1.

Consider the partition $t_{2n+1} = 0$ and $t_j = \dfrac{1}{j}$ where $1 \leq j \leq 2n$. Then we have that

$$\begin{aligned}
\sum_{j=1}^{2n} \|t_{j+1} - t_j\| &= \|t_{2n+1} - t_{2n}\| + \sum_{j=1}^{2n-1} \sqrt{(1-1)^2 + \left(\frac{\cos((j+1)\pi)}{j+1} - \frac{\cos(j\pi)}{j}\right)^2} \\
&= \frac{\cos(2n\pi)}{2n} + \sum_{j=1}^{2n-1} \left|\frac{1}{j+1} + \frac{1}{j}\right| \cdot |\cos((j+1)\pi)| \\
&> \frac{1}{2n} + \sum_{j=1}^{2n-1} \frac{1}{j} \\
&= \sum_{j=1}^{2n} \frac{1}{j}
\end{aligned}$$

However, we know that the harmonic series $\sum_{j=1}^{\infty} \dfrac{1}{j}$ diverges. Therefore, the supremum of $\sum_{j=1}^{2n} \|t_{j+1} - t_j\|$ of all partitions does not exists and hence the curve is not rectifiable.

2.

$$\begin{aligned}
\alpha'(t) &= J_{\Phi\circ\gamma}(t) \\
&= J_\Phi(\gamma(t))J_\gamma(t) \\
&= \begin{bmatrix} \frac{\partial \Phi_1}{\partial x}(\gamma(t)) & \frac{\partial \Phi_1}{\partial y}(\gamma(t)) \\ \frac{\partial \Phi_2}{\partial x}(\gamma(t)) & \frac{\partial \Phi_2}{\partial y}(\gamma(t)) \\ \frac{\partial \Phi_3}{\partial x}(\gamma(t)) & \frac{\partial \Phi_3}{\partial y}(\gamma(t)) \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial t}(t) \\ \frac{\partial y}{\partial t}(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial \Phi_1}{\partial x}(\gamma(t))\frac{\partial x}{\partial t}(t) + \frac{\partial \Phi_1}{\partial y}(\gamma(t))\frac{\partial y}{\partial t}(t) \\ \frac{\partial \Phi_2}{\partial x}(\gamma(t))\frac{\partial x}{\partial t}(t) + \frac{\partial \Phi_2}{\partial y}(\gamma(t))\frac{\partial y}{\partial t}(t) \\ \frac{\partial \Phi_3}{\partial x}(\gamma(t))\frac{\partial x}{\partial t}(t) + \frac{\partial \Phi_3}{\partial y}(\gamma(t))\frac{\partial y}{\partial t}(t) \end{bmatrix}
\end{aligned}$$

Let $x_i = \frac{\partial \Phi_i}{\partial x}(\gamma(t)), y_i = \frac{\partial \Phi_i}{\partial y}(\gamma(t)), a = \frac{\partial x}{\partial t}(t), b = \frac{\partial y}{\partial t}(t)$, then

$$\begin{aligned}
\alpha'(t) \cdot N(\gamma(t)) &= \alpha'(t) \cdot \left(\frac{\partial \Phi}{\partial x}(\gamma(t)) \times \frac{\partial \Phi}{\partial y}(\gamma(t)) \right) \\
&= \det \begin{bmatrix} x_1 \cdot a + y_1 \cdot b & x_2 \cdot a + y_2 \cdot b & x_3 \cdot a + y_3 \cdot b \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \\
&= 0
\end{aligned}$$

as the first row is a linear combination of the second and third row. Therefore, the two vectors are orthogonal.

3.

Since K is a normal domain, there exists a piecewise $\mathcal{C}^1$ curve $\gamma : [a, b] \to \mathbb{R}^2$ on the positive oriented boundary of K, and since $\Phi|_{\partial K} = \Psi|_{\partial K}$.

$$\int_{\Phi} \mathrm{curl} f \cdot n d\sigma = \int_{\Phi \circ \gamma} P dx + Q dy + R dz = \int_{\Psi \circ \gamma} P dx + Q dy + R dz = \int_{\Psi} \mathrm{curl} f \cdot n d\sigma$$

4.

a.

Apply the Gauss theorem to the vector field $f\nabla g$ then

$$\begin{aligned}\int_S fD_n g d\sigma &= \int_S f\nabla g \cdot n d\sigma \\ &= \int_V \nabla(f\nabla g) \\ &= \int_V f\nabla^2 g + \nabla f \cdot \nabla g \\ &= \int_V f\Delta g + \int_V \nabla f \cdot \nabla g\end{aligned}$$

b.

From part a, we have that

$$\int_S fD_n g d\sigma = \int_V f\Delta g + \int_V \nabla f \cdot \nabla g$$

$$\int_S gD_n f d\sigma = \int_V g\Delta f + \int_V \nabla g \cdot \nabla f$$

Therefore,

$$\int_S (fD_n g - gD_n f) d\sigma = \int_V (f\Delta g - g\Delta f)$$

5.

a.

From 4a, let $g = 1$ then $\nabla g = \Delta g = 0$, we have that

$$0 = \int_V (\nabla g) \cdot (\nabla f) + \int_V g \nabla f = \int_S g D_n f d\sigma$$

b.

From 4a, let $g = f$, we have that

$$\int_S f D_n f d\sigma = \int_V f \Delta f + \int_V \nabla f \cdot \nabla f = \int_V |\nabla f|^2$$