1.

a.

$$
\begin{aligned}
&-\sum_{i=1}^{n}\int_{a_j}^{a_{j+1}}\frac{f((g(\gamma(t)))^{-1})}{(g(\gamma(t)))^2}g'(\gamma(t))\gamma'(t)dt \\
&=-\sum_{i=1}^{n}\int_{a_j}^{a_{j+1}}\frac{f(\gamma(t))}{(g(\gamma(t)))^2}g'(\gamma(t))\gamma'(t)dt \\
&=\sum_{j=1}^{n}\int_{b_j}^{b_{j+1}}f(u)du
\end{aligned}
$$

$u=\frac{1}{g(\gamma(t))}=\gamma(t)$, $du=\frac{1}{(g(\gamma(t)))^2}g'(\gamma(t))\gamma'(t)$

b.

For any z that satisfies $|z|=1$, we have that

$$\frac{1}{z}=\frac{\overline{z}}{z\overline{z}}=\frac{\overline{z}}{|z|^2}=\overline{z}$$

Let

$$\gamma:[0,2\pi]\to\mathbb{C},\qquad t\to e^{it}$$

Then

$$g\circ\gamma=e^{-it}$$

which means that

$$\frac{d}{dt}(g\circ\gamma)(t)=-\frac{d}{dt}\gamma(t)$$

Therefore,

$$
\begin{aligned}
&\int_\gamma f(z)dz \\
&=-\int_{g\circ\gamma}\frac{f(z^{-1})}{z^2}dz \\
&=-\int_{2\pi}^{0}\frac{f(z^{-1})}{z^2}dz \\
&=\int_{0}^{2\pi}\frac{f(z^{-1})}{z^2}dz \\
&=\int_\gamma\frac{f(z^{-1})}{z^2}dz
\end{aligned}
$$

2.

For any point $z_0 = x + iy$, we have that

$$\begin{aligned} & z_0 \exp(\overline{z_0}) \\ =& (x+iy)\exp(x+iy) \\ =& (x+iy)e^x e^{iy} \end{aligned}$$

Define

$$\gamma_1 : [0,1] \to \mathbb{C}, \qquad t \to t$$
$$\gamma_2 : [0,1] \to \mathbb{C}, \qquad t \to 1 + ti$$
$$\gamma_3 : [0,1] \to \mathbb{C}, \qquad t \to (1-t)(1+i)$$

Then

$$\int_0^1 f(\gamma_1(t))\gamma_1'(t)dt = \int_0^1 t\exp(t)dt = (t-1)e^t|_{t=0}^1 = 1$$

Similarly,

$$\begin{aligned} \int_0^1 f(\gamma_2(t))\gamma_2'(t)dt &= \int_0^1 (1+ti)e^{1-it}idt = -\left(\mathrm{it}+2\right)\mathrm{e}^{1-\mathrm{it}}|_{t=0}^1 \\ &= \mathrm{ei}\cdot\left(\left(\mathrm{i}+2\right)\sin\left(1\right)+\left(2\mathrm{i}-1\right)\cos\left(1\right)-2\mathrm{i}\right) \end{aligned}$$

$$\begin{aligned} \int_0^1 f(\gamma_3(t))\gamma_3'(t)dt &= \int_0^1 (1-t)(1+i)e^{(1-t)(1-i)}(-(1+i))dt \\ &= -\mathrm{e}\sin\left(1\right)-\mathrm{ei}\cos\left(1\right)+1 \end{aligned}$$

Hence, the result is

$$2 + 2e + \sin(1)(-2e + 2ie) - \cos(1)(2ie + 2e)$$

3.

We want to prove that

$$\left|\lim_{R\to\infty}\int_{L_R}\frac{dz}{zf(\exp(-iz))}\right| = 0$$

As f is a complex polynomial with degree larger than 1. $|f(\exp(-iz))| = |\exp$. Thus we have that

$$\left|\lim_{R\to\infty}\int_{L_R}\frac{dz}{zf(\exp(-iz))}\right| \leq \lim_{R\to\infty}\int_{L_R}\frac{dz}{|z||f(\exp(-iz))|} \leq \lim_{R\to\infty}\int_{L_R}\frac{dz}{|z\exp(-iz)|}$$

However, we can prove that

$$\begin{aligned}
\lim_{R\to\infty}\int_{L_R}\frac{dz}{|z\exp(-iz)|} &= \lim_{R\to\infty}\int_0^{\pi}\left|\frac{e^{iRe^{i\theta}}}{Re^{i\theta}}iRe^{i\theta}\right|d\theta \\
&= \lim_{R\to\infty}\int_0^{\pi}|e^{iRe^{i\theta}}|d\theta \\
&= \lim_{R\to\infty}\int_0^{\pi}|e^{iR\cos(\theta)-R\sin(\theta)}|d\theta \\
&= \lim_{R\to\infty}\int_0^{\pi}|e^{iR\cos(\theta)}||e^{-R\sin(\theta)}|d\theta \\
&= \lim_{R\to\infty}\int_0^{\pi}|e^{-R\sin(\theta)}|d\theta \\
&= 2\lim_{R\to\infty}\int_0^{\pi/2}|e^{-R\sin(\theta)}|d\theta \\
&\leq 2\lim_{R\to\infty}\int_0^{\pi/2}|e^{-\frac{2\pi}{R}\theta}|d\theta \\
&= \lim_{R\to\infty}-\frac{\pi}{R}(e^{-R}-1) = 0
\end{aligned}$$

As $\sin(\theta) \geq \frac{2\theta}{\pi}$ for $\theta \in [0, \pi/2]$

4.

a

Consider any point in $\{z : |z| < 2\} \cup \{z : |z-3| < 2\}$. It is the center of the star shaped domain D as a ball is star shaped.

b.

Consider any point in $\{z : |z| < 2\} \cup \{z : |z-3| < 2\}$. It is the center of the star shaped domain D as a ball is star shaped.

c.

Consider the point $x_R = (0, R)$. Then for any point $z_0 = (x_0, y_0)$ where $x_0 \neq 0$ and $x_0^2 + y_0^2 > 1.$, we have the line go through x_R, z_0 being $y = \frac{y_0 - R}{x_0} x + R$.
Consider every point (x', y') satisfies the line equation and $-x_0 < x' < x_0$

$$\begin{aligned}
&x'^2 + y'^2 \\
=&x'^2 + \left(\frac{y_0 - R}{x_0} x' + R\right)^2 \\
=&x'^2 + \frac{y_0^2 + 2y_0 R + R^2}{x_0^2} x'^2 + 2\frac{R(y_0 - R)}{x_0} x' + R^2 \\
=&R^2 \left(\left(\frac{x'}{x_0}\right)^2 - \frac{2x'}{x_0} + 1\right) + h \\
=&R^2 \left(\frac{x'}{x_0} - 1\right)^2 + h \to \infty \text{ as } R \to \infty
\end{aligned}$$

where h is a function where degree of R is less than 2. Hence, for every point $(x_0, y_0) \in D^+$, we can find $(0, R)$ such that $(0, R)$ is the center of the star shaped domain $D^+ \cap \{(x, y) : y \geq \frac{y_0 - R}{x_0} x + R\}$. Hence, every analytic has a complex antiderivative on D^+. We also have that for every closed curve γ

$$\int_{\overline{\gamma}} f(z) dz = \int_{\gamma} f(z) \overline{dz} = \overline{\int_{\gamma} \overline{f}(z) dz} = 0$$

Which means every anayltic function has a complex antiderivative on D. For every closed curves in D,the curve is bounded and hence we can find R such that we can find a star shaped with center $(0, R)$ contained the close curves which means its integral is 0.

d.

Consider the point $(0, 0)$. Then for every point $(x_0, y_0) \in D$. The line between the two points

$$\gamma : [0, 1] \to \mathbb{C}, \qquad t \to tx_0 + ity_0$$

is completely contained inside D as $|x_0 - y_0| < 1 \leq \frac{1}{t}$ oc $|x_0 + y_0| < 1 \leq \frac{1}{t}$

5.

Consider

$D_1 = \{y > x, x \leq 0\} \cup \{y > -x, 0 \leq x \leq 1\} \cup \{y > x-2, 1 \leq x \leq 2\} \cup \{y > 2-x, x \geq 2\}$

and

$D_2 = \{y < -x, x \leq 0\} \cup \{y < x, 0 \leq x \leq 1\} \cup \{y < 2-x, 1 \leq x \leq 2\} \cup \{y < x-2, x \geq 2\}$

so that $\mathbb{C} \backslash \{0, 2\} = D_1 \cup D_2$.
Also, let

$$G_1 = D_1 \cap D_2 \cap \{x \leq 0\}$$

$$G_2 = D_1 \cap D_2 \cap \{0 \leq x \leq 2\}$$

$$G_3 = D_1 \cap D_2 \cap \{x \geq 2\}$$

so that $G_1 \cup G_2 \cup G_3 = D_1 \cap D_2$. Since D_1 and D_2 are star shaped and open, there exists antiderivative F_1, F_2 of f on D_1 and D_2 respectively However,

$$\int_{|z|=1} f(z)dz = \int_{\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4} f(z)dz = 0$$

where γ_1 be the path of $|z| = 1$ in $D_1 \backslash D_2$, γ_2 be the path of that in G_2, γ_3 be the path of that in $D_2 \backslash D_1$, γ_4 be the path of that in G_1. Hence, $F_1 = F_2$ on G_1 and G_2. Doing similarly for the integral along $|z| = 3$, we have that $F_1 = F_2$ on G_1 and G_3.
Therefore, $F_1 = F_2$ on $D_1 \cup D_2$ and hence there exists antiderivative on $\mathbb{C} \backslash \{0, 2\}$.
If there is antiderivative, the line integral along closed path must be 0 hence proved.