1.

From definition, we have that

$$P(Y > y) = \int_y^\infty f_Y(x)dx$$

and

$$P(Y > y) = \int_{-\infty}^y f_Y(x)dx$$

Therefore,

$$\begin{aligned}
E[Y] &= \int_0^\infty P(Y > y)dy - \int_{-\infty}^0 P(Y < y)dy \\
&= \int_0^\infty \int_y^\infty f_Y(x)dxdy - \int_{-\infty}^0 \int_{-\infty}^y f_Y(x)dxdy \\
&= \int_0^\infty \int_0^x f_Y(x)dydx - \int_{-\infty}^0 \int_{-x}^0 f_Y(x)dydx \\
&= \int_0^\infty y f_Y(x)|_{y=0}^x dx - \int_{-\infty}^0 y f_Y(x)|_{y=-x}^0 dx \\
&= \int_0^\infty x f_Y(x)dx + \int_{-\infty}^0 x f_Y(x)dx \\
&= \int_0^\infty y f_Y(y)dy + \int_{-\infty}^0 y f_Y(y)dy
\end{aligned}$$

2.

$$f_Y(y) = \int_0^1 \frac{x^y}{\ln 2} dx = \frac{1}{(y+1)\ln 2}$$

$$f_X(x) = \int_0^1 \frac{x^y}{\ln 2} dy = \frac{x-1}{\ln x \ln 2}$$

Hence,

$$f_{XY}(x,y) \neq f_Y(y) \cdot f_X(x)$$

and hence they are independent

$$\begin{aligned}
E[XY] &= \int_0^1 \int_0^1 \frac{yx^{y+1}}{\ln 2} dxdy \\
&= \int_0^1 \frac{y}{(y+2)\ln 2} dy \\
&= \frac{1}{\ln 2}(y - 2\ln(y+2))|_0^1 \\
&= \frac{1 - 2\ln 3 + 2\ln 2}{\ln 2} \\
&\equiv 0.273
\end{aligned}$$

3.

$$\int_0^{0.5} 0.25 \cdot e^{-0.25t} dt = 0.1175 \implies 11.75\%$$

4.

a.

Suppose the location of the fire station is x, then the expected distance is

$$\begin{aligned}\int_0^A \frac{1}{A}|x-y|dy &= \frac{1}{A}\left(\int_0^x (x-y)dy + \int_x^A (y-x)dy\right)\\ &= \frac{1}{A}\left(x^2 - \frac{x^2}{2} + \frac{A^2}{2} - \frac{x^2}{2} - Ax + x^2\right)\\ &= \frac{1}{A}\left(x^2 - Ax + \frac{A^2}{2}\right)\\ &= \frac{1}{A}\left(\left(x - \frac{A}{2}\right)^2 + \frac{A^2}{4}\right)\end{aligned}$$

Therefore, the optimal location is at $\frac{A}{2}$.

b.

Similarly, the expected distance is

$$\int_0^{A/2} \frac{1}{A}|x-y|dy + \int_{5A/8}^{A} \frac{1}{A}|x-y|dy$$

Here, we consider two cases, the first cases is the fire station is located between 0 and $A/2$, the second case is the fire station is located between $5A/8$ and A. In the first case, the expceted distance is

$$\begin{aligned}&\frac{1}{A}\left(\int_0^x (x-y)dy + \int_x^{A/2} (y-x)dy + \int_{5A/8}^{A} (y-x)dy\right)\\ =&\frac{1}{A}\left(x^2 - \frac{x^2}{2} + \frac{A^2}{8} - \frac{x^2}{2} - \frac{Ax}{2} + x^2 + \frac{A^2}{2} - \frac{25A^2}{128} - Ax + \frac{5Ax}{8}\right)\\ =&\frac{1}{A}\left(x^2 - \frac{7Ax}{8} + \frac{55A^2}{128}\right)\\ =&\frac{1}{A}\left(\frac{61A^2}{256} + \frac{1}{256}(7A - 16x)^2\right)\end{aligned}$$

Therefore, the optimal location in the interval between 0 and $A/2$ is $7A/16$ with the expected value being $\frac{61A}{256}$.

In the second case, the expected distance is

$$
\begin{aligned}
&\frac{1}{A}\left(\int_0^{A/2}(x-y)dy+\int_{5A/8}^{x}(x-y)dy+\int_x^A(y-x)dy\right)\\
=&\frac{1}{A}\left(\frac{Ax}{2}-\frac{A^2}{8}+x^2-\frac{5Ax}{8}-\frac{x^2}{2}+\frac{25A^2}{128}+\frac{A^2}{2}-\frac{x^2}{2}-Ax+x^2\right)\\
=&\frac{1}{A}\left(\frac{73A^2}{128}-\frac{9Ax}{8}+x^2\right)\\
=&\frac{1}{A}\left(\frac{65A^2}{256}+\frac{1}{256}(9A-16x)^2\right)
\end{aligned}
$$

Hence, the optimal location in the interval between $5A/8$ and A is at A or $5A/8$ as $9/16A < 5A/8$.
If the fire station location is A, then the expected distance is $57A/128$.
If the fire station location is $5A/8$, then the expected distance is $33A/128$.
Therefore, the best location for the fire station is at $A/2$.

c.

The expected distance is

$$
\begin{aligned}
\int_0^\infty |x-y|\cdot\lambda e^{-\lambda y}dy &= \int_0^x (x-y)\lambda e^{-\lambda y}dy+\int_x^\infty (y-x)\lambda e^{-\lambda y}dy\\
&= -xe^{-\lambda y}|_0^x+\int_0^x -y\lambda e^{-\lambda y}dy + xe^{-\lambda y}|_x^\infty+\int_x^\infty y\lambda e^{-\lambda y}dy\\
&= -2xe^{-\lambda x}+x+\int_0^x -y\lambda e^{-\lambda y}dy+\int_x^\infty y\lambda e^{-\lambda y}dy\\
&= -2xe^{-\lambda x}+x-\left(-y\cdot e^{-\lambda y}\right)\Big|_0^x+\left(-y\cdot e^{-\lambda y}\right)\Big|_x^\infty-\int_0^x e^{-\lambda y}dy+\int_x^\infty e^{-\lambda y}dy\\
&= x+\frac{e^{-\lambda x}-1}{\lambda}+\frac{e^{-\lambda x}}{\lambda}\\
&= \frac{\lambda x+2e^{-\lambda x}-1}{\lambda}
\end{aligned}
$$

Consider $f(x)=\lambda x+2e^{-\lambda x}-1$
$f'(x)=\lambda-2\lambda e^{-\lambda x}=0 \iff e^{-\lambda x}=1/2 \iff x=\frac{\ln 2}{\lambda}$.
We also have that $f''(x)=2\lambda^2 e^{-\lambda x}\geq 0 \quad \forall x\geq 0$.
Hence, f reaches its global minimum at $x=\frac{\ln 2}{\lambda}$ and therefore the best place to place the fire station is at $\frac{\ln 2}{\lambda}$