1 p.208

$$
\begin{aligned}
& y'' - y = 0 \\
\implies & \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} a_k x^k = 0 \\
\implies & \sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2}x^k - a_k x^k = 0 \\
\implies & (k+1)(k+2)a_{k+2} - a_k = 0 \\
\implies & a_{k+2} = \frac{a_k}{(k+1)(k+2)}
\end{aligned}
$$

and hence

$$a_{2k} = a_0 \frac{1}{(2k)!} \text{ and } a_{2k+1} = a_1 \frac{1}{(2k+1)!}$$

and hence

$$y(x) = a_0 \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}}_{y_1} + a_1 \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}}_{y_2}$$

Hence,

$$W(y_1, y_2, 0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

3 p.208

$$
\begin{aligned}
& y'' - xy' - y = 0 \\
\implies & \sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}x^k - x\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k - \sum_{k=0}^{\infty}a_k x^k = 0 \\
\implies & \sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}x^k - \sum_{k=1}^{\infty}ka_k x^k - \sum_{k=0}^{\infty}a_k x^k = 0 \\
\implies & 2a_2 - a_0 + \sum_{k=1}^{\infty}((k+1)(k+2)a_{k+2} - ka_k - a_k)x^k = 0 \\
\implies & \begin{cases} a_0 = 2a_2 \\ (k+1)(k+2)a_{k+2} - ka_k - a_k = 0 \forall k > 0 \end{cases} \\
\implies & \begin{cases} a_0 = 2a_2 \\ a_{k+2} = \dfrac{a_k}{k+2} \forall k > 0 \end{cases}
\end{aligned}
$$

Hence,

$$a_{2k} = \frac{a_{2k-2}}{2k} = \frac{a_2}{2^{k-1} \cdot k!} = \frac{a_0}{2^k \cdot k!}$$

for $k > 0$, note that it also works for $k = 1$.

$$a_{2k+1} = \frac{a_1}{(2k+1)(2k-1)\dots 1} = \frac{a_1 \prod_{i=1}^{k} 2i}{\prod_{i=0}^{k}(2i+1)\prod_{i=1}^{k} 2i} = \frac{a_1 2^{k+1} k!}{(2k+1)!}$$

Hence,

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} x^k + a_1 \sum_{k=1} \frac{2^{k+1} k!}{(2k+1)!} x^k$$

4 p.214

There is no singularity hence the radius of convergence is ∞.

5 p.214

We can rewrite the equations as

$$y'' + \frac{x}{x^2 - 2x - 3}y' + \frac{4}{x^2 - 2x - 3}y = 0$$

The singularities are $3. - 1$. Hence, the radius of convergence is

$$R = \begin{cases} 1, & \text{if } x_0 = 4 \\ 3, & \text{if } x_0 = -4 \\ 1, & \text{if } x_0 = 0 \end{cases}$$

6 p.214

We can rewrite the equation as

$$y'' + \frac{4x}{1 + x^3}y' + \frac{1}{1 + x^3}y = 0$$

and hence the singularities can be found by letting $1 + x^3 = 0$ which yields $x = -1, e^{i\pi/3}, e^{2i\pi/3}$, which are the three points dividing the unit circle into 3 equal sections. Hence, for $x_0 = 0$, the radius of convergence is 1. For $x_0 = 2$, the radius of convergence will be

$$\left| 2 - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) \right| = \sqrt{3}$$