Fall 2021, Math 328, Homework 3

Due: End of day on 2022-10-17

1 (10 points)

Let G be a group and X a set. Suppose that $G \times G$ acts on X, and let $\rho : G \times G \to \mathrm{Per}(X)$ denote the associated permutation representation.

1. Let $\sigma : G \times G \to G \times G$ be the map $\sigma(x, y) = (y, x)$. Show that σ is an automorphism of $G \times G$.
2. Suppose that for all $x \in X$ and $g \in G \times G$, one has $g \cdot x = \sigma(g) \cdot x$. Show that the image of ρ is abelian.

2 (10 points)

Let G be a group and X a set. Suppose that the group G acts on the set X. For $x, y \in X$, write $x \sim y$ provided that there exists some $g \in G$ such that $g \cdot x = y$. Show that $\sim$ is an equivalence relation on X.

3 (10 points)

Let G be a group and consider the action of G on itself by conjugation, so that $g \in G$ acts on $h \in G$ by sending it to $g \cdot h \cdot g^{-1}$. Let $\rho : G \to \mathrm{Per}(G)$ denote the permutation representation.

1. Show that $\mathrm{Aut}(G)$ is a subgroup of $\mathrm{Per}(G)$ and that the image of ρ is contained in $\mathrm{Aut}(G)$. Let $\gamma : G \to \mathrm{Aut}(G)$ denote the induced homomorphism (part of the exercise is to explain precisely what this means).
2. Prove that $\ker(\rho) = \ker(\gamma) = Z(G)$.
3. Let $\sigma \in \mathrm{im}(\gamma)$ and $\tau \in \mathrm{Aut}(G)$ be given. Prove that $\tau \circ \sigma \circ \tau^{-1} \in \mathrm{im}(\gamma)$.

4 (10 points)

1. Let G be an abelian group, and let $g_1, \ldots, g_n \in G$ be generators of G. Put $k_i := \mathrm{ord}_G(g_i)$, and assume that k_i is finite for every i. Prove that $\#G \leq k_1 \cdots k_n$. Give an explicit example showing that this can fail if G is nonabelian.

2. Let G be an abelian group. Prove that the set of elements of finite order in G is a subgroup of G. Give an explicit example showing that this can fail if G is nonabelian.

3. Prove that $(\mathbb{Q}, +)$ is noncyclic. Prove that every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic. Find an explicit *proper* subgroup of $(\mathbb{Q}, +)$ which is noncyclic.

5 (10 points)

1. Let G be a group and H a subgroup of G. Show that H is contained in $N_G(H)$. Show that H is contained in $C_G(H)$ if and only if H is abelian.

2. Let H be a subgroup of order 2 in a group G. Show that $N_G(H) = C_G(H)$. Show that $H \subset Z(G)$ provided that $N_G(H) = G$.

6 (10 points)

This exercise is about the Heisenberg group $H(\mathbb{R})$, which is the group of 3×3 real matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{R},$$

with respect to matrix multiplication.

1. Verify that $H(\mathbb{R})$ is indeed a subgroup of $\mathrm{GL}_3(\mathbb{R})$.

2. Prove that $Z(H(\mathbb{R}))$ is isomorphic to $(\mathbb{R}, +)$.

3. Prove that the map $H(\mathbb{R}) \to \mathbb{R} \times \mathbb{R}$ defined by

   $$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto (a, b)$$

   is a surjective group homomorphism. What is the kernel of this homomorphism?