1.

a.

Since $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by all open sets in $\mathbb{R}$, and Y is contained only open sets. We have that $\sigma(Y) \subseteq \mathcal{B}(\mathbb{R})$. Thus to prove that $\sigma(Y) = \mathbb{R}$, we need to prove that all open sets in $\mathbb{R} \in Y$. There are 3 types of open set in $\mathbb{R}$. For $a, b \in \mathbb{R}$

- (a, ∞)
  Choose any $x \in \mathbb{R}$ such that $x > a$, since D is dense, there is infinitely countable number $d \in \mathbb{D}$ such that $a < d < x$. Thus there is a stricly increasing $(d_n)_{n=1}^{\infty}$ such that $\lim d_n = a$. Therefore, $\cup_{n=1}^{\infty}(d, \infty) = (a, \infty) \in \sigma(\mathbb{Y})$.

- $(-\infty, a)$
  Choose any $x \in \mathbb{R}$ such that $x < a$, since D is dense, there is infinitely countable number $d \in \mathbb{D}$ such that $a > d > x$. Thus there is a stricly decreasing $(d_n)_{n=1}^{\infty}$ such that $\lim d_n = a$. Therefore, $\cup_{n=1}^{\infty}(d, \infty)^c = \cup_{n=1}^{\infty}(-\infty, d] = (-\infty, a) \in \sigma(\mathbb{Y})$.

- (a, b)
  We have that $(a, b) = (a, \infty) \cap (-\infty, b) \in \sigma(Y)$

Thus $\sigma(Y) = \mathcal{B}(\mathbb{R})$.

b.

For any interval $(a, \infty) \in Y$, notice that since D is dense, we can construct a strictly decreasing $(a_n)_{n=1}^{\infty} \in D$ and strictly increasing sequence $(b_n)_{n=1}^{\infty} \in D$ such that $\lim(a_n, b_n) = (a, \infty)$. And thus for any $a \in D$, we have that $(a, \infty) = \cup_{n=1}^{\infty}(a_n, b_n) \in \sigma(Z)$. Therefore, $\sigma(Y) \subseteq \sigma(Z)$.
For any $a, b \in D$, we have that $(-\infty, b) \cap (-\infty, a)^c = [a, b) \in \sigma(Y)$. Thus $\sigma(Z) \subseteq \sigma(Y)$ and $\sigma(Y) = \sigma(Z) = \mathcal{B}(\mathbb{R})$.

c.

2.

a.

For any open sets $G = (a, b) \in \mathbb{R}$, we have that

$$(a, b) = \cup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

b.

For each $x \in \mathbb{R} \backslash E$, f is continuous at x thus for all $n \in \mathbb{N}$, there is $\delta_{x,n}$ such that $|f(y) - f(x)| < \dfrac{1}{n}$ for all $y \in (x - \delta_{x,n}, x + \delta_{x,n})$. Thus let $\mathcal{O}_n = \cup_{x \in E^c}(x - \delta_{x,n}, x + \delta_{x,n})$. We can see that

$$\bigcap_{n=1}^{\infty} \mathcal{O}_n = \bigcap_{n=1}^{\infty} \bigcup_{x \in E^c} (x - \delta_{x,n}, x + \delta_{x,n}) = \bigcup_{x \in E^c} \underbrace{\bigcap_{n=1}^{\infty} (x - \delta_{x,n}, x + \delta_{x,n})}_{x} = E^c$$

And therefore, E^c is a G_δ set and thus E is a F_σ set.

3.

a.

If μ is σ-finite, then there exists some $X_n \in \mathcal{M}$ such that $X = \cup_{n=1}^{\infty} X_n$, $X_n \subseteq X_{n+1}$ and $\mu(X_n) < \infty$. Thus for each $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $N \in \mathbb{N}$ such that

$$\mu(X_N \cap E) > 0$$

as else $\mu(E \cap \bigcup_{j=1}^{n}) = \mu(E \cap A) = 0$. But since $\mu(X_N \cap E) \leq \mu(X_N) < \infty$. X_N satisfies $X_N \in \mathcal{M}, X_N \subseteq E, 0 < \mu(X_N) < \infty$.

b.

Let

$$S = \{F \in \mathcal{M} : F \subseteq E, \mu(F) < \infty\}$$

Then we know that $\sup_{F \in S} \mu(F)$ must exists. Suppose it is less than ∞, that is $\sup_{F \in S} \mu(F) = L$ for some $L \in \mathbb{R}$, then we can choose $(F_n)_{n=1}^{\infty}$ such that $\lim \mu(F_n) = L$. Then, we have that $\mu(\cup_{F \in S} F) = \mu(\cup_{n=1}^{\infty} F_n) = L$. Therefore, $\mu(E \backslash \cup_{F \in S} F) = \infty$, thus there exists $F' \subset E \backslash F$, so that $0 < \mu(F') < \infty$. But $F \cup F' \subset E$ and $\infty > \mu(F \cup F') = \mu(F) + \mu(F') > L$ which is a contradiction and therefore $\sup_{F \in S} \mu(F) = \infty$ and there is some set $F \subseteq E$ such that $C < \mu(F) < \infty$.

c.

First,

$$u_0(\varnothing) = \sup\{\mu(F) : F \subseteq \varnothing, \mu(F) < \infty\} = 0$$

as the only subset of empty set is itself. If $E_j \in \mathcal{M}$ for all $j \in \mathbb{N}$ and E_j are pairwise disjoint then in case where there is j such that $\mu(E_j) = \infty$ then from part b, we have

$$\sum_{j=1}^{\infty} \mu_0(E_j) = \infty = \mu_0(\sqcup_{j=1}^{\infty} E_j)$$

If $\mu_0(E_j) = L_j$ are finite for every $j \in \mathbb{N}$ then we can choose $(F_{j,n})_{n=1}^{\infty} \subseteq E_j$ such that $\lim \mu(F_{j,n}) = L_j$ and since E_j are pairwise disjoint, $F_{j_1,n} \cap F_{j_2,n} = \varnothing$ for $j_1 \neq j_2$. Therefore, there is a sequence $F_n = \sqcup_{j=1}^{\infty} F_{j,n} \subset \sqcup_{j=1}^{\infty} E_j$ such that $\lim \mu_0(F_n) = \sum_{j=1}^{\infty} L_j$ thus $\mu_0(\sqcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} L_j = \sum_{j=1}^{\infty} \mu_0(E_j)$.

4.

a.

- $\varnothing \in \mathcal{E}$. Choosing $a < b$ so that $([-\infty, a] \cap \mathbb{Q}) \cap (\mathbb{Q} \cap (b, \infty]) \in \mathcal{E}$.
- If $E = (a_1, b_1] \cap \mathbb{Q}, F = (a_2, b_2] \cap \mathbb{Q} \in \mathcal{E}$ then in case $(a_1, b_1] \cap (a_2, b_2] = \varnothing, E \cap F = \varnothing \in \mathcal{E}$. In case $(a_1, b_1] \cap (a_2, b_2] \neq \varnothing$ then there exists a_3, b_3 such that $(a_1, b_1] \cap (a_2, b_2] = (a_3, b_3]$ thus $E \cap F = (a_3, b_3] \cap \mathbb{Q} \in \mathcal{E}$.
- If $E = (a, b] \cap \mathbb{Q} \in \mathcal{E}$ where $a < b$, then $E^c = ((-\infty, a] \cup (b, \infty]) \cap \mathbb{Q}$ $= \underbrace{((-\infty, a] \cap \mathbb{Q})}_{\in \mathcal{E}} \cap \underbrace{((b, \infty] \cup \mathbb{Q})}_{\in \mathcal{E}}$, which are disjoint as $a < b$

b.

From definition, $\mathcal{A} \subseteq \mathbb{Q}$ thus $\sigma(\mathcal{A}) \subseteq \mathcal{P}(\mathbb{Q})$.
For any $E \subseteq \mathcal{P}(Q)$, we can write $E = \{x_1, x_2, \dots, \}$ as rationals are countable. Then for any x_j, we can define

$$E_{j,n} = \left(x_1 - \frac{1}{n}, x_1\right] \cap \mathbb{Q} \in \mathcal{A} \subseteq \sigma(\mathcal{A})$$

so that

$$\bigcap_{n=1}^{\infty} E_{j,n} = x_j \cap \mathbb{Q} = x_j$$

Thus

$$\bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} E_{j,n} = E$$

c.

We have that $u_0(\varnothing) = 0$ and for $E_j \in \mathcal{A}$, where E_j are pairwise disjoint, there is 2 cases

- if there is non-empty E_k then $\sqcup_{j=1}^{n} E_j \neq \varnothing$ and thus

$$\mu_0(\sqcup_{j=1}^{\infty} E_j) = \infty = \sum_{j=1}^{\infty} E_j = \mu_0(E_k) + \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \mu_0(E_j) = \infty$$

- if all of them are empty, then simply

$$\mu_0(\sqcup_{j=1}^{\infty} E_j) = \mu_0(\varnothing) = 0 = \sum_{j=1}^{n} \mu_0(E_j)$$

5.

a.

If $\sum_{j=1}^n |R_j^o| < 1$ then there is some rectangle $[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d]$ such that $a_j < b_j$ for all j and since $\mathbb{Q}$ is dense. There is a rational $c_j \in [a_j, b_j]$ and thus $(c_1, c_2, \ldots, c_d) \notin A$ thus contradiction. Therefore, $\sum_{j=1}^n |R_j^o| \geq 1$

b.

Since $m(E_j) = 1$, we have that $m(E_j^c) = 0$ and thus

$$m(\cap_{j=1}^\infty E_j) = 1 - m(\cup_{j=1}^\infty E_j^c) \geq 1 - \sum_{j=1}^\infty m(E_j^c) = 1$$

But since $\cap_{j=1}^\infty E_j \subseteq [0,1]$ thus $m(\cap_{j=1}^\infty E_j) \leq 1$. Therefore, $m(\cap_{j=1}^\infty E_j) = 1$.

c.

Suppose that $m(\cap_{j=1}^n A_n) = 0$, then $m(\cup_{j=1}^n A_j^c) = 1$. However, we have that $m(\cup_{j=1}^n A_n^c) \leq \sum_{j=1}^n m(A_j^c)$. Now we know that

$$\sum_{j=1}^n m(A_j^c) + \sum_{j=1}^n m(A_j) > n - 1 + 1 = n$$

But

$$\sum_{j=1}^n m(A_j^c) + \sum_{j=1}^n m(A_j) = \sum_{j=1}^n m(A_j^c) + m(A_j) = \sum_{j=1}^n m([0,1]) = n$$