1.

a.

Since $\frac{1}{x^2} \to \infty$ as $x \to 0$ there is infinitely $x \in (0,1)$ such that

$$\frac{1}{x^2} = n\pi + \frac{\pi}{2} = \frac{2n\pi + \pi}{2}$$

for some $n \in \mathbb{N}$. Let

$$x_n = \frac{\sqrt{2}}{\sqrt{2n\pi + \pi}}$$

Notice that

$$\sin\left(\frac{1}{x^2}\right) = 1 \text{ if n is even}$$

and

$$\sin\left(\frac{1}{x^2}\right) = -1 \text{ if n is odd}$$

and we can define a sequence of partition $\Gamma_n = \{1, x_0, x_1, x_2, \ldots x_n, 0\}$, note that we have $1 > x_0 > x_1 > \ldots > x_n > 0$ instead. Then we have

$$\begin{aligned}
V(f, \Gamma_n) >& \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \\
=& \sum_{j=1}^{n} \left|(-1)^n (x_j^2 + x_{j-1}^2)\right| \\
=& \sum_{j=1}^{n} \frac{2}{2j\pi + \pi} + \frac{2}{2j\pi - \pi} \\
>& \sum_{j=1}^{n} \frac{2}{2j\pi + \pi} \\
>& \frac{1}{\pi} \sum_{j=1}^{n} \frac{2}{2j + 2} \\
=& \frac{1}{\pi} \sum_{j=2}^{n} \frac{1}{j}
\end{aligned}$$

Thus $V(f, \Gamma_n) \to \infty$ as $n \to \infty$ and thus f is not a bounded variation.

b.

If $V_I(f) > \liminf_{n\to\infty} V_I(f_n)$, then there is $\varepsilon > 0$ such that for all $n \in \mathbb{N}, n_0 > n$ such that $V_I(f_{n_0}) < V_I(f) + \varepsilon$. Thus there is a partition Γ such that $V_I(f_{n_0}, \Gamma) < V_I(f, \Gamma) + \varepsilon/2$.
However, we have $f_n \to f$ on Γ, therefore, there is n_0' such that for all $n > n_0'$

$$||f_n(x_j) - f_n(x_{j-1})| - |f(x_j) - f(x_{j-1})|| < \frac{|\Gamma|\varepsilon}{2}$$

and hence

$$|V_I(f_n,\Gamma) - V_I(f,\Gamma)| < \varepsilon/2$$

which is a contradiction.

c.

If $\inf_{x\in\mathbb{R}} |f(x)| = 0$, then for every $\varepsilon > 0$, there is $x \in \mathbb{R}$ such that $|f(x)| < \varepsilon$. Thus for every $M > 0$, there is x such that $|\frac{1}{f(x)}| > M$ and for any $N > 0$, we can fix x_1 and choose x_2 so that

$$\left|\frac{1}{f(x_1)} - \frac{1}{f(x_2)}\right| > N$$

Therefore, $V(f, \{x_1, x_2\}) > N$ and $\dfrac{1}{f}$ is not of bounded variation.
If $\inf_{x\in\mathbb{R}} |f(x)| > 0$ then there is some $\varepsilon > 0$ such that $|f(x)| > \varepsilon$ and $\dfrac{1}{|f(x)|} < \dfrac{1}{\varepsilon}$. For every partition Γ,

$$\begin{aligned}
V(1/f,\Gamma) &= \sum_{j=1}^{n} |1/f(x_j) - 1/f(x_{j-1})| \\
&= \sum_{j=1}^{n} \left|\frac{f(x_{j-1}) - f(x_j)}{f(x_j)f(x_{j-1})}\right| \\
&\le \frac{1}{\varepsilon^2}\sum_{j=1}^{n} |f(x_{j-1}) - f(x_j)| \\
&= \frac{1}{\varepsilon^2}V(f)
\end{aligned}$$

2.

If f is absolutely continuous on $[a, b]$ and $f(x) \neq 0$ for all $x \in [a, b]$, there is $M > 0$ such that $|f(x)| > M$ for all $x \in [a, b]$.
Also, since f is absolutely continuous, for every $\varepsilon > 0$, there is $\delta > 0$ such that there is x_j, y_j such that $\sum_{j=1}^{n} |x_j - y_j| < \delta$. and

$$\sum_{j=1}^{n} |f(y_j) - f(x_j)| < M^2/\varepsilon$$

Thus

$$\sum_{j=1}^{n} |1/f(y_j) - 1/f(x_j)| < \frac{1}{M^2} \sum_{j=1}^{n} |f(y_j) - f(x_j)| < \varepsilon$$

3.

a.

Since $m(E) = 0$, for all $\delta > 0$, there is a set of intervals $I_n := [a_n, b_n]$ such that $E \subseteq \cup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} m(I_n) < \delta$, hence $\sum_{n=1}^{\infty} |b_n - a_n| < \delta$. Therefore, because of f being absolutely continuous, for every $\varepsilon > 0$, the finite collection $(a_j, b_j)_{j=1}^{n}$ satisfies

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \varepsilon$$

But f is an increasing continuous function, thus $m(f(I_j)) = f(b_j) - f(a_j)$. Hence,

$$\sum_{j=1}^{n} m(f(I_j)) < \varepsilon$$

finally since $E \subseteq \cup_{j=1}^{\infty} I_j$, $f(E) \subseteq \cup_{j=1}^{\infty} f(I_j)$ and the inequality works for all $n \in \mathbb{N}$.

$$m(F(E)) \leq \sum_{j=1}^{\infty} m(f(I_j)) \leq \varepsilon$$

b.

Since F is Lesbegue measurable and $m^*(F) = b - a < \infty$, for all $\varepsilon > 0$, there is a compact set $K \subseteq E$ such that $m^*(F \backslash K) < \varepsilon$, and $F = (F \backslash K) \cup K$. From part a, we know that $f(F \backslash K)$ has measure zero and $f(K)$ is measurable since K is compact. Then, we have that

$$f(F) = f(F \backslash K) \cup f(K)$$

is measurable.

4.

If f is Lipschitz continuous then

$$|f'(x_0)| = \lim_{x \to x_0} \left| \frac{f(x_0) - f(x)}{x_0 - x} \right| \leq M$$

and for all $\varepsilon > 0$, there is $\delta = \varepsilon / M > 0$ such that

$$\sum_{j=1}^{n} |f(y_j) - f(x_j)| \leq M \sum_{j=1}^{n} |y_j - x_j| = \varepsilon$$

for all finite collection $\{(x_j, y_j)\}_{j=1}^{n}$ that satisfies $\sum_{j=1}^{n} |y_j - x_j| < \delta$.
Now suppose f is not Lipschitz continuous, for every $N > 0$, there is x_0, y_0 such that

$$|f(x_0) - f(y_0)| > |x_0 - y_0| N$$

but since $f' \in L_\infty([a, b])$, there is $\delta > 0$ and $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $|x_0 - y_0| < \delta$. Thus, for all $N > 0$ and x_0 we can find y_0 such that

$$|f(x_0) - f(y_0)| > |x_0 - y_0| N > N\delta$$

Thus for all $\varepsilon > 0$, there is a finite collection $\{(x_{j-1}, x_j)\}_{j=1}^{n}$, with $x_a = x_0, x_b = y_0$ and $|x_{j-1} - x_j| < \varepsilon$ such that

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| > |f(x_0) - f(y_0)| > \delta N$$

for all $N > 0$, since δ is fixed that is a contradiction because f is absolutely continuous.