1.

$$J_f(r,\theta,\phi) = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\cos\phi & r\cos\theta\sin\phi & r\cos\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

and

$$J_g(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.

Proof. Using the fact that the matrix $X(N \times N)$ is invertible if and only if its determinant is 0. Let each entries of the matrix be the entries of the function

$$g : \mathbb{R}^{N\times N} \to \mathbb{R}, \qquad X \to \det(X)$$

Since $\det(X) = \sum_\sigma \operatorname{sgn}(\sigma) \prod_{i=1}^n x_{\sigma i,i}$, which is a polynomial of the matrix's coordinate. Hence g is continuous. Since $\mathbb{R}\backslash\{0\}$ is open, there is an open set $V \in \mathbb{R}^{N\times N}$ such that $f(V) = \mathbb{R}\backslash\{0\}$. That set V is the set of all invertible matrices.
By Cramer's rule, $X^{-1} = \frac{1}{\det(X)}\operatorname{adj}(X)$. The function g is continuous, and the function h that maps the matrix X to its adjugate matrix is also continuous. Therefore,

$$f = h \cdot \frac{1}{g} : U \to M_N(\mathbb{R}), \qquad X \to X^{-1} = \frac{1}{\det(X)}\operatorname{adj}(X)$$

is continuous
We have as $H \to 0, -HX_0^{-1} \to 0$, and hence

$$\begin{aligned}
f(X_0 + H) &= (X_0 + H)^{-1} \\
&= ((I + HX_0^{-1})X_0)^{-1} \\
&= X_0^{-1}(I + HX_0^{-1})^{-1} \\
&= X_0^{-1}\left(I - HX_0^{-1} + \sum_{i=2}^{\infty}(-HX_0^{-1})^i\right) \\
&= f(X_0) - X_0^{-1}HX_0^{-1} + X_0^{-1}\sum_{i=2}^{\infty}(-HX_0^{-1})^i
\end{aligned}$$

Therefore, define

$$T : M_N(\mathbb{R}) \to M_N(\mathbb{R}), \qquad X \to -X_0^{-1}XX_0^{-1}$$

so that as $H \to 0$

$$\frac{\|f(X_0 + H) - f(X_0) - T(H)\|}{\|H\|} = \frac{\|X_0^{-1}\sum_{i=2}^{\infty}(-HX_0^{-1})^i\|}{\|H\|} \to 0$$

which proves that f is (totally) differentiable and that $Df(X_0)X = -X_0^{-1}XX_0^{-1}$ $\square$

3.

$$
\begin{aligned}
\frac{\partial (f\circ p)}{\partial r}(r,\theta) &= \frac{\partial f}{\partial r}(p(r,\theta))\cdot \frac{\partial p}{\partial r}(r,\theta)\\
&= \left(\frac{\partial f}{\partial r}\circ p\right)(r,\theta)\cdot(\cos\theta,\sin\theta)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 (f\circ p)}{\partial r^2}(r,\theta) &= \frac{\partial}{\partial r}\left(\left(\left(\frac{\partial f}{\partial r}\circ p\right)(r,\theta)\right)\cdot(\cos\theta,\sin\theta)\right)\\
&= \left(\frac{\partial^2 f}{\partial r^2}\circ p\right)(r,\theta)\cdot\frac{\partial p}{\partial r}(r,\theta)\cdot(\cos\theta,\sin\theta)\\
&= \left(\frac{\partial^2 f}{\partial r^2}\circ p\right)(r,\theta)\cdot(\cos^2\theta+\sin^2\theta)\\
&= \left(\frac{\partial^2 f}{\partial r^2}\circ p\right)(r,\theta)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 (f\circ p)}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}(p(r,\theta))\cdot\frac{\partial p}{\partial \theta}(r,\theta)\right)\\
&= \frac{\partial}{\partial \theta}\left(\left(\frac{\partial f}{\partial \theta}\circ p\right)(r,\theta)\cdot(-r\sin\theta, r\cos\theta)\right)\\
&= \left(\frac{\partial^2}{\partial \theta^2}\circ p\right)(r,\theta)\cdot\frac{\partial p}{\partial \theta}(r,\theta)\cdot(-r\sin\theta,r\cos\theta)+\left(\frac{\partial f}{\partial \theta}\circ p\right)(r,\theta)\cdot(-r\cos\theta,-r\sin\theta)\\
&= \left(\frac{\partial^2 f}{\partial \theta^2}\circ p\right)(r,\theta)\cdot(-r\sin\theta,r\cos\theta)\cdot(-r\sin\theta,r\cos\theta)-r\cdot\left(\frac{\partial f}{\partial \theta}\circ p\right)(r,\theta)\cdot(\cos\theta,\sin\theta)\\
&= \left(\frac{\partial^2 f}{\partial \theta^2}\circ p\right)(r,\theta)\cdot r^2 - r\frac{\partial (f\circ p)}{\partial r}
\end{aligned}
$$

Therefore,

$$
\frac{\partial^2 (f\circ p)}{\partial r^2}+\frac{1}{r}\frac{\partial (f\circ p)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 (f\circ p)}{\partial \theta^2} = \frac{\partial^2 f}{\partial r^2}\circ p+\frac{\partial^2 f}{\partial \theta^2}\circ p = (\Delta f)\circ p
$$

4.

We have that

$$\left|\frac{xy^3}{x^2+y^4}\right| = \frac{|x^2y|\cdot|y|}{x^2+y^4} \leq \frac{\frac{x^2+y^4}{2}\cdot|y|}{x^2+y^4} = \frac{|y|}{2}$$

Therefore, $\forall\epsilon > 0 : \forall(x,y) \in B_\epsilon(x,y) : |f(x,y)| \leq \dfrac{|y|}{2} < \dfrac{\epsilon}{2} < \epsilon$. Hence, f is continuous at (0,0).
$\forall v = (v_1, v_2) \in \mathbb{R}^2 : \|v\| = 1 :$

$$\begin{aligned}
D_v f(0,0) &= \lim_{\substack{h\to 0\\ h\neq 0}} \frac{f((0,0)+hv) - f(0,0)}{h} \\
&= \lim_{\substack{h\to 0\\ h\neq 0}} \frac{hv_1\cdot h^3v_2^3}{h\cdot(h^2v_1^2 + h^4v_2^4)} \\
&= \lim_{\substack{h\to 0\\ h\neq 0}} \frac{v_1v_2^3}{\frac{v_1^2}{h} + hv_2^4} \\
&= 0
\end{aligned}$$

Therefore, $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$. Which means that if f is totally differentiable, $T = (0,0)$ is the jacobian matrix.
Consider the function

$$g : \mathbb{R} \to \mathbb{R}, \qquad t \to f(t^2, t)$$

$g(t^2,t) = \frac{t}{2}$, hence $g'(0) = g'(t) = \frac{1}{2}$. But given the function

$$h : \mathbb{R} \to \mathbb{R}^2, \qquad t \to (t^2, t)$$

We have $f \circ h = g$ but $D(f\circ h)(0) = Df(h(0))Dh(0) = T\cdot Dh(0) = 0 \neq \frac{1}{2}$
Therefore, f is not totally differentiable at (0,0).

5.

U is open and convex, U also contains 0. Hence, $\{tx : t \in [0,1]\} \subset U$. By Taylor's theorem, there is $\theta \in [0,1]$ such that

$$\begin{aligned} f(x) &= \sum_{|\alpha|\leq n} \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(0)x^\alpha + \sum_{|\alpha|=n+1} \frac{1}{\alpha!}\underbrace{\frac{\partial^\alpha f}{\partial x^\alpha}}_{0}(0+\theta x) \\ &= \sum_{|\alpha|\leq n} \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(0)x^\alpha \end{aligned}$$

Hence, for $|a| \leq n$, there is

$$c_\alpha = \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(0)$$

that satisfies

$$f(x) = \sum_{|a|\leq n} c_\alpha x^\alpha$$

6.

f is differentiable such that $\Delta f = 0$, hence f is totally differentiable and $\forall v : D_v f = 0$. If c is convex, then $\forall x, y \in C$, there exsits a continuous function

$$g : [0,1] \to C, \qquad t \to tx + (1-t)y$$

Let $h = f \circ g$. Then $\forall t \in [0,1]$:

$$\begin{aligned} h'(t) &= Dh(t) \\ &= D(f \circ g)(t) \\ &= Df(g(t))Dg(t) \\ &= (x-y)\sum_{j=1}^{N} \frac{\partial f}{\partial x_j}(tx + (1-t)y) \\ &= 0 \end{aligned}$$

Hence, h is constant and that $y = h(0) = h(1) = x$.
Since x, y are arbitary, f is also constant.
For general C, if f is not contant then $\exists x, y \in C : f(x) \neq f(y)$. Consider the set

$$U := \{z \in C : f(z) = f(x)\}$$

$$V := \{z \in C : f(z) \neq f(x)\}$$

It is obvious that $x \in U \cap C$ and $y \in V \cap C$, hence

$$U \cap C \neq \varnothing \neq V \cap C$$

Straight from the definition of U and V, we also have that

$$(U \cap C) \cap (V \cap C) = \varnothing$$

$$(U \cap C) \cup (V \cap C) = C$$

We also have that if $z \in U$, then $\exists \epsilon > 0 : B_\epsilon(z) \in C$ as C is open. $B_\epsilon(z)$ is convex hence $\forall z' \in B_\epsilon(z) : f(z') = f(z) = f(x)$ which means that $B_\epsilon(z) \in U$ and hence U is open.
If $z \in V$, then similarly, $\exists \epsilon > 0 : B_\epsilon(z) \in C$ as C is open. $B_\epsilon(z)$ is convex hence $\forall z' \in B_\epsilon(z) : f(z') = f(z) \neq f(x)$ which means that $B_\epsilon(z) \in V$ and hence V is open.
Hence, $\{U, V\}$ is a disconnection for C, therefore f is constant.