1.

$$\begin{aligned}
\Delta(fg) &= \sum_{i=1}^{N} \frac{\partial^2 (fg)}{\partial x_i^2} \\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_i} \cdot g + \frac{\partial g}{\partial x_i} \cdot f \right) \\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_i} \cdot g \right) + \frac{\partial}{\partial x_i} \left(\frac{\partial g}{\partial x_i} \cdot f \right) \\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_i} \cdot g + \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} + \frac{\partial}{\partial x_i} \frac{\partial g}{\partial x_i} \cdot f + \frac{\partial g}{\partial x_i} \cdot \frac{\partial f}{\partial x_i} \\
&= \Delta f \cdot g + 2(\nabla f)(\nabla g) + f \cdot \Delta g
\end{aligned}$$

2.

$$\nabla f \times \nabla g = \left(\frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_3} - \frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_2}, \frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_3}, \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1} \right)$$

Hence,

$$\begin{aligned}
\operatorname{div}(\nabla f \times \nabla g) &= \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_3} - \frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_2} \right) \\
&+ \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_3} \right) \\
&+ \frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1} \right) \\
&= \frac{\partial^2 f}{\partial x_1 x_2}\frac{\partial g}{\partial x_3} + \frac{\partial f}{\partial x_2}\frac{\partial^2 g}{\partial x_1 x_3} - \frac{\partial^2 f}{\partial x_1 x_3}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_3}\frac{\partial^2 g}{\partial x_1 x_2} \\
&+ \frac{\partial^2 f}{\partial x_2 x_3}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_3}\frac{\partial^2 g}{\partial x_2 x_1} - \frac{\partial^2 f}{\partial x_2 x_1}\frac{\partial g}{\partial x_3} - \frac{\partial f}{\partial x_1}\frac{\partial^2 g}{\partial x_2 x_3} \\
&+ \frac{\partial^2 f}{\partial x_3 x_1}\frac{\partial g}{\partial x_2} + \frac{\partial f}{\partial x_1}\frac{\partial^2 g}{\partial x_3 x_2} - \frac{\partial^2 f}{\partial x_3 x_2}\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_2}\frac{\partial^2 g}{\partial x_3 x_1} \\
&= 0
\end{aligned}$$

as f and g are twice continuously partially differentiable hence

$$\forall i, j \in \{1, 2, 3\} : \frac{\partial^2 f}{\partial x_i x_j} = \frac{\partial^2 f}{\partial x_j x_i} \text{ and } \frac{\partial^2 g}{\partial x_i x_j} = \frac{\partial^2 g}{\partial x_j x_i}$$

3.

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right) \\
&= \frac{-1\cdot(x^2+y^2+z^2)^{3/2} - (-x)\cdot\frac{3}{2}\cdot(x^2+y^2+z^2)^{1/2}\cdot 2x}{(x^2+y^2+z^2)^3} \\
&= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}
\end{aligned}$$

Similarly,

$$\frac{\partial^2 f}{\partial y^2} = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

Hence,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

4.

$$\frac{\partial f}{\partial x_i} = \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right) \cdot \left(\frac{-2x_i}{4t}\right)$$

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right) \cdot \frac{4x_i^2}{16t^2} + \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right) \cdot \left(\frac{-2}{4t}\right) = \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right) \left(\frac{4x_i^2 - 8t}{16t^2}\right)$$

$$\frac{\partial f}{\partial t} = -\frac{N}{2} \frac{1}{t^{\frac{N}{2}+1}} \exp\left(-\frac{\|x\|^2}{4t}\right) + \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right) \left(\frac{\|x\|^2}{4t^2}\right)$$

$$\begin{aligned}
\Delta f = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} &= \sum_{i=1}^{N} \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right) \left(\frac{4x_i^2 - 8t}{16t^2}\right) \\
&= \sum_{i=1}^{N} \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right) \left(\frac{x_i^2}{4t^2}\right) + \sum_{i=1}^{N} \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right) \left(\frac{-8t}{16t^2}\right) \\
&= -\frac{N}{2} \frac{1}{t^{\frac{N}{2}+1}} \exp\left(-\frac{\|x\|^2}{4t}\right) + \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right) \left(\frac{\|x\|^2}{4t^2}\right) \\
&= \frac{\partial f}{\partial t}
\end{aligned}$$

5.

Since

$$F(x,t) = f(x \cdot v - wt)$$

We have that $\forall i \in \{1, 2, \dots, N\}$:

$$\frac{\partial F}{\partial x_i}(x,t) = v_i f'(x \cdot v - wt)$$
$$\implies \frac{\partial^2 F}{\partial x_i^2}(x,t) = v_i^2 f'(x \cdot v - wt)$$

$$\implies (\Delta F)(x,t) = \sum_{i=1}^{N} \frac{\partial^2 F}{\partial x_i^2} = f''(x \cdot v - wt) \sum_{i=1}^{N} v_i^2 = \|v\|^2 f''(x \cdot v - wt)$$

We also have that

$$\frac{\partial^2 F}{\partial t^2}(x,t) = w^2 f''(x \cdot v - wt) = c^2 \|v\|^2 f''(x \cdot v - wt)$$

Therefore,

$$\frac{1}{c^2} \frac{\partial^2 F}{\partial t^2}(x,t) = (\Delta F)(x,t)$$

6.

$\forall (x,y) \neq (0,0) \in \mathbb{R}^2$, f is twice partially differentiable at (x,y) with

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{-4y^3x(x^2-3y^2)}{(x^2+y^2)^3}$$
$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{4x^3y(y^2-3x^2)}{(x^2+y^2)^3}$$
$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{x^6+9y^2x^4-9y^4x^4-y^6}{(x^2+y^2)^3}$$
$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = -\frac{y^6+9x^2y^4-9x^4y^2-x^6}{(y^2+x^2)^3}$$

At $(0,0)$:

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \left(\frac{\partial}{\partial x}\left(\frac{-x(y^4+4x^4y^2-x^4)}{(y^2+x^2)^2}\right)\right)(0,0) = \lim_{\substack{h\to 0\\ h\neq 0}} \frac{\dfrac{-h(0^4+4h^40^2-h^4)}{(0^2+h^2)^2}-0}{h} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \left(\frac{\partial}{\partial y}\left(\frac{y(x^4+4y^4x^2-y^4)}{(x^2+y^2)^2}\right)\right)(0,0) = \lim_{\substack{h\to 0\\ h\neq 0}} \frac{\dfrac{h(0^4+4h^40^2-h^4)}{(0^2+h^2)^2}-0}{h} = -1$$

$$\frac{\partial^2 f}{\partial y^2}(0,0) = \left(\frac{\partial}{\partial y}\left(\frac{-x(y^4+4x^4y^2-x^4)}{(y^2+x^2)^2}\right)\right)(0,0) = \lim_{\substack{h\to 0\\ h\neq 0}} \frac{0-0}{h} = 0$$

$$\frac{\partial^2 f}{\partial y^2}(0,0) = \left(\frac{\partial}{\partial y}\left(\frac{-x(y^4+4x^4y^2-x^4)}{(y^2+x^2)^2}\right)\right)(0,0) = \lim_{\substack{h\to 0\\ h\neq 0}} \frac{0-0}{h} = 0$$

Hence, it is twice differentiable everywhere but $\dfrac{\partial^2 f}{\partial y \partial x} \neq \dfrac{\partial^2 f}{\partial x \partial y}$.
We have that $\forall (x,y) \neq (0,0)$

$$-1 = \frac{-x^2-y^2}{x^2+y^2} \leq \frac{x^2-y^2}{x^2+y^2} \leq \frac{x^2+y^2}{x^2+y^2} = 1$$

Therefore, if $xy \geq 0$, $-xy \leq xy\dfrac{x^2-y^2}{x^2+y^2} \leq xy$ and
$-xy \leq xy\dfrac{x^2-y^2}{x^2+y^2} \leq xy$ if $xy < 0$.
We have that $\lim_{(x,y)\to(0,0)} xy = 0 = \lim_{(x,y)\to(0,0)} -xy$.
Hence, $\lim_{(x,y)\to(0,0)} xy\dfrac{x^2-y^2}{x^2+y^2} = 0$ which means that f is continuous at $(0,0)$