1.

Let $S \subset \mathbb{R}^N$ be any set. To prove that ∂S is closed, we need to prove it contains all of its cluster point.
Let x be a cluster point of ∂S, we need to prove $x \in \partial S$, that is x is boundary point of S.
Since x is a cluster point of $\partial S, \forall \epsilon > 0 : \exists y \in \partial S \cap B_\epsilon(x) \backslash \{x\} \neq \varnothing$.
Therefore, $\exists \delta > 0 : B_\delta(y) \subset B_\epsilon(x)$and y is a boundary point of S, which means that $B_\delta(y) \cap S \neq \varnothing \wedge B_\delta(y) \cap S^c \neq \varnothing$ and hence $B_\epsilon(x) \cap S \neq \varnothing \wedge B_\epsilon(x) \cap S^c \neq \varnothing$.

2.

Suppose $x \in \partial(S_1 \cup S_2 \cup \ldots \cup S_n) = \partial\left(\bigcup_{k=1}^{n} S_k\right)$, which means that $\forall \epsilon > 0 :$ $B_\epsilon(x) \cap \left(\bigcup_{k=1}^{n} S_k\right)^c \neq \varnothing \wedge B_\epsilon(x) \cap \bigcup_{k=1}^{n} S_k \neq \varnothing$.
Let $y \in B_\epsilon(x) \cap \bigcup_{k=1}^{n} S_k \neq \varnothing$ then $y \in \bigcup_{k=1}^{n} S_k$ and hence $\exists j \in \{1, 2, \ldots, n\} : y \in S_j$. Therefore, $B_\epsilon(x) \cap S_j \neq \varnothing$.
Let $z \in B_\epsilon(x) \cap \left(\bigcup_{k=1}^{n} S_k\right)^c$, then as $\forall i \in \{1, 2, \ldots, n\} : S_i \subset \bigcup_{k=1}^{n} S_k \implies \left(\bigcup_{k=1}^{n} S_k\right)^c \subset S_i^c, z \in S_i^c$. Therefore, $B_\epsilon(x) \cap S_j^c \neq \varnothing$, which proves that x is a boundary points of S_j and finally,

$$\partial(S_1 \cup S_2 \cup \ldots \cup S_n) \subset \partial S_1 \cup \partial S_2 \cup \ldots \cup \partial S_n$$

Equal does not necessarily hold because: consider [-1,0] and [0,1]

$$\partial([-1, 0] \cup [-0, 1]) = \partial[-1, 1] = \{-1, 1\}$$

and

$$\partial[-1, 0] \cup \partial[0, 1] = \{-1, 0\} \cup \{1, 0\} = \{-1, 0, 1\}$$

3.

a.

Let $A = \{x \in \mathbb{R}^N : r \leq \|x\| \leq R\}$
For all $0 \leq r \leq R : B_R[0] \backslash B_r(0) = B_R[0] \cup B_r^c(0) = \{x \in \mathbb{R}^N : r \leq \|x\| \leq R\}$. Since $B_R[0]$ is closed and $B_r(0)$ is open hence $B_r^c(0)$ is closed, the union of the two sets are closed and hence A is closed. Since $x \in A \implies \|x\| \leq R, \forall i \in \{1, 2, \ldots, N\} : x_i \leq R$. A is bounded and therefore compact.

b.

Let $B = \{x \in \mathbb{R}^N : r < \|x\| \leq R\}$
Consider the point $x = (r, 0, \ldots, 0)$ then x is a cluster point of B because:

$$\forall \epsilon > 0 : \{y = (r', 0, \ldots, 0) | r < r' < r + \epsilon\} \in B_\epsilon(x)$$

Hence, $\forall m : r < m < r + \epsilon : \exists y \in B_\epsilon(x) \wedge \|y\| = m$ and therefore $(B_\epsilon(x) \cap B) \backslash \{x\} \neq \varnothing$ Therefore, B is not closed and not compact.

c.

Closure of any set is closed.
$0 < t \leq 2022$ and $-1 \leq \sin\frac{1}{t} \leq 1$ Hence, for all boundary point $x = (x_1, x_2)$ of the set: $-1 < x_1 < 2023$ and $-2 < x_2 < 2$. Because else, $B_{1/2}(x) \cap \left\{\left(t, \sin\frac{1}{t}\right) : t \in (0, 2022]\right\} = \varnothing$ which means that x is not a boundary point. Therefore, the set is bounded and closed and hence compact.

d.

Let $D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$We have

$$\forall \epsilon > 0 : \exists n \in \mathbb{N} : \frac{1}{n} < \epsilon \implies \frac{1}{n} \in B_\epsilon(0) \implies (B_\epsilon(0) \cap D) \backslash \{0\} \neq \varnothing$$

which means that 0 is a cluster point and therefore D is not closed and therefore not compact.

e.

If there exists a cluster point s not in E. Then
If $s < 0$ then let $s = -\epsilon$ where $\epsilon > 0$, then $\left(-\epsilon - \frac{\epsilon}{2}, -\epsilon + \frac{\epsilon}{2}\right) \cap E = \varnothing$ which is a contradiction.
If $s > 1$ then let $s = 1 + \epsilon$ where $\epsilon > 0$, then $\left(1 + \epsilon - \frac{\epsilon}{2}, 1 + \epsilon + \frac{\epsilon}{2}\right) \cap E = \varnothing$ which is a contradiction.
If $0 < s < 1$ then $\exists! n \in \mathbb{N} : s \in \left(\dfrac{1}{n+1}, \dfrac{1}{n}\right)$, which $\left(\dfrac{1}{n+1}, \dfrac{1}{n}\right) \cap E = \varnothing$

Let $\epsilon = \min\left\{s - \frac{1}{n+1}, \frac{1-n}{s}\right\}$, then $(s-\epsilon, s+\epsilon) \cap \left\{\frac{1}{n+1}, \frac{1}{n}\right\} = \varnothing$ and $(s-\epsilon, s+\epsilon) \subset \left(\frac{1}{n+1}, \frac{1}{n}\right)$ which have no common elements with E and therefore E contains all its cluster point.
Therefore, E is closed. We have

$$\forall n \in \mathbb{N} : \frac{1}{n} \geq 0 \implies \forall e \in E : e \geq 0 > -1$$

Also,

$$\forall n \in \mathbb{N} : 1 \leq n \implies 2 > 1 = \frac{1}{1} \geq \frac{1}{n} \implies \forall e \in E : e \leq 2$$

Therefore E is bounded and compact.

4.

a.

For an arbitary point $x = (x_1, x_2) \in U_1 \times U_2$.
Because $x_1 \in U_1$, which is open, $\exists \epsilon_1 > 0 : B_{\epsilon_1}(x_1) \subset U_1$.
Similarly, $\exists \epsilon_2 > 0 : B_{\epsilon_2}(x_2) \subset U_2$.
Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then

$$\forall y = (y_1, y_2) \in B_\epsilon(x) : |y_1 - x_1| < \epsilon \leq \epsilon_1 \wedge |y_2 - x_2| < \epsilon \leq \epsilon_2 \text{ else } \|x - y\| \geq \epsilon$$

Therefore $y_1 \in U_1 \wedge y_2 \in U_2$ and hence $B_\epsilon(x) \subset U_1 \times U_2$. Which means that $U_1 \times U_2$ is open.

b.

Since F_1 is closed, F_1^c is open and therefore $F_1^c \times \mathbb{R}^M$ is open. Similarly, $\mathbb{R}^N \times F_2^c$ is open and therefore because of

$$(F_1 \times F_2)^c = (F_1^c \times \mathbb{R}^M) \cup (\mathbb{R}^N \times F_2^c)$$

$(F_1 \times F_2)^c$ is open and $F_1 \times F_2$ is closed.

c.

We know from part b that $K_1 \times K_2$ is also closed. And since K_1 and K_2 is bounded, $K_1 \times K_2$ is also bounded hence compact.

5.

If K is compact and there is a family of closed set $\{F_i : i \in I\}$ in $\mathbb{R}^N$ such that

$$K \cap \bigcap_{i\in I} F_i = \varnothing$$

then we have

$$\begin{aligned} K \cap \left(\bigcup_{i\in I} F_i^c \right)^c &= \varnothing \\ \implies K \cap \bigcup_{i\in I} F_i^c &= K \\ \implies K \subset \bigcup_{i\in I} F_i^c \end{aligned}$$

K is compact, therefore $\exists i_1, i_2, \ldots, i_N : K \subset F_{i_1}^c \cup F_{i_2}^c \cup \ldots \cup F_{i_n}^c = F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_n}$ which proves that K has the finite intersection property.
If K has the finite intersection property, suppose that K is not closed, that is there is a cluster point $s \notin K$ then

$$\forall \epsilon > 0 : (B_\epsilon(s) \cap K) \backslash \{s\} \neq \varnothing$$

Therefore, create a family $\{B_{\frac{1}{n}}[s] \,|n \in \mathbb{N}\}$. If $K \cap \bigcap_{n\in\mathbb{N}} B_{\frac{1}{n}}[x] \neq \varnothing$, then there exists a point $y \in K \cap \bigcap_{n\in\mathbb{N}} B_{\frac{1}{n}}[x]$. Since $x \notin K, y \neq x$.

Moreover, we can find a $n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < \|x - y\|$ which means that $y \notin B_{\frac{1}{n_0}}[x]$ which contradicts with $K \cap \bigcap_{n\in\mathbb{N}} B_{\frac{1}{n}}[x] \neq \varnothing$ and hence

$$K \cap \bigcap_{n\in\mathbb{N}} B_{\frac{1}{n}}[x] = \varnothing \tag{1}$$

For all $i_1, i_2, \ldots, i_n \in \mathbb{N}$, let $j = \max\{i_1, i_2, \ldots, i_n\}$ then $j >= i_t \,\forall t \in \{1, 2, \ldots, n\}$ which means that $B_j[x] \subseteq B_{i_t}[x] \,\forall t \in \{1, 2, \ldots, n\}$ and hence $B_j[x] \cap B_{i_t}[x] = B_j[x] \,\forall t \in \{1, 2, \ldots, n\}$ .Therefore,

$$K \cap B_{\frac{1}{i_1}}[x] \cap B_{\frac{1}{i_2}}[x] \cap \ldots \cap B_{\frac{1}{i_n}}[x] = K \cap B_{\frac{1}{j}}[x] \neq \varnothing \tag{2}$$

(1) and (2) contradicts with the finite intersection property, and hence K is closed.

6.

Let

$$X_{s_i} \text{ be any point } \begin{cases} \in I_i & \text{if } s_i = 1 \\ \in \{a_i, b_i\} & \text{if } s_i = 2 \end{cases}$$

and $X_{s_1,s_2,\ldots,s_N} = X_{s_1} \times X_{s_2} \times \ldots \times X_{s_N}$. Now claim that

$$\partial I = X := \bigcup_{\exists i : s_i = 2} X_{s_1,s_2,\ldots,s_n}$$

For every point $x \in X$, because $\exists i : s_i = 2$, suppose $x_i = b_i$ which means that $\forall \epsilon > 0 : y = (x_1, x_2, \ldots, x_i + \epsilon/2, \ldots, x_n) \in B_\epsilon(x) \cap I^c$ and $z = (x_1, x_2, \ldots, x_i - \epsilon/2, \ldots, x_n) \in B_\epsilon(x) \cap I$.
Similarly, if $x_i = a_i$ then $\forall \epsilon > 0 : y = (x_1, x_2, \ldots, x_i - \epsilon/2, \ldots, x_n) \in B_\epsilon(x) \cap I^c$ and $z = (x_1, x_2, \ldots, x_i + \epsilon/2, \ldots, x_n) \in B_\epsilon(x) \cap I$.
Hence x is a boundary points.

For every point $t = (t_1, t_2, \ldots, t_n) \notin X$. Suppose $t \in I$, then

$$\nexists i \in \{1, 2, \ldots, N\} : t_i \in \{a_i, b_i\}$$

Let

$$\epsilon := \min(\{|t_1 - a_i| \,|i \in \{1, 2, \ldots, N\}\} \cup \{|t_1 - b_i| \,|i \in \{1, 2, \ldots, N\}\})$$

Which means that $\epsilon > 0$ because $\forall i \in \{1, 2, \ldots, N\} : a_i - t_i \neq 0 \wedge b_i - t_i \neq 0$.
If $t \in I$ then $B_\epsilon(t) \subset I$ as $\forall i \in \{1, 2, \ldots, N\} : t_i + \epsilon \leq b_i$ and $t_i - \epsilon \geq a_i$ and hence $B_\epsilon(t) \cap I^c = \varnothing$, which means t is not a boundary points. If $t \notin I$ then

$$\exists i \in \{1, 2, \ldots, N\} : t_i \notin I_i$$

If $t_i > b_i$ then let

$$\epsilon := t_i - b_i$$

Then $\forall m = (m_1, m_2, \ldots, m_N) \in B_\epsilon(t) : m_i > t_i - \epsilon = b_i$, which means that $m \notin I$ and therefore $B_\epsilon(t) \cap I = \varnothing$, which means t is not a boundary point.
Similarly, if $t_i < a_i$ then let

$$\epsilon := a_i - t_i$$

Then $\forall m = (m_1, m_2, \ldots, m_N) \in B_\epsilon(t) : m_i < t_i + \epsilon = a_i$, which means that $m \notin I$ and therefore $B_\epsilon(t) \cap I = \varnothing$, which means t is not a boundary point.
In all cases $t \notin X, t$ is not a boundary point.