1.

Let $A = \{(x, y, z) \in \mathbb{R}^3 : r^2 \leq x^2 + z^2 \leq R^2, |y| \in [\epsilon, 1]\}$.

The set is not open: Consider $m = \left(0, \dfrac{\epsilon+1}{2}, r\right)$, then

$$\forall \epsilon' > 0 : \exists \delta : \epsilon' > \delta > 0 : \left(0, \frac{\epsilon+1}{2}, R+\delta\right) \in B_{\epsilon'}(m)$$

But $\left(0, \dfrac{\epsilon+1}{2}, R+\delta\right) \notin A$ because $x^2 + z^2 \geq R^2$.

The set in closed:

From the homework 3:

$\{(x, z) \in \mathbb{R}^2 : r \leq \|(x, z)\| \leq R\} = \{(x, z) \in \mathbb{R}^2 : r^2 \leq x^2 + z^2 \leq R^2\}$ is closed. $[\epsilon, 1], [-1, -\epsilon]$ are closed, therefore its union is also closed and therefore, $\{y \in \mathbb{R} : |y| \in [\epsilon, 1]\}$ is closed. Hence, A is closed.

The set is bounded: $-R \leq x, z \leq R, -1 \leq y \leq 1$, hence compact. The set is not connected:

$$\begin{aligned} A = &\{(x, y, z) \in \mathbb{R}^3 : r^2 \leq x^2 + z^2 \leq R^2, |y| \in [\epsilon, 1]\} = \\ &\{(x, y, z) \in \mathbb{R}^3 : r^2 \leq x^2 + z^2 \leq R^2, y \in [\epsilon, 1]\} \cup \\ &\{(x, y, z) \in \mathbb{R}^3 : r^2 \leq x^2 + z^2 \leq R^2, y \in [-1, -\epsilon]\}\}. \end{aligned}$$

Consider

$U = \{(x, y, z) \in \mathbb{R}^3 : y > 0\} \supset \{(x, y, z) \in \mathbb{R}^3 : r^2 \leq x^2 + z^2 \leq R^2, y \in [\epsilon, 1]\}$

and

$V = \{(x, y, z) \in \mathbb{R}^3 : y < 0\} \supset \{(x, y, z) \in \mathbb{R}^3 : r^2 \leq x^2 + z^2 \leq R^2, y \in [-1, -\epsilon]\}$

$(0, \infty), (-\infty, 0), \mathbb{R}^2$ are open, hence U and V are also open, but

$$U \cap A \neq \varnothing \neq V \cap A$$

$$(U \cap A) \cap (V \cap A) = \varnothing$$

$$(U \cap A) \cup (V \cap A) = A$$

2.

Definition of a path connected set: Let $C \subset \mathbb{R}^N$. We say that $x_0, x_1 \in C$ can be joined by a path if there is a continuous function $\gamma : [0,1] \to \mathbb{R}^N$ with $\gamma([0,1]) \subset C, \gamma(0) = x_0$, and $\gamma(1) = x_1$. We call C path connected if any two points in C can be joined by a path.
First, we prove that every path connected set is connected. Suppose S is not a connected set, then $\exists U, V$ open such that

$$U \cap S \neq \varnothing \neq V \cap S \tag{1}$$
$$(U \cap S) \cap (V \cap S) = \varnothing \tag{2}$$
$$(U \cap S) \cup (V \cap S) = S \tag{3}$$

then from (1), $\exists x \in U \cap S \wedge \exists y \in V \cap S$. Then as a path connected set, there exists a continuous function $\gamma : [0,1] \to S$ such that $\gamma(0) = x, \gamma(1) = y$.
Then, as [0,1] is connected, img(γ) is also connected. However, we have that

$$(1) \implies U \cap \operatorname{img}(\gamma) \neq \varnothing \neq \varnothing = V \cap \operatorname{img}(\gamma)$$
$$(2) \wedge \operatorname{img}(\gamma) \subset S \implies (U \cap \operatorname{img}(\gamma)) \cap (V \cap \operatorname{img}(\gamma)) = \varnothing$$
$$(3) \wedge \operatorname{img}(\gamma) \subset S \implies (U \cap \operatorname{img}(\gamma)) \cup (V \cap \operatorname{img}(\gamma)) = \operatorname{img}(\gamma)$$

Therefore, img(γ) is not connected, which is a contradiction. Hence, if S is a path connected set then S is connected.
It is obvious that a stap-shaped set S is connected:
Firstly, $\exists x_0 : \forall x \in S \wedge t \in [0,1] : tx_0 + (1-t)x \in S$
Then, $\forall x, y \in S$, we can construct a path as follows:

$$\gamma_1 : [0,1] \to S, \qquad t_1 \to t_1 x_0 + (1-t)x$$
$$\gamma_2 : [0,1] \to S, \qquad t_2 \to t_2 y + (1-t_2)x_0$$
$$\gamma : [0,1] \to S, \qquad t \to \begin{cases} \gamma_1(2t) & \text{if } t \leq 1/2 \\ \gamma_2(2t-1) & \text{otherwise} \end{cases}$$

We have that γ_1, γ_2 is continuous and the functions mapping t to $2t$ and t to $2t-1$ are continuous. Also, $\lim_{t \to \frac{1}{2}^+} \gamma(t) = x_0 = \gamma(\frac{1}{2}) = \lim_{t \to \frac{1}{2}^-} \gamma(t)$. Hence, γ is continuous and it maps an arbitary point x to y, which means that stap-shaped sets is path connected and therefore connected.
Consider the set $A = \{(x,y) \in \mathbb{R}^2 : y = x \vee y = -x\}$. Then we have

$$\forall (a,b) \in A : \forall t \in \mathbb{R} : (0,0)t + (1-t)(a,b) = ((1-t)a, (1-t)b)$$

But since $(a,b) \in A : a = b \vee a = -b, (1-t)a = (1-t)b \vee (1-t)a = -(1-t)b$. Which means that $\forall t \in [0,1] \subset \mathbb{R} : (0,0)t + (1-t)(a,b) = ((1-t)a, (1-t)b) \in A$. Hence, the set is stap shaped. However, consider (1,1) and (-1,1),

$$\frac{1}{2} \cdot (1,1) + (1 - \frac{1}{2}) \cdot (-1,1) = (0,1) \notin A$$

Therefore, A is not convex but is stap shaped.

3.

Suppose C is connected. If $\overline{C}$ is not connected then, exists U, V such that

$$U \cap \overline{C} \neq \varnothing \neq V \cap \overline{C} \tag{4}$$

$$(U \cap \overline{C}) \cap (V \cap \overline{C}) = \varnothing \tag{5}$$

$$(U \cap \overline{C}) \cup (V \cap \overline{C}) = \overline{C} \tag{6}$$

Then it is obvious that as $U \cap C \subset U \cap \overline{C}$ and $V \cap C \subset V \cap \overline{C}$

$$(U \cap C) \cap (V \cap C) = \varnothing \tag{7}$$

$$(U \cap C) \cup (V \cap C) = C \tag{8}$$

Since, $U \cap \overline{C} \neq \varnothing$, take $x \in \overline{C} \cap U$, then $x \in U$ and $x \in \overline{C}$. Then $\forall \epsilon > 0$: $B_\epsilon(x) \cap C \neq \varnothing \implies U \cap C \neq \varnothing$. Hence $\overline{C}$ is connected.

4.

Proof: $x \in \overline{S} \implies$ there is a sequence $(x_n)_{n=1}^{\infty} \in S$ such that $x = \lim_{n\to\infty} x_n$
If $x \in S$ then it is obvious that there exists a sequence $(x)_{n=1}^{\infty}$ such that $x = \lim_{n\to\infty} x = \lim_{n\to\infty} x_n$.
If $x \in \partial S$ then since $\forall \epsilon > 0 : B_\epsilon(x) \cap S \neq \varnothing$. We can construct a sequnce: (x_n) where x_n is a random point in $B_{\frac{1}{n}}(x)$, which means $\forall \delta > 0 : \exists n_0 : \forall n > n_0 : \frac{1}{n} < \delta \implies \|x_n - x\| < \delta$. Therefore, x is the limit of the sequence.
If $x \notin \overline{S}$, x is not a cluster point of S. Then $\exists \epsilon : B_\epsilon \subset S^c$, which means that there don't exist a sequence in S such that its limit is x.

5.

Let S be the set. For every open cover $\{U_i : i \in I\}$ of S, since $x \in S$, $\exists i_1 \in I$ such that $x \in U_{i_0}$. Hence, $\exists \epsilon > 0 : B_\epsilon(x) \in U_{i_0}$. We also have that $x = \lim_{n\to\infty} x_n$, therefore $\exists n_0 \in \mathbb{N} : \forall m > m_0 : \|x - x_m\| < \epsilon \implies x_m \in B_\epsilon(x)$. $\forall m \leq m_0 : x_m \in S \implies \exists i_1, i_2, \ldots, i_{m_0} \in I : x_m \in U_{i_m}$.
Hence, $\{x_n : n \in \mathbb{N}\} \cup \{x\} \subset U_{i_0} \cup U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_m}$. Therefore, S is compact.

6.

Proof. $\forall N > 1 : \mathbb{R}^N \backslash \{0\}$ is connected.
Definition of a path connected set: Let $C \subset \mathbb{R}^N$. We say that $x_0, x_1 \in C$ can be joined by a path if there is a continuous function $\gamma : [0,1] \to \mathbb{R}^N$ with $\gamma([0,1]) \subset C, \gamma(0) = x_0$, and $\gamma(1) = x_1$. We call C path connected if any two points in C can be joined by a path.
First, we prove that every path connected set is connected (this is just a repeat of what proved in question 2). Suppose S is not a connected set, then $\exists U, V$ open such that

$$U \cap S \neq \varnothing \neq V \cap S \tag{9}$$

$$(U \cap S) \cap (V \cap S) = \varnothing \tag{10}$$

$$(U \cap S) \cup (V \cap S) = S \tag{11}$$

then from (9), $\exists x \in U \cap S \wedge \exists y \in V \cap S$. Then as a path connected set, there exists a continuous function $\gamma : [0,1] \to S$ such that $\gamma(0) = x, \gamma(1) = y$.
Then, as [0,1] is connected, $\mathrm{img}(\gamma)$ is also connected. However, we have that

$$(9) \implies U \cap \mathrm{img}(\gamma) \neq \varnothing \neq \varnothing = V \cap \mathrm{img}(\gamma)$$

$$(10) \wedge \mathrm{img}(\gamma) \subset S \implies (U \cap \mathrm{img}(\gamma)) \cap (V \cap \mathrm{img}(\gamma)) = \varnothing$$

$$(11) \wedge \mathrm{img}(\gamma) \subset S \implies (U \cap \mathrm{img}(\gamma)) \cup (V \cap \mathrm{img}(\gamma)) = \mathrm{img}(\gamma)$$

Therefore, $\mathrm{img}(\gamma)$ is not connected, which is a contradiction. Hence, if S is a path connected set then S is connected.
Next, we will prove that $\forall N > 1 : S = \mathbb{R}^N \backslash \{0\}$ is a path connected set. $\forall x, y \in S$
If $\nexists t \in [0,1] : tx + (1-t)y = 0$ then the function

$$\gamma : [0,1] \to S, \qquad t \to tx + (1-t)y$$

is continuous, $\gamma(0) = y, \gamma(1) = x$ and $tx + (1-t)y \neq 0 \forall t \in [0,1]$
Else if $\exists t_0 \in (0,1)$ (because $x, y \neq 0$) : $t_0 x + (1-t_0)y = 0$ which means that $y = \dfrac{t_0 x}{(t_0 - 1)}$.
Then let $\overrightarrow{u} = \overrightarrow{xy} = \dfrac{x}{t_0 - 1}$.
Based on $\overrightarrow{u}$, we can construct a basis for $\mathbb{R}^N \backslash \{0\}$. And because of $N > 1$, we have that $\exists \overrightarrow{v} \neq \overrightarrow{u}$ in that basis of $\mathbb{R}^N \backslash \{0\}$. Hence, $\overrightarrow{v} \neq 0$ and $\overrightarrow{u}$ is linearly dependent, which means that we can have a point $z = (v_1, v_2, \ldots, v_N) \neq 0$ such that $\{\overrightarrow{zx}, \overrightarrow{xy}\}$ is linearly indepndent and $\{\overrightarrow{zy}, \overrightarrow{xy}\}$ is linearly independent. Therefore, the two functions:

$$\gamma_1 : [0,1] \to S, \qquad t \to tz + (1-t)x$$

$$\gamma_2 : [0,1] \to S, \qquad t \to ty + (1-t)z$$

does not pass through 0 because else if:

$$
\begin{aligned}
&\exists t_1 \in (0,1)(\text{beacuse } x, z \neq 0) : t_1 z + (1-t_1)x = 0 \\
&\qquad\qquad\qquad\qquad \implies z = \frac{x(t_1-1)}{t_1} \\
&\qquad\qquad\qquad\qquad \implies \overrightarrow{xz} = \frac{-x}{t_1} = \frac{x}{t_0-1} \cdot \frac{-(t_0-1)}{t_1} = \frac{1-t_0}{t_1}\overrightarrow{xy} \text{ (contradiction)} \\
&\exists t_2 \in (0,1)(\text{beacuse } x, z \neq 0) : t_2 y + (1-t_2)z = 0 \\
&\qquad\qquad\qquad\qquad \implies z = \frac{yt_2}{t_2-1} \\
&\qquad\qquad\qquad\qquad \implies \overrightarrow{yz} = \frac{y}{t_2-1} = \frac{t_0 x}{(t_2-1)(t_0-1)} = \frac{t_0}{t_2-1}\overrightarrow{xy} \text{ (contradiction)}
\end{aligned} \tag{12}
$$

As a result, $\nexists t \in [0,1] : tz + (1-t)x = 0 \wedge ty + (1-t)z = 0$. Hence, we can construct a new function based on the two funcions γ_1, γ_2:

$$
\gamma_3 : [0,1] \to S, \qquad t \to \begin{cases} \gamma_1(2t) & \text{if } t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{if } t > \frac{1}{2} \end{cases}
$$

We have that γ_1, γ_2 is continuous and the functions mapping t to $2t$ and t to $2t-1$ are continuous. Also, $\lim_{t \to \frac{1}{2}^+} \gamma(t) = z = \gamma(\frac{1}{2}) = \lim_{t \to \frac{1}{2}^-} \gamma(t)$. Hence, γ_3 is continuous and $\gamma_3(0) = x, \gamma_3(1) = y$ means that every 2 point $x, y \in \mathbb{R}^N \backslash \{0\}$ for $N > 1$ is path connected, therefore $\mathbb{R}^N \backslash \{0\}$ for $N > 1$ is connected. $\square$