

**1.**

Since  $f$  is a  $\mathcal{C}^1$  function, the graph of  $f$

$$\gamma : [a, b] \rightarrow [a, b] \times f([a, b]), \quad t \rightarrow (t, f(t))$$

is also a  $\mathcal{C}^1$  function and hence is rectifiable, therefore has the length

$$\int_a^b \sqrt{\left(\frac{dt}{dt}\right)^2 + \left(\frac{df(t)}{dt}\right)^2} dt = \int_a^b \sqrt{1 + f'(t)} dt$$

## 2.

Since  $x$  and  $y$  plays the same role in the domain  $K$ , if  $K$  is normal with respect to  $x$ -axis means that  $K$  is also normal with respect to the  $y$ -axis. We considier

$$\phi_1 : \mathbb{R} \rightarrow \mathbb{R}, \quad x \rightarrow 0$$

and

$$\phi_2 : \mathbb{R} \rightarrow \mathbb{R}, \quad x \rightarrow 1 - x$$

Then let

$$\sigma : [a, b] \rightarrow [a, b], \quad x \rightarrow x$$

We have the curve  $\gamma_1$  and  $\gamma_2$

$$\gamma_1 : [a, b] \rightarrow \mathbb{R}^2, \quad t \rightarrow (\sigma(t), \phi_1(\sigma(t))) = (t, 0)$$

$$\gamma_2 : [a, b] \rightarrow \mathbb{R}^2, \quad t \rightarrow (\sigma(t), \phi_2(\sigma(t))) = (t, 1 - t)$$

which is clearly a  $\mathcal{C}^1$  curves and hence  $K$  is a normal domain.

### 3.

a.

$S_1(1) = 2\pi$  since it is all the point in the set  $\{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ , which is a circle with radius 1.

$S_2(1) = 4\pi$  since it is all the point in the set  $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ , which is a sphere with radius 1.

b.

Consider  $\phi(x) = rx$  then apply the change of variables theorem, we have that

$$\begin{aligned} S_n(r) &= 2^n \int_0^r \int_0^{\sqrt{r^2-x_1^2}} \dots \int_0^{\sqrt{r^2-x_1^2-x_2^2-\dots-x_{n-1}^2}} |N| dx_n \dots dx_2 dx_1 \\ &= 2^n |\det J_\phi|^n \int_0^1 \int_0^{\sqrt{1-(\frac{x_1}{r})^2}} \dots \int_0^{\sqrt{1-(\frac{x_1}{r})^2-(\frac{x_2}{r})^2-\dots-(\frac{x_{n-1}}{r})^2}} |N| d\left(\frac{x_n}{r}\right) \dots d\left(\frac{x_2}{r}\right) d\left(\frac{x_1}{r}\right) \\ &= r^n S_n(1) \end{aligned}$$

c.

Since swapping columns of a determinant return the same answer or the negative value of that. That is for a specific  $(i_1, i_2, \dots, i_n)$  and a composition  $\sigma$  mapping  $\{1, 2, \dots, n\}$  to itself then

$$\left( \frac{\partial(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n})}{\partial(x_1, x_2, \dots, x_n)} \right)^2 = \left( \frac{\partial(\phi_{i_{\sigma(1)}}, \phi_{i_{\sigma(2)}}, \dots, \phi_{i_{\sigma(n)}})}{\partial(x_1, x_2, \dots, x_n)} \right)^2$$

and since there is exactly  $n!$  ways of ways to shuffle the columns and there is only 1 ways to put them in a non-ascending order. We have that

$$\begin{aligned} & \frac{1}{n!} \sum_{i_1=1}^{n+1} \sum_{i_2=1}^{n+1} \dots \sum_{i_n=1}^{n+1} \left| \frac{\partial(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n})}{\partial(x_1, x_2, \dots, x_n)} \right|^2 \\ &= \sum_{\substack{i_1, i_2, \dots, i_n=1 \\ i_1 \leq i_2 \leq \dots \leq i_n}}^n \left( \frac{\partial(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n})}{\partial(x_1, x_2, \dots, x_n)} \right)^2 \\ &= \sum_{\substack{i_1, i_2, \dots, i_n=1 \\ i_1 < i_2 < \dots < i_n}}^n \left( \frac{\partial(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n})}{\partial(x_1, x_2, \dots, x_n)} \right)^2 + \underbrace{\sum_{\substack{i_1, i_2, \dots, i_n=1 \\ i_1 \leq i_2 \leq \dots \leq i_n \\ \exists j_1, j_2: i_{j_1} = i_{j_2}}}^n \left( \frac{\partial(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n})}{\partial(x_1, x_2, \dots, x_n)} \right)^2}_{=0} \\ &= \sum_{j=1}^n \left| \frac{\partial(\phi_1, \phi_2, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_{n+1})}{\partial(x_1, x_2, \dots, x_n)} \right|^2 + \left| \frac{\partial(\phi_1, \phi_2, \dots, \phi_n)}{\partial(x_1, x_2, \dots, x_n)} \right|^2 \\ &= \sum_{j=1}^n \left| \frac{\partial(\phi_1, \phi_2, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_{n+1})}{\partial(x_1, x_2, \dots, x_n)} \right|^2 + 1 \end{aligned}$$

d.

$$\begin{aligned}
|N(x_1, x_2, \dots, x_n)| &= \left( \sum_{j=1}^n \left| \frac{\partial(\phi_1, \phi_2, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_{n+1})}{\partial(x_1, x_2, \dots, x_n)} \right|^2 + 1 \right)^{1/2} \\
&= \left( \sum_{j=1}^n \left| \frac{\partial \phi_{n+1}}{x_j} \right|^2 + 1 \right)^{1/2} \\
&= \left( 1 + \sum_{j=1}^n \left| \frac{-x_j}{\sqrt{r^2 - x_1^2 - x_2^2 - \dots - x_n^2}} \right|^2 \right)^{1/2} \\
&= \left( 1 + \sum_{j=1}^n \frac{x_j^2}{r^2 - \sum_{i=1}^n x_i^2} \right)^{1/2} \\
&= \left( 1 + \frac{\sum_{j=1}^n x_j^2}{r^2 - \sum_{i=1}^n x_i^2} \right)^{1/2} \\
&= \left( \frac{r^2}{r^2 - \sum_{i=1}^n x_i^2} \right)^{1/2} \\
&= \frac{r}{\sqrt{r^2 - \sum_{i=1}^n x_i^2}}
\end{aligned}$$

e.

We have that

$$\begin{aligned}
&\frac{S_{n-2}(\sqrt{1-x_1^2-x_2^2})}{\sqrt{1-x_1^2-x_2^2}} \\
&= 2^{n-2} \int_0^{\sqrt{1-x_1^2-x_2^2}} \int_0^{\sqrt{1-x_1^2-x_2^2-x_3^2}} \dots \int_0^{\sqrt{1-x_1^2-x_2^2-\dots-x_{n-1}^2}} \frac{|N(x_3, x_4, \dots, x_n)|}{\sqrt{1-x_1^2-x_2^2}} dx_n \dots dx_4 dx_3 \\
&= 2^{n-2} \int_0^{\sqrt{1-x_1^2-x_2^2}} \int_0^{\sqrt{1-x_1^2-x_2^2-x_3^2}} \dots \int_0^{\sqrt{1-x_1^2-x_2^2-\dots-x_{n-1}^2}} \frac{1}{\sqrt{1-x_1^2-x_2^2-\dots-x_n^2}} dx_n \dots dx_4 dx_3
\end{aligned}$$

Hence

$$\begin{aligned}
&4 \int_0^1 \int_0^{\sqrt{1-x_1^2}} \frac{S_{n-2}(\sqrt{1-x_1^2-x_2^2})}{\sqrt{1-x_1^2-x_2^2}} \\
&= 4 \int_0^1 \int_0^{\sqrt{1-x_1^2}} \dots \int_0^{\sqrt{1-x_1^2-x_2^2-\dots-x_n^2}} 2^{n-2} \frac{1}{\sqrt{1-\sum_{j=1}^n x_j^2}} dx_n \dots dx_2 dx_1 \\
&= 2^n \int_0^1 \int_0^{\sqrt{1-x_1^2}} \dots \int_0^{\sqrt{1-x_1^2-x_2^2-\dots-x_n^2}} |N(x_1, x_2, \dots, x_n)| dx_n \dots dx_2 dx_1 \\
&= S_n(1)
\end{aligned}$$

f.

Consider the spherical coordinates, let

$$\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad (r, \theta, \sigma) \rightarrow (r \cos \theta \cos \sigma, r \cos \theta \sin \sigma, r \sin \theta)$$

and  $K = [0, 1] \times [0, \pi/2] \times [0, \pi/2]$  so that

$$\phi(K) = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, x, y, z > 0\}$$

$$\begin{aligned} S_n(1) &= 4 \int_0^1 \int_0^{\sqrt{1-x_1^2}} \frac{S_{n-2}(\sqrt{1-x_1^2-x_2^2})}{\sqrt{1-x_1^2-x_2^2}} dx_2 dx_1 \\ &= 4 \int_0^1 \int_0^{\sqrt{1-x_1^2}} \frac{S_{n-2}(1) \cdot \left(\sqrt{1-x_1^2-x_2^2}\right)^{n-2}}{\sqrt{1-x_1^2-x_2^2}} dx_2 dx_1 \\ &= 4 \int_0^1 \int_0^{\sqrt{1-x_1^2}} S_{n-2}(1) \cdot \left(\sqrt{1-x_1^2-x_2^2}\right)^{n-3} dx_2 dx_1 \\ &= 4S_{n-2}(1) \int_0^1 \int_0^{\sqrt{1-x_1^2}} \int_0^{\sqrt{1-x_1^2-x_2^2}} (n-3) \cdot x_3^{n-4} dx_3 dx_2 dx_1 \\ &= 4(n-3)S_{n-2}(1) \int_0^1 \int_0^{\sqrt{1-x_1^2}} \int_0^{\sqrt{1-x_1^2-x_2^2}} x_3^{n-4} dx_3 dx_2 dx_1 \\ &= 4(n-3)S_{n-2}(1) \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} (r \sin \theta)^{n-4} \cdot r^2 \cos \theta d\sigma d\theta dr \\ &= 4\frac{\pi}{2}(n-3)S_{n-2}(1) \int_0^1 r^{n-2} \int_0^{\pi/2} (\sin \theta)^{n-4} \cdot \cos \theta d\theta dr \\ &= 4\frac{\pi}{2}(n-3)S_{n-2}(1) \int_0^1 r^{n-2} \left( \frac{(\sin \theta)^{n-3}}{n-3} \right) \Big|_{\theta=0}^{\pi/2} dr \\ &= 2\pi S_{n-2}(1) \frac{n-3}{n-3} \int_0^1 r^{n-2} dr \\ &= 2\pi S_{n-2}(1) \frac{r^{n-1}}{n-1} \Big|_{r=0}^1 \\ &= 2\pi S_{n-2}(1) \frac{1}{n-1} \end{aligned}$$

First we prove the case for odd  $n = 2m - 1$  for all natural  $m$ . We have the base case:

$$S_1(1) = \frac{2\pi^1}{(1-1)!} = 2\pi$$

which is the same as the answers we have in part a and the inductive steps:

Given that  $S_{2m-1}(1) = \frac{2\pi^m}{(m-1)!}$ , we have that

$$\begin{aligned} S_{2(m+1)-1}(1) &= S_{2m+1}(1) = 2\pi S_{2m-1}(1) \cdot \frac{1}{2m} \\ &= \pi \cdot \frac{1}{m} \cdot \frac{2\pi^m}{(m-1)!} \\ &= \frac{2\pi^{m+1}}{m!} \end{aligned}$$

For even  $n = 2m$ , we have the base case:

$$S_2(1) = \frac{(4\pi)^1 \cdot (1-1)!}{(2-1)!} = 4\pi$$

which is the same as the answers we have in part a and the inductive steps:

Given that  $S_{2m}(1) = \frac{(4\pi)^m \cdot (m-1)!}{(2m-1)!}$ , we have that

$$\begin{aligned} S_{2(m+1)}(1) &= S_{2m+2}(1) = 2\pi S_{2m}(1) \cdot \frac{1}{2m+1} \\ &= 2\pi \cdot \frac{1}{2m+1} \cdot \frac{(4\pi)^m \cdot (m-1)!}{(2m-1)!} \cdot \frac{2 \cdot m}{2m} \\ &= 4\pi \cdot \frac{(4\pi)^m \cdot (m-1)! \cdot m}{2m(2m+1)(2m-1)!} \\ &= \frac{(4\pi)^{m+1} m!}{(2m+1)!} \end{aligned}$$

**g.**

From part f, we know that

$$\lim_{m \rightarrow \infty} S_{2m-1}(1) = 0$$

as

$$\lim_{m \rightarrow \infty} \frac{S_{2m-1}(1)}{S_{2m+1}(1)} = \lim_{m \rightarrow \infty} \frac{\pi}{m} = 0$$

and

$$\lim_{m \rightarrow \infty} S_{2m}(1) = 0$$

as

$$\lim_{m \rightarrow \infty} \frac{S_{2m+2}(1)}{S_{2m}(1)} = \lim_{m \rightarrow \infty} \frac{2\pi}{2m+1} = 0$$

Hence,

$$\lim_{n \rightarrow \infty} S_n(1) = 0$$

4.