1.

Apply seperation of variables, $u(x,t) = X(x)T(t)$, we have

$$\begin{aligned}
&\rho_0 X(x)T''(t) = T_0 X''(x)T(t) - \beta X(x)T'(t) \\
\implies & X(x)(\rho_u T''(t) + \beta T'(t)) = T_0 X''(x)T(t) \\
\implies & \frac{\rho_u T''(t) + \beta T'(t)}{T_0 T(t)} = \frac{X''(x)}{X(x)} = -\lambda
\end{aligned}$$

and

$$\begin{cases} u(0,t) = 0 \implies X(0) = 0 \\ u(L,t) = 0 \implies X(L) = 0 \end{cases}$$

Thus for every integer $n \geq 1$,

$$X_n(x) = \sin \frac{n\pi x}{L}$$

and

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

Now we need to solve for

$$\rho_u T''(t) + \beta T'(t) + T_0 \frac{n^2\pi^2}{L^2} T(t) = 0$$

which has no solution as

$$\beta^2 - 4\rho_u \frac{T_0 n^2 \pi^2}{L^2} < \beta^2 - \frac{4\rho_u \pi^2 T_0}{L^2} < 0$$

2.

Apply seperation of variables, $u(x,t) = X(x)T(t)$, we have

$$\begin{cases} u(0,t) = 0 \implies X(0) = 0 \\ u(L,t) = 0 \implies X(L) = 0 \\ u_t(x,0) = 0 \implies T'(0) = 0 \end{cases}$$

and

$$X(x)T''(t) = c^2 X''(x)T(t)$$
$$\implies \frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = -\lambda$$

and Thus for every integer $n \geq 1$,

$$X_n(x) = \sin \frac{n\pi x}{L}$$

and

$$\lambda_n = \frac{c^2 n^2 \pi^2}{L^2}$$

and thus

$$T''(t) + \lambda T(t) = 0$$

and

$$T(t) = c_1 \cos \frac{cn\pi t}{L} + c_2 \sin \frac{cn\pi t}{L}$$
$$T'(t) = \frac{cn\pi}{L} \left(-c_1 \sin \frac{cn\pi t}{L} + c_2 \cos \frac{cn\pi t}{L} \right)$$

Therefore,

$$T'(0) = \frac{cn\pi}{L} c_2 \cos \frac{cn\pi 0}{L} = \frac{cn\pi c_2}{L} = 0 \implies c_2 = 0$$

Thus $c_1 \neq 0$ to avoid trivial solution

$$T_n(t) = \cos \frac{cn\pi t}{L}$$

and

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L} \\ &= \sum_{n=1}^{\infty} \frac{A_n}{2} \left(\sin \frac{n\pi(x+ct)}{L} + \sin \frac{n\pi(x-ct)}{L} \right) \\ &= \frac{F(x+ct) - F(x-ct)}{2} \end{aligned}$$

As

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

from

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = f(x)$$

3.

Apply seperation of variables, $u(x,t) = X(x)T(t)$, we have

$$\begin{cases} u(0,t) = 0 \implies X(0) = 0 \\ u(L,t) = 0 \implies X(L) = 0 \\ u(x,0) = 0 \implies T(0) = 0 \end{cases}$$

and

$$\begin{aligned} X(x)T''(t) &= c^2 X''(x)T(t) \\ \implies \frac{T''(t)}{T(t)} &= c^2 \frac{X''(x)}{X(x)} = -\lambda \end{aligned}$$

and Thus for every integer $n \geq 1$,

$$X_n(x) = \sin \frac{n\pi x}{L}$$

and

$$\lambda_n = \frac{c^2 n^2 \pi^2}{L^2}$$

and thus

$$T''(t) + \lambda T(t) = 0$$

and

$$T(t) = c_1 \cos \frac{cn\pi t}{L} + c_2 \sin \frac{cn\pi t}{L}$$

Therefore,

$$T(0) = c_1 \cos \frac{cn\pi 0}{L} = 0 \implies c_1 = 0$$

Thus $c_2 \neq 0$ to avoid trivial solution

$$T_n(t) = \sin \frac{cn\pi t}{L}$$

and

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sin \frac{cn\pi t}{L} \\ &= \sum_{n=1}^{\infty} \frac{A_n}{2} \left(\cos \frac{n\pi(x-ct)}{L} - \cos \frac{n\pi(x+ct)}{L} \right) \\ &= \sum_{n=1}^{\infty} \frac{A_n}{2} \cos \frac{n\pi \overline{x}}{L} \bigg|_{x+ct}^{x-ct} \\ &= \sum_{n=1}^{\infty} \frac{A_n}{2} \int_{x-ct}^{x+ct} \frac{n\pi}{L} \sin \frac{n\pi \overline{x}}{L} d\overline{x} \\ &= \frac{1}{2c} \int_{x-ct}^{x+ct} \underbrace{\sum_{n=1}^{\infty} A_n \frac{n\pi c}{L} \sin \frac{n\pi \overline{x}}{L}}_{G(\overline{x})} d\overline{x} \end{aligned}$$

as we can get A_n to match

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} A_n \sin\frac{n\pi x}{L} \cos\frac{cn\pi t}{L} d\overline{x}$$

Thus as $u_t(x,0) = f(x)$, we have

$$\frac{n\pi c}{L} A_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L} dx$$

4.

1.

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int_0^L u_t^2(x,t)dx + \frac{c^2}{2}\int_0^L u_x^2(x,t)dx \\
&= \frac{1}{2}\int_0^L u_t^2(x,t) + c^2 u_x^2(x,t)dx
\end{aligned}
$$

Thus

$$
\begin{aligned}
E'(t) &= \frac{1}{2}\int_0^L 2u_t u_{tt} + 2c^2 u_x u_{xt} dx \\
&= c^2 \int_0^L u_t u_{xx} + u_x u_{xt} dx \\
&= c^2 \left. (u_x u_t) \right|_0^L
\end{aligned}
$$

2.

a.

Since $u(0,t) = u(L,t) = 0$, $u_t(0,t) = u_t(L,t) = 0$

$$
E'(t) = c^2 u_x(L,t) u_t(L,t) - c^2 u_x(0,t) u_t(0,t) = 0
$$

Thus energy is conserved.

b.

Since $u(L,t) = 0$, $u_t(L,t) = 0$

$$
E'(t) = c^2 u_x(L,t) u_t(L,t) - c^2 u_x(0,t) u_t(0,t) = 0
$$

Thus energy is conserved.

c, d.

Since $u(0,t) = 0$, $u_t(0,t) = 0$

$$
E'(t) = -c^2 \gamma u(L,t) u_t(L,t) = -\frac{c^2\gamma}{2}\frac{d}{dt} u_t^2(L,t)
$$

Thus integrating

$$
E(t) - E(0) = -\frac{\gamma c^2}{2} \left. u^2(L,t') \right|_{t'=0}^{t}
$$

and therefore

$$
E(t) = E(0) + \frac{\gamma c^2}{2}(u^2(L,0) - u^2(L,t))
$$

If $\gamma > 0$, we have that E increase at time t if $u(L,0) > u(L,t)$ and decrease if $u(L,0) < u(L,t)$. Similarly, for $\gamma < 0$, E decrease at time t if $u(L,0) > u(L,t)$ and increase if $u(L,0) < u(L,t)$.

5.

We have that

$$H\phi'' + \alpha H\phi' + (H\lambda\beta + H\gamma)\phi = 0$$

We want to find H so that the DE is of the form

$$p'(x)\phi'(x) + p(x)\phi''(x) + (\lambda\sigma + q)\phi = 0$$

Thus we have the conditions

$$\begin{cases} p' = \alpha H \\ p = H \\ H\lambda\beta = \lambda\sigma \\ H\gamma = q \end{cases}$$

The first 2 equations give us

$$H'(x) = \alpha(x)H(x)$$

Thus

$$H(x) = \exp\left(\int \alpha(x)dx\right)$$

and

$$p(x) = \exp\left(\int \alpha(x)\right), q(x) = \gamma(x)\exp\left(\int \alpha(x)\right), \sigma(x) = \beta(x)\exp\left(\int \alpha(x)\right)$$

6.

a.

We know that the eigenfunction and eigenvalue are

$$\phi_n(x) = \cos\frac{n\pi x}{L}$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Thus the smallest eigenvalue is $\lambda_0 = 0$ and there is no largest eigenvalue as

$$\lim_{n\to\infty} \lambda_n = \infty$$

b.

$$\cos\frac{n\pi x}{L} = 0 \implies \frac{n\pi x}{L} = \frac{\pi}{2} + k\pi$$

where $k \in \mathbb{Z}$. Thus

$$x = \left(\frac{\pi}{2} + k\pi\right)\frac{L}{n\pi} = \frac{L}{2n} + \frac{kL}{n} = L\left(\frac{1+2k}{2n}\right)$$

which means that as $0 < x < L$,

$$0 < \frac{1+2k}{2n} < 1 \implies 0 \leq k \leq n-1$$

But since the eigenfunction starts at $n = 0$, we have that this is the $n+1$-th eigenfunction which has n zeros.

c.

Since, the domain is $0 < x < L$, every function $f(x)$ can be represented by the Fourier cosine series thus the eigenfunctions are complete. It is also obivously orthogonal from the material in class.

d.

Since $\phi'(0) = \phi'(L) = 0$, and we can see that

$$p(x) = 1, \qquad q(x) = 0, \qquad \sigma(x) = 1$$

Thus the Rayleigh quotient is

$$\lambda = \frac{\int_0^L \phi'(x)^2 dx}{\int_0^L \phi(x)^2 dx} \geq 0$$

0 can be an eigenvalue as else $\phi''(x) = \phi'(x) = 0$ for all $x \in (0, L)$ and therefore $\phi(x) = C$ for some constant C since this satisfies the equation

$$\phi''(x) = -\lambda\phi(x)$$

7.

This is a Sturm-Liouville problem where

$$p(x) = 1, \qquad q(x) = -x^2, \qquad \sigma(x) = 1$$

Thus we have the Rayleigh quotient

$$\frac{-\phi(x)\phi'(x)|_0^1 + \int_0^1 \phi'(x)^2 + x^2\phi^2(x)dx}{\int_0^1 \phi^2(x)dx} = \frac{\int_0^1 \phi'(x)^2 + x^2\phi^2(x)dx}{\int_0^1 \phi^2(x)dx}$$

If the quotient $= 0$ then $\phi'(x)^2 + x^2\phi^2(x) = 0$ a.e. in (0,1), which means that $\phi'(x) = \phi(x) = 0$ in (0,1) as $\phi(x)$ and $\phi'(x)$ is continuous. But that is a trivial solution thus 0 is not an eigenvalue.

8.

a.

Multiplying, we have

$$x\phi''(x) + \phi'(x) + \frac{\lambda}{x}\phi(x) = (\phi'(x)x)' + \frac{\lambda}{x}\phi(x) = 0$$

is indeed a Sturm-Liouville with

$$p(x) = x, \qquad q(x) = 0, \qquad \sigma(x) = \frac{1}{x}$$

b.

The Rayleigh quotient thus is

$$\frac{-x\phi(x)\phi'(x)|_1^b + \int_1^b x\phi'(x)^2 dx}{\int_1^b \phi^2(x)\frac{1}{x}dx} = \frac{\int_1^b x\phi'(x)^2 dx}{\int_1^b \phi^2(x)\frac{1}{x}dx}$$

We have that for all $1 < x < b$

$$x > 0, \phi'(x)^2 \geq 0, \phi^2(x) \geq 0, \frac{1}{x} > 0$$

thus

$$\frac{\int_1^b x\phi'(x)^2 dx}{\int_1^b \phi^2(x)\frac{1}{x}dx} \geq 0$$

c.

If 0 is an eigenvalue then there is $\phi(x)$ such that $\phi'(x) = 0$ for all $x \in (1, b)$ but since $\phi(1) = \phi(b) = 0$. This means that $\phi(x) = 0$ and is a trivial solution thus 0 is not an eigenvalue. Since it is equidimensional, let $\phi(x) = x^r$. Plugging in we have that

$$x^2 r(r-1)x^{r-2} + xrx^{r-1} + \lambda x^r = x^r(r(r-1) + r + \lambda)) = 0$$

which has the characterisic function

$$r^2 + \lambda = 0$$

Thus

$$r = \pm i\sqrt{\lambda}$$

Since $\lambda > 0$, we have that

$$\phi(x) = x^{\pm i\sqrt{\lambda}} = \exp(\pm i\sqrt{\lambda}\ln(x))$$

The solution is thus

$$\phi(x) = c_1 \cos(\sqrt{\lambda}\ln x) + c_2 \sin(\sqrt{\lambda}\ln x)$$

Applying the boundary conditions, we have $c_1 = 0$ and

$$\sin(\sqrt{\lambda}\ln(b)) = 0$$

Therefore, for all $n \geq 1$,

$$\lambda_n = \left(\frac{n\pi}{\ln b}\right)^2$$

and

$$\phi_n(x) = \sin\left(\frac{n\pi}{\ln b}\ln x\right)$$

d.

The weight is found to be

$$\sigma(x) = \frac{1}{x}$$

And let $y = \ln x$, we have $dy = \frac{1}{x}dx$ and thus

$$\begin{aligned}
&\int_1^b \phi_n(x)\phi_m(x)\sigma(x)dx \\
=&\int_1^b \sin\left(\frac{n\pi}{\ln b}\ln x\right)\sin\left(\frac{m\pi}{\ln b}\ln x\right)\frac{1}{x}dx \\
=&\int_0^{\ln b} \sin\left(\frac{n\pi}{\ln b}y\right)\sin\left(\frac{m\pi}{\ln b}y\right)dy \\
=&0
\end{aligned}$$

if $n \neq m$.

e.

Let the n-th eigenfunction be 0

$$\phi_n(x) = \sin\left(\frac{n\pi}{\ln b}\ln x\right) = 0$$

We have

$$\frac{n\pi}{\ln b}\ln x = k\pi \implies x = \exp\left(\frac{k\ln b}{n}\right) = b^{k/n}$$

where $k \in \mathbb{Z}$. Since $1 < x < b$, we have that

$$0 < \frac{k}{n} < 1$$

and

$$1 \leq k \leq n-1$$

Thus $n-1$ zeros.

9.

a.

Assume solution of the form $u(x,t) = X(x)T(t)$ then

$$c\rho X T' = (K_0 X')'T + \alpha X T \implies \frac{(K_0 X')'}{c\rho X} + \frac{\alpha}{c\rho} = \frac{T'}{T} = -\lambda$$

Thus we have the ODE for X to be

$$(K_0 X')' + (c\rho\lambda + \alpha)X = 0$$

which is a Sturm-Liouville with

$$p(x) = K_0, \qquad q(x) = \alpha, \qquad \sigma(x) = c\rho$$

Thus, the Rayleigh quotient is

$$\lambda = \frac{-K_0 \phi(x)\phi'(x)|_0^L + \int_0^L K_0 \phi'(x)^2 - \alpha\phi(x)^2 dx}{\int_0^L \phi(x)^2 \rho c dx} = \frac{\int_0^L K_0 \phi'(x)^2 - \alpha\phi(x)^2 dx}{\int_0^L \phi(x)^2 \rho c dx}$$

which is ≥ 0 if $\alpha < 0$.

b.

We have that the eigenfunction for t is

$$T_n(t) = \exp(-\lambda_n t)$$

Assume the appropriate eigenfunctions for x is X_n. Then

$$u(x,t) = \sum_{n=1}^{\infty} A_n \exp(-\lambda_n t) X_n(x)$$

To solve for A_n, we have that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n X_n(x)$$

Multiplying by $X_m(x)c\rho$ and integrating on $[0, L]$. Then

$$\int_0^L f(x) X_m(x) c\rho dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L X_n(x) X_m(x) \sigma(x)\, dx}_{=0 \text{ if } m \neq n}$$

Thus

$$A_n = \frac{\int_0^L f(x)\phi_n(x) c\rho dx}{\int_0^L \phi_n^2(x) c\rho dx}$$

10.

Apply seperation of varibles $u(x,t) = X(x)T(t)$, we have that

$$\begin{cases} u(0,t) = 0 \implies X(0) = 0 \\ u(L,t) = 0 \implies X(L) = 0 \\ u(x,0) = f(x) \implies X(x)T(0) = f(x) \\ u_t(x,0) = g(x) \implies X(x)T'(0) = g(x) \end{cases}$$

and

$$\rho X(x)T''(t) = T_0 X''(x)T(t) + \alpha X(x)T(t)$$
$$\implies \frac{T''(t)}{T(t)} = \frac{T_0 X''(x) + \alpha X(x)}{\rho X(x)} = -\lambda$$

Let's first rewrite the equation for x,

$$T_0 X''(x) + X(x)(\alpha + \lambda\rho) = 0$$

Thus is a Sturm-Liouville equation with

$$p(x) = T_0, \qquad q(x) = \alpha, \qquad \sigma(x) = \rho$$

As $X(0) = X(L) = 0$, the Rayleigh quotient will then be

$$\lambda = \frac{\int_0^L T_0 X'(x)^2 - \alpha X(x)^2 dx}{\int_0^L X(x)^2 \rho dx} \geq 0$$

as $\alpha < 0, \rho > 0$ and the assumption we made $T_0 > 0$. Also note that if 0 is an eigenvalue then $X'(x) = X(x) = 0$ and thus is 0 is not an eigenvalue. Since $\lambda > 0$,

$$T(t) = c_1 \sin(\sqrt{\lambda}t) + c_2 \cos(\sqrt{\lambda}t)$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} A_n X_n(x) \sin(\sqrt{\lambda_n}t) + \sum_{n=1}^{\infty} B_n X_n(x) \cos(\sqrt{\lambda_n}t)$$

and

$$u_t(x,t) = \sum_{n=1}^{\infty} A_n X_n(x)\sqrt{\lambda_n} \cos(\sqrt{\lambda_n}t) - \sum_{n=1}^{\infty} B_n X_n(x)\sqrt{\lambda_n} \sin(\sqrt{\lambda_n}t)$$

Now, we need to solve for A_n, B_n

$$\begin{cases} u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n X_n(x) \\ u_t(x,0) = g(x) = \sum_{n=1}^{\infty} A_n X_n(x)\sqrt{\lambda_n} \end{cases}$$

Thus we get

$$\begin{cases} \int_0^L f(x)X_m(x)\rho(x)dx = \sum_{n=1}^{\infty} B_n \int_0^L X_n(x)X_m(x)\rho(x)dx \\ \int_0^L g(x)X_m(x)\rho(x)dx = \sum_{n=1}^{\infty} A_n\sqrt{\lambda_n} \int_0^L X_n(x)X_m(x)\rho(x)dx \end{cases}$$

and therefore,

$$\begin{cases} A_n = \dfrac{\int_0^L g(x)X_n(x)\rho(x)dx}{\sqrt{\lambda_n}\int_0^L X_n(x)^2\rho(x)dx} \\ B_n = \dfrac{\int_0^L f(x)X_n(x)\rho(x)dx}{\int_0^L X_n(x)^2\rho(x)dx} \end{cases}$$