1.

We have that
$$\frac{n^2+a^2}{(n^2-a^2)^2} = \frac{1}{2(n+a)^2} + \frac{1}{2(n-a)^2}$$

Let $f = \dfrac{1}{(z+a\pi)^2\tan(z)}$. Then f has a simple pole at $-a\pi$ and $N\pi$ for all $N \in \mathbb{Z}$. We have

$$\operatorname{Res}(f, N\pi) = \lim_{z\to N\pi}(z-N\pi)f(z) = \frac{1}{(N+a)^2\pi^2}\lim_{z\to N\pi}\frac{z-N\pi}{\tan(z)} = \frac{1}{(N+a)^2\pi^2}$$

and
$$\operatorname{Res}(f, -a\pi) = \frac{d}{dz}\left((z+a\pi)^2 f(z)\right)_{z=-a\pi} = -\frac{1}{\sin^2(-a\pi)}$$

We also have that
$$\lim_{n\to\infty}\left|\int_{\partial D_n}\frac{dz}{(z+a\pi)^2\tan(z)}\right| = 0$$

as the circumference of a square is $4(N\pi+\pi/2)$ and for $z = x+iy$, we have

$$\lim_{x\to\pm\infty}|(z+a\pi)^2| = \infty(\text{"with degree 2"}) \text{ and } \lim_{x\to\pi/2}|\tan(z)| = \infty(\text{"with degree 1"})$$

and similarly for y.
Thus

$$-\frac{1}{\sin^2(-a\pi)} + \sum_{N=-\infty}^{\infty}\frac{1}{(N+a)^2\pi^2} = 0$$

Thus
$$\sum_{N=1}^{\infty}\frac{1}{\pi^2}\left(\frac{1}{(N+a)^2}+\frac{1}{(N-a)^2}\right) + \frac{1}{a^2\pi^2} = \frac{1}{\sin^2(a\pi)}$$

and hence
$$\sum_{n=1}^{\infty}\frac{n^2+a^2}{(n^2-a^2)^2} = \frac{1}{2}\left(\frac{1}{\sin^2(a\pi)} - \frac{1}{a^2\pi^2}\right)\pi^2 = \frac{\pi^2}{2}\left(\frac{1}{\sin^2(a\pi)} - \frac{1}{a^2\pi^2}\right)$$

2.

Suppose that $f(z)$ has an essential singularity at 0. Then by open mapping theorem, there exists $r > 0$ such that

$$f(B) \supset D \text{ for } B = \left\{\left|z - \frac{1}{2}\right| < \frac{1}{4}\right\} \text{ and } D = \left\{\left|w - f\left(\frac{1}{2}\right)\right| < r\right\}$$

Let $U = \{0 < |z| < 1/4\}$. Since $B \cap U = \varnothing$ and f is 1-to-1,

$$f(B) \cup f(U) = \varnothing$$

and hence

$$f(U) \subset \mathbb{C} \backslash f(B) \subset \mathbb{C} \backslash D$$

and

$$\overline{f(U)} \subset \overline{\mathbb{C} \backslash D} = \mathbb{C} \backslash D$$

But by Casorati-Weierstrass, $\overline{f(U)} = \mathbb{C}$ which is a contradiction.

3.

If $f/g \circ \gamma$ is positive and real at z_0 then we have that $f/g(z_0) = c$

$$|a_1 f(z)+b_1 g(z)|+|a_2 f(z)+b_2 g(z)| = |f(z)|(|a_1+b_1c+a_2+b_2c|) = |(a_1+a_2)f(z)+(b_1+b_2)g(z)|$$

which is a contradiction, hence $f/g \circ \gamma$ is contained in $\mathbb{C}\backslash[0,\infty)$. Then applying the argument principle to f/g on the curve γ, we have that

$$\sum_{p \in Z_f} \nu(\gamma, p)\text{mult}_p f = \sum_{p \in Z_g} \nu(\gamma, p)\text{mult}_p g$$

4.

Let $h = 1 + \frac{f}{g}$, hence $h(D) \subseteq \{z : \mathrm{Re}(z) > 0\}$. Thus by the argument principle, the zeros of $f + g$ is the same as the zeros of g. Since $f + g$ does not have zero because $|f| \neq |g|$ in D, g does not have zero in D. Then $|f/g|$ is holomorphic on D thus attain a local maximum on ∂D which is less than 1, which confirms that $|f| < |g|$

5.

First, we can rewrite $f'(z) = na_n(z - z_1)(z - z_2)\dots(z - z_{n-1})$ and since $f'(z) \neq 0 \forall z \in D$, we have that $|z_k| \geq 1$, then $f'(0) = na_n(-z_1)(-z - 2)\dots(-z_{n-1}) = 1$ thus $|a_n| < \frac{1}{n}$.
On the other hand

$$f''(z) = \sum_{k=1}^{n-1} \frac{f'(z)}{z - z_k}$$

and hence

$$2|a_2| = |f''(0)| = \left| \sum_{k=1}^{n-1} \frac{1}{-z_k} \right| \leq n - 1 \implies |a_2| \leq \frac{n-1}{2}$$