1.

a.

We have

$$\int_{-\infty}^{\infty} f_Y(y)dy = \int_0^6 f_Y(y)dy = \int_0^6 c(8-y)dy = c\left(8y - \frac{y^2}{2}\right)\Big|_0^6 = 30c = 1$$

Therefore, $c = \dfrac{1}{30}$

b.

$$E[Y] = \int_0^6 y f_Y(y)dy = \int_0^6 y\left(\frac{4}{15} - \frac{y}{30}\right)dy = \left(\frac{4y^2}{30} - \frac{y^3}{90}\right)\Big|_0^6 = 2.4$$

Similarly, we have that

$$E[Y] = \int_0^6 y^2 f_Y(y)dy = \int_0^6 y^2\left(\frac{4}{15} - \frac{y}{30}\right)dy = \left(\frac{4y^3}{45} - \frac{y^4}{120}\right)\Big|_0^6 = 8.4$$

Hence,

$$V[Y] = 8.4 - 2.4^2 = 2.64$$

c.

$$F_Y(y) = \begin{cases} 0, & \text{if } y \le 0 \\ 1, & \text{if } y \ge 6 \\ \int_0^y \left(\frac{4}{15} - \frac{t}{30}\right)dt = \frac{8y}{30} - \frac{y^2}{60}, & \text{otherwise} \end{cases}$$

For $y \le 0$:

$$F_Y(y) = 0$$

For $0 < y < 6$:

$$F_Y(y) = \int_0^y \left(\frac{4}{15} - \frac{t}{30}\right)dt = \frac{8y}{30} - \frac{y^2}{60}$$

For $y \ge 6$:

$$F_Y(y) = 1$$

2.

a.

$$E[Y] = \int_{-1}^{0} \frac{y}{16} dy + \int_{0}^{1} \left(\frac{y}{16} + y^4 \right) dy + \int_{1}^{3} \frac{5y^2}{32} dy = \frac{373}{240}$$

b.

$$P(-0.75 < Y < 1.25) = \int_{-0.75}^{1.25} f_Y(y) dy = 1 - \int_{-1}^{-0.75} \frac{dy}{16} - \int_{1.25}^{3} \frac{5y}{32} dy = \frac{413}{1024}$$

c.

For $y < 0$:

$$F_Y(y) = 0$$

For $-1 < y < 0$:

$$F_Y(y) = \int_{-1}^{y} \frac{1}{16} dt = \frac{y}{16} + C$$

Since the cumulative distribution function is continuous, we also have

$$F_Y(-1) = \frac{-1}{16} + C = 0 \implies C = \frac{1}{16}$$

And hence

$$F_Y(y) = \frac{y+1}{16}$$

For $0 < y < 1$:

$$F_Y(y) = \int_{0}^{y} \left(\frac{1}{16} + t^3 \right) dt = \frac{y}{16} + \frac{y^4}{4} + C$$

Since the cumulative distribution function is continuous, we also have

$$F_Y(0) = C = \left. \frac{y+1}{16} \right|_{y=0} = \frac{1}{16}$$

And hence

$$F_Y(y) = \frac{1+y}{16} + \frac{y^4}{4}$$

For $1 < y < 3$:

$$F_Y(y) = \int_{0}^{y} \frac{5t}{32} dt = \frac{5y^2}{64} + C$$

Since the cumulative distribution function is continuous, we also have

$$F_Y(1) = \frac{5}{64} + C = \left. \frac{y+1}{16} + \frac{y^4}{4} \right|_{y=1} = \frac{3}{8} \implies C = \frac{19}{64}$$

And hence

$$F_Y(y) = \frac{5y^2}{64} + \frac{19}{64}$$

For $y > 1$:

$$F_Y(y) = 1$$

d.

$F_Y(1) = \frac{24}{64} < 0.8$, hence the 80^{th} percentile of Y is at x where x satisfies:

$$F_Y(x) = \frac{5x^2 + 19}{64} = 0.8 \implies x = 2.5377$$

3.

We have

$$\int_0^4 ay^b dy = \left.\frac{ay^{b+1}}{b+1}\right|_0^4 = \frac{a4^{b+1}}{b+1} = 1$$

$$E[Y] = \int_0^4 y \cdot ay^b dy = \left.\frac{ay^{b+2}}{b+2}\right|_0^4 = \frac{a4^{b+2}}{b+2} = \frac{a4^{b+1}}{b+1} \cdot \frac{4(b+1)}{b+2} = \frac{4(b+1)}{b+2} = 3$$

Hence, $b = 2$ and $a = \dfrac{3}{64}$

4.

$$E[Y] = \frac{d}{dt}\left(e^t(1+12t)^{-2}\right)\bigg|_{t=0} = -23$$

$$E[Y^2] = \frac{d^2}{dt^2}\left(e^t(1+12t)^{-2}\right)\bigg|_{t=0} = 817$$

$$E[Y^3] = \frac{d^3}{dt^3}\left(e^t(1+12t)^{-2}\right)\bigg|_{t=0} = -38951$$

$$E[(Y-\mu_Y)^3] = E[Y^3]-3E[Y^2\mu_Y]+3E[Y\mu_Y^2]-E[\mu_Y^3] = E[Y^3]-3\mu_Y E[Y^2]+2\mu_Y^3 = -6912$$

$$\sigma^3 = (V[Y])^{3/2} = (817-(-23)^2)^{3/2} = 3456\sqrt{2}$$

Hence, the skewness coefficient is

$$\frac{-6912}{3456\sqrt{2}} = -\sqrt{2}$$

5.

$m_X(t) = E[e^{t(3Y-6)}] = E[e^{-6t}] \cdot (E[e^{(3t)Y}]) = e^{-6t}e^{6t}(1-9t)^{-1} = (1-9t)^{-1}$

which is a exponential distribution with $\theta = 9$.

6.

a.

Since Y is uniform, $f_Y(y) = \frac{1}{10}$ if $0 < y < 10$ is given.

$$E[X] = \int_0^{10} 10\sqrt{2}(2^{\frac{-y}{2}})\frac{1}{10}dy = 3.953$$

$$E[X^2] = \int_0^{10} \left(10\sqrt{2}(2^{\frac{-y}{2}})\right)^2 \frac{1}{10}dy = 28.8257$$

Hence

$$V[X] = 28.8257 - 3.953^2 = 13.1995$$

b.

$X > 4 \iff Y < 3.6439$
$X < 10 \iff Y > 1$

$$P(X > 4|X < 10) = \frac{P(10 > X > 4)}{P(X < 10)} = \frac{P(1 < Y < 3.6439)}{P(Y > 1)} = \frac{2.6439}{9} = 0.29377$$

7.

From the table, we have that

$$P(Y > -0.5) = P(Z > -1.56) \text{ and } P(Y < 1.25) = P(Z < 1.04)$$

Hence, we have the equations

$$\frac{-0.5 - \mu}{\sigma} = -1.56 \text{ and } \frac{1.25 - \mu}{\sigma} = 1.04$$

Therefore, $\sigma = \frac{35}{52}$ and $\mu = 0.55$, hence $\sigma^2 = \frac{1225}{2704}$

8.

1.

We need to find X such that $P(Y > X) = 0.12$.

We have that $P(Z > 1.175) = 0.12$ hence $\frac{X - 2.25}{0.49} = 1.175$ and therefore $X = 2.82575$

2.

Consider $U = Y_1 - 2Y_2$.

$$\mu_U = -2.25 \text{ and } \sigma_U = \sqrt{0.49^2 + 4 * 0.49^2} = 1.0957$$

Hence, $P(U > 0) = P(Z > 2.05) = 0.0202$

9.

The possibility that any person achieves more than 80% is

$$\int_{0.8}^{1} 12y^2(1-y)dy = \left(4y^3 - 3y^4\right)\Big|_{0.8}^{1} = 0.1808$$

and hence the possibility that at least three of them achieve more than 80% is

$$1-\binom{10}{0}0.1808^0\cdot(1-0.1808)^{10}-\binom{10}{1}0.1808^1\cdot(1-0.1808)^9-\binom{10}{2}0.1808^2\cdot(1-0.1808)^8 = 0.26513$$

10.

Median of Y is 8, that means that

$$P(Y > 8) = P(Y < 8) = 0.5$$

Hence,

$$\int_8^\infty \frac{1}{\beta} e^{\frac{-y}{\beta}}\, dy = -e^{\frac{-y}{\beta}}\Big|_0^8 = 1 - e^{\frac{-8}{\beta}} = 0.5 \implies \beta = \frac{8}{\ln 2}$$

$$P(a < Y < 2a) = e^{\frac{-a}{\beta}} - e^{\frac{-2a}{\beta}} = 0.16 \implies e^{\frac{-a}{\beta}} \in \{0.2, 0.8\} \implies a \in \{2.5754, 18.5754\}$$

11.

We have the relationship between B(bonus) and H(hour segment) is

$$B = 1000 - 100H$$

Also,

$$P_H(h) = P_Y(h < Y < h+1) = e^{\frac{-h}{6}} - e^{\frac{-h-1}{6}}$$

Hence,

$$E[H] = (1 - e^{\frac{-1}{6}}) \sum_{h=0}^{\infty} he^{\frac{-h}{6}} = \frac{e^{\frac{-1}{6}}}{1 - e^{\frac{-1}{6}}} = 5.51388$$

and

$$E[H^2] = (1 - e^{\frac{-1}{6}}) \sum_{h=0}^{\infty} \left(h^2 e^{\frac{-h}{6}}\right) = \frac{e^{\frac{-2}{6}} + e^{\frac{-1}{6}}}{\left(1 - e^{\frac{-1}{6}}\right)^2} = 66.31968$$

Therefore,

$$E[B] = 1000 - 100 \cdot E[B] = 448.6118$$

$$V[B] = E[B^2] - (E[B])^2 = E[1000000 - 200000H + 10000H^2] - 448.6118^2 = 359168.253$$

12.

Probability that any claim is less than 4k is

$$\int_0^{4000} \frac{1}{5000} e^{\frac{-y}{5000}} dy = 1 - e^{-0.8} = 0.5507$$

Sum of 2 identical exponentially distributed, hence it is a gamma distribution.
Probability that any 2 claim has a total less than 4k is

$$\int_0^{4000} \frac{1}{\Gamma(2)5000^2} t^1 e^{\frac{-t}{5000}} dt = 0.1912$$

Hence, the possibility that a person claim less than 4k is

$$0.1912 \cdot 0.1 + 0.5507 \cdot 0.25 + 0.65 = 0.806795$$