1.

Let B be the event where committees can be formed without any restriction and C be the event where committes are formed with both Charlotte and Leonard.

$$n_B = \binom{6}{3} \cdot \binom{8}{5} \cdot \binom{3}{2} = 3360$$

$$n_C = \binom{6}{3} \cdot \underbrace{\binom{7}{4}}_{\substack{\text{Charlotte} \\ \text{is chosen}}} \cdot \underbrace{\binom{2}{1}}_{\substack{\text{Leonard} \\ \text{is chosen}}} = 1400$$

$$n_A = n_B - n_C = 1960$$

2.

a. Since A is the event where there is at least one ace chosen, $\overline{A}$ is the event where there is no aces chosen.

$$n_{\overline{A}} = \binom{48}{5} = 1712304$$

where 48 is the number of non-ace card in the deck. Also

$$n = \binom{52}{5} = 2598960$$

Therefore, the probability that there are no ace chosen is

$$P(\overline{A}) = \frac{n_{\overline{A}}}{n} = \frac{1712304}{2598960}$$

and the probability that there is at least one ace chosen is

$$P(A) = 1 - p(\overline{A}) = \frac{886656}{2598960}$$

b. Since we are choosing 5 cards with different denominations, we are choosing 5 denominations from the 13 denominations. For each denominations, we can choose a suit out of 4 suits. Therefore,

$$n_D = \binom{13}{5} \cdot \binom{4}{1}^5 = 1317888$$

Hence,

$$P(D) = \frac{n_D}{n} = \frac{1317888}{2598960}$$

c. The probability that a hand of 5 different denominations are dealt and 1 one of them is an ace is similar to part b with a minor change

$$P(A \cap D) = \frac{n_{A \cap D}}{n} = \frac{\overbrace{\binom{12}{4}}^{\text{Ace is chosen}} \cdot \binom{4}{1}^5}{\binom{52}{5}} = \frac{2112}{10829}$$

We have

$$P(A \cup D) = P(A) + P(D) - P(A \cap D) = \frac{35368}{54145}$$

3.

The number of ways the deck can be dealt is

$$n = \binom{9}{3 \quad 3 \quad 3} = 1680$$

Let T_i be the event where player i is dealt three of a kind.

$$n_{T_1 \cap T_2 \cap T_3} = \binom{3}{1 \quad 1 \quad 1} = 6$$

because there is 3 denominations and each player choose 1.

$$n_{\overline{T}_1 \cap T_2 \cap T_3} = n_{T_1 \cap \overline{T}_2 \cap T_3} = n_{T_1 \cap T_2 \cap \overline{T}_3} = 0$$

because if 2 player are dealt three of a kind then the rest 3 cards are of the same denomination, which means it is impossible for the third player not being dealt three of a kind.

The number of ways player 1 is dealt three of a kind is

$$n_{T_1} = \underbrace{3 \cdot \binom{3}{3 \quad 0 \quad 0}}_{\substack{\text{There is 3 different} \\ \text{denominations player 1 can get}}} \cdot \underbrace{\binom{6}{0 \quad 3 \quad 3}}_{\text{Player 2 and 3}} = 3 \cdot 1 \cdot 20 = 60$$

$$\begin{aligned} n_{T_1 \cap \overline{T}_2 \cap \overline{T}_3} &= n_{T_1} - n_{T_1 \cap \overline{T}_2 \cap T_3} - n_{T_1 \cap T_2 \cap \overline{T}_3} - n_{T_1 \cap T_2 \cap T_3} \\ &= 60 - 0 - 0 - 6 = 54 \end{aligned}$$

WLOG,

$$n_{\overline{T}_1 \cap T_2 \cap \overline{T}_3} = n_{\overline{T}_1 \cap \overline{T}_2 \cap T_3} = 54$$

Therefore, the total number of ways one player is dealt three of a kind is

$$\begin{aligned} n_A &= n_{T_1 \cap \overline{T}_2 \cap \overline{T}_3} + n_{\overline{T}_1 \cap T_2 \cap \overline{T}_3} + n_{\overline{T}_1 \cap \overline{T}_2 \cap T_3} \\ &+ n_{\overline{T}_1 \cap T_2 \cap T_3} + n_{T_1 \cap \overline{T}_2 \cap T_3} + n_{T_1 \cap T_2 \cap \overline{T}_3} \\ &+ n_{T_1 \cap T_2 \cap T_3} \\ &= 54 + 54 + 54 + 0 + 0 + 0 + 6 = 168 \end{aligned}$$

Hence the probability is

$$P(A) = \frac{168}{1680} = \frac{1}{10}$$

4.

Let A, B respectively be the event where the product is manufacture from machine A and B.
Let D be the event where the product is defective. We have

$$P(B|D) = 0.65 \implies P(A|D) = 1 - 0.65 = 0.35$$

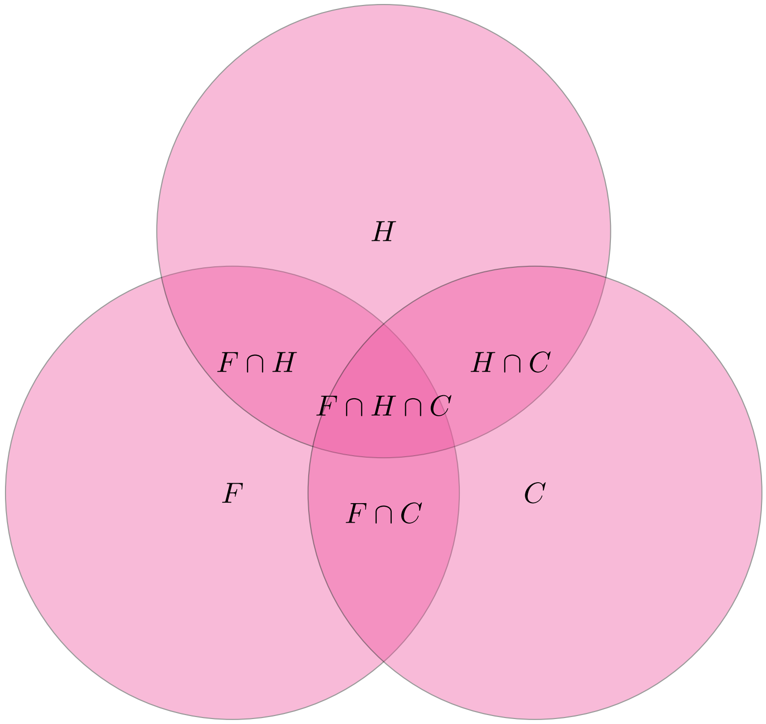
and

$$\begin{aligned} P(D|A) &= 3 \cdot P(D|B) \\ \implies P(A|D) \cdot \frac{P(D)}{P(A)} &= 3 \cdot P(B|D) \cdot \frac{P(D)}{P(B)} \\ \implies 0.35 \cdot \frac{1}{P(A)} &= 3 \cdot 0.65 \cdot \frac{1}{P(B)} \\ \implies P(A) &= \frac{7}{39} P(B) \end{aligned}$$

We also have

$$\begin{aligned} P(A) + P(B) &= 1 \\ \implies \frac{7}{39} P(B) + P(B) &= 1 \\ \implies P(B) = \frac{39}{46} &= 84.783\% \end{aligned}$$

5.



We know that $P(F) = 0.25, P(H) = 0.4, P(C) = 0.3, P(F \cap H) = 0.15, P(H \cap C) = 0.2, P(F \cap C) = 0.1, P(F \cap H \cap C) = 0.05$
From the diagram, we can see that
$P(F \cap \overline{H} \cap C) = P(F \cap C) - P(F \cap C \cap H) = 0.1 - 0.05 = 0.05$
$P(\overline{F} \cap H \cap C) = P(H \cap C) - P(F \cap C \cap H) = 0.2 - 0.05 = 0.15$
$P(F \cap H \cap \overline{C}) = P(F \cap H) - P(F \cap C \cap H) = 0.15 - 0.05 = 0.1$

$$\begin{aligned} P(F \cap \overline{H} \cap \overline{C}) &= P(F) - P(F \cap H \cap \overline{C}) - P(F \cap \overline{H} \cap C) - P(F \cap H \cap C) \\ &= 0.25 - 0.1 - 0.05 - 0.05 = 0.05 \end{aligned}$$

$$\begin{aligned} P(\overline{F} \cap \overline{H} \cap C) &= P(C) - P(F \cap \overline{H} \cap C) - P(\overline{F} \cap H \cap C) - P(F \cap H \cap C) \\ &= 0.3 - 0.05 - 0.15 - 0.05 = 0.05 \end{aligned}$$

$$\begin{aligned} P(\overline{F} \cap H \cap \overline{C}) &= P(H) - P(F \cap H \cap \overline{C}) - P(\overline{F} \cap H \cap C) - P(F \cap H \cap C) \\ &= 0.4 - 0.1 - 0.15 - 0.05 = 0.1 \end{aligned}$$

Therefore, the proportion of individuals watch exactly one of the sports is $0.05 + 0.05 + 0.1 = 0.2 = 20\%$

6.

a. For the game to end at exactly 10-th turn, the winner has to win 8 times and loses 2 times, the the loser loses 8 times and win 2 times. Also, the loser must not go bankrupt before round 10 and the loser must lose the 10-th round.

First, we consider the first 6 rounds, if the winner win all 6 round, the loser goes bankrupt which means the game ends at the 6-th round, therefore, the loser must win at least 1 game, and at most 2 game.

If the loser win 1 game, then at the end of round 6, the loser has 2 dollars, and the loser must win on round 7 or 8 so that the game does not end because if the loser lose both round 7 and 8 he goes bankrupt at round 8. Hence, the number of ways the game ends at round 10 if the loser win 1 game in the first 6 round is

$$\binom{6}{1} \cdot \binom{2}{1} = 12$$

If the loser win 2 game before round 6, then the loser is safe to not go bankrupt before round 10. Hence the probability the game ends at round 10 if the loser win 2 games in the first 6 round is

$$\binom{6}{2} = 15$$

Therefore, the probability the game end in exactly 10 round is

$$2 \cdot (15 + 12) \cdot \left(\frac{1}{2}\right)^8 \cdot \left(\frac{1}{2}\right)^2 = \frac{54}{1024}$$

as any player can be the winner.

b. Consider a one player game where the player has i dollars where i is less than or equal to 5, each round, the player has a probability of p winning 1 dollar and if he does not win, the player loses 1 dollar. The player win the game when he reached 5 dollars and loses when he go bankrupt, that is he reached 0 dollar. Let P(i) be the possibility that the player win the game when he starts with i dollars.

Then we have the following:

$$
\begin{aligned}
& P(0) = 0, P(5) = 1 \\
& P(1) = (1-p)P(0) + pP(2) = pP(2) \\
& P(4) = pP(5) + (1-p)P(3) = p + (1-p)P(3) \\
& P(3) = (1-p)P(4) + pP(2) = (1-p)(p + (1-p)P(3)) + pP(2) \\
\implies & P(3) = P(2)(1-p) + p^2 + (p-p^2)P(3) \\
\implies & P(3)(1-p+p^2) = P(2)(1-p) + p^2 \\
\implies & P(3) = \frac{P(2)(1-p)+p^2}{1-p+p^2} \\
& P(2) = (1-p)P(1) + pP(3) = (1-p)pP(2) + p \cdot \frac{P(2)(1-p)+p^2}{p^2-p+1} \\
\implies & P(2)\left(1-p+p^2+\frac{p^2-p}{p^2-p+1}\right) = \frac{p^3}{p^2-p+1} \\
\implies & P(2)\left(\frac{(p^2-p+1)^2+p^2-p}{p^2-p+1}\right) = \frac{p^3}{p^2-p+1} \\
\implies & P(2) = \frac{p^3}{(p^2-p+1)^2+p^2-p} \\
\implies & P(2) = \frac{p^3}{p^4-2p^3+4p^2-3p+1}
\end{aligned}
$$

P(2) is the possibility that the player starts from 2 and then reaches 5 dollars without going bankrupt, which means that the possibility player A reaches 5 dollars without going bankrupt is the same, and since when player A reaches 5 dollars, player B has 0 dollar and go bankrupt. The possibility player A wins the game is similar to P(2)

7.

Let D_i be the event where i is rolled and Q be the event where both the applicants are qualified

$$P(D_4 \cup D_5 \cup D_6) = \frac{3}{6} = \frac{1}{2}$$

$$P(Q|D_4 \cup D_5 \cup D_6) = \frac{8}{11} \cdot \frac{7}{11} = \frac{56}{121}$$

$$P(D_1 \cup D_2) = \frac{2}{6} = \frac{1}{3}$$

$$P(Q|D_1 \cup D_2) = \frac{8}{11} \cdot \frac{7}{10} = \frac{28}{55}$$

$$P(D_3) = \frac{1}{6}$$

$$P(Q|D_3) = \frac{7}{11} \cdot \frac{6}{10} = \frac{21}{55}$$

$$\begin{aligned} P(Q) &= P(Q \cap (D_1 \cup D_2)) + P(Q \cap D_3) + P(Q \cap (D_4 \cup D_5 \cup D_6)) \\ &= P(Q|D_1 \cup D_2)P(D_1 \cup D_2) + P(Q|D_1 \cup D_3)P(D_3) \\ &+ (Q|D_4 \cup D_5 \cup D_6)P(D_4 \cup D_5 \cup D_6) \\ &= \frac{1}{2} \cdot \frac{56}{121} + \frac{1}{6} \cdot \frac{21}{55} + \frac{1}{3} \cdot \frac{28}{55} \\ &= \frac{1687}{3630} \end{aligned}$$

$$\begin{aligned} P(D_4 \cup D_5 \cup D_6|Q) &= \frac{P(D_4 \cup D_5 \cup D_6)}{P(Q)} \cdot P(Q|D_4 \cup D_5 \cup D_6) \\ &= \frac{\frac{1}{2}}{\frac{1687}{3630}} \cdot \frac{56}{121} = \frac{120}{241} \end{aligned}$$

8.

Let F_i be the event where the product is defective by factor i and T_i be the event where the test for the i factor is positive. We have

$P(F_1) = \frac{1}{36}, P(F_2) = \frac{8}{36}, P(F_3) = \frac{27}{36}$

$P(T_1|F_1) = \frac{1}{5}, P(T_2|F_2) = \frac{2}{5}, P(T_3|F_3) = \frac{3}{5}$

$P(\overline{T}_m|F_n) = \begin{cases} 1 \text{ if } m \neq n \\ \frac{5-m}{5} \text{ otherwise} \end{cases}$

a.

$$\begin{aligned}
P(\overline{T}_1) &= P(\overline{T}_1 \cap F_1) + P(\overline{T}_1 \cap F_2) + P(\overline{T}_1 \cap F_3) \\
&= P(\overline{T}_1|F_1)P(F_1) + P(\overline{T}_1|F_2)P(F_2) + P(\overline{T}_1|F_3)P(F_3) \\
&= \frac{4}{5} \cdot \frac{1}{36} + 1 \cdot \frac{8}{36} + 1 \cdot \frac{27}{36} = \frac{179}{180} \\
P(F_1|\overline{T}_1) &= P(\overline{T}_1|F_1) \cdot \frac{P(F_1)}{P(\overline{T}_1)} \\
&= \frac{4}{5} \cdot \frac{\frac{1}{36}}{\frac{179}{180}} = \frac{4}{179}
\end{aligned}$$

b.

$$\begin{aligned}
P(\overline{T}_2) &= P(\overline{T}_2 \cap F_1) + P(\overline{T}_2 \cap F_2) + P(\overline{T}_2 \cap F_3) \\
&= P(\overline{T}_2|F_1)P(F_1) + P(\overline{T}_2|F_2)P(F_2) + P(\overline{T}_2|F_3)P(F_3) \\
&= 1 \cdot \frac{1}{36} + \frac{3}{5} \cdot \frac{8}{36} + 1 \cdot \frac{27}{36} = \frac{41}{45} \\
P(F_3|\overline{T}_2) &= P(\overline{T}_2|F_3) \cdot \frac{P(F_3)}{P(\overline{T}_2)} \\
&= 1 \cdot \frac{\frac{27}{36}}{\frac{41}{45}} = \frac{135}{164}
\end{aligned}$$

9.

In case Bert goes first, Bert can only win on odd turns, which means that he can only win on 2k+1 turn where k ≥ 0. The possibility Bert wins on the 2k+1 turn is

$$\underbrace{(0.8)^k(0.7)^k}_{\substack{\text{each player}\\ \text{lose k turns}}} \underbrace{(0.2)}_{\substack{\text{Bert wins}\\ \text{the last turn}}} = (0.56)^k(0.2)$$

Therefore, the possibility Bert wins if Bert goes first is

$$\sum_{k=0}^{\infty}(0.56)^k(0.2) = \frac{0.2}{1-0.56} = \frac{5}{11}$$

In case Earnie goes first, Bert can only win on even turns, which means that he can only on 2k+2 turn where k ≥ 0. The possibility Bert wins on the 2k+2 turn is

$$\underbrace{(0.8)^k(0.7)^k}_{\substack{\text{each player}\\ \text{lose k turns}}} \underbrace{(0.7)}_{\substack{\text{Earnie losess the}\\ \text{2nd last turn}}} \underbrace{(0.2)}_{\substack{\text{Bert wins}\\ \text{the last turn}}} = (0.56)^k(0.14)$$

Therefore, the possibility Bert wins if Earnie goes first is

$$\sum_{k=0}^{\infty}(0.56)^k(0.14) = \frac{0.14}{1-0.56} = \frac{7}{22}$$

Since the coin is fair, the possibility that Bert goes first is 0.5 and the possibility that Earnie goes first is 0.5, hence the possibility that Bert wins is

$$\frac{1}{2}\cdot\frac{5}{11}+\frac{1}{2}\cdot\frac{7}{22}=\frac{17}{44}$$

10.

a. Without replacement, player A gets 1 straws, player B gets 2 straws and player C gets 2 straws in the end. Therefore,

$$P(A) = \frac{1}{5}$$

$$P(B) = \frac{2}{5}$$

$$P(C) = \frac{2}{5}$$

b. With replacement, the probability player A must go on the mission at turn $5k+1$ for $k \geq 0$ is

$$\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{5k}$$

Hence, the probability player A win is

$$P(A) = \left(\frac{1}{5}\right)\sum_{k=0}^{\infty}\left(\frac{4}{5}\right)^{5k} = \frac{625}{2101}$$

Similarly, the probability player B and C win are

$$P(B) = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)\sum_{k=0}^{\infty}\left(\frac{4}{5}\right)^{5k} + \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{4}\sum_{k=0}^{\infty}\left(\frac{4}{5}\right)^{5k} = \frac{500}{2101} + \frac{256}{2101} = \frac{756}{2101}$$

$$P(C) = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{2}\sum_{k=0}^{\infty}\left(\frac{4}{5}\right)^{5k} + \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{3}\sum_{k=0}^{\infty}\left(\frac{4}{5}\right)^{5k} = \frac{400}{2101} + \frac{320}{2101} = \frac{720}{2101}$$