3.2

iii.

a.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda + 1 \implies \lambda = \frac{1 \pm \sqrt{3}i}{2}$$

For $\lambda = \dfrac{1+\sqrt{3}i}{2}$, the eigenvector is $\begin{pmatrix} \dfrac{-\sqrt{3}i - 1}{2} \\ 1 \end{pmatrix}$.

For $\lambda = \dfrac{1-\sqrt{3}i}{2}$, the eigenvector is $\begin{pmatrix} \dfrac{\sqrt{3}i - 1}{2} \\ 1 \end{pmatrix}$

b.

Thus

$$T = \begin{pmatrix} \frac{\sqrt{3}i-1}{2} & \frac{-\sqrt{3}i-1}{2} \\ 1 & 1 \end{pmatrix}$$

c.

First, let's look at the solution of

$$X' = AX$$

Based on the eigenvector the general solution is

$$X(t) = c_1 e^{t/2} \begin{pmatrix} \cos(\sqrt{3}t/2) \\ -\sin(\sqrt{3}t/2) \end{pmatrix} + c_2 e^{t/2} \begin{pmatrix} \sin(\sqrt{3}t/2) \\ \cos(\sqrt{3}t/2) \end{pmatrix}$$

and we have

$$T^{-1}AT = \begin{pmatrix} \frac{1-i\sqrt{3}}{2} & 0 \\ 0 & \frac{1+i\sqrt{3}}{2} \end{pmatrix}$$

which yields

$$Y(t) = c_1 e^{t/2} \begin{pmatrix} \cos(\sqrt{3}t/2) \\ -\sin(\sqrt{3}t/2) \end{pmatrix} + c_2 e^{t/2} \begin{pmatrix} \sin(\sqrt{3}t/2) \\ \cos(\sqrt{3}t/2) \end{pmatrix}$$

d.

Spiral source for both system (taken from book):

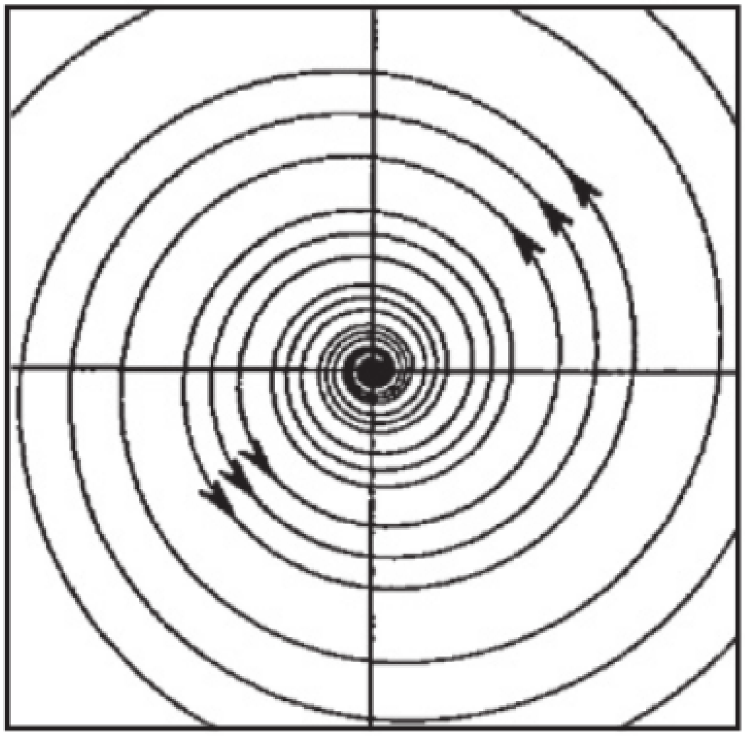

Figure 1:

iv.

a.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 \implies \lambda = 2$$

For $\lambda = 2$, the eigenvector is $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b.

Thus

$$T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

c.

First, let's look at the solution of

$$X' = AX$$

Based on the eigenvector the general solution is

$$X(t) = c_1 e^{2t} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + c_2 \left(te^{2t} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

and we have

$$T^{-1}AT = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

which has eigenvalue 2 with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, thus

$$Y(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \left(te^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

d.

Portrait graphs for both system, (graph is taken from book and is wrong, to make it right, change the direction of the arrow since it is a source instead of sink as the eigenvalue is positive).

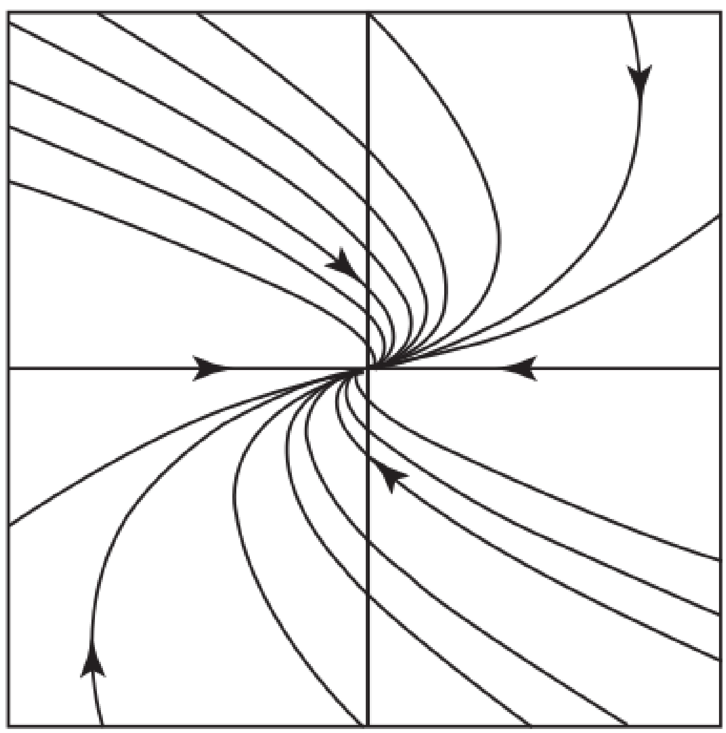

Figure 2:

v.

a.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 4 \implies \lambda = -1 \pm \sqrt{3}$$

For $\lambda = -1 - \sqrt{3}$, the eigenvector is $\begin{pmatrix} -2+\sqrt{3} \\ 1 \end{pmatrix}$.

For $\lambda = -1 + \sqrt{3}$, the eigenvector is $\begin{pmatrix} -2-\sqrt{3} \\ 1 \end{pmatrix}$

b.

Thus

$$T = \begin{pmatrix} -2+\sqrt{3} & -2-\sqrt{3} \\ 1 & 1 \end{pmatrix}$$

c.

First, let's look at the solution of

$$X' = AX$$

Based on the eigenvector the general solution is

$$X(t) = c_1 e^{t(-1-\sqrt{3})} \begin{pmatrix} -2+\sqrt{3} \\ 1 \end{pmatrix} + c_2 e^{t(-1+\sqrt{3})} \begin{pmatrix} -2-\sqrt{3} \\ 1 \end{pmatrix}$$

and we have

$$T^{-1}AT = \begin{pmatrix} -1-\sqrt{3} & 0 \\ 0 & -1+\sqrt{3} \end{pmatrix}$$

which yields

$$Y(t) = c_1 e^{t(-1-\sqrt{3})} \begin{pmatrix} -1+\sqrt{3} \\ 1 \end{pmatrix} + c_2 e^{t(-1+\sqrt{3})} \begin{pmatrix} -1-\sqrt{3} \\ 1 \end{pmatrix}$$

d.

For both systems,

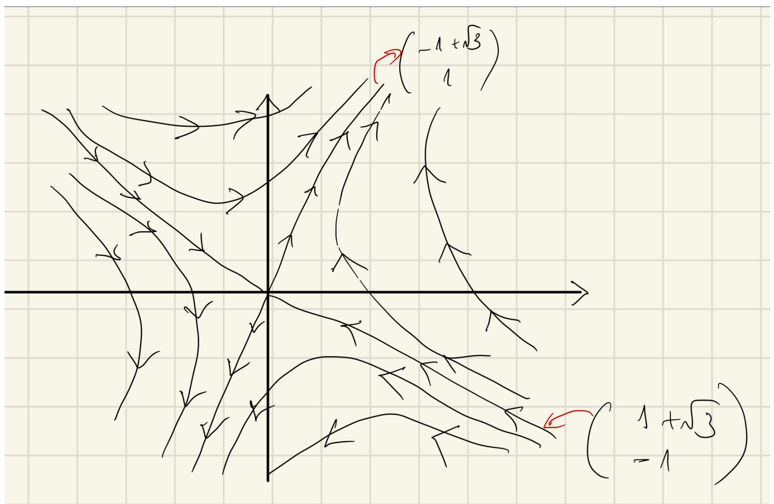
$\begin{pmatrix} -1+\sqrt{3} \\ 1 \end{pmatrix}$
$\begin{pmatrix} 1+\sqrt{3} \\ -1 \end{pmatrix}$

Figure 3:

3.3

a.

$$x'' + x' + x = 0$$

The charateristics equation is $r^2 + r + 1 = 0$ thus

$$r = \frac{-1 \pm \sqrt{3}i}{2}$$

and the general solution is

$$x(t) = c_1 e^{-t/2} \cos(\sqrt{3}t/2) + c_2 e^{-t/2} \sin(\sqrt{3}t/2)$$

b.

$$x'' + 2x' + x = 0$$

The charateristics equation is $r^2 + 2r + 1 = 0$ thus

$$r = -1$$

and the general solution is

$$x(t) = c_1 e^{-t} + c_2 t e^{-t}$$

3.5

The charateristics polynomial is

$$\lambda^2 - \lambda(2+a) = 0 \implies \lambda = 0, 2+a$$

For eigenvalue 0, the eigenvector is

$$\begin{pmatrix} -1 \\ a \end{pmatrix}$$

For eigenvalue $2+a$, the eigenvector is

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus the bifurcation point is $a = -2$.

- $a > -2$ As the eigenvector $(-1, a)$ is a constant in the solution.

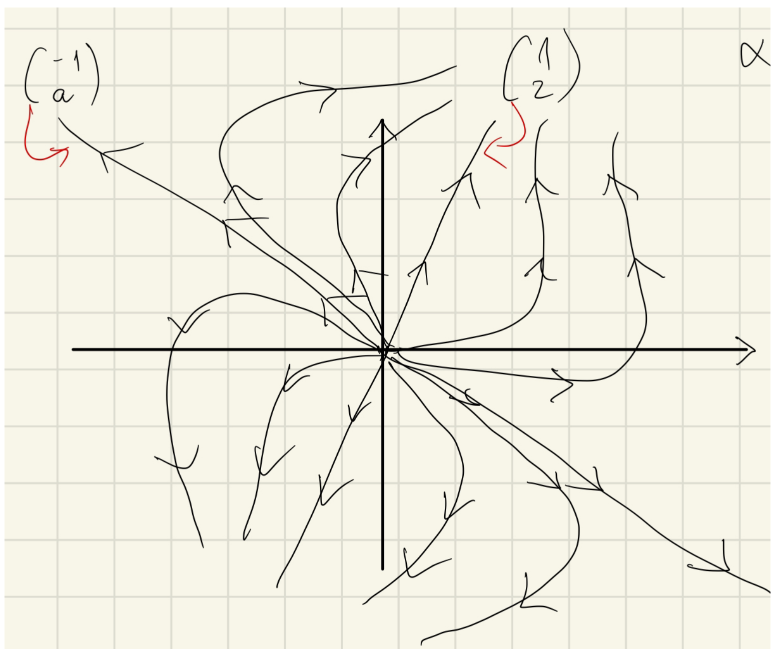

Figure 4:

- $a = -2$ Let $(1, 0)$ be the other eigenvector, then the solution is just $c_1(1,0) + c_2(1,2)$ which is just constant everywhere.

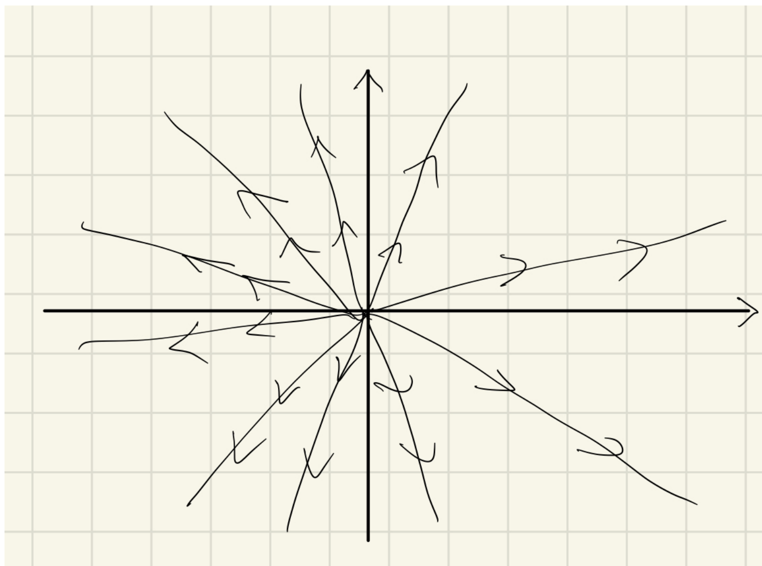

Figure 5:

- $a < -2$

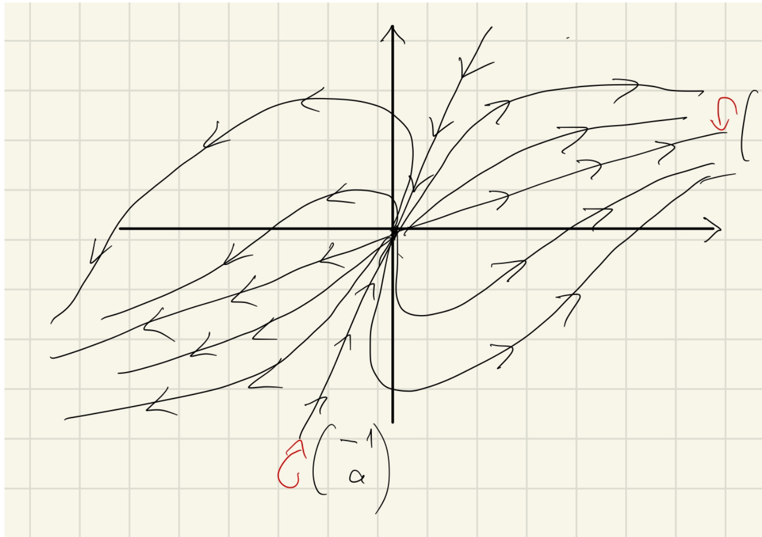

Figure 6:

3.11

Let A be the matrix in the equation

$$|A - \lambda I| = (a-\lambda)(d-\lambda) - bc = -\lambda(a+d) + \lambda^2 = 0$$

Thus

$$\lambda \in \{0, a+d\}$$

For $\lambda = 0$, the eigenvector will be $\begin{pmatrix} d \\ -c \end{pmatrix}$

For $\lambda = a+d$, the eigenvector will be $\begin{pmatrix} b \\ d \end{pmatrix}$ which we can use to obtain the general solution

$$X(t) = c_1 \begin{pmatrix} d \\ -c \end{pmatrix} + c_2 e^{a+d} \begin{pmatrix} b \\ d \end{pmatrix}$$

Since $c_1 \begin{pmatrix} d \\ -c \end{pmatrix}$ is constant, the curves should eventually converge to a line parallel with $\operatorname{sign}(c_2) \cdot \begin{pmatrix} b \\ d \end{pmatrix}$.

Thus, if $a+d > 0$, it is a source, and every curves should eventually converges to said vector.

If $a+d = 0$, then it is a source and every curves is a line.

If $a+d < 0$, then it is a sink and every curves should eventually converges to said vector.

3.13

Let the matrix be

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Since we know the charateristics polynomial is

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

We can substitute $\alpha = -a - d, \beta = ad - bc$ in and have

$$\begin{aligned}
&A^2 - (a+d)A + (ad-bc)I \\
=&\begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix} + (ad-bc)I \\
=&\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

4.1

$T = a,\ D = \dfrac{a^2}{4} - 2$ and the charateristics polynomial is

$$-a\lambda + \lambda^2 - 2 + \frac{a^2}{4} = 0$$

Thus, there is 2 real eigenvalue as

$$a^2 - 4\left(-2 + \frac{a^2}{4}\right) = 8 > 0$$

Now the eigenvalue are

$$\lambda_{1,2} = \frac{a \pm \sqrt{8}}{2}$$

where $\lambda_1 > \lambda_2$. Thus there are three section.

- $\lambda_1 > \lambda_2 > 0,\ a > \sqrt{8}$.
- $\lambda_1 > 0 > \lambda_2,\ -\sqrt{8} < a < \sqrt{8}$
- $0 > \lambda_1 > \lambda_2 > 0,\ a < -\sqrt{8}$

and 2 subsection

- $\lambda_1 > \lambda_2 = 0, a = \sqrt{8}$, then the solution has a constant vector and the all curves should converges to parallel with the other one.
- $\lambda_2 < \lambda_1 = 0, a = -\sqrt{8}$, then the solution has a constant vector and the all curves should converges to parallel with the other one.

Hence, the trace determinant plane is

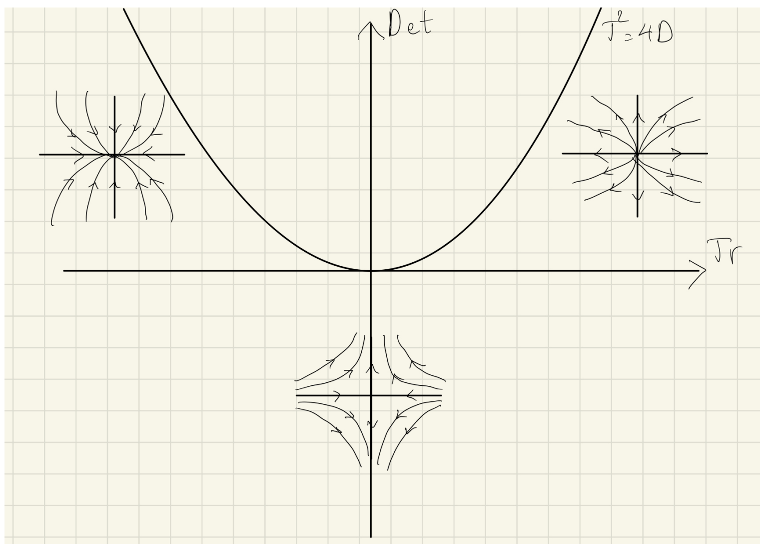

Figure 7:

4.2

$$T = 2a, D = a^2 - b^2$$

$$T^2 - 4D = 4a^2 - 4a^2 + 4b^2 = 4b^2 \geq 0$$

and the charateristics polynomial is

$$a^2 - 2a\lambda + \lambda^2 - b^2 = 0$$

Thus if $b = 0$, then there is only 1 eigenvalue which is a (which the corresponding eigenvector is $(1, 0), (0, 1)$.
If $b \neq 0$, then there is 2 eigenvalue

$$\lambda_{1,2} = a \pm |b|$$

If $b > 0$,

$$\lambda_1 = a + b > \lambda_2 = a - b$$

If $b < 0$,

$$\lambda_1 = a - b > \lambda_2 = a + b$$

The eigenvector for eigenvalue $a + b$ is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and the eigenvector for eigenvalue $a - b$ is

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus we have 4 regions in the ab plane

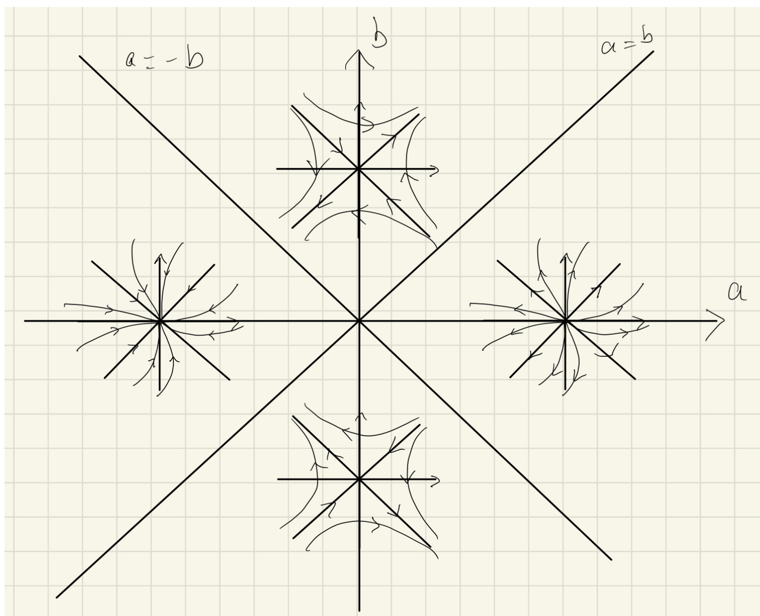

Figure 8:

4.3

The charateristics equation is $r^2 + br + k$ and thus have

- 2 real solutions if $b^2 > 4k$
  We know that $r_{1,2} = -b \pm \sqrt{b^2 - 4k}$, $r_1 = -b - \sqrt{b^2 - 4k} < 0$ and $r_2 = -b + \sqrt{b^2 - 4k} < 0$. Thus $b^2 > 4k$ has similar portraits.

- 1 real duplicated solution if $b = 2\sqrt{k}$, and. There is obviously 1 portraits here.

- 2 complex solution if $b^2 < 4k$. Since $b > 0$, the region $b^2 < 4k$ also has similar portraits as $b > 0$. The real parts of the solution of the charateristics polynomial is always positive.

4.5

a.

We first put them into canonical form.
The first matrix A has eigenvalue 2 with eigenvector

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and eigenvalue -1 with eigenvector

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus we can find the canonical form

$$T_1^{-1}AT = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

where

$$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

The second matrix B has eigenvalue -2 with eigenvector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and eigenvalue 1 with eigenvector

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Thus we can find the canonical form

$$T_2^{-1}AT_2 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

where

$$T_2 = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$$

Thus we can obtain the conjugacy

$$H(x,y) = T_2T_1^{-1}H'(x,y)$$

with H' being the conjugacy between the 2 canonical form of the 2 matrices and

$$H'(x,y) = (h_1(x), h_2(y))$$

where

$$h_1(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

and

$$h_2(y) = \begin{cases} y^2, & \text{if } y \geq 0 \\ -y^2, & \text{if } y < 0 \end{cases}$$

The reason this works is explained in first 6 lines in 4.6.

b.

We first put them into canonical form.
The first matrix A has eigenvalue $2i$ with eigenvector

$$\begin{pmatrix} -i \\ 2 \end{pmatrix}$$

and eigenvalue $-2i$ with eigenvector

$$\begin{pmatrix} i \\ 2 \end{pmatrix}$$

Thus we can find the canonical form

$$T_1^{-1} A T_1 = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}$$

where

$$T_1 = \begin{pmatrix} i & -i \\ 2 & 2 \end{pmatrix}$$

The second matrix B has eigenvalue $2i$ with eigenvector

$$\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

and eigenvalue $-2i$ with eigenvector

$$\begin{pmatrix} i \\ 1 \end{pmatrix}$$

Thus we can find the canonical form

$$T_2^{-1} A T_2 = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}$$

where

$$T_2 = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

Thus we can obtain the conjugacy

$$H(x, y) = T_2 \circ T_1^{-1}$$

since they have the same Jordan canonical form.

4.6

If 2 linear system $X' = AX$ and $Y' = BY$ have the same eigenvalue $\pm i\beta \neq 0$, then we know that both matrix have the same canonical form

$$C = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

Thus the conjugacy is $T_2 T_1^{-1}$ since T_1^{-1} send the solution of $X' = AX$ to $Z' = CZ$ and T_2 send the solution of $Z' = CZ$ to $Y' = BY$ (this can also be applied to 4.5).
If they have different eigenvalue $\pm i\beta$ and $\pm i\gamma$ then WLOG assume $|\gamma| > |\beta|$ then the solution of both consists of sin and cos with period $\frac{2\pi}{|\gamma|}$ or $\frac{2\pi}{|\beta|}$

$$\phi^A(t, X_0) = \phi^A(t + \frac{2\pi}{\beta}, X_0)$$

$$\phi^B(t, X_0) = \phi^B(t + \frac{2\pi}{\gamma}, X_0)$$

Thus there is no conjugacy.
$\gamma = -\beta$ means that the systems have the same eigenvalue which is the first scenario.