1.

a.

From definition,

$$\liminf_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} (\inf_{m\geq n} \mu(E_m)) = \sup_{n\geq 0} \inf_{m\geq n} \mu(E_m)$$

$$\mu(\liminf_{n\to\infty} E_n) = \mu(\cup_{n=1}^{\infty} \cap_{j=n}^{\infty} E_j)$$

Also, notice that $\cap_{j=n}^{\infty} E_n \subseteq \cap_{j=n+1}^{\infty} E_n$ for all $n \in \mathbb{N}$ thus

$$\mu(\liminf_{n\to\infty} E_n) = \mu(\lim_{n\to\infty} \cap_{j=n}^{\infty} E_j) = \lim_{n\to\infty} \mu(\cap_{j=n}^{\infty} E_j)$$

We also have that $\mu(\cap_{j=n}^{\infty} E_n) \leq \inf_{m\geq n} \mu(E_m)$, therefore

$$\mu(\liminf_{n\to\infty} E_n) = \lim_{n\to\infty} \mu(\cap_{j=n}^{\infty} E_j) \leq \liminf_{n\to\infty} \mu(E_n)$$

b.

From definition,

$$\limsup_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} (\sup_{m\geq n} \mu(E_m)) = \inf_{n\geq 0} \sup_{m\geq n} \mu(E_m)$$

$$\mu(\limsup_{n\to\infty} E_n) = \mu(\cap_{n=1}^{\infty} \cup_{j=n}^{\infty} E_n)$$

Also, notice that $\cup_{j=n}^{\infty} E_n \supseteq \cup_{j=n+1}^{\infty} E_n$ for all $n \in \mathbb{N}$ thus

$$\mu(\limsup_{n\to\infty} E_n) = \mu(\lim_{n\to\infty} \cup_{j=n}^{\infty} E_j) = \lim_{n\to\infty} \mu(\cup_{j=n}^{\infty} E_j)$$

We also have that $\mu(\cup_{j=n}^{\infty} E_n) \geq \sup_{m\geq n} \mu(E_m)$, therefore

$$\mu(\limsup_{n\to\infty} E_n) = \lim_{n\to\infty} \mu(\cup_{j=n}^{\infty} E_n) \geq \limsup_{n\to\infty} \mu(E_n)$$

2.

a.

From definition, it is obvious that $E \subset O_n$ for all $n \in \mathbb{N}$, thus we have that

$$m(E) \leq \lim_{n\to\infty} m(O_n)$$

Now, for every $x \in \mathbb{R}^d$, if $x \in \bigcap_{n=1}^{\infty} O_n$, then for every $n \in \mathbb{N}$, $\operatorname{dist}(x, E) = 0$ as if $\operatorname{dist}(x, E) = \varepsilon$ for some $\varepsilon > 0$ then there exists n_0 such that for all $n > n_0$, $1/n < \varepsilon$ and $x \notin O_n$. Thus $x \in E$ as E is closed and therefore $\cap_{n=1}^{\infty} O_n \subseteq E$. Finally, as $O_n \supseteq O_{n+1}$,

$$m(\cap_{n=1}^{\infty} O_n) = \lim_{n\to\infty} m(O_n) \leq m(E)$$

b.

We have

$$\begin{aligned} m(E) &= m(\cup_{j=1}(r_j - 4^{-j}, r_j + 4^{-j})) \\ &\leq \sum_{j=1}^{\infty} m(r_j - 4^{-j}, r_j + 4^{-j}) \\ &= \sum_{j=1}^{\infty} 2 \cdot 4^{-j} \\ &= \frac{2}{3} \end{aligned}$$

However, for every $n \in \mathbb{N}$, since rationals are dense, we can find a partition $\{r_{x_0}, r_{x_1}, \ldots, r_{x_{2n}}\}$ of $[0, 1]$ from the sequence (r_n) such that $r_{x_0} = 0, r_{x_{2n}} = 1$ and $0 < r_{x_{n+1}} - r_{x_n} < \dfrac{1}{n}$.

Thus for every $x \in [0, 1]$, there exists n_0 such that $x \in [r_{x_n}, r_{x_{n+1}}]$, thus $|r_{x_{n_0}} - x| < \dfrac{1}{n}$ and thus $x \in O_n$, which means that $m(O_n) \geq 1$ and $m(O_n) > 1$ if we extend the interval $[0, 1]$ to $[0, 1 + \dfrac{1}{n}]$.

Therefore, $\lim_{n\to\infty} m(O_n) \geq 1 > m(E)$.

3.

First, let

$$S_{n,\varepsilon} = \{x \in E : \sup_{k \geq n} |f_k(x) - f(x)| \geq \epsilon\}$$

since $\sup_{k \geq n} |f_k(x) - f(x)| \geq \sup_{k \geq n+1} |f_k(x) - f(x)|$, if $x \in S_{n+1,\varepsilon}$ then $x \in S_{n,\varepsilon}$. Thus $S_{n,\varepsilon} \supseteq S_{n+1,\varepsilon}$ for all $n \in \mathbb{N}$, and

$$\lim_{n \to \infty} m(S_{n,\epsilon}) = m(\cap_{j=1}^{\infty} S_{j,\varepsilon})$$

Suppose that

$$\lim_{n \to \infty} f_n(x) = f(x), \qquad \text{a.e. } x \in E$$

Then there is a null set $E_0 \subset E$ such that $m(E_0) = 0$ and $\lim_{n \to \infty} f_n(x) = f(x)$ on $E \backslash E_0$. Then for every $x \in E \backslash E_0$, for every $\varepsilon > 0$, there is n_0 such that for $n > n_0$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}$$

which means that

$$\sup_{k \geq n} |f_n(x) - f(x)| < \varepsilon$$

and

$$\lim_{n \to \infty} \sup_{k \geq n} |f_n(x) - f(x)| < \varepsilon$$

But, if $x \in \cap_{j=1}^{n} S_{n,\epsilon}$ then $x \in S_{n,\varepsilon}$ for all $n \in \mathbb{N}$, thus $\sup_{k \geq n} |f_k(x) - f(x)| \geq \varepsilon$ for all $n \in \mathbb{N}$ and consequently $x \in E \backslash (E \backslash E_0) = E_0$. Thus $\cap_{j=1}^{n} S_{n,\varepsilon} \subseteq E_0$ and

$$\lim_{n \to \infty} m(S_{n,\varepsilon}) = m(\cap_{j=1}^{\infty} S_{j,\varepsilon}) \leq m(E_0) = 0$$

Suppose that for every $\varepsilon > 0$,

$$m(\cap_{j=1}^{\infty} S_{j,\varepsilon}) = 0$$

then let denote $E_0 := \cap_{j=1}^{\infty} S_{j,\varepsilon}$, if $x \notin E_0$ then there is $n_0 \in \mathbb{N}$ such that $x \notin S_{n_0,\varepsilon}$ but $S_{n_0,\varepsilon} \supseteq S_{n_0+1,\varepsilon}$ for all $n_0 \in \mathbb{N}$, thus $x \notin S_{n,\varepsilon}$ for all $n \geq n_0$.
Therefore, we can conclude that if $x \notin E_0$ then there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\sup_{k \geq n} |f_k(x) - f(x)| < \varepsilon$ which implies $|f_n(x) - f(x)| < \varepsilon$ and $\lim_{n \to \infty} f_n(x) = f(x)$ for $x \notin E_0$. Since $m(E_0) = 0$,

$$\lim_{n \to \infty} f_n(x) = f(x), \qquad \text{a.e. } x \in E$$

4.

1.

For all $a \in \mathbb{R}$, let S $= \{f > a\}$ then if $x \in S$, then $y \in S$ whenever $y > x$. Thus we can rewrite S is $\varnothing, \mathbb{R}, [\inf S, \infty)$ or $(\inf S, \infty)$which are all $\mathcal{B}(\mathbb{R})$-measurable. Thus f is Borel measurable.

2.

In the case where E is a measure zero set. For all $\varepsilon > 0$, $|f| \leq M$ except on a set of measure less than $\varepsilon > 0$ is already satisfied.
In case where E is not a measure zero set. Then for every $\varepsilon > 0$ if for all $M > 0$, $m(\{|f| > M\}) \geq \varepsilon$ then $m\{|f| = \infty\} \geq \varepsilon$ which is a contradiction. Thus there must exists some $M > 0$ such that $m\{|f| > M\} < \varepsilon$.

3.

Suppose there is a a function f such that $f(x) = \xi_{(a,b)}(x)$ a.e. $x \in \mathbb{R}$. Then for every $\varepsilon > 0$ there is $x_1 \in [b, b + \varepsilon/2)$ such that $f(x_1) = 0$ and $x_2 \in (b - \varepsilon/2, b)$ such that $f(x_2) = 1$. Thus for every $\varepsilon > 0$ there is x_1, x_2 such that $f(x_2) - f(x_1) = 1$ but $x_2 - x_1 < \varepsilon$.

5.

Let X_f, X_g be the set of points that is finite in f and g so that $X_f \cap X_g = X_0$. Then as $X, X_f, X_g \in \mathcal{M}$, we have that $X_0 \in \mathcal{M}$ and thus $X \backslash X_0 \in \mathcal{M}$. We also have that X_f^c, X_g^c are null sets thus

$$\mu(X \backslash X_0) = \mu(X \cap (X_f^c \cup X_g^c)) = \mu(X_f^c \cup X_g^c) \leq \mu(X_f^c) + \mu(X_g^c) = 0$$

and therefore $\mu(X \backslash X_0) = 0$.