1.

a.

From definition,

$$\liminf_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} (\inf_{m\geq n} \mu(E_m)) = \sup_{n\geq 0} \inf_{m\geq n} \mu(E_m)$$

$$\mu(\liminf_{n\to\infty} E_n) = \mu(\cup_{n=1}^{\infty} \cap_{j=n}^{\infty} E_n)$$

Also, notice that $\cap_{j=n}^{\infty} E_n \subseteq \cap_{j=n+1}^{\infty} E_n$ and $\mu(\cap_{j=n+1}^{\infty} E_n) \leq \inf_{m\geq n} \mu(E_m)$

$$\mu(\liminf_{n\to\infty} E_n) = \lim_{n\to\infty} \mu(\cap_{j=n}^{\infty} E_n) \leq \liminf_{n\to\infty} \mu(E_n)$$

b.

From definition,

$$\limsup_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} (\sup_{m\geq n} \mu(E_m)) = \inf_{n\geq 0} \sup_{m\geq n} \mu(E_m) \leq \sup_{m\geq n} \mu(E_m)$$

$$\mu(\limsup_{n\to\infty} E_n) = \mu(\cap_{n=1}^{\infty} \cup_{j=n}^{\infty} E_n)$$

Notice that $\cup_{j=n}^{\infty} E_n \subseteq \cup_{j=n+1}^{\infty} E_n$ and $\mu(\cup_{j=n+1}^{\infty} E_n) \geq \sup_{m\geq n} \mu(E_m)$

$$\mu(\limsup_{n\to\infty} E_n) = \lim_{n\to\infty} \mu(\cup_{j=1}^{\infty} E_j) \geq \limsup_{no\to\infty} \mu(E_n)$$

2.

a.

Since, $E \subset O_n$ for all $n \in \mathbb{N}$. We have that

$$m(E) \leq \lim_{n\to\infty} m(O_n)$$

Now, for every $x \in \mathbb{R}^d$, if $x \in \bigcap_{n=1}^{\infty} O_n$, then for every $n \in \mathbb{N}$, $\mathrm{dist}(x, E) = 0$ as if $\mathrm{dist}(x, E) = \varepsilon$ for some $\varepsilon > 0$ then there exists n_0 such that for all $n > n_0$, $1/n < \varepsilon$ and $x \notin O_n$. Thus $\cap_{n=1}^{\infty} O_n \subseteq E$ and

$$m(\cap_{n=1}^{\infty} O_n) = \lim_{n\to\infty} m(O_n) \leq m(E)$$

b.

We have

$$\begin{aligned} m(E) &= m(\cup_{j=1}(r_j - 4^{-j}, r_j + 4^{-j})) \\ &\leq \sum_{j=1}^{\infty} m(r_j - 4^{-j}, r_j + 4^{-j}) \\ &= \sum_{j=1}^{\infty} 2 \cdot 4^{-j} \\ &= \frac{2}{3} \end{aligned}$$

However, for every $n \in \mathbb{N}$, since rationals are dense, we can find a partition $\{r_{x_0}, r_{x_1}, \ldots, r_{x_{2n}}\}$ of $[0, 1]$ from the sequence (r_n) such that $r_{x_0} = 0, r_{x_{2n}} = 1$ and $0 < r_{x_{n+1}} - r_{x_n} < \dfrac{1}{n}$. Thus for every $x \in [0, 1]$, there exists n_0 such that $|r_{x_{n_0}} - x| < \dfrac{1}{n}$ and thus $x \in O_n$, which means that $m(O_n) \geq 1 > m(E)$.

3.

4.

1.

For all $a \in \mathbb{R}$,

- if there don't exists $x \in \mathbb{R}$ such that $f(x) > a$, then $m(\{f > a\}) = 0$.
- if $\exists x \in \mathbb{R}$ such that $f(x) = a$ but for all $x' > x, f(x') = a$ then $m(\{f > a\}) = 0$.
- if there exists $x \in \mathbb{R}$ such that $f(x) = a$ and there exists x' such that $f(x') > a$. Then $m(\{f > a\}) \geq m((x', \infty)) = \infty$

2.

In the case where E is a measure zero set. For all $\varepsilon > 0$, $|f| \leq M$ except on a set of measure less than $\varepsilon > 0$ is already satisfied. In case where E is not a measure zero set. For every $\varepsilon > 0$, suppose that for all M, $|f| > M$ on a set having measure $> \varepsilon$ then $|f| = \infty$ on a set having measure $> \varepsilon$ thus contradict.

3.

Suppose there is a a function f such that $f(x) = \xi_{(a,b)}(x)$ a.e. $x \in \mathbb{R}$. Then for every $\varepsilon > 0$ there is $x_1 \in [b, b + \varepsilon/2)$ such that $f(x_1) = 0$ and $x_2 \in (b - \varepsilon/2, b)$ such that $f(x_2) = 1$. Thus for every $\varepsilon > 0$ there is x_1, x_2 such that $f(x_2) - f(x_1) = 1$ but $x_2 - x_1 < \varepsilon$.

5.

Let X_f, X_g be the set of points that is finite in f and g so that $X_f \cap X_g = X_0$. Then as $X, X_f, X_g \in \mathcal{M}$, we have that $X_0 \in \mathcal{M}$ and thus $X \backslash X_0 \in \mathcal{M}$. We also have that X_f^c, X_g^c have measure zero thus

$$\mu(X \backslash X_0) = \mu(X \cap (X_f^c \cup X_g^c)) = \mu(X_f^c \cup X_g^c) \leq \mu(X_f^c) + \mu(X_g^c) = 0$$

and therefore $\mu(X \backslash X_0) = 0$.