1 Preliminary

1.1 Basic on sets

1.2 Countable sets

- Bernstein's theorem: if $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(Y) \leq \text{card}(X)$ then $\text{card}(X) = \text{card}(Y)$
- $\text{card}(\mathcal{P}(\mathbb{N})) = \text{card}(\mathbb{R})$

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1.3 Properties of the real line $\mathbb{R}$

Definition 1.1. *The set of extended real* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$.
For $x \in \mathbb{R}, x \pm \infty = \pm\infty$ *and*

$$x \cdot \infty = \begin{cases} \infty, & \text{if } x > 0 \\ -\infty, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

But $\infty - \infty$ *is undefined.*

Theorem 1.1 (Representation of open sets in $\mathbb{R}$) *Every nonempty open set* $\mathcal{O}$ *in* $\mathbb{R}$ *can be written as at most countable union of pairwise disjoin open intervals. That is* $\mathcal{O} = \sqcup_{j=1}^{\infty}(a_j, b_j)$ *such that* $(a_i, b_i) \cap (a_j, b_j) = \varnothing$ *for all* $i \neq j$ *(some of the intervals may be empty. If such are ignored, the representation is unique).*

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Let D denote the set of all points $x \in \mathbb{R}$ such that f is not continuous at x, that is $D = \{x \in \mathbb{R} : f(x-) \neq f(x+)\}$. Then D is countable.