

1.

Solve $u_t + 3tu_x = u$ with $u(0, x) = h(x)$ for $x \in \mathbb{R}$. Follow the method, we have

$$\frac{dT(\sigma; s)}{ds} = 1, \frac{dX(\sigma; s)}{ds} = 3t, \frac{dU(\sigma; s)}{ds} = U$$

with

$$T(\sigma; 0) = 0, X(\sigma; 0) = \sigma, U(\sigma; 0) = h(\sigma)$$

Thus,

$$T(\sigma; s) = s, X(\sigma; s) = \sigma + s, U(\sigma; s) = e^{s+\ln(h(\sigma))}$$

Next, we can obtain the inverse function,

$$\begin{cases} t = T(\sigma; s) = s \\ x = X(\sigma; s) = \sigma + s \end{cases} \implies \begin{cases} \sigma = \Sigma(t, x) = x - t \\ s = S(t, x) = t \end{cases}$$

Finally, the solution is obtained as

$$u(t, x) = U(\Sigma(t, x), S(t, x)) = e^{t+\ln(h(x-t))}$$

2.

Solve $u_t + uu_x = 2t$ with $u(0, x) = x$. Follow the method, we have

$$\frac{dT(\sigma; s)}{ds} = 1, \frac{dX(\sigma; s)}{ds} = U, \frac{dU(\sigma; s)}{ds} = 2t$$

with

$$T(\sigma; 0) = 0, X(\sigma; 0) = \sigma, U(\sigma; 0) = \sigma$$

Thus,

$$T(\sigma; s) = s, X(\sigma; s) = ts^2 + \sigma s + \sigma, U(\sigma; s) = 2ts + \sigma$$

Next, we can obtain the inverse function,

$$\begin{cases} t = T(\sigma; s) = s \\ x = X(\sigma; s) = ts^2 + \sigma s + \sigma \end{cases} \implies \begin{cases} \sigma = \Sigma(t, x) = \frac{x - t^3}{t + 1} \\ s = S(t, x) = t \end{cases}$$

Finally, the solution is obtained as

$$u(t, x) = U(\Sigma(t, x), S(t, x)) = 2t^2 + \frac{x - t^3}{t + 1}$$

3.

Solve $yu_x + xu_y = u$ with $u(x, 0) = x^3, u(0, y) = y^3$ for $x, y > 0$.
First, let's solve for $u(x, 0) = x^3$. We have

$$\frac{dX(\sigma; s)}{ds} = Y, \frac{dY(\sigma; s)}{ds} = X, \frac{dU(\sigma; s)}{ds} = U$$

with

$$X(\sigma; 0) = \sigma, Y(\sigma; 0) = 0, U(\sigma; 0) = \sigma^3$$

Thus,

$$X(\sigma; s) = \frac{\sigma}{2}(e^{-s} + e^s), Y(\sigma; s) = \frac{\sigma}{2}(e^s - e^{-s}), U(\sigma; s) = e^{s+3\ln(\sigma)}$$

Next, we can obtain the inverse function,

$$\begin{cases} x = X(\sigma; s) = \frac{\sigma}{2}(e^{-s} + e^s) \\ y = Y(\sigma; s) = \frac{\sigma}{2}(e^s - e^{-s}) \end{cases} \implies \begin{cases} x^2 = \frac{\sigma^2}{4}(e^{-2s} + 2 + e^{2s}) \\ y^2 = \frac{\sigma^2}{4}(e^{-2s} - 2 + e^{2s}) \\ x + y = \sigma e^s \\ x - y = \sigma e^{-s} \end{cases}$$

Hence,

$$\begin{cases} \sigma = \Sigma(x, y) = \sqrt{x^2 - y^2} \\ s = S(x, y) = \frac{1}{2} \ln \left(\frac{x+y}{x-y} \right) \end{cases}$$

Finally, the solution is obtained as

$$u_1(x, y) = U(\Sigma(x, y), S(x, y)) = e^{\frac{1}{2} \ln \left(\frac{x+y}{x-y} \right) + \frac{3}{2} \ln(x^2 - y^2)} = (x+y)^2 + x - y$$

and similarly, on $u_2(0, y) = y^3$, we have

$$u_2(x, y) = (x+y)^2 + y - x$$

which means that $u_1(x, y) = u_2(x, y)$ on $x = y$ thus

$$u(x, y) = \begin{cases} (x+y)^2 + x - y, & \text{if } x > y \\ (x+y)^2 + y - x, & \text{if } y > x \end{cases}$$

4.

Solve $u_x^2 + u_y = u$ with $u(x, 0) = x^2$.

First, let $F(x, y, u, p, q) = p^2 + q - u$, we further have

$$X_0(\sigma) = \sigma, Y_0(\sigma) = 0, U_0(\sigma) = \sigma^2 \quad (1)$$

We have

$$\begin{cases} P_0^2 + Q_0^2 - U_0 = P_0^2 + Q_0^2 - \sigma^2 = 0 \\ 2\sigma = U'_0(\sigma) = P_0 \end{cases} \implies \begin{cases} P_0 = 2\sigma \\ Q_0 = \sigma\sqrt{5} \end{cases}$$

Then we have the ODE system

$$\begin{cases} \frac{dX}{ds} = F_p = 2P \\ \frac{dY}{ds} = F_q = 1 \\ \frac{dU}{ds} = PF_p + QF_q = 2P^2 + Q \\ \frac{dP}{ds} = -F_x - F_zP = P = 2\sigma \\ \frac{dQ}{ds} = -F_y - F_zQ = Q = \sqrt{5}\sigma \end{cases}$$

Therefore,

$$\begin{cases} P(\sigma, s) = 2\sigma s + 2\sigma \\ Q(\sigma, s) = \sqrt{5}\sigma s + \sqrt{5}\sigma \end{cases}$$

We have

$$\begin{cases} \frac{dX}{ds} = 4\sigma s + 4\sigma \\ \frac{dY}{ds} = 1 \\ \frac{dU}{ds} = 2P^2 + Q = 8\sigma^2(s+1)^2 + \sqrt{5}\sigma(s+1) \end{cases}$$

and with the initial conditions (1), we have

$$\begin{cases} X(\sigma, s) = 2\sigma s^2 + 4\sigma s + \sigma \\ Y(\sigma, s) = s \\ U(\sigma, s) = \frac{8\sigma^2}{3}(s+1)^3 + \frac{\sqrt{5}\sigma}{2}(s+1)^2 - \frac{5\sigma^2}{3} - \frac{\sqrt{5}\sigma}{2} \end{cases}$$

Hence, we can obtain the inverse function,

$$\begin{cases} \sigma = \frac{x}{4y+1} \\ s = y \end{cases}$$

and the solution

$$u(x, y) = U(\sigma, s) = \left(\frac{x}{4y+1}\right)^2 \left(\frac{8}{3}(y+1)^3 - \frac{5}{3}\right) + \frac{\sqrt{5}x}{8y+2}((y+1)^2 - 1)$$

5.

Solve $u_x u_y = xy$ with $u(x, 0) = x$.

First, let $F(x, y, u, p, q) = pq - xy$, we further have that

$$X_0(\sigma) = \sigma, Y_0(\sigma) = 0, U_0(\sigma) = \sigma \quad (2)$$

We have

$$\begin{cases} P_0 Q_0 - X_0 Y_0 = P_0 Q_0 = 0 \\ 1 = U'_0(\sigma) = X'_0(\sigma) P_0 + Y'_0(\sigma) Q_0 = P_0 \end{cases} \implies \begin{cases} P_0 = 1 \\ Q_0 = 0 \end{cases}$$

Then we have the ODE system

$$\begin{cases} \frac{dX}{ds} = F_p = Q \\ \frac{dY}{ds} = F_q = P \\ \frac{dU}{ds} = P F_p + Q F_q = 2PQ \\ \frac{dP}{ds} = -F_x - F_z P = Y \\ \frac{dQ}{ds} = -F_y - F_z Q = X \end{cases}$$

From which, we have

$$\frac{d^2 Q}{ds^2} = \frac{dX}{ds} = Q$$

thus

$$Q(\sigma, s) = C_1(\sigma) e^s + C_2(\sigma) e^{-s} \text{ with } Q(\sigma, 0) = 0$$

and

$$Q(\sigma, s) = C_5(\sigma) (e^s - e^{-s}) = C_5(\sigma) \sinh(s)$$

and similarly,

$$P(\sigma, s) = C_3(\sigma) e^s + C_4(\sigma) e^{-s} \text{ with } P(\sigma, 0) = 1$$

and hence $C_3 + C_4 = 1$.

Finally, we can obtain

$$X = C_5 \cosh(s)$$

and

$$Y = C_3(\sigma) e^s - C_4(\sigma) e^{-s} = C_3(\sigma) e^s + (C_3(\sigma) - 1) e^{-s} = C_3(\sigma) \cosh(s) - e^{-s}$$

Now, using the initial conditions, we can get

$$X = \sigma \cosh(s) \text{ and } Y = \cosh(s) - e^{-s} = \sinh(s)$$

and the inverse functions

$$\begin{cases} \sigma = \frac{x}{\cosh(\ln(y + \sqrt{1 + y^2}))} = \frac{x}{\sqrt{y^2 + 1}} \\ s = \sinh^{-1}(y) = \ln(y + \sqrt{1 + y^2}) \end{cases}$$

We can also obtain that

$$\begin{aligned}
\frac{dU}{ds} &= 2C_5(\sigma) \sinh(s) (C_3(\sigma)e^s + C_4(\sigma)e^{-s}) \\
&= 2C_5(\sigma) \sinh(s)(C_3(\sigma)e^s + (1 - C_3(\sigma))e^{-s}) \\
&= 2C_5(\sigma) \sinh(s)(e^{-s} + C_3(\sigma)2 \sinh(s)) \\
&= C_6(\sigma) \sinh(s)e^{-s} + C_7(\sigma) \sinh^2(s)
\end{aligned}$$

Thus

$$U = C_6(\sigma) \frac{e^{-2s} + 2s}{4} + C_7(\sigma) \frac{\sinh(2s) - 2s}{4}$$

with the initial conditions

$$\sigma = U_0(\sigma) = C_6(\sigma) \frac{3}{4}$$

Thus

$$U(\sigma, s) = \frac{\sigma(e^{-2s} + 2s)}{3} + C_7(\sigma) \frac{\sinh(2s) - 2s}{4}$$

Therefore, the solution is

$$\begin{aligned}
u(x, y) &= \frac{x((y + \sqrt{1 + y^2})^{-2} + 2 \ln(y + \sqrt{1 + y^2}))}{3\sqrt{y^2 + 1}} \\
&+ C \left(\frac{x}{\sqrt{y^2 + 1}} \right) \frac{\sinh(2 \ln(y + \sqrt{1 + y^2})) - 2 \ln(y + \sqrt{1 + y^2})}{4}
\end{aligned}$$