

1.

We have that

$$\frac{n^2 + a^2}{(n^2 - a^2)^2} = \frac{1}{2(n + a)^2} + \frac{1}{2(n - a)^2}$$

Let $f = \frac{1}{(z + a\pi)^2 \tan(z)}$. Then f has a simple pole at $-a\pi$ and $N\pi$ for all $N \in \mathbb{Z}$. We have

$$\text{Res}(f, N\pi) = \lim_{z \rightarrow N\pi} (z - N\pi)f(z) = \frac{1}{(N + a)^2 \pi^2} \lim_{z \rightarrow N\pi} \frac{z - N\pi}{\tan(z)} = \frac{1}{(N + a)^2 \pi^2}$$

and

$$\text{Res}(f, -a\pi) = \frac{d}{dz} ((z + a\pi)^2 f(z))_{z=-a\pi} = -\frac{1}{\sin^2(-a\pi)}$$

We also have that

$$\lim_{n \rightarrow \infty} \left| \int_{\partial D_n} \frac{dz}{(z + a\pi)^2 \tan(z)} \right| = 0$$

as the circumference of a square is $4(N\pi + \pi/2)$ and for $z = x + iy$, we have

$$\lim_{x \rightarrow \pm\infty} |(z + a\pi)^2| = \infty (\text{"with degree 2"}) \text{ and } \lim_{x \rightarrow \pi/2} |\tan(z)| = \infty (\text{"with degree 1"})$$

and similarly for y .

Thus

$$-\frac{1}{\sin^2(-a\pi)} + \sum_{N=-\infty}^{\infty} \frac{1}{(N + a)^2 \pi^2} = 0$$

Thus

$$\sum_{N=1}^{\infty} \frac{1}{\pi^2} \left(\frac{1}{(N + a)^2} + \frac{1}{(N - a)^2} \right) + \frac{1}{a^2 \pi^2} = \frac{1}{\sin^2(a\pi)}$$

and hence

$$\sum_{n=1}^{\infty} \frac{n^2 + a^2}{(n^2 - a^2)^2} = \frac{1}{2} \left(\frac{1}{\sin^2(a\pi)} - \frac{1}{a^2 \pi^2} \right) \pi^2 = \frac{\pi^2}{2} \left(\frac{1}{\sin^2(a\pi)} - \frac{1}{a^2 \pi^2} \right)$$

2.

Suppose that $f(z)$ has an essential singularity at 0. Then by open mapping theorem, there exists $r > 0$ such that

$$f(B) \supset D \text{ for } B = \left\{ \left| z - \frac{1}{2} \right| < \frac{1}{4} \right\} \text{ and } D = \left\{ \left| w - f\left(\frac{1}{2}\right) \right| < r \right\}$$

Let $U = \{0 < |z| < 1/4\}$. Since $B \cap U = \emptyset$ and f is 1-to-1,

$$f(B) \cup f(U) = \emptyset$$

and hence

$$f(U) \subset \mathbb{C} \setminus f(B) \subset \mathbb{C} \setminus D$$

and

$$\overline{f(U)} \subset \overline{\mathbb{C} \setminus D} = \mathbb{C} \setminus D$$

But by Casorati-Weierstrass, $\overline{f(U)} = \mathbb{C}$ which is a contradiction.

3.

If $f/g \circ \gamma$ is positive and real at z_0 then we have that $f/g(z_0) = c$

$$|a_1 f(z) + b_1 g(z)| + |a_2 f(z) + b_2 g(z)| = |f(z)|(|a_1 + b_1 c + a_2 + b_2 c|) = |(a_1 + a_2) f(z) + (b_1 + b_2) g(z)|$$

which is a contradiction, hence $f/g \circ \gamma$ is contained in $\mathbb{C} \setminus [0, \infty)$. Then applying the argument principle to f/g on the curve γ , we have that

$$\sum_{p \in Z_f} \nu(\gamma, p) \text{mult}_p f = \sum_{p \in Z_g} \nu(\gamma, p) \text{mult}_p g$$

4.

Let $h = 1 + \frac{f}{g}$, hence $h(D) \subseteq \{z : \operatorname{Re}(z) > 0\}$. Thus by the argument principle, the zeros of $f + g$ is the same as the zeros of g . Since $f + g$ does not have zero because $|f| \neq |g|$ in D , g does not have zero in ∂D . Then $|f/g|$ is holomorphic on D thus attain a local maximum on ∂D which is less than 1, which confirms that $|f| < |g|$

5.

First, we can rewrite $f'(z) = na_n(z - z_1)(z - z_2) \dots (z - z_{n-1})$ and since $f'(z) \neq 0 \forall z \in D$, we have that $|z_k| \geq 1$, then $f'(0) = na_n(-z_1)(-z_2) \dots (-z_{n-1}) = 1$ thus $|a_n| < \frac{1}{n}$.

On the other hand

$$f''(z) = \sum_{k=1}^{n-1} \frac{f'(z)}{z - z_k}$$

and hence

$$2|a_2| = |f''(0)| = \left| \sum_{k=1}^{n-1} \frac{1}{-z_k} \right| \leq n - 1 \implies |a_2| \leq \frac{n-1}{2}$$