1.

We know that $\frac{1}{4}\left(Z_1 + Z_2 + Z_3 + Z_4\right) \sim Normal(0, \frac{1}{4})$
and hence $\frac{1}{2}\left(Z_1 + Z_2 + Z_3 + Z_4\right) \sim Normal(0, 1)$
We also have that $W = Z_5^2 + Z_6^2 + Z_7^2 + Z_8^2 + Z_9^2 + Z_{10}^2 \sim \chi_6^2$
Therefore, with $c = \sqrt{6}/2$

$$\frac{\sqrt{6}}{2} \cdot \frac{Z_1 + Z_2 + Z_3 + Z_4}{\sqrt{Z_5^2 + Z_6^2 + Z_7^2 + Z_8^2 + Z_9^2 + Z_{10}^2}} \sim t_6$$

2.

We have that

$$Z_1^2 + Z_2^2 + \ldots + Z_n^2 \sim \chi_n^2$$

and

$$Z_{n+1}^2 + Z_{n+2}^2 + \ldots + Z_{3n}^2 \sim \chi_{2n}^2$$

Therefore, with $c = 2$

$$2 \cdot \frac{Z_1^2 + Z_2^2 + \ldots + Z_n^2}{Z_{n+1}^2 + Z_{n+2}^2 + \ldots + Z_{3n}^2 \sim \chi_{2n}^2} \sim F_{2n}^n$$

3.

$$\overline{X} = 2\overline{Y} + 35 > 60 \iff \overline{Y} > 12.5$$

$$\sum_{i=1}^{52} Y_i \sim Gamma(\alpha = 3 \cdot 52 = 160, \beta = 5)$$

$$\overline{Y} = \frac{1}{52}\sum_{i=1}^{52} Y_i \sim Gamma(\alpha = 160/52 = 3, \beta = 5 \cdot 52 = 260)$$

and hence

4.

Consider $Y = \sum_{i=1}^{100} Y_i \sim Normal(\mu_Y = 100 \cdot 2540 = 254000, \sigma_Y = 100 \cdot 2100 = 210000)$ Then $Z = \dfrac{300000 - 254000}{210000} = \dfrac{23}{105}$ and hence the proability that the total of 100 claims will be over 300000 dollars is 0.4129

5.

For each bulb, the probability that it is not a dud is

$$1 - \int_0^{2.5} 11 \cdot e^{-11x} dx = e^{-\frac{55}{2}}$$

Then the probability that there is less than 45 duds follows a normal distribution with $\mu = 200 \cdot e^{-\frac{55}{2}}$ and $\sigma = \sqrt{200 \cdot e^{-\frac{55}{2}} \cdot (1 - e^{-\frac{55}{2}})}$, which hence is

$$\frac{45 - 200 \cdot e^{-\frac{55}{2}}}{\sqrt{200 \cdot e^{-\frac{55}{2}} \cdot (1 - e^{-\frac{55}{2}})}}$$

6.

We have that

$$\begin{aligned} E[\overline{Y}]^2 &= E[\overline{Y}^2] - V[\overline{Y}] \\ &= E[\overline{Y}^2] - \frac{\beta^2}{m} \\ &= E[\overline{Y}^2] - \frac{E[\overline{Y}]^2}{m} \end{aligned}$$

Therefore,

$$E[\overline{Y}]^2 = E[\overline{Y}^2] \cdot \frac{m}{m+1}$$

Hence,

$$\begin{aligned} E[C] &= E[2Y^2 - 4Y] \\ &= 2E[Y^2] - 4E[Y] \\ &= 2(V[Y] + E[Y]^2) - 4E[Y] \\ &= 2(\beta^2 + \beta^2) - 4\beta \\ &= 4\beta^2 - 4\beta \\ &= 4E[\overline{Y}]^2 - 4E[\overline{Y}] \\ &= 4\frac{m}{m+1}E[\overline{Y}^2] - 4E[\overline{Y}] \end{aligned}$$

Therefore, an unbiased estimator is $\dfrac{4m\overline{Y}^2}{m+1} - 4\overline{Y}$

7.

$X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$. Therefore,

$$\begin{aligned} F_{X_{(n)}}(x) &= P(X_{(n)} \leq x) = P(X_1, X_2, \ldots, X_n < x) \\ &= \left(\frac{x}{\theta}\right)^n \\ f_{X_{(n)}}(x) &= n \cdot \frac{1}{\theta} \cdot \left(\frac{x}{\theta}\right)^{n-1} \end{aligned}$$

Therefore,

$$E[X_{(n)}] = \int_0^\theta x \frac{n}{\theta} \left(\frac{x}{\theta}\right)^{n-1} dx = \frac{n}{\theta^n} \int_0^\theta x^n dx = \frac{\theta n}{n+1}$$

and similarly

$$E[X_{(n)}^2] = \int_0^\theta x^2 \frac{n}{\theta} \left(\frac{x}{\theta}\right)^{n-1} dx = \frac{n}{\theta^n} \int_0^\theta x^{n+1} dx = \frac{\theta^2 n}{n+2}$$

We have that

$$\begin{aligned} V[Y] &= V[E[Y|X]] + E[V[Y|X]] \\ &= V\left[\frac{X}{3}\right] + E\left[\frac{X^2}{9}\right] \\ &= \frac{\theta^2}{108} + \frac{1}{9}(E[X]^2 - V[X]) \\ &= \frac{\theta^2}{108} + \frac{1}{9}\left(\frac{\theta^2}{4} - \frac{\theta^2}{12}\right) \\ &= \frac{\theta^2}{36} \\ &= \frac{n+2}{36n} E[X_{(n)}^2] \end{aligned}$$

8.

$$F_{Y_{(n)}}(y) = \left(\frac{5y^4}{(\beta+1)^5}\right)^n$$

$$f_{Y_{(n)}}(y) = n\left(\frac{5y^4}{(\beta+1)^5}\right)^{n-1} \cdot \frac{20y^3}{(\beta+1)^5}$$

$$E[Y_{(n)}] = \int_0^{\beta+1} y \cdot n\left(\frac{5y^4}{(\beta+1)^5}\right)^{n-1} \cdot \frac{20y^3}{(\beta+1)^5}$$

$$=$$

$$E[\hat{\beta}_1] =$$