1.

(a) It is obvious from definition that $\varnothing$ and the compact set I has content and is a subset of I. Hence, $\varnothing, I \in \mathcal{A}$
(b) If $A \in \mathcal{A}$ then both I and A has a zero measure boundary,

$$\partial(I \backslash A) \subset \partial(A \cup I) \subset \partial A \cup \partial I$$

which means that $I \backslash A \subset I$ also have zero measure boundary and hence has content. Hence $I \backslash A \in \mathcal{A}$.
(c) If $A_1, \ldots, A_n \in \mathcal{A}$ then $A_1, \ldots A_n \subset I$ and has measure zero boundary. Hence $A_1 \cup \ldots \cup A_n \subset I$ also have boundary and therefore $A_1 \cup \ldots \cup A_n \in \mathcal{A}$.

2.

We create a sequence of partition $P_n = P_{x_n} \times P_{y_n}$ as follows:

$$P_{x_n} = \left\{\frac{1}{n}, \frac{2}{n}, \dots \frac{n-1}{n}, 1\right\}$$

$$P_{y_n} = \left\{\frac{1}{n}, \frac{2}{n}, \dots \frac{n-1}{n}, 1\right\}$$

For each I_v, we choose (x_v, y_v) such that

$$\|(x_v, y_v)\| = \max\{\|(x_v, y_v)\| : (x_v, y_v) \in I_v$$

Hence,

$$\begin{aligned}
S(P_n, f) &= \sum_v \mu(I_v) \cdot f(x_v, y_v) \\
&= \frac{1}{n^2} \cdot \left(\frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \dots + 1 \cdot \frac{1}{n}\right) \\
&+ \frac{1}{n^2} \cdot \left(\frac{1}{n} \cdot \frac{2}{n} + \frac{2}{n} \cdot \frac{2}{n} + \dots + 1 \cdot \frac{2}{n}\right) \\
&+ \dots \\
&+ \frac{1}{n^2} \cdot \left(\frac{1}{n} \cdot \frac{n}{n} + \frac{2}{n} \cdot \frac{n}{n} + \dots + 1 \cdot \frac{n}{n}\right) \\
&= \frac{1}{n^2} \cdot \left(\frac{1}{n} + \frac{2}{n} + \dots + \frac{n}{n}\right)^2 \\
&= \frac{1}{n^2} \cdot \left(\frac{n(n-1)}{2n}\right)^2 \\
&= \frac{n^4 - 2n^3 + n^2}{4n^4} \\
&= \frac{1}{4} \text{ as } n \to \infty
\end{aligned}$$

Hence, $\int_{[0,1]^2} f = \frac{1}{4}$

3.

$$\begin{aligned}
\int_{[0,1]^3} f &= \int_0^1 \int_0^1 \int_0^1 f(x,y,z)dzdxdy \\
&= \int_0^1 \int_0^1 \int_0^{xy} f(x,y,z)dzdxdy + \int_0^1 \int_0^1 \int_{xy}^1 f(x,y,z)dzdxdy \\
&= \int_0^1 \int_0^1 \int_0^{xy} xydzdxdy + \int_0^1 \int_0^1 \int_{xy}^1 zdzdxdy \\
&= \int_0^1 \int_0^1 x^2y^2dxdy + \int_0^1 \int_0^1 \frac{1-x^2y^2}{2}dxdy \\
&= \int_0^1 \frac{y^2}{3}dy + \int_0^1 \frac{1}{2} - \frac{y^2}{6}dy \\
&= \frac{5}{9}
\end{aligned}$$

4.

$$
\begin{aligned}
\int_D f &= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{4y^3}{(x+1)^2} dydx \\
&= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{4y^3}{(x+1)^2} dydx \\
&= \int_0^1 \frac{(1-x^2)^2}{(x+1)^2} dx \\
&= \int_0^1 (1-2x+x^2) dx \\
&= \frac{1}{3}
\end{aligned}
$$

5.

$$\begin{aligned}\mu(D) &= \int_D \chi_D \\ &= \int_a^b \int_0^{f(x)} 1 dy dx \\ &= \int_a^b (f(x) - 0) dx \\ &= \int_a^b f(x) dx\end{aligned}$$

6.

We have that

$$\int_0^1 \left(\int_{1/2}^1 f(x,y)dx \right) dy$$
$$= \left(1 - \frac{1}{2}\right)^2 \cdot 2^2 = 1$$

and for all $n \in \mathbb{N}$ such that $n > 1$:

$$\int_0^1 \left(\int_{2^{-n}}^{2^{-n+1}} f(x,y)dx \right) dy$$
$$= (2^{-n+1} - 2^{-n})^2 \cdot 2^{2n} + (2^{-n} - 2^{-n-1}) \cdot (2^{-n+1} - 2^{-n}) \cdot (-2^{2n+1})$$
$$= 2^{-2n} \cdot 2^{2n} + 2^{-n-1} \cdot 2^{-n} \cdot (-2^{2n+1})$$
$$= 1 - 1 = 0$$

which means that

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy$$
$$= \int_0^1 \left(\int_{1/2}^1 f(x,y)dx \right) dy + \int_0^1 \left(\int_{1/4}^{1/2} f(x,y)dx \right) dy$$
$$+ \int_0^1 \left(\int_{1/8}^{1/4} f(x,y)dx \right) dy + \dots$$
$$= 1 + \sum_{n=2}^{\infty} \int_0^1 \left(\int_{2^{-n}}^{2^{-n+1}} f(x,y)dx \right) dy$$
$$= 1 + 0 = 1$$

We also have that for all natural number n:

$$\int_0^1 \left(\int_{2^{-n}}^{2^{-n+1}} f(x,y)dy \right) dx$$
$$= (2^{-n+1} - 2^{-n})^2 \cdot 2^{2n} + (2^{-n} - 2^{-n-1}) \cdot (2^{-n+1} - 2^{-n}) \cdot (-2^{2n+1})$$
$$= 2^{-2n} \cdot 2^{2n} + 2^{-n-1} \cdot 2^{-n} \cdot (-2^{2n+1})$$
$$= 1 - 1 = 0$$

which means that

$$
\begin{aligned}
&\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx \\
&= \int_0^1 \left(\int_{1/2}^1 f(x,y)dy\right) dx + \int_0^1 \left(\int_{1/4}^{1/2} f(x,y)dy\right) dx \\
&+ \int_0^1 \left(\int_{1/8}^{1/4} f(x,y)dy\right) dx + \dots \\
&= \sum_{n=2}^{\infty} \int_0^1 \left(\int_{2^{-n}}^{2^{-n+1}} f(x,y)dy\right) dx \\
&= 0
\end{aligned}
$$

Therefore,

$$
\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx \neq \int_0^1 \left(\int_0^1 f(x,y)dx\right) dy
$$

However, it does not contradicts the Fubini's theorem because when we write out the whole sequence

$$
\begin{aligned}
&\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx \\
&= (1-1) + (1-1) + (1-1) + \dots \\
&= 1 - 1 + 1 - 1 + 1 - 1 + \dots
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy \\
&= 1 + ((1-1) + (1-1) + (1-1) + \dots) \\
&= 1 + (1 - 1 + 1 - 1 + 1 - 1 + \dots)
\end{aligned}
$$

We can see that both $1-1+1-1+1-1+\dots$ and $1+(1-1+1-1+\dots)$ diverges. And hence we cannot rearrange the terms while we should be able to rearrange terms in the riemann integral.