1.

Without loss of generality, we can assume $\dim(\text{Image}(g)) = 1$. Assume that g is only defined in the set $[-N, N] \times [-M, M]$ where N, M are arbitary. We have that

$$\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} g(x^*,y^*)\Delta F_{i,j}$$

where with $x_i = \frac{-(n-i)N+iN}{n} = \frac{2iN-nN}{n}$, $y_i = \frac{-(m-i)M+iM}{m} = \frac{2iM-mM}{m}$, we have

$$\begin{aligned}\Delta F_{i,j} &= F(x_i,y_j) - F(x_{i-1},y_j) - F(x_i,y_{j-1}) + F(x_{i-1},y_{j-1}) \\ &= \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(x,y)dxdy\end{aligned}$$

Hence,

$$\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} g(x^*,y^*)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(x,y)dxdy$$

$$\begin{aligned}E[g(X,Y)] &= \int_{[-N,N]\times[-M,M]} g(x,y)f(x,y)dxdy \\ &= \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} g(x,y)f(x,y)dxdy\end{aligned}$$

We know that g is uniformly continuous as it is continuous in a compact set, we have that with large enough n,m

$$|g(x,y) - g(x^*,y^*)| < \epsilon$$

Hence,

$$\begin{aligned}&\left|\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) - E[g(X,Y)]\right| \\ =&\left|\lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} (g(x,y)-g(x^*,y^*))f(x,y)dxdy\right| \\ <&\left|\lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} \epsilon f(x,y)dxdy\right| \\ =&\epsilon\left|\lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(x,y)dxdy\right| \\ =&\epsilon\end{aligned}$$

Since, ϵ is arbitary, $\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = E[g(X,Y)]$, and since N, M are arbitary,

$$\int_{\mathbb{R}^2} g(x,y)dF(x,y) = E[g(X,Y)]$$