1.

Since f is measurable, then we know that for any open interval (x, y), $f^{-1}(x, y)$ is measurable, then for an arbitary $a \in \mathbb{R}$

- If $a \geq 0$, $\{0 < f < 1/a\}$ is measurable thus $\{g > a\}$ is also measurable.
- Now if $a < 0$, we have that $\{g > a\} = \{g > 0\} \cup \{g = 0\} \cup \{a < g < 0\}$, but we have

  $$\{g = 0\} = f^{-1}(\{0, \infty, -\infty\})$$

  and

  $$\{a < g < 0\} = \{a < 1/f < 0\} = \{f < 1/a\}$$

  are measurable.

Therefore,

$$\{g > a\}$$

is measurable for all $a \in \mathbb{R}$. Thus g is measurable.

2.

Suppose $m(F) = 0$ then we can find n_0 such that $\cup_{k=n_0}^{\infty} E_k < \infty$ thus WLOG we assume that $\cup_{k=1}^{\infty} E_k < \infty$.

$$0 = m(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} m(E_n)$$

Therefore,

$$\limsup_{n\to\infty} m(E_n) = \lim_{n\to\infty} \sup_{m\geq n} m(E_n) = 0$$

and thus $m(E_n) \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} \chi_{E_n}(x) = 0$$

a.e. $x \in \mathbb{R}^d$.
In the other direction, first let $G_n = \cup_{k=n}^{\infty} E_k$.
Suppose $m(F) > 0$ then if $m\left(\cup_{k=j}^{\infty} E_k\right) = \infty$ for all $j \in \mathbb{N}$ then obviously, $m(G_n) = \infty > a$ for all $a \in \mathbb{R}$.
Suppose $m(F) > 0$ and $m\left(\cup_{k=j}^{\infty} E_k\right) < \infty$ for some j then

$$\lim_{j\to\infty} m(\cup_{k=j}^{\infty} E_k) = m(F) > 0$$

Thus there is $\varepsilon > 0$ and n_0 such that for all $n > n_0$, $m(\cup_{k=n}^{\infty} E_k) > \varepsilon$.
Therefore, in both cases there is some $\varepsilon > 0$ and n_0 such that for all $n > n_0$,

$$m(G_n) > \varepsilon$$

But for every $x \in G_n$, there is some $j \geq n$ such that $x \in E_j$ and thus $\chi_{E_j}(x) = 1$. However, if

$$\lim_{n\to\infty} \chi_{E_n}(x) = 0$$

for all $x \in \mathbb{R}^d \backslash G$ where $m(G) = 0$ which means that $\{x : \exists n' > n, \chi_{E_n'}(x) \neq 0\} \to 0$ as $n \to \infty$, which is a contradiction.

3.

a.

We know from notes 2 there is a nonmeasurable set $\mathcal{N} \subset [0, 1]$. Define

$$g : \mathbb{R} \to \mathbb{R}, \qquad x \to \begin{cases} x, & \text{if } x \in \mathcal{N} \\ -x, & \text{if } x \notin \mathcal{N} \end{cases}$$

$g^{-1}(x)$ has at most 2 elements thus is measurable. But $\{g \geq 0\} \backslash (-\infty, 0] = \mathcal{N}$ is nonmeasurable.

b.

We first have that

$$g^{-1}(a, \infty) = \begin{cases} f'^{-1}(a, \infty), & \text{if } a \geq 0 \\ f'^{-1}(a, \infty) \cup \mathbb{R} \backslash B, & \text{if } a < 0 \end{cases}$$

Thus, we only need to prove that f' is measurable as $\mathbb{R} \backslash B$ is measurable. We have that

$$f' = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Let $g_h = \dfrac{f(x + h) - f(x)}{h}$, we can see that since $f(x+h)$ and $f(x)$ are both measurable, g_h is measurable and thus f' is measurable.

4.

Since μ is σ-finite, there is some $X_n \in \mathcal{M}$ such that $X_n \subseteq X_{n+1}$ and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$.
Thus for every X_m and every $k \in \mathbb{N}$, we can apply the erogov's theorem on the set X_m to get there is a subset $E_{m,k}$ such that $\mu(X_m \backslash E_{m,k}) < \varepsilon/2^{mk}$ and $f_n \to f$ uniformly on E_m.
Now we have

$$\begin{aligned}
\mu((\cup_{n,k=1}^{\infty} E_{n,k})^c) &= \mu(\cup_{n=1}^{\infty} X_n \backslash \cup_{n,k=1}^{\infty} E_{j,k}) \\
&= \mu(\cup_{n=1}^{\infty} (X_n \backslash \cup_{k=1}^{\infty} E_{n,k})) \\
&\leq \sum_{n=1}^{\infty} \mu(X_n \backslash \cup_{k=1}^{\infty} E_{n,k})) \\
&= \sum_{n=1}^{\infty} \mu(\cap_{k=1}^{\infty} (X_n \backslash E_{n,k})) \\
&\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(X_n \backslash E_{n,k}) \\
&< \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{1}{2^k} \\
= \varepsilon
\end{aligned}$$

5.

First, note that $f(x) < \infty$ as $\int_X f < \infty$. Let $Y_n = (1/n, n)$ and $X_n = f^{-1}(Y_n \cup \{0\})$ so that

$$\cup_{n=1}^{\infty} X_n = \cup_{n=1}^{\infty} f^{-1}(Y_n \cup \{0\}) = f^{-1}(\{0\} \cup \cup_{n=1}^{\infty} Y_n) = f^{-1}([0, \infty)) = X$$

$$Y_n \subset Y_{n+1} \implies X_n \subset X_{n+1}$$

and

$$\frac{1}{n} \cdot \mu(X_n \backslash f^{-1}(0)) < \int_{X_n \backslash f^{-1}(0))} f \leq \int_X f \leq \infty \implies \mu(X_n \backslash f^{-1}(0)) < \infty$$

Then we can define the sequence of function

$$f_n = f \cdot \chi_{X_n}$$

that is

- non-negative as $f, \chi_{X_n} > 0$
- $f_n(x) \uparrow f(x)$ for all $x \in X$ because of
    1. $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$ as $X_n \subset X_{n+1}$
    2. For all $x \in X$, $f(x) < \infty$, thus there exists $N \in \mathbb{N}$ such that $f(x) \in Y_N$ and thus $x \in X_N \subseteq X_{N+1} \subseteq \ldots$. Therefore, $f_n(x) \to f(x)$ as $n \to \infty$.

Therefore, the monotone convergence theorem states that

$$\lim_{n \to \infty} \int_{X_n} f dx = \lim_{n \to \infty} \int_X f \cdot \chi_{X_n} dx = \int_X f dx$$

and noted that it is monotone increasing as well, therefore, for all $\varepsilon > 0$, we can find $F = X_{n_0}$ such that

$$\int_X f - \int_F f < \varepsilon$$

and thus let $E = F \backslash f^{-1}(0))$, we have $\mu(E) < \infty$ and

$$\int_X f - \int_E f < \varepsilon$$

as

$$\int_E f = \int_F f$$