1.

a.

For any constant c_1, c_2 and u_1, u_2, we have that

$$\begin{aligned}
L(c_1u_1 + c_2u_2) &= \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial(c_1u_1 + c_2u_2)}{\partial x}\right] \\
&= \frac{\partial}{\partial x}\left[K_0(x)c_1\frac{\partial u_1}{\partial x} + K_0(x)c_2\frac{\partial u_2}{\partial x}\right] \\
&= c_1\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u_1}{\partial x}\right] + c_2\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u_2}{\partial x}\right] \\
&= c_1L(u_1) + c_2L(u_2)
\end{aligned}$$

b.

Similarly, we have that

$$L(u) = c_1\frac{\partial}{\partial x}\left[K_0(x, c_1u_1 + c_2u_2)\frac{\partial u_1}{\partial x}\right] + c_2\frac{\partial}{\partial x}\left[K_0(x, c_1u_1 + c_2u_2)\frac{\partial u_2}{\partial x}\right]$$

which is different from $c_1L(u_1) + c_2L(u_2)$, thus not a linear operator.

2.

a.

$$L(u_p + c_1u_1 + c_2u_2) = L(u_p) + c_1L(u_1) + c_2L(u_2) = f$$

b.

Since we have that $L(u_{p_1}) = f_1$ and $L(u_{p_2}) = f_2$,

$$L(u_{p_1} + u_{p_2}) = f_1 + f_2$$

Thus $u_{p_1} + u_{p_2}$ is a solution.

3.

a.

$$
\begin{aligned}
u_t(r,t) &= \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \\
R(r)\frac{\partial T(t)}{\partial t} &= \frac{k}{r}\frac{\partial}{\partial r}(rR'(r)T(t)) \\
\frac{T'(t)}{kT(t)} &= \frac{1}{rR(r)}\frac{d}{dr}(rR'(r)) = \lambda
\end{aligned}
$$

b.

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x} \\
X(x)T'(t) &= kT(t)X''(x) - v_0T(t)X'(x) \\
\frac{T'(t)}{T(t)} &= \frac{kX''(x) - v_0X'(x)}{X(x)} = \lambda
\end{aligned}
$$

c.

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2}(u(x,y)) + \frac{\partial^2}{\partial y^2}(u(x,y)) = 0 \\
X''(x)Y(y) + X(x)Y''(y) = 0 \\
\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda
\end{aligned}
$$

d.

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) \\
R(r)T'(t) &= \frac{kT(t)}{r^2}\frac{\partial}{\partial r}\left(r^2R'(r)\right) \\
\frac{T'(t)}{kT(t)} &= \frac{1}{r^2R(r)}\frac{d}{dr}(r^2R'(r))
\end{aligned}
$$

e.

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{\partial^4 u}{\partial x^4} \\
X(x)T'(t) &= kX''''(x)T(t) \\
\frac{X(x)}{X''''(x)} &= \frac{T'(t)}{kT(t)} = \lambda
\end{aligned}
$$

f.

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial t^2} \\
X(x)T''(t) &= c^2 X''(x)T(t) \\
\frac{X(x)}{X''(x)} &= c^2 \frac{T(t)}{T''(t)} = \lambda
\end{aligned}
$$

4.

There is three cases for λ

- $\lambda > 0$
  From
  $$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$
  general solution can be written
  $$\phi(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$
  Thus
  $$\phi'(x) = \sqrt{\lambda}(-C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x))$$
  Plugging in the $\phi(0) = 0$ and $\phi'(L) = 0$, we have that $C_1 = 0$ and
  $$\phi'(L) = C_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$$
  Thus let $C_2 \neq 0$, we have that
  $$\cos(\sqrt{\lambda}L) = 0$$
  Therefore,
  $$\sqrt{\lambda}L = \frac{(2n-1)\pi}{2}$$
  where $n \in \mathbb{N}$, and
  $$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}$$
  where the eigenfunctions are
  $$\phi_n(x) = \sin\left(\frac{(2n-1)\pi}{2L}\right)$$

- $\lambda = 0$, then
  $$\frac{d^2\phi}{dx^2} = 0$$
  Thus as $\phi'(L) = 0$,
  $$\frac{d\phi}{dx} = 0$$
  and similarly as $\phi(0) = 0$,
  $$\phi = 0$$
  which is a trivial solution and thus 0 is not an eigenvalue.

- $\lambda < 0$, then similarly to the case $\lambda > 0$, we have that
  $$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

general solution can be written

$$\phi(x) = C_1 \cosh(\sqrt{-\lambda}x) + C_2 \sinh(\sqrt{-\lambda}x)$$

Thus

$$\phi'(x) = \sqrt{-\lambda}(C_1 \sinh(\sqrt{-\lambda}x) + C_2 \cosh(\sqrt{-\lambda}x))$$

Plugging in the $\phi(0) = 0$ and $\phi'(L) = 0$, we have that $C_1 = 0$ and

$$\phi'(L) = C_2\sqrt{-\lambda}\cosh(\sqrt{-\lambda}L) = 0$$

which means

$$\cosh(\sqrt{-\lambda}L) = 0$$

which has no real solution thus λ cannot have negative values.

5.

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \sin\left(\frac{nx\pi}{t}\right)$$

where B_n can be determined using the initial condition where

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

a.

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 6\sin\left(\frac{9\pi x}{L}\right)$$

Thus, let $B_9 = 6$ and $B_n = 0$ for all $n \neq 9$. Therefore,

$$u(x,t) = 6\exp\left(-\frac{81\pi^2 kt}{L^2}\right) \sin\left(\frac{9\pi x}{L}\right)$$

b.

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$$

Thus, let $B_1 = 3, B_3 = -1$ and $B_n = 0$ for all $n \notin \{1,3\}$. Therefore,

$$u(x,t) = 3\exp\left(-\frac{\pi^2 kt}{L^2}\right) \sin\left(\frac{\pi x}{L}\right) - \exp\left(-\frac{9\pi^2 kt}{L^2}\right) \sin\left(\frac{3\pi x}{L}\right)$$

c.

We have that

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 2\cos\left(\frac{3\pi x}{L}\right)$$

Thus

$$\int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 2\int_0^L \cos\left(\frac{3\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

From question 6, we get

$$\begin{aligned} B_m \frac{L}{2} &= \int_0^L B_m \sin^2\left(\frac{m\pi x}{L}\right) dx = 2\int_0^L \cos\left(\frac{3\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \\ &= \int_0^L \left[\sin\left(\frac{(m+3)\pi x}{L}\right) - \sin\left(\frac{(-m+3)\pi x}{L}\right)\right] \\ &= \frac{2mL}{(m^2-9)\pi}(1 + (-1)^m) \end{aligned}$$

Thus

$$B_m = \begin{cases} 0, & \text{if n is odd} \\ \dfrac{8m}{(m^2-9)\pi}, & \text{if n is even} \end{cases}$$

Thus we can get a general solution,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{kn^2\pi^2 t}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \frac{16n}{(4n^2-9)\pi} \exp\left(-\frac{k4n^2\pi^2 t}{L^2}\right)\sin\left(\frac{2n\pi x}{L}\right)$$

d.

Similar to part c, we have that

$$\begin{aligned} B_m\frac{L}{2} &= \int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L} 2\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2L}{n\pi}\sin^2\left(\frac{n\pi}{4}\right) + \frac{2L}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right] \end{aligned}$$

Thus,

$$B_n = \frac{4}{n\pi}\left(\sin^2\left(\frac{n\pi}{4}\right) + \cos\left(\frac{n\pi}{2}\right) - (-1)^n\right)$$

plugging that in the supposed solution for $u(x,t)$ will give us the final answer.

6.

In case $n \neq m$,

$$\begin{aligned}
&\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx \\
=&\frac{1}{2}\int_0^L \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right]dx \\
=&\frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right) - \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\right]\Bigg|_{x=0}^{L} \\
=&\frac{L}{2\pi}\left[\frac{\sin(n\pi - m\pi)}{n-m} - \frac{\sin(n\pi + m\pi)}{n+m}\right] \\
=&0
\end{aligned}$$

as n, m are integers. If $n = m$, then

$$\begin{aligned}
&\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx \\
=&\int_0^L \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{L}\right)dx \\
=&\frac{1}{2}\left(L - \frac{L}{2n\pi}(\sin(2n\pi) - \sin(0))\right) \\
=&\frac{L}{2}
\end{aligned}$$

7.

In case $n \neq m$,

$$\begin{aligned}
&\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx\\
=&\frac{1}{2}\int_0^L \left[\cos\left(\frac{(n-m)\pi x}{L}\right)+\cos\left(\frac{(n+m)\pi x}{L}\right)\right]dx\\
=&\frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)+\frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\right]\Bigg|_{x=0}^{L}\\
=&\frac{L}{2\pi}\left[\frac{\sin(n\pi-m\pi)}{n-m}+\frac{\sin(n\pi+m\pi)}{n+m}\right]\\
=&0
\end{aligned}$$

as n, m are integers. If $n = m$, then

$$\begin{aligned}
&\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx\\
=&\int_0^L \frac{1}{2}\left(1+\cos\frac{2n\pi x}{L}\right)dx\\
=&\frac{1}{2}\left(L+\frac{L}{2n\pi}(\sin(2n\pi)-\sin(0))\right)\\
=&\frac{L}{2}
\end{aligned}$$

8.

Let $u(x,y) = X(x)Y(y)$, then We have

$$\begin{cases} u_x(0,y) = X'(0)Y(y) = 0 \implies X'(0) = 0 \\ u_x(L,y) = X'(L)Y(y) = 0 \implies X'(L) = 0 \\ u(x,0) = X(x)Y(0) = 0 \implies Y(0) = 0 \end{cases}$$

From the laplace equation, we know that

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

for some λ.

- if $\lambda > 0$, then as $X'' = \lambda X$,

$$X(x) = C_1 \cosh(\sqrt{\lambda}x) + C_2 \sinh(\sqrt{\lambda}x)$$

  thus

$$X'(x) = C_1\sqrt{\lambda}\sinh(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cosh(\sqrt{\lambda}x)$$

  which we plugging in $X'(0) = 0$ and $X'(L) = 0$ gives us $C_2 = 0$ and

$$C_1\sqrt{\lambda}\sinh(\sqrt{\lambda}L) = 0$$

  which has no solution thus λ cannot be positive.

- if $\lambda = 0$ then as $X'' = 0$, and $X'(0) = X'(L) = 0$,

$$X(x) = C_3$$

  Similarly, $Y'' = 0$, then

$$Y(y) = C_4 y + C_5$$

  which we will get $C_5 = 0$ as $Y(0) = 0$, thus

$$Y(y) = C_4 y$$

- if $\lambda < 0$, then

$$X(x) = C_6 \cos(\sqrt{-\lambda}x) + C_7 \sin(\sqrt{-\lambda}x)$$

  thus

$$X'(x) = -C_6\sqrt{-\lambda}\sin(\sqrt{-\lambda}x) + C_7\sqrt{-\lambda}\cos(\sqrt{-\lambda}x)$$

  which we will get $C_7 = 0$ and

$$X'(L) = -C_6\sqrt{-\lambda}\sin(\sqrt{-\lambda}L) = 0$$

Let $C_6 \neq 0$, we have that $\sqrt{-\lambda}L = n\pi$ for all $n \in \mathbb{N}$. Therefore,

$$\lambda_n = -\frac{n^2\pi^2}{L^2}$$

Thus

$$Y'' = \frac{n^2\pi^2}{L^2}Y$$

and

$$Y = C_8 \cosh\left(\frac{n\pi y}{L}\right) + C_9 \sinh\left(\frac{n\pi y}{L}\right)$$

Plugging in $Y(0) = 0$, we have $C_8 = 0$ and

$$Y = C_9 \sinh\left(\frac{n\pi y}{L}\right)$$

Thus we have the solution,

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \sin\frac{n\pi y}{L}$$

Plugging in $y = H$, we get

$$u(x, H) = A_0 H + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \sin\frac{n\pi H}{L} = f(x)$$

Integrating both sides,

$$\int_0^L A_0 H + \sum_{n=1}^{\infty} A_n \int_0^L \cos\frac{n\pi x}{L} dx \sin\frac{n\pi H}{L} = \int_0^L f(x)dx$$

and hence

$$A_0 = \frac{1}{HL}\int_0^L f(x)dx$$

We can also get

$$\int_0^L \cos\frac{m\pi x}{L} A_0 H + \sum_{n=1}^{\infty} A_n \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L} \sin\frac{n\pi H}{L} dx = f(x)\cos\frac{m\pi x}{L} dx$$

which thus gives us

$$A_n \sinh\frac{n\pi H}{L} \int_0^L \cos^2\frac{n\pi x}{L} dx = \int_0^L f(x)\cos\frac{n\pi x}{L} dx$$

and

$$A_n = \frac{2}{L\sinh\frac{n\pi H}{L}} \int_0^L f(x)\cos\frac{n\pi x}{L} dx$$

9.

Let $u(x,y) = X(x)Y(y)$, then We have

$$\begin{cases} u_x(L,y) = X'(L)Y(y) = 0 \implies X'(L) = 0 \\ u(x,0) = X(x)Y(0) = 0 \implies Y(0) = 0 \\ u(x,H) = X(x)Y(H) = 0 \implies Y(H) = 0 \end{cases}$$

From the laplace equation, we know that

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

for some λ.

- if $\lambda > 0$, then

  $$Y(y) = C_1\cos(\sqrt{\lambda}y) + C_2\sin(\sqrt{\lambda}y)$$

  Then as $Y(0) = Y(H) = 0$, we get $C_1 = 0$ and

  $$C_2\sin(\sqrt{\lambda}H) = 0$$

  Let $C_2 \neq 0$, we have that $\sqrt{\lambda}L = n\pi$ for all $n \in \mathbb{N}$. Therefore,

  $$\lambda_n = \frac{n^2\pi^2}{H^2}$$

  Thus

  $$X'' = \frac{n^2\pi^2}{H^2}X$$

  and

  $$X(x) = C_3\cosh\left(\frac{n\pi x}{H}\right) + C_4\sinh\left(\frac{n\pi x}{H}\right)$$

  Thus

  $$X'(x) = \frac{n\pi}{H}\left(C_3\sinh\left(\frac{n\pi x}{H}\right) + C_4\cosh\left(\frac{n\pi x}{H}\right)\right)$$

  Since $X'(L) = 0$, plugging in we can get

  $$C_4 = -C_3\frac{\sinh\frac{n\pi L}{H}}{\cosh\frac{n\pi L}{H}}$$

  and thus

  $$X(x) = \frac{C_3}{\cosh\frac{n\pi L}{H}}\left(\frac{n\pi}{H}(x-L)\right)$$

  and

  $$X_n(x) = \cosh\left(\frac{n\pi}{H}(x-L)\right)$$

- if $\lambda = 0$, then $Y'' = 0$ and $Y(0) = 0$, $Y(H) = 0$ implies that

$$Y(y) = C_5 y + C_6$$

where there is a system of equations and we can solve for $C_5 = C_6 = 0$, which is a trivial solution thus there is no zero eigenvalue.

- if $\lambda < 0$, then as

$$Y'' = -\lambda Y$$
$$Y(y) = C_7 \cosh(\sqrt{-\lambda} y) + C_8 \sinh(\sqrt{-\lambda} y)$$

then as $Y(0) = Y(H) = 0$, we can get that $C_7 = 0$ and

$$C_8 \sinh(\sqrt{-\lambda} H) = 0$$

which is only true when $C_8 = 0$ which leads to a trivial solution. Thus there is no solution in this case.

Thus we have the solution,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi(x - L)}{H}\right) \sin\frac{n\pi y}{L}$$

and

$$u_x(x, y) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{H} \cosh\left(\frac{n\pi(x - L)}{H}\right) \sin\frac{n\pi y}{L}$$

therefore,

$$u_x(0, y) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{H} \sinh\left(\frac{n\pi}{H}(-L)\right) \sin\frac{n\pi y}{H} = g(y)$$

Afterwards, we can get

$$\sum_{n=1}^{\infty} -A_n \frac{n\pi}{H} \sinh\left(\frac{n\pi L}{H}\right) \int_0^H \sin\frac{n\pi y}{H} \sin\frac{m\pi y}{H} dy = \int_0^H g(y) \sin\frac{m\pi y}{H} dy$$

which we can solve for

$$A_n = -\frac{2}{n\pi \sinh\frac{n\pi L}{H}} \int_0^H g(y) \sin\frac{n\pi y}{H} dy$$

10.

We first use seperation of variables, let $u(r,\theta) = \phi(\theta)R(r)$, thus

$$\begin{cases} \phi(0)R(0) \text{ is bounded } \implies R(0) \text{ is bounded} \\ \phi(-\pi)R(r) = \phi(\pi)R(r) \implies \phi(\pi) = \phi(-\pi) \\ \phi'(-\pi)R(r) = \phi'(\pi)R(r) \implies \phi'(\pi) = \phi'(-\pi) \end{cases}$$

We also have that

$$\frac{r}{R(r)}\frac{\partial}{\partial r}(rR'(r)) = -\frac{\phi''(\theta)}{\phi(\theta)} = \lambda$$

- if $\lambda = \alpha^2 > 0$, where $\alpha \in \mathbb{N}$, then

  $$\phi(\theta) = C_1\cos(\alpha\theta) + C_2\sin(\alpha\theta)$$

  and

  $$\phi'(\theta) = -C_1\alpha\sin(\alpha\theta) + C_2\alpha\cos(\alpha\theta)$$

  Thus we have the system of equations from $\phi(\pi) = \phi(-\pi)$ and $\phi'(\pi) = \phi'(-\pi)$.

  $$\begin{cases} C_1\cos(\pi\alpha) + C_2\sin(\alpha\pi) = C_1\cos(-\pi\alpha) + C_2\sin(-\alpha\pi) \\ -C_1\sin(\pi\alpha) + C_2\cos(\alpha\pi) = -C_1\sin(-\pi\alpha) + C_2\cos(-\alpha\pi) \end{cases}$$

  and in the first equation,

  $$C_2\sin(\alpha\pi) = C_2\sin(-\alpha\pi) \implies \sin(\alpha\pi) = 0 \implies \alpha \in \mathbb{N}$$

  we get the similar result from the second equation. Thus

  $$\phi_n(\theta) = C_1\cos(n\theta) + C_2\sin(n\theta)$$

  Then we have that

  $$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2$$

  Let $R(r) = r^k$, we have that

  $$k(k-1)r^k + kr^k - n^2r^k = 0$$

  $$k(k-1) + k - n^2 = 0$$

  Thus

  $$k = \pm n$$

  and

  $$R(r) = C_3r^n + C_4r^{-n}$$

  which is bounded thus $C_3 = 0$ and

  $$R(r) = C_4r^{-n}$$

- if $\lambda = 0$, then
$$\phi'' = 0$$
Thus
$$\phi(\theta) = C_5 y + C_6$$
Then similar to when $\lambda > 0$, we get
$$\begin{cases} C_5\pi + C_6 = C_5(-\pi) + C_6 \\ C_5 = C_5 \end{cases}$$
Thus $C_5 = 0$ and C_6 is arbitary. Then
$$\phi(\theta) = C_6$$
thus the function is $\phi_0(\theta) = 1$. Now
$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$
which leads to
$$R(r) = C_7 \ln(r) + C_8 = C_8$$
as R is bounded.

- if $\lambda = -\alpha^2 < 0$, then
$$\phi(\theta) = C_9 \cosh(\alpha\theta) + C_{10} \sinh(\alpha\theta)$$
and
$$\phi'(\theta) = C_9\alpha \sinh(\alpha\theta) + C_{10}\alpha \cosh(\alpha\theta)$$
Then we get
$$\begin{cases} C_9 \cosh(\alpha\pi) + C_{10} \sinh(\alpha\pi) = C_9 \cosh(-\alpha\pi) + C_{10} \sinh(-\alpha\pi) \\ C_9\alpha \sinh(\alpha\pi) + C_{10}\alpha \cosh(\alpha\pi) = C_9\alpha \sinh(-\alpha\theta) + C_{10}\alpha \cosh(-\alpha\theta) \end{cases}$$
Thus $C_9 = C_{10} = 0$ and $\phi(\theta) = 0$ which is trivial thus no negative eignevalues.

Therefore, we get the solution
$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos(n\theta) + B_n \sin(n\theta))$$
Then we have that
$$u(a,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos(n\theta) + B_n \sin(n\theta)) = f(\theta)$$
The part after this was done in class. Integrating on $[-\pi, \pi]$,
$$A_0 2\pi = \int_{-\pi}^{\pi} f(\theta)d\theta \implies A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

Multiply by $\cos(m\theta)$ and integrating on $[-\pi, \pi]$, we have that

$$A_m = \frac{1}{a^m \pi} \int_{-\pi}^{\pi} f(\theta) \cos(m\theta) d\theta$$

Multiply by $\sin(m\theta)$ and integrating on $[-\pi, \pi]$, we have that

$$B_m = \frac{1}{a^m \pi} \int_{-\pi}^{\pi} f(\theta) \sin(m\theta) d\theta$$