1.

Since $u(t,x,y) = T(t)f(x,y)$, then

$$T'(t)f(x,y) = kT(t)(f_{xx}(x,y) + f_{yy}(x,y))$$

thus

$$\frac{T'(t)}{T(t)} = k\frac{f_{xx}(x,y) + f_{yy}(x,y)}{f(x,y)} = -\lambda$$

We have that

$$T'(t) + \lambda T(t) = 0 \implies T_n(t) = Ce^{-\lambda t}$$

and then apply $f(x,y) = X(x)Y(y)$, we get the 2 dimensional eigenvalue problem:

$$\begin{cases} f_{xx}(x,y) + f_{yy}(x,y) = -\frac{\lambda}{k}f(x,y) \\ u(x,0,t) = X(x)Y(0)T(t) = 0 \implies Y(0) = 0 \\ u(x,H,t) = X(x)Y(H)T(t) = 0 \implies Y(H) = 0 \\ u_x(0,y,t) = X'(0)Y)T(t) = 0 \implies X'(0) = 0 \\ u_x(L,y,t) = X'(L)Y(y)T(t) = 0 \implies X'(L) = 0 \end{cases}$$

We have

$$X''(x)Y(y) + X(x)Y''(y) = -\frac{\lambda}{k}X(x)Y(y)$$

thus

$$\frac{Y''(y)}{Y(y)} = -\frac{\frac{\lambda_1}{k}X(x) + X''(x)}{X(x)} = -\mu$$

Therefore,

$$\mu = \left(\frac{n\pi}{H}\right)^2$$

for $n = 1, 2, \ldots$ and the eigenfunctions are

$$Y_n(y) = \sin\frac{n\pi y}{H}$$

Now, we solve for

$$-\frac{\lambda}{k}X(x) + X''(x) = -\left(\frac{n\pi}{H}\right)^2 X(x)$$

which can be rewritten as

$$X''(x) = -X(x)\left(\mu - \frac{\lambda}{k}\right)$$

Thus the eigenvalue for the equations with is for $m = 0, 1, 2, \ldots$

$$\mu - \frac{\lambda}{k} = \left(\frac{m\pi}{L}\right)^2 \implies \lambda = k\left(\left(\frac{n\pi}{H}\right)^2 - \left(\frac{m\pi}{L}\right)^2\right)$$

and

$$X_m(x) = \cos\left(\frac{m\pi x}{L}\right)$$

Therefore, we have that

$$f_{m,n}(x, y) = \sin\frac{n\pi y}{H}\cos\frac{m\pi x}{L}$$

and

$$u(x, y, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}e^{-\lambda_{mn}t}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)$$

where

$$\lambda_{mn} = -k\left(\left(\frac{n\pi}{H}\right)^2 - \left(\frac{m\pi}{L}\right)^2\right)$$

We have that

$$u(x, y, 0) = \sum_{m=0}^{\infty}\left(\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi y}{H}\right)\right)\cos\left(\frac{m\pi x}{L}\right) = \alpha(x, y)$$

Thus for $m > 0$,

$$\sum_{n=1}^{\infty} A_{mn}\sin\frac{n\pi y}{H} = \frac{2}{L}\int_0^L \alpha(x, y)\cos\frac{n\pi x}{L}dx$$

and

$$\sum_{n=1}^{\infty} A_{0n}\sin\frac{n\pi y}{H} = \frac{1}{L}\int_0^L \alpha(x, y)dx$$

Therefore, for $m > 0$,

$$\begin{aligned} A_{mn} &= \frac{2}{H}\int_0^H \sin\frac{n\pi y}{H}\cdot\left(\frac{2}{L}\int_0^L \alpha(x, y)\cos\frac{n\pi x}{L}dx\right)dy \\ &= \frac{4}{HL}\int_0^H \sin\frac{n\pi y}{H}\cdot\left(\int_0^L \alpha(x, y)\cos\frac{n\pi x}{L}dx\right)dy \end{aligned}$$

and similarly,

$$A_{0n} = \frac{2}{HL}\int_0^H \sin\frac{n\pi y}{H}\cdot\left(\int_0^L \alpha(x, y)dx\right)dy$$

2.

Separation of variables $u(x, y, z) = f(x, y)Z(z)$, we have

$$f_{xx}(x, y)Z(z) + f_{yy}(x, y)Z(z) + f(x, y)Z''(z) = 0$$

thus

$$\frac{Z''(z)}{Z(z)} = -\frac{f_{xx}(x, y) + f_{yy}(x, y)}{f(x, y)} = -\lambda$$

and thus we have the system of equations

$$\begin{cases} Z''(z) = -\lambda Z(z) \\ u_z(x, y, 0) = X(x)Y(y)Z'(0) = 0 \implies Z'(0) = 0 \end{cases}$$

Applying $f(x, y) = X(x)Y(y)$, the other equation is

$$X''(x)Y(y) + Y''(y)X(x) = \lambda X(x)Y(y) \implies \frac{X''(x)}{X(x)} = \lambda - \frac{Y''(y)}{Y(y)} = -\mu$$

Thus we have

$$\begin{cases} X''(x) = -\mu X(x) \\ u(x, y, z) = 0 \text{ for } (x, y) \in \partial\Gamma \implies X(0) = X(L) = 0 \end{cases}$$

and

$$\begin{cases} Y''(x) = -(-\lambda - \mu)X(x) \\ u(x, y, z) = 0 \text{ for } (x, y) \in \partial\Gamma \implies X(0) = X(L) = 0 \end{cases}$$

hence

$$\mu = \left(\frac{n\pi}{L}\right)^2$$

for $n = 1, 2, \ldots$ and

$$-\lambda - \mu = \left(\frac{m\pi}{H}\right)^2$$

for $m = 1, 2, \ldots$, thus

$$\lambda_{mn} = -\left(\frac{n\pi}{L}\right)^2 - \left(\frac{m\pi}{H}\right)^2 < 0$$

Therefore,

$$Z(z) = c_1 \exp(\sqrt{-\lambda}z) + c_2 \exp(-\sqrt{-\lambda}z)$$

and

$$Z'(z) = \sqrt{-\lambda}(c_1 \exp(\sqrt{-\lambda}z) - c_2 \exp(-\sqrt{-\lambda}z))$$

and plugging in $Z'(0) = 0$, we have $c_1 = c_2$ thus

$$Z(z) = c_2 \cosh(\sqrt{-\lambda}z)$$

Therefore,

$$u(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \cosh\left(z\sqrt{-\lambda_{mn}}\right)$$

where

$$\lambda_{mn} = -\left(\frac{n\pi}{L}\right)^2 - \left(\frac{m\pi}{H}\right)^2$$

Applying $u(x, y, H) = \alpha(x, y)$, then using orthogonality we have

$$\begin{aligned}
&\int_0^L \int_0^H \alpha(x,y) \sin\frac{n\pi x}{L} \sin\frac{n\pi y}{H} dxdy \\
=&A_{mn}\cosh(H\sqrt{-\lambda_{mn}}) \int_0^L \int_0^H \sin^2\frac{n\pi x}{L} \sin^2\frac{m\pi y}{H} dxdy \\
=&A_{mn}\cosh(H\sqrt{-\lambda_{mn}})\frac{LH}{4}
\end{aligned}$$

Therefore,

$$A_{mn} = \frac{4}{LH\cosh(H\sqrt{-\lambda_{mn}})} \int_0^L \int_0^H \alpha(x,y) \sin\frac{n\pi x}{L} \sin\frac{n\pi y}{H} dxdy$$

3.

Apply separation of variables $u(r,\theta,t) = f(r,\theta)T(t)$, we have

$$T''(t)f(r,\theta) = c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rf_r(r,\theta)T(t)\right) + \frac{1}{r^2}f_{\theta\theta}(r,\theta)T(t)\right)$$

Thus

$$\frac{T''(t)}{c^2T(t)} = \frac{\frac{1}{r}\frac{\partial}{\partial r}(rf_r(r,\theta)) + \frac{1}{r^2}f_{\theta\theta}(r,\theta)}{f(r,\theta)} = -\lambda$$

Now, apply separation of variables $f(r,\theta) = R(r)\phi(\theta)$, we have

$$\frac{1}{r}\frac{\partial}{\partial r}(rR'(r)\phi(\theta)) + \frac{1}{r^2}R(r)\phi''(\theta) = -\lambda R(r)\phi(\theta)$$

and thus

$$\frac{1}{r^2}R(r)\phi''(\theta) = -\phi(\theta)\left(\lambda R(r) + \frac{1}{r}(rR''(r) + R'(r))\right)$$

and

$$\frac{\phi''(\theta)}{\phi(\theta)} = -\frac{r^2\lambda R(r) + r^2R''(r) + rR'(r))}{R(r)} = -\mu$$

Now since

$$u(r,-\pi,t) = u(r,\pi,t) \implies \phi(-\pi) = \phi(\pi)$$

and

$$u_\theta(r,-\pi,t) = u_\theta(r,\pi,t) \implies \phi'(-\pi) = \phi'(\pi)$$

We have for $n = 0, 1, 2, \ldots$,

$$\mu_n = n^2$$

and for $n = 1, 2, \ldots$

$$\phi_n(\theta) = \sin(n\theta), \cos(n\theta)$$

with

$$\phi_0(\theta) = 1$$

Now to solve for R, we look at the equation

$$r^2R''(r) + rR'(r) + \left(\lambda r^2 - \mu\right)R(r) = 0$$

Let $z = \sqrt{\lambda}r$, we have

$$z^2\frac{\partial^2 R}{\partial z^2} + z\frac{\partial R}{\partial z} + \left(z^2 - n^2\right)R(z) = 0$$

which is a bessel DE of order n. Thus

$$R(z) = c_1J_n(z) + c_2Y_n(z)$$

But since $R(0)$ is bounded,

$$R(r) = c_1 J_n(\sqrt{\lambda} r)$$

But $R(a) = 0$ so $\sqrt{\lambda} a$ is zeros of the J_n function. Thus let z_{mn} be the m-th zeros, we get

$$\lambda_{mn} = \frac{z_{mn}^2}{a^2} > 0$$

Since $\lambda_{mn} > 0$, we can find

$$T(t) = cos(c\sqrt{\lambda} t), \sin(c\sqrt{\lambda} t)$$

Thus

$$\begin{aligned} u(r,\theta,t) = &\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}} r)\cos(n\theta)\cos(c\sqrt{\lambda_{mn}} t) \\ &+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}} r)\cos(n\theta)\sin(c\sqrt{\lambda_{mn}} t) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}} r)\sin(n\theta)\cos(c\sqrt{\lambda_{mn}} t) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}} r)\sin(n\theta)\sin(c\sqrt{\lambda_{mn}} t) \end{aligned}$$

4.

Apply separation of variables $u(r,\theta,t) = f(r,\theta)T(t)$, we have

$$T''(t)f(r,\theta) = c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rf_r(r,\theta)T(t)\right) + \frac{1}{r^2}f_{\theta\theta}(r,\theta)T(t)\right)$$

Thus

$$\frac{T''(t)}{c^2T(t)} = \frac{\frac{1}{r}\frac{\partial}{\partial r}(rf_r(r,\theta)) + \frac{1}{r^2}f_{\theta\theta}(r,\theta)}{f(r,\theta)} = -\lambda$$

Now, apply separation of variables $f(r,\theta) = R(r)\phi(\theta)$, we have

$$\frac{1}{r}\frac{\partial}{\partial r}(rR'(r)\phi(\theta)) + \frac{1}{r^2}R(r)\phi''(\theta) = -\lambda R(r)\phi(\theta)$$

and thus

$$\frac{1}{r^2}R(r)\phi''(\theta) = -\phi(\theta)\left(\lambda R(r) + \frac{1}{r}(rR''(r) + R'(r))\right)$$

and

$$\frac{\phi''(\theta)}{\phi(\theta)} = -\frac{r^2\lambda R(r) + r^2R''(r) + rR'(r))}{R(r)} = -\mu$$

Now since

$$u(r,-\pi,t) = u(r,\pi,t) \implies \phi(-\pi) = \phi(\pi)$$

and

$$u_\theta(r,-\pi,t) = u_\theta(r,\pi,t) \implies \phi'(-\pi) = \phi'(\pi)$$

We have for $n = 0,1,2,\ldots$,

$$\mu_n = n^2$$

and for $n = 1,2,\ldots$

$$\phi_n(\theta) = \sin(n\theta), \cos(n\theta)$$

with

$$\phi_0(\theta) = 1$$

Now to solve for R, we look at the equation

$$r^2R''(r) + rR'(r) + \left(\lambda r^2 - \mu\right)R(r) = 0$$

Let $z = \sqrt{\lambda}r$, we have

$$z^2\frac{\partial^2 R}{\partial z^2} + z\frac{\partial R}{\partial z} + \left(z^2 - n^2\right)R(z) = 0$$

which is a bessel DE of order n. Thus

$$R(z) = c_1J_n(z) + c_2Y_n(z)$$

But since $R(0)$ is bounded,

$$R(r) = c_1 J_n(\sqrt{\lambda} r)$$

But $R'(a) = 0$ so $\sqrt{\lambda} a$ is the extrema of the J_n function. Thus let z_{mn} be the m-th extrema, we get

$$\lambda_{mn} = \frac{z_{mn}^2}{a^2} > 0$$

Since $\lambda_{mn} > 0$, we can find

$$T(t) = cos(c\sqrt{\lambda}t), \sin(c\sqrt{\lambda}t)$$

Thus

$$\begin{aligned}
u(r,\theta,t) = &\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}} r)\cos(n\theta)\cos(c\sqrt{\lambda_{mn}} t) \\
&+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}} r)\cos(n\theta)\sin(c\sqrt{\lambda_{mn}} t) \\
&+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}} r)\sin(n\theta)\cos(c\sqrt{\lambda_{mn}} t) \\
&+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}} r)\sin(n\theta)\sin(c\sqrt{\lambda_{mn}} t)
\end{aligned}$$

5.

If equilibrium exists then

$$ku_{xx} = -Q(x)$$

Thus if $Q = 0$

$$u_E = c_1 x + c_2$$

and

$$u_E' = c_1$$

a.

Substituting the boundary conditions, we have

$$u_E = Bx + A$$

Now let

$$u(x,t) = v(x,t) + u_E(x)$$

Thus as u_E is independent of t and $u_E'' = 0$

$$v_t(x,t) = kv_{xx}(x,t)$$

with boundary and initial conditions

$$\begin{cases} v(0,t) = 0 \\ v_x(L,t) = 0 \\ v(x,0) = u(x,0) - u_{eq}(x) = f(x) - Bx - A \end{cases}$$

$v(x,t) = X(x)T(t)$, we have

$$X(x)T'(t) = kX''(x)T(t) \implies \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

with boundary conditions

$$X(0) = 0 \text{ and } X'(L) = 0$$

which has the eigenfunctions with $n = 1, 3, 5, \dots$

$$X_n(x) = \sin(\sqrt{\lambda_n} x)$$

where

$$\lambda_n = \left(\frac{n\pi}{2L}\right)^2$$

and

$$T_n(t) = \exp(-k\lambda_n t)$$

Thus

$$v(x,t) = \sum_{n=1,3,\dots}^{\infty} A_n \sin(\sqrt{\lambda_n} x) \exp(-k\lambda_n t)$$

Applying the initial conditions, we have

$$v(x,0) = \sum_{n=1,3,\ldots}^{\infty} A_n \sin(\sqrt{\lambda_n}x) = f(x) - Bx - A$$

Therefore,

$$A_n = \frac{2}{L}\int_0^L (f(x) - Bx - A)\sin(\sqrt{\lambda_n}x)dx$$

and the final solution is

$$u(x,t) = v(x,t) + Bx + A$$

where $v(x,t)$ is given above.

b.

The equilibrium does not exists as $u_x(0,t) = 0 < B = u_x(L,t)$, $u_{xx}(x_0,t) \neq 0$ for some x_0 thus $u_t(x_0,t) \neq 0$. Since equilibrium does not exists and the known boundary conditions for u, we can find the function

$$r(x) = \frac{B}{2L}x^2$$

so that

$$r'(0) = 0 \text{ and } r'(L) = B$$

Then let

$$u(x,t) = v(x,t) + r(x)$$

we find that

$$\begin{cases} v_t(x,t) = u_t(x,t) = ku_{xx}(x,t) = k(v_{xx}(x,t) + \dfrac{B}{L}) \\ v_t(0,t) = 0 \\ v_t(L,t) = 0 \\ u(x,0) = v(x,0) + r(x) \implies v(x,0) = f(x) - \dfrac{B}{2L}x^2 \end{cases}$$

c.

If equilibrium exists then

$$u_E''(x) = \frac{-Q(x)}{k}$$

Thus

$$u_E(x) = \frac{L^2}{k4\pi^2}\sin\frac{2\pi x}{L} + c_1 x$$

Thus,

$$u_E'(x) = \frac{L}{2k\pi}\cos\frac{2\pi x}{L} + c_1$$

and substitute the boundary conditions in

$$u_E'(0) = u_E'(L) = c_1 + \frac{L}{2k\pi} = 0 \implies c_1 = -\frac{L}{2k\pi}$$

Thus,

$$v_t(x,t) = u_t(x,t) = ku_{xx}(x,t) + Q(x,t) = kv_{xx}(x,t) + ku_E''(x) + Q(x,t)$$

Therefore,

$$v_t(x,t) = kv_{xx}(x,t)$$

with boundary and initial conditions

$$\begin{cases} v_x(0,t) = 0 \\ v_x(L,t) = 0 \\ v(x,0) = f(x) - u_E(x) \end{cases}$$

$v(x,t) = X(x)T(t)$, then

$$X'(0) = X'(L) = 0$$

Thus for $n = 0, 1, 2, \ldots$

$$X_n(x) = \cos(\sqrt{\lambda_n} x)$$

where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and thus similar to part a

$$v(x,t) = \sum_{n=0}^{\infty} A_n \cos(\sqrt{\lambda_n} x) \exp(-k\lambda_n t)$$

and by using the initial conditions, we get

$$A_n = \frac{2}{L} \int_0^L (f(x) - u_E(x)) \cos(\sqrt{\lambda_n} x) dx$$

and the final solution is

$$u(x,t) = v(x,t) + u_E(x)$$

where $v(x,t)$ and $u_E(x)$ are given above.

6.

Let

$$u(x,t) = v(x,t) + r(x,t)$$

a.

Let

$$r(x,t) = c_1(t)x^2 + c_2(t)x$$

then

$$r_x(x,t) = 2c_1(t)x + c_2(t)$$

and thus using the boundary conditions,

$$r(x,t) = \frac{B(t) - A(t)}{2L}x^2 + A(t)x$$

We also have

$$v_t(x,t) - r_t(x,t) = kv_{xx}(x,t) - kr_{xx}(x,t) + Q(x,t)$$

Thus the equation is

$$v_t(x,t) = kv_{xx}(x,t) + Q(x,t) + k\frac{B(t) - A(t)}{L} + r_t(x,t)$$

where

$$r_t(x,t) = \frac{B'(t) - A'(t)}{2L}x^2 + A'(t)x$$

with the boundary conditions

$$v_t(0,t) = v_t(L,t) = 0$$

and initial conditions

$$v(x,0) = u(x,0) - r(x,0) = f(x) + \frac{B(0) - A(0)}{2L}x^2 - A(0)x$$

b.

Let

$$r(x,t) = c_1(t)x + c_2(t)$$

then

$$r_x(x,t) = c_1(t)$$

and thus using the boundary conditions

$$r(x,t) = A(t)x + B(t) - A(t)L$$

Therefore,

$$v_t(x,t) = kv_{xx}(x,t) + Q(x,t) + r_t(x,t)$$

with the boundary conditions

$$v_t(0,t) = v_t(L,t) = 0$$

and initial conditions

$$v(x,0) = u(x,0) - r(x,0) = f(x) - A(0)x + B(0) - A(0)L$$

c.

Let

$$r(x,t) = c_1(t)x + c_2(t)$$

then

$$r_x(0,t) = c_1(t) = 0$$

and

$$h(c_3(t) - B(t)) = 0 \implies c_2(t) = B(t)$$

Thus

$$v_t(x,t) = kv_{xx}(x,t) + Q(x,t) + r_t(x,t) = kv_{xx}(x,t) + Q(x,t) + B'(t)$$

with the boundary conditions

$$v_t(0,t) = v_t(L,t) = 0$$

and initial conditions

$$v(x,0) = u(x,0) - r(x,0) = f(x) - B(0)$$

7.

a.

Suppose there is an equilibrium

$$u_E''(x) = -\frac{Q(x)}{k} = -\frac{1}{k}$$

Then let

$$u_E(x) = -\frac{1}{2k}x^2 + c_1 x + c_2$$

Plugging in the boundary conditions $u_E(0) = A, u_E(L) = B$, we have

$$u_E(x) = -\frac{1}{2k}x^2 + \left(\frac{L}{2k} + \frac{B-A}{L}\right)x + A$$

Then let

$$v(x,t) = u(x,t) - u_E(x)$$

We have that

$$\begin{cases} v_t = kv_{xx} \\ v(0,t) = u(0,t) - u_E(0) = 0 \\ v(L,t) = u(L,t) - u_E(L) = 0 \\ v(x,0) = u(x,0) - u_E(x) = f(x) - u_E(x) \\ v_t(x,0) = u_t(x,0) = g(x) \end{cases}$$

Let $v(x,t) = X(x)T(t)$, we know the boundary conditions for X are

$$X(0) = 0 \text{ and } X(L) = 0$$

Thus the eigenfunctions are for $n = 1, 2, 3, \ldots$,

$$X_n(x) = \sin(\sqrt{\lambda_n}x)$$

where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and

$$T_n(t) = \exp(-k\lambda_n t)$$

Thus

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}x)\exp(-k\lambda_n t)$$

where we can find A_n using the initial conditions $v(x,0) = f(x) - u_E(x)$

$$A_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\frac{n\pi x}{L}dx$$

However, we have another initial conditions to satisfy

$$v_t(x,0) = -\sum_{n=1}^{\infty} A_n k\lambda_n \sin(\sqrt{\lambda_n}x) = g(x)$$

Thus

$$-A_n k\lambda_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

which means

$$A_n = -\frac{2}{Lk\lambda_n}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

Therefore, there is a solution only when the two version of A_n are equal to each other, that is

$$\int_0^L (f(x) - u_E(x))\sin\frac{n\pi x}{L}dx = -\frac{1}{k\lambda_n}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

and the solution is

$$u(x,t) = v(x,t) + u_E(x)$$

where $v(x,t)$ and $u_E(x)$ are mentioned above.

b.

Suppose that there is an equilibrium solution, that is

$$u_E''(x) = -\frac{\sin\frac{\pi x}{L}}{k}$$

Then let

$$u_E(x) = \frac{L^2}{k\pi^2}\sin\frac{\pi x}{L} + c_1 x$$

Plugging in the boundary conditions,

$$\begin{cases} u_E(0) = \dfrac{L^2}{k\pi^2}\sin(0) = 0 \\ u_E(L) = \dfrac{L^2}{k\pi^2}\sin(\pi) + c_1 L = 0 \end{cases}$$

Thus $c_1 = 0$ and we have

$$\begin{cases} v_t(x,t) = kv_{xx}(x,t) \\ v(0,t) = u(0,t) + u_E(0) = u(0,t) = 0 \\ v(L,t) = u(L,t) + u_E(L) = u(L,t) = 0 \\ v(x,0) = u(x,0) - u_E(x) = f(x) - u_E(x) \\ u_t(x,0) = v_t(x,0) = g(x) \end{cases}$$

Let $v(x,t) = X(x)T(t)$, we know the boundary conditions for X are

$$X(0) = 0 \text{ and } X(L) = 0$$

Thus the eigenfunctions are for $n = 1, 2, 3, \ldots$,

$$X_n(x) = \sin(\sqrt{\lambda_n}x)$$

where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and

$$T_n(t) = \exp(-k\lambda_n t)$$

Thus

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}x)\exp(-k\lambda_n t)$$

where we can find A_n using the initial conditions $v(x,0) = f(x) - u_E(x)$

$$A_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\frac{n\pi x}{L}dx$$

However, we have another initial conditions to satisfy

$$v_t(x,0) = -\sum_{n=1}^{\infty} A_n k\lambda_n \sin(\sqrt{\lambda_n}x) = g(x)$$

Thus

$$-A_n k\lambda_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

which means

$$A_n = -\frac{2}{Lk\lambda_n}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

Therefore, there is a solution only when the two version of A_n are equal to each other, that is

$$\int_0^L (f(x) - u_E(x))\sin\frac{n\pi x}{L}dx = -\frac{1}{k\lambda_n}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

and the solution is

$$u(x,t) = v(x,t) + u_E(x)$$

where $v(x,t)$ and $u_E(x)$ are mentioned above.

8.

a.

Let $r(x)$ satisfy the $r(0) = 0, r'(L) = 0$, thus let

$$r(x) = c_1x^2 + c_2x + c_3$$

then applying the boudary conditions, we have

$$r(x) = c_1x^2 - 2c_1Lx$$

Then let

$$v(x,t) = u(x,t) - r(x)$$

and

$$r''(x) = -\frac{Q}{k} \implies c_1 = -\frac{Q}{k}$$

and therefore,

$$\begin{cases} v_t(x,t) = kv_{xx}(x,t) \\ v(0,t) = u(0,t) - r(0) = 0 \\ v_x(L,t) = u_x(L,t) - r'(L) = 0 \\ v(x,0) = u(x,0) - r(x) = f(x) - r(x) \end{cases}$$

Thus let $v(x,t) = X(x)T(t)$, we have the boundary conditions for X

$$\begin{cases} v(0,t) = X(0)T(t) = 0 \implies X(0) = 0 \\ v_x(L,t) = X'(L)T(t) = 0 \implies X'(L) = 0 \end{cases}$$

Then we know that for $n = 1, 3, 5, \ldots$

$$X_n(x) = \sin(\sqrt{\lambda_n}x)$$

where $\lambda_n = \left(\frac{n\pi}{2L}\right)^2$ and

$$T_n(t) = \exp(-k\lambda_nt)$$

Therefore,

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} A_n \sin(\sqrt{\lambda_n}x)\exp(-k\lambda t)$$

Applying the initial conditions, we have

$$A_n = \frac{2}{L}\int_0^L (f(x) - r(x))\sin\frac{n\pi x}{L}dx$$

and the solution is

$$u(x,t) = v(x,t) + r(x)$$

where $v(x,t)$ and $r(x)$ are mentioned above.

b.

Let $r(x,t)$ satisfy the $r(0,t) = A(t), r_x(L,t) = 0$, thus let

$$r(x,t) = c_1 x^2 + c_2 x + c_3$$

then applying the boudary conditions, we have

$$r(x,t) = c_1 x^2 - 2c_1 Lx + A(t)$$

Then let

$$v(x,t) = u(x,t) - r(x,t)$$

and

$$r_{xx}(x,t) = -\frac{Q}{k} \implies c_1 = -\frac{Q}{k}$$

and therefore,

$$\begin{cases} v_t(x,t) = kv_{xx}(x,t) \\ v(0,t) = u(0,t) - r(0) = 0 \\ v_x(L,t) = u_x(L,t) - r'(L) = 0 \\ v(x,0) = u(x,0) - r(x) = f(x) - r(x) \end{cases}$$

Thus let $v(x,t) = X(x)T(t)$, we have the boundary conditions for X

$$\begin{cases} v(0,t) = X(0)T(t) = 0 \implies X(0) = 0 \\ v_x(L,t) = X'(L)T(t) = 0 \implies X'(L) = 0 \end{cases}$$

Then we know that for $n = 1, 3, 5, \ldots$

$$X_n(x) = \sin(\sqrt{\lambda_n} x)$$

where $\lambda_n = \left(\frac{n\pi}{2L}\right)^2$ and

$$T_n(t) = \exp(-k\lambda_n t)$$

Therefore,

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} A_n \sin(\sqrt{\lambda_n} x) \exp(-k\lambda t)$$

Applying the initial conditions, we have

$$A_n = \frac{2}{L} \int_0^L (f(x) - r(x)) \sin \frac{n\pi x}{L} dx$$

and the solution is

$$u(x,t) = v(x,t) + r(x)$$

where $v(x,t)$ and $r(x)$ are mentioned above.

9.

Let

$$r(x) = -\frac{x}{\pi} + 1$$

so that $r(0) = 1$ and $r(\pi) = 0$, then let

$$v(x,t) = u(x,t) - r(x)$$

so that we have

$$\begin{cases} v_t = v_{xx} + e^{-2t}\sin(5x) \\ v(0,t) = 0 \\ v(\pi,t) = 0 \\ v(x,0) = u(x,0) - r(x) = \dfrac{\pi}{x} - 1 \end{cases}$$

Let's first solve the homogenous part $v_t = v_{xx}$, let $v(x,t) = X(x)T(t)$, then we know v has the solution

$$v(x,t) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n} x) A_n(t)$$

where $\lambda_n = n^2$ for $n = 1, 2, 3, \ldots$, plugging this back in the equations,

$$\sum_{n=1}^{\infty} A_n'(t)\sin(\sqrt{\lambda_n} x) = -\sum_{n=1}^{\infty} \lambda_n \sin(\sqrt{\lambda_n} x) A_n(t) + \exp(-2t)\sin(5x)$$

Thus

$$\sum_{n=1}^{\infty} \sin(nx)\left(A_n'(t) + n^2 A_n(t)\right) = \exp(-2t)\sin(5x)$$

which means that

$$A_n'(t) + n^2 A_n(t) = \begin{cases} 0, & \text{if } n \neq 5 \\ \exp(-2t), & \text{if } n = 5 \end{cases}$$

In case $n \neq 5$,

$$A_n'(t) + n^2 A_n(t) = 0 \implies A_n(t) = ce^{-n^2 t}$$

and thus

$$c = A_n(0)$$

In case $n = 5$,

$$A_5'(t) + 25A_5(t) = \exp(-2t) \implies A_5(t) = \frac{\exp(-2t)}{23} + ce^{-25t}$$

and since $A_5(0) = \frac{1}{23} + c \implies c = A_5(0) - \frac{1}{23}$.
Now all we need to do is find $A_n(0)$, we have

$$\sum_{n=1}^{\infty} \sin(nx) A_n(0) = v(x,0) = \frac{\pi}{x} - 1$$

Thus

$$A_n(0) = \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{x} - 1 \right) \sin(nx) dx = \frac{-2}{n\pi}$$

and therefore the solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin(nx) + 1 - \frac{x}{\pi}$$

where

$$A_n(t) = \begin{cases} \frac{1}{23} e^{-2t} - \left(\frac{2}{5}\pi + \frac{1}{23} \right) e^{-25t}, & \text{if } n = 5 \\ -\frac{2}{n\pi} e^{-n^2 t}, & \text{otherwise} \end{cases}$$

10.

Let's first solve the homogenous part $u_{xx} = u_{yy}$, then we know u has the solution

$$u(x,y) = \sum_{n=1}^{\infty} A_n(y)\sin(xn)$$

Plugging back in the equation,

$$\sum_{n=1}^{\infty} -A_n(y)\sin(xn) + \sum_{n=1}^{\infty} A_n''(y)\sin(xn) = e^{2y}\sin(x)$$

Thus

$$\sum_{n=1}^{\infty} \sin(nx)(A_n''(y) - A_n(y)) = e^{2y}\sin(x)$$

Therefore,

$$A_n''(y) - A_n(y) = \begin{cases} e^{2y}\sin(x), & \text{if } n = 1 \\ 0, & \text{if } n \neq 1 \end{cases}$$

In case $n = 1$,

$$A_1''(y) - A_1(y) = e^{2y}$$

and therefore,

$$A_1(y) = C_1 e^y + C_2 e^{-y} + \frac{1}{3}e^{2y}$$

We have $A_1(0) = 0$, thus $C_1 + C_2 + \frac{1}{3} = 0$ and

$$A_1(y) = K_1 e^y - (K_1 + 1/3)e^{-y} + \frac{1}{3}e^{2y}$$

Then we can rewrite it as

$$A_1(y) = C_n \sinh(y) + \frac{1}{3}(e^{-y} + e^{2y})$$

In case $n \neq 1$,

$$A_n(y) = K_1 e^y + K_2 e^{-y}$$

Since $A_1(y) = 0$, $K_1 = -K_2$ and $A_n(y) = C_n \sinh(y)$. Then all we need to do is find C_n, we have

$$\sum_{n=1}^{\infty} \sin(nx) A_n(L) = f(x)$$

Thus for $n \neq 1$,

$$A_n(L) = C_n \sinh(L) = \frac{2}{L}\int_0^L f(x)\sin(nx)dx$$

and

$$C_n = \frac{2}{L\sinh(L)}\int_0^L f(x)\sin(nx)dx$$

and for $n = 1$,

$$A_n(L) = C_1\sinh(L) + \frac{1}{3}(e^{-L} + e^{2L}) = \frac{2}{L}\int_0^L f(x)\sin(nx)dx$$

and

$$C_1 = \frac{2}{L}\left(\frac{2}{L}\int_0^L f(x)\sin(nx)dx - \frac{e^{-L} + e^{2L}}{3}\right)$$