Fall 2022, Math 328, Homework 5

Due: End of day on 2021-11-28

1 10 points

Let G be a group and let X and Y be two sets. Suppose that G acts on X and Y. A function $f : X \to Y$ is called G-*equivariant* provided that for all $g \in G$ and $x \in X$, one has $f(g \cdot x) = g \cdot f(x)$. A G-equivariant *isomorphism* between X and Y is a G-equivariant function $f : X \to Y$ such that there exists a G-equivariant function $g : Y \to X$ where $f \circ g = \mathbf{1}_Y$ and $g \circ f = \mathbf{1}_X$.

1. Show that a G-equivariant function $f : X \to Y$ is bijective if and only if it is an isomorphism.

2. Let G act on X, and let $x \in X$ be given. Prove that there is a unique action of G on $\mathrm{Orb}_G(x)$ such that the inclusion $\mathrm{Orb}_G(x) \hookrightarrow X$ is G-equivariant.

3. For a subgroup H of G, let G act on G/H by left multiplication. Show that the map $G/\operatorname{Stab}_G(x) \to \mathrm{Orb}_G(x)$, defined by $g \cdot \operatorname{Stab}_G(x) \mapsto g \cdot x$, is a G-equivariant isomorphism.

4. Let H be a subgroup of G. Show that the action of G on G/H is transitive.

5. Suppose that X is nonempty and that the action of G on X is transitive. Show that there is a G-equivariant isomorphism between X and G/H for some subgroup H of G.

6. Let X_i be a collection of sets where each X_i is endowed with a G-action. Show that there is a unique action of G on the disjoint union

$$\coprod_i X_i$$

such that the natural inclusions $X_i \hookrightarrow \coprod_i X_i$ are all G-equivariant.

7. Suppose now that X is any set on which G acts. Show that there is a G-equivariant isomoprhism between X and a disjoint union of the form

$$\coprod_i G/H_i,$$

for some (possibly infinite) collection of subgroups H_i of G. Here, the action of G on the disjoint union is the one induced by item (6) and the left multiplication action of G on G/H_i as in part (3).

2 10 points

Let G be a group acting transitively on a nonempty finite set X, and let H be a normal subgroup of G. Let H act on X via the inclusion $H \hookrightarrow G$, and let $A_1, \ldots, A_r$ be the distinct orbits of H acting on X.

1. Prove that for all $g \in G$ and $i = 1, \ldots, r$, there is some j such that $g \cdot A_i = A_j$. Show that this induces an action of G on $\{A_1, \ldots, A_r\}$, and that this action is transitive. Show that A_i all have the same size.

2. Suppose that $a \in A_1$. Show that $\#A_1 = [H : H \cap \mathrm{Stab}_G(a)]$. Prove that $r = [G : \mathrm{Stab}_G(a) \cdot H]$.

3 10 points

1. Let G be a group and N a normal subgroup of order 2. Show that N is contained in the centre of G.

2. Prove that every nonabelian group of order 6 has a nonnormal subgroup of order 2.

3. Classify all groups of order 6, up-to isomorphism.

4 10 points

1. Find all finite groups (up-to isomorphism) which have exactly two conjugacy classes.

2. Find all finite groups (up-to isomorphism) which have exactly three conjugacy classes.

5 10 points

1. Let G be a group, and suppose that $[G : Z(G)] = n$ with n finite. Prove that every conjugacy class of G has at most n elements.

2. Let G be a finite group and let $g_1, \ldots, g_k$ be representatives of the conjugacy classes of G. Suppose that for all i, j, one has $g_i \cdot g_j = g_j \cdot g_i$. Prove that G is abelian.

6 10 points

Let G be a group, and H a subgroup of G. We say that H is *characteristic* provided that for all automorphisms φ of G, one has $\varphi(H) = H$.

1. Show that any characteristic subgroup of G is normal.

2. Give an explicit example of a normal subgroup of a group which is not characteristic.

3. Suppose that H is a subgroup of G with $\#H = n$, and that H is the unique subgroup of G of order n. Prove that H is characteristic, hence normal.

4. Suppose that H is a subgroup of G such that $[G : H] = n$ and that H is the unique subgroup of G of index n. Prove that H is characteristic, hence normal.