1.

a.

$$
\begin{aligned}
\mathcal{F}[c_1 f + c_2 g](\xi) &= \frac{1}{2\pi}\int_{\mathbb{R}} (c_1 f + c_2 g)(x) e^{ix\cdot\xi} dx \\
&= \frac{1}{2\pi} c_1 \int_{\mathbb{R}} f(x) e^{ix\cdot\xi} dx + \frac{1}{2\pi} c_2 \int_{\mathbb{R}} g(x) e^{ix\cdot\xi} dx \\
&= c_1 \mathcal{F} f + c_2 \mathcal{F} g
\end{aligned}
$$

b.

$$
\mathcal{F}(fg) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x) g(x) e^{-ix\cdot\xi} dx \neq \frac{1}{4\pi^2}\left(\int_{\mathbb{R}} f(x) e^{-ix\cdot\xi}\right)\left(\int_{\mathbb{R}} g(x) e^{-ix\cdot\xi}\right) = \mathcal{F} f \mathcal{F} g
$$

2.

a.

$$\begin{aligned}
(\mathcal{F}f)(\xi) &= \frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{ix\xi}dx \\
&= \frac{1}{2\pi}\int_{-a}^{a} e^{ix\xi}dx \\
&= \frac{1}{2\pi}\left.\frac{e^{ix\xi}}{i\xi}\right|_{x=-a}^{a} \\
&= \frac{1}{2\pi}\frac{e^{ia\xi}-e^{-ia\xi}}{i\xi} \\
&= \frac{2\sinh(ia\xi)}{2\pi i\xi} \\
&= \frac{-i\sin(-a\xi)}{i\xi\pi} \\
&= \frac{sin(a\xi)}{\xi\pi}
\end{aligned}$$

b.

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}} e^{-|\xi|\alpha}e^{-i\xi x}d\xi \\
&= \int_{0}^{\infty} e^{-\xi(\alpha+ix)}d\xi + \int_{-\infty}^{0} e^{\xi(\alpha-ix)}d\xi \\
&= -\left.\frac{e^{-\xi(\alpha+ix)}}{\alpha+ix}\right|_{\xi=0}^{\infty} + \left.\frac{e^{\xi(\alpha-ix)}}{\alpha-ix}\right|_{\xi=-\infty}^{0} \\
&= \frac{1}{\alpha+ix}+\frac{1}{\alpha-ix} \\
&= \frac{\alpha-ix+\alpha+ix}{(\alpha+ix)(\alpha-ix)} \\
&= \frac{2\alpha}{\alpha^2+x^2}
\end{aligned}$$

c.

$$\begin{aligned}
&\int_{\mathbb{R}} -iF'(\xi)e^{-i\xi x}d\xi \\
=& -ie^{-i\xi x}|_{-\infty}^{\infty} - \int_{\mathbb{R}} F(\xi)(-i\cdot(-ix)e^{-i\xi x})d\xi \\
=& x\int_{\mathbb{R}} F(\xi)e^{-i\xi x}d\xi \\
=& \mathcal{F}[xf(x)]
\end{aligned}$$

3.

We have that

$$\begin{aligned}\mathcal{F}[u_t] = U_t &= k\mathcal{F}[u_{xx}] + c[u_x] \\ &= -k\xi^2 U - ci\xi U\end{aligned}$$

Thus we can find

$$U(\xi, t) = C(\xi)e^{-k\xi^2 t - ci\xi t}$$

and since $u(x, 0) = f(x)$ and $U(\xi, 0) = F(\xi)$,

$$U(\xi, t) = F(\xi)e^{-k\xi^2 t - ci\xi t}$$

Let $G(\xi) = e^{-k\xi^2 t}$, $H(\xi) = F(\xi)e^{-ci\xi t}$ we have

$$U(\xi, t) = G(\xi)H(\xi)$$

And the inverse fourier of G, H are

$$g(x) = \frac{1}{\sqrt{2kt}}e^{-x^2/4kt}$$

$$h(x) = f(x - ct)$$

Thus the solution is

$$u(x, t) = \frac{1}{2\pi}\left(f(x - ct) * \frac{1}{\sqrt{2kt}}e^{-x^2/4kt}\right)$$

4.

Apply the fourier transform, we have that

$$\begin{cases} U_t = -k\xi^2 U - \gamma U \\ U(\xi, 0) = F(\xi) \end{cases}$$

Then, we can solve for

$$U(\xi, t) = C(\xi)e^{-(k\xi^2+\gamma)t}$$

and using the initial condition,

$$U(\xi, t) = F(\xi)e^{-(k\xi^2+\gamma)t} = e^{-\gamma t}F(\xi)e^{-k\xi^2 t}$$

And apply the inverse, we have

$$u(x, t) = e^{-\gamma t}\left(f(x) * \frac{1}{\sqrt{2kt}}e^{-x^2/4kt}\right)$$

5.

Apply the fourier transform on y, we have that

$$\begin{cases} U_{xx} - \xi^2 U = 0 \\ U(0,\xi) = G_1(\xi) \\ U(L,\xi) = G_2(\xi) \end{cases}$$

Thus

$$U(x,\xi) = C_1(\xi)e^{-\xi x} + C_2(\xi)e^{\xi x}$$

To ensure the boundedness of the solution, we must have that

$$C_1(\xi) = 0 \text{ if } \xi < 0 \text{ and } C_2(\xi) = 0 \text{ if } \xi > 0$$

Thus, the solution can be rewrite as

$$U(x,\xi) = C(\xi)e^{-|\xi|x}$$

The initial conditions state that

$$C(\xi) = G_1(\xi)$$

and

$$U(L,\xi) = G_1(\xi)e^{-|\xi|L} = G_2(\xi)$$

Thus,

$$u(x,y) = \frac{1}{2\pi}\left(g_1(y) * \frac{2L}{y^2 + L^2}\right)$$

6.

Apply fourier transform, we have

$$\begin{cases} U_{tt}(\xi, t) = -c^2\xi^2 U(\xi, t) \\ U(\xi, 0) = F(\xi) \\ U_t(\xi, 0) = 0 \end{cases}$$

Thus

$$U(\xi, t) = C_1(\xi)\cos(c\xi t) + C_2(\xi)\sin(c\xi t)$$

Apply the boundary conditions, we have that

$$C_2(\xi)c\xi\cos(c\xi 0) = 0 \implies C_2(\xi) = 0$$

and

$$C_1(\xi) = F(\xi)$$

Thus

$$U(\xi, t) = F(\xi)\cos(c\xi t)$$

and therefore,

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}} F(\xi)\cos(c\xi t)e^{-i\xi x}d\xi \\ &= \frac{1}{2}\int_{\mathbb{R}} F(\xi)(e^{-i\xi(x-ct)} + e^{-i\xi(x+ct)})d\xi \\ &= \frac{1}{2}\left(f(x - ct) + f(x + ct)\right) \end{aligned}$$

7.

Apply the fourier cosine transform on y and $u_y(x,0)=0$,

$$\begin{cases} U_{xx}(x,\xi)-\xi^2U(x,\xi)=0 \\ U(0,\xi)=G_1(\xi) \\ U_x(L,\xi)=0 \end{cases}$$

We have that

$$U(x,\xi)=C_1(\xi)e^{-\xi x}+C_2(\xi)e^{\xi x}$$

Apply the boundary conditions,

$$-\xi C_1(\xi)e^{-\xi L}+\xi C_2(\xi)e^{\xi L}=0 \implies C_1(\xi)=C_2(\xi)e^{-2\xi L}$$

Thus, we can rewrite

$$U(x,\xi)=C(\xi)\cosh(\xi(L-x))$$

Apply the other boundary conditions give us

$$C(\xi)=\frac{G_1(\xi)}{\cosh(\xi L)}$$

Thus, solution is

$$u(x,y)=\frac{1}{\pi}\int_0^\infty g_1(\overline{x})(f(x,y-\overline{x})+f(x,y+\overline{x}))dx$$

where

$$f(x,y)=\int_0^\infty \frac{\cosh(\xi(L-x))}{\cosh(\xi L)}\cos(\xi y)d\xi$$

8.

Apply fourier transform on y,

$$\begin{cases} U_{xx}(x,\xi) - \xi^2 U(x,\xi) = 0 \\ U(0,\xi) = G(\xi) \end{cases}$$

Then

$$U(x,\xi) = C_1(\xi)e^{\xi x} + C_2(\xi)e^{-\xi x}$$

To ensure U is bounded for $x < 0$,

$$C_1(\xi) = 0 \text{ if } \xi < 0 \text{ and } C_2(\xi) = 0 \text{ if } \xi > 0$$

Thus, we can rewrite

$$U(x,\xi) = C(\xi)e^{|\xi|x}$$

and apply the boundary condition gives

$$U(x,\xi) = G(\xi)e^{|\xi|x}$$

Thus let $t = y - \overline{x}$ so that $dt = -d\overline{x}$, we have

$$\begin{aligned} u(x,y) &= \frac{1}{2\pi}\int_{\mathbb{R}} g(\overline{x})\frac{-2(y-\overline{x})}{(y-\overline{x})^2 + x^2}d\overline{x} \\ &= \frac{1}{2\pi}\int_{y+1}^{y-1} \frac{2t}{t^2 + x^2}dt \\ &= \frac{1}{2\pi}\ln(x^2 + t^2)|_{t=y+1}^{y-1} \\ &= \frac{1}{2\pi}\left(\ln\left(y^2 - 2y + x^2 + 1\right) - \ln\left(y^2 + 2y + x^2 + 1\right)\right) \end{aligned}$$

9.

Since both a,b have $u(0,y) = 0$, apply the fourier sine transform on x, we have

$$U_{yy}(\xi, y) - \xi^2 U(\xi, y) = 0$$

Therefore,

$$U(\xi, y) = C_1(\xi)e^{\xi y} + C_2(\xi)e^{-\xi y}$$

To ensure the boundedness of U on $y > 0$,

$$C_1(\xi) = 0$$

Thus

$$U(\xi, y) = C(\xi)e^{-\xi y}$$

a.

$$u_y(x, 0) = f(x) \implies U_y(\xi, 0) = F(\xi)$$

Hence,

$$U_y(\xi, 0) = -\xi C(\xi)e^{-\xi 0} = F(\xi)$$

and

$$U(\xi, y) = -\frac{F(\xi)}{\xi}e^{-\xi y}$$

and

$$u(x, y) = \frac{1}{\pi}\int_0^\infty f(\overline{x})[g(x - \overline{x}, y) + g(x + \overline{x}, y)]d\overline{x}$$

where $g(x, y)$ is the inverse fourier cosine of $-\dfrac{e^{-\xi y}}{\xi}$, which is

$$g(x, y) = -\int_0^y \frac{x}{x^2 + \overline{x}^2}d\overline{x} = -\arctan\frac{y}{x}$$

b.

$$u(x, 0) = f(x) \implies U(\xi, 0) = F(\xi)$$

Hence,

$$U(\xi, y) = F(\xi)e^{-\xi y}$$

and

$$u(x, y) = \frac{1}{\pi}\int_0^\infty f(\overline{x})[g(x - \overline{x}, y) + g(x + \overline{x}, y)]d\overline{x}$$

where

$$g(x, y) = \frac{y}{x^2 + y^2}$$

10.

Apply the fourier transform on x,

$$\begin{cases} U_{tt}(\xi,t) = -c^2\xi^2 U(\xi,t) \\ U(\xi,0) = 0 \\ U_t(\xi,0) = G(\xi) \end{cases}$$

Thus

$$U(\xi,t) = C_1(\xi)\cos(c\xi t) + C_2(\xi)\sin(c\xi t)$$

Apply the boundary condition $C_1(\xi) = 0$ and

$$U(\xi,t) = C(\xi)\sin(c\xi t)$$

Apply the other boundary conditions,

$$U_t(\xi,0) = C(\xi)c\xi\cos(c\xi 0) = G(\xi) \implies C(\xi) = \frac{G(\xi)}{c\xi}$$

Hence,

$$U(\xi,t) = G(\xi)\frac{\sin(c\xi t)}{c\xi}$$

Thus

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} g(\overline{x})f(x-\overline{x},t)d\overline{x}$$

where

$$f(x,t) = \begin{cases} \dfrac{\pi}{c} & \text{if } |x| < ct \\ 0 & \text{if } |x| > ct \end{cases}$$