1.

a.

Since $y > \theta$, we assume $z > \theta$ and hence

$$F_{Y_{(1)}}(z) = P(Y_{(1)} < z) = 1 - P(Y_{(1)} > z) = 1 - (P(Y > z))^n$$

$$= 1 - \left(\int_z^\infty \frac{2\theta^2}{y^3} dy \right)^n$$

$$= 1 - \left(-\frac{\theta^2}{y^2} \Big|_{y=z}^\infty \right)^n$$

$$= 1 - \left(\frac{\theta^2}{z^2} \right)^n$$

Hence,

$$f_{Y_1}(y) = -n \cdot \left(\frac{\theta^2}{y^2} \right)^{n-1} \cdot \left(-\frac{2\theta^2}{y^3} \right) = \frac{2n}{y} \left(\frac{\theta^2}{y^2} \right)^n$$

We have

$$y_{(1)} = h^{-1}(u) = \theta \cdot u \implies \frac{d}{du} h^{-1}(u) = \theta$$

Therefore,

$$f_U(u) = f_{Y_{(1)}}(\theta \cdot u) \cdot \theta = \frac{2n}{\theta \cdot u} \left(\frac{1}{u^2} \right)^n \cdot \theta = \frac{2n}{u^{2n+1}}$$

Hence, U is a pivotal quantity.

b.

We have for $1 < u$,

$$F_U(u) = 1 - \frac{1}{u^{2n}}$$

Hence,

$$P(U < a) = F_U(a) = 1 - \frac{1}{a^{2n}} \implies a = \sqrt[2n]{\frac{2}{2-\alpha}}$$

$$P(U > b) = 1 - F_U(b) = 1 - \left(1 - \frac{1}{b^{2n}} \right) = \frac{1}{b^{2n}} \implies b = \sqrt[2n]{\frac{2}{\alpha}}$$

$$P(a < U < b) = P \left(a < \frac{Y_{(1)}}{\theta} < b \right) = P \left(\frac{Y_{(1)}}{b} < \theta < \frac{Y_{(1)}}{a} \right) = 1 - \alpha$$

Hence, the $(1-\alpha)100\%$ CI for θ is $\left(\frac{Y_{(1)}}{\sqrt[2n]{\frac{2}{\alpha}}} < \theta < \frac{Y_{(1)}}{\sqrt[2n]{\frac{2}{2-\alpha}}} \right)$

c.

The 99% CI for θ is

$$\left(\frac{10.55}{\sqrt[50]{\frac{2}{2-0.01}}} < \theta < \frac{10.55}{\sqrt[50]{\frac{2}{0.01}}} \right) = (9.48925, 10.5489)$$

d.

Since p-value $= 0.095$, $\alpha = 0.01$ for the two-sided confidence interval which means that

$$\frac{y_{(1)}}{\sqrt[40]{\frac{2}{2-0.01}}} = 5 \implies y_{(1)} \simeq 5$$

e.

2.

a.

$$F_{Y_{(n)}}(z) = P(Y_{(n)} < z) = P(Y_{(n)} < z) = (P(Y < z))^n$$
$$= \left(\int_0^z \frac{2y}{\theta^2} dy \right)^n$$
$$= \left(\frac{y^2}{\theta^2} \Big|_{y=0}^z \right)^n$$
$$= \left(\frac{z^2}{\theta^2} \right)^n$$

Hence,

$$f_{Y_1}(y) = \frac{2n \cdot z^{2n-1}}{\theta^{2n}}$$

We have

$$y_{(1)} = h^{-1}(u) = \theta \cdot u \implies \frac{d}{du} h^{-1}(u) = \theta$$

Therefore,

$$f_U(u) = f_{Y_{(n)}}(\theta \cdot u) \cdot \theta = \frac{2n \cdot (\theta \cdot u)^{2n-1}}{\theta^{2n}} \cdot \theta = 2n \cdot u^{2n-1}$$

Hence, U is a pivotal quantity.

b.

We have for $0 < u < 1$,

$$F_U(u) = u^{2n}$$

Hence,

$$P(U < a) = F_U(a) = a^{2n} \implies a = \sqrt[2n]{\frac{\alpha}{2}}$$

$$P(U > b) = 1 - F_U(b) = 1 - (b^{2n}) \implies b = \sqrt[2n]{\frac{2-\alpha}{2}}$$

$$P(a < U < b) = P\left(a < \frac{Y_{(n)}}{\theta} < b \right) = P\left(\frac{Y_{(n)}}{b} < \theta < \frac{Y_{(n)}}{a} \right) = 1 - \alpha$$

Hence, the $(1-\alpha)100\%$ CI for θ is $\left(\frac{Y_{(n)}}{\sqrt[2n]{\frac{2-\alpha}{2}}}, \frac{Y_{(n)}}{\sqrt[2n]{\frac{\alpha}{2}}} \right)$

c.

The 95% CI for θ is

$$\left(\frac{6.5}{\sqrt[30]{\frac{2-0.05}{2}}}, \frac{6.5}{\sqrt[30]{\frac{0.05}{2}}} \right) = (6.50549, 7.35047)$$

d.

3.

We know that $\overline{Y} \sim \text{Gamma}(cn, \beta/n)$, which means that

$$f_{\overline{Y}}(y) = \frac{y^{cn-1}e^{-yn/\beta}}{\Gamma(cn)(\beta/n)^{cn}} = \frac{n}{\beta\Gamma(cn)} \cdot \left(\frac{yn}{\beta}\right)^{cn-1} e^{-yn/\beta}$$

Consider $T = \frac{\overline{Y}}{\beta}$, then as $\overline{y} = h^{-1}(t) = \beta \cdot t \implies \frac{d}{dt}h^{-1}(t) = \beta$

$$f_T(t) = f_{\overline{Y}}(\beta \cdot t) \cdot \beta = \frac{n}{\beta\Gamma(cn)} \cdot (tn)^{cn-1} \cdot e^{-tn} \cdot \beta = \frac{n(tn)^{cn-1}}{\Gamma(cn)e^{tn}}$$

which is independent of β hence is a pivotal quantity.
We have that $T = \frac{\overline{Y}}{\beta} \sim \text{Gamma}(cn, 1/n) \sim \frac{1}{2n} \text{Gamma}(cn,2) \sim \frac{1}{2n}\chi^2_{2cn}$
Hence, $U = \frac{2n\overline{Y}}{\beta} \sim \chi^2_{2cn}$ is also a pivotal quantity.

$$1 - \alpha = P(a < U < b)$$
$$= P\left(a < \frac{2n\overline{Y}}{\beta} < b\right)$$
$$= P\left(\frac{2n\overline{Y}}{b} < \beta < \frac{2n\overline{Y}}{a}\right)$$

Hence, a $(1-\alpha)100\%$ CI is

$$\left(\frac{2n\overline{Y}}{\chi^{2*}_{2cn,\alpha/2}}, \frac{2n\overline{Y}}{\chi^{2*}_{2cn,1-\alpha/2}}\right)$$

b.

$$P(-z^*_{\alpha/2} < Z < -z^*_{\alpha/2})$$

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