1.

a.

We have that

$$\sqrt[n]{5^n} = 5$$

and

$$\lim \frac{(n+1)^5}{n^5} = 1$$

Hence,

$$\sum_{n=0}^{\infty} 5^n z^n$$

converges with $R = \frac{1}{5}$ and

$$\sum_{n=0}^{\infty} n^5 z^n$$

converges with $R = 1$. Hence, the radius of convergence of the original power series is $\frac{1}{5}$.

b.

We have that

$$\lim \left(\left(1 - \frac{1}{n}\right)^{n^2} \right)^{1/n} = \lim \left(1 - \frac{1}{n}\right)^n = \lim_{x \to 0} (1+x)^{-1/x} = \frac{1}{e}$$

Hence, the radius of convergence is e.

c.

Let $a_n = 0$ if n is odd and $a_n = 2^{n/2}$ if n is even. Then

$$\sum_{n=0}^{\infty} 2^n z^{2n} = \sum_{n=0}^{\infty} a_n z^n$$

Then

$$R = \frac{1}{\limsup |a_n|^{1/n}} = \left(\frac{1}{\limsup |2^n|^{1/n}} \right)^{1/2} = \frac{1}{\sqrt{2}}$$

d.

Let $a_n = 0$ if n is not a power of 2, $a_n = 2\log_2(n)$ otherwise. Then

$$\sum_{n=0}^{\infty} 2n z^{2^n} = \sum_{n=0}^{\infty} a_n z^n$$

Then

$$R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{\limsup |2\log_2(n)|^{1/n}} = \frac{1}{\limsup |2^{\frac{1}{n}} 2^{\frac{1}{n}\log_2(\log_2(n))}|} = 1$$

2.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$f'(z) = \sum_{n=2}^{\infty} n a_n z^{n-1} + a_1$$

As $a_1 \neq 0, f'(0) = a_1 \neq 0$, hence we can find $r_1 > 0$ such that $f'(z) \neq 0$ in $\{|z| < r_1\}$. Thus from the inverse function theorem, f is injective in $\{|z| < r_1\}$. Hence, choose $r = \min(R, r_1)$, we have that f is bijective on $\{|z| < r\}$.

3.

$$\begin{aligned}
\cos(\overline{z}) &= \sum_{n=0}^{\infty}(-1)^n\frac{\overline{z}^{2n}}{2n!} \\
&= \overline{\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{2n!}} \\
&= \overline{\cos(z)}
\end{aligned}$$

Hence,

$$\begin{aligned}
&|\cos(z)|^2 \\
=&(\cos(x+iy))(\cos(x-iy)) \\
=&(\cos(x)\cos(iy)-\sin(x)\sin(iy))(\cos(x)\cos(iy)+\sin(x)\sin(iy)) \\
=&(\cos(x))^2(\cosh(y))^2+(\sin(x))^2(\sinh(y))^2 \\
=&(\cos(x))^2(1+\sinh(y))^2+(\sin(x))^2(\sinh(y))^2 \\
=&(\cos(x))^2+(\sinh(y))^2
\end{aligned}$$

because of $\cos(iy) = \cosh(y)$ and $\sin(iy) = i\sinh(y)$. Similarly,

$$\begin{aligned}
\sin(\overline{z}) &= \sum_{n=0}^{\infty}(-1)^n\frac{\overline{z}^{2n+1}}{(2n+1)!} \\
&= \overline{\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}} \\
&= \overline{\sin(z)}
\end{aligned}$$

Hence,

$$\begin{aligned}
&|\sin(z)|^2 \\
=&(\sin(x+iy))(\sin(x-iy)) \\
=&(\sin(x)\cos(iy)+\cos(x)\sin(iy))(\sin(x)\cos(iy)-\cos(x)\sin(iy)) \\
=&(\sin(x))^2(\cosh(y))^2+(\cos(x))^2(\sinh(y))^2 \\
=&(\sin(x))^2(1+\sinh(y))^2+(\cos(x))^2(\sinh(y))^2 \\
=&(\sin(x))^2+(\sinh(y))^2
\end{aligned}$$

4.

a.

We have that $|a_n| \leq p_n, |b_n| \leq q_n$. Therefore,

$$|a_n \pm b_n| \leq |a_n| + |b_n| \leq p_n + q_n$$

and hence

$$\sum_{n=0}^{\infty} a_n z^n \pm \sum_{n=0}^{\infty} b_n z^n \prec \sum_{n=0}^{\infty} p_n z^n + \sum_{n=0}^{\infty} q_n z^n$$

We have that

$$\left| \sum_{l=0}^{k} a_l b_{k-l} \right| \leq \sum_{l=0}^{k} |a_l||b_{k-l}| \leq \sum_{l=0}^{k} p_l q_{k-l}$$

Hence,

$$\left(\sum_{n=0}^{\infty} a_n z^n \right) \left(\sum_{n=0}^{\infty} b_n z^n \right) \prec \left(\sum_{n=0}^{\infty} p_n z^n \right) + \left(\sum_{n=0}^{\infty} q_n z^n \right)$$

b.

$$\frac{M}{r-z} = \frac{M/r}{1-z/r} = \sum_{n=0}^{\infty} \frac{M}{r} \left(\frac{z}{r} \right)^n = \sum_{n=0}^{\infty} \frac{M}{r^{n+1}} z^n$$

We know that for every $r \in [0, R)$,

$$\limsup \sqrt[n]{|a_n|} = \frac{1}{R} < \frac{1}{r}$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that $|a_n| < \left(\frac{1}{r}\right)^n$, and hence we can choose $M \geq r$ such that $|a_n| < \frac{M}{r^{n+1}}$. Thus,

$$\sum_{n=0}^{\infty} a_n z^n \prec \frac{M}{r-z}$$

5.

From A3.4, we know that there exists $M > 0$ such that

$$\sum_{m=1}^{\infty} a_m z^m \prec \frac{M}{r-z}$$

Hence,

$$\left|\frac{1}{f(z)}\right| = \left|\sum_{n=0}^{\infty} \frac{(-1)^n}{a_0^{n+1}} \left(\sum_{m=1}^{\infty} a_m z^m\right)^n\right| \leq \sum_{n=0}^{\infty} \frac{1}{|a_0^{n+1}|} \left|\frac{M}{r-z}\right|^n$$

which means that

$$\left|\frac{1}{f((rz+M)/z)}\right| \leq \sum_{n=0}^{\infty} \frac{1}{|a_0^{n+1}|} |z|^n$$

However,

$$\left|\frac{\frac{1}{a_0^{n+1}}}{\frac{1}{a_0^{n+2}}}\right| = |a_0|$$

which means $1/f((rz+M)/z)$ has a positive radius of convergence, thus also $1/f(z)$