1.

(i) the set $A = \{(x, y) \in \mathbb{R}^2 : x > y\}$ is convex:

$$\begin{aligned}\forall m = (x_1, y_1), n = (x_2, y_2) \in A : tm + (1-t)n &= (tx_1, ty_1) + (x_2 - tx_2, y_2 - ty_2) \\ &= (tx_1 + x_2 - tx_2, ty_1 + y_2 - ty_2)\end{aligned}$$

$$\forall t \in [0,1] : tx_1 + x_2 - tx_2 - ty_1 - y_2 + ty_2 = \underbrace{t}_{>0}\underbrace{(x_1 - y_1)}_{>0} + \underbrace{(1-t)}_{>0}\underbrace{(x_2 - y_2)}_{>0} > 0$$

$$\implies x_1 + x_2 - tx_2 > ty_1 + y_2 - ty_2 \implies tm + (1-t)n \in A$$

(ii) the set $B = \{x \in \mathbb{R}^N : \|x\| > 2\}$ is not convex:
Consider $x = (3, 0, \dots, 0), y = (-3, 0, \dots, 0) \in B$ as $\|x\| = \|y\| = 3$
If $t = \frac{1}{2} \in [0, 1]$, then

$$\begin{aligned}\|tx + (1-t)y\| &= \left\| \left(\frac{1}{2} \cdot 3, 0, \dots, 0\right) + \left(\frac{1}{2} \cdot (-3), 0, \dots, 0\right) \right\| \\ &= \|(0, 0, \dots, 0)\| = 0.\end{aligned}$$

Therefore, $tx + (1-t)y \notin B$.
(iii) the set $C = \mathbb{R} \backslash \mathbb{Q}$ is not convex: Consider $\pi, -\pi \in C$
If $t = \frac{1}{2}$, then $t\pi + (1-t)\pi = 0 \notin C$.
(iv) the set $D = \{(x, y, z) \in \mathbb{R}^3 : x + y + z \geq 2022\}$ is convex:

$$\forall m = (x_1, y_1, z_1), n = (x_2, y_2, z_2) \in A :$$

$$\begin{aligned}\forall t \in [0,1] : tm + (1-t)n &= (tx_1, ty_1, tz_2) + (x_2 - tx_2, y_2 - ty_2, z_2 - ty_2) \\ &= (tx_1 + x_2 - tx_2, ty_1 + y_2 - ty_2, tz_1 + z_2 - tz_2) \\ &= tx_1 + x_2 - tx_2 + ty_1 + y_2 - ty_2 + tz_1 + z_2 - tz_2 \\ &= \underbrace{t}_{>0}\underbrace{(x_1 + y_1 + z_1)}_{\geq 2022} + (1-t)\underbrace{(x_2 + y_2 + z_2)}_{>0} \\ &\geq t \cdot 2022 + (1-t) \cdot 2022 = 2022 \\ &\implies tm + (1-t)n \in D\end{aligned}$$

2.

Let $\mathcal{C} = \{C_i | i \in I\}$ be the family of convex sets, then

$$\forall x, y \in \bigcap_{C \in \mathcal{C}} C : x, y \in C_i \forall i \in I$$

and since C_i is convex for all $i \in I$, which means that

$$\forall t \in [0, 1] : tx + (1 - t)y \in C_i \, \forall i \in I$$

Therefore,

$$tx + (1 - t)y \in \bigcap_{C \in \mathcal{C}} C$$

and hence $\bigcap_{C \in \mathcal{C}} C$ is also convex.
However, $\bigcup_{C \in \mathcal{C}} C$ is not necessarily convex:
Consider the two convex sets $B_1[1, 0]$ and $B_1[-1, 0] \in \mathbb{R}^2$.
The point $x = (1, -1) \in B_1[1, 0]$ since $\|(1, -1) - (1, 0)\| = 1$
and the point $y = (-1, -1) \in B_1[-1, 0]$ since $\|(-1, -1) - (-1, 0)\| = 1$ but given $t = \frac{1}{2}$, $tx + (1 - t)y = \left(\frac{1}{2}, -\frac{1}{2}\right) + \left(-\frac{1}{2}, -\frac{1}{2}\right) = (0, -1)$
$\|(1, 0) - (0, -1)\| = \sqrt{2}$ and $\|(-1, 0) - (0, -1)\| = \sqrt{2}$. Therefore $tx + (1-t)y \notin B_1[1, 0] \cup B_1[-1, 0]$

3.

Consider an open interval $\forall n \in \mathbb{Z} : (n, n+1)$ is open, then $\bigcup_{n\in\mathbb{Z}}(n, n+1)$ is also open. We have

$$\forall r \in \mathbb{R} : \exists n \in \mathbb{Z}, \exists 0 \leq m < 1 \in \mathbb{R} : r = n + m$$

If $m \neq 0$ then $r \in (n, n+1)$
else if $m = 0$ then $r \in \mathbb{Z}$ as $n = r$.
Therefore, $R \setminus \bigcup_{n\in\mathbb{Z}}(n, n+1) = \mathbb{Z}$ and consequently, $\mathbb{Z}$ is closed in $\mathbb{R}$.
Consider $0 \in \mathbb{Z}, \forall \epsilon > 0 : \exists n : \frac{1}{n} < \epsilon \wedge \frac{1}{n} < 1$ and therefore, $\frac{1}{n} \in B_\epsilon(0)$ which means that $\forall \epsilon > 0 : B_\epsilon 0 \not\subset \mathbb{Z}$ and hence $\mathbb{Z}$ is not open.
Suppose $\mathbb{Q}$ is open, then given $q \in \mathbb{Q} : \exists \epsilon > 0 : (q - \epsilon, q + \epsilon) \subset \mathbb{Q}$.
But $\forall \epsilon > 0$, there exists an n large enough so that $\dfrac{\sqrt{2}}{n} < \epsilon$ and therefore $q + \dfrac{\sqrt{2}}{n}$ is irrational and is an element of $(q - \epsilon, q + \epsilon)$ which is a contradiction.
Therefore $\mathbb{Q}$ is not open.
Suppose $\mathbb{Q}$ is closed, then $\mathbb{R}\setminus\mathbb{Q}$ is open, which means for a given $x \in \mathbb{R}\setminus\mathbb{Q} : \exists \epsilon : (x - \epsilon, x + \epsilon) \subset \mathbb{R}\setminus\mathbb{Q}$.
But for all open interval, there exists a rational number in that interval, which means that there is $y \in \mathbb{Q} \wedge y \in (x - \epsilon, x + \epsilon)$ which is a contradiction.
Therefore $\mathbb{Q}$ is not closed.

4.

$\forall t \in S + U : t = x + y$ where $x \in S \wedge y \in U$.
Since $y \in U : \exists \epsilon > 0 : B_\epsilon(y) \in U$
$\forall t' \in B_\epsilon(t),$ let $t' = t + d$, then

$$\|t' - t\| = \|d\| < \epsilon$$

Let $y' = y + d$, then

$$\|y' - y\| = \|d\| < \epsilon$$

which means that $y' \in U$ and $t' = x + y'$ where $x \in S \wedge y' \in U$.
As a result, $\forall t' \in B_\epsilon(t) : t' \in S + U$ and hence $B_\epsilon(t) \in S + U$.
Therefore, $S + U$ is open

5.

Given $x \in \mathbb{R}^N$ is a cluser point of x, if there is a neighborhood of x contains finite points of S, which means that

$$\exists \epsilon > 0 : B_\epsilon(x) \cap S = \{x_1, x_2, \ldots, x_n\}$$

Since the set has finite element, the set $T = \{\|x - x_i\| | i \in \{1, 2, \ldots, n\}\}$ also has finite element and therefore has a minimum which we denote δ. Then

$$(B_\delta(x) \cap S) \backslash \{x\} = \emptyset$$

which means that x is not a cluster point and therefore leads to a contradiction. Therefore, the neighborhood of x must contain infinite elements.
If each neighborhood of x contains an infinte number of points in S then x is obviously a cluster point

6.

a.

(i)

$$\|x\|_1 = \underbrace{|x_1|}_{\geq 0} + \underbrace{|x_2|}_{\geq 0} + \ldots + \underbrace{|x_N|}_{\geq 0} \geq 0$$

$$\|x\|_\infty = \max\{\underbrace{|x_1|}_{\geq 0}, \underbrace{|x_2|}_{\geq 0}, \ldots, \underbrace{|x_N|}_{\geq 0}\} \geq 0$$

If $x \neq 0$, then $\exists i : x_i \neq 0$, which means that

$$\|x\|_1 = \underbrace{|x_1|}_{\geq 0} + \underbrace{|x_2|}_{\geq 0} + \ldots + \underbrace{|x_i|} + \ldots + \underbrace{|x_N|}_{\geq 0} \geq |x_i| > 0$$

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_N|\} \geq |x_i| > 0$$

Therefore, $\|x\|_1 = 0 \implies x = 0$ and $\|x\|_\infty = 0 \implies x = 0$
It is also obvious that if $x = 0$, then

$$\|x\|_1 = \underbrace{|x_1|}_{=0} + \underbrace{|x_2|}_{=0} + \ldots + \underbrace{|x_N|}_{=0} = 0$$

and

$$\|x\|_\infty = \max\{\underbrace{|x_1|}_{=0}, \underbrace{|x_2|}_{=0}, \ldots, \underbrace{|x_N|}_{=0}\} = 0$$

(ii)

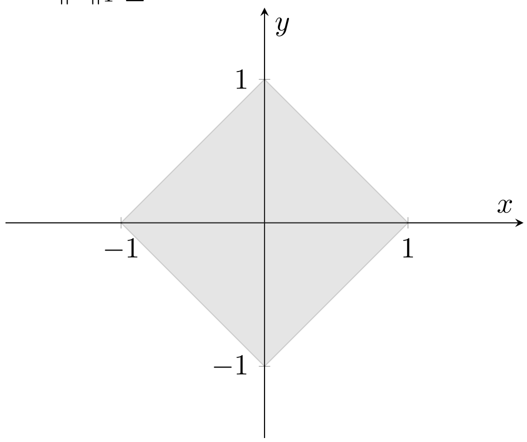
$$\begin{aligned}
\|\lambda x\|_1 &= |\lambda x_1| + |\lambda x_2| + \ldots + |\lambda x_N| \\
&= |\lambda||x_1| + |\lambda||x_1| + \ldots + |\lambda||x_1| \\
&= |\lambda|(|x_1| + |x_2| + \ldots + |x_N|) = |\lambda|\|x\|_1 \\
\|\lambda x\|_\infty &= \max\{|\lambda x_1|, |\lambda x_2|, \ldots, |\lambda x_N|\} \\
&= \max\{|\lambda||x_1|, |\lambda||x_1|, \ldots, |\lambda||x_1|\} \\
&= |\lambda| \max\{|x_1|, |x_2|, \ldots, |x_N|\} \\
&= |\lambda|\|x\|_\infty
\end{aligned}$$

(iii)

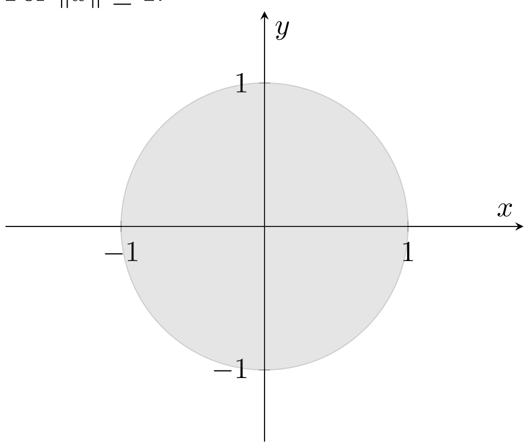
$$\begin{aligned}
\|x + y\|_1 &= |x_1 + y_1| + |x_2 + y_2| + \ldots + |x_N + y_N| \\
&\leq |x_1| + |y_1| + |x_2| + |y_2| + \ldots + |x_N| + |y_N| \\
&= \|x\|_1 + \|y\|_1 \\
\|x + y\|_\infty &= \max\{|x_1 + y_1| + |x_2 + y_2| + \ldots + |x_N + y_N|\} \\
&\leq \max\{|x_1| + |y_1|, |x_2| + |y_2|, \ldots, |x_N| + |y_N|\} \\
&\leq \max\{|x_1|, |x_2|, \ldots, |x_N|\} + \max\{|y_1|, |y_2|, \ldots, |y_N|\} \\
&= \|x\|_\infty + \|y\|_\infty
\end{aligned}$$

b.

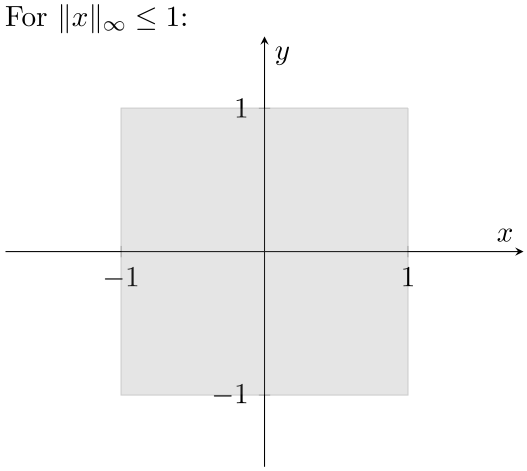
For $\|x\|_1 \leq 1$:



For $\|x\| \leq 1$:



For $\|x\|_\infty \leq 1$:



c.

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_N|$$

$$\begin{aligned}
\sqrt{N} \cdot \|x\| &= \sqrt{\sum_{i=1}^{N} 1^2} \cdot \|x\| \\
&= \|(1, 1, \ldots, 1)\| \cdot \|x\| \\
&\geq \sum_{i=1}^{N} |x_i \cdot 1| \\
&= \|x\|_1
\end{aligned}$$

Let $|x_i|$ be the maximum element of $\{|x_1|, |x_2|, \ldots, |x_N|\}$, then

$$\begin{aligned}
\sqrt{N} \cdot \|x\| &= \sqrt{N} \cdot \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2} \\
&\leq \sqrt{N} \cdot \sqrt{x_i^2 + x_i^2 + \ldots + x_i^2} \\
&= \sqrt{N} \cdot \sqrt{N \cdot x_i^2} = N \cdot |x_i| = N\|x\|_\infty
\end{aligned}$$