\boxed{12}$

$\dfrac{1}{\frac{\ln a}{\ln 1024}} + \dfrac{1}{\frac{\ln a}{\ln 729}} - \dfrac{1}{\frac{\ln a}{\ln 36}} = 4$

$\log_a 1024 + \log_a 729 - \log_a 36 = 4$

$\dfrac{\ln 1024}{\ln a} + \dfrac{\ln 729}{\ln a} - \dfrac{\ln 36}{\ln a} = 4$

$\log_a \dfrac{1024 \cdot 729}{36} = 4$

$\log_a 20736 = 4$

$\cancel{20736} =$

$20736 = a^4$

7. If $\sec\theta = B$, $\pi<\theta\le 2\pi$, find $\cos(2\theta)\cos\theta+\sin(2\theta)\sin\theta$

$\dfrac{1}{\cos\theta} = B$

$\cos\theta = \dfrac{1}{B}$

$\cos^2\theta = \dfrac{1}{B^2}$

$1-\sin^2\theta = \dfrac{1}{B^2}$ $\quad 1-\dfrac{1}{B^2} = \sin^2\theta$

$(\cos^2(\theta) - \sin^2\theta)\cos\theta + 2\sin^2\theta\cos\theta$

$\left(\dfrac{1}{B^2} - \left(1-\dfrac{1}{B^2}\right)\right)\dfrac{1}{B} + 2\left(1-\dfrac{1}{B^2}\right)\left(\dfrac{1}{B}\right)$

$-1+\dfrac{1}{B^2}$

$\left(\dfrac{2}{B^2}-1\right)\left(\dfrac{1}{B}\right) + \left(\dfrac{2}{B}-\dfrac{2}{B^3}\right)$ $\quad \boxed{\dfrac{1}{B}}$

8. Find all numbers x, the sum of whose distances from 1 and from -1 is less than 4

$(-2, 2)$

9. After drinking a caffeinated soda, the caffeine level in an adult's bloodstream reaches a peak of about 30 mg. The level then decays exponentially and two hours later there will be 21.675 mg remaining. What percent was lost in the first hour?

Pe^{kt}

$30e^{k(2)} = 21.675$

$30e^{-0.1625(1)} = 25.5$

$30 - x(30) = 25.5$

$30(1-x) = 25.5$

$\dfrac{25.5}{30}$ $\quad 0.15$

-0.3660254 $\quad \boxed{15\%}$

$1 - \dfrac{25.5}{30} = x$

10. The quadratic equation $2ax^2-4ax+a+1=0$ has two rational roots. If one root is three times the second root, what is the value of a?

1.3660254

$(2a^2+2a) \quad -8a^2-8a$

$\dfrac{a+1}{2a}$

$\dfrac{4a \pm \sqrt{16a^2-4(2a)(a+1)}}{2(2a)} = \dfrac{4a\pm\sqrt{8a^2-8a}}{4a}$

$1 \pm \dfrac{2\sqrt{2a^2-2a}}{4a}$

$\left(1+\dfrac{\sqrt{2a^2-2a}}{2}\right) = 3\left(1-\dfrac{\sqrt{2a^2-2a}}{2}\right)$

$1 \pm \dfrac{\sqrt{2a^2-2a}}{2a}$

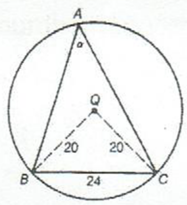
11. Solve for x if $\log_2\left(\log_3\left(\log_4\left(x^{3x}\right)\right)\right)=0$

$\log_3(\log_4(x^{3x})) = 2^0 = 1$ $\boxed{x=2}$

$\log_4(x^{3x}) = 3^1 = 3$

$x^{3x} = 4^3 = 2^{3(2)}$

12. Triangle ABC is inscribed in a circle of radius 20 having center Q as shown. Find the measure of $\angle BAC$ to the nearest tenth of a degree



$24^2 = 20^2 + 20^2 - 2(20)(20)\cos C$

73.739

36.869

$\boxed{36.9}$

13. Find the term of $\left(3p+q^3\right)^7$ that contains q^9

$\frac{7!}{4!3!}(3p)^4(q^3)^3$

$35 \cdot 81\, p^4 q^9$ $\boxed{2835p^4q^9}$

$2^3 \cdot 3^2$

14. In order, the first four terms of a sequence are 2, 6, 12 and 72, where each term, beginning with the third term, is the product of the two preceding terms. If the ninth term is 2^a3^b, what is the value of $a+b$?

2	$2\cdot 3$	$2^2\cdot 3$	$2^3\cdot 3^2$	$2^5\cdot 3^3$	$2^8\cdot 3^5$	$2^{13}\cdot 3^8$	$2^{21}\cdot 3^{13}$	$2^{34}\cdot 3^{21}$
1	2	3	4	5	6	7	8	9

$34+21 = \boxed{55}$

15. If $\int_{-1}^{1} f(x)dx = 3$, $\int_{2}^{3} f(x)dx = -2$ and $\int_{1}^{3} f(x)dx = 5$, evaluate $\int_{-1}^{2} f(x)dx$

$\int_{-1}^{1} + \int_{1}^{3} = \int_{-1}^{3}$ $3+5=8$

$\int_{-1}^{2} + \int_{2}^{3} = \int_{-1}^{3}$ $u + -2 = 8$

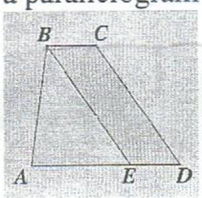
$\boxed{u=10}$

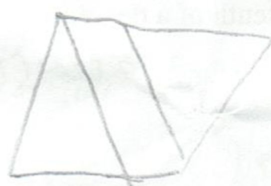
16. Find the rectangular equation of a curve having parametric equations $x = 3\cos t + 7$ and $y = 3\sin t + 7$

$\frac{x-7}{3} = \cos t$ $\quad \frac{y-7}{3} = \sin t$

$\frac{(x-7)^2}{9} + \frac{(y-7)^2}{9} = 1$

17. Trapezoid $ABCD$ has side BC parallel to side AD. $\overline{BE}$, which is parallel to $\overline{CD}$, creates a parallelogram and a triangle of equal area. If $AD = 10$, find the length of $\overline{BC}$.





$25 \cdot 4 = 100$

18. An eccentric math teacher presents her problems by matching the base of the number system she uses to the hour of the day. For example, in the hour beginning at 11:00 a.m., the product of 25 and 4 would be written as $23 \cdot 4 = 91$ (because $23_{11} \cdot 4_{11} = 91_{11}$)
If the teacher poses the problem $18 \cdot 6$ at 9:30 a.m., what is the answer in the appropriate base?

9

$18 \cdot 6 = \cancel{108}$

130

$18_9 \cdot 6_9 = ??_9$

$81 \cdot 1 + 9 \cdot 3 + 1 \cdot 0$

$(8 + 9 \cdot 7) \cdot (6)$ $\quad 81 + 9(2) + 3$

123_{11}

$17 \cdot 6 = 102$ $\quad 123$

19. The reciprocal of $2x^2 - 5x + 3$ can be written in the form $\frac{m}{x+p} + \frac{n}{x+q}$. Find the value of $m + n + p + q$

$\frac{1}{2x^2 - 5x + 3} = \frac{1}{(2x-3)(x-1)} = \frac{2}{2x-3} + \frac{-1}{x-1}$

$\frac{2 \cdot \frac{1}{2}}{2x-3 \cdot \frac{1}{2}} = \frac{1}{x - \frac{3}{2}} + \frac{-1}{x-1}$

$\cancel{2x} - 2 - \cancel{2x} + 3$

1

$1 + -\frac{3}{2} + (-1) + (-1)$

$1 - 1.5 - 2$

-2.5

20. Find the absolute maximum value attained by the function $f(x) = A\sin(x) + B\cos(x)$ on $[0, \pi]$, where A and B are positive constants

7. CIS

$f'(x) = A\cos x - B\sin x = 0$

$A\cos x = B\sin x$

$(0, B)$

$(\pi, -B)$

$\sqrt{A^2 + B^2}$