Fall 2022, Math 328, Homework 6

Due: End of day on 2021-12-11

1 10 points

Let m and n be two nonnegative integers. Prove that $\mathbb{Z}^m$ and $\mathbb{Z}^n$ are isomorphic if and only if $m = n$.

2 10 points

Suppose that G is a group of order 1575 which has a normal subgroup of order 9.

1. Prove that G has a normal subgroup of order 25.
2. Prove that G has a normal subgroup of order 7.
3. Prove that G is abelian.
4. Classify all groups of order 1575 which have a normal subgroup of order 9.

3 10 points

1. Prove that any group of order 56 has a normal subgroup of order 7 or 8.
2. Suppose that p, q and r are distinct primes. Prove that any group of order $p \cdot q \cdot r$ has a normal subgroup of order p, q or r.
3. Classify all groups of order $49 \cdot 11$.
4. Classify all groups of order 315 which have a normal subgroup of order 9.

4 10 points

Let $n \geq 3$ be an integer. How many Sylow 2-subgroups does D_{2n} have? Justify your answer!

5 10 points

Let G be a finite group, p a prime number and P a Sylow p-subgroup of G.

1. Assume that $n_p(G) = 1$. Show that $n_p(H) = 1$ for any subgroup H of G. Give an example showing that this can fail if $n_p(G) \neq 1$.
2. Let H be any subgroup of G. Show that there exists some $g \in G$ such that $(g \cdot P \cdot g^{-1}) \cap H$ is a Sylow p-subgroup of H.
3. Assume that N is a normal subgroup of G. Show that $P \cap N$ is a Sylow p-subgroup of N and that $(P \cdot N)/N$ is a Sylow p-subgroup of G/N.

6 10 points

1. Suppose that G is a simple group of order 168. How many elements of order 7 does G have? Justify your answer.
2. Classify all simple groups of order < 100.