1

Without loss of generality, we can assume $\dim(\text{Image}(g)) = 1$. Assume that g is only defined in the set $[-N, N] \times [-M, M]$ where N, M are arbitary. We have that

$$\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} g(x^*,y^*)\Delta F_{i,j}$$

where with $x_i = \frac{-(n-i)N+iN}{n} = \frac{2iN-nN}{n}$, $y_i = \frac{-(m-i)M+iM}{m} = \frac{2iM-mM}{m}$, we have

$$\begin{aligned}\Delta F_{i,j} &= F(x_i,y_j) - F(x_{i-1},y_j) - F(x_i,y_{j-1}) + F(x_{i-1},y_{j-1}) \\ &= \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(x,y)dxdy\end{aligned}$$

Hence,

$$\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} g(x^*,y^*)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(x,y)dxdy$$

$$\begin{aligned}E[g(X,Y)] &= \int_{[-N,N]\times[-M,M]} g(x,y)f(x,y)dxdy \\ &= \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} g(x,y)f(x,y)dxdy\end{aligned}$$

We know that g is uniformly continuous as it is continuous in a compact set, we have that with large enough n,m

$$|g(x,y) - g(x^*,y^*)| < \epsilon$$

Hence,

$$\begin{aligned}&\left|\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) - E[g(X,Y)]\right| \\ =&\left|\lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} (g(x,y) - g(x^*,y^*))f(x,y)dxdy\right| \\ <&\left|\lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} \epsilon f(x,y)dxdy\right| \\ =&\epsilon\left|\lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(x,y)dxdy\right| \\ =&\epsilon\end{aligned}$$

Since, ϵ is arbitary, $\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = E[g(X,Y)]$, and since N, M are arbitary,

$$\int_{\mathbb{R}^2} g(x,y)dF(x,y) = E[g(X,Y)]$$

2

Because $s_i > a_i$ for all i, $X(0)$ does not affect stationary distributions.

$$\begin{aligned}
\pi(0) &= \left(1 + \frac{a_0}{s_1} + \frac{a_0 a_1}{s_1 s_2} + \dots\right)^{-1} \\
&= \left(1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 4 \cdot 5} + \dots\right)^{-1} \\
&= \left(1 + \sum_{i=1}^{\infty} \frac{2 \cdot i!}{(i+2)!}\right)^{-1} \\
&= \left(1 + \sum_{i=1}^{\infty} \frac{2}{(i+1)(i+2)}\right)^{-1} \\
&= \left(1 + 2\sum_{i=1}^{\infty} \left(\frac{1}{i+1} - \frac{1}{i+2}\right)\right)^{-1} \\
&= \left(1 + 2\left(\frac{1}{2}\right)\right)^{-1} \\
&= \frac{1}{2}
\end{aligned}$$

Hence, we can calculate

$$\pi(i) = \pi(0)\frac{a_0 a_1 \dots a_{i-1}}{s_1 s_2 \dots s_i} = \frac{1}{2} \cdot \frac{2 \cdot i!}{(i+2)!} = \frac{1}{(i+1)(i+2)}$$

3

3.1

$$f^{-1}(\varnothing) = \{x \in S : f(x) \in \varnothing\} = \varnothing$$

3.2

$$f^{-1}(B^C) = \{x \in S : f(x) \notin B\} = S \backslash \{x \in S : f(x) \in B\} = S \backslash (f^{-1}(B)) = f^{-1}(B)^C$$

3.3

$$\begin{aligned} f^{-1}\left(\bigcap_\beta B_\beta\right) &= \{x \in S : f(x) \in \bigcap_\beta B_\beta\} \\ &= \bigcap_\beta \{x \in S : f(x) \in B_\beta\} \\ &= \bigcap_\beta f^{-1}(B_\beta) \end{aligned}$$

3.4

$$\begin{aligned} f^{-1}\left(\bigcup_\beta B_\beta\right) &= \{x \in S : f(x) \in \bigcup_\beta B_\beta\} \\ &= \bigcup_\beta \{x \in S : f(x) \in B_\beta\} \\ &= \bigcup_\beta f^{-1}(B_\beta) \end{aligned}$$

4

Let $\{Y_i\}_{i=1}^N$ be geometric distributions, then $\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)$ is a filtration. We will use the geometric distributions to estimate $\sin(X \ln(X))$, where X is a $(2, 1/2)$-negative binomial.

$$\alpha_i = \frac{p(Y_i)}{q(Y_i)} = \frac{\binom{Y_i - 1}{1} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{Y_i - 2}}{\frac{1}{2} \left(\frac{1}{2}\right)^{Y_i - 1}} = Y_i - 1$$

and with

$$L_n = \prod_{i=1}^{n} \alpha_i = \prod_{i=1}^{n} (Y_i - 1)$$

We have

$$\begin{aligned}
&E[L_n g(Y_1, Y_2, \ldots, Y_n)] \\
=&E[E[L_n|\mathcal{F}_n] g(Y_1, Y_2, \ldots, Y_n)] \\
=&E\left[\prod_{i=1}^{n} \frac{p(Y_i)}{q(Y_i)} g(Y_1, Y_2, \ldots, Y_n)\right] \\
=&\sum_{j_1, j_2, \ldots, j_n = 1}^{\infty} \prod_{i=1}^{n} \frac{p(j_i)}{q(j_i)} g(j_1, j_2, \ldots, j_n) q(j_1) q(j_2) \ldots q(j_n) \\
=&\sum_{j_1, j_2, \ldots, j_n = 1}^{\infty} g(j_1, j_2, \ldots, j_n) p(j_1) p(j_2) \ldots p(j_n) \\
=&E[g(X_1, X_2, \ldots, X_n)]
\end{aligned}$$

where X_i is a $(2, 1/2)$-negative binomial. Therefore, to estimate $E[\sin(X \ln(X))]$, calculate

$$g(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} (Y_i - 1) g(Y_1, Y_2, \ldots, Y_n)$$

where $g(X_1, X_2, \ldots, X_n) = \frac{1}{n} \sum_{m=1}^{n} \sin(X_m \ln(X_m))$ because as $n \to \infty$, we have

$$g(X_1, X_2, \ldots, X_n) = E[g(X_1, X_2, \ldots, X_n)]$$

and

$$g(Y_1, Y_2, \ldots, Y_n) = E[g(Y_1, Y_2, \ldots, Y_n)]$$