Fall 2022, Math 328, Homework 1

Due: End of day on 2022-09-19

1 (10 points)

Let G be a group, $g \in G$ an element, and $a \in \mathbb{Z}$ an integer. Define g^a inductively as follows:

1. $g^0 = 1$ and $g^{n+1} = g \cdot g^n$ for all $n \geq 0$.
2. g^{-1} is the inverse of g and $g^{-n} = (g^{-1})^n$ for $n \geq 0$.

Prove the following assertions: For a group G, $g \in G$, and two integers a, b, one has

$$g^{a+b} = g^a \cdot g^b, \quad g^{ab} = (g^a)^b.$$

Assume furthermore that $h \in G$ is another element which commutes with g, i.e. $g \cdot h = h \cdot g$. Prove that $(gh)^a = g^a \cdot h^a$ for all $a \in \mathbb{Z}$.

2 (10 points)

Let G be a group and $g, h \in G$ two elements.

1. Prove that g and g^{-1} have the same order.
2. Prove that g and $h \cdot g \cdot h^{-1}$ have the same order.
3. Prove that $g \cdot h$ and $h \cdot g$ have the same order.
4. Prove that $g = g^{-1}$ if and only if g has order 1 or 2.

3 (10 points)

A group G is called *abelian* provided that its binary operation is commutative, so that $g \cdot h = h \cdot g$ for all $g, h \in G$. Suppose G is a group such that $g^2 = 1$ for all $g \in G$. Prove that G is abelian. Give an example showing that this fails if 2 is replaced by 6.

Remark: In fact this fails if 2 is replaced by any positive integer n which is strictly larger than 2. More on this later.

4 (10 points)

Let G be a group and $g \in G$ an element.

1. Suppose g has finite order n. Prove that $1, g, g^2, \ldots, g^{n-1}$ are all distinct.
2. Suppose g has infinite order. Prove that the map $\mathbb{Z} \to G$ given by $a \mapsto g^a$ is injective with image $\{g^a \mid a \in \mathbb{Z}\}$.
3. Suppose that g has finite order n and let $a \in \mathbb{Z}$ be given. Prove that $g^a = g^r$ where r is the remainder when a is divided by n. Prove that this yields a well-defined injective map $\mathbb{Z}/n \to G$ given by $(a \bmod n) \mapsto g^a$, and that its image is $\{g^a \mid a \in \mathbb{Z}\}$.
4. Prove that the size of the subset $\{g^a \mid a \in \mathbb{Z}\}$ of G agrees with the order of g.

5 (10 points)

Let G be a finite group of even order. Prove that G contains an element of order 2.

Hint: What can you say about the size of the set $\{g \in G \mid g \neq g^{-1}\}$?

6 (10 points)

Suppose that G is a set endowed with an associative binary operation $\cdot : G \times G \to G$ which satisfies the following properties:

1. There exists some $e \in G$ such that for all $x \in G$, one has $e \cdot x = x$.
2. For all $g \in G$, there exists some $h \in G$ such that $h \cdot g = e$, where e is the element from item (1).

Show that $(G, \cdot)$ is a group.