Contents

1 8 Sep **2**

1.1 intuitive counting formula 2

1.2 problem 3

2 11 Sep **3**

2.1 HW8.1.2 3

2.2 CR.11.1 4

2.3 HW8.1.3 4

2.4 CR.11.4 4

3 15 Sep **4**

3.1 Events and Marginals 4

3.2 Generating σ-algebra 4

4 18 Sep **5**

4.1 PP 8.8.1 5

4.2 HW 8.3.2 5

4.3 HW 8.2.1 6

4.4 HW 8.2.5 6

5 25 Sep **6**

5.1 HW 8.4.2 6

5.2 PP 8.5.1 7

5.3 HW 8.4.3 7

5.4 HW 8.5.2 8

6 2 Oct **8**

6.1 HW 8.7.2 8

6.2 PP 8.7.7 8

6.3 PP 8.6.7 8

6.4 PP 8.6.8 8

1 8 Sep

1.1 intuitive counting formula

Let N be a λ-Poisson process and f be any continuous function in $\mathbb{R}$.

We will prove that Note: $f(0) = f(N_0)$. We have,

$$f(N_1) - f(0) = \sum_{i=1}^{N_1} (f(i) - f(i-1))$$

By right contiuity of the poisson process

$$\int_0^1 (f(N_{s^-} + 1) - f(N_{s^-}))\, dN_s = \lim_{M\to\infty} \left(f\left(N_{\frac{j-1}{M}} + 1\right) - f\left(N_{\frac{j-1}{M}}\right)\right)\left(N_{\frac{j}{M}} - N_{\frac{j-1}{M}}\right)$$

For large M, we have that

$$N_{\frac{j}{M}} - N_{\frac{j-1}{M}} = \begin{cases} 1, \text{at the jump where } N \text{ increases} \\ 0, \text{otherwise} \end{cases}$$

Hence,

$$\int_0^1 (f(N_{s^-} + 1) - f(N_{s^-}))\, dN_s = \sum_{i=1}^{N_1} (f(i) - f(i-1)) = f(N_1) - f(0)$$

1.2 problem

problem link
Guess

$$a_t = a_0 \exp\left(\int_0^1 f_s ds\right)$$

which starts right. The exponential rule gives

$$\frac{d}{dt} a_t = f_t a_t$$

Unique? Consider second solution satisfies $\frac{d}{dt}\alpha_t = f_t \alpha_t$ such that $\alpha_0 = a_0$. Set $e_t = \frac{\alpha_t}{a_t}$, then $e_0 = \frac{a_0}{\alpha_0} = 1$ hence by product rule

$$\frac{d}{dt} e^t = \frac{f_t \alpha_t}{a_t} - \frac{\alpha_1 f_t \exp\left(-\int_0^1 f_s ds\right)}{a_0} = 0$$

Hence, $e_t = 1$, hence $\alpha_t = a_t$ for all t.

2 11 Sep

2.1 HW8.1.2

Let $a_t = E[X_t]$ and $\beta = \lambda - \mu$. Then solve

$$a_t = a_0 + \int_0^t \beta a_s ds + \theta t$$

or

$$\dot{a}_t - \beta a_t = \theta$$

Multiplying by

$$\begin{aligned} & I(t) = e^{-\beta t} \\ & \frac{d}{ds}\left(I(s) a_s\right) = I(s) a_s - \underbrace{I(s) p}_{something} a_s = I(s)\theta \end{aligned}$$

and

$$\int_0^t I(t) a_t - I(0) a_0 = \theta \int_0^t I(s) ds$$

so

$$a_t = I^{-1}(t)(a_0 + \theta \int_0^t I(s) ds)$$

Solving

$$a_t = e^{\beta t} a_0 + \theta e^{\beta t}(e^{-\beta t} - 1) = e^{\beta t} a_0 + \frac{\theta}{\beta}(e^{\beta t} - 1)$$

Uniqueness
Let α_1 be anothre solution

$$= alpha_t = \alpha_0 + \int_0^t \beta \alpha_s ds + \theta t$$

then

$$e_t = \alpha_t - a_t$$

satisfies $e_t = \int_0^t \beta e_s ds$ such that $e_0 = 0$ but $e_t = e^{\beta t} e_0 = 0$ is the unique solution.

2.2 CR.11.1

2.3 HW8.1.3

$$E[X_0] = \frac{1}{p} = 10$$

and

$$E[X_0^2] = \frac{2-p}{p^2} = 190$$

Substitute $f(x) = x$ in, we get

$$E[X_t] = 10 + \int_0^t \frac{1}{10} E[X_s(X_s + 1 + X_s - 1 - 2X_s)]ds = 10 + \int_0^t 0ds = 10$$

and substitute $f(x) = x^2$ in,

$$\begin{aligned} E[X_t^2] &= 190 + \int_0^t \frac{1}{10} E[X_s((X_s+1)^2 + (X_s-1)^2 - 2X_s^2)]ds \\ &= 190 + \int_0^t \frac{1}{10} E[2X_s]ds \\ &= 190 + 2t \end{aligned}$$

Therefore,

$$Var[X_t] = 90 + 2t$$

2.4 CR.11.4

3 15 Sep

3.1 Events and Marginals

Constraining Random vectors

$$\{w : (x_1, \ldots, x_n, x_{n+1}) \in A \times \mathbb{R}\} = \{w : (x_1, \ldots, x_n, x_{n+1}) \subset A, \underbrace{x_{n+1} \in \mathbb{R}}_{\text{always true}}\}$$

3.2 Generating σ-algebra

Borel σ-algebra ar egenrated, which means smallest containing a class of sets.
Given a collection $\mathcal{C}$ of $\mathbb{R}$ or Ω say e.g. $\mathcal{C}_1 = \{(-\infty, x) : x \in \mathbb{R}\}$ and $\mathcal{C}_2 = \{X^{-1}(-\infty, x) : x \in \mathbb{R}\}$. Want the smallest σ-alg containing $\mathcal{C}_1$ or $\mathcal{C}_2$.

Note: $X^{-1}(-\infty, x) \subset \Omega \implies \mathcal{C}_2 \subset 2^\Omega . X^{-1}(-\infty, x] = \{w : X(w) \leq x\}$
Note : $\sigma(\mathbb{R}) = \sigma\mathcal{C}_1$ and $\sigma(X) = \sigma\mathcal{C}_2$.
Definition: the σ-alg generated by $\mathcal{C}, \sigma\mathcal{C}$, is the smallest σ-alg containing $\mathcal{C}$ makes sense since

- 2^Ω is a σ-alg conataining $\mathcal{C}$
- If $\{\mathcal{F}_\alpha\}$ are σ-alg, then $\bigcup_\alpha \mathcal{F}_\alpha$ is also a σ-alg

Step 1. Let $\{\mathcal{F}_\alpha\}$ be all σ-algs that contain $\mathcal{C}$. Not empty by fact 1.
Step 2. Let $\sigma(\mathcal{C}) = \bigcup_\alpha \mathcal{F}_\alpha$

4 18 Sep

4.1 PP 8.8.1

4.2 HW 8.3.2

Taking out knows

$$
\begin{aligned}
E[L_n|\mathcal{F}_{n-1}] &= E\left[\prod_{j=1}^{n} \frac{p(x_j)}{q(x_j)}\middle| \mathcal{F}_{n-1}\right] \\
&= \prod_{j-1}^{n-1} \frac{p(x_j)}{q(x_j)} E\left[\frac{p(x_)}{q(x_n)}\middle| \mathcal{F}_{n-1}\right]
\end{aligned}
$$

But, by independence,

$$
E\left[\frac{p(x_n)}{q(x_n)}\middle| \mathcal{F}_{n-1}\right] = E\left[\frac{p(x_n)}{q(x_n)}\right] = \sum_{x \in A_x} \frac{p(x)}{q(x)} q(x) = 1
$$

Hence, $E[L_n|\mathcal{F}_{n-1}] = L_{n-1}$ and a $\{\mathcal{F}_n\}$-martingale.

$$
\begin{aligned}
E[X_1 X_3 L_n] &= E[E[X_1 X_3 L_n | \mathcal{F}_3]] \\
&= E[X_1 X_3 E[L_n|\mathcal{F}_3]] \\
&= E[X_1 X_3 L_3] \\
&= E[E[X_1 X_3 L_3|\mathcal{F}_2]] \\
&= E\left[X_1 L_2 E\left[X_3 \frac{p(X_3)}{q(X_3)}\middle| \mathcal{F}_2\right]\right]
\end{aligned}
$$

But by independence,

$$
\begin{aligned}
E\left[X_3 \frac{p(X_3)}{q(X_3)}\middle| \mathcal{F}_2\right] &= E\left[X_3 \frac{p(X_3)}{q(X_3)}\right] \\
&= \sum_x x \frac{p(x)}{q(x)} q(x) \\
&= \frac{1}{2}
\end{aligned}
$$

Similarly

$$E[E[X_1L_2|\mathcal{F}_1]] = E[E[X_1L_1|\mathcal{F}_1]] = E[X_1L_1] = \sum_x x\frac{p(x)}{q(x)}q(x) = \frac{1}{2}$$

so $E[X_1X_3L_n] = \frac{1}{4}$.

4.3 HW 8.2.1

a.

Trivially, it includes $\varnothing$

b.

If $A \in 2^\Omega$ then A is a subset of Ω and A^C is a subset of Ω so $A^C \in 2^\Omega$

c.

If $\{A_i\}_{i=1}^\infty \subset 2^\Omega$ then each A_i is a subset of Ω and so is $\bigcup_i A_i$. Hence, $\bigcup_i A_i \in 2^\Omega$

4.4 HW 8.2.5

$\sigma(\mathbb{R})$ is defined as:
i. $(-\infty, x] \in \sigma(\mathbb{R}) \forall x \in \mathbb{R}$
ii. $\sigma(\mathbb{R})$ is a σ-algebra.
Basic sets $(-\infty, x]$ included so $(a, b] = (-\infty, b] \cap (-\infty, a]^C$ is also included.
Hence,

$$\left(0, 1 - \frac{1}{n+1}\right] \in \sigma(\mathbb{R}) \qquad \forall n \in \mathbb{N}$$

Hence,

$$(0,1) = \bigcup_{n=1}^\infty \left(0, 1 - \frac{1}{n+1}\right] \in \sigma(\mathbb{R})$$

5 25 Sep

5.1 HW 8.4.2

Suppose to contrary $P(Z > 0) > 0$. then $\exists \epsilon > 0$ such that

$$P(Z > 0) > \epsilon$$

But by continuous of measure

$$\lim_{n\to\infty} P(Z > 1/n) = P(\bigcup_{n=1}^\infty \{Z > 1/n\}) = P(Z > 0) > \epsilon$$

Hence, $\exists n > 0$ such that
$$P(Z > n) > \epsilon/2$$
and
$$E[Z1_{Z>0}] > \frac{n\epsilon}{2} > 0$$
Proved hint. Suppose 2 solutions: Y, Z. Then

$$\begin{aligned} 0 &= E[X1_G] - E[X1_G] \\ &= E[Y1_G] - E[Z1_G] \\ &= E[(Y-Z)1_G] \end{aligned}$$

but $G = \{Y - Z > 0\}$ and $G = \{Y - Z < 0\}$ are in $\mathcal{F}$ so

$$E[(Y-Z)1_{Y-Z>0}] = 0 = E[(Y-Z)1_{Y-Z<0}]$$

and $Y = Z$ by hint.

5.2 PP 8.5.1

5.3 HW 8.4.3

$$E[X|Y] = \sum_y \underbrace{E[X|Y=y]}_{\text{value it takes}} \underbrace{1_{y=Y}}_{\text{where takes}}$$

and $\sigma(y)$ is the collection of unions of $\{w : Y(w) = y\}$ for distinct $y \in A_y$
Have to show 3 properties of cond. exp.

i. $E[|E[X|Y]|] \leq E[E[|X|\,|Y]] = E[|X|] < \infty$

ii. Suppose $B \in \sigma\mathbb{R}$. **Then** $E[X|Y]^{-1}(B) = \bigcup_{y:E[X|Y]\in B}\{w : Yw = y\} \in \sigma(Y)$

iii. Suppose $G \in \sigma(y)$. **Then** $G = Y^{-1}(B)$ **for** $B \in \sigma(\mathbb{R})$

$$\begin{aligned} E[X1_G] &= E[X1_B(Y)] = E[X1_{\bigcup_{y\in B}\{Y=y\}}] \\ &= \sum_{y\in B} E[X1_{\{Y=y\}}] \\ &= \sum_{y\in B}\sum_{x,y} x1_{z=y} \underbrace{P_{xy}(x,y)}_{P_{x|y}(x|z)P_y(x)} \\ &= \sum_{y\in B} \underbrace{\sum_x xP_{x|y}(x|y)}_{E[X|Y=y]} P_y(y) \\ &= \sum_{y\in A_y} E[X|Y=y]1_B y P_y(y) \\ &= E[E[X|Y]\underbrace{1_B(Y)}_{1_G}] \end{aligned}$$

5.4 HW 8.5.2

Let N^+, N^- be independent PP with rates $\frac{N^2}{2}$. Take $X_t = \frac{N^+ - N^-}{N}$. Then, by MP for PP and role of independent.

$$f(X_t) = f\left(\frac{N_t^+ - N_t^-}{N}\right) = g(N_t^+, N_t^-)$$
$$= \int_0^t \underbrace{\frac{N^2}{2}}_{\lambda}\left(f\left(\frac{N_u^+ + 1 - N_u^-}{N}\right) - 2f\left(\frac{N_u^+ - N_u^-}{N}\right) + f\left(\frac{N_u^+ - 1 - N_u^-}{N}\right)\right) \quad = \int_0^t \underbrace{\frac{N^2}{2}}_{\lambda}\left(f\left(X_u\right.\right.$$

6 2 Oct

6.1 HW 8.7.2

MP for B is

$$f(B_t) - f(0) - \int_9^t \frac{1}{2} f''(B_s) ds$$

IF $f(x) = g(x^3)$, then by chain rule

$$g''(x) = \frac{d}{dx} 3x^2 q'(x^3) = 6xq'(x^3) + 9x^4 q''(x^3)$$

and hence

$$g(B_t)^3 - g(0) - \int_0^t \frac{1}{2} g''(B_s) ds = g(B_t^3) - g(0) - \int_0^t 3X_s^{1/3} g'(X_3) + \frac{9}{2} X_s^{4/3} q''(X_s) ds$$

So

$$Lg(x) = 3x^{1/3} q'(x) + \frac{9}{2} x^{4/3} q''(x)$$

and SDE is

$$dX_t = 3X_t^{1/3} + 3X_t^{2/3} dB_t$$

6.2 PP 8.7.7

6.3 PP 8.6.7

6.4 PP 8.6.8