Contents

Chapter 1

HW1

1.1

Without loss of generality, we can assume $\dim(\text{Image}(g)) = 1$. Assume that g is only defined in the set $[-N, N] \times [-M, M]$ where N, M are arbitary. We have that

$$\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} g(x^*,y^*)\Delta F_{i,j}$$

where with $x_i = \frac{-(n-i)N+iN}{n} = \frac{2iN-nN}{n}$, $y_i = \frac{-(m-i)M+iM}{m} = \frac{2iM-mM}{m}$, we have

$$\Delta F_{i,j} = F(x_i, y_j) - F(x_{i-1}, y_j) - F(x_i, y_{j-1}) + F(x_{i-1}, y_{j-1})$$
$$= \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)dxdy$$

Hence,

$$\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} g(x^*,y^*) \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)dxdy$$

$$E[g(X,Y)] = \int_{[-N,N]\times[-M,M]} g(x,y)f(x,y)dxdy$$
$$= \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} g(x,y)f(x,y)dxdy$$

We know that g is uniformly continuous as it is continuous in a compact set, we have that with large enough n,m

$$|g(x,y) - g(x^*,y^*)| < \epsilon$$

Hence,

$$\left| \int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) - E[g(X,Y)] \right|$$

$$= \left| \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} (g(x,y) - g(x^*,y^*))f(x,y)dxdy \right|$$

$$< \left| \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} \epsilon f(x,y)dxdy \right|$$

$$= \epsilon \left| \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)dxdy \right|$$

$$= \epsilon$$

Since, ϵ is arbitary, $\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = E[g(X,Y)]$, and since N, M are arbitary,

$$\int_{\mathbb{R}^2} g(x,y)dF(x,y) = E[g(X,Y)]$$

1.2

Because $s_i > a_i$ for all i, $X(0)$ does not affect stationary distributions.

$$
\begin{aligned}
\pi(0) &= \left(1 + \frac{a_0}{s_1} + \frac{a_0 a_1}{s_1 s_2} + \dots \right)^{-1} \\
&= \left(1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 4 \cdot 5} + \dots \right)^{-1} \\
&= \left(1 + \sum_{i=1}^{\infty} \frac{2 \cdot i!}{(i+2)!} \right)^{-1} \\
&= \left(1 + \sum_{i=1}^{\infty} \frac{2}{(i+1)(i+2)} \right)^{-1} \\
&= \left(1 + 2 \sum_{i=1}^{\infty} \left(\frac{1}{i+1} - \frac{1}{i+2} \right) \right)^{-1} \\
&= \left(1 + 2 \left(\frac{1}{2} \right) \right)^{-1} \\
&= \frac{1}{2}
\end{aligned}
$$

Hence, we can calculate

$$
\pi(i) = \pi(0) \frac{a_0 a_1 \dots a_{i-1}}{s_1 s_2 \dots s_i} = \frac{1}{2} \cdot \frac{2 \cdot i!}{(i+2)!} = \frac{1}{(i+1)(i+2)}
$$

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1.3

1.3.1

$$f^{-1}(\varnothing) = \{x \in S : f(x) \in \varnothing\} = \varnothing$$

1.3.2

$$f^{-1}(B^C) = \{x \in S : f(x) \notin B\} = S\backslash\{x \in S : f(x) \in B\} = S\backslash(f^{-1}(B)) = f^{-1}(B)^C$$

1.3.3

$$f^{-1}\left(\bigcap_\beta B_\beta\right) = \{x \in S : f(x) \in \bigcap_\beta B_\beta\}$$
$$= \bigcap_\beta \{x \in S : f(x) \in B_\beta\}$$
$$= \bigcap_\beta f^{-1}(B_\beta)$$

1.3.4

$$f^{-1}\left(\bigcup_\beta B_\beta\right) = \{x \in S : f(x) \in \bigcup_\beta B_\beta\}$$
$$= \bigcup_\beta \{x \in S : f(x) \in B_\beta\}$$
$$= \bigcup_\beta f^{-1}(B_\beta)$$

1.4

Let $\{X_i\}_{i=1}^{N}$ be geometric distributions, which is also a filtration. We will use the geometric distributions to estimate a, where X is a $(2, 1/2)$-negative binomial.

$$\alpha_i = \frac{p(X_i)}{q(X_i)} = \frac{\binom{X_i+1}{X_i}\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{X_i-2}}{\frac{1}{2}\left(\frac{1}{2}\right)^{X_i-1}} = X_i + 1$$

and with

$$L_n = \prod_{i=1}^{n} \alpha_i = \prod_{i=1}^{n}(X_i + 1)$$

We have

$$
\begin{aligned}
&E[L_n g(X_1, X_2, \ldots, X_n)] \\
=&E[E[L_n|\mathcal{F}_n]g(X_1, X_2, \ldots, X_n)] \\
=&E\left[\prod_{i=1}^{n}\frac{p(X_i)}{q(X_i)}g(X_1, X_2, \ldots, X_n)\right] \\
=&\sum_{j_1,j_2,\ldots,j_n=1}^{\infty}\prod_{i=1}^{n}\frac{p(j_i)}{q(j_i)}g(j_1, j_2, \ldots, j_n)q(j_1)q(j_2)\ldots q(j_n) \\
=&\sum_{j_1,j_2,\ldots,j_n=1}^{\infty}g(j_1, j_2, \ldots, j_n)p(j_1)p(j_2)\ldots p(j_n) \\
=&E[g(Y_1, Y_2, \ldots, Y_n)]
\end{aligned}
$$

where Y_i is a $(2, 1/2)$-negative binomial. Therefore, to estimate $E[\sin(X\ln(X))]$, we calculate

$$g(Y_1, Y_2, \ldots, Y_n) = \frac{1}{N}\sum_{m=1}^{N}\sin(Y_m\ln(Y_m)) = \frac{1}{N}\prod_{i=1}^{n}(X_i+1)\sum_{m=1}^{N}\sin(X_m\ln(X_m))$$