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CHAPTER 1: Functions and Models

1.1 Four Ways to Represent a Function

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

1.2 Mathematical Models: A Catalog of Essential Functions

- Linear Model – Graph of the function is a line. The formula for a linear model can be written in standard, point-slope, or slope-intercept form.
- Polynomial

  $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

  - Degree – the highest power present in a polynomial; in the polynomial, the degree is n
  - Quadratic and Cubic Functions are both polynomials
- Power Function

  $$f(x) = x^a$$

  - a is a constant; x is the independent variable
  - Root Function when $a = 1/n$
- Rational Function

  $$f(x) = \frac{P(x)}{Q(x)}$$

- Exponential Function

  $$f(x) = a^x$$

- Logarithmic Function

  $$f(x) = \log_a x$$

1.3 New Functions from Old Functions

<u>Transformations</u>

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Composite Functions

Algebra of Functions Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows:

$$(f+g)(x) = f(x) + g(x) \qquad \text{domain} = A \cap B$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{domain} = A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{domain} = A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{domain} = \{x \in A \cap B \mid g(x) \neq 0\}$$

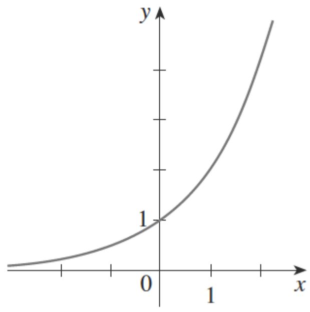
Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

1.5 Exponential Functions

Exponential Form: $$f(x) = a^x$$

Exponential Graph:



Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

1. $a^{x+y} = a^x a^y$ **2.** $a^{x-y} = \dfrac{a^x}{a^y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

1.6 Inverse Functions and Logarithms

One-to-One Functions

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Inverse Functions

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

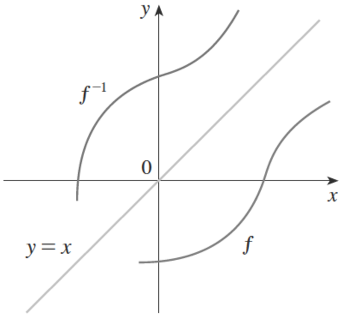
STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

The following equations cancel out and produce x:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Graphing Inverse Functions:



The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Logarithmic Functions

$$\log_a x = y \iff a^y = x$$

Cancellation Equations:

$$\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$$

$$a^{\log_a x} = x \quad \text{for every } x > 0$$

Laws of Logarithms If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r\log_a x$ (where r is any real number)

Natural Logarithms

$$\log_e x = \ln x$$

10 **Change of Base Formula** For any positive number a ($a \neq 1$), we have

$$\log_a x = \frac{\ln x}{\ln a}$$

Inverse Trigonometric Functions

Cancellation Equations:

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

CHAPTER 2: Limits and Derivatives

2.1 The Tangent and Velocity Problems

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{time elapsed}}$$

2.2 The Limit of a Function

3 $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^-} f(x) = L$ and $\lim\limits_{x \to a^+} f(x) = L$

6 **Definition** The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

2.3 Calculating Limits and Using the Limit Laws

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$
2. $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$
3. $\lim\limits_{x \to a} [cf(x)] = c \lim\limits_{x \to a} f(x)$
4. $\lim\limits_{x \to a} [f(x)g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$
5. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ if $\lim\limits_{x \to a} g(x) \neq 0$

6. $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$ where n is a positive integer

7. $\lim\limits_{x \to a} c = c$

8. $\lim\limits_{x \to a} x = a$

9. $\lim\limits_{x \to a} x^n = a^n$ where n is a positive integer

10. $\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

11. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim\limits_{x \to a} f(x) > 0$.]

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

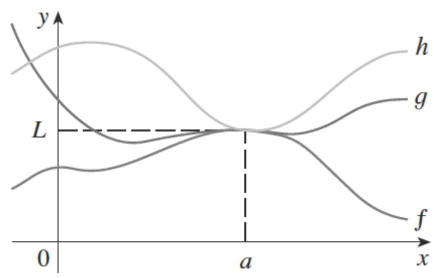
1 Theorem $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^-} f(x) = L = \lim\limits_{x \to a^+} f(x)$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$



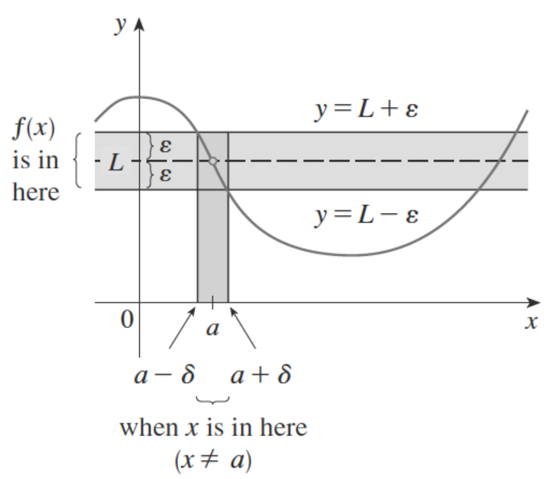
2.4 The Precise Definition of a Limit

2 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$



2.5 Continuity

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$

7 Theorem The following types of functions are continuous at every number in their domains:

polynomials rational functions root functions

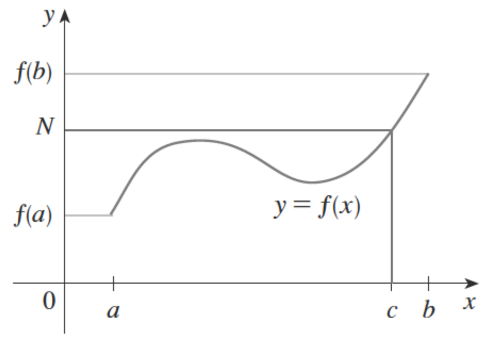
trigonometric functions inverse trigonometric functions

exponential functions logarithmic functions

Requirements for Continuity:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

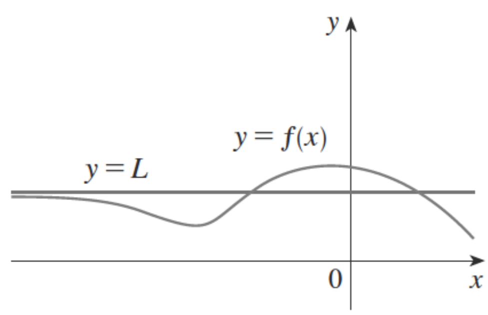
10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



2.6 Limits at Infinity; Horizontal Asymptotes

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$



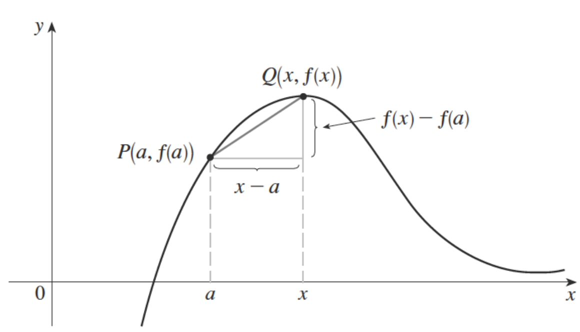
2.7 Tangents, Velocities, and Other Rates of Change

Tangents

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.



Above, *a* becomes progressively smaller as the slope *m* becomes closer to that of point *P*.

To find the equation of the slope of a tangent line, we use the following equation:

$$\boxed{m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}}$$

Velocity

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a + h) - f(a)}{h}$$

Since the velocity at a point in time *v(a)* is analogous to the slope of a tangent at an x-value *f(a)*, the equation to find instantaneous velocity is similar.

$$\boxed{v(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}}$$

2.8 Derivatives

2 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

or

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

2.9 The Derivative as a Function

Equation of a Derivative

Compare the following to the equation in the previous section. While the preceding equations find the value of a derivative at a certain point a, the following equation finds the equation of the derivative at point x.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Notations

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

Differentiability

3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

4 Theorem If f is differentiable at a, then f is continuous at a.

CHAPTER 3: Differentiation Rules

3.1 Derivatives of Polynomials and Exponential Functions

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x) = 1$$

The Power Rule: $$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Constant Multiple Rule: $$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

The Sum Rule: $$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Derivative of the Natural Exponential Function: $$\frac{d}{dx}(e^x) = e^x$$

3.2 The Product and Quotient Rules

The Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

3.3 Rates of Change in the Natural and Social Sciences

Velocity
- *s(t)* – **position** is the base quantity
- |*s(t)*| – **total distance** is the sum of all segments traveled (sum of all partial distances)
- *v(t)* – **velocity** is the rate of change of position (derivative of position)
- *a(t)* – **acceleration** is the rate of change of velocity (derivative of velocity) or the rate of change of the rate of change of position (second derivative of position)

Chemistry
- *C(t)* – **concentration** is the base quantity
- *C'(t)* – **instantaneous rate of reaction** measures the change in concentration over time, how fast the concentration changes

Biology
- *P(t)* – **population** is the base quantity
- *P'(t)* – **instantaneous rate of growth** is the rate at which the population changes

Economics
- *C(n)* – **cost** is the base quantity; *C* is the cost of producing *n* units
- *C'(n)* – **marginal cost** is the cost of each additional unit

Generic Definition
- *f(t)* – expresses the base **quantity** *f* as *t* varies
- *f'(t)* – expresses the **rate of change in quanity** as *t* varies

3.4 Derivatives of Trigonometric Functions

$$\boxed{\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0} \qquad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

3.5 The Chain Rule

The Chain Rule If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively, $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

3.6 Implicit Differentiation

"**Implicit differentiation** consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'."

Example: Find y' if $x^3 + y^3 = 6xy$. →

$$3x^2 + 3y^2y' = 6y + 6xy'$$

$$x^2 + y^2y' = 2y + 2xy'$$

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

"Two curves are called **orthogonal** if at each point of intersection their tangent lines are perpendicular."

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

3.8 Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

By the chain rule,

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

3.10 Related Rates

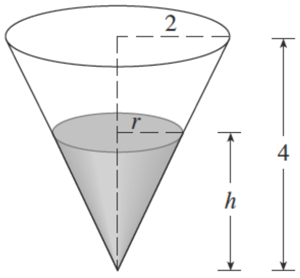


FIGURE 3

EXAMPLE 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

SOLUTION We first sketch the cone and label it as in Figure 3. Let V, r, and h be the volume of the water, the radius of the surface, and the height at time t, where t is measured in minutes.

We are given that $dV/dt = 2$ m^3/min and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \tfrac{1}{3}\pi r^2 h$$

but it is very useful to express V as a function of h alone. In order to eliminate r, we use the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \qquad r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now we can differentiate each side with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3$ m and $dV/dt = 2$ m^3/min, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}\cdot 2 = \frac{8}{9\pi}$$

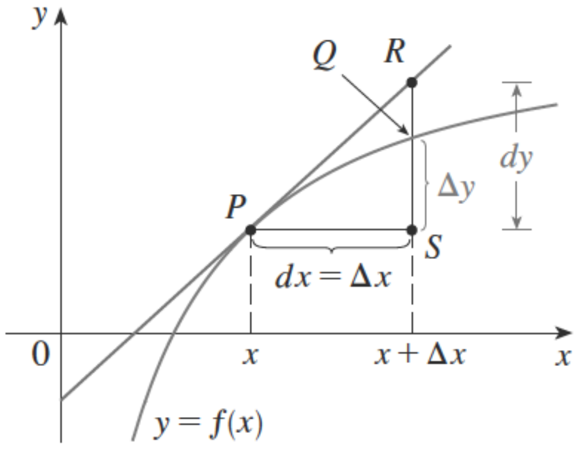
The water level is rising at a rate of $8/(9\pi) \approx 0.28$ m/min.

3.11 Linear Approximations and Differentials

<u>Linearization</u>
The standard form for linear approximations is $L(x) = f(a) + f'(a)(x-a)$, which is analogous to point-slope form, $y = y_1 + m(x-x_1)$. The Linear approximation $L(x)$ is equivalent to the extrapolated value y, and the rate of change $f'(a)$ is equivalent to the instantaneous slope m.

Differentials

- dx, dy, dz, etc. are differentials
- Difference between dy and Δy:



- $\frac{dy}{dx} = 2x^2 \rightarrow dy = 2x^2\, dx$

CHAPTER 4: Applications of Differentiation

4.1 Maximum and Minimum Values

There can either be one (or none) absolute maximum or absolute minimum. However, there can be several (or none) relative minima or relative maxima.

1 Definition A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geqslant f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** at c if $f(c) \leqslant f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme values** of f.

2 Definition A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geqslant f(x)$ when x is near c. [This means that $f(c) \geqslant f(x)$ for all x in some open interval containing c.] Similarly, f has a **local minimum** at c if $f(c) \leqslant f(x)$ when x is near c.

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

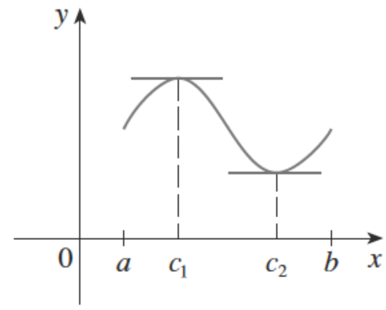
4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

4.2 The Mean Value Theorem



Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

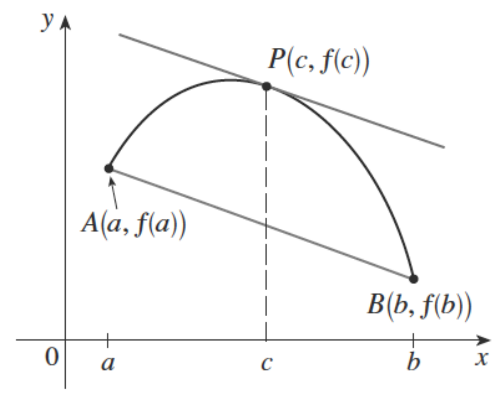
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

1 $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

2 $$f(b) - f(a) = f'(c)(b - a)$$



7 **Corollary** If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

4.3 How Derivatives Affect the Shape of a Graph

<u>Finding Maxima and Minima: Method I</u>

The **Increasing/Decreasing Test** tests for the direction in which *f(x)* is changing, wheras the **First Derivative Test** applies the prior to finding relative maxima and minima.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

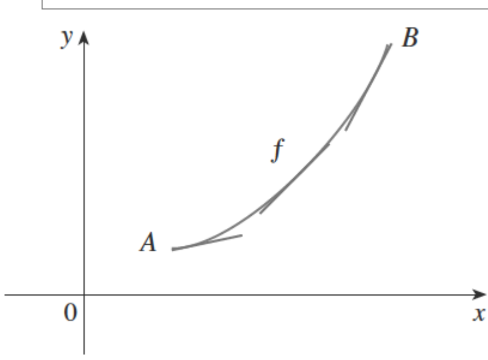
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

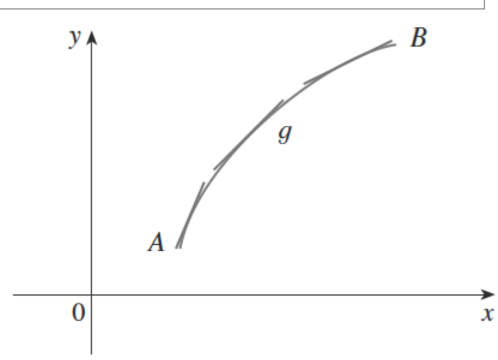
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

<u>Concavity</u>

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.



(a) Concave upward



(b) Concave downward

Concavity Test
(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

The inflection point is the point at which *f''(x)* is equal to zero or is not defined.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Finding Maxima and Minima: Method II

The Second Derivative Test Suppose f'' is continuous near c.
(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

4.4 Indeterminate Forms and L'Hospital's Rule

Indeterminate Forms

The limit of a function approaches: $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or 0^0 or ∞^0 or 1^∞

L'Hospital's Rule: take the derivative of the numerator and denominator (fractions only) until the limit of the function no longer produces an indeterminate form.

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Indeterminate Product: $0 \bullet \infty$
Indeterminate Difference: $\infty - \infty$

4.5 Summary of Curve Sketching

A. Domain
B. Intercepts
C. Symmetry
D. Asymptotes
   a. Horizontal Asymptotes – numerator degree equal to or less than that of denominator
   b. Slant (Oblique) Asymptotes – numerator degree greater than that of the denominator
   c. Vertical Asymptotes – function approaches infinity at an x-value
E. Intervals of Increase or Decrease
F. Local Maximum and Minimum Values
G. Cocavity and Points of Inflection
H. Sketch the Curve

4.7 Optimization Problems

A. Understand the Problem
B. Diagram the Problem
C. Substituting with constants provided, equate the quantity to be optimized Q to an expression using a variable quantity t.
D. Find the relative maximum or minimum desired.

4.8 Applications to Business and Economics

- *C(x)* – **cost function** expresses the total price of producing *x* units
- *c(x)* – **average cost function** expresses the price per unit *x* and is calculated by the expression *C(x)/x*
- *C'(x)* – **marginal cost** is the cost per additional unit
- average cost *c(x)* is at a minimum when *C'(x) = c(x)*

Similarly,

- *R(x)* – **revenue (sales) function** expresses the total amount sold when selling *x* units
- *p(x)* – **price (demand) function** expresses the price per unit when selling *x* units
- *R(x) = x•p(x)* – expresses the **revenue** *R(x)* from selling *x* units at price *p(x)* each
- *P(x) = R(x) - C(x)* – expresses the total **profit** *P(x)* from selling *x* units
- Therefore, *P'(x) = R'(x) - C'(x)* and profit is at a maximum when marginal revenue is equal to marginal cost, that is it costs as much to produce another unit as the amount of revenue received for it.

4.9 Newton's Method

Newton's method is used to find the root of a function.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

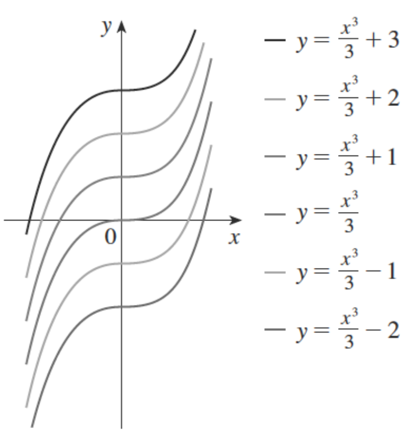
Start at an estimate. From the estimate , subtract the value of the function at the estimate divided by the value of the derivative at the estimate. Store this value as your new estimate. Repeat until your "estimate" stops changing.

4.10 Antiderivatives

Antiderivatives

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Effect of varying constants of integration *C*:



1 Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Rectilinear Motion

Recall that:

- *s(t)* – **position** is the base quantity
- *v(t)* – **velocity** is the rate of change of position (derivative of position)
- *a(t)* – **acceleration** is the rate of change of velocity (derivative of velocity) or the rate of change of the rate of change of position (second derivative of position)

Similarly,

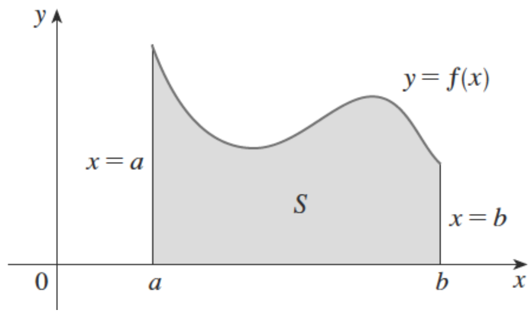
- *s(t)* – **position** measures the total change in velocity, the antiderivative of velocity (plus a constant) or the antiderivative of the antiderivative of acceleration (plus two constants)
- *v(t)* – **velocity** measures the total change in acceleration, the antiderivative of acceleration (plus a constant)
- *a(t)* – **acceleration** is the rate of change of velocity

Taking the antiderivate of a quantity does not tell you about the "starting point" of the base quantity. For example, taking the antiderivative of velocity does not define from where a particle begins moving. The **constant of integration** *C* added adjusts the expression for the starting point. If the problem is worded, "The particle begins at *s(0) = 10*", then *s(t)* should be evaluated for *t = 0* and the constant *C* should be adjusted so that *s(0)* equals 10.

CHAPTER 5: Integrals

5.1 Areas and Distance

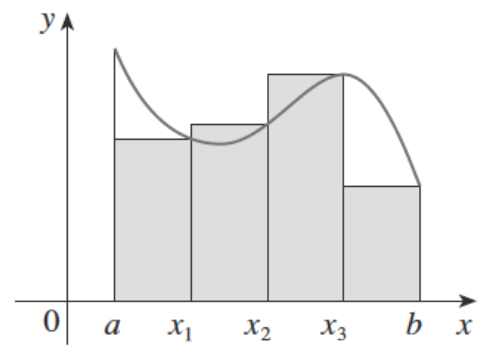
The integral of a function finds the area under its graph.



As the derivative is to the division of small parts, the integral is to multiplication of small parts.

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

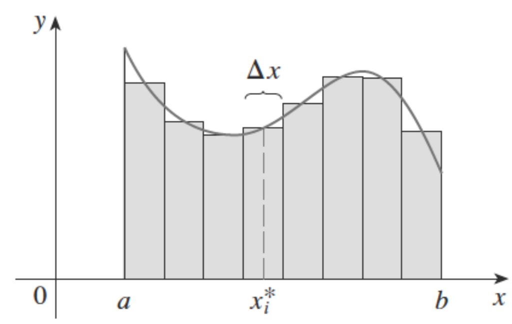
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$



5.2 The Definite Integral

2 Definition of a Definite Integral If f is a continuous function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$



A **Riemann Sum** sums a set number of approximating rectangles under the curve. The rectangles may be of constant or variable width. The height of each rectangle dependent upon the height of the graph at either the *rightmost* or *leftmost* corner. The notation for a Riemann Sum is R_n or L_n, respectively.

A **Midpoint Sum** is similar to Riemann Sum, except that the height of each rectangle is dependent upon the height of the graph at the rectangle's horizontal midpoint rather than at a corner. The notation for a Midpoint Sum is M_n. The following is a method for computing a midpoint sum using rectangles of fixed width.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x\,[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where $\Delta x = \dfrac{b-a}{n}$

and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

<u>Properties of Integral</u>

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5.3 The Fundamental Theorem of Calculus

<u>FTC I</u>
In short, the integral of the derivative of a function is the function itself.

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

When the upper limit of the integral does not equal x, then follow the method demonstrated in the following example.

$$\begin{aligned}\frac{d}{dx}\int_1^{x^4} \sec t\,dt &= \frac{d}{dx}\int_1^{u} \sec t\,dt \\ &= \frac{d}{du}\left(\int_1^u \sec t\,dt\right)\frac{du}{dx} && \text{(by the Chain Rule)} \\ &= \sec u\,\frac{du}{dx} && \text{(by FTC1)} \\ &= \sec(x^4)\cdot 4x^3\end{aligned}$$

<u>FTC II</u>

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

5.4 Indefinite Integrals and the Net Change Theorem

Indefinite integrals do not have limits of integration.

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

Just as in FTC I,

The Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Just as in FTC II,

5.5 Substitution Rule

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Typical Example:

EXAMPLE 2 Evaluate $\int \sqrt{2x+1}\,dx$.

SOLUTION 1 Let $u = 2x + 1$. Then $du = 2\,dx$, so $dx = du/2$. Thus, the Substitution Rule gives

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \int \sqrt{u}\,\frac{du}{2} = \tfrac{1}{2}\int u^{1/2}\,du \\ &= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}u^{3/2} + C \\ &= \tfrac{1}{3}(2x+1)^{3/2} + C\end{aligned}$$

6 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

With definite integrals, the substitution rule for indefinite integrals can still be used. However, *g(x)* must be substituted back in place of *u* immediately before applying the limits of integration. Essentially, this is the same process at the substitution property for definite integrals.

7 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

5.6 The Logarithm Defined as an Integral

1 Definition The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

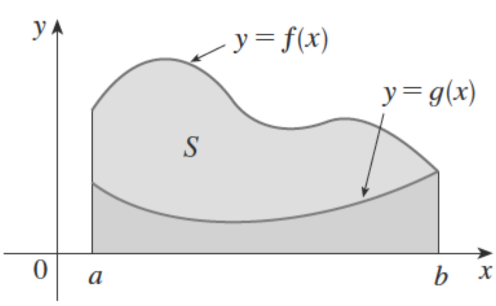
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

CHAPTER 6: Applications of Integration

6.1 Areas Between Curves

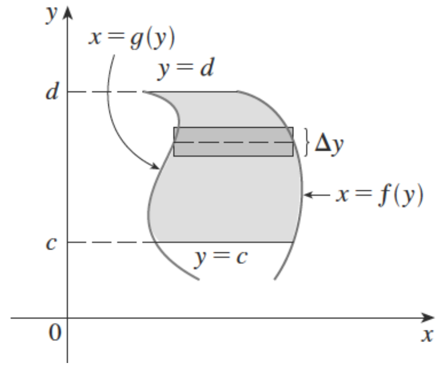
2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$



In the expression above, *f(x)* refers to the uppermost function.

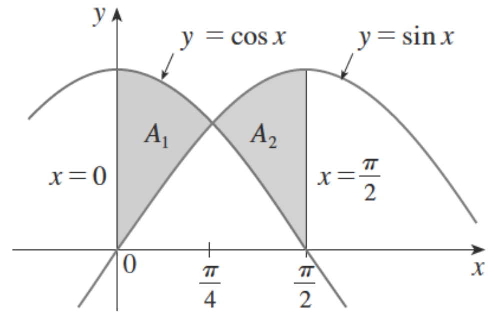
$$A = \int_c^d [f(y) - g(y)]\,dy$$



Similarly, *f(y)* refers to the rightmost function.

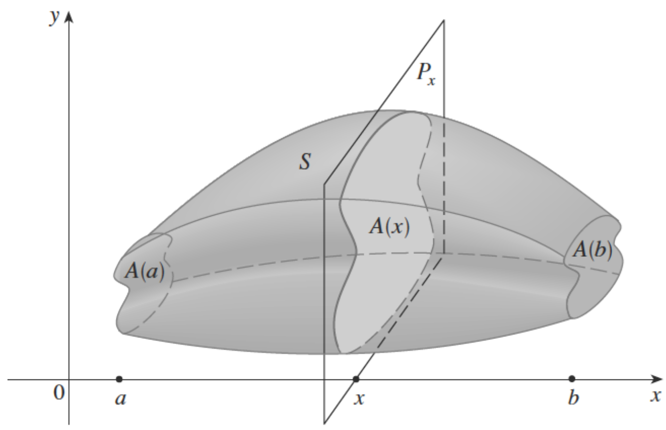
When the superlative function alternates, then the area of each segment must be taken:

> **3** The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is
>
> $$A = \int_a^b |f(x) - g(x)|\,dx$$



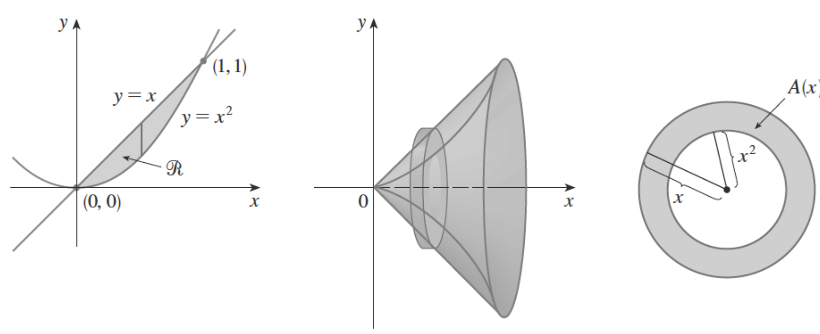
6.2 Volumes

The volume of a solid can be determined by the summation of the cross-sectional areas of the solid *A(x)*, multiplied by the thickness of each cross-section *dx*.

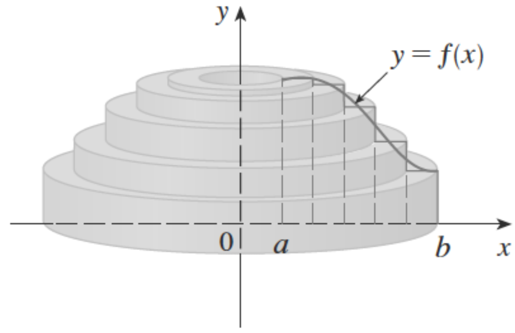


When revolving functions about an axis or line, each cross-section will represent the area of a **disk** or **washer**. By revolving a function about an axis at a certain point, a disk is produced. When determining the volume produced with the combination of two functions, the area of each smaller disk is subtracted from the area of each larger one at every point, creating a washer. It is easier to view a volume derived from a rotated function as sums of cross-sections, rather than viewing it as a separate form entirely.

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx$$



6.3 Volumes by Cylindrical Shells



> **2** The volume of the solid in Figure 3, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is
>
> $$V = \int_a^b 2\pi x f(x)\,dx \qquad \text{where } 0 \leq a < b$$

The method of using cylindrical shells is analogous to summing the volumes of "circumfrences" of radius x, height $f(x)$, and width dx.

If there are two functions involved, then $f(x)$ above simply becomes the difference in height between the functions.

6.4 Work

$$W = Fd \qquad \text{work} = \text{force} \times \text{distance}$$

Work is the product of force and distance.
Integrating force with respect to distance will produce a measure of work.

Force is prodcued by multiplying mass m by acceleration. Recall that acceleration $a(t)$ is the second derivative of position $s''(t)$. Therefore, force can be expressed by the following expression:

$$F = m\frac{d^2s}{dt^2}$$

A common work problem involves the use of a spring. The force on a spring is expressed by $f(x) = kx$, where f is the force exerted when a spring with **spring constant** k is stretched a distance x from its **natural length**.

6.5 Average Value of a Function

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

> **The Mean Value Theorem for Integrals** If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that
>
> $$\int_a^b f(x)\,dx = f(c)(b-a)$$

CHAPTER 7: Techniques of Integration

7.1 Integration by Parts

The following two forms are equivalent.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

With the implementation of limits to produce definite integrals:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$$

7.2 Trigonometric Integrals

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

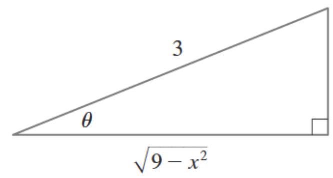
Then substitute $u = \sec x$.

7.3 Trigonometric Substitution

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example:

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta$$



7.4 Integration of Rational Functions by Partial Fractions

CASE I ▫ **The denominator $Q(x)$ is a product of distinct linear factors.**

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

CASE II ▫ **$Q(x)$ is a product of linear factors, some of which are repeated.**

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

CASE III ▫ **$Q(x)$ contains irreducible quadratic factors, none of which is repeated.**

$$\frac{Ax + B}{ax^2 + bx + c}$$

CASE IV ▫ **$Q(x)$ contains a repeated irreducible quadratic factor.**

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

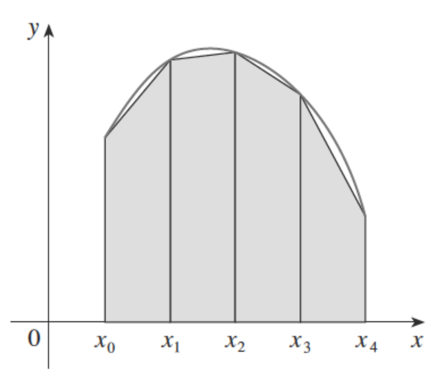
7.7 Approximate Integrals

A **Trapezoidal Sum** is attained through the summation of trapezoids, whose bases are located at the end points of each step-width. The trapezoidal sum can be taken with constant or variable step sizes; the trapezoidal rule below uses constant-width steps.

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.



Another method for estimating the value of an integral is **Simpson's Rule**.

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = (b - a)/n$.

7.8 Improper Integrals

2 $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 74).

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

CHAPTER 8: Further Applications of Integrals

8.1 Arc Length

Horizontally,

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Vertically,

$$\boxed{L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

8.2 Area of Surface of Revolution

$$S = \int 2\pi y\, ds \qquad \text{OR} \qquad S = \int 2\pi x\, ds$$

WHERE

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

8.3 Applications to Physics and Engineering

<u>Hydrostatic Pressure and Force</u>

$$P = \frac{F}{A} = \rho g d \qquad F = mg = \rho g A d$$

<u>Moments and Centers of Mass</u>

The **moment of the system** about the axis indicated in each subscript can be found through the following:

$$M_y = \sum_{i=1}^{n} m_i x_i \qquad M_x = \sum_{i=1}^{n} m_i y_i$$

Similarly, the moment can be found with the following equations, where ρ represents an appropriate measure of density Notice that the integral of $\rho f(x)$ with respect to x simply represents area.

$$\boxed{M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\, \Delta x = \rho \int_a^b x f(x)\, dx} \qquad \boxed{M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \tfrac{1}{2}[f(\bar{x}_i)]^2\, \Delta x = \rho \int_a^b \tfrac{1}{2}[f(x)]^2\, dx}$$

The **center of mass**, located at $(\bar{x}, \bar{y})$, can be found using the following:

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

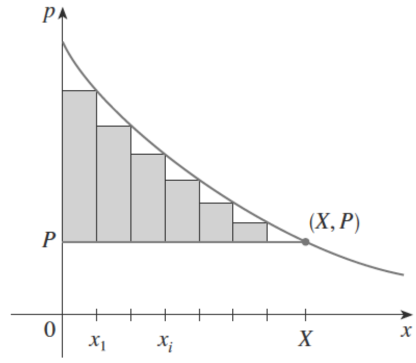
Combining the moment of the system and the center of mass, we arrive at the following:

$$\boxed{\bar{x} = \frac{1}{A} \int_a^b x f(x)\, dx \qquad \bar{y} = \frac{1}{A} \int_a^b \tfrac{1}{2}[f(x)]^2\, dx}$$

8.4 Applications to Economics and Biology

Consumer Surplus

$$\int_0^X [p(x) - P]\,dx$$



"The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P."

CHAPTER 9: Differential Equations

9.1 Modeling with Differfntial Equations

Population Growth

$$\frac{dP}{dt} = kP \qquad \frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

Above, t represents time and P represents the number of individuals in a population at a point in time. While the first expression represents the rate of growth of an uninhibited species, the second models the rate of growth of a population with a **carrying capacity** (maxium inhabitants) K.

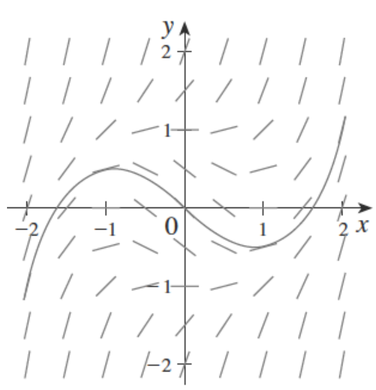
Motion of a Spring
Recall that **Hooke's Law** $F = kx$ is used for springs. In combining Hooke's Law with the laws of motion, we can obtain the following, which expresses a relationship between acceleration and a spring's properties.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

.

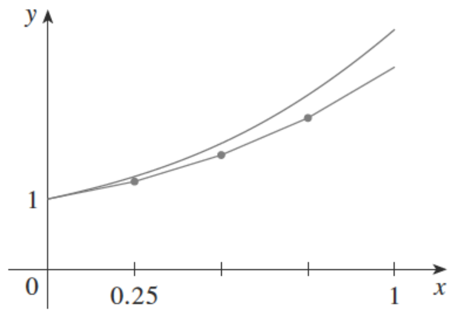
9.2 Direction Fields and Euler's Method

Direction Fields
When direction fields are used, the initial value of the solution is unknown (constant C). Therefore, the solution from the direction field may be offset vertically.



Euler's Method

Euler's Method approximates the shape of a curve by making incremental linear approximations based on a differential function. The increments are called **steps** and their magnitudes are called **step sizes**.



The following represents the general form for Euler's method, where $f'(x_n)$ represents the value of the differential at the n[th] step.

$$y_{n+1} = y_n + \Delta x[f'(x_n)]$$

9.3 Separable Equations

When given a differential *dy/dx*, *dx* and *dy* can be separated onto opposition sides. If the expression contains x-values on one side of the equals sign and y-values on the other, the expression can be integration on both sides.

Example:

$$\frac{dy}{y} = x^2\,dx \qquad y \neq 0$$

$$\int \frac{dy}{y} = \int x^2\,dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

9.4 Exponential Growth and Decay

The following can be adapted to model population growth, radioactive decay, Newton's Law of Cooling, or continuously compounded interest.

2 The solution of the initial-value problem

$$\frac{dy}{dt} = ky \qquad y(0) = y_0$$

is

$$y(t) = y_0 e^{kt}$$

9.5 The Logistic Equation

The logistic equation is analogous model of growth of a population with carrying capacity *K*.

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)}$$

The general solution of the logistic equation:

$$P(t) = \frac{K}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{K - P_0}{P_0}$$

9.6 Linear Equations

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\,dx}$ and integrate both sides.

CHAPTER 10: Curves in Parametric, Vector, and Polar Form

10.1 Curves Defined by Parametric Equations

Parametric equations are split into *x* and *y* equations, each in terms of a **parameter** *t*. Similarly, the rates of change of *x* and *y* are calculated separately, each in terms of the third variable *t*.

$$x = f(t) \qquad y = g(t)$$

10.2 Calculus with Parametric Curves

Tangents

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

After understanding the first derivative of a parametric function, the second can be derived repeatedly.

Area

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$$

It may be more intuitive to view *g(t)* as *y* and *f'(t)* as *dx/dt*. The integrand would then be the expanded form of the previous term.

Arc Length

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

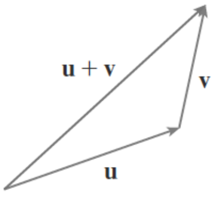
Surface Area
Recall that to calculate the surface area around a solid of rotation, the integrand used is $2\pi y\,ds$. Below, ds has been replaced with the parametric expression for arc length.

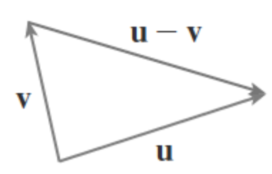
$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

10.3 Vectors in Two Dimensions

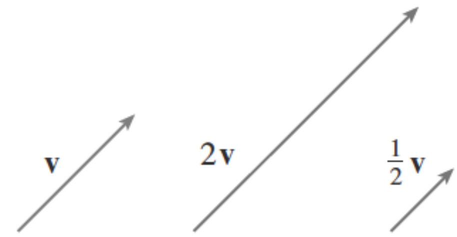
Vectors are added by joining the **head** (end with arrow) of the first vector to the **tail** (end without arrow) of the next. When subtracting vectors, change the direction of the negated vector.

Definition of Vector Addition If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.





Definition of Scalar Multiplication If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.



The **components** of a vector are its horizontal and vertical magnitudes. The components of vector v from point $A(a, b)$ to point $B(c, d)$ would be expressed as $v = \langle c - a, d - b \rangle$. A vector can also be written as the sum of horizontal and vertical components i and j.

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\,\mathbf{i} + a_2\,\mathbf{j}$$

The length of the vetor $|v|$ would be calculated with the pythagoream theorem, using the components of the vector as the legs of a triangle.

<u>Properties</u>

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	**2.** $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$	**4.** $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$	**6.** $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$	**8.** $1\mathbf{a} = \mathbf{a}$

10.4 Vector Functions and Their Derivatives

The limit of a vector is computed by taking the limit of its components.

The **unit tangent vector** T(t) is expressed by the following, where r'(t) represents the derivative of the vector and |r'(t)| represents the derivative of the length of the vector.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The following describes the process for taking derivatives of three-dimensional vectors, but the same methodology holds for 2D vectors.

$\boxed{2}$ **Theorem** If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

Integral of a vector:

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j}$$

<u>Vector Equation of a Line</u>
To create an equation of a line with vectors, the initial vector r_0 is added to an increment vector v, t number of times.

$$r = r_0 + tv$$

If v = <*a, b*>, then the equation can be broken into the following two vectors.

$$x = x_0 + at$$
$$y = y_0 + bt$$

10.5 Curvilinear Motion: Velocity and Acceleration

Motion can be described in terms of vectors. The curvilinear representation of motion incorporates the *x* and *y* equations of the rectilinear approach as components of a vector.

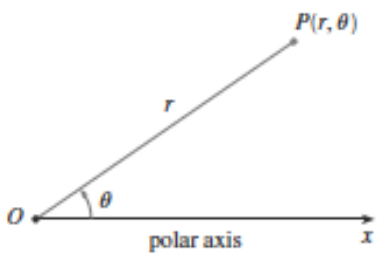
The principles of motion hold:

$$\mathrm{a}(t) = \mathrm{v}'(t) = \mathrm{r}''(t)$$
$$\mathrm{r}(t) = \int \mathrm{v}(t)\,dt = \int \left[\int \mathrm{a}(t)\,dt\right] dt$$

Given the initial velocity of a projectile, the motion (the **trajectory**) of the object can be separeted into components of a vector or parametric equations. Below, *g* is the metric **gravitational constant**.

$$x = (v_0 \cos\alpha)t$$
$$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \quad \text{where } g = 9.8$$

10.6 Polar Coordinates



$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

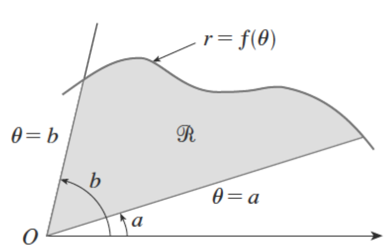
Tangents to Polar Curves

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

10.7 Areas and Lengths in Polar Coordinates

Areas

$$A = \int_a^b \frac{1}{2} r^2\, d\theta$$



Arc Length

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

CHAPTER 11: Infinite Sequences and Series

11.1 Sequences

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

11.2 Series

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

7 The Test for Divergence If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

8 Theorem If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

11.3 The Integral Test and Estimates of Sums

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

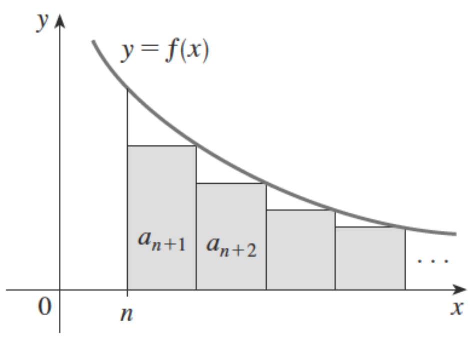
(i) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

1 The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

<u>Remainders</u>

$$R_n = s - s_n$$



2 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_n^{\infty} f(x)\,dx$$

11.4 The Comparison Test

The Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

(i) If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.

(ii) If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

11.5 Alternating Series

The Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies

(i) $b_{n+1} \leqslant b_n$ for all n

(ii) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

11.6 Absolute Convergence and the Ratio and Root Tests

1 Definition A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

2 Definition A series $\Sigma\, a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

3 Theorem If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

The Ratio Test

(i) If $\lim_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

The Root Test

(i) If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

11.8 Power Series

A **power series** is a series in the following form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\displaystyle\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

11.10 Taylor and Maclaurin Series

Taylor Series

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

6
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Maclaurin Series

A **Maclaurin** series is a special instance of a Taylor Series in which the series is centered at 0. That is, $a = 0$.

7
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leqslant M$ for $|x-a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leqslant d$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x$$

11.11 The Binomial Series

2 The Binomial Series If k is any real number and $|x| < 1$, then

$$(1 + x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where $\displaystyle \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \quad (n \geqslant 1)$ and $\displaystyle \binom{k}{0} = 1$

Reference

Trigonometry

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

2 To evaluate the integrals (a) $\int \sin mx \cos nx\,dx$, (b) $\int \sin mx \sin nx\,dx$, or (c) $\int \cos mx \cos nx\,dx$, use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Integrals

12. $\int \tan u\,du = \ln|\sec u| + C$

13. $\int \cot u\,du = \ln|\sin u| + C$

14. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

15. $\int \csc u\,du = \ln|\csc u - \cot u| + C$

16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

17. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

19. $\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

20. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

1 **Table of Indefinite Integrals**

$\int cf(x)\,dx = c\int f(x)\,dx$ $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

$\int k\,dx = kx + C$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$ $\int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$ $\int a^x\,dx = \frac{a^x}{\ln a} + C$

$\int \sin x\,dx = -\cos x + C$ $\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$ $\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$ $\int \csc x \cot x\,dx = -\csc x + C$

$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$ $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (-\infty, \infty)$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad [-1, 1]$$

All clippings are from the following source:

Stewart, James. *Single Variable Calculus Early Transcendentals*. 5th ed. Belmont: Thomson/Brooks/Cole, 2003. Print.