1.

a.

$$
\begin{aligned}
\mathcal{F}[c_1 f + c_2 g](\xi) &= \frac{1}{2\pi}\int_{\mathbb{R}} (c_1 f + c_2 g)(x) e^{ix\cdot\xi} dx \\
&= \frac{1}{2\pi} c_1 \int_{\mathbb{R}} f(x) e^{ix\cdot\xi} dx + \frac{1}{2\pi} c_2 \int_{\mathbb{R}} g(x) e^{ix\cdot\xi} dx \\
&= c_1 \mathcal{F} f + c_2 \mathcal{F} g
\end{aligned}
$$

b.

$$
\mathcal{F}(fg) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x) g(x) e^{-ix\cdot\xi} dx \neq \frac{1}{4\pi^2}\left(\int_{\mathbb{R}} f(x) e^{-ix\cdot\xi}\right)\left(\int_{\mathbb{R}} g(x) e^{-ix\cdot\xi}\right) = \mathcal{F} f \mathcal{F} g
$$

2.

a.

$$
\begin{aligned}
(\mathcal{F}f)(\xi) &= \frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{ix\xi}dx \\
&= \frac{1}{2\pi}\int_{-a}^{a} e^{ix\xi}dx \\
&= \frac{1}{2\pi}\left.\frac{e^{ix\xi}}{i\xi}\right|_{x=-a}^{a} \\
&= \frac{1}{2\pi}\frac{e^{ia\xi}-e^{-ia\xi}}{i\xi} \\
&= \frac{2\sinh(ia\xi)}{2\pi i\xi} \\
&= \frac{-i\sin(-a\xi)}{i\xi\pi} \\
&= \frac{sin(a\xi)}{\xi\pi}
\end{aligned}
$$

b.

$$
\begin{aligned}
f(x) &= \int_{\mathbb{R}} e^{-|\xi|\alpha}e^{-i\xi x}d\xi \\
&= \int_0^{\infty} e^{-\xi(\alpha+ix)}d\xi + \int_{-\infty}^{0} e^{\xi(\alpha-ix)}d\xi \\
&= -\left.\frac{e^{-\xi(\alpha+ix)}}{\alpha+ix}\right|_{\xi=0}^{\infty} + \left.\frac{e^{\xi(\alpha-ix)}}{\alpha-ix}\right|_{\xi=-\infty}^{0} \\
&= \frac{1}{\alpha+ix}+\frac{1}{\alpha-ix} \\
&= \frac{\alpha-ix+\alpha+ix}{(\alpha+ix)(\alpha-ix)} \\
&= \frac{2\alpha}{\alpha^2+x^2}
\end{aligned}
$$

c.

$$
\begin{aligned}
&\int_{\mathbb{R}} -iF'(\xi)e^{-i\xi x}d\xi \\
=& -ie^{-i\xi x}|_{-\infty}^{\infty} - \int_{\mathbb{R}} F(\xi)(-i\cdot(-ix)e^{-i\xi x})d\xi \\
=& x\int_{\mathbb{R}} F(\xi)e^{-i\xi x}d\xi \\
=& \mathcal{F}[xf(x)]
\end{aligned}
$$

3.

We have that

$$\begin{aligned}\mathcal{F}[u_t] = U_t &= k\mathcal{F}[u_{xx}] + c[u_x] \\ &= -k\xi^2 U - ci\xi U\end{aligned}$$

Thus we can find

$$U(\xi, t) = C(\xi)e^{-k\xi^2 t - ci\xi t}$$

and since $u(x, 0) = f(x)$ and $U(\xi, 0) = F(\xi)$,

$$U(\xi, t) = F(\xi)e^{-k\xi^2 t - ci\xi t}$$

Let $G(\xi) = e^{-k\xi^2 t}$, $H(\xi) = F(\xi)e^{-ci\xi t}$ we have

$$U(\xi, t) = G(\xi)H(\xi)$$

And the inverse fourier of G, H are

$$g(x) = \frac{1}{\sqrt{2kt}}e^{-x^2/4kt}$$

$$h(x) = f(x - ct)$$

Thus the solution is

$$u(x, t) = \frac{1}{2\pi}\left(f(x - ct) * \frac{1}{\sqrt{2kt}}e^{-x^2/4kt}\right)$$

4.

Apply the fourier transform, we have that

$$\begin{cases} U_t = -k\xi^2 U - \gamma U \\ U(\xi, 0) = F(\xi) \end{cases}$$

Then, we can solve for

$$U(\xi, t) = C(\xi)e^{-(k\xi^2+\gamma)t}$$

and using the initial condition,

$$U(\xi, t) = F(\xi)e^{-(k\xi^2+\gamma)t} = e^{-\gamma t}F(\xi)e^{-k\xi^2 t}$$

And apply the inverse, we have

$$u(x, t) = e^{-\gamma t}\left(f(x) * \frac{1}{\sqrt{2kt}}e^{-x^2/4kt}\right)$$

5.

Apply the fourier transform on y, we have that

$$\begin{cases} U_{xx} - \xi^2 U = 0 \\ U(0,\xi) = G_1(\xi) \\ U(L,\xi) = G_2(\xi) \end{cases}$$

Thus

$$U(x,\xi) = C_1(\xi)e^{-\xi x} + C_2(\xi)e^{\xi x}$$

To ensure the boundedness of the solution, we must have that

$$C_1(\xi) = 0 \text{ if } \xi < 0 \text{ and } C_2(\xi) = 0 \text{ if } \xi > 0$$

Thus, the solution can be rewrite as

$$U(x,\xi) = C(\xi)e^{-|\xi|x}$$

The initial conditions state that

$$C(\xi) = G_1(\xi)$$

and

$$U(L,\xi) = G_1(\xi)e^{-|\xi|L} = G_2(\xi)$$

Thus,

$$u(x,y) = \frac{1}{2\pi}\left(g_1(y) * \frac{2L}{y^2 + L^2}\right)$$

6.