# 1.

## a.

Since $y > \theta$, we assume $z > \theta$ and hence

$$\begin{aligned}
F_{Y_{(1)}}(z) = P(Y_{(1)} < z) = 1 - P(Y_{(1)} > z) = 1 - (P(Y > z))^n \\
&= 1 - \left(\int_z^\infty \frac{2\theta^2}{y^3} dy\right)^n \\
&= 1 - \left(-\frac{\theta^2}{y^2}\Big|_{y=z}^\infty\right)^n \\
&= 1 - \left(\frac{\theta^2}{z^2}\right)^n
\end{aligned}$$

Hence,

$$f_{Y_1}(y) = -n \cdot \left(\frac{\theta^2}{y^2}\right)^{n-1} \cdot \left(-\frac{2\theta^2}{y^3}\right) = \frac{2n}{y}\left(\frac{\theta^2}{y^2}\right)^n$$

We have

$$y_{(1)} = h^{-1}(u) = \theta \cdot u \implies \frac{d}{du}h^{-1}(u) = \theta$$

Therefore,

$$f_U(u) = f_{Y_{(1)}}(\theta \cdot u) \cdot \theta = \frac{2n}{\theta \cdot u}\left(\frac{1}{u^2}\right)^n \cdot \theta = \frac{2n}{u^{2n+1}}$$

Hence, $U$ is a pivotal quantity.

## b.

We have for $1 < u$,

$$F_U(u) = 1 - \frac{1}{u^{2n}}$$

Hence,

$$P(U < a) = F_U(a) = 1 - \frac{1}{a^{2n}} \implies a = \sqrt[2n]{\frac{2}{2-\alpha}}$$

$$P(U > b) = 1 - F_U(b) = 1 - \left(1 - \frac{1}{b^{2n}}\right) = \frac{1}{b^{2n}} \implies b = \sqrt[2n]{\frac{2}{\alpha}}$$

$$P(a < U < b) = P\left(a < \frac{Y_{(1)}}{\theta} < b\right) = P\left(\frac{Y_{(1)}}{b} < \theta < \frac{Y_{(1)}}{a}\right) = 1 - \alpha$$

Hence, the $(1-\alpha)100\%$ CI for $\theta$ is $\left(\frac{Y_{(1)}}{\sqrt[2n]{\frac{2}{\alpha}}} < \theta < \frac{Y_{(1)}}{\sqrt[2n]{\frac{2}{2-\alpha}}}\right)$

**c.**

The 99% CI for $\theta$ is

$$\left( \frac{10.55}{\sqrt[50]{\frac{2}{2-0.01}}} < \theta < \frac{10.55}{\sqrt[50]{\frac{2}{0.01}}} \right) = (9.48925, 10.5489)$$

**d.**

Since p-value $= 0.095$, $\alpha = 0.01$ for the two-sided confidence interval which means that

$$\frac{y_{(1)}}{\sqrt[40]{\frac{2}{2-0.01}}} = 5 \implies y_{(1)} \simeq 5$$

**e.**

## 2.

### a.

$$\begin{aligned}F_{Y_{(n)}}(z) = P(Y_{(n)} < z) = P(Y_{(n)} < z) &= (P(Y < z))^n \\ &= \left(\int_0^z \frac{2y}{\theta^2} dy\right)^n \\ &= \left(\left.\frac{y^2}{\theta^2}\right|_{y=0}^{z}\right)^n \\ &= \left(\frac{z^2}{\theta^2}\right)^n\end{aligned}$$

Hence,

$$f_{Y_1}(y) = \frac{2n \cdot z^{2n-1}}{\theta^{2n}}$$

We have

$$y_{(1)} = h^{-1}(u) = \theta \cdot u \implies \frac{d}{du} h^{-1}(u) = \theta$$

Therefore,

$$f_U(u) = f_{Y_{(n)}}(\theta \cdot u) \cdot \theta = \frac{2n \cdot (\theta \cdot u)^{2n-1}}{\theta^{2n}} \cdot \theta = 2n \cdot u^{2n-1}$$

Hence, $U$ is a pivotal quantity.

### b.

We have for $0 < u < 1$,

$$F_U(u) = u^{2n}$$

Hence,

$$P(U < a) = F_U(a) = a^{2n} \implies a = \sqrt[2n]{\frac{\alpha}{2}}$$

$$P(U > b) = 1 - F_U(b) = 1 - \left(b^{2n}\right) \implies b = \sqrt[2n]{\frac{2-\alpha}{2}}$$

$$P(a < U < b) = P\left(a < \frac{Y_{(n)}}{\theta} < b\right) = P\left(\frac{Y_{(n)}}{b} < \theta < \frac{Y_{(n)}}{a}\right) = 1 - \alpha$$

Hence, the $(1-\alpha)100\%$ CI for $\theta$ is $\left(\frac{Y_{(n)}}{\sqrt[2n]{\frac{2-\alpha}{2}}}, \frac{Y_{(n)}}{\sqrt[2n]{\frac{\alpha}{2}}}\right)$

### c.

The 95% CI for $\theta$ is

$$\left(\frac{6.5}{\sqrt[30]{\frac{2-0.05}{2}}}, \frac{6.5}{\sqrt[30]{\frac{0.05}{2}}}\right) = (6.50549, 7.35047)$$

### d.

## 3.

We know that $\overline{Y} \sim \text{Gamma}(cn, \beta/n)$, which means that

$$f_{\overline{Y}}(y) = \frac{y^{cn-1}e^{-yn/\beta}}{\Gamma(cn)(\beta/n)^{cn}} = \frac{n}{\beta\Gamma(cn)} \cdot \left(\frac{yn}{\beta}\right)^{cn-1} e^{-yn/\beta}$$

Consider $T = \frac{\overline{Y}}{\beta}$, then as $\overline{y} = h^{-1}(t) = \beta \cdot t \implies \frac{d}{dt}h^{-1}(t) = \beta$

$$f_T(t) = f_{\overline{Y}}(\beta \cdot t) \cdot \beta = \frac{n}{\beta\Gamma(cn)} \cdot (tn)^{cn-1} \cdot e^{-tn} \cdot \beta = \frac{n(tn)^{cn-1}}{\Gamma(cn)e^{tn}}$$

which is independent of $\beta$ hence is a pivotal quantity. Therefore,

$$P(T < a) = \int_0^a \frac{n(tn)^{cn-1}}{\Gamma(cn)e^{tn}} dt = \frac{n^{cn}}{\Gamma(cn)} \int_0^a t^{cn-1}e^{-tn} dt$$