1.

a.

Let $y = 2\pi m + n$, where $0 \leq n < 2\pi$

$$\exp(x + iy) = \exp(x)(\cos(n) + i\sin(n))$$

Hence, $f(D) = B_{\exp(1)}(0)\backslash B_{\exp(-1)}(0)$

b.

Since $(x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3 = x(x^2 - 3y^2) + iy(3x^2 - y^2)$. Then for every $m = tn \in \mathbb{R}$, consider the system of equations

$$\begin{cases} x^3 - 3xy^2 = m \\ 3x^2y - y^3 = n \end{cases}$$

Then if $x = ky$ where $k \in \mathbb{R}_{>0}$, we have that

$$\begin{cases} k^3 - 3k = \dfrac{m}{y^3} \\ 3k^2 - 1 = \dfrac{n}{y^3} \end{cases}$$

We have that for every $a, b \in \mathbb{R}$, we can find y such that $a = tb = \dfrac{m}{y^3} = \dfrac{tn}{y^3} = \dfrac{n}{y^3}$ and hence

$$\begin{cases} k^3 - 3k - a = 0 \\ 3k^2 - 1 - b = 0 \end{cases}$$

And hence we can find a, b satisfies the system of equations above. Since m, n is arbitary. $f(D) = \mathbb{C}$

c.

Consider the equations $z^{2018} + z + 1 = c$, where $c \in \mathbb{C}$. By FTA, the equations always has roots and hence $f(D) = \mathbb{C}$.

d.

We have that

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

and

$$\left|\frac{1}{z}\right| = \frac{1}{|z|}$$

which means that $f(D) = \mathbb{C}\backslash B_{1/2}(0)$.

2.

a.

The function
$$f : \mathbb{C} \to \mathbb{C}, \qquad z \to z^2 - z + 1$$
is holomorphic. Hence, $|f|$ maximum is attained in ∂G. we know that
$$\partial G = [0,1] \times \{0,1\} \cup \{0,1\} \times [0,1]$$
In cases $z = t$, where $0 \le t \le 1$
$$|z^2 - z + 1| = t^2 - t + 1 = \left(t - \frac{1}{2}\right)^2 + \frac{3}{4}$$

In cases $z = ti$, where $0 \le t \le 1$
$$|z^2 - z + 1| = |-t^2 - ti + 1| = \sqrt{(t^2-1)^2 + t^2} = \sqrt{t^4 - t^2 + 1} = \sqrt{\left(t^2 - \frac{1}{2}\right)^2 + \frac{3}{4}}$$
In cases $z = i + t$, where $0 \le t \le 1$
$$|z^2 - z + 1| = |t^2 + 2it - 1 - i - t + 1| = |t^2 - t + i(2t-1)| = \sqrt{(t^2-t)^2 + (2t-1)^2}$$
$$= \sqrt{t^4 - 2t^3 + 5t^2 - 4t + 1}$$
Consider $4t^3 - 6t^2 + 10t - 4 = 2(2x-1)(x^2 - x + 2)$, which means that $t^4 - 2t^3 + 5t^2 - 4t + 1$ has 1 minima at $\frac{1}{2}$. In cases $z = 1 + ti$, where $0 \le t \le 1$,
$$|z^2 - z + 1| = |-t^2 - 2ti + 1 - 1 - ti + 1| = |-t^2 + 1 + i(-3t)| = \sqrt{(t^2-1)^2 + (3t)^2}$$
$$= \sqrt{t^4 + 7t^2 + 1} = \sqrt{\left(t^2 + \frac{7}{2}\right)^2 - \frac{45}{4}}$$
Hence, the maximum is attained in $0, i, 1, 1+i$.
At 0, $|z^2 - z + 1| = 1$.
At 1, $|z^2 - z + 1| = 1$.
At i, $|z^2 - z + 1| = 1$.
At $1+i$, $|z^2 - z + 1| = 1$.
Thus the maximum is 1.

b.

We have that

$$|\cos z| = \sqrt{(\cos x)^2 + (\sinh y)^2}$$

Notice that the function $\cos(x)$ monotone decreasing in $[0, 1]$ and $\sinh(y)$ monotone increasing in $[0, 1]$. We also know that the maximum is obtained in the boundary. Thus the maximum of $|\cos z|$ is $\sqrt{1 + \left(\frac{e-e^{-1}}{2}\right)^2}$ at $z = i$

3.

Suppose f is not constant and there is a local maximum. Let $z_0 \in D$ such that $|u| + |v|$ attains its local maximum at z_0. That is there exists an $\epsilon > 0$ such that $B_\epsilon(z_0) \subset D$ and $|u(x_0, y_0)| + |v(x_0, y_0)| > |u(x, y)| + |v(x, y)|$ for all $z = x + iy \in B_\epsilon(z_0)$. However, $f(B_\epsilon(z_0))$ is open and connected. Therefore, there is a point $z' \in B_\epsilon(z_0)$ such that $|u(x', y')| + |v(x', y')| > |u(x_0, y_0)| + |v(x_0, y_0)|$, which is a contradiction.

4.

If f_1 and f_2 are non constant, then $f_1(f_2(D_2))$ is open and connected, which means that it cannot be constant.

5.

If $|f| + |g|$ has local maximum at z_0, then we can find constant $a, b \in \mathbb{C}$ with norm 1 such that $|f(z_0)| = af(z_0), |g(z_0)| = bg(z_0)$. Since f, g are continuous, $\left|f + \frac{b}{a}g\right|$ also has a local maximum at z_0. Therefore, $\left|f + \frac{b}{a}g\right|$ and $f + \frac{b}{a}g$ are constant, which means that $f \equiv c - \frac{b}{a}g$ for some $c \in \mathbb{C}$. We also have that $c = |f(z_0)| + \frac{b}{a}|g(z_0)| \geq |c - \frac{b}{a}g(z)| + \frac{b}{a}|g(z)|$ for some neighborhood around z_0. However, by the triangle equality,

$$|c - \frac{b}{a}g(z)| + \frac{b}{a}|g(z)| \geq c$$

and hence $c = \frac{b}{a}|g(z)| + \left|c - \frac{b}{a}g(z)\right|$ which is only true when $\frac{b}{a}g(z)$ is non-negative and real. Thus g is constant and so does f.