

1.

a.

Suppose  $m(F) = 0$ , then for any  $x \notin F$ ,  $x \notin E_n$  for finitely many  $n \in \mathbb{N}$ , thus

$$\lim_{n \rightarrow \infty} \chi_{E_n}(x) = 0$$

a.e.  $x \in \mathbb{R}^d$ .

Now suppose  $m(F) > 0$ , then let  $x \in F$ , thus  $x \in E_n$  for infinitely many  $n \in \mathbb{N}$ , which means  $\limsup_{n \rightarrow \infty} \chi_{E_n}(x) = 1$  for all  $x \in F$ . Thus

$$\lim_{n \rightarrow \infty} \chi_{E_n}(x) = 0$$

for some set  $X \subseteq F^c$  thus contradiction as  $m(F) > 0$ .

b.

Apply fatou's lemma, we have that

$$\int_{\mathbb{R}^d} \liminf_{n \rightarrow \infty} f \chi_{E_n} dm = 0$$

which means that

$$m(f \liminf \chi_{E_n} \neq 0) = 0$$

hence

$$m(\liminf \chi_{E_n} \neq 0) = 0$$

Therefore,  $\liminf \chi_{E_n}(x) = 1$  on a set  $X$  where  $m(X) = 0$ . But for every  $x \in G$ ,  $\liminf \chi_{E_n}(x) = 1$  thus  $m(G) = 0$ .



### 3.

If  $\lim_{n \rightarrow \infty} \int_E |f_n - f| = 0$  then for every  $\varepsilon > 0$  there is  $n_0$  such that for all  $n > n_0$ ,

$$\left| \int_E |f_n| - \int_E |f| \right| \leq \left| \int_E (|f_n| - |f|) \right| \leq \int_E |f_n - f| \leq \varepsilon$$

Thus  $\int_E |f_n| \rightarrow \int_E |f|$ .

Now suppose  $\int_E |f_n| \rightarrow \int_E |f|$ , then we know that

$$|f_n| + |f| \rightarrow 2|f|$$

a.e.  $x \in E$

$$\int_E |f_n - f| \leq \int_E |f_n| + |f|$$

$$\int_E |f_n - f| = 0$$

as  $f_n \rightarrow f$  a.e.  $x \in E$  and

$$\lim_{n \rightarrow \infty} \int_E |f_n| + |f| = \int_E 2|f|$$

Thus applying the Generalized Dominance Convergence Theorem on  $|f_n - f|$  and  $|f_n| + |f|$ , we have that

$$\lim_{n \rightarrow \infty} \int_E |f_n - f| = \int_E 0 = 0$$

4.

For all  $\varepsilon > 0$  we can find a function  $g$  such that  $\int_{\mathbb{R}^d} |f - g_n| < \varepsilon/3$  and small enough  $t > 0$  such that  $|g(x-t) - g(x)| < \varepsilon/3m(E)$ . Then

$$\begin{aligned}
& \int_{\mathbb{R}^d} |f_t(x) - f(x)| \\
& \leq \int_{\mathbb{R}^d} |f(x-t) - g(x-t)| + |g(x-t) - g(x)| + |g(x) - f(x)| \\
& < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
& = \varepsilon
\end{aligned}$$

Thus

$$\int_{\mathbb{R}^d} |f_t(x) - f(x)| = 0$$

b.

Since  $\chi_E \in \mathcal{L}_1(\mathbb{R}^d)$ , for all  $\varepsilon > 0$ , there is a continuous function  $h$  such that  $\int_{\mathbb{R}^d} |\chi_E - h| < \varepsilon/3$ , then let the sequence  $x_n \rightarrow x$  and thus there is an  $n_0$  such that for all  $n > n_0$ ,  $|h(x_n) - h(x)| < \varepsilon/3m(E)$ . Then

$$\begin{aligned}
& |\phi(x) - \phi(x_n)| \\
& = \left| \int_{\mathbb{R}^d} \chi_E(x+t) \chi_E(t) - \chi_E(x_n+t) \chi_E(t) dt \right| \\
& = \left| \int_{\mathbb{R}^d} \chi_E(t) (\chi_E(x+t) - \chi_E(x_n+t)) dt \right| \\
& \leq \int_{\mathbb{R}^d} |\chi_E(x+t) - \chi_E(x_n+t)| dt \\
& \leq \int_{\mathbb{R}^d} |\chi_E(x+t) - h(x+t)| + |h(x+t) - h(x_n+t)| + |h(x_n+t) - \chi_E(x_n+t)| \\
& < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
& = \varepsilon
\end{aligned}$$

c.

We first have that for  $x \in E$ ,

$$\phi(x) = \int_{\mathbb{R}^d} \chi_E(x+t) \chi_E(t) dt = m(E \cap (E-x)) = m(E_x)$$

where  $E_x = \{y : y \in E, x+y \in E\}$ .

Notice that if  $y \in E_x$  then  $y \in E$  and  $x+y \in E$  thus  $x \in E - E$ . Thus if  $m(E_x) > 0$  then  $x \in E - E$ .

Now since  $m(E) > 0$ , we have that there is  $B_\varepsilon(x_0) \subseteq E$  and thus for any  $\delta < \varepsilon/2$ , we have that  $\phi(x) = m(E_x) > 0$  for all  $x \in B_{\delta/2}(0)$ . Thus  $B_\delta(0) \subseteq E - E$

## 5.

For every  $\varepsilon > 0$ , we can find a respective integrable step function  $\phi$  such that  $\int_{\mathbb{R}} |f - \phi| < \varepsilon/2$ , where

$$\phi = \sum_{k=1}^N a_k \chi_{R_k}$$

where  $a_k \in \mathbb{R}$  and  $R_k$  are bounded intervals. Thus we can find an interval  $R$  such that  $\cup_{k=1}^N R_k \subseteq R$ . Let  $M = \max_R \phi$ . Now for any  $x \in R$ , since  $\sin(\lambda x) \rightarrow 0$  as  $\lambda \rightarrow \infty$ , there is  $\delta > 0$  such that for all  $\lambda < \delta$ ,

$$|\sin(\lambda x)| < \frac{m(R)\varepsilon}{2M}$$

Then

$$\begin{aligned} & \left| \int_{\mathbb{R}} f(x) \sin(\lambda x) dx \right| \\ & \leq \int_{\mathbb{R}} |f(x) - \phi(x)| |\sin(\lambda x)| dx + \int_{\mathbb{R}} |\phi(x) \sin(\lambda x)| dx \\ & \leq \int_{\mathbb{R}} |f(x) - \phi(x)| dx + \int_R |\phi(x)| |\sin(\lambda x)| dx \\ & \leq \frac{\varepsilon}{2} + \int_R \frac{m(R)\varepsilon}{2M} \cdot M dx \\ & \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ & = \varepsilon \end{aligned}$$