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Without loss of generality, we can assume $\dim(\text{Image}(g)) = 1$. Assume that g is only defined in the set $[-N, N] \times [-M, M]$ where N, M are arbitrary. We have that

$$\int_{[-N, N] \times [-M, M]} g(x, y) dF(x, y) = \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m g(x_i^*, y_j^*) \Delta F_{i,j}$$

where with $x_i = \frac{-(n-i)N+iN}{n} = \frac{2iN-nN}{n}$, $y_i = \frac{-(m-i)M+iM}{m} = \frac{2iM-mM}{m}$, we have

$$\begin{aligned} \Delta F_{i,j} &= F(x_i, y_j) - F(x_{i-1}, y_j) - F(x_i, y_{j-1}) + F(x_{i-1}, y_{j-1}) \\ &= \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x, y) dx dy \end{aligned}$$

Hence,

$$\begin{aligned} \int_{[-N, N] \times [-M, M]} g(x, y) dF(x, y) &= \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m g(x_i^*, y_j^*) \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x, y) dx dy \\ E[g(X, Y)] &= \int_{[-N, N] \times [-M, M]} g(x, y) f(x, y) dx dy \\ &= \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} g(x, y) f(x, y) dx dy \end{aligned}$$

We know that g is uniformly continuous as it is continuous in a compact set, we have that with large enough n, m

$$|g(x, y) - g(x^*, y^*)| < \epsilon$$

Hence,

$$\begin{aligned} &\left| \int_{[-N, N] \times [-M, M]} g(x, y) dF(x, y) - E[g(X, Y)] \right| \\ &= \left| \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} (g(x, y) - g(x_i^*, y_j^*)) f(x, y) dx dy \right| \\ &< \left| \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} \epsilon f(x, y) dx dy \right| \\ &= \epsilon \left| \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x, y) dx dy \right| \\ &= \epsilon \end{aligned}$$

Since, ϵ is arbitrary, $\int_{[-N,N] \times [-M,M]} g(x,y) dF(x,y) = E[g(X,Y)]$, and since N, M are arbitrary,

$$\int_{\mathbb{R}^2} g(x,y) dF(x,y) = E[g(X,Y)]$$

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Because $s_i > a_i$ for all i , $X(0)$ does not affect stationary distributions.

$$\begin{aligned}
\pi(0) &= \left(1 + \frac{a_0}{s_1} + \frac{a_0 a_1}{s_1 s_2} + \dots\right)^{-1} \\
&= \left(1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 4 \cdot 5} + \dots\right)^{-1} \\
&= \left(1 + \sum_{i=1}^{\infty} \frac{2 \cdot i!}{(i+2)!}\right)^{-1} \\
&= \left(1 + \sum_{i=1}^{\infty} \frac{2}{(i+1)(i+2)}\right)^{-1} \\
&= \left(1 + 2 \sum_{i=1}^{\infty} \left(\frac{1}{i+1} - \frac{1}{i+2}\right)\right)^{-1} \\
&= \left(1 + 2 \left(\frac{1}{2}\right)\right)^{-1} \\
&= \frac{1}{2}
\end{aligned}$$

Hence, we can calculate

$$\pi(i) = \pi(0) \frac{a_0 a_1 \dots a_{i-1}}{s_1 s_2 \dots s_i} = \frac{1}{2} \cdot \frac{2 \cdot i!}{(i+2)!} = \frac{1}{(i+1)(i+2)}$$

3

3.1

$$f^{-1}(\emptyset) = \{x \in S : f(x) \in \emptyset\} = \emptyset$$

3.2

$$f^{-1}(B^C) = \{x \in S : f(x) \notin B\} = S \setminus \{x \in S : f(x) \in B\} = S \setminus (f^{-1}(B)) = f^{-1}(B)^C$$

3.3

$$\begin{aligned} f^{-1}\left(\bigcap_{\beta} B_{\beta}\right) &= \{x \in S : f(x) \in \bigcap_{\beta} B_{\beta}\} \\ &= \bigcap_{\beta} \{x \in S : f(x) \in B_{\beta}\} \\ &= \bigcap_{\beta} f^{-1}(B_{\beta}) \end{aligned}$$

3.4

$$\begin{aligned} f^{-1}\left(\bigcup_{\beta} B_{\beta}\right) &= \{x \in S : f(x) \in \bigcup_{\beta} B_{\beta}\} \\ &= \bigcup_{\beta} \{x \in S : f(x) \in B_{\beta}\} \\ &= \bigcup_{\beta} f^{-1}(B_{\beta}) \end{aligned}$$

4

Let $\{Y_i\}_{i=1}^N$ be geometric distributions, then $\mathcal{F}_n = \sigma(Y_1, Y_2, \dots, Y_n)$ is a filtration. We will use the geometric distributions to estimate $\sin(X \ln(X))$, where X is a $(2, 1/2)$ -negative binomial.

$$\alpha_i = \frac{p(Y_i)}{q(Y_i)} = \frac{\binom{Y_i+1}{Y_i} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{Y_i-2}}{\frac{1}{2} \left(\frac{1}{2}\right)^{Y_i-1}} = Y_i + 1$$

and with

$$L_n = \prod_{i=1}^n \alpha_i = \prod_{i=1}^n (Y_i + 1)$$

We have

$$\begin{aligned} & E[L_n g(Y_1, Y_2, \dots, Y_n)] \\ &= E[E[L_n | \mathcal{F}_n] g(Y_1, Y_2, \dots, Y_n)] \\ &= E \left[\prod_{i=1}^n \frac{p(Y_i)}{q(Y_i)} g(Y_1, Y_2, \dots, Y_n) \right] \\ &= \sum_{j_1, j_2, \dots, j_n=1}^{\infty} \prod_{i=1}^n \frac{p(j_i)}{q(j_i)} g(j_1, j_2, \dots, j_n) q(j_1) q(j_2) \dots q(j_n) \\ &= \sum_{j_1, j_2, \dots, j_n=1}^{\infty} g(j_1, j_2, \dots, j_n) p(j_1) p(j_2) \dots p(j_n) \\ &= E[g(X_1, X_2, \dots, X_n)] \end{aligned}$$

where X_i is a $(2, 1/2)$ -negative binomial. Therefore, to estimate $E[\sin(X \ln(X))]$, calculate

$$g(X_1, X_2, \dots, X_n) = \prod_{i=1}^n (Y_i + 1) g(Y_1, Y_2, \dots, Y_n)$$

where $g(X_1, X_2, \dots, X_n) = \frac{1}{n} \sum_{m=1}^n \sin(X_m \ln(X_m))$ because as $n \rightarrow \infty$, we have

$$g(X_1, X_2, \dots, X_n) = E[g(X_1, X_2, \dots, X_n)]$$

and

$$g(Y_1, Y_2, \dots, Y_n) = E[g(Y_1, Y_2, \dots, Y_n)]$$