5.2

b.

$|B - \lambda I| = \lambda^2(2-\lambda) - 3(2-\lambda) = -\lambda^3 + 2\lambda^2 + 3\lambda - 6 = (\lambda - 2)(\lambda^2 - 3) = 0$

Thus $\lambda \in \{\pm\sqrt{3}, 2\}$.

- $\lambda = 2$

$$B \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_3 \\ 2v_2 \\ 3v_1 \end{pmatrix} = \begin{pmatrix} 2v_1 \\ 2v_2 \\ 2v_3 \end{pmatrix}$$

  Thus

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = k \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- $\lambda = \sqrt{3}$

$$B \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_3 \\ 2v_2 \\ 3v_1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}v_1 \\ \sqrt{3}v_2 \\ \sqrt{3}v_3 \end{pmatrix}$$

  Thus

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ \sqrt{3} \end{pmatrix}$$

- $\lambda = -\sqrt{3}$

$$B \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_3 \\ 2v_2 \\ 3v_1 \end{pmatrix} = \begin{pmatrix} -\sqrt{3}v_1 \\ -\sqrt{3}v_2 \\ -\sqrt{3}v_3 \end{pmatrix}$$

  Thus

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ -\sqrt{3} \end{pmatrix}$$

c.

$$|C - \lambda I| = (1-\lambda)^3 + 1 + 1 - 3(1-\lambda) = -\lambda^3 + 3\lambda^2$$

Thus $\lambda \in \{0, 3\}$.

- $\lambda = 3$

$$C \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 \end{pmatrix} = \begin{pmatrix} 3v_1 \\ 3v_2 \\ 3v_3 \end{pmatrix}$$

  Thus

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- $\lambda = 0$

$$C \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 \end{pmatrix} = O$$

  Thus

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

  and

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

5.4.

$$
\begin{aligned}
|A - \lambda I| &= (a-\lambda)^4 - (a-\lambda)^2 bc + a^2 bc - (a-\lambda)^2 a^2 \\
&= ((a-\lambda)^2 - a^2)((a-\lambda)^2 - bc)
\end{aligned}
$$

Thus

$$
\lambda \in \left\{0, 2a, a \pm \sqrt{bc}\right\}
$$

So the eigenvalues are real when $bc \geq 0$.
The eigenvalues are complex if $bc < 0$.
If $a = bc = 0$ then the eigenvalue is 0 with multiplicity 4.
If $a = \pm\sqrt{bc} \neq 0$ then $0 = a \mp \sqrt{bc}$ and $2a = a \pm \sqrt{bc}$. Thus the eigenvalue 0 and $2a$ are with multiplicity 2.

5.5

c.

$$\det(A - \lambda I) = -(\lambda - 1)(\lambda^2 + 1)$$

Thus $\lambda \in \{1, \pm i\}$.

- For $\lambda = 1$, the eigenvector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- For $\lambda = i$, the eigenvector is $\begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}$
- For $\lambda = -i$, the eigenvector is $\begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix}$

Thus we let

$$T = \begin{pmatrix} -i & i & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

so that

$$T^{-1}AT = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d.

$$\det(B - \lambda I) = -(\lambda - 1)^2(\lambda + 1)$$

Thus $\lambda \in \{-1, 1\}$.

- For $\lambda = -1$, the eigenvector is $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
- For $\lambda = 1$, the eigenvectors are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Thus we let

$$T = \begin{pmatrix} -1 & 0 & 1/2 \\ 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

so that

$$T^{-1}BT = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

g.

$$\det(C - \lambda I) = -(\lambda - 1)^3$$

Thus $\lambda = 1$.

- For $\lambda = 1$, the eigenvector are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Thus we let

$$T = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that

$$T^{-1}CT = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

5.8

a.

$$Tv = \begin{pmatrix} v_1 + 2v_2 \\ 2v_1 + 4v_2 \end{pmatrix}$$

Thus

$$\text{Ker } T = \left\{ k \begin{pmatrix} 2 \\ -1 \end{pmatrix} \middle| k \in \mathbb{R} \right\}$$

$$\text{Range } T = \left\{ k \begin{pmatrix} 1 \\ 2 \end{pmatrix} \middle| k \in \mathbb{R} \right\}$$

b.

$$Tv = \begin{pmatrix} v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 \end{pmatrix}$$

Thus

$$\text{Ker } T = \left\{ k_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \middle| k_1, k_2 \in \mathbb{R} \right\}$$

$$\text{Range } T = \left\{ k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| k \in \mathbb{R} \right\}$$

c.

The RREF form is

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus

$$\text{Ker } T = \left\{ k_1 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \middle| k_1, k_2 \in \mathbb{R} \right\}$$

Since the matrix has 2 linearly independent columns, we can choose any 2 as the basis of our range

$$\text{Range } T = \left\{ k_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} \middle| k_1, k_2 \in \mathbb{R} \right\}$$

5.14

a.

Is open since we can find a neighborhood with small enough radius compare to $\det A$ so that $\det A$ remains $\neq 0$.
Is not dense, since determinant is continuous.

b.

Is not open and not dense since the rational set in $\mathbb{R}$ is not open and dense.

c.

Is not open and is dense since the integers in $\mathbb{R}$ is not open and is dense.

d.

Is not open since continuous map matrix to determinant is continuous maps the compact set to compact sets. Is not dense since there is a small enough neighborhood around $\det = 0$ such that no elements of the said set is in.

e.

Is open since the sets of of matrices that has distinct eigenvalue are open and dense, which we can start at some value $\lambda_1, \lambda_2, \lambda_3$ satisfies the initial condition and start expanding until it reaches the condition some eigenvalue $= 1$.
Is obviously not dense since we can find matrix with characteristic polynomial $\lambda^n - 100^n$ which results in all eigenvalue > 99 in the neighborhood.

f.

Is obviously not dense since we can find matrix with characteristic polynomial $\lambda^n - 100^n$ which results in all eigenvalue > 99 in the neighborhood.
Is open since if a matrix has only complex eigenvalue $a + bi$ then $a - bi$ are also an eigenvalue, thus we can rewrite the characteristic polynomial as a product of $x^2 + cx + d$, the sets $\{(c, d) \in \mathbb{R}^2 : c^2 - 4d < 0\}$ is open. Thus the set is open.

g.

The sets of real eigenvalue with multiplicity > 1 is closed since $\{(c, d) \in \mathbb{R}^2 : c^2 = 4d\}$ is closed. The set is dense since matrix with all complex eigenvalue satisfies the condition, thus let's consider matrix with some real eigenvalue. If there is a neighborhood then supposed if there is some matrix in that neighborhood with real eigenvalue with multiplicity > 1 by $(\lambda - r_i)^{k_i}$, then we can find another matrix in such neighborhood such that all real eigenvalue has multiplicity 1 as we can break $(\lambda - r_i)^k$ into k eigenvalue with multiplicity 1.

6.1

a.

$$\det(A - \lambda I) = \lambda^2(1-\lambda) - (1-\lambda) = -(1-\lambda)^2(1+\lambda)$$

Thus $\lambda \in \{-1, 1\}$.

- For $\lambda = 1$, the eigenvectors are $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
- For $\lambda = -1$, the eigenvector is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Thus we can choose

$$T = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

so that

$$T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and thus the solution is

$$X(t) = T\left(c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right)$$

b.

$$\det(B - \lambda I) = (1-\lambda)^3 - (1-\lambda) = (1-\lambda)(\lambda^2 - 2\lambda)$$

Thus $\lambda \in \{0, 1, 2\}$.

- For $\lambda = 0$, the eigenvector is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
- For $\lambda = 1$, the eigenvector is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
- For $\lambda = 2$, the eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Thus we can choose

$$T = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

so that

$$T^{-1}AT = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and thus the solution is

$$X(t) = T\left(c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2e^t\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3e^{2t}\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

c.

$$\det(C - \lambda I) = (1-\lambda)(\lambda^2+1)$$

Thus $\lambda \in \{1, \pm i\}$.

- For $\lambda = 1$, the eigenvector is $\begin{pmatrix}0\\0\\1\end{pmatrix}$
- For $\lambda = i$, the eigenvector is $\begin{pmatrix}i\\-1\\1\end{pmatrix}$
- For $\lambda = -i$, the eigenvector is $\begin{pmatrix}-i\\-1\\1\end{pmatrix}$

Thus we can choose

$$T = \begin{pmatrix} 0 & i & -i \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

so that

$$T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$$

and thus the solution is

$$X(t) = T\left(c_1e^t\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\\cos(t)\\-\sin(t)\end{pmatrix} + c_3\begin{pmatrix}0\\\sin(t)\\\cos(t)\end{pmatrix}\right)$$

6.6

a.

$$\det(A - \lambda I) = (a-\lambda)^2(b-\lambda) + b^2(b-\lambda) = (b-\lambda)(\lambda^2 - 2a\lambda + a^2 + b^2)$$

Thus $\lambda \in \{b, a \pm bi\}$.

- For $\lambda = b$, the eigenvector is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
- For $\lambda = a - bi$, the eigenvector is $\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$
- For $\lambda = a + bi$, the eigenvector is $\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$

Thus we can choose

$$T = \begin{pmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

so that

$$T^{-1}AT = \begin{pmatrix} b & 0 & 0 \\ 0 & a-bi & 0 \\ 0 & 0 & a+bi \end{pmatrix}$$

and thus the solution is

$$X(t) = T\left(c_1 e^{bt}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{at}\begin{pmatrix} 0 \\ \cos(bt) \\ -\sin(bt) \end{pmatrix} + c_3 e^{at}\begin{pmatrix} 0 \\ \sin(bt) \\ \cos(bt) \end{pmatrix}\right)$$

b.

On the x-axis, it is a sink if $b < 0$ and is a source if $b > 0$.
On the yz-plane, it is a sink if $a < 0$ and is a source if $a > 0$.
Note that it is a sink on the x-axis does not means it is a sink on yz-plane and vice versa. Thus we can divide the ab-plane into 4 regions which have different phase portraits using the axis.

6.7.

Let $y_1 = x_1'$ and $y_2 = x_2'$ then we get the system

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -(k_1+k_2) & 0 & k_2 & 0 \\ 0 & 0 & 0 & 1 \\ k_2 & 0 & -(k_1+k_2) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{pmatrix}$$

Thus the characteristic polynomial is

$$\det(A - \lambda I) = (\lambda^2 + k_1)(\lambda^2 + k_1 + 2k_2)$$

Thus the eigenvalues are $\pm i\sqrt{k_1}, \pm i\sqrt{k_1 + 2k_2}$ and

- For $\lambda = -i\sqrt{k_1}$, the eigenvector is $\begin{pmatrix} \frac{i}{\sqrt{k_1}} \\ 1 \\ \frac{i}{\sqrt{k_1}} \\ 1 \end{pmatrix}$

- For $\lambda = i\sqrt{k_1}$, the eigenvector is $\begin{pmatrix} -\frac{i}{\sqrt{k_1}} \\ 1 \\ -\frac{i}{\sqrt{k_1}} \\ 1 \end{pmatrix}$

- For $\lambda = -i\sqrt{k_1 + 2k_2}$, the eigenvector is $\begin{pmatrix} -\frac{i}{\sqrt{k_1+2k_2}} \\ -1 \\ \frac{i}{\sqrt{k_1+2k_2}} \\ 1 \end{pmatrix}$

- For $\lambda = i\sqrt{k_1 + 2k_2}$, the eigenvector is $\begin{pmatrix} \frac{i}{\sqrt{k_1+2k_2}} \\ -1 \\ -\frac{i}{\sqrt{k_1+2k_2}} \\ 1 \end{pmatrix}$

Thus we can let

$$T = \begin{pmatrix} \frac{i}{\sqrt{k_1}} & -\frac{i}{\sqrt{k_1}} & -\frac{i}{\sqrt{k_1+2k_2}} & \frac{i}{\sqrt{k_1+2k_2}} \\ 1 & 1 & -1 & -1 \\ \frac{i}{\sqrt{k_1}} & -\frac{i}{\sqrt{k_1}} & \frac{i}{\sqrt{k_1+2k_2}} & -\frac{i}{\sqrt{k_1+2k_2}} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

so that

$$T^{-1}AT = \begin{pmatrix} -i\sqrt{k_1} & 0 & 0 & 0 \\ 0 & i\sqrt{k_1} & 0 & 0 \\ 0 & 0 & -i\sqrt{k_1+2k_2} & 0 \\ 0 & 0 & 0 & i\sqrt{k_1+2k_2} \end{pmatrix}$$

and thus the general solution is

$$\begin{aligned} X(t) = T & \left(c_1 \begin{pmatrix} \cos(\sqrt{k_1}t) \\ -\sin(\sqrt{k_1}t) \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \sin(\sqrt{k_1}t) \\ \cos(\sqrt{k_1}t) \\ 0 \\ 0 \end{pmatrix} \right. \\ & \left. + c_3 \begin{pmatrix} 0 \\ 0 \\ \cos(\sqrt{k_1+2k_2}t) \\ -\sin(\sqrt{k_1+2k_2}t) \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ \sin(\sqrt{k_1+2k_2}t) \\ \cos(\sqrt{k_1+2k_2}t) \end{pmatrix} \right) \end{aligned}$$

d.

The periodicity of the first 2 dimensions of the solution is w_1 while the periodicity of the last 2 dimensions of the solutions is w_2.

6.12

a.

The characteristic polynomial is

$$\det(A - \lambda I) = (\lambda + 1)(\lambda - 2)$$

Thus the eigenvalue is $-1, 2$ with respective eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Thus we can find

$$T = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \text{ and } T^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

such that

$$T^{-1}AT = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

and thus

$$A^k = T \begin{pmatrix} (-1)^k & 0 \\ 0 & 2^k \end{pmatrix} T^{-1} = \begin{pmatrix} 2^{k+1} - (-1)^k & 2^k - (-1)^k \\ 2(-1)^k - 2^{k+1} & 2(-1)^k - 2^k \end{pmatrix}$$

and

$$\exp(A) = \begin{pmatrix} 2e^2 - e^{-1} & e^2 - e^{-1} \\ 2e^{-1} - 2e^2 & 2e^{-1} - e^2 \end{pmatrix}$$

c.

Let's inspect

$$C^2 = \begin{pmatrix} 4 & -4 \\ 0 & 4 \end{pmatrix}$$

We can see that the $C^k_{1,1} = C^k_{2,2} = 2^k, C^k_{2,1} = 0$. Now we claim that $C^k_{1,2} = -2^{k-1} + 2C^{k-1}_{1,2}$. Indeed, the base is $k = 2$,

$$C^2_{1,2} = -2 + 2(-1) = -4$$

and the inductive steps

$$C^kC = \begin{pmatrix} 2^k & C^k_{1,2} \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & -2^k + 2C^k_{1,2} \\ 0 & 2^{k+1} \end{pmatrix}$$

Now let's look at the sequence $a_n = -2^{n-1} + 2a_{n-1}$ with $a_1 = -1$.

$$a_n = -2^{n-1} + 2a_{n-1} = -2^{n-1} + 2(-2^{n-2} + a_{n-2}) = \ldots = -n2^{n-1}$$

and Thus

$$\exp(C) = \begin{pmatrix} 2e^2 & -\sum_{k=0}^{\infty} \dfrac{k2^{k-1}}{k!} \\ 0 & 2e^2 \end{pmatrix} = \begin{pmatrix} 2e^2 & -1 - \sum_{k=1}^{\infty} \dfrac{2^{k-1}}{(k-1)!} \\ 0 & 2e^2 \end{pmatrix} = \begin{pmatrix} 2e^2 & -e^2 \\ 0 & 2e^2 \end{pmatrix}$$

e.

Since the matrix is nilpotent,

$$E^2 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$E^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have that

$$\exp(E) = I + E + E^2/2 = \begin{pmatrix} 1 & 1 & 2+3/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3.5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

h.

$$H^k = \begin{pmatrix} i^k & 0 \\ 0 & (-i)^k \end{pmatrix}$$

Thus

$$\exp(H) = \begin{pmatrix} e^i & 0 \\ 0 & e^{-i} \end{pmatrix} = \begin{pmatrix} \cos(1) + i\sin(1) & 0 \\ 0 & \cos(-1) + i\sin(-1) \end{pmatrix}$$