1.

a.

$$f_Y(y) = \int_0^2 \frac{1}{6} dx = \frac{1}{3}$$

Since $0 < y < 3$, $2 < y^2 + 2 < 11$, and hence if $u \leq 2, F_U(u) = 0$ and if $u \geq 11, F_U(u) = 1$. For $2 < u < 11$, we have that

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(Y^2 + 2 \leq u) \\ &= P(Y^2 \leq u - 2) = P(Y \leq \sqrt{U-2}) \\ &= \int_0^{\sqrt{U-2}} \frac{1}{3} dy = \frac{\sqrt{U-2}}{3} \end{aligned}$$

b.

Since $0 < x < 2, 0 < y < 3, -3 < x - y < 2$. Therefore, if $v \leq -3, F_V(v) = 0$ and if $v \geq 2, F_V(v) = 1$. For $-3 < v < 2$, we have that

$$\begin{aligned} F_V(v) &= P(V \leq v) = P(X - Y \leq v) \\ &= P(Y \geq v + X) \end{aligned}$$

For $0 < v < 2$,

$$\begin{aligned} F_V(v) &= 1 - \int_0^{2-v} \int_{v+x}^{2} \frac{1}{6} dx dy \\ &= 1 - \int_0^{2-v} \frac{2-v-x}{6} dy = 1 - \left(\frac{(2-v)^2}{6} - \frac{(2-v)^2}{12} \right) \\ &= 1 - \frac{(2-v)^2}{12} \end{aligned}$$

For $-3 < v < 0$,

$$\begin{aligned} F_V(v) &= \int_{-v}^{3} \int_0^{v+x} \frac{1}{6} dx dy \\ &= \int_{-v}^{3} \frac{1}{6}(v+x) dx \\ &= \frac{3v+v^2}{6} + \frac{9-v^2}{12} \\ &= \frac{(v+3)^2}{12} \end{aligned}$$

2.

Since $16 \geq U = Y^4 \geq 0$. We have that if $0 \leq u, F_U(u) = 0$ and if $u \geq 16, F_U(u) = 1$. For $0 < u < 16$, we have that

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(Y^4 \leq u) = P(-\sqrt[4]{u} \leq Y \leq \sqrt[4]{u}) \\ &= \int_0^{\sqrt[4]{u}} \frac{y}{6} dy + \left| \int_{-\sqrt[4]{u}}^{0} \frac{y^2}{4} dy \right| \\ &= \frac{\sqrt{u}}{12} + \frac{\sqrt[4]{u^3}}{12} \end{aligned}$$

3.

$$f_Y(y) = \int_y^{\infty} \frac{1}{9} e^{-x/3} dx = \frac{1}{3} e^{-y/3}$$

$$\begin{aligned} F_U(u) &= P(U \le u) = P(1 - Y^2 \le u) = P(Y^2 \ge 1 - u) = P(Y \ge \sqrt{1-u}) \\ &= \int_{\sqrt{1-u}}^{\infty} \frac{1}{3} e^{-y/3} dy = e^{-\sqrt{1-u}/3} \end{aligned}$$

4.

a.

$$x = h^{-1}(u) = \frac{u+3}{2} \implies \frac{d}{du}h^{-1}(u) = \frac{1}{2}$$

If $1 > \dfrac{u+3}{2} > 0 \iff -1 > u > -3$

$$f_U(u) = f_X\left(\frac{u+3}{2}\right) \cdot \frac{1}{2} = \frac{-u-1}{2}$$

If $u \leq -3$ or $u \geq -1$ then $f_U(u) = 0$

b.

$$x = h^{-1}(v) = \sqrt[3]{v} \implies \frac{d}{dv}h^{-1}(v) = \frac{1}{3}v^{-2/3}$$

If $1 < \sqrt[3]{v} < 0 \iff 1 < v < 0$

$$f_V(v) = f_X(\sqrt[3]{v}) \cdot \frac{1}{3}x^{-2/3} = \frac{2}{3}v^{-2/3}(1 - \sqrt[3]{v})$$

If $v \geq 1$ or $v \leq 0$ then $f_V(v) = 0$.

5.

$$y = h^{-1}(u) = \frac{2-u}{4} \implies \frac{d}{du}h^{-1}(u) = -\frac{1}{4}$$

If $-1 < \dfrac{2-u}{4} < 1 \iff -2 < u < 6$

$$f_U(u) = f_X\left(\frac{2-u}{4}\right) \cdot \left|-\frac{1}{4}\right| = \frac{6-u}{32}$$

6.

Consider $V = Y$

$$x = h_1^{-1}(u,v) = \frac{v}{u}, \qquad y = h_2^{-1}(u,v) = v$$

$$\det \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial u} & \frac{\partial h_1^{-1}}{\partial v} \\ \frac{\partial h_2^{-1}}{\partial u} & \frac{\partial h_2^{-1}}{\partial v} \end{vmatrix} = \det \begin{vmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ 0 & 1 \end{vmatrix} = -\frac{v}{u^2}$$

As $y > 0$, we have that $v > 0$ and hence

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{v}{u}, v\right) \cdot \left|-\frac{v}{u^2}\right| = \frac{v}{u} e^{-(u/v+v)} \cdot \frac{v}{u^2} = \frac{v^2}{u^3} e^{-(u+v^2)/v}$$

7.

$$x = h_1^{-1}(u,v) = \frac{u-v}{2}, \qquad y = h_2^{-1}(u,v) = \frac{u+v}{2}$$

$$\det \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial u} & \frac{\partial h_1^{-1}}{\partial v} \\ \frac{\partial h_2^{-1}}{\partial u} & \frac{\partial h_2^{-1}}{\partial v} \end{vmatrix} = \det \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u-v}{2}, \frac{u+v}{2}\right) \cdot \frac{1}{2} = 6 \cdot \frac{u-v}{2} \cdot \frac{1}{2} = \frac{3(u-v)}{2}$$