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$x = 0$ is the singular point, hence let $y = x^r$, we have that

$$\begin{aligned} & x^2y'' - xy' + y = 0 \\ \implies & r(r-1)x^r - rx^r + x^r = 0 \\ \implies & r^2x^r - 2rx^r + x^r = 0 \\ \implies & x^r(r^2 - 2r + 1) = 0 \\ \implies & r = 1 \end{aligned}$$

Hence, the solution is

$$y = c_1x + c_2x\ln(x)$$

for $x > 0$

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$x = 0$ is the singular point, hence let $y = x^r$, we have that

$$\begin{aligned}
&2x^2y'' - 4xy' + 6y = 0 \\
\implies &2r(r-1)x^r - 4rx^r + 6x^r = 0 \\
\implies &2r^2x^r - 6rx^r + 6x^r = 0 \\
\implies &x^r(2r^2 - 6r + 6) = 0 \\
\implies &r = \frac{6 \pm \sqrt{6^2 - 4 \cdot 2 \cdot 6}}{4} = \frac{3 \pm i\sqrt{3}}{2}
\end{aligned}$$

Hence, the solution is

$$y = c_1 x^{3/2} \cos\left(\frac{\sqrt{3}}{2}\ln(x)\right) + c_2 x^{3/2} \sin\left(\frac{\sqrt{3}}{2}\ln(x)\right)$$

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$x = 0$ is the singular point, hence let $y = x^r$, we have that

$$\begin{aligned} & 4x^2y'' + 8xy' + 17y = 0 \\ \implies & 4r(r-1)x^r + 8rx^r + 17x^r = 0 \\ \implies & 4r^2x^r + 4rx^r + 17x^r = 0 \\ \implies & x^r(4r^2 + 4r + 17) = 0 \\ \implies & r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4 \cdot 17}}{4} = -1 \pm 4i \end{aligned}$$

Hence, the solution is

$$y = c_1 \frac{1}{x} \cos(4\ln(x)) + c_2 \frac{1}{x} \sin(4\ln(x))$$

Then we know that

$$y(1) = c_1 = 2$$

and

$$y'(1) = c_1 \frac{-1}{x^2} \cos(4\ln(x)) + c_2 \frac{1}{x} \cos(4\ln(x)) \cdot \frac{4}{x} \bigg|_{x=1} = -c_1 + 4c_2 = -3$$

Solving the equations we have that

$$y = \frac{2}{x} \cos(4\ln(x)) - \frac{1}{4x} \sin(4\ln(x))$$

As $x \to 0$, $y(x)$ fluctuates around 0 where the oscillation diverges to 0.

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we have that

$$\lim_{x\to 0} x\frac{1}{2x} = \frac{1}{2}$$

and

$$\lim_{x\to 0} x\frac{x}{2x} = 0$$

Hence, $x = 0$ is a regular singular point. Assume $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$. Then

$$\begin{aligned}
&2xy'' + y' + xy = 0 \\
\implies &\sum_{n=0}^{\infty} 2a_n(r+n)(r+n-1)x^{n+r-1} + a_n(r+n)x^{r+n-1} + a_n x^{r+n+1} = 0 \\
\implies &\sum_{n=0}^{\infty} 2a_n(r+n)(r+n-1)x^{n+r-1} + a_n(r+n)x^{r+n-1} + \sum_{n=2}^{\infty} a_{n-2}x^{r+n-1} = 0
\end{aligned}$$

Therefore,

$$2a_0 r(r-1)x^{r-1} + a_0 r x^{r-1} + 2a_1(r+1)r x^r + a_1(r+1)x^r = 0$$

which can be simplify to

$$a_0 x^{r-1}(2r^2 - r) + a_1 x^r(2r^2 + 3r + 1) = 0$$

Hence, the indicial equation is

$$2r^2 - r = 0$$

Thus $r = 1/2$ is the larger root and the smaller root is 0. Hence, $a_1 = 0$ and for $n \geq 2$, the recurrence relation is

$$2a_n(r+n)(r+n-1) + a_n(r+n) = a_n(1+2n)n = -a_{n-2}$$

which means that

$$a_n = \frac{-1}{n(2n+1)}a_{n-2}$$

Thus, for every n,

$$a_{2n} = a_0 \frac{(-1)^n}{\prod_{i=1}^{n} 2i(4i+1)}$$

and $a_{2n+1} = 0$ so we can write the power series based on a_n.
If we choose $r = 0$, we have that $a_1 = 0$ and the recurrente relation is

$$2a_n(r+n)(r+n-1) + a_n(r+n) = a_n n(2n-1) = -a_{n-2}$$

which means that

$$a_n = \frac{-1}{n(2n-1)}a_{n-2}$$

Thus, for every n,

$$a_{2n} = a_0 \frac{(-1)^n}{\prod_{i=1}^{n} 2i(4i-1)}$$

and $a_{2n+1} = 0$ so we can write the power series based on a_n.