1.

b.

Since $f_n(x) \uparrow f(x)$ for all $x \in X$, we have that

$$\int_X f = \int_X \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_X f_n$$

But we also know that $\sup_{n\geq k} \int_X f_n \leq \int_X f$ for all $n \in \mathbb{N}$, thus

$$\int_X f \geq \lim_{n\to\infty} \sup_{n\geq k} \int_X f_n$$

Thus we have that

$$\limsup_{n\to\infty} \int_X f_n = \liminf_{n\to\infty} \int_X f_n = \lim_{n\to\infty} \int_X f_n = \int_X f$$

b.

Define a sequence of function

$$f_n(x) = f(x) \cdot \chi_{x\leq n}$$

Thus $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and is nonnegative as f is nonnegative. Then from part a, we know that

$$\int_N f d\mu = \lim_{n\to\infty} \int_N f_n = \lim_{n\to\infty} \int_{\{1,2,\ldots,n\}} f_n d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} f(i)$$

2.

For any measurable subset E, we have that

$$\int_E f = \int_E \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_E f_n \leq \limsup_{n\to\infty} \int_E f_n$$

We also have that

$$\begin{aligned}
\int_E f &= \int_X f - \int_{E^c} f \\
&= \int_X f - \int_{E^c} \liminf f_n \\
&\geq \int_X f - \liminf \int_{E^c} f_n \\
&= \int_X f + \limsup \int_{E^c} -f_n \\
&= \limsup \left(\int_X f - \int_{E^c} f \right) \\
&= \limsup \int_E f
\end{aligned}$$

Thus

$$\int_E f = \limsup_{n\to\infty} \int_E f_n = \liminf_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \int_E f_n$$

3.

a.

Let $\varphi = \sum_{j=0}^{n} c_j \chi_{E_j}$, where $c_0 = 0$ and E_j are pairwise disjoint and $\cup_{j=0}^{n} E_j = X$.

$$\int_X \varphi d\nu = \sum_{j=0}^{n} \int_{E_j} c_j f d\mu = \int_X d\mu = \int_X \sum_{j=0}^{n} c_j \chi_{E_j} f d\mu = \int_X \varphi f d\mu$$

b.

Since X is a nonnegative measurable function, there is a sequence of non-negative simple function ϕ_n such that $\phi_n \uparrow g$ for all $x \in X$. Then

$$\int_X \phi_n d\nu = \int_X \phi_n f d\mu$$

and

$$\lim_{n\to\infty} \int_X \phi_n d\nu = \lim_{n\to\infty} \int_X \phi_n f d\mu$$

Since $\phi_n \uparrow g$ andthus $\phi_n f \uparrow gf$, we have that

$$\int_X g d\nu = \int_X g f d\mu$$

4.

Definition: If $f_n \to f$ in measure then for an arbitary $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\underbrace{\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}}_{X_n} = 0$$

Therefore, for all $\delta > 0$, we can find n_0 such that for all $n > n_0$, $\mu(X_n) < \delta/2$ and $|f_n - f| < \varepsilon$ for all $x \in X \backslash X_n$.

$$\begin{aligned}
&\lim_{n\to\infty} \rho(f_n, f) \\
&= \lim_{n\to\infty} \int_X \frac{|f_n - f|}{|f_n - f| + 1} d\mu \\
&\leq \lim_{n\to\infty} \int_X d\mu + \int_{X\backslash X_n} \frac{\varepsilon}{\varepsilon + 1} d\mu \\
&\leq \frac{\delta}{2} + \mu(X) \cdot \frac{\varepsilon}{\varepsilon + 1}
\end{aligned}$$

Since ε is arbitary, choose $\varepsilon = \dfrac{\delta}{2\mu(X) - \delta}$ so that

$$\lim_{n\to\infty} \rho(f_n, f) \leq \frac{\delta}{2} + \mu(X) \cdot \frac{\delta}{\delta + 2\mu(X) - \delta} > \delta$$

If $f_n \to f$ in measure is false then there is some $\varepsilon, \delta > 0$ such that for all $n_0 > 0$, there is $n > n_0$ such that

$$\mu(\underbrace{\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}}_{X_n} = \delta$$

and therefore

$$\begin{aligned}
&\rho(f_n, f) \\
&= \int_X \frac{|f_n - f|}{|f_n - f| + 1} d\mu \\
&\geq \int_{X_n} \frac{|f_n - f|}{|f_n - f| + 1} d\mu \\
&> \delta \frac{\varepsilon}{\varepsilon + 1}
\end{aligned}$$

since $|f_n - f| \geq \varepsilon$ and $\dfrac{x}{1 + x}$ is an increasing function. Thus

$$\lim_{n\to\infty} \rho(f_n, f) \geq \frac{\delta\varepsilon}{\varepsilon + 1} > 0$$

for some $\varepsilon, \delta > 0$, thus is a contradiction.

5.

Applying Fatou's, we have that

$$\lim_{n\to\infty}\int_{E\setminus E_0}[f(x)]^{1/n}dx = \liminf_{n\to\infty}\int_{E\setminus E_0}[f(x)]^{1/n}dx \geq \int_{E\setminus E_0}\liminf_{n\to\infty}[f(x)]^{1/n} = \int_{E\setminus E_0}1 = m(E\setminus E_0)$$

and also

$$\lim_{n\to\infty}\int_{E\setminus E_0}[f(x)]^{1/n}dx \leq \limsup_{n\to\infty}\int_{E\setminus E_0}[f(x)]^{1/n}dx$$

We first prove the reverse Fatou's lemma: Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions and g an integrable function such that $f_n \leq g$ for all $n \in \mathbb{N}$. Then $\limsup_{n\to\infty}\int_X f_n \leq \int_X \limsup_{n\to\infty} f_n$.
We can apply the fatou's lemma to $g - f_n \geq 0$,

$$\int_X \liminf_{n\to\infty}(g - f_n) \leq \liminf_{n\to\infty}\int_X (g - f_n)$$

Thus

$$\int_X \liminf_{n\to\infty} -f_n \leq \liminf_{n\to\infty}\int_X -f_n$$

and therefore,

$$-\int_X \limsup_{n\to\infty} f_n \leq -\limsup_{n\to\infty}\int_X f_n$$

which concludes the proof for the reverse version. Now apply the lemma with the function g on the domain D of f

$$g : D \to \mathbb{R}, \qquad x \to f(x) + 1$$

so that $g \geq f_n$ for all $n \in \mathbb{N}$ as

- if $f(x) \geq 1$, then $f_n(x) \leq f(x) < g(x)$
- if $f(x) < 1$, then $f_n(x) < 1 \leq f(x) + 1$

thus we have

$$\lim_{n\to\infty}\int_{E\setminus E_0}[f(x)]^{1/n}dx \leq \int_{E\setminus E_0}\limsup_{n\to\infty}[f(x)]^{1/n}dx = \int_{E\setminus E_0}1dx = m(E\setminus E_0)$$

Thus,

$$\lim_{n\to\infty}\int_E [f(x)]^{1/n} = \lim_{n\to\infty}\int_{E\setminus E_0}[f(x)]^{1/n}dx = m(E\setminus E_0)$$