1.

For each k, we can consider the matrix A as a block matrix as follows:

$$\begin{pmatrix} A_k & B \\ C & D \end{pmatrix}$$

Then we have that

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \|x^2\| > 0 \text{ for all } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} A_k & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = A_k x_1 \cdot x_1 > 0$$

which means that upper left submatrices are all positive definite.

2.

As $a > 0$ and $ad > b^2 > 0$, d is also larger then 0.
If λ is an eigenvalue of A, then

$$\chi_A(\lambda) = (a - \lambda)(d - \lambda) - b^2 = ad - b^2 - \lambda(a + d) + \lambda^2 = 0$$

And hence,

$$ad - b^2 + \lambda^2 = \lambda(a + d)$$

We know that $ad - b^2 > 0$, hence $ad - b^2 + \lambda^2 > 0$. Therefore, $\lambda > 0$ as $(a + d) > 0$.
Therefore, A is positive definite.

3.

$$\frac{\partial f}{\partial y}(x,y) = (3+2\cos x)(-\sin y) = 0 \iff \sin y = 0 \iff y \in \{0,\pi\}$$

$$\frac{\partial f}{\partial x}(x,y) = -2\sin x\cos y = 0 \iff \cos y = 0 \vee \sin x = 0 \iff x = 0 \vee \cos y = 0$$

As $\forall y : \sin y \neq \cos y$, we can conclude that the stationary points are $(0,0),(0,\pi)$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 2\sin x \sin y$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = 2\sin x \sin y$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = (3+2\cos x)(-\cos y)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -2\cos x\cos y$$

$\det(\text{Hess } f)(0,\pi) = (3+2\cos 0)(-\cos\pi)(-2\cos 0\cos\pi) - (2\sin 0\sin\pi)^2 = 10 > 0$
where $(3+2\cos 0)(-\cos\pi) = 5 > 0$.
Therefore $(0,\pi)$ is a local minimum. $f(0,\pi) = -5$
$\det(\text{Hess } f)(0,0) = (3+2\cos 0)(-\cos 0)(-2\cos 0\cos 0) - (2\sin 0\sin 0)^2 = 10 > 0$
where $(3+2\cos 0)(-\cos\pi) = -5 < 0$.
Therefore $(0,0)$ is a local maximum. $f(0,0) = 5$

4.

$$\frac{\partial f}{\partial y}(x,y,z) = 3x^2 - 3 = 0 \iff x \in \{1,-1\}$$
$$\frac{\partial f}{\partial y}(x,y,z) = -3y^2 + 9 = 0 \iff y \in \{\sqrt{3}, \sqrt{-3}\}$$
$$\frac{\partial f}{\partial z}(x,y,z) = 2z = 0 \iff z = 0$$

As $\forall y : \sin y \neq \cos y$, we can conclude that the stationary points must be in the set $\{1,-1\} \times \{\sqrt{3}, \sqrt{-3}\} \times \{0\}$. $\forall a \neq b \in \{x,y,z\}$:

$$\frac{\partial^2 f}{\partial a \partial b} = 0$$

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 6x$$
$$\frac{\partial^2 f}{\partial y^2}(x,y,z) = -6y$$
$$\frac{\partial^2 f}{\partial z^2}(x,y,z) = 2$$

Hence, as $(\text{Hess } f)(x,y,z)$ is a diagonal matrix, its eigenvalue are $\{6x, 6y, 2\}$. Which means that $(\text{Hess } f)(x,y,z)$ be definite, and positive definite in this case because $2 > 0$. $6x, 6y > 0$. Hence, $(1, \sqrt{3}, 2\}$ is the only local minimum point where the others are saddle points. $f(1, \sqrt{3}, 2) = 6\sqrt{3} + 2$

5.

$$
\begin{aligned}
&\frac{\partial f}{\partial x}(x,y) = 2x \cdot e^{-(x^2+y^2)} + (x^2+2y^2)\cdot(-2x)\cdot e^{-(x^2+y^2)} = 0\\
\Longrightarrow\ & e^{-(x^2+y^2)}(2x)(1-x^2-2y^2) = 0\\
\Longrightarrow\ & x = 0 \vee x^2+2y^2 = 1\\
&\frac{\partial f}{\partial y}(x,y) = 4y \cdot e^{-(x^2+y^2)} + (x^2+2y^2)\cdot(-2y)\cdot e^{-(x^2+y^2)} = 0\\
\Longrightarrow\ & e^{-(x^2+y^2)}(2y)(2-x^2-2y^2) = 0\\
\Longrightarrow\ & y = 0 \vee x^2+2y^2 = 2
\end{aligned}
$$

Hence, we have that $\nabla f(x,y) = 0$ if and only if
1. $x = y = 0$
2. $x = 0$ and $x^2+2y^2 = 2$. Which means that $x = 0$ and $y \in \{1,-1\}$
3. $y = 0$ and $x^2+2y^2 = 1$. Which means that $x \in \{1,-1\}$ and $y = 0$
4. $x^2+2y^2 = 2$ and $x^2+2y^2 = 1$ which is impossible. Therefore, the set of stationary points is $(\{0\}\times\{0,1,-1\}) \cup (\{0,1,-1\}\times\{0\})$.

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(x,y) &= (2-6x^2-4y^2)\cdot e^{-(x^2+y^2)} + (2x-2x^3-4xy^2)\cdot(-2x)\cdot e^{-(x^2+y^2)}\\
&= e^{-(x^2+y^2)}(2-10x^2-4y^2+4x^4+8x^2y^2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y^2}(x,y) &= (4-12y^2-2x^2)\cdot e^{-(x^2+y^2)} + (4y-4y^3-2yx^2)\cdot(-2y)\cdot e^{-(x^2+y^2)}\\
&= e^{-(x^2+y^2)}(4-20y^2-2x^2+8y^4+4x^2y^2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y\partial x}(x,y) &= (2x)(-4y)e^{-(x^2+y^2)} + (2x)(-2y)(1-x^2-2y^2)e^{-(x^2+y^2)}\\
&= -4xye^{-(x^2+y^2)}(3-x^2-2y^2)\\
\frac{\partial^2 f}{\partial x\partial y}(x,y) &= (2y)(-2x)e^{-(x^2+y^2)} + (2y)(2-x^2-2y^2)(-2x)e^{-(x^2+y^2)}\\
&= -4xye^{-(x^2+y^2)}(1+2-x^2-2y^2)
\end{aligned}
$$

We have that

$\frac{\partial^2 f}{\partial x^2}(0,0) > 0, \frac{\partial^2 f}{\partial x^2}(1,0) = \frac{\partial^2 f}{\partial x^2}(-1,0) = -4e^{-1} < 0, \frac{\partial^2 f}{\partial x^2}(0,1) = \frac{\partial^2 f}{\partial x^2}(0,-1) < 0$

$\frac{\partial^2 f}{\partial y^2}(0,0) > 0, \frac{\partial^2 f}{\partial y^2}(1,0) = \frac{\partial^2 f}{\partial y^2}(-1,0) = 2e^{-1} > 0, \frac{\partial^2 f}{\partial x^2}(0,1) = \frac{\partial^2 f}{\partial x^2}(0,-1) < 0$

For all stationary point (x,y), we have

$$
\frac{\partial^2 f}{\partial x\partial y}(x,y) = \frac{\partial^2 f}{\partial y\partial x}(x,y) = 0
$$

Every $(\text{Hess } f)(x,y)$ of a stationary points are in the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Hence, (0,0) is a local minimum and (0,1), (0,-1) are local maximums while (1,0) and (-1,0) are saddle points. $f(0,0) = 0, f(0,1) = f(0,-1) = \frac{2}{e}$

6.

$$\frac{\partial f}{\partial x}(x,y) = \cos(x) + \cos(x+y)$$
$$\frac{\partial f}{\partial y}(x,y) = \cos(y) + \cos(x+y)$$

Therefore, $\nabla f = 0 \iff \cos x = \cos y$ which means that $x = y$ as it is bounded between 0 and $\frac{\pi}{2}$.
We also have that $\cos x + \cos(x+x) = 0 \iff \cos x + 2\cos^2(x) - 1 = 0 \iff x = \frac{\pi}{3}$.

$$\frac{\partial^2 f}{\partial x^2}(\frac{\pi}{3}, \frac{\pi}{3}) = -\sin(\frac{\pi}{3}) - \sin(\frac{2\pi}{3}) = -\sqrt{3}$$
$$\frac{\partial^2 f}{\partial y^2}(\frac{\pi}{3}, \frac{\pi}{3}) = -\sin(\frac{\pi}{3}) - \sin(\frac{2\pi}{3}) = -\sqrt{3}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$$

Hence, $\frac{\partial^2 f}{\partial x \partial y}(\frac{\pi}{3}, \frac{\pi}{3}) = \frac{\partial^2 f}{al x \partial y}(\frac{\pi}{3}, \frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ Therefore, $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ and hence $(\frac{\pi}{3}, \frac{\pi}{3})$ is a local (and thus global) maximum. Since there is no local minimum, the global mimumum is on the boundary of the domain, that is $x = 0$ or $y = 0$ or $x = \frac{\pi}{2}$ or $y = \frac{\pi}{2}$.
If $x = 0$ then $\frac{\partial f}{\partial y}(0, y) = 2\cos(y) = 0 \iff y = \frac{\pi}{2}$
If $y = 0$ then $\frac{\partial f}{\partial x}(x, 0) = 2\cos(y) = 0 \iff x = \frac{\pi}{2}$
If $x = \frac{\pi}{2}$ then $\frac{\partial f}{\partial y}(\frac{\pi}{2}, y) = \cos(y) + \cos(\frac{\pi}{2} + y) = 0 \iff y = \frac{\pi}{4}$
If $y = \frac{\pi}{2}$ then $\frac{\partial f}{\partial x}(x, \frac{\pi}{2}) = \cos(x) + \cos(\frac{\pi}{2} + x) = 0 \iff x = \frac{\pi}{4}$ Therefore, the minimum point must be at one of these points: $(0, \frac{\pi}{2}), (\frac{\pi}{2}, 0), (0, 0), (\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{4}, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{\pi}{4})$

$$f(0,0) = 0, f\left(0, \frac{\pi}{2}\right) = f\left(\frac{\pi}{2}, 0\right) = 2, f\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = f\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = 1 + \sqrt{2}, f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 2$$

Hence, the global minimum is at $(0, 0)$ with the value of 0