1.

$\forall (x_1, y_1), (x_2, y_2) \in G \times G$

$$\begin{aligned}
\sigma((x_1, y_1)) \cdot (x_2, y_2)) &= \sigma(x_1 x_2, y_1 y_2) \\
&= (y_1 y_2, x_1 x_2) = (y_1, x_1) \cdot (y_2, x_2) \\
&= \sigma(x_1, y_1) \cdot \sigma(x_2, y_2)
\end{aligned}$$

Therefore, σ is an homomorphism.
We have that

$$\forall (x, y) \in G \times G : (\sigma \circ \sigma)(x, y) = \sigma(y, x) = (x, y)$$

Therefore, σ is an automorphism.

$$\begin{aligned}
\forall g_1, g_2 \in G : \forall x \in X : &(g_1, g_2) \cdot x = (g_2, g_1) \cdot x \\
&\implies ((g_1, 1) \cdot (1, g_2)) \cdot x = ((g_2, 1) \cdot (1, g_1)) \cdot x \\
&\implies ((1, g_1) \cdot (1, g_2)) \cdot x = ((1, g_2) \cdot (1, g_1)) \cdot x \\
&\implies (1, g_1 g_2) \cdot x = (1, g_2 g_1) \cdot x \\
\text{Similarly,} &(g_1 g_2, 1)(x) = (g_2 g_1, 1)(x)
\end{aligned}$$

$\forall g_1 = (g_{1_1}, g_{1_2}), g_2 = (g_{2_1}, g_{2_2}) \in G \times G, \forall x \in X :$

$$\begin{aligned}
(\rho(g_1) \circ \rho(g_2))(x) &= (\rho(g_{1_1}, g_{1_2}) \circ \rho(g_{2_1}, g_{2_2}))(x) \\
&= (g_{1_1}, g_{1_2}) \cdot (g_{2_1}, g_{2_2}) \cdot x \\
&= (g_{1_1} \cdot g_{2_1}, g_{1_2} \cdot g_{2_2}) \cdot x \\
&= (g_{1_1} \cdot g_{2_1}, 1)(1, g_{1_2} \cdot g_{2_2}) \cdot x \\
&= (g_{2_1} \cdot g_{1_1}, 1)(1, g_{2_2} \cdot g_{1_2}) \cdot x \\
&= (g_{2_1} \cdot g_{1_1}, g_{2_2} \cdot g_{1_2}) \cdot x \\
&= (g_{2_1}, g_{2_2}) \cdot (g_{1_1}, g_{1_2}) \cdot x \\
&= (\rho(g_{2_1}, g_{2_2}) \circ \rho(g_{1_1}, g_{1_2}))(x) \\
&= (\rho(g_2) \circ \rho(g_1))(x)
\end{aligned}$$

2.

G is a group, hence $1 \in G$, which means that $\forall x \in X : 1 \cdot x = x \implies x \sim x$
If $x \sim y$, then $\exists g \in G : g \cdot x = y$.
Hence, $g^{-1} \in G$ and $g^{-1} \cdot y = g^{-1} \cdot g \cdot x = 1 \cdot x = x$ which means that $y \sim x$
If $x \sim y$ and $y \sim z$ then $\exists g_1, g_2 \in G : g_1 \cdot x = y$ and $g_2 \cdot y = z$.
Let $g = g_2 g_1$, then $g \cdot x = g_2 \cdot g_1 \cdot x = g_2 \cdot y = z$, hence $x \sim z$. Therefore, $\sim$ is an equivalence relation on X.

3.

Since $\mathrm{Aut}(G)$ contains all isomorphism function that maps $G \to G$ which means it is bijective and therefore is an element of $\mathrm{Per}(G)$. Therefore, $\mathrm{Aut}(G) \subset \mathrm{Per}(G)$. Since $\mathrm{Aut}(G)$ is a group (hw2), it is a subgroup of $\mathrm{Per}(G)$

$\forall g \in G : \exists g^{-1} \in G : (\rho(g) \circ \rho(g^{-1}))(h) = g \cdot g^{-1} \cdot h \cdot (g^{-1})^{-1} \cdot g^{-1} = h$. Therefore, $\mathrm{im}(\rho) \subset \mathrm{Aut}(G)$.

Hence, we can have an induced homomorphisms γ such that $\mathrm{range}(\gamma) \subset \mathrm{range}(\rho)$ and $\forall g \in G : \rho(g) = \gamma(g)$.

Obviously, $\mathrm{Ker}(\gamma) = \mathrm{Ker}(\rho)$ since $\forall g \in G : \rho(g) = \gamma(g)$.

If $g \in \ker(\gamma)$ then $\gamma(g) = 1$, which means that $\forall g_1 \in G$:

$$g \cdot g_1 \cdot g^{-1} = g_1 \iff g \cdot g_1 \cdot g^{-1} \cdot g = g_1 \cdot g \iff g \cdot g_1 = g_1 \cdot g$$

Hence $\ker(\gamma) = Z(G)$

$\forall \sigma \in \mathrm{im}(\gamma) \implies \exists g_1 \in G : \gamma(g_1) = \sigma_1$. $g_1 \in G$. Let $g_2 = \tau(g_1)$, then

$$g_1 \tau^{-1}(h) g_1^{-1} = \tau^{-1}(g_2)\tau^{-1}(h)\tau^{-1}(g_2^{-1}) = \tau^{-1}(g_2 h g_2^{-1})$$

and hence

$$\rho(g_1)(\tau^{-1}(h)) = \tau^{-1}(\rho(g_2)(h)) \implies (\rho(g_1) \circ \tau^{-1})(h) = (\tau^{-1} \circ \rho(g_2))(h)$$

which means that

$$(\tau \circ \rho \circ \tau^{-1})(g_1) = \tau(\rho(g_1)(\tau^{-1}(h))) = \tau(\tau^{-1} \circ \rho(g_2))(h) = \rho(g_2)(h)$$

Therefore, $\tau \circ \sigma \circ \tau^{-1} \in \mathrm{im}(\gamma)$

4.

1. Since $g_1, g_2, \ldots, g_n$ are generators of G. We have that, $\forall g \in G$: $\exists i_1, i_2, \ldots i_n$such that : $\forall j \in \{1, 2, \ldots, n\} : 0 \leq i_j < \operatorname{ord}(g_j) = k_j$

$$g = g_1^{i_1} \cdot g_2^{i_2} \cdot \ldots \cdot g_n^{i_n}$$

Hence, there exists at most $k_1 k_2 \ldots k_n$ elements in G.
Quaternion group has 8 elements. But we have $\operatorname{ord}(i) = 4$, $\operatorname{ord}(-i) = 4$ hence the products of order of all elements in $Q_8 \geq 16 > 8$.
2. Suppose g_1 is an element in G such that the order of it is finite: $\operatorname{ord}(g_1) = k_1$. Then, $\operatorname{ord}(g_1) = \operatorname{ord}(g_1^{-1}) = k_1$ which means that each element has an inverse.
Suppose g_1, g_2 are element in G of finite order, then $\exists m, n \in \mathbb{N} : g_1^m = 1$ and $g_2^n = 1$, then $(g_1 g_2)^{mn} = 1$ where mn is also finite.
1 has order 1 hence also in the set. Hence, the set of all elements of finite order in G is a subgroup of G. Consider the set of all 2 by 2 matrices.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But as $\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ has eigenvalue 1 which means that $\forall x : \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} x = x$, which means that for all naturanl number k

$$\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}^k x = x$$

As x is arbitary, $\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}^k \neq 0$ and hence has infinite order which means it is not an element of the set of finite order elements.
3. If $(\mathbb{Q}, +)$ is a cyclic group then $\exists g > 0$ be the generator since if $g < 0$ then $\exists g' = -g > 0$ and $g \neq 0$.
Therefore, we have $\forall n \in \mathbb{Z} : g \cdot n < g \cdot (n + 1)$
Hence, $\nexists h \in (g^n, g^{n+1}) \cap \mathbb{Q}$. But since Q is dense, that is there exists a rational number in all open interval, hence contradiction and $\mathbb{Q}$ is a non-cyclic group.
Every subgroup of a cyclic group is cyclic. Suppose, there is a finite generated subgroup generated by $\{g_1, g_2, \ldots, g_n\}$. Since, each g_i is a generators of $\mathbb{Q}$, it is also an element of $\mathbb{Q}$. Let $g_i = \frac{p_i}{q_i}$ then we can find a cyclic group generated by $\{\prod_{i=1}^{n} \frac{1}{q_i}\}$.

Hence, we have $\forall g \in G : \exists j_1, j_2, \ldots, j_n \in \mathbb{Z}$:

$$g = \sum_{i=1}^{n} j_i \cdot \frac{p_i}{q_i} = \sum_{i=1}^{n} \left(j_i \cdot p_i \cdot \prod_{\substack{k=1 \\ k \neq i}}^{n} q_k \prod_{k=1}^{n} \frac{1}{q_k} \right) = \left(\prod_{k=1}^{n} \frac{1}{q_k} \right) \sum_{i=1}^{n} \left(j_i \cdot p_i \cdot \prod_{\substack{k=1 \\ k \neq i}}^{n} q_k \right)$$

which means that the subgroup generated by $\{g_1, g_2, \ldots, g_n\}$ is a subgroup of the subgroup generated by $\{\prod_{i=1}^{n} \frac{1}{q_i}\}$, which is a cyclic group. Hence, the subgroup generated by $\{g_1, g_2, \ldots, g_n\}$ is cyclic.
Consider the subgroup H generated by $\{\frac{1}{2^n} | n \in \mathbb{N}\}$. If it is cyclic then there is an element h which can generated the whole subgroup. We have that $\nexists h_1 \in (0, h_1) \cap H$, but $\exists n_0 : \frac{1}{2^{n_0}} \in (0, h_1)$ which leads a contradiction. Hence, it is not cyclic.

5.

1. If $h \in H$, then $ghg^{-1} = h$. Which means that $H \subseteq N_G(H)$
$H \subseteq C_G(H) \iff \forall h_1, h_2 \in H : h_1 h_2 h_1^{-1} = h_2 \iff h_1 h_2 = h_2 h_1 \iff H$ is abelian.
2. Since H has order 2, let it elements be 1 and h. Since has order 2, one element is 1 and each elements inverse is itself. Then we have that $H \subseteq N_G(H)$, which means that $\forall g \in N_G(H)$, either

$$ghg^{-1} = h \text{ and } g1g^{-1} = 1$$

or

$$ghg^{-1} = 1 \text{ and } g1g^{-1} = h$$

In the first case, it is obvious that $N_G(H) = C_G(H)$.
In the second case, $g1g^{-1} = h \implies 1 = h$ which is a contradiction. Hence, $N_G(H) = C_G(H)$.
If $C_G(H) = N_G(H) = G$ then $\forall g \in G : ghg^{-1} = h$ which means that h is a central element, 1 is also a central element. Hence $H \subset Z(G)$.

6.

1. Every matrix in $H(\mathbb{R})$ is upper triangular hence is invertible.

The identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H(\mathbb{R})$

$\forall a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R} : \begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 + a_2 & c_1 + c_2 + a_1 b_2 \\ 0 & 1 & b_1 + b_2 \\ 0 & 0 & 1 \end{pmatrix}$

which is also an element of $H(\mathbb{R})$

$\forall a, b, c \in \mathbb{R} : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -a & -c + ab \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

which means that every matrix in $H(\mathbb{R})$ has an inverse. Therefore, $H(\mathbb{R})$ is a subgroup of $GL_3(\mathbb{R})$

2. Suppose $\begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix}$ is an element of $Z(H(\mathbb{R}))$ then $\forall a_2, b_2, c_2 \in \mathbb{R}$:

$$\begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} 1 & a_1 + a_2 & c_1 + c_2 + a_1 b_2 \\ 0 & 1 & b_1 + b_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 + a_2 & c_1 + c_2 + a_2 b_1 \\ 0 & 1 & b_1 + b_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, $\forall a_2, b_2 \in \mathbb{R} : a_1 b_2 = a_2 b_1 \implies a_2 = b_2 = 0$.

And hence, $Z(H(\mathbb{R}))$ consists of matrixes in the form: $\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We can construct a function

$$f : (\mathbb{R}, +) \to Z(H(\mathbb{R})), \qquad x \to \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

f is surjective: $\forall \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Z(H(\mathbb{R})) : f(x) = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

f is injective: $\forall x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$:

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies x_1 = x_2$$

f is hommorphic:

$$f(x_1{+}x_2) = \begin{pmatrix}1 & 0 & x_1+x_2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & x_1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & x_2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = f(x_1)\cdot f(x_2)$$

Therefore, $Z(H(\mathbb{R}))$ is isomorphic to $(\mathbb{R},+)$

3. Let the map be g

$\forall(a,b)\in\mathbb{R} : \forall c\in\mathbb{R} : g\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} = (a,b)$ which means it is surjective.

$\forall a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}:$

$$\begin{aligned} g\left(\begin{pmatrix}1 & a_1 & c_1\\ 0 & 1 & b_1\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & a_2 & c_2\\ 0 & 1 & b_2\\ 0 & 0 & 1\end{pmatrix}\right) &= (a_1+a_2, b_1+b_2)\\ &= (a_1,b_1)+(a_2+b_2)\\ &= g\begin{pmatrix}1 & a_1 & c_1\\ 0 & 1 & b_1\\ 0 & 0 & 1\end{pmatrix} + g\begin{pmatrix}1 & a_2 & c_2\\ 0 & 1 & b_2\\ 0 & 0 & 1\end{pmatrix}\end{aligned}$$

Which proves that g is homomrphic and hence is a surjective group homomorphism. The identity element of $(\mathbb{R}^2,+)$ is $(0,0)$ hence

$$\ker(g) = \left\{\begin{pmatrix}1 & 0 & c\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\middle|\, c\in\mathbb{R}\right\}$$