1.

a.

$$f_Y(y) = \int_0^2 \frac{1}{6} dx = \frac{1}{3}$$

Since $0 < y < 3$, $2 < y^2 + 2 < 11$, and hence if $u \leq 2, F_U(u) = 0$ and if $u \geq 11, F_U(u) = 1$. For $2 < u < 11$, we have that

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(Y^2 + 2 \leq u) \\ &= P(Y^2 \leq u - 2) = P(Y \leq \sqrt{U - 2}) \\ &= \int_0^{\sqrt{U-2}} \frac{1}{3} dy = \frac{\sqrt{U-2}}{3} \end{aligned}$$

b.

Since $0 < x < 2, 0 < y < 3, -3 < x - y < 2$. Therefore, if $v \leq -3, F_V(v) = 0$ and if $v \geq 2, F_V(v) = 1$. For $-3 < v < 2$, we have that

$$\begin{aligned} F_V(v) &= P(V \leq v) = P(X - Y \leq v) \\ &= P(Y \geq -v + X) \end{aligned}$$

For $-3 < v < -1$,

$$\begin{aligned} F_V(v) &= \int_{-v}^{3} \int_0^{y+v} \frac{1}{6} dx dy \\ &= \int_{-v}^{3} \frac{y+v}{6} dy = \frac{9}{12} + \frac{3v}{6} - \frac{v^2}{12} + \frac{v^2}{6} \\ &= \frac{(3+v)^2}{12} \end{aligned}$$

For $0 < v < 2$,

$$\begin{aligned} F_V(v) &= 1 - \int_v^2 \int_0^{-v+x} \frac{1}{6} dy dx \\ &= 1 - \int_v^2 \frac{1}{6}(-v + x) dx \\ &= 1 + \frac{2v}{6} - \frac{1}{3} - \frac{v^2}{6} + \frac{v^2}{12} \\ &= 1 - \frac{(v-2)^2}{12} \\ \implies f_V(v) &= \frac{-2v + 4}{12} \end{aligned}$$

For $-1 < v < 0$

$$\begin{aligned} F_V(v) &= \int_0^2 \int_{-v+x}^{3} \frac{1}{6} dy dx \\ &= \frac{4v + 8}{12} \end{aligned}$$

2.

Since $16 \geq U = Y^4 \geq 0$. We have that if $0 \leq u, F_U(u) = 0$ and if $u \geq 16, F_U(u) = 1$. For $0 < u < 16$, we have that

$$\begin{aligned}
F_U(u) = P(U \leq u) = P(Y^4 \leq u) &= P(-\sqrt[4]{u} \leq Y \leq \sqrt[4]{u}) \\
&= \int_0^{\sqrt[4]{u}} \frac{y}{6} dy + \left| \int_{-\sqrt[4]{u}}^{0} \frac{y^2}{4} dy \right| \\
&= \frac{\sqrt{u}}{12} + \frac{\sqrt[4]{u^3}}{12}
\end{aligned}$$

3.

$$f_Y(y) = \int_y^\infty \frac{1}{9} e^{-x/3} dx = \frac{1}{3} e^{-y/3}$$

$$\begin{aligned} F_U(u) &= P(U \le u) = P(1 - Y^2 \le u) = P(Y^2 \ge 1 - u) = P(Y \ge \sqrt{1-u}) \\ &= \int_{\sqrt{1-u}}^\infty \frac{1}{3} e^{-y/3} dy = e^{-\sqrt{1-u}/3} \end{aligned}$$

4.

a.

$$x = h^{-1}(u) = \frac{u+3}{2} \implies \frac{d}{du}h^{-1}(u) = \frac{1}{2}$$

If $1 > \dfrac{u+3}{2} > 0 \iff -1 > u > -3$

$$f_U(u) = f_X\left(\frac{u+3}{2}\right) \cdot \frac{1}{2} = \frac{-u-1}{2}$$

If $u \leq -3$ or $u \geq -1$ then $f_U(u) = 0$

b.

$$x = h^{-1}(v) = \sqrt[3]{v} \implies \frac{d}{dv}h^{-1}(v) = \frac{1}{3}v^{-2/3}$$

If $1 < \sqrt[3]{v} < 0 \iff 1 < v < 0$

$$f_V(v) = f_X(\sqrt[3]{v}) \cdot \frac{1}{3}x^{-2/3} = \frac{2}{3}v^{-2/3}(1 - \sqrt[3]{v})$$

If $v \geq 1$ or $v \leq 0$ then $f_V(v) = 0$.

5.

$$y = h^{-1}(u) = \frac{2-u}{4} \implies \frac{d}{du}h^{-1}(u) = -\frac{1}{4}$$

If $-1 < \dfrac{2-u}{4} < 1 \iff -2 < u < 6$

$$f_U(u) = f_X\left(\frac{2-u}{4}\right) \cdot \left|-\frac{1}{4}\right| = \frac{6-u}{32}$$

6.

Consider $V = Y$

$$x = h_1^{-1}(u,v) = \frac{v}{u}, \qquad y = h_2^{-1}(u,v) = v$$

$$\det \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial u} & \frac{\partial h_1^{-1}}{\partial v} \\ \frac{\partial h_2^{-1}}{\partial u} & \frac{\partial h_2^{-1}}{\partial v} \end{vmatrix} = \det \begin{vmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ 0 & 1 \end{vmatrix} = -\frac{v}{u^2}$$

As $y > 0$, we have that $v > 0$ and hence

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{v}{u}, v\right) \cdot \left|-\frac{v}{u^2}\right| = \frac{v}{u} e^{-(u/v+v)} \cdot \frac{v}{u^2} = \frac{v^2}{u^3} e^{-(u+v^2)/v}$$

7.

a.

For $0 < \dfrac{u-v}{2} < y, x < \dfrac{u+v}{2} < 1 \iff 2 > u > v > 0, u < 2-v, v < 1.$

$$x = h_1^{-1}(u,v) = \frac{u-v}{2}, \qquad y = h_2^{-1}(u,v) = \frac{u+v}{2}$$

$$\det \begin{vmatrix} \dfrac{\partial h_1^{-1}}{\partial u} & \dfrac{\partial h_1^{-1}}{\partial v} \\ \dfrac{\partial h_2^{-1}}{\partial u} & \dfrac{\partial h_2^{-1}}{\partial v} \end{vmatrix} = \det \begin{vmatrix} \dfrac{1}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u-v}{2}, \frac{u+v}{2}\right) \cdot \frac{1}{2} = 6 \cdot \frac{u-v}{2} \cdot \frac{1}{2} = \frac{3(u-v)}{2}$$

b.

For $0 < u < 1$

$$f_U(u) = \int_0^u \frac{3(u-v)}{2} dv$$

For $1 < u < 2$

$$f_U(u) = \int_0^{2-u} \frac{3(u-v)}{2} dv$$

$$f_V(v) = \int_u^{2-v} \frac{3(u-v)}{2} du =$$

8.

$$x = h_1^{-1}(u,v) = \frac{uv}{u-1}, \qquad y = h_2^{-1}(u,v) = \frac{v}{u-1}$$

For $0 < \dfrac{uv}{u-1} < 2, 0 < \dfrac{v}{u-1} < 3$

$$\det \begin{vmatrix} \dfrac{\partial h_1^{-1}}{\partial u} & \dfrac{\partial h_1^{-1}}{\partial v} \\ \dfrac{\partial h_2^{-1}}{\partial u} & \dfrac{\partial h_2^{-1}}{\partial v} \end{vmatrix} = \det \begin{vmatrix} \dfrac{-v}{(u-1)^2} & \dfrac{u}{u-1} \\ \dfrac{-v}{(u-1)^2} & \dfrac{1}{u-1} \end{vmatrix} = \frac{v}{(u-1)^2}$$

If $u > 1$ then for $v < \dfrac{2u-2}{u}, v < 3u-3, v > 0$

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{uv}{u-1}, \frac{v}{u-1}\right) \cdot \frac{v}{(u-1)^2} = \frac{v}{6(u-1)^2}$$

If $0 < u < 1$ then for $v > \dfrac{2u-2}{u}, v > 3u-3, v < 0$

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{uv}{u-1}, \frac{v}{u-1}\right) \cdot \frac{v}{(u-1)^2} = \frac{v}{6(u-1)^2}$$

For $0 < v < 2$,

$$f_V(v) = \int_{-2/(v-2)}^{\infty} \frac{v}{6(u-1)^2} du = -\frac{v-2}{6}$$

which is the same as the answer in question 1b

9.

We have that

$$m_U(t) = E[e^{tu}] = E[e^{tcY}] = m_Y(tc) = (1 - \beta tc)^{-n/2}$$

If $U \sim \chi^2_n$ then

$$m_U(t) = m_\chi(t) \implies (1 - \beta tc)^{-n/2} = (1 - 2t)^{-n/2} \implies c = \frac{2}{\beta}$$

10.

$$\begin{aligned} m_U(t) &= E[e^{t(3y^2-2)}] = e^{-2t}E[e^{3ty^2}] \\ &= e^{-2t}\int_0^3 e^{3ty^2}\cdot\frac{2y}{9}dy \\ &= \frac{e^{-2t}(e^{27t}-1)}{27t} \\ &= \frac{e^{25t}-e^{-2t}}{27t} \end{aligned}$$

which is the moment generating function of Uniform$(25t, -2t)$ if $t < 0$ or Uniform$(-2t, 25t)$ if $t > 0$.

11.

$$\begin{aligned}
m_U(t) &= E[e^{-at\ln(y/x)}] = E\left[\left(\frac{y}{x}\right)^{-at}\right] \\
&= \int_0^\infty \int_0^x \left(\frac{y}{x}\right)^{-at} \cdot \frac{e^{-x/2}}{4} dy dx \\
&= \int_0^\infty \frac{x^{-at+1}}{-at+1} \cdot \frac{1}{x^{-at}} \cdot \frac{e^{-x/2}}{4} dx \\
&= \frac{1}{4(-at+1)} \int_0^\infty x \cdot e^{-x/2} \\
&= \frac{1}{4(-at+1)} \cdot 4 \\
&= \frac{1}{-at+1}
\end{aligned}$$

which is the moment generating function of Exponential(a)

12.

We have that

$$F_Y(y) = y^5$$

and hence

$$f_{Y_7}(y) = 7(F_Y(y))^6 f_Y(y) = 35y^{34}$$

Therefore, the expected value of the biggest bonus is

$$\int_0^1 y \cdot 35y^{34} dy = \frac{35}{36} y^{36}|_0^1 = \frac{35}{36}$$

13.

$$f_X(x) = \int_0^\infty \frac{1}{\beta\theta} e^{-(x/\beta+y/\theta)} dy = \frac{e^{-x/\beta}}{\beta}$$

$$f_Y(y) = \int_0^\infty \frac{1}{\beta\theta} e^{-(x/\beta+y/\theta)} dx = \frac{e^{-y/\theta}}{\theta}$$

and hence X and Y is independent. Let $U = \min(X, Y)$, then

$$\begin{aligned}
F_U(u) &= P(U \leq u) = 1 - P(X \geq u, Y \geq u) \\
&= 1 - P(X \geq u)P(Y \geq u) \\
&= 1 - \int_u^\infty \frac{e^{-x/\beta}}{\beta} dx \cdot \int_u^\infty \frac{e^{-y/\theta}}{\theta} dy \\
&= 1 - e^{-u/\beta} \cdot e^{-u/\theta}
\end{aligned}$$

Therefore,

$$f_U(u) = -e^{-u/\beta - u/\theta} \left(-\frac{1}{\beta} - \frac{1}{\theta}\right)$$

and hence

$$\begin{aligned}
E[U] &= \int_0^\infty u \cdot -e^{-u/\beta-u/\theta} \left(-\frac{1}{\beta} - \frac{1}{\theta}\right) du \\
&= \left(-\frac{1}{\beta} - \frac{1}{\theta}\right) \cdot \left(-\frac{1}{\left(\frac{1}{\beta} + \frac{1}{\theta}\right)^2}\right)
\end{aligned}$$

$$= \frac{1}{\frac{1}{\beta} + \frac{1}{\theta}} = \frac{\beta\theta}{\beta + \theta}$$

14.

Let $U = \min(X, Y)$, then

$$\begin{aligned}F_U(u) &= P(U \leq u) = 1 - P(X \geq u, Y \geq u)\\&= 1 - \int_u^1 \int_u^1 \frac{4}{3}x + \frac{2}{3}y dx dy\\&= -u^3 + u^2 + u\end{aligned}$$

and hence

$$f_U(u) = -3u^2 + 2u + 1$$

Therefore,

$$E[U] = \int_0^1 u(-3u^2 + 2u + 1)du = \frac{5}{12}$$

15.

We have that

$$\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= \frac{2!}{0!}[F_{Y_2}(y_2) - F_{Y_1}(y_1)]^0 f_Y(Y_2) \cdot f_Y(y_1) \\
&= 2 \cdot \frac{1}{\beta}e^{-y_1/\beta} \cdot \frac{1}{\beta}e^{-y_2/\beta} \\
&= 2 \cdot \frac{1}{\beta^2}e^{-(y_1+y_2)/\beta}
\end{aligned}$$