1.

a.

Consider the function $f(z) = \sin(z)$

$$\begin{aligned}\int_{|z|=2} \frac{\sin(z)}{z^2+1} dz &= \int_{|z|=2} \frac{1}{2i} \left(\frac{\sin(z)}{z-i} - \frac{\sin(z)}{z+i} \right) dz \\ &= \frac{1}{2i} \left(\int_{|z|=2} \frac{\sin(z)}{z-i} dz - \int_{|z|=2} \frac{\sin(z)}{z+i} dz \right) \\ &= \frac{1}{2i} \left(\sin(i) - \sin(-i) \right) \\ &= \frac{1}{i} \sin(i)\end{aligned}$$

b.

Consider the constant function $f(z) = 1$, then

$$\begin{aligned}&\int_\gamma \frac{z}{z^3-1} dz \\ =&\int_\gamma \left(\frac{\frac{1}{3}}{z-1} + \frac{-\frac{1}{6} + \frac{i}{2\sqrt{3}}}{z - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)} + \frac{-\frac{1}{6} - \frac{i}{2\sqrt{3}}}{z - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)} \right) dz \\ =&2\pi i \left(\frac{1}{3} - \frac{1}{6} + \frac{i}{2\sqrt{3}} - \frac{1}{6} - \frac{i}{2\sqrt{3}} \right) \\ =&0\end{aligned}$$

2.

$$\cot(z) = \cos(z)/\sin(z)$$

and since $\sin(z) \neq 0 \iff z = n\pi$ for some $n \in \mathbb{Z}$. $\cot(z)$ has a complex antiderivative on D as D is starshaped with center $\pi/2$. We know that

$$f(\pi/2) = \ln|\sin(\pi/2)| + C = 0 \implies C = 0$$

Hence,

$$f(i) = \ln|sin(i)|$$

3.

We have that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|z|=R}\frac{z^m f'(z)}{f(z)}dz \\
=&\frac{1}{2\pi i}\int_{|z|=R}\sum_{i=1}^{n}\frac{z^m}{z-r_1}dz \\
=&\sum_{i=1}^{n}\frac{1}{2\pi i}\int_{|z|=R}\frac{z^m}{z-r_1}dz \\
=&\sum_{i=1}^{n} z_i^m = b_m
\end{aligned}$$

and

$$\begin{aligned}
&b_{m+n} + a_1 b_{m+n-1} + \ldots + a_n b_m \\
=&\sum_{i=1}^{n} r_i^{m+n} + a_1\sum_{i=1}^{n} r_i^{m+n-1} + \ldots + a_n\sum_{i=1}^{n} r_i^m \\
=&\sum_{i=1}^{n} r_i^{m+n} + a_1 r_i^{m+n-1} + \ldots + a_n r_i^m \\
=&\sum_{i=1}^{n} r_i^m(r_i^n + a_1 r_i^{n-1} + \ldots + a_n) \\
=&0
\end{aligned}$$

4.

Let

$$h(z) = f(\exp(z)) = g(\exp(iz))$$

It is obvious that

$$f(\exp(z + i2\pi)) = f(\exp(z)) \text{ and } g(\exp(iz)) = g(\exp(i(z + 2\pi)))$$

which means that

$$h(z) = h(z + 2\pi) = h(z + 2\pi i)$$

Consider $D = \{x + iy : 0 \leq x \leq 2\pi, 0 \leq y \leq 2\pi\}$. Then there exists M such that

$$|h(z)| \leq M \text{ for every } z \in D$$

Then for every $z \in \mathbb{C}$, there is a $z_0 \in D$ such that $h(z) = h(z_0)$ and hence h is bounded. Therefore, h is constant.

5.

Let $f(z) = u(z) + iv(z)$, then $f^2 = u^2 - v^2 + 2uvi$. We have that

$$\frac{\partial}{\partial x}(u^2 - v^2) = \frac{\partial}{\partial y}(2uv)$$

and

$$\frac{\partial}{\partial y}(u^2 - v^2) = -\frac{\partial}{\partial x}(2uv)$$

Therefore, we can get the system of equations

$$\begin{cases} u\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - v\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \\ u\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + v\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = 0 \end{cases}$$

Hence if $u \neq 0$ or $v \neq 0$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, which means f is analytic. If $u = v = 0$, then if $f^{(n)}(z) = 0$ for all n, $f \equiv 0$ in any ball around it. In the other case, $f'(z)$ does not vanish. Hence, we can find r such that $f(z) \neq 0$ for $0 < |z - z_0| < r$, which means that the Morera condition holds (Lemma 5.1).