

**1.**

**1.**

If  $x \in \liminf_{n \rightarrow \infty} E_n$  then, there exists  $j \in \mathbb{N}$  such that  $\forall k \geq j, x \in E_k$  which means that  $x \in \cup_{k=j}^{\infty} E_k \subseteq \cup_{k=j-1}^{\infty} E_k \subseteq \dots \subseteq \cup_{k=1}^{\infty} E_k$  thus  $x \in \cap_{j=1}^{\infty} \cup_{k=j}^{\infty} E_k =: \limsup_{n \rightarrow \infty} E_n$ .

**2.**

$x \in \liminf_{n \rightarrow \infty} (E_n \cap F_n) \iff$  there exists  $j \in \mathbb{N}$  such that  $\forall k \geq j, x \in E_k \cap F_k$  which is equivalent to  $x \in E_k$  and  $x \in F_k \iff x \in \liminf_{n \rightarrow \infty} E_n$  and  $x \in \liminf_{n \rightarrow \infty} F_n \iff x \in \liminf_{n \rightarrow \infty} E_n \cap \liminf_{n \rightarrow \infty} F_n$

**3.**

Base case:  $n=1$

$$\cup_{j=1}^1 E_j := E_1 = E_1 \setminus \underbrace{\cup_{k=1}^0 E_k}_{\emptyset} := F_1$$

Inductive steps: if the equation holds for  $n$  then it also holds for  $n+1$ .

$$\cup_{j=1}^{n+1} E_j = \cup_{j=1}^n E_j \cup E_{n+1} = \cup_{j=1}^n F_j \cup E_{n+1} \stackrel{(1)}{=} \cup_{j=1}^n F_j \cup \underbrace{E_{n+1} \setminus (\cup_{k=1}^n E_k)}_{F_{n+1}} = \cup_{j=1}^{n+1} F_j$$

$$\text{as } \cup_{k=1}^n E_k = \cup_{k=1}^n F_k \implies E_{n+1} \setminus (\cup_{k=1}^n E_k) = F_{n+1}.$$

## 2.

Assume  $X$  is non-empty, then there exists  $z_0 \in X$ , thus fixing  $z_0 = (x_0, y_0)$ , we can define an equivalence class,  $\xi_r$  as follows:  $p$  is in the class  $\xi_r$  if

$$|p - x_0| = r$$

where  $r$  is some rational number so that the set containing all classes  $\xi_r$  is countable. Next, define the function

$$f_r : [0, 2\pi] \rightarrow \mathbb{R}^2, \quad t \rightarrow (x_0 + r \cos(t), x_0 + r \sin(t))$$

such that  $\xi_r \subseteq \text{Img}(f_r)$ .

WLOG, assume that  $f_r(0) \in X$ , then as the function  $g(\theta) = |f_r(\theta) - f_r(0)|$  is a continuous function that strictly increasing on  $[0, \pi]$  from 0 to  $2r$  and then strictly decreasing on  $[\pi, 2\pi]$  back to 0, there is a bijective function that maps  $[0, \pi] \rightarrow [0, 2r]$  and  $[\pi, 2\pi] \rightarrow [0, 2r]$ . Thus, the set of all points in each  $\xi_r$  is countable because  $\mathbb{Q}_{[0, 2r]}$  is countable. And since the set containing all classes  $\xi_r$  is also countable,  $X$  is countable.

### 3.

#### 1.

Since  $A \subseteq A \cup B$ ,  $\text{card}(A) \leq \text{card}(A \cup B)$ . Now suppose  $A$  is countable, then we can write  $A$  and  $B$  as  $\{a_1, a_2, \dots\}$  and  $\{b_1, b_2, \dots\}$ . Therefore, it is possible to construct a bijection function between  $A$  and  $A \cup B$

$$\phi(a_i) = \begin{cases} b_{i/2}, & \text{if } i \text{ is even} \\ a_{(i+1)/2}, & \text{if } i \text{ is odd} \end{cases}$$

and thus having the same cardinality. We can extend that to the case where  $A$  is not countable as there is an infinite countable subset  $\{a_1, a_2, \dots\} = \tilde{A} \subset A$  and therefore we can construct a bijection between  $A$  and  $A \cup B$  based on the previous bijection.

$$\psi(a) = \begin{cases} \phi(a), & \text{if } a \in \tilde{A} \\ a, & \text{if } a \in A \setminus \tilde{A} \end{cases}$$

where in the case  $a \in \tilde{A}$ , there is  $a_i$  such that  $a = a_i$

$$\phi(a) = \phi(a_i) = \begin{cases} b_{i/2}, & \text{if } i \text{ is even} \\ a_{(i+1)/2}, & \text{if } i \text{ is odd} \end{cases}$$

#### 2.

For every  $x \in E$ , there exists  $\delta_x > 0$  such that  $(x - \delta_x, x) \cap B = \emptyset$ , and since  $\mathbb{Q}$  is dense, we can create a function  $f$  that maps  $x$  to a rational number in  $(x - \delta_x, x)$ . We claim that  $f$  is injective.

For any  $x_1 \neq x_2 \in E$ , if  $f(x_1) = f(x_2)$ , then  $f(x_1) \in (x_1 - \delta_{x_1}, x_1)$  and  $f(x_2) \in (x_2 - \delta_{x_2}, x_2)$ . But we have  $(x_1 - \delta_{x_1}, x_1) \cap (x_2 - \delta_{x_2}, x_2) = \emptyset$  else  $B \ni x_1 \in (x_2 - \delta_{x_2}, x_2)$  or  $B \ni x_2 \in (x_1 - \delta_{x_1}, x_1)$ . Therefore,  $f(x_1) \neq f(x_2)$  and hence a contradiction. Thus  $f$  is injective.

4.

a.

For any set  $E_k = \{x \in E : x \geq \frac{1}{k}\}$ , we can see that  $\sum_{x \in E} x \geq \sum_{x \in E_k} x \geq \frac{j}{k}$  where  $j$  is the number of elements in  $E_k$ . Thus as  $\sum_{x \in E} x < \infty$ , there is finite countable elements in  $E_k$ . We also have that

$$E = \bigcup_{k \in \mathbb{N}} E_k$$

Thus  $E$  is at most countable.

b.

Let  $E_k = \{x_i : i \leq k\}$  and since every element in  $E$  is positive, the series strictly increasing and  $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i = \infty$  or  $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i = L$  for some  $L > 0 \in \mathbb{R}$ .

If  $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i = \infty$ , then for every  $M > 0$ , there exists  $k_0$  such that for every  $k > k_0$ ,  $\sum_{x \in E_k} x = \sum_{i=1}^k x_i > M$  thus  $\sup_{F \in \mathcal{F}} s_F = \sum_{x \in E} x = \infty = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i$ .

If  $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i = L$ , then we consider two cases

- If  $\sup_{F \in \mathcal{F}} s_F > L$ , then there is a subset  $F$  such that  $s_F \geq L$  and since  $E$  is infinitely countable, we can let  $F' = F \cup \{x_0\}$  so that  $s_{F'} > L$ . Since  $F'$  is finite, and the mapping which we called  $f$  is bijective. We can find the largest index  $k = \max\{f^{-1}(x) : x \in F'\}$ , and thus  $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i > \sum_{i=1}^k x_i > s_{F'} > L$  which is a contradiction.
- If  $\sup_{F \in \mathcal{F}} s_F < L$ , then there exists an  $\epsilon > 0$  so that for every  $F \in \mathcal{F}$ ,  $L - s_F > \epsilon$  but this contradicts with  $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i = L$  as for every  $\epsilon > 0$  there exists  $k_0$  such that for every  $k > k_0$ ,  $L - \underbrace{\sum_{i=1}^k x_i}_{s_{E_k}} < \epsilon$ .

Therefore,  $\sup_{F \in \mathcal{F}} s_F = L = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i$

## 5.

1.  $\emptyset$  is countable.

2. If  $E \subseteq S$ , then either  $E$  or  $E^c$  is countable, therefore  $E^c \subseteq S$ .

3. If  $E_k \subseteq S$  are all countable then  $\bigcup_{n=1}^{\infty} E_n \in S$ . If one or more of  $E_k$  are not countable then  $\bigcap_{n=1}^{\infty} E_n$  is countable thus  $\bigcup_{n=1}^{\infty} E_n \in S$

Therefore,  $S$  is a  $\sigma$ -algebra.

Since every singleton is contained in  $S$  as it is countable and every countable set in  $S$  is a union of singleton,  $\sigma(P) \subseteq S$  and every countable set is in  $\sigma(P)$ .

Therefore, for every  $E \in S$ ,

- if  $E$  is countable then  $E \in \sigma(P)$ .
- if  $E^c$  is countable, then  $X \setminus E^c = E \in \sigma(P)$

Thus,  $S \subseteq \sigma(P)$  and therefore  $S = \sigma(P)$