1.

We first have that

$$u_t = ku_{xx}$$

where $k = \frac{K}{cp}$. Then because it is an equilibrium temperature distribution

$$u_{xx} = 0$$

which implies that

$$u_x = C_1$$

and

$$u(x) = C_1 x + C_2$$

Plugging in $u(0) = T$ and $u_x(L) = \alpha$, we have that

$$u(x) = \alpha x + T$$

2.

We have the heat equation

$$cpu_t = K_0 u_{xx} + K_0 = K_0(u_{xx} + 1)$$

and thus

$$\frac{cp}{K} u_t = u_{xx} + 1$$

Since $u_t = 0$, we have that $u_{xx} = -1$, therefore

$$u_x(x) = -x + C_1$$

and

$$u(x) = -\frac{x^2}{2} + C_1 x + C_2$$

Thus, we have the system of equations

$$\begin{cases} u(0) = C_2 = T_1 \\ u(L) = \dfrac{-L^2}{2} + C_1 L + C_2 = T_2 \end{cases}$$

Hence,

$$\begin{cases} C_1 = \dfrac{1}{L}\left(T_2 - T_1 + \dfrac{L^2}{2}\right) \\ C_2 = T_1 \end{cases}$$

and we obtain the solution

$$u(x) = -\frac{x^2}{2} + \frac{x}{L}\left(T_1 - T_1 + \frac{L^2}{2}\right) + T_1$$

3.

We have that

$$K_0 u_{xx} + x^2 K_0 = K_0(u_{xx} + x^2) = 0 \implies u_{xx} = -x^2$$

Thus $u_x(x) = -\dfrac{x^3}{3} + C_1$ where $C_1 = -\dfrac{L^3}{3}$ as $u_x(L) = 0$. Therefore, we have

$$u(x) = -\frac{x^4}{12} - \frac{L^3}{3}x + \underbrace{T_2}_{C_2}$$

as $u(0) = T_2$

4.

We have that

$$u_{xx} = 0$$

Therefore, since $u_x(L) = \alpha$, $u_x = \alpha$, and thus we get

$$\alpha - [u(0) - T] = 0 \implies u(0) = \alpha + T$$

and

$$u(x) = \alpha x + C$$

where $u(0) = \alpha + T$ thus

$$u(x) = \alpha x + \alpha + T$$

5.

a.

Assuming there is a equilibrium, all the environment variables must be constant therefore

$$u_{xx} = -\frac{Q_0}{K_0}$$

Thus

$$u_x(x) = -\frac{Q_0}{K_0}x + C_1$$

Therefore, $u_x(0) = C_1 = 0$ and $u_x(L) = \underbrace{-\frac{Q_0}{K_0}L}_{\neq 0} + C_1 = 0$ which is a contradiction. Therefore, there is no equilibrium when the environmen has constant physical properties and both sides of the rod are insulated.

b.

First, notice that

$$\frac{d}{dt}\int_V e dV = \int_V \frac{\partial e}{\partial t} dV = \int_0^L \frac{\partial e}{\partial t} A dx$$

where A is the area of the rod at the end.
We know that

$$\frac{\partial e}{\partial t} = \frac{\partial (c\rho u)}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q_0$$

and therefore,

$$\begin{aligned}\frac{d}{dt}\int_V e dV &= \int_0^L \left(K_0 \frac{\partial^2 u}{\partial x^2} + Q_0\right) A dx \\ &= A\left(K_0 \frac{\partial u}{\partial x}(x,t) + Q_0 x\right)\Big|_0^L \\ &= AK_0\left(u_x(L,t) - u_x(0,t)\right) + AQ_0 L \\ &= AQ_0 L\end{aligned}$$

as both ends are insulated thus $u_x(L,t) = u_x(0,t) = 0$. Thus we get

$$\int_V e dV = AQ_0 L t + C_2$$

Thus,

$$\rho c A \int_0^L u(x,0) dx = AQ_0 L 0 + C_2 = C_2$$

6.

a.

$$
\begin{aligned}
\frac{d}{dt}\int_V e dV &= \int_0^L A\rho c u_t(x,t)dx \\
&= \int_0^L A\rho c(u_{xx}(x,t)+x)dx \\
&= A\rho c\left(u_x(x,t)+\frac{x^2}{2}\right)\Big|_0^L \\
&= A\rho c\left(7-\beta+\frac{L^2}{2}\right)
\end{aligned}
$$

Thus,

$$\int_V e dV = A\rho c\left(7-\beta+\frac{L^2}{2}\right)t + C$$

where

$$C = \int_0^L A\rho c\,\underbrace{u(x,0)}_{f(x)}\,dx$$

b.

Since $u_t = 0$, $\int_0^L \frac{\partial u}{\partial t}dx = 0$ and thus

$$
\begin{aligned}
\frac{d}{dt}\int_0^L u(x,t)dx = \underbrace{\int_0^L \frac{\partial u}{\partial t}dx}_{0} &= \int_0^L \frac{\partial^2 u}{\partial x^2} + xdx \\
&= \frac{\partial u}{\partial x}\Big|_0^L + \frac{L^2}{2} \\
&= 7-\beta+\frac{L^2}{2}
\end{aligned}
$$

Therefore, $\beta = 7+\frac{L^2}{2}$ and $\int_0^L u(x,t)dx$ is constant over time which means that

$$\lim_{t\to\infty}\int_0^L u(x,t)dx = \int_0^L u(x,0)dx = \int_0^L f(x)dx$$

We also have that $u_{xx} = -x$ thus $u_x = -\frac{x^2}{2}+7+\frac{L^2}{2}$ and hence

$$u(x) = -\frac{x^3}{6}+7x+\frac{L^2x}{2}+C$$

where

$$C = \frac{1}{L}\left(\frac{L^4}{24}-\frac{7L^2}{2}-\frac{L^4}{4}+\int_0^L f(x)dx\right)$$

as

$$\int_0^L f(x)dx = \int_0^L \left(-\frac{x^3}{6} + 7x + \frac{L^2x}{2} + C\right) dx$$

7.

a.

Since there is no source,

$$\frac{d}{dt}\int_0^L e(x,t)Adx = A[\phi(0,t) - \phi(L,t)] = A[\alpha - \beta]$$

b.

Thus, we have that the total amount of chemical based on time is

$$\int_0^L c\rho u(x,t)Adx = A(\alpha - \beta)t + C$$

where $C = \int_0^L c\rho f(x)Adx$ after substituting $t = 0$ in the equation.

c.

If there is an equilibrium then the total amount of chemical must be a constant thus from part a, $\alpha = \beta$.

8.

$$
\begin{aligned}
&\nabla^2 u = 0 \\
\implies &\iiint\limits_{\mathbb{R}} \nabla \cdot \nabla u = 0 \\
\implies &\oiint \nabla u \cdot \hat{n} dS = 0 \\
\implies &\oiint \underbrace{-K_0 \nabla u}_{\phi} \cdot \hat{n} dS = 0
\end{aligned}
$$

Thus there is no heat flow through any surface in the object.

9.

Assuming the angle does not affect the temperature, that is $u_\theta = 0$, thus

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$$

and hence

$$\frac{\partial u}{\partial r} = \frac{C_1}{r}$$

a.

$$u(r) = C_1 \ln(r) + C_2$$

$$\begin{cases} C_1 \ln(r_1) + C_2 = T_1 \\ C_1 \ln(r_2) + C_2 = T_2 \end{cases}$$

Thus

$$\begin{cases} C_1 = \frac{T_1 - T_2}{\ln(r_1/r_2)} \\ C_2 = T_1 - C_1 \ln(r_1) \end{cases}$$

and therefore,

$$u(r) = T_1 + (T_2 - T_1)\frac{\ln \dfrac{r}{r_1}}{\ln \dfrac{r_2}{r_1}}$$

b.

Following the same steps as part a, since the outer radius is insulated, we get that

$$\frac{\partial u}{\partial r} = 0$$

Thus

$$u(r) = C_3$$

where $C_3 = T_1$ as $u(r_1) = T_1$ thus

$$u(r) = T_1$$

10.

Since there is no source, we have that

$$0 = \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

But it is axially symmetric, and therefore,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$$

and thus

$$r\frac{\partial u}{\partial r} = C_1$$

for some constant C_1, and therefore

$$r_1\frac{\partial u}{\partial r}(r_1) = 0 = C_1$$

and thus $C_1 = 0$.
Therefore,

$$u(r, \theta) = u(r) = C_2 = T_0$$

as $u(r_0) = T_0$