1.

Consider the function

$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \to f(x, y) - 1$$

Then consider the set U satisfies $f(x, y) = 1 \iff g(x, y) = 1$.
Let $(x_0, y_0) = (0, 1)$. We have that

$$\det \left[\frac{\partial g}{\partial y}(x_0, y_0) \right] = 2 \neq 0$$

Therefore, there exists $(-\epsilon, \epsilon)$ and a unique function ϕ such that $\phi(0) = 1$ and

$$\phi'(x) = -\left(2\phi(x)\right)^{-1} \cdot 2x = -\frac{x}{\phi(x)}$$

2.

a.

If $(\alpha, \beta), (\xi, \eta)$ are respectively the points on f and g such that the distance between them reaches an extremum then
Consider the function

$$\phi_1 : \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \to |(x, y) - (\xi, \eta)|$$

Then we have ϕ_1 have an extremum at (α, β) under the constraint that $f(x, y) = 0$. Hence,

$$f_x(\alpha, \beta) = \frac{\partial \phi}{\partial x}(\alpha, \beta) \cdot f_y(\alpha, \beta) \cdot \left(\frac{\partial \phi}{\partial y}\right)^{-1}(\alpha, \beta) = \frac{\alpha - \xi}{\beta - \eta} \cdot f_y(\alpha, \beta)$$

and therefore,

$$f_y(\alpha, \beta)(\alpha - \xi) = f_x(\alpha, \beta)(\beta - \eta)$$

Following the same process, we can get that

$$g_y(\xi, \eta)(\alpha - \xi) = g_x(\xi, \eta)(\beta - \eta)$$

b.

Consider two smooth curves $f(x, y) = x + y - 2 = 0$ and $g(x, y) = x^2 + 2y^2 - 1 = 0$. We have that

$$f_x(x, y) = f_y(x, y) = 1, g_x(x, y) = 2x, g_y(x, y) = 4y$$

Then there is two points $(\alpha, \beta), (\xi, \eta)$ lying on the respective curves such that the distance between those two points is the minimum and hence is the distance between those curves.
Substitute what we know into the results obtained from part a, we have that

$$\alpha - \xi = \beta - \eta$$

and

$$4\beta(\alpha - \xi) = 2\alpha(\beta - \eta) = 2\alpha(\alpha - \xi)$$

Therefore, $\alpha = 2\beta$. Substitute that into $f(x, y) = 0$, we have

$$\alpha + \beta - 2 = 0 \implies \beta + 2\beta - 2 = 0 \implies \beta = \frac{2}{3} \implies \alpha = \frac{4}{3}$$

Then we have that

$$\frac{4}{3} - \xi = \frac{2}{3} - \eta \implies \xi = \frac{2}{3} + \eta$$

Substitute that into the $g(x, y) = 0$, we have

$$\left(\frac{2}{3} + \eta\right)^2 + 2\eta^2 - 1 = 3\eta^2 + \frac{4\eta}{3} - \frac{5}{9} = 0$$

Hence,

$$\eta = \frac{1}{9}(-2-\sqrt{19}) \implies \xi = \frac{1}{9}(4-\sqrt{19})$$

in which case the distance between (α, β) and (ξ, η) is approximately 1.942.
or

$$\eta = \frac{1}{9}(-2+\sqrt{19}) \implies \xi = \frac{1}{9}(4+\sqrt{19})$$

in which case the distance between (α, β) and (ξ, η) is approximately 0.572.
Taking the minimum of those distance we have that the distance between the two curves is 0.572

3.

As K is compact and f is continuous, f attains both a minimum and a maximum on K. First we consider the interior of k. We have

$$\nabla f(x,y,z) = (2x-2, y, 2z+2) = 0 \iff (x,y,z) = (1,0,-1)$$

Since

$$(\text{Hess}f)(1,0,-1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

the Hessian is positive definite and hence f reaches its minimum at $(1,0,-1)$ with the value of $f(1,0,-1) = -1$ Therefore, f attains its maximum on ∂K. Let

$$\phi : \mathbb{R}^3 \to \mathbb{R}, \qquad (x,y,z) \to x^2+y^2+z^2-9$$

so that $\partial K = \{(x,y,z) \in \mathbb{R}^3 : \phi(x,y,z) = 0\}$. We then have that

$$y = 2\lambda y \implies \lambda = \frac{1}{2} \text{ or } y = 0$$

In case $\lambda = \frac{1}{2}$, we have that

$$2x - 2 = 2\lambda x \implies x = 2$$

$$2z + 2 = 2\lambda z \implies z = -2$$

and

$$x^2 + y^2 + z^2 = 9 \implies y = 1$$

In case $y = 0$, as $2x-2 = 2\lambda x$ and $2z+2 = 2\lambda z$, $x \neq 0$ and $z \neq 0$. Therefore, we have that

$$\frac{2x-2}{2x} = 2\lambda = \frac{2z+2}{2z} \implies x = -z$$

Therefore,

$$x^2+y^2+z^2 = 9 \implies (x,z) = \left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) \text{ or } (x,z) = \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$$

To sum everything up

$$f(2,1,-2) = 2, f\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) = 1.515, f\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) = 18.485$$

and hence f attains its maximum at $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ with the value of 18.485 and its minimum at $(1,0,-1)$ with the value of -1.

4.

a.

Consider the function

$$\phi : \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \to 1 - xy$$

then at the point (x_0, y_0) where f reaches its extremum, we have

$$\begin{aligned}
\frac{\partial f}{\partial x}(x_0, y_0) &= \frac{\partial f}{\partial y}(x_0, y_0) \cdot \left(\frac{\partial \phi}{\partial y}(x_0, y_0)\right)^{-1} \cdot \frac{\partial f}{\partial x}(x_0, y_0) \\
\implies x_0^{p-1} &= y_0^{q-1} \cdot \frac{1}{-x_0} \cdot (-y_0) \\
\implies x_0^p &= y_0^q \\
\implies x_0^{p+q} &= 1 \\
\implies x_0 &= 1 \text{ (p+q} \neq 0) \\
\implies y_0 &= 1
\end{aligned}$$

Therefore, the minimum is

$$f(1, 1) = \frac{1}{p} + \frac{1}{q} = 1$$

as it cannot be the maximum because $\lim_{x \to \infty} f(x, \frac{1}{x}) = \infty$

b.

Changing ϕ

$$\phi : \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \to ab - xy$$

If (x_0, y_0) is where f reaches its extremum, then applying the same process, we have

$$x_0^p = y_0^q \implies \quad (ab)^p = y_0^{p+q} = (y_0^q)^p \implies ab = y_0^q$$

Hence, f reaches its extremum at (x_0, y_0) and

$$f(x_0, y_0) = \frac{x_0^p}{p} + \frac{y_0^q}{q} = y_0^q \left(\frac{1}{p} + \frac{1}{q}\right) = ab$$

as it cannot be the maximim because $\lim_{x \to \infty} f(x, \frac{ab}{x}) = \infty$ and hence we have that

$$f(a, b) = \frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

c.

Let $x_k = \dfrac{a_k}{\left(\sum_{k=1}^n a_k^p\right)^{1/p}}, y_k = \dfrac{b_k}{\left(\sum_{k=1}^n b_k^q\right)^{1/q}}$. We have that

$$\begin{aligned}\sum_{k=1}^n x_k y_k &= \frac{\sum_{k=1}^n a_k}{\left(\sum_{k=1}^n a_k^p\right)^{1/p}} + \frac{\sum_{k=1}^n b_k}{\left(\sum_{k=1}^n b_k^q\right)^{1/q}} \\ &\leq \frac{\sum_{k=1}^n a_k^p}{p\left(\sum_{k=1}^n a_k^p\right)} + \frac{\sum_{k=1}^n b_k^q}{q\left(\sum_{k=1}^n b_k^q\right)} \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1\end{aligned}$$

Multiplying both sides of the inequality by $\left(\sum_{k=1}^n a_k^p\right)^{1/p}\left(\sum_{k=1}^n b_k^q\right)^{1/q}$, we get the results

$$\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k^p\right)^{1/p}\left(\sum_{k=1}^n b_k^q\right)^{1/q}$$

d.

Since $p \geq 1$, there exists a $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Therefore,

$$\begin{aligned}\sum_{k=1}^n |a_k + b_k|^p &= \sum_{k=1}^n |a_k + b_k| \cdot |a_k + b_k|^{p-1} \\ &= \left(\sum_{k=1}^n |a_k + b_k|^p\right)^{1/p} \cdot \left(\sum_{k=1}^n (|a_k + b_k|^{p-1})^q\right)^{1/q} \\ &\leq \left(\sum_{k=1}^n |a_k|^p\right)^{1/p} \cdot \left(\sum_{k=1}^n (|a_k + b_k|^{p-1})^q\right)^{1/q} \\ &+ \left(\sum_{k=1}^n |b_k|^p\right)^{1/p} \cdot \left(\sum_{k=1}^n (|a_k + b_k|^{p-1})^q\right)^{1/q} \\ &= \left(\left(\sum_{k=1}^n |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^n |b_k|^p\right)^{1/p}\right)\left(\sum_{k=1}^n |a_k + b_k|^p\right)^{1/q}\end{aligned}$$

Dividing both sides of the inequality by $\left(\sum_{k=1}^n |a_k + b_k|^p\right)^{1/q}$, we get the desired results

$$\left(\sum_{k=1}^n |a_k + b_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^n |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^n |b_k|^p\right)^{1/p}$$

5.

Proof. Consider the function F:

$$F : U \times f_1(U) \to \mathbb{R}, \qquad (x, y, t) \to f_1(x, y) - t$$

Then we have that

$$\frac{\partial F}{\partial x}(x_0, y_0, f_1(x_0, y_0)) = \frac{\partial f_1}{\partial x}(x_0, y_0) \neq 0$$

and

$$F(x_0, y_0, f_1(x_0, y_0)) = 0$$

Therefore, there exists neighborhoods $V \subset \mathbb{R}^2$ of $(y_0, f_1(x_0, y_0))$ and $W \subset \mathbb{R}$ of x_0 such that $W \times V \subset U \times \mathbb{R}$ and a unique $\phi \in \mathcal{C}^1(V, \mathbb{R})$ such that for all $(x, y, t) \in U \times f_1(U)$:

$$x = \phi(y, t) \iff f_1(x, y) = t$$

Thus, we have that

$$f_1(\phi(y, t), y) = t$$

and hence, taking the derivative with respect to y, we get

$$f_{1_x}(\phi(y, t), y) \cdot \phi_y(y, t) + f_{1_y}(\phi(y, t), y) = 0$$

$$\implies \phi_y(y, t) = -\frac{f_{1_y}(\phi(y, t), y)}{f_{1_x}(\phi(y, t), y)} = -\frac{f_{2_y}(\phi(y, t), y)}{f_{2_x}(\phi(y, t), y)}$$

as rank $J_f(x, y) = 1$ for all $(x, y) \in U$. Now consider the function:

$$\psi : V \to f_2(U), \qquad (y, t) \to f_2(\phi(y, t), y)$$

We can see that g is not dependent on y as

$$g_y = f_{2_x}(\phi(y, t), y) \cdot \phi_y(y, t) + f_{2_y}(\phi(y, t), y) = 0$$

It is possible then to rewrite ψ as

$$\psi : V_y \to f_2(U), \qquad (y, t) \to f_2(\phi(y, t), y)$$

Let (x, y) be an arbitary point in $W \times V_y$ then we can find a t such that $f_1(x, y) = t \iff x = \phi(y, t)$ and therefore

$$f_2(x, y) = f_2(\phi(y, t), y) = \psi(t) = \psi(f_1(x, y))$$

which finishes the proof. $\square$

6.