1.

Since f is a $\mathcal{C}^1$ function, the graph of f

$$\gamma : [a,b] \to [a,b] \times f([a,b]), \qquad t \to (t, f(t))$$

is also a $\mathcal{C}^1$ function and hence is rectifiable, therefore has the length

$$\int_a^b \sqrt{\left(\frac{dt}{dt}\right)^2 + \left(\frac{df(t)}{dt}\right)^2} dt = \int_a^b \sqrt{1 + f'(t)} dt$$

2.

Since x and y plays the same role in the domain K, if K is normal with respect to x-axis means that K is also normal with respect to the y-axis. We consdier

$$\phi_1 : \mathbb{R} \to \mathbb{R}, \qquad x \to 0$$

and

$$\phi_2 : \mathbb{R} \to \mathbb{R}, \qquad x \to 1 - x$$

Then let

$$\sigma : [a, b] \to [a, b], \qquad x \to x$$

We have the curve γ_1 and γ_2

$$\gamma_1 : [a, b] \to \mathbb{R}^2, \qquad t \to (\sigma(t), \phi_1(\sigma(t))) = (t, 0)$$

$$\gamma_1 : [a, b] \to \mathbb{R}^2, \qquad t \to (\sigma(t), \phi_2(\sigma(t))) = (t, 1 - t)$$

which is clearly a $\mathcal{C}^1$ curves and hence K is a normal domain.

3.

a.

$S_1(1) = 2\pi$ since it is all the point in the set $\{(x_1, x_2) : x_1^2 + x_2^2 = 1$, which is a circle with radius 1.
$S_2(1) = 4\pi$ since it is all the point in the set $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$, which is a sphere with radius 1.

b.

Consider $\phi(x) = rx$ then apply the change of variables theorem, we have that

$$\begin{aligned}
S_n(r) &= 2^n \int_0^r \int_0^{\sqrt{r^2 - x_1^2}} \dots \int_0^{\sqrt{r^2 - x_1^2 - x_2^2 - \dots - x_{n-1}^2}} |N| dx_n \dots dx_2 dx_1 \\
&= 2^n |\det J_\phi|^n \int_0^1 \int_0^{\sqrt{1-(\frac{x_1}{r})^2}} \dots \int_0^{\sqrt{1-(\frac{x_1}{r})^2 - (\frac{x_2}{r})^2 - \dots - (\frac{x_{n-1}}{r})^2}} |N| d\left(\frac{x_n}{r}\right) \dots d\left(\frac{x_2}{r}\right) d\left(\frac{x_1}{r}\right) \\
&= r^n S_n(1)
\end{aligned}$$

c.

Since swapping columns of a determinant return the same answer or the negative value of that. That is for a specific $(i_1, i_2, \dots, i_n)$ and a composition σ mapping $\{1, 2, \dots, n\}$ to itself then

$$\left(\frac{\partial(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n})}{\partial(x_1, x_2, \dots, x_n)}\right)^2 = \left(\frac{\partial(\phi_{i_{\sigma(1)}}, \phi_{i_{\sigma(2)}}, \dots, \phi_{i_{\sigma(n)}})}{\partial(x_1, x_2, \dots, x_n)}\right)^2$$

and since there is exactly $n!$ ways of ways to shuffle the columns and there is only 1 ways to put them in a non-ascending order. We have that

$$\begin{aligned}
&\frac{1}{n!} \sum_{i_1=1}^{n+1} \sum_{i_2=1}^{n+1} \dots \sum_{i_n=1}^{n+1} \left|\frac{\partial(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n})}{\partial(x_1, x_2, \dots, x_n)}\right|^2 \\
&= \sum_{\substack{i_1, i_2, \dots, i_n = 1 \\ i_1 \le i_2 \le \dots \le i_n}}^{n} \left(\frac{\partial(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n})}{\partial(x_1, x_2, \dots, x_n)}\right)^2 \\
&= \sum_{\substack{i_1, i_2, \dots, i_n = 1 \\ i_1 < i_2 < \dots < i_n}}^{n} \left(\frac{\partial(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n})}{\partial(x_1, x_2, \dots, x_n)}\right)^2 + \underbrace{\sum_{\substack{i_1, i_2, \dots, i_n = 1 \\ i_1 \le i_2 \le \dots \le i_n \\ \exists j_1, j_2 : i_{j_1} = i_{j_2}}}^{n} \left(\frac{\partial(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n})}{\partial(x_1, x_2, \dots, x_n)}\right)^2}_{=0} \\
&= \sum_{j=1}^{n} \left|\frac{\partial(\phi_1, \phi_2, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_{n+1})}{\partial(x_1, x_2, \dots, x_n)}\right|^2 + \left|\frac{\partial(\phi_1, \phi_2, \dots, \phi_n)}{\partial(x_1, x_2, \dots, x_n)}\right|^2 \\
&= \sum_{j=1}^{n} \left|\frac{\partial(\phi_1, \phi_2, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_{n+1})}{\partial(x_1, x_2, \dots, x_n)}\right|^2 + 1
\end{aligned}$$

d.

$$
\begin{aligned}
|N(x_1, x_2, \ldots, x_n)| &= \left(\sum_{j=1}^{n} \left| \frac{\partial(\phi_1, \phi_2, \ldots, \phi_{j-1}, \phi_{j+1}, \ldots, \phi_{n+1})}{\partial(x_1, x_2, \ldots, x_n)} \right|^2 + 1 \right)^{1/2} \\
&= \left(\sum_{j=1}^{n} \left| \frac{\partial \phi_{n+1}}{x_j} \right|^2 + 1 \right)^{1/2} \\
&= \left(1 + \sum_{j=1}^{n} \left| \frac{-x_j}{\sqrt{r^2 - x_1^2 - x_2^2 - \ldots x_n^2}} \right|^2 \right)^{1/2} \\
&= \left(1 + \sum_{j=1}^{n} \frac{x_j^2}{r^2 - \sum_{i=1}^{n} x_i^2} \right)^{1/2} \\
&= \left(1 + \frac{\sum_{j=1}^{n} x_j^2}{r^2 - \sum_{i=1}^{n} x_i^2} \right)^{1/2} \\
&= \left(\frac{r^2}{r^2 - \sum_{i=1}^{n} x_i^2} \right)^{1/2} \\
&= \frac{r}{\sqrt{r^2 - \sum_{i=1}^{n} x_i^2}}
\end{aligned}
$$

e.

We have that

$$
\begin{aligned}
&\frac{S_{n-2}\left(\sqrt{1 - x_1^2 - x_2^2}\right)}{\sqrt{1 - x_1^2 - x_2^2}} \\
=&2^{n-2} \int_0^{\sqrt{1-x_1^2-x_2^2}} \int_0^{\sqrt{1-x_1^2-x_2^2-x_3^2}} \ldots \int_0^{\sqrt{1-x_1^2-x_2^2-\ldots-x_{n-1}^2}} \frac{|N(x_3, x_4, \ldots, x_n)|}{\sqrt{1 - x_1^2 - x_2^2}} dx_n \ldots dx_4 dx_3 \\
=&2^{n-2} \int_0^{\sqrt{1-x_1^2-x_2^2}} \int_0^{\sqrt{1-x_1^2-x_2^2-x_3^2}} \ldots \int_0^{\sqrt{1-x_1^2-x_2^2-\ldots-x_{n-1}^2}} \frac{1}{\sqrt{1 - x_1^2 - x_2^2 - \ldots - x_n^2}} dx_n \ldots dx_4 dx_3
\end{aligned}
$$

Hence

$$
\begin{aligned}
&4 \int_0^1 \int_0^{\sqrt{1-x_1^2}} \frac{S_{n-2}(\sqrt{1 - x_1^2 - x_2^2})}{\sqrt{1 - x_1^2 - x_2^2}} \\
=&4 \int_0^1 \int_0^{\sqrt{1-x_1^2}} \ldots \int_0^{\sqrt{1-x_1^2-x_2^2-\ldots-x_n^2}} 2^{n-2} \frac{1}{\sqrt{1 - \sum_{j=1}^{n} x_j^2}} dx_n \ldots dx_2 dx_1 \\
=&2^n \int_0^1 \int_0^{\sqrt{1-x_1^2}} \ldots \int_0^{\sqrt{1-x_1^2-x_2^2-\ldots-x_n^2}} |N(x_1, x_2, \ldots, x_n)| dx_n \ldots dx_2 dx_1 \\
=&S_n(1)
\end{aligned}
$$

f.

Consider the spherical coordinates, let

$$\phi : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (r, \theta, \sigma) \to (r\cos\theta\cos\sigma, r\cos\theta\sin\theta, r\sin\theta)$$

and $K = [0,1] \times [0, \pi/2] \times [0, \pi/2]$ so that
$\phi(K) = \{(x, y, z) : x^2 + y^2 + z^2 \le 1, x, y, z > 0\}$

$$\begin{aligned}
S_n(1) &= 4\int_0^1 \int_0^{\sqrt{1-x_1^2}} \frac{S_{n-2}(\sqrt{1-x_1^2-x_2^2})}{\sqrt{1-x_1^2-x_2^2}} dx_2 dx_1 \\
&= 4\int_0^1 \int_0^{\sqrt{1-x_1^2}} \frac{S_{n-2}(1) \cdot \left(\sqrt{1-x_1^2-x_2^2}\right)^{n-2}}{\sqrt{1-x_1^2-x_2^2}} dx_2 dx_1 \\
&= 4\int_0^1 \int_0^{\sqrt{1-x_1^2}} S_{n-2}(1) \cdot \left(\sqrt{1-x_1^2-x_2^2}\right)^{n-3} dx_2 dx_1 \\
&= 4S_{n-2}(1) \int_0^1 \int_0^{\sqrt{1-x_1^2}} \int_0^{\sqrt{1-x_1^2-x_2^2}} (n-3) \cdot x_3^{n-4} dx_3 dx_2 dx_1 \\
&= 4(n-3)S_{n-2}(1) \int_0^1 \int_0^{\sqrt{1-x_1^2}} \int_0^{\sqrt{1-x_1^2-x_2^2}} x_3^{n-4} dx_3 dx_2 dx_1 \\
&= 4(n-3)S_{n-2}(1) \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} (r\sin\theta)^{n-4} \cdot r^2 \cos\theta d\sigma d\theta dr \\
&= 4\frac{\pi}{2}(n-3)S_{n-2}(1) \int_0^1 r^{n-2} \int_0^{\pi/2} (\sin\theta)^{n-4} \cdot \cos\theta d\theta dr \\
&= 4\frac{\pi}{2}(n-3)S_{n-2}(1) \int_0^1 r^{n-2} \left(\frac{(\sin\theta)^{n-3}}{n-3}\right)\Bigg|_{\theta=0}^{\pi/2} dr \\
&= 2\pi S_{n-2}(1) \frac{n-3}{n-3} \int_0^1 r^{n-2} dr \\
&= 2\pi S_{n-2}(1) \left.\frac{r^{n-1}}{n-1}\right|_{r=0}^{1} \\
&= 2\pi S_{n-2}(1) \frac{1}{n-1}
\end{aligned}$$

First we prove the case for odd $n = 2m - 1$ for all natural m. We have the base case:

$$S_1(1) = \frac{2\pi^1}{(1-1)!} = 2\pi$$

which is the same as the answers we have in part a and the inductive steps:

Given that $S_{2m-1}(1) = \frac{2\pi^m}{(m-1)!}$, we have that

$$\begin{aligned}
S_{2(m+1)-1}(1) = S_{2m+1}(1) &= 2\pi S_{2m-1}(1) \cdot \frac{1}{2m} \\
&= \pi \cdot \frac{1}{m} \cdot \frac{2\pi^m}{(m-1)!} \\
&= \frac{2\pi^{m+1}}{m!}
\end{aligned}$$

For even $n = 2m$, we have the base case:

$$S_2(1) = \frac{(4\pi)^1 \cdot (1-1)!}{(2-1)!} = 4\pi$$

which is the same as the answers we have in part a and the inductive steps:

Given that $S_{2m}(1) = \frac{(4\pi)^m \cdot (m-1)!}{(2m-1)!}$, we have that

$$\begin{aligned}
S_{2(m+1)}(1) = S_{2m+2}(1) &= 2\pi S_{2m}(1) \cdot \frac{1}{2m+1} \\
&= 2\pi \cdot \frac{1}{2m+1} \cdot \frac{(4\pi)^m \cdot (m-1)!}{(2m-1)!} \cdot \frac{2 \cdot m}{2m} \\
&= 4\pi \cdot \frac{(4\pi)^m \cdot (m-1)! \cdot m}{2m(2m+1)(2m-1)!} \\
&= \frac{(4\pi)^{m+1} m!}{(2m+1)!}
\end{aligned}$$

g.

From part f, we know that

$$\lim_{m\to\infty} S_{2m-1}(1) = 0$$

as

$$\lim_{m\to\infty} \frac{S_{2m-1}(1)}{S_{2m+1}(1)} = \lim_{m\to\infty} \frac{\pi}{m} = 0$$

and

$$\lim_{m\to\infty} S_{2m}(1) = 0$$

as

$$\lim_{m\to\infty} \frac{S_{2m+2}(1)}{S_{2m}(1)} = \lim_{m\to\infty} \frac{2\pi}{2m+1} = 0$$

Hence,

$$\lim_{n\to\infty} S_n(1) = 0$$

4.

We know that the surface area of the frostum is $\pi(r_1+r_2)l$, then we partition the x-axis into $\{x_1, x_2, \ldots, x_n\}$.
Consider the area of the surface with $x_i \leq x \leq x_{i+1}$. Then $r_1 = f(x_i), r_2 = f(x_{i+1}), \exists x_i^* : 2f(x_i^*) = f(x_i) + f(x_{i+1})$ and from question 1,
$l = \int_{x_i}^{x_{i+1}} \sqrt{1 + f'(t)^2} dt$.
Hence, we have that the area having x between x_i and x_{i+1} which we denote A_i is

$$2\pi f(x_i^*) \int_{x_i}^{x_{i+1}} \sqrt{1 + f'(t)^2} dt = 2\pi \int_{x_i}^{x_{i+1}} f(x_i^*) \sqrt{1 + f'(t)^2} dt$$

Hence, the total area of the surface is

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^{n-1} 2\pi \int_{x_i}^{x_{i+1}} f(x_i^*) \sqrt{1 + f'(t)^2} dt \\
=&2\pi \lim_{n\to\infty} \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} f(x_i^*) \sqrt{1 + f'(t)^2} dt \\
=&2\pi \int_a^b f(t) \sqrt{1 + f'(t)^2} dt
\end{aligned}$$

as both $f(x_i^*)$ and $f'(t)$ is bounded because f is uniformly continuous.