

# 1.

## a.

Since  $\mathcal{B}(\mathbb{R})$  is the  $\sigma$ -algebra generated by all open sets in  $\mathbb{R}$ , and  $Y$  is contained only open sets. We have that  $\sigma(Y) \subseteq \mathcal{B}(\mathbb{R})$ . Thus to prove that  $\sigma(Y) = \mathbb{R}$ , we need to prove that all open sets in  $\mathbb{R} \in Y$ . Since every open sets in  $\mathbb{R}$  can be rewritten as at most countable of pairwise disjoint open intervals and there are 3 types of open intervals in  $\mathbb{R}$ . For  $a, b \in \mathbb{R}$

- $(a, \infty)$   
Choose any  $x \in \mathbb{R}$  such that  $x > a$ , since  $D$  is dense, there is infinitely countable number  $d \in \mathbb{D}$  such that  $a < d < x$ . Thus there is a stricly increasing  $(d_n)_{n=1}^\infty$  such that  $\lim d_n = a$ . Therefore,  $\cup_{n=1}^\infty (d, \infty) = (a, \infty) \in \sigma(Y)$ .
- $(-\infty, a)$   
Choose any  $x \in \mathbb{R}$  such that  $x < a$ , since  $D$  is dense, there is infinitely countable number  $d \in \mathbb{D}$  such that  $a > d > x$ . Thus there is a stricly decreasing  $(d_n)_{n=1}^\infty$  such that  $\lim d_n = a$ . Therefore,  $\cup_{n=1}^\infty (d, \infty)^c = \cup_{n=1}^\infty (-\infty, d] = (-\infty, a) \in \sigma(Y)$ .
- $(a, b)$   
We have that  $(a, b) = (a, \infty) \cap (-\infty, b) \in \sigma(Y)$
- $(-\infty, \infty) = \mathbb{R}$   
Choosing  $a < b$ , we have that  $(-\infty, \infty) = (-\infty, b) \cup (a, \infty) \in \sigma(Y)$

Thus, every open sets in  $\mathbb{R}$  is in  $\sigma(Y)$ , thus  $\sigma(Y) = \mathcal{B}(\mathbb{R})$ .

## b.

For any interval  $(a, \infty) \in Y$ , notice that since  $D$  is dense, we can construct a strictly decreasing  $(a_n)_{n=1}^\infty \in D$  and strictly increasing sequence  $(b_n)_{n=1}^\infty \in D$  such that  $\lim(a_n, b_n) = (a, \infty)$ . And thus for any  $a \in D$ , we have that  $(a, \infty) = \cup_{n=1}^\infty (a_n, b_n) \in \sigma(Z)$ . Therefore,  $\sigma(Y) \subseteq \sigma(Z)$ . For any  $a, b \in D$ , we have that  $(-\infty, b) \cap (-\infty, a)^c = [a, b) \in \sigma(Y)$ . Thus  $\sigma(Z) \subseteq \sigma(Y)$  and  $\sigma(Y) = \sigma(Z) = \mathcal{B}(\mathbb{R})$ .

## c.

From definition, for any  $E \in \mathcal{B}(\overline{\mathbb{R}})$ ,  $E$  can be rewritten as  $E = E' \cup B'$  or  $E' \cup B'^c$  where  $E' \in \mathcal{B}(\mathbb{R})$  and  $B' \in \{(a, \infty], [-\infty, a), [-\infty, a) \cup (b, \infty]\}$ . Thus we have that  $E \setminus \{-\infty, \infty\} = (E' \setminus \{-\infty, \infty\}) \cup (B' \setminus \{-\infty, \infty\}) = E' \cup B''$ , where  $B'' \in \{(a, \infty), (-\infty, a), (-\infty, a) \cup (b, \infty)\} \subset \mathcal{B}(\mathbb{R})$  thus  $E \setminus \{-\infty, \infty\} \in \mathcal{B}(\mathbb{R})$ .

**2.**

**a.**

For any open sets  $G = (a, b) \in \mathbb{R}$ , let

$$a_n = \begin{cases} a + \frac{1}{n}, & \text{if } a \neq -\infty \\ C - n, & \text{if } a = -\infty \end{cases}$$

and similarly

$$b_n = \begin{cases} b - \frac{1}{n}, & \text{if } b \neq \infty \\ C + n, & \text{if } b = \infty \end{cases}$$

where  $C$  is some constant in  $\mathbb{R}$ . Then we have that

$$(a, b) = \cup_{n=1}^{\infty} [a_n, b_n]$$

**b.**

For each  $x \in \mathbb{R} \setminus E$ ,  $f$  is continuous at  $x$  thus for all  $n \in \mathbb{N}$ , there is  $\delta_{x,n}$  such that  $|f(y) - f(x)| < \frac{1}{n}$  for all  $y \in (x - \delta_{x,n}, x + \delta_{x,n})$ . Thus let  $\mathcal{O}_n = \cup_{x \in E^c} (x - \delta_{x,n}, x + \delta_{x,n})$ . We can see that

$$\bigcap_{n=1}^{\infty} \mathcal{O}_n = \bigcap_{n=1}^{\infty} \bigcup_{x \in E^c} (x - \delta_{x,n}, x + \delta_{x,n}) = \bigcup_{x \in E^c} \underbrace{\bigcap_{n=1}^{\infty} (x - \delta_{x,n}, x + \delta_{x,n})}_x = E^c$$

And therefore,  $E^c$  is a  $G_\delta$  set and thus  $E$  is a  $F_\sigma$  set.

### 3.

#### a.

If  $\mu$  is  $\sigma$ -finite, then there exists some  $X_n \in \mathcal{M}$  such that  $X = \bigcup_{n=1}^{\infty} X_n$ ,  $X_n \subseteq X_{n+1}$  and  $\mu(X_n) < \infty$ . Thus for each  $E \in \mathcal{M}$  with  $\mu(E) = \infty$ , there exists  $N \in \mathbb{N}$  such that

$$\mu(X_N \cap E) > 0$$

as else  $\mu(E \cap \bigcup_{j=1}^n X_j) = \mu(E \cap X_N) = 0$ . But since  $\mu(X_N \cap E) \leq \mu(X_N) < \infty$ .  $X_N$  satisfies  $X_N \in \mathcal{M}$ ,  $X_N \subseteq E$ ,  $0 < \mu(X_N) < \infty$ .

#### b.

Let

$$S = \{F \in \mathcal{M} : F \subseteq E, \mu(F) < \infty\}$$

Then we know that  $\sup_{F \in S} \mu(F)$  must exist. Suppose it is less than  $\infty$ , that is  $\sup_{F \in S} \mu(F) = L$  for some  $L \in \mathbb{R}$ , then we can choose  $(F_n)_{n=1}^{\infty}$  such that  $\lim \mu(F_n) = L$ . Then, we have that  $\mu(\bigcup_{F \in S} F) = \mu(\bigcup_{n=1}^{\infty} F_n) = L$ . Therefore,  $\mu(E \setminus \bigcup_{F \in S} F) = \infty$ , thus there exists  $F' \subset E \setminus F$ , so that  $0 < \mu(F') < \infty$ . But  $F \cup F' \subset E$  and  $\infty > \mu(F \cup F') = \mu(F) + \mu(F') > L$  which is a contradiction and therefore  $\sup_{F \in S} \mu(F) = \infty$  and there is some set  $F \subseteq E$  such that  $C < \mu(F) < \infty$ .

#### c.

First,

$$u_0(\emptyset) = \sup\{\mu(F) : F \subseteq \emptyset, \mu(F) < \infty\} = 0$$

as the only subset of empty set is itself. If  $E_j \in \mathcal{M}$  for all  $j \in \mathbb{N}$  and  $E_j$  are pairwise disjoint then in case where there is  $j$  such that  $\mu(E_j) = \infty$  then from part b, we have

$$\sum_{j=1}^{\infty} \mu_0(E_j) = \infty = \mu_0(\bigsqcup_{j=1}^{\infty} E_j)$$

If  $\mu_0(E_j) = L_j$  are finite for every  $j \in \mathbb{N}$  then we can choose  $(F_{j,n})_{n=1}^{\infty} \subseteq E_j$  such that  $\lim \mu(F_{j,n}) = L_j$  and since  $E_j$  are pairwise disjoint,  $F_{j_1,n} \cap F_{j_2,n} = \emptyset$  for  $j_1 \neq j_2$ . Therefore, there is a sequence  $F_n = \bigsqcup_{j=1}^{\infty} F_{j,n} \subset \bigsqcup_{j=1}^{\infty} E_j$  such that  $\lim \mu_0(F_n) = \sum_{j=1}^{\infty} L_j$  thus  $\mu_0(\bigsqcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} L_j = \sum_{j=1}^{\infty} \mu_0(E_j)$ . If  $\mu_0(E) = \infty$  for some  $E \in \mathcal{M}$ , then for any constant  $C$ , there exists  $F \subseteq E$  such that  $\mu(F) < \infty$ . Now if for every  $F \subseteq E$  satisfying  $\mu(F) < \infty$ ,  $\mu(F) = 0$  then  $\mu_0(E) = 0$  which is a contradiction. Thus there exists some  $F$  such that  $F \subseteq E$  and  $0 < \mu(F) < \infty$ .

4.

a.

- $\emptyset \in \mathcal{E}$ . Choosing  $a = b$ , we have that  $(a, b] \cap \mathbb{Q} = \emptyset \in \mathcal{E}$ .
- If  $E = (a_1, b_1] \cap \mathbb{Q}, F = (a_2, b_2] \cap \mathbb{Q} \in \mathcal{E}$  then in case  $(a_1, b_1] \cap (a_2, b_2] = \emptyset, E \cap F = \emptyset \in \mathcal{E}$ . In case  $(a_1, b_1] \cap (a_2, b_2] \neq \emptyset$  then there exists  $a_3, b_3$  such that  $(a_1, b_1] \cap (a_2, b_2] = (a_3, b_3]$  thus  $E \cap F = (a_3, b_3] \cap \mathbb{Q} \in \mathcal{E}$ .
- If  $E = (a, b] \cap \mathbb{Q} \in \mathcal{E}$  where  $a < b$ , then  $E^c = ((-\infty, a] \cup (b, \infty]) \cap \mathbb{Q} = \underbrace{((-\infty, a] \cap \mathbb{Q})}_{\in \mathcal{E}} \cup \underbrace{((b, \infty] \cap \mathbb{Q})}_{\in \mathcal{E}}$ , which are disjoint as  $a < b$

b.

From definition,  $\mathcal{A} \subseteq \mathbb{Q}$  thus  $\sigma(\mathcal{A}) \subseteq \mathcal{P}(\mathbb{Q})$ .

For any  $E \subseteq \mathcal{P}(\mathbb{Q})$ , we can write  $E = \{x_1, x_2, \dots\}$  as rationals are countable. Then for any  $x_j$ , we can define

$$E_{j,n} = \left(x_1 - \frac{1}{n}, x_1\right] \cap \mathbb{Q} \in \mathcal{A} \subseteq \sigma(\mathcal{A})$$

so that

$$\bigcap_{n=1}^{\infty} E_{j,n} = x_j \cap \mathbb{Q} = x_j$$

Thus

$$\bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} E_{j,n} = E$$

c.

We have that  $\mu_0(\emptyset) = 0$  and for  $E_j \in \mathcal{A}$ , where  $E_j$  are pairwise disjoint, there is 2 cases

- if there is non-empty  $E_k$  then  $\sqcup_{j=1}^n E_j \neq \emptyset$  and thus

$$\mu_0(\sqcup_{j=1}^{\infty} E_j) = \infty = \sum_{j=1}^{\infty} \mu_0(E_j) = \mu_0(E_k) + \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \mu_0(E_j) = \infty$$

- if all of them are empty, then simply

$$\mu_0(\sqcup_{j=1}^{\infty} E_j) = \mu_0(\emptyset) = 0 = \sum_{j=1}^n \mu_0(E_j)$$

**5.**

**a.**

Since  $Q \cap [0, 1] \subseteq \cup_{j=1}^{\infty} R_j^o$ , taking closure, we have  $[0, 1] \subseteq \cup_{j=1}^{\infty} \overline{R_j^o}$ . Thus

$$\sum_{j=1}^{\infty} |R_j^o| = \sum_{j=1}^{\infty} |\overline{R_j^o}| \geq \left| \cup_{j=1}^{\infty} \overline{R_j^o} \right| \geq 1$$

**b.**

Since  $m(E_j) = 1$ , we have that  $m(E_j^c) = 0$  and thus

$$m(\cap_{j=1}^{\infty} E_j) = 1 - m(\cup_{j=1}^{\infty} E_j^c) \geq 1 - \sum_{j=1}^{\infty} m(E_j^c) = 1$$

But since  $\cap_{j=1}^{\infty} E_j \subseteq [0, 1]$  thus  $m(\cap_{j=1}^{\infty} E_j) \leq 1$ . Therefore,  $m(\cap_{j=1}^{\infty} E_j) = 1$ .

**c.**

Suppose that  $m(\cap_{j=1}^n A_n) = 0$ , then  $m(\cup_{j=1}^n A_j^c) = 1$ . However, we have that  $m(\cup_{j=1}^n A_n^c) \leq \sum_{j=1}^n m(A_j^c)$ . Now we know that

$$\sum_{j=1}^n m(A_j^c) + \sum_{j=1}^n m(A_j) > n - 1 + 1 = n$$

But

$$\sum_{j=1}^n m(A_j^c) + \sum_{j=1}^n m(A_j) = \sum_{j=1}^n m(A_j^c) + m(A_j) = \sum_{j=1}^n m([0, 1]) = n$$