1.

We have that

$$L^* = \begin{pmatrix} -\lambda_{1\to} & \lambda_{2\to 1} \\ \lambda_{1\to} & -\lambda_{2\to} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$$

Hence, we have

$$\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \pi(0) \\ \pi(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which can be solved to $\pi(0) = \frac{2}{3}$ and $\pi(1) = \frac{1}{3}$. We also know that

$$P_t(1 \to 1) = \frac{2}{3} + \frac{1}{3}e^{-3t}$$

$$P_t(2 \to 1) = \frac{2}{3} - \frac{2}{3}e^{-3t}$$

$$P_t(1 \to 2) = \frac{1}{3} - \frac{1}{3}e^{-3t}$$

$$P_t(2 \to 2) = \frac{1}{3} + \frac{2}{3}e^{-3t}$$

and thus

$$\lim_{t\to\infty} P_t(1 \to 1) = \frac{2}{3}$$

$$\lim_{t\to\infty} P_t(2 \to 1) = \frac{2}{3}$$

$$\lim_{t\to\infty} P_t(1 \to 2) = \frac{1}{3}$$

$$\lim_{t\to\infty} P_t(2 \to 2) = \frac{1}{3}$$

2.

First, define a markov chain $A(t) = \begin{pmatrix} X(t) \\ \lambda(t) \end{pmatrix}$. Then let's redefine the states

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is state } 0$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is state } 1$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ is state } 2$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is state } 3$$

$$L_A^*/\Delta = \begin{pmatrix} \Delta^2 - 2\Delta & 2\Delta - 2\Delta^2 & 0 & 8\Delta^2 \\ \Delta - \Delta^2 & 2\Delta^2 - 3\Delta & 0 & 4\Delta - 8\Delta^2 \\ \Delta - \Delta^2 & 2\Delta^2 & -\Delta & 2\Delta - 8\Delta^2 \\ \dots \end{pmatrix} \cdot \frac{1}{\Delta} \to \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 0 & 4 \\ 1 & 0 & -1 & 2 \\ \dots \end{pmatrix}$$

as $\Delta \to 0$. Hence, we have that

$$\begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 0 & 4 \\ 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi(0) \\ \pi(1) \\ \pi(2) \\ \pi(3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

which can be solved

$$\begin{pmatrix} \pi(0) \\ \pi(1) \\ \pi(2) \\ \pi(3) \end{pmatrix} = \begin{pmatrix} \frac{2}{9} \\ \frac{2}{9} \\ \frac{4}{9} \\ \frac{1}{9} \end{pmatrix}$$

Thus $\pi(X = 0) = \frac{4}{9}$ and $\pi(X = 1) = \frac{5}{9}$

3.

First, we have that

$$\gamma_{X_u \to j}(u) = \frac{\lambda^2_{X_u \to j} u}{1 + \lambda_{X_u \to j} u}$$

Then we can define

$$m(X_u, j, u) = \frac{\gamma_{X_u \to j}}{\lambda_{X_u \to j}} - 1 = \frac{\lambda_{X_u \to j} u}{1 + \lambda_{X_u \to j} u} - 1 = -\frac{1}{1 + \lambda_{X_u \to j} u}$$

and

$$l(X_u, u) = \lambda_{X_u \to} - \gamma_{X_u \to}(u) = \sum_{j \neq X_u} \lambda_{X_u \to j} - \frac{\lambda^2_{X_u \to j} u}{1 + \lambda_{X_u \to j} u} = \sum_{j \neq X_u} \frac{\lambda_{X_u \to j}}{1 + \lambda_{X_u \to j} u}$$

Thus we can find

$$dA_t = A_{t^-} m(X_t^-, X_t, t) dN_t + A_t l(X_t, t) dt$$

And hence

$$\begin{aligned}
A_t &= \exp\left(\int_0^t l(X_s, s) ds\right) \prod_{0 < s \leq t} \left(1 + m(X_t^-, X_t, t) \Delta N_s\right) \\
&= \exp\left(\int_0^t \sum_{j \neq X_s} \frac{\lambda_{X_s \to j}}{1 + \lambda_{X_s \to j} s} ds\right) \prod_{0 < s \leq t} \left(1 - \frac{\Delta N_s}{1 + \lambda_{X_{s-} \to X_s} s}\right) \\
&= \exp\left(\sum_{X_t \neq j} \ln(1 + \lambda_{X_t \to j} t)\right) \prod_{0 < s \leq t} \left(1 - \frac{\Delta N_s}{1 + \lambda_{X_{s-} \to X_s} s}\right) \\
&= \prod_{X_t \neq j} (1 + \lambda_{X_t \to j} t) \prod_{0 < s \leq t} \left(1 - \frac{\Delta N_s}{1 + \lambda_{X_{s-} \to X_s} s}\right)
\end{aligned}$$

so that

$$E^Q[A_t f(X_t) | \mathcal{F}_s] - A_s f(X_s) - \int_s^t E^Q[A_u L_u f(X_u) | \mathcal{F}_s] du = 0$$

4.

We have that

$$
\begin{aligned}
E^Q[E^P[Z_t|\mathcal{G}_s]A_s 1_G] &= E^Q[E^Q[A_s E^P[Z_t|\mathcal{G}_s]1_G|\mathcal{G}_s]] \\
&= E^Q[E^Q[A_T|\mathcal{G}_s]E^P[Z_t|\mathcal{G}_s]1_G] \\
&= E^Q[E^Q[A_T E^P[Z_t|\mathcal{G}_s]1_G|\mathcal{G}_s]] \\
&= E^Q[A_T E^P[Z_t|\mathcal{G}_s]1_G] \\
&= E^P[E^P[Z_t|\mathcal{G}_s]1_G] \\
&= E^P[E^P[Z_t 1_G|\mathcal{G}_s]] \\
&= E^P[Z_t 1_G] \\
&= E^Q[A_t Z_t 1_G]
\end{aligned}
$$