1.

a.

Let $x, y \in U$ such that $f(x) = f(y)$ and $\xi = y - x$. From the Taylor's theorem and the fact that $x + \theta\xi$ is in the convex set U for $\theta \in [0,1]$, we have that for each $j = 1, \dots, N$, there exists a number $\theta_j \in [0,1]$ such that

$$f_j(y) = f_j(x + \xi) = f_j(x) + \sum_{k=1}^{N} \frac{\partial f_j}{\partial x_k}(x + \theta_j \xi)\xi_k = f_j(x)$$

It follows that

$$\sum_{k=1}^{N} \frac{\partial f_j}{\partial x_k}(x + \theta_j \xi)\xi_k = 0$$

for $j = 1, \dots, N$. Let

$$A := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x + \theta_1 \xi) & \dots & \frac{\partial f_1}{\partial x_N}(x + \theta_1 \xi) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1}(x + \theta_N \xi) & \dots & \frac{\partial f_1}{\partial x_N}(x + \theta_N \xi) \end{bmatrix}$$

so that $A\xi = 0$. However, since the points in the set $\{x + \theta\xi | \theta \in [0,1]\}$ are collinear points. We have that $\det(A) \neq 0$ and therefore $\xi = 0$ which means that $x = y$.

b.

For any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have that

$$\det \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(x_1, y_1) & \dfrac{\partial f_1}{\partial x_2}(x_1, y_1) \\ \dfrac{\partial f_2}{\partial x_1}(x_1, y_1) & \dfrac{\partial f_2}{\partial x_2}(x_1, y_1) \end{bmatrix} = \det \begin{bmatrix} 3x_1^2 & -1 \\ e^{x_1+y_1} & e^{x_2+y_2} \end{bmatrix} = 3x_1^2 \cdot e^{x_2+y_2} + e^{x_1+y_1}$$

which is > 0 for all $(x_1, y_1), (x_2, y_2)$ and hence f is injective.

2.

a.

$$\det J_f(x,y) = \det \begin{bmatrix} \dfrac{\sqrt{x^2+y^2} - x \cdot \dfrac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} & \dfrac{-xy}{(x^2+y^2)^{3/2}} \\ \dfrac{-xy}{x^2+y^2} & \dfrac{\sqrt{(x^2+y^2)^{3/2}} - y \cdot \dfrac{2y}{2\sqrt{x^2+y^2}}}{x^2+y^2} \end{bmatrix}$$

$$= \det \begin{bmatrix} \dfrac{y^2}{(x^2+y^2)^{3/2}} & \dfrac{-xy}{(x^2+y^2)^{3/2}} \\ \dfrac{-xy}{(x^2+y^2)^{3/2}} & \dfrac{x^2}{(x^2+y^2)^{3/2}} \end{bmatrix} = 0$$

b.

For all (x, y), we have that

$$|f(x,y)| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = 1$$

Therefore, $f(U)$ is the circle around the origin with radius 1. It does not contain any non-empty open subset as for all point $(x, y) \in f(U) : \forall \epsilon > 0 : (x + \epsilon, y) \notin f(U)$

3.

Consider

$$f : \mathbb{R} \to \mathbb{R}, \qquad x \to x^3$$

Then $f \in \mathcal{C}^1(U, \mathbb{R}^N)$ where $U = \mathbb{R}$ and $N = 1$.
$\det J_f(0) = 0$ but every neighborhood $(-\epsilon, \epsilon)$ is mapped to $(-\epsilon^3, \epsilon^3)$ which is open.

4.

a.

Consider the function

$$h : \mathbb{R}^N \to \mathbb{R}, \qquad x \to |x|$$

It is obvious that for each $j = 1, \ldots, N$, $\dfrac{\partial h}{\partial x_j}(x) = \dfrac{x_j}{|x|}$.
If there is a local maximum at $f(x_0) \neq (0, \ldots, 0)$, then applying the chain rule, we have that for each $j = 1, \ldots, N$

$$0 = \frac{\partial g}{\partial x_j}(x_0) = \frac{\partial (h \circ f)}{\partial x_j}(x_0) = \frac{\partial h}{\partial x_j}(f(x_0)) \cdot \underbrace{\frac{\partial f}{\partial x_j}(x_0)}_{\neq 0}$$

and hence $f(x_0) = (0, \ldots, 0)$, which is a contradiction. At $f(x_0) = (0, \ldots, 0)$, $g(x_0) = h(f(x_0)) = 0$ and hence it is clearly the global minimum as norm is always ≥ 0. Therefore, there is no local maximum.

b.

Since $\overline{U}$ is compact $g(\overline{U})$ is also compact and hence $g(\overline{U})$ must attain its maximum since every converges sequence converges to a point in the set and hence the supremum is the maximum.
Since $\tilde{f}$ does not attain its local maximum and hence global maximum on int $\overline{U}$, the global maximum must be attained on the boundary $\partial U = \partial \overline{U}$.