1.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. As $\mathcal{G}_u \subseteq \mathcal{G}_t$, we have that

$$\begin{aligned}
E[E[X_t|\mathcal{G}_t]|\mathcal{G}_u] &= \int_{G_u} E[X_t|\mathcal{G}_t] dP(\mathcal{G}_u) \\
&= \int_{G_u}\int_{G_t} X dP(\mathcal{G}_t) dP(\mathcal{G}_u) \\
&= \int_{G_t}\int_{G_u} X dP(\mathcal{G}_u) dP(\mathcal{G}_t) \\
&= E[X_t|\mathcal{G}_u]
\end{aligned}$$

Similarly,

$$\begin{aligned}
E\left[\int_0^t E\left[Y_s|\,\mathcal{G}_s\right] ds \middle|\, \mathcal{G}_u\right] &= E\left[\int_0^t \int_{G_s} Y dP(G_s) ds \middle|\, \mathcal{G}_u\right] \\
&= \int_{G_u}\int_0^t \int_{G_s} Y dP(G_s) ds dP(G_u) \\
&= \int_0^t \int_{G_u}\int_{G_s} Y dP(G_s) dP(G_u) ds \\
&= \int_0^t E[Y_s|\mathcal{G}_u] ds
\end{aligned}$$

We also know that

$$\begin{aligned}
\max\left(\int_{G_t} X_+ dP(G_t), \int_{G_t} X_- dP(G_t)\right) &\leq \max\left(\int_\Omega X_+ dP, \int_\Omega X_- dP\right) \\
&= \max(E[X_+], E[X_-]) < \infty
\end{aligned}$$

Therefore, $E[|X_t||G_t] < \infty$. Similarly, using Fubini and the steps above, we can also show that

$$\int_0^t E[|Y_s||\mathcal{G}_s] ds < \infty$$

Therefore,

$$\begin{aligned}
&E[E[X_t|\mathcal{G}_t]|\mathcal{G}_u] - E\left[\int_0^t E\left[Y_s|\,\mathcal{G}_s\right] ds \middle|\, \mathcal{G}_u\right] \\
=&E[X_t|\mathcal{G}_u] - \int_0^t E[Y_s|\mathcal{G}_u] ds
\end{aligned}$$

which confirms it is indeed a martingale.

2.

To match the state equations, we have that

$$a^1_{i,j} = iK^1, \qquad s^1_{i,j} = i^2$$

and

$$a^2_{i,j} = rj, \qquad s^2_{i,j} = \frac{r}{K^2}(j^2 + \alpha_{21} ij)$$

and

$$\begin{aligned} Lf(i,j) = a^1_{i,j}[f(i+1,j) - f(i,j)] + a^2_{i,j}[f(i,j+1) - f(i,j)] \\ + s^1_{i,j}[f(i-1,j) - f(i,j)] + s^2_{i,j}[f(i,j-1) - f(i,j)] \end{aligned}$$

Then the 2 state equations are consistent with the martingale problem:

$$f(X^1_t, X^2_t) - f(X^1_0, X^2_0) - \int_0^t Lf(X^1_u, X^2_u)du$$

which is a $\sigma(X^1_s, X^2_s, s \leq t)$-martingale.

3.

We know that

$$F_{X_1 \vee X_2}(x) = (F_{X_i}(x))^2$$

Hence,

$$\begin{aligned} f_{X_1 \vee X_2}(x) &= 2F_{X_i}(x) f_{X_i}(x) \\ &= 2\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \\ &= \frac{e^{-\frac{x^2}{2}}}{\pi} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \end{aligned}$$

Therefore, we can calculate

$$f_{X_1 \vee X_2}(0) = \frac{1}{\pi} \underbrace{\int_{-\infty}^{0} e^{-\frac{u^2}{2}} du}_{\sqrt{\pi/2}} = \frac{1}{\sqrt{2\pi}}$$

$$\begin{aligned} &\int_0^1 f_{X_1 \vee X_2}(x) dx \\ =&F_{X_1 \vee X_2}(1) - F_{X_1 \vee X_2}(0) \\ =&F_{X_i}(1)^2 - F_{X_i}(0)^2 \\ =&0.84134475^2 - 0.5^2 \\ =&0.45786098835 \end{aligned}$$

4.

Let $g(B_t, t) = f\left(\frac{\sigma^2}{2}t\right)$, hence we have that

$$\frac{\partial}{\partial x} g(B, x) = \frac{\sigma^2}{2} e^{\frac{\sigma^2 x}{2}} \left(f' \circ e^{\frac{\sigma^2 x}{2}}\right)$$

$$\frac{\Delta}{2} g(B, s) = 0$$

and that

$$g(B_t, t) - g(0,0) - \int_0^t \frac{\Delta}{2} g(B_s, s) + \frac{\partial}{\partial s} g(B_s, s) ds$$

is a $\sigma(B_u, u \leq t)$-martingale problem. However, we have that

$$\frac{\Delta}{2} g(B_s, s) + \frac{\partial}{\partial s} g(B_s, s) = \frac{\sigma^2}{2} \exp\left(\frac{\sigma^2 s}{2}\right) f'\left(\exp\left(\frac{\sigma^2 s}{2}\right)\right)$$

Hence, if we let $X_t = \exp\left(\frac{\sigma^2}{2}t\right)$, then we get

$$f(X_t) - f(X_0) - \int_0^t \frac{\sigma^2 X_u}{2} f'(X_u) du$$

is a martingale. If there is another solution Y_t that has $Y_0 = X_0$. Let $f(x) = \ln(x)$ so that $f'(x) = \frac{1}{x}$ and therefore, $X_u f'(X_u) = 1$. Thus,

$$\ln(X_t) - \ln(X_0) - \int_0^t \frac{\sigma^2}{2} du = \ln(Y_t) - \ln(Y_0) - \int_0^t \frac{\sigma^2}{2} du$$

and hence $\ln(X_t) = \ln(Y_t) \implies X_t = Y_t$ for all t which proves its uniqueness.