1 Preliminary

1.1 Basic on sets

1.2 Countable sets

- Bernstein's theorem: if $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(Y) \leq \text{card}(X)$ then $\text{card}(X) = \text{card}(Y)$
- $\text{card}(\text{P}(\text{N})) = \text{card}(\text{R})$

1.3 Properties of the real line $\mathbb{R}$

Definition 1.1. *The set of extended real* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$.
For $x \in \mathbb{R}, x \pm \infty = \pm\infty$ *and*

$$x \cdot \infty = \begin{cases} \infty, & \text{if } x > 0 \\ -\infty, & \text{if } x < 0 \text{ But} \\ 0, & \text{if } x = 0 \end{cases}$$

$\infty - \infty$ *is undefined.*

Theorem 1.1 (Representation of open sets in $\mathbb{R}$) *Every nonempty open set $\mathcal{O}$ in $\mathbb{R}$ can be written as at most countable union of pairwise disjoin open intervals. That is $\mathcal{O} = \sqcup_{j=1}^{\infty}(a_j, b_j)$ such that $(a_i, b_i) \cap (a_j, b_j) = \varnothing$ for all $i \neq j$ (some of the intervals may be empty. If such are ignored, the representation is unique).*

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Let D denote the set of all points $x \in \mathbb{R}$ such that f is not continuous at x, that is $D = \{x \in \mathbb{R} : f(x-) \neq f(x+)$. Then D is countable.