

1.

b.

Since $f_n(x) \uparrow f(x)$ for all $x \in X$, we have that

$$\int_X f = \int_X \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int_X f_n$$

But we also know that $\sup_{n \geq k} \int_X f_n \leq \int_X f$ for all $n \in \mathbb{N}$, thus

$$\int_X f \geq \lim_{n \rightarrow \infty} \sup_{n \geq k} \int_X f_n$$

Thus we have that

$$\limsup_{n \rightarrow \infty} \int_X f_n = \liminf_{n \rightarrow \infty} \int_X f_n = \lim_{n \rightarrow \infty} \int_X f_n = \int_X f$$

b.

Define a sequence of function

$$f_n(x) = f(x) \cdot \chi_{x \leq n}$$

Thus $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and is nonnegative as f is nonnegative. Then from part a, we know that

$$\int_N f d\mu = \lim_{n \rightarrow \infty} \int_N f_n = \lim_{n \rightarrow \infty} \int_{\{1,2,\dots,n\}} f_n d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(i)$$

2.

For any measurable subset E , we have that

$$\int_E f = \int_E \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int_E f_n \leq \limsup_{n \rightarrow \infty} \int_E f_n$$

We also have that

$$\begin{aligned} \int_E f &= \int_X f - \int_{E^c} f \\ &= \int_X f - \int_{E^c} \liminf_{n \rightarrow \infty} f_n \\ &\geq \int_X f - \liminf_{n \rightarrow \infty} \int_{E^c} f_n \\ &= \int_X f + \limsup_{n \rightarrow \infty} \int_{E^c} -f_n \\ &= \limsup_{n \rightarrow \infty} \left(\int_X f - \int_{E^c} f \right) \\ &= \limsup_{n \rightarrow \infty} \int_E f_n \end{aligned}$$

Thus

$$\int_E f = \limsup_{n \rightarrow \infty} \int_E f_n = \liminf_{n \rightarrow \infty} \int_E f_n = \lim_{n \rightarrow \infty} \int_E f_n$$

3.

a.

Let $\varphi = \sum_{j=0}^n c_j \chi_{E_j}$, where $c_0 = 0$ and E_j are pairwise disjoint and $\cup_{j=0}^n E_j = X$.

$$\int_X \varphi d\nu = \sum_{j=0}^n \int_{E_j} c_j f d\mu = \int_X d\mu = \int_X \sum_{j=0}^n c_j \chi_{E_j} f d\mu = \int_X \varphi f d\mu$$

b.

Since X is a nonnegative measurable function, there is a sequence of non-negative simple function ϕ_n such that $\phi_n \uparrow g$ for all $x \in X$. Then

$$\int_X \phi_n d\nu = \int_X \phi_n f d\mu$$

and

$$\lim_{n \rightarrow \infty} \int_X \phi_n d\nu = \lim_{n \rightarrow \infty} \int_X \phi_n f d\mu$$

Since $\phi_n \uparrow g$ and thus $\phi_n f \uparrow gf$, we have that

$$\int_X g d\nu = \int_X g f d\mu$$

4.

Definition: If $f_n \rightarrow f$ in measure then for an arbitrary $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mu(\underbrace{\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}}_{X_n}) = 0$$

Therefore, for all $\delta > 0$, we can find n_0 such that for all $n > n_0$, $\mu(X_n) < \delta/2$ and $|f_n - f| < \varepsilon$ for all $x \in X \setminus X_n$.

$$\begin{aligned} & \lim_{n \rightarrow \infty} \rho(f_n, f) \\ &= \lim_{n \rightarrow \infty} \int_X \frac{|f_n - f|}{|f_n - f| + 1} d\mu \\ &\leq \lim_{n \rightarrow \infty} \int_X d\mu + \int_{X \setminus X_n} \frac{\varepsilon}{\varepsilon + 1} d\mu \\ &\leq \frac{\delta}{2} + \mu(X) \cdot \frac{\varepsilon}{\varepsilon + 1} \end{aligned}$$

Since ε is arbitrary, choose $\varepsilon = \frac{\delta}{2\mu(X) - \delta}$ so that

$$\lim_{n \rightarrow \infty} \rho(f_n, f) \leq \frac{\delta}{2} + \mu(X) \cdot \frac{\delta}{\delta + 2\mu(X) - \delta} = \delta$$

If $f_n \rightarrow f$ in measure is false then there is some $\varepsilon, \delta > 0$ such that for all $n_0 > 0$, there is $n > n_0$ such that

$$\mu(\underbrace{\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}}_{X_n}) = \delta$$

and therefore

$$\begin{aligned} & \rho(f_n, f) \\ &= \int_X \frac{|f_n - f|}{|f_n - f| + 1} d\mu \\ &\geq \int_{X_n} \frac{|f_n - f|}{|f_n - f| + 1} d\mu \\ &\geq \delta \frac{\varepsilon}{\varepsilon + 1} \end{aligned}$$

since $|f_n - f| \geq \varepsilon$ and $\frac{x}{1+x}$ is an increasing function. Thus

$$\lim_{n \rightarrow \infty} \rho(f_n, f) \geq \frac{\delta\varepsilon}{\varepsilon + 1} > 0$$

for some $\varepsilon, \delta > 0$, thus is a contradiction.

5.

Applying Fatou's, we have that

$$\lim_{n \rightarrow \infty} \int_{E \setminus E_0} [f(x)]^{1/n} dx = \liminf_{n \rightarrow \infty} \int_{E \setminus E_0} [f(x)]^{1/n} dx \geq \int_{E \setminus E_0} \liminf_{n \rightarrow \infty} [f(x)]^{1/n} = \int_{E \setminus E_0} 1 = m(E \setminus E_0)$$

and also

$$\lim_{n \rightarrow \infty} \int_{E \setminus E_0} [f(x)]^{1/n} dx \leq \limsup_{n \rightarrow \infty} \int_{E \setminus E_0} [f(x)]^{1/n} dx$$

We first prove the reverse Fatou's lemma: Suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence of measurable functions and g an integrable function such that $f_n \leq g$ for all $n \in \mathbb{N}$. Then $\limsup_{n \rightarrow \infty} \int_X f_n \leq \int_X \limsup_{n \rightarrow \infty} f_n$.

We can apply the fatou's lemma to $g - f_n \geq 0$,

$$\int_X \liminf_{n \rightarrow \infty} (g - f_n) \leq \liminf_{n \rightarrow \infty} \int_X (g - f_n)$$

Thus

$$\int_X \liminf_{n \rightarrow \infty} -f_n \leq \liminf_{n \rightarrow \infty} \int_X -f_n$$

and therefore,

$$-\int_X \limsup_{n \rightarrow \infty} f_n \leq -\limsup_{n \rightarrow \infty} \int_X f_n$$

which concludes the proof for the reverse version. Now apply the lemma with the function g on the domain D of f

$$g : D \rightarrow \mathbb{R}, \quad x \mapsto f(x) + 1$$

so that $g \geq f_n$ for all $n \in \mathbb{N}$ as

- if $f(x) \geq 1$, then $f_n(x) \leq f(x) < g(x)$
- if $f(x) < 1$, then $f_n(x) < 1 \leq f(x) + 1$

thus we have

$$\lim_{n \rightarrow \infty} \int_{E \setminus E_0} [f(x)]^{1/n} dx \leq \int_{E \setminus E_0} \limsup_{n \rightarrow \infty} [f(x)]^{1/n} dx = \int_{E \setminus E_0} 1 dx = m(E \setminus E_0)$$

Thus,

$$\lim_{n \rightarrow \infty} \int_E [f(x)]^{1/n} = \lim_{n \rightarrow \infty} \int_{E \setminus E_0} [f(x)]^{1/n} dx = m(E \setminus E_0)$$