Fall 2022, Math 328, Homework 4

Due: End of day on 2021-10-31

1 10 points

Let $\varphi : G \to H$ be a surjective homomorphism of groups, and assume that G is cyclic. Prove that H is cyclic. Deduce that any group which is isomorphic to a cyclic group is again cyclic. Deduce that any quotient of a cyclic group is again cyclic.

2 10 points

Describe (with proof) the lattice of subgroups of D_8, S_3 and Q_8. In these lattices, identify the subgroups which are cyclic.

3 10 points

Let n be a positive integer and G a cyclic group of order n with generator g.

1. Suppose that k is any integer. Prove that there is a unique homomorphism $\varphi_k : G \to G$ satisfying $\varphi_k(g) = g^k$.
2. Prove that φ_k is an automorphism if and only if $\gcd(k, n) = 1$.
3. Suppose that $a, b \in \mathbb{Z}$ are two integers. Show that $\varphi_a \circ \varphi_b = \varphi_{a \cdot b}$.
4. Show that the map $\mathbb{Z}/n \to \mathrm{End}(G)$ defined by $(a \bmod n) \mapsto \varphi_a$ is well defined, and that it restricts to a bijection

   $$(\mathbb{Z}/n)^\times := \{a \bmod n \mid \gcd(a, n) = 1\} \cong \mathrm{Aut}(G).$$

   Here $\mathrm{End}(G)$ denotes the collection of *endomorphisms* of G, i.e. the homomorphisms from G to itself.
5. Deduce that $(\mathbb{Z}/n)^\times$ is a group with respect to the operation

   $$(a \bmod n,\ b \bmod n) \mapsto a \cdot b \bmod n,$$

   and that this group is isomorphic to $\mathrm{Aut}(G)$.
6. If H is an infinite cyclic group, describe $\mathrm{Aut}(H)$.

4 10 points

Let $\pi : G \to H$ be a surjective homomorphism of groups and let $\psi : G \to K$ be any homomorphism of groups.

1. Suppose that $\delta : H \to K$ is a homomorphism satisfying $\psi = \delta \circ \pi$. Show that the kernel of π is contained in the kernel of ψ.

2. Suppose that the kernel of π is contained in the kernel of ψ. Show that there exists a unique homomorphism $\delta : H \to K$ which satisfies $\psi = \delta \circ \pi$.

Use the above in conjunction with the universal property of quotients to (re)prove the *first isomorphism theorem*:

Theorem. *Let $\varphi : G \to H$ be a homomorphism of groups. Then $\ker(\varphi)$ is a normal subgroup of G, and φ induces an isomorphism $\bar{\varphi} : G/\ker(\varphi) \cong \operatorname{im}(\varphi)$ where $\bar{\varphi}(g \cdot \ker(\varphi)) = \varphi(g)$. In particular, if φ is surjective, then H is isomorphic to $G/\ker(\varphi)$.*

5 10 points

Let H be a subgroup of a group G and assume that $[G : H] = 2$. Prove that H is normal in G. Give an example showing that this can fail if 2 is replaced by a larger integer.

6 10 points

Let G be a group.

1. Show that $Z(G)$ is normal in G.

2. Show that the following are equivalent:

   (a) G is abelian.

   (b) $G = Z(G)$.

   (c) $G/Z(G)$ is cyclic.

   (d) $G/Z(G)$ is trivial.

3. Assume that G is a finite group of order $p \cdot q$ where p and q are (not necessarily distinct) primes. Show that G is either abelian or $Z(G)$ is trivial.

4. Assume that $\operatorname{Aut}(G)$ is cyclic. Prove that G is abelian.

   Hint: Revisit Problem 3 on Homework 3.