1.

Let $I = I_1 \times I_2 \times \ldots \times I_N, I_i = [a_i, b_i]$, from homework 3, we know that

$$\partial I = J_1 \cup J_2 \cup \ldots \cup J_N$$

where

$$J_j = I_1 \times \ldots \times \{a_j, b_j\} \times \ldots I_N$$

It is straight from definition that $\mu(J_j) = 0$ for all $j \in \{1, 2, \ldots, N\}$.
and as any subset F of $\{1, 2, \ldots, N\} : \mu(\bigcup_{j \in F} J_j) = 0$, we have that

$$\mu(\partial I) = \sum_{j=1}^{N} \mu(J_j) = 0$$

2.

If $\int_I f_i$ is integrable for all $i \in \{1, 2, \ldots, M\}$. Let $\int_I f_i = y_i$. Hence, $\forall i : \forall \epsilon > 0$, there exists a partition P_{ϵ_1} such that for all refinement of P_{ϵ_1}:

$$\|S(P, f) - y_i\| < \frac{\epsilon}{\sqrt{M}}$$

which means that

$$\|S(P, (f_1, f_2, \ldots, f_M)) - (y_1, y_2, \ldots, y_M)\| < \sqrt{\frac{\epsilon^2}{M} \cdot M} = \sqrt{\epsilon^2} = \epsilon$$

which is equivalent to

$$\|S(P, f) - y\| < \epsilon$$

and from what we got,

$$\int_I f = y = (y_1, y_2, \ldots, y_M) = \left(\int_I f_1, \int_I f_2, \ldots, \int_I f_M\right)$$

If $\int_I f_i$ is not integrable for some i, that means that there exists $\epsilon > 0$, for all partition P_i,

$$\|S(P_i, f_i)\| > \epsilon$$

Hence, for any partition $P = P_1 \times P_2 \times \ldots \times P_N$

$$\|S(P, f)\| \geq \|S(P_i, f_i) > \epsilon$$

3.

First, we will prove that if f is integrable on D, f^2 is also integrable on D. We have that f is bounded, that is $\forall x \in D : \|f(x)\| < M$

$$|(f(x))^2 - (f(y))^2| = |f(x) - f(y)||f(x) + f(y)| \leq 2M|f(x) - f(y)|$$

Since f is integrable on D, let $\int_D f = y = f(x_0)\mu(D)$ for some $x_0 \in D$ and $\forall \epsilon > 0$, there exists a partition P_ϵ such that for all refinement P of P_ϵ

$$\|S(P, f) - y\| < \epsilon$$

and hence

$$\mathcal{U}(P, f) - \mathcal{L}(P, f) < \frac{\epsilon}{2M}$$

Therefore, it is obvious that with x_1, x_2 be the maximum and minimum value in the subdivision, we have

$$\begin{aligned}
S(P, f^2) - U(P, f^2) &= \sum_v \mu(I_v)((f(x_1))^2 - (f(x_2))^2) \\
&= \sum_v \mu(I_v)(f(x_1) - f(x_2))(f(x_1) + f(x_2)) \\
&\leq 2M \cdot (\mathcal{U}(P, f) - \mathcal{L}(P, f) \\
&\leq 2M \cdot \frac{\epsilon}{2M} = \epsilon
\end{aligned}$$

Hence, f^2 is also integrable, and therefore, g^2 and $(f+g)^2$ are integrable. Consider a set $S \subset \mathbb{R}$, where $\mu(S) = 10$ and the function

$$f : S \to \mathbb{R}, \qquad x \to 1$$

We have that

$$\left(\int_S f\right)^2 = (1 \cdot \mu(S))^2 = \mu(S)^2 \neq \mu(S) = \mu(S) \cdot 1^2 = \int_S f^2$$

Therefore,

$$\int_D fg = \frac{1}{2} \int_D \left((f+g)^2 - f^2 - g^2\right)$$

is also integrable but

$$\int_D fg \neq \left(\int_D f\right)\left(\int_D g\right)$$

4.

Since f is bounded, $\forall x \in D : \exists M \in \mathbb{R} : \|f(x)\| < M$.
Since D has content zero, $\forall \epsilon > 0$: for all compact interval $I_1, \ldots, I_n \in R^M$ satisfies

$$D \subset \bigcup_{j=1}^{n} I_j \text{ and } \sum_{j=1}^{n} \mu(I_j) < \frac{\epsilon}{M}$$

and hence

$$S(P, f) = \sum_{v} f(x_v)\mu(I_v) \leq \frac{\epsilon}{M} \cdot M = \epsilon$$

which means that

$$\int_D f = 0$$

5.

If $\exists x_0 \in U : f(x_0) = \delta > 0$, then since f is continuous, we have $\exists \epsilon > 0 : B_\epsilon(x_0) \in U : \forall x \in B_\epsilon(x_0) : f(x) > \frac{\delta}{2}$
then

$$\int_{B_\epsilon(x_0)} f > \frac{\delta}{2} \cdot \mu(B_\epsilon(x_0)) > 0$$

We also know that since $\forall x \in U : f(x) \geq 0$

$$\int_{U \setminus B_\epsilon(x_0)} f \geq 0$$

and hence

$$\int_U f = \int_{B_\epsilon(x_0)} f + \int_{U \setminus B_\epsilon(x_0)} f > 0$$

which is a contradiction. Therefore, $f \equiv 0$ on U

6.

Suppose f is not bounded, we have that $\exists (x_n) \in I : \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \|f(x_n)\| = \infty$.
Hence, for all partition P:

$$\exists v_0 : \exists (x'_n) \in I_{v_0} : \lim_{n\to\infty} x'_n = x \text{ and } \lim_{n\to\infty} \|f(x'_n)\| = \infty$$

Therefore, $\forall M > 0 : \forall y \in \mathbb{R}^N : \exists n_0 : \|f(x_{n_0})\| > \dfrac{M + \|y\|}{\mu(I_{v_0})}$ and hence

$$\begin{aligned}
\|S(P, f) - y\| &= \left\| \sum_v \mu(I_v) \cdot f(x_{n_0}) - y \right\| \\
&\geq \left\| \sum_v \mu(I_v) \cdot f(x_{n_0}) \right\| - \|y\| \\
&= \sum_v \mu(I_v) \cdot \|f(x_{n_0})\| - \|y\| \\
&\geq \frac{M + \|y\|}{\mu(I_{v_0})} \cdot \mu(I_{v_0}) - \|y\| = M
\end{aligned}$$

which means that $S(P, f)$ diverges and hence the riemann sum does not exists, which is a contradiction.
Therefore, f is bounded