A2.1

a. $\Rightarrow$

If f is anti-holomorphic at $z_0 = x_0 + iy_0$ then consider the function $g = \overline{f}$, we know that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{\overline{z - z_0}}$$

exists. Hence,

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}$$

also exists. Therefore, g is complex differentiable at z_0. And hence f is complex differentiable at z_0. Therefore,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Hence at z_0,

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y} = i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial f}{\partial y}$$

Define

$$T : \mathbb{C} \to \mathbb{C}, \qquad z \to \overline{zg'(z_0)}$$

$$\frac{|f(z) - f(z_0) - T(z - z_0)|}{|z - z_0|} = \left|\frac{f(z) - f(z_0)}{\overline{z - z_0}} - \overline{g'(z_0)}\right| \to 0$$

as $z \to z_0$. Therefore, f is totally differentiable.

b. $\Leftarrow$

We know that $g = \overline{f}$ is totally differentiable at z_0 as f is totally differentiable at z_0 and since

$$\frac{\partial f}{\partial x} = i\frac{\partial f}{\partial y} \implies \frac{\partial g}{\partial x} = -i\frac{\partial g}{\partial y}$$

We know that g is complex differentiable at z_0 and hence

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}$$

exists which means that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{\overline{z - z_0}}$$

exists. Thus, f is anti-holomorphic.

A2.2

a.

$$
\begin{aligned}
\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} &= \lim_{z\to z_0}\frac{a(z^2-z_0^2)+b(z\overline{z}-z_0\overline{z_0})+c(\overline{z}^2-\overline{z_0}^2)}{z-z_0}\\
&= 2az_0+\lim_{z\to z_0}\frac{b(z\overline{z}-z_0\overline{z_0})+c(\overline{z}^2-\overline{z_0}^2)}{z-z_0}\\
&= 2az_0+\lim_{z\to z_0}\frac{bz\overline{z}-b\overline{z}z_0+b\overline{z}z_0-bz_0\overline{z_0}+c(\overline{z}^2-\overline{z_0}^2)}{z-z_0}\\
&= 2az_0+b\overline{z}+\lim_{z\to z_0}\frac{bz_0(\overline{z}-\overline{z_0})+c(\overline{z}-\overline{z_0})(\overline{z}+\overline{z_0})}{z-z_0}\\
&= 2az_0+b\overline{z}+\lim_{z\to z_0}\frac{(\overline{z}-\overline{z_0})(bz_0+c(\overline{z}+\overline{z_0}))}{z-z_0}\\
&= 2az_0+b\overline{z}+\lim_{z\to z_0}\frac{(\overline{z}-\overline{z_0})(bz_0+c(\overline{z}+\overline{z_0}))}{z-z_0}
\end{aligned}
$$

Since the map $z\to\overline{z}$ is not complex differentiable, $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ exists iff $\lim_{z\to z_0}\frac{(\overline{z}-\overline{z_0})(bz_0+c(\overline{z}+\overline{z_0}))}{z-z_0}$ iff $bz_0+2c(\overline{z_0})=0$

b.

Let $z_0=x_0+iy_0$, then f is differentiable at z_0 if

$$bx_0+2cx_0=0 \qquad by_0-2cy_0=0$$

If $b=-2c, bx_0+2cx_0=0$ is true for all x_0. However, $by_0-2cy_0=2by_0=0\implies y_0=0$.
If $b=2c, by_0-2cy_0=0$ is true for all y_0. However, $bx_0+2cx_0=2bx_0=0\implies x_0=0$.
In other cases, $x_0=y_0=0$ must be satisfied so that f is differentiable at z_0.

A2.3

$g(z) = zf(z)$ is analytic in D, hence for all $z \in D$

$$\frac{\partial g}{\partial x} = -i\frac{\partial g}{\partial y}$$

We also have

$$\frac{\partial g}{\partial x} = \frac{\partial z}{\partial x}f(z) + \frac{\partial f}{\partial x}z = f(z) + \frac{\partial f}{\partial x}z$$

$$-i\frac{\partial g}{\partial y} = -i\left(\frac{\partial z}{\partial y}f(z) + \frac{\partial f}{\partial y}z\right) = -i\left(if(z) + \frac{\partial f}{\partial y}z\right) = f(z) - i\frac{\partial f}{\partial y}z$$

Therefore,

$$\frac{\partial f}{\partial x}z = -i\frac{\partial f}{\partial y}z$$

However, $\overline{f(z)}$ is analytic in D, hence for all $z \in D$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

Thus f is constant on D.

A2.4

Since f and g are anti-holomorphic at z_0 and $g(z_0)$, they are totally differentiable at z_0 and $g(z_0)$, hence $f \circ g$ is also totally differentiable. We have

$$\begin{aligned} J_{f\circ g}(x_0) &= J_f(g(z_0))J_g(x_0) \\ &= \begin{bmatrix} \frac{\partial u_f}{\partial x}(g(z_0)) & \frac{\partial u_f}{\partial y}(g(z_0)) \\ \frac{\partial v_f}{\partial x}(g(z_0)) & \frac{\partial v_f}{\partial y}(g(z_0)) \end{bmatrix} \begin{bmatrix} \frac{\partial u_g}{\partial x}(z_0) & \frac{\partial u_g}{\partial y}(z_0) \\ \frac{\partial v_g}{\partial x}(z_0) & \frac{\partial v_g}{\partial y}(z_0) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \frac{\partial u_{f\circ g}}{\partial x} &= \frac{\partial u_f}{\partial x}(g(z_0))\frac{\partial u_g}{\partial x}(z_0) + \frac{\partial u_f}{\partial y}(g(z_0))\frac{\partial v_g}{\partial x}(z_0) \\ \frac{\partial u_{f\circ g}}{\partial y} &= \frac{\partial u_f}{\partial x}(g(z_0))\frac{\partial u_g}{\partial y}(z_0) + \frac{\partial u_f}{\partial y}(g(z_0))\frac{\partial v_g}{\partial y}(z_0) \\ \frac{\partial v_{f\circ g}}{\partial x} &= \frac{\partial v_f}{\partial x}(g(z_0))\frac{\partial u_g}{\partial x}(z_0) + \frac{\partial v_f}{\partial y}(g(z_0))\frac{\partial v_g}{\partial x}(z_0) \\ \frac{\partial v_{f\circ g}}{\partial y} &= \frac{\partial v_f}{\partial x}(g(z_0))\frac{\partial u_g}{\partial y}(z_0) + \frac{\partial v_f}{\partial y}(g(z_0))\frac{\partial v_g}{\partial y}(z_0) \end{aligned}$$

Since f and g are anti-holomorphic, we have that

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

for both function at z_0 and $g(z_0)$ respectively. Hence,

$$\frac{\partial u_{f\circ g}}{\partial x} = \frac{\partial v_{f\circ g}}{\partial y}, \qquad \frac{\partial u_{f\circ g}}{\partial y} = -\frac{\partial v_{f\circ g}}{\partial x}$$

Therefore, $f \circ g$ is complex differentiable.

A2.5

We have that

$$\frac{\partial g}{\partial x}(z) = \frac{\partial}{\partial x}\int_0^1 f(t,z)dt = \int_0^1 \frac{df}{dx}(t,z)dt$$

And since $\frac{\partial f}{\partial x}$ exists and are continuous on $[0,1]\times D$ for $z = x + iy$, $\frac{\partial g}{\partial x}$ and similarly $\frac{\partial g}{\partial y}$ exists and are continuous on $[0,1]\times D$.
Therefore, g is totally differentiable. Notice that

$$\frac{\partial g}{\partial x}(z) = \int_0^1 \frac{df}{dx}(t,z)dt = \int_0^1 -i\frac{df}{dy}(t,z)dt = -i\int_0^1 \frac{df}{dy}(t,z)dt = -i\frac{\partial g}{\partial y}$$

which confirms that g is indeed complex differentiable.