1.

a.

Since the time to wrap 1 present follows an exponential distribution with a mean of 5 minutes, the time to wrap 4 presents follow an gamma distribution with a mean of 20 and .
Therefore, the probability that 4 presents are wrapped in less than 15 minutes is

$$\int_0^{15} \frac{1}{\Gamma(4)5^4} y^3 e^{-\frac{y}{5}} dy = 0.35276811121$$

b.

We have that the probability that 1 present is wrapped in under 3 minutes is

$$\int_0^{3} \frac{1}{5} e^{-\frac{y}{5}} dy = 0.4511883639059736$$

Hence the probability that none of the three presents is wrapped under 3 minutes, which is also the probability that the fastest present in over 3 minutes is

$$(1 - 0.4511883639059736)^3 = 0.16529888822$$

and hence the probability that the present wrapped fastest is under 3 minutes is

$$1 - 0.16529888822 = 0.83470111177$$

2.

a.

We have that $X - Y < 1.5 \iff X < 1.5 + Y$. Therefore,

$$P(X - Y < 1.5) = \int_0^1 \int_0^{1.5+y} \frac{1}{4} dx dy = 0.5$$

b.

We have that $XY < 2 \iff X < \frac{2}{Y}$. As $0 < X < 4$, we only need to consider $0.5 < Y < 1$. Therefore,

$$P(XY < 2) = \int_0^{0.5} \int_0^4 \frac{1}{4} dx dy + \int_{0.5}^1 \int_0^{2/Y} \frac{1}{4} dx dy = 0.84657$$

3.

a.

We have that

$$\int_{-2}^{2}\int_{x^2}^{4} c\,dydx = \frac{32c}{3} = 1 \iff c = \frac{3}{32}$$

b.

$x^2 < 2 - x \iff x^2 + x - 2 = (x-1)(x+2) < 0 \iff -2 < x < 1.$

$$P(Y < 2 - X) = \int_{-2}^{1}\int_{x^2}^{2-x} \frac{3}{32} dydx = \frac{27}{64}$$

c.

Since $y = 2.25$, we have that $x^2 < y \iff x^2 < 2.25 \iff -1.5 < x < 1.5$. Therefore, as it is uniformly distributed

$$\begin{aligned} P(X < 1|Y = 2.25) &= \frac{P((X < 1) \cap (-1.5 < X < 1.5), Y = 2.25)}{P(-1.5 < X < 1.5, Y = 2.25)} \\ &= \frac{1 - (-1.5)}{1.5 - (-1.5)} \\ &= \frac{5}{6} \end{aligned}$$

d.

We have

$$f_X(x) = \int_{x^2}^{4} \frac{3}{32} dy = \frac{12 - 3x^2}{32}$$

and

$$f_Y(y) = \int_{-2}^{2} \frac{3}{32} dx = \frac{3}{8}$$

Hence, X and Y are not independent as

$$f_X(x) \cdot f_Y(y) = \frac{36 - 9x^2}{256} \neq f(x, y)$$

4.

a.

We have

$$E[XY] = \int_0^2 \int_0^3 \frac{(x+y)}{15} \cdot xydydx = 2$$

$$E[X] = \int_0^2 \int_0^3 \frac{(x+y)}{15} \cdot xdydx = \frac{17}{15}$$

$$E[Y] = \int_0^2 \int_0^3 \frac{(x+y)}{15} \cdot ydydx = \frac{9}{5}$$

Hence,

$$\mathrm{Cov}(X,Y) = 2 - \frac{9}{5} \cdot \frac{17}{15} = -\frac{1}{25}$$

b.

We have that $f(1.5, y) = \frac{y}{15} + \frac{1}{10}$. Therefore,

$$P(X = 1.5) = \int_0^3 \frac{1.5+y}{15} dy = 0.6$$

$$E[Y|X = 1.5] = \frac{\int_0^3 \frac{(1.5+y)}{15} \cdot ydy}{0.6} = \frac{1.05}{0.6} = \frac{7}{4}$$

$$E[Y^2|X = 1.5] = \frac{\int_0^3 \frac{(1.5+y)}{15} \cdot y^2dy}{0.6} = \frac{2.25}{0.6} = \frac{15}{4}$$

And hence,

$$V[Y|X = 1.5] = \frac{15}{4} - \left(\frac{7}{4}\right)^2 = \frac{11}{16}$$

5.

We have that

$$f_X(x) = \frac{1}{4\pi} e^{-\frac{1}{8}(x^2-2x+1)} \int_0^\infty \sqrt{y} e^{-\frac{y}{2}} dy = \frac{\sqrt{2\pi}}{4\pi} e^{-\frac{1}{8}(x^2-2x+1)}$$

$$f_Y(y) = \frac{\sqrt{y}}{4\pi} e^{-\frac{y}{2}} \int_{-\infty}^\infty e^{-\frac{(x-1)^2}{8}} dx = \frac{\sqrt{2y\pi}}{2\pi} e^{-\frac{y}{2}}$$

and hence

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

which means that X and Y is independent, therefore, we can see that X follows the normal distribution with a mean of 1 and variance of 2 and Y follows the chi-square distribution with a mean of 3 and variance of 6. We have that Therefore,

$$V[3X - 5Y + 100] = 9 \cdot 2^2 + 25 \cdot 6 = 186$$

6.

For $y \geq 4$:

$$f_Y(y) = \int_0^{8-y} \frac{3}{128}(8-x-y)dx = \frac{3\left(y^2-16y+64\right)}{256}$$

For $y < 4$:

$$f_Y(y) = \int_0^{y} \frac{3}{128}(8-x-y)dx = \frac{3\left(16y-3y^2\right)}{256}$$

7.

a.

We have that X follows an exponential distribution with a mean of 1.75 (thousands of dollars). Which means that $f_X(x) = \dfrac{1}{1.75}e^{-x/1.75}$. And hence since y is uniformly distributed over the range x to $2.25x$, we have that

$$f(x,y) = \begin{cases} \dfrac{1}{1.75 \cdot 1.25x}e^{-x/1.75} & , \quad 0 < x < y < 2.25x \\ 0 \quad , \quad \text{else} \end{cases}$$

and therefore,

$$E[X+Y] = \int_0^{\infty}\int_x^{2.25x}(x+y)\cdot\frac{e^{-x/1.75}}{2.1875x}dydx = 4.59375$$

b.

Given that $x > 0$, we have that

$$x^2 < x \iff x < 1$$

$$x < x^2 < 2.25x \iff 1 < x < 2.25$$

$$x^2 > 2.25x \iff x > 2.25$$

Therefore, we can partition

$$\begin{aligned} P(Y > X^2) &= \int_0^1\int_x^{2.25x}\frac{1}{1.75\cdot 1.25x}e^{-x/1.75}dydx \\ &+ \int_1^{2.25}\int_x^{x^2}\frac{1}{1.75\cdot 1.25x}e^{-x/1.75}dydx + 0 \\ &= 0.43528 + 0.16114 = 0.59642 \end{aligned}$$

8.

We have that

$$f_X(x) = \int_x^\infty \frac{1}{16} e^{-y/4} dy = \frac{1}{4} e^{-\frac{x}{4}}$$

Therefore,

$$\begin{aligned}
P(6 < Y < 9 | X > 3) &= \frac{P(6 < Y < 9, X > 3)}{P(X > 3)} \\
&= \frac{\int_6^9 \int_3^y \frac{1}{16} e^{-y/4} dx dy}{\int_3^\infty \frac{1}{4} e^{-\frac{x}{4}} dx} \\
&= \frac{0.12697}{0.47236} = 0.2688
\end{aligned}$$

9.

We have that

$$E[Y|X] = \frac{2}{x}$$

and

$$V[Y|X] = \frac{1-p}{p^2} = \frac{4-2x}{x^2}$$

Therefore,

$$\begin{aligned} V[Y] &= E[V[Y|X]] + V[E[Y|X]] \\ &= \int_0^2 \frac{4-2x}{x^2} \cdot \frac{5x^4}{32} dx + \int_0^2 \frac{4}{x^2} \cdot \frac{5x^4}{32} dx - \left(\int_0^2 \frac{2}{x} \cdot \frac{5x^4}{32} dx \right)^2 \\ &= \frac{5}{12} + \frac{5}{3} - \frac{25}{16} = \frac{25}{48} \end{aligned}$$