MATH 217 (Fall 2022)

Honors Advanced Calculus, I

ASssignment #2

1. Which of the following sets are convex:

   (i) $\{(x,y) \in \mathbb{R}^2 : x > y\}$;

   (ii) $\{x \in \mathbb{R}^N : \|x\| > 2\}$;

   (iii) $\mathbb{R} \setminus \mathbb{Q}$;

   (iv) $\{(x,y,z) \in \mathbb{R}^3 : x + y + z \geq 2022\}$?

   Justify your answers.

2. Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^N$. Show that $\bigcap_{C \in \mathcal{C}} C$ is again convex. Is $\bigcup_{C \in \mathcal{C}} C$ necessarily convex?

3. Show that $\mathbb{Z}$ is closed in $\mathbb{R}$, but not open, and that $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed.

4. Let $\varnothing \neq S \subset \mathbb{R}^N$ be arbitrary, and let $\varnothing \neq U \subset \mathbb{R}^N$ be open. Show that

   $$S + U := \{x + y : x \in S,\ y \in U\}$$

   is open.

5. Let $S \subset \mathbb{R}^N$. Show that $x \in \mathbb{R}^N$ is a cluster point of S if and only if each neighbourhood of x contains an infinite number of points in S.

6* For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, set

   $$\|x\|_1 := |x_1| + \cdots + |x_N| \qquad \text{and} \qquad \|x\|_\infty := \max\{|x_1|, \ldots, |x_N|\}.$$

   (a) Show that the following are true for $j = 1, \infty$, $x, y \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$:

      (i) $\|x\|_j \geq 0$ and $\|x\|_j = 0$ if and only if $x = 0$;

      (ii) $\|\lambda x\|_j = |\lambda| \|x\|_j$;

      (iii) $\|x + y\|_j \leq \|x\|_j + \|y\|_j$.

   (b) For $N = 2$, sketch the sets of those x for which $\|x\|_1 \leq 1$, $\|x\| \leq 1$, and $\|x\|_\infty \leq 1$.

   (c) Show that

   $$\|x\|_1 \leq \sqrt{N} \|x\| \leq N \, \|x\|_\infty$$

   for all $x \in \mathbb{R}^N$.

Due Monday, September 26, 2020, at 10:00 a.m.; no late assignments.