1

Suppose $\exp(f(z))$ has a pole at 0 , then for all $M > 0$, there is some $\epsilon > 0$ such that $|\exp(f(z))| = \exp(Re(f(z))) > M$ for all $|z| < \epsilon$. But that means that 0 is not essential and hence is a pole. But if 0 is a pole at 0 for $\exp(f(z))$, 0 is removeable for $\exp(-f(z))$ and hence 0 is removable for $-f$ and hence f. If 0 is a removeable singularity for f then it is also a removable for $\exp(f)$.

2.

If $f(z)$ has an essential singularity at 0, then there is a closed neighborhood B_ϵ around 0 such that $f(B_\epsilon)$ is dense in C but then there is $z \in D$ such that $f(z) > 1$ which is a contradiction as f is 1-to-1

3.

If all of f_k are polynomials, then obviously $f_1 \circ f_2 \circ \dots f_n$ is a polynomial. Consider a non-constant polynomial $f \circ g$, then $(f \circ g)(1/z)$ does not have essential singularity at 0. Thus there is a $U = \{z \in \mathbb{C} : |z| > r\}$ such that $(f \circ g)(U)$ is not dense in $\mathbb{C}$. Hence, there is a disk $B_\epsilon(z_0) \not\subset (f \circ g)(U)$
If $g(\mathbb{C})$ is dense in $\mathbb{C}$, then $B_\epsilon(z_0) \not\subset f(\mathbb{C})$ which means that $\frac{1}{f(z)-z_0}$ is bounded and entire thus f and consequently $f \circ g$ are constant. Therefore, $g(\mathbb{C})$ is not dense in $\mathbb{C}$ and hence $g(1/z)$ does not have a essential singularity at 0 and hence g is a non-constant polynomial. Thus, there is $g(U) = \mathbb{C} \backslash B_{r'}(0)$ and since $(f \circ g)(U)$ is not dense $f(1/z)$ has no essential singularity at 0 and thus is also a non-constant polynomial.

4.

Consider the family of closed smooth curve

$$\gamma_n : [0, 2\pi] \to \mathbb{C}, \qquad t \to cos(t)/n + 1 - 1/n + isin(t)/n$$

which is $D_n := \partial B_{1/n}(1 - 1/n)$ when materialized. Then we can see that $D_1 \supset D_2 \supset \dots$.
Note that $\gamma := \gamma_1 \oplus \gamma_2 \oplus \dots \gamma_{2022}$ is a a piecewise smooth curve.
Then, as for all $z \in D_n$, if $i \leq n$

$$\frac{1}{2\pi i} \int_{\gamma_i} \frac{1}{\zeta - z} d\zeta = 1$$

and 0 if $i > n$. We have that for all $z \in D_n$

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} \frac{1}{\zeta - z} d\zeta &= \frac{1}{2\pi i} \int_{\gamma_1 \oplus \gamma_2 \oplus \dots \oplus \gamma_{2022}} \frac{1}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \sum_{i=1}^{2022} \int_{\gamma_i} \frac{1}{\zeta - z} d\zeta \\ &= n \end{aligned}$$

If $z \notin B_1(0)$ then $\nu(\gamma, z) = 0$.

5.

a.

We have that $q(z) = z^2 + 2z + 2 = 0 \iff z = -1 \pm i$, and

$$\operatorname{res}\left(\frac{e^{iz}}{z^2+2z+2}, -1+i\right) = \lim_{z\to -1+i} \frac{e^{iz}}{z+1+i} = \frac{e^{-1-i}}{2i}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+2x+2}dx &= \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+2x+2}dx\right) \\ &= \operatorname{Re}\left(2\pi i \operatorname{res}\left(\frac{e^{iz}}{q}, -1+i\right)\right) \\ &= \operatorname{Re}(\pi e^{-1-i}) \\ &= \pi e^{-1}\cos(-1)\end{aligned}$$

b.

Let

$$\gamma_1 : [0, 2\pi/2023] \to \mathbb{C}, \qquad t \to Re^{it}$$

and

$$\gamma_2 : [0, R] \to \mathbb{C}, \qquad t \to te^{i2\pi/2023}$$

We have simple pole at $z_0 = e^{i2\pi/2023}$. Hence,

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz + \int_0^{\infty} \frac{x}{x^{2023}+1}dx = 2\pi i \operatorname{res}(f, z_0) = -\frac{2\pi i}{2023}e^{2i\pi/2023}$$

We have that

$$\left|\int_{\gamma_1} f(z)dz\right| \le \frac{R}{R^{2023}+1}\cdot\frac{2\pi}{2023} \to 0$$

as $R \to 0$, and by letting $z = xe^{i2\pi/2023}$

$$\int_{\gamma_2} f(z)dz = e^{4\pi i/2023} I$$

Then

$$I - e^{4\pi i/2023} I = -\frac{2\pi i}{2023}e^{2i\pi/2023}$$

Hence,

$$I = \frac{2\pi i}{2023(e^{2\pi i/2023} + e^{-2\pi i/2023})} = \frac{\pi}{2023\sin(2\pi/2023)}$$