1.

a.

Let $M = \sup\{|z| : z \in B_\epsilon(z_0)\}$. Then for every $z \in B_\epsilon(z_0)$, for $n \geq M+1$, we have that

$$\left|\frac{z^n}{n^n}\right| \leq \frac{M^n}{(M+1)^n}$$

and since

$$\frac{M}{M+1} < 1$$

$f_n(z)$ converges compactly.

b.

For every $z_0 \in \mathbb{C}$, we can choose any $\epsilon > 0$, then let $M = \sup\{|z| : z \in B_\epsilon(z_0)\}$. Then for every $z \in B_\epsilon(z_0)$, we have that

$$\left|\frac{1}{m^2}\exp\left(\frac{z}{m}\right)\right| \leq \frac{1}{m^2}\exp\left(\frac{M}{m}\right)$$

and since

$$\int_0^\infty \frac{1}{m^2}\exp\left(\frac{M}{m}\right) = -\frac{1}{M}e^{-\frac{M}{m}}\Big|_0^\infty = \frac{1}{M}$$

$f_n(z)$ converges compactly.

2.

We have that

$$\sum_{n=1}^{\infty} f\left(\frac{z}{n}\right) = \sum_{n=1}^{\infty}\sum_{k=2}^{\infty} a_k \left(\frac{z}{n}\right)^k$$
$$= \sum_{k=2}^{\infty}\left(a_k \sum_{n=1}^{\infty}\frac{1}{n^k}\right) z^k$$
$$\leq \left(\sum_{n=2}^{\infty}\frac{1}{n^2}\right)\left(\sum_{k=2}^{\infty} a_k z^k\right)$$

which converges. Hence, the series converges compactly on D.
If $f(0) \neq 0$ then there exists some $\delta, \epsilon > 0$ such that $|f(z)| > \epsilon$ for $|z| < \delta$, let $n_0 = |z|/\delta$, we have

$$\left|\sum_{n=1}^{\infty} f\left(\frac{z}{n}\right)\right| \leq \left|\sum_{n=1}^{n_0} f\left(\frac{z}{n}\right)\right| + \sum_{n=n_0}^{\infty} \epsilon$$

diverges. If $f(0) = 0$ and $f'(0) \neq 0$ then there exists $\delta, \epsilon > 0$ such that $|f'(z)| > \epsilon$ for $|z| < \delta$,

$$\left|f\left(\frac{z}{n}\right)\right| = \left|f\left(\frac{z}{n}\right) - f(0)\right| > \epsilon\left|\frac{z}{n}\right|$$

and hence let $n_0 = |z|/\delta$, we have

$$\left|\sum_{n=1}^{\infty} f\left(\frac{z}{n}\right)\right| \leq \left|\sum_{n=1}^{n_0} f\left(\frac{z}{n}\right)\right| + \sum_{n=n_0}^{\infty} \epsilon\left|\frac{z}{n}\right|$$

diverges. Hence, if the summation converges compactly, $f(0) = f'(0) = 0$.

3.

Let $\deg f(z) = n > 1$,

$$\lim_{z\to\infty} \frac{f(z)}{z^n} = c \neq 0$$

Hence, for all $\epsilon > 0$, there exists R such that

$$|c| - \epsilon \leq \frac{|f(z)|}{|z|^n} \leq |c| + \epsilon$$

for $|z| > R$. Fix $0 < \epsilon < |c|$, we have

$$c_1|z|^n - M_1 \leq |f(z)| \leq c_2|z|^n + M_2$$

for all z, where $c_1 = |c| - \epsilon, c_2 = |c| + \epsilon, M_1 = c_1 R^n, M_2 = \max_{|z|\leq R} |h(z)|$. By the fundamental theorem of algebra, for every $w_0 \in \mathbb{C}$, there exists $z_0 \in \mathbb{C}$ such that

$$g(w_0) = z_0$$

Then there exists a, b, c and d,

$$f(z_0) = f(g(w_0)) \leq a|w_0|^m + b$$

$$f(z_0) \geq c|z_0|^n + d$$

$$|g(w_0)| = |z_0| \leq \left(\frac{f(z_0) - d}{c}\right)^{1/n} \leq \left(\frac{a|w_0|^m + b - d}{c}\right)^{1/n}$$

By generalized cauchy integral formula

$$g^{(k)}(0) = \frac{k!}{2\pi i}\int_{|z|=R} \frac{g(z)}{z^{k+1}}dz$$

then

$$|g^{(k)}(0)| \leq k!\frac{\left(\frac{aR^m+b-d}{c}\right)^{1/n}}{R^k}$$

so for all $k > n/m$,

$$\lim_{R\to\infty} |g^{(k)}(0)| = 0$$

Therefore,

$$g(z) = \sum_{0\leq k\leq n/m} \frac{f^{(k)}(0)}{k!} z^k$$

is a polynomial

4.

Let Z be the set of zeroes of g. If Z has an accumulation point, then by the identity theorem $f = g = 0$. Otherwise, $D \backslash Z$ is an open connected set. Hence, we can define

$$h : D \backslash Z \to \mathbb{C}, \qquad z \to \frac{f(z)}{g(z)}$$

$(h(z))^n = 1$ for each $z \in D \backslash Z$, $h(z) \subset \{\text{nth root of } 1\}$ but $h(D \backslash Z)$ is open and connected which means there exists a constant k such that $h(z) = k$ for all $z \in D \backslash Z$. Thus, $f \equiv kg$