1.

Solve $u_t + 3tu_x = u$ with $u(0,x) = h(x)$ for $x \in \mathbb{R}$. Follow the method, we have

$$\frac{dT(\sigma;s)}{ds} = 1, \frac{dX(\sigma;s)}{ds} = 3t, \frac{dU(\sigma;s)}{ds} = U$$

with

$$T(\sigma;0) = 0, X(\sigma;0) = \sigma, U(\sigma;0) = h(\sigma)$$

Thus,

$$T(\sigma;s) = s, X(\sigma;s) = \sigma + s, U(\sigma;s) = e^{s+\ln(h(\sigma))}$$

Next, we can obtain the inverse function,

$$\begin{cases} t = T(\sigma;s) = s \\ x = X(\sigma;s) = \sigma + s \end{cases} \implies \begin{cases} \sigma = \Sigma(t,x) = x - t \\ s = S(t,x) = t \end{cases}$$

Finally, the solution is obtained as

$$u(t,x) = U(\Sigma(t,x), S(t,x)) = e^{t+\ln(h(x-t))}$$

2.

Solve $u_t + uu_x = 2t$ with $u(0,x) = x$. Follow the method, we have

$$\frac{dT(\sigma;s)}{ds} = 1, \frac{dX(\sigma;s)}{ds} = U, \frac{dU(\sigma;s)}{ds} = 2t$$

with

$$T(\sigma;0) = 0, X(\sigma;0) = \sigma, U(\sigma;0) = \sigma$$

Thus,

$$T(\sigma;s) = s, X(\sigma;s) = ts^2 + \sigma s + \sigma, U(\sigma;s) = 2ts + \sigma$$

Next, we can obtain the inverse function,

$$\begin{cases} t = T(\sigma;s) = s \\ x = X(\sigma;s) = ts^2 + \sigma s + \sigma \end{cases} \implies \begin{cases} \sigma = \Sigma(t,x) = \dfrac{x - t^3}{t+1} \\ s = S(t,x) = t \end{cases}$$

Finally, the solution is obtained as

$$u(t,x) = U(\Sigma(t,x), S(t,x)) = 2t^2 + \frac{x - t^3}{t+1}$$

3.

Solve $yu_x + xu_y = u$ with $u(x,0) = x^3, u(0,y) = y^3$ for $x, y > 0$.
First, let's solve for $u(x,0) = x^3$. We have

$$\frac{dX(\sigma;s)}{ds} = Y, \frac{dY(\sigma;s)}{ds} = X, \frac{dU(\sigma;s)}{ds} = U$$

with

$$X(\sigma;0) = \sigma, Y(\sigma;0) = 0, U(\sigma;0) = \sigma^3$$

Thus,

$$X(\sigma;s) = \frac{\sigma}{2}(e^{-s} + e^s), Y(\sigma;s) = \frac{\sigma}{2}(e^s - e^{-s}), U(\sigma;s) = e^{s+3\ln(\sigma)}$$

Next, we can obtain the inverse function,

$$\begin{cases} x = X(\sigma;s) = \frac{\sigma}{2}(e^{-s} + e^s) \\ y = Y(\sigma;s) = \frac{\sigma}{2}(-e^{-s} + e^s) \end{cases} \implies \begin{cases} x^2 = \frac{\sigma^2}{4}(e^{-2s} + 2 + e^{2s}) \\ y^2 = \frac{\sigma^2}{4}(e^{-2s} - 2 + e^{2s}) \\ x + y = \sigma e^s \\ x - y = \sigma e^{-s} \end{cases}$$

Hence,

$$\begin{cases} \sigma = \Sigma(x,y) = \sqrt{x^2 - y^2} \\ s = S(x,y) = \frac{1}{2}\ln\left(\frac{x+y}{x-y}\right) \end{cases}$$

Thus the domain for this is $\{(x,y) : x > y\}$ as $x^2 - y^2 > 0$ Finally, the solution is obtained as

$$u_1(x,y) = U(\Sigma(x,y), S(x,y)) = e^{\frac{1}{2}\ln\left(\frac{x+y}{x-y}\right) + \frac{3}{2}\ln(x^2-y^2)} = (x+y)^2(x-y)$$

and similarly, on $u_2(0,y) = y^3$, we have

$$u_2(x,y) = (x+y)^2(y-x)$$

which means that $u_1(x,y) = u_2(x,y)$ on $x = y$ thus

$$u(x,y) = \begin{cases} (x+y)^2(x-y), & \text{if } x > y \\ (x+y)^2(y-x), & \text{if } y > x \end{cases}$$

4.

Solve $u_x^2 + u_y = u$ with $u(x, 0) = x^2$.
First, let $F(x, y, u, p, q) = p^2 + q - u$, we further have

$$X_0(\sigma) = \sigma, Y_0(\sigma) = 0, U_0(\sigma) = \sigma^2 \tag{1}$$

We have

$$\begin{cases} P_0^2 + Q_0^2 - U_0 = P_0^2 + Q_0^2 - \sigma^2 = 0 \\ 2\sigma = U_0'(\sigma) = P_0 \end{cases} \implies \begin{cases} P_0 = 2\sigma \\ Q_0 = \sigma\sqrt{5} \end{cases}$$

Then we have the ODE system

$$\begin{cases} \dfrac{dX}{ds} = F_p = 2P \\ \dfrac{dY}{ds} = F_q = 1 \\ \dfrac{dU}{ds} = PF_p + QF_q = 2P^2 + Q \\ \dfrac{dP}{ds} = -F_x - F_z P = P = 2\sigma \\ \dfrac{dQ}{ds} = -F_y - F_z Q = Q = \sqrt{5}\sigma \end{cases}$$

Therefore,

$$\begin{cases} P(\sigma, s) = 2\sigma s + 2\sigma \\ Q(\sigma, s) = \sqrt{5}\sigma s + \sqrt{5}\sigma \end{cases}$$

We have

$$\begin{cases} \dfrac{dX}{ds} = 4\sigma s + 4\sigma \\ \dfrac{dY}{ds} = 1 \\ \dfrac{dU}{ds} = 2P^2 + Q = 8\sigma^2(s+1)^2 + \sqrt{5}\sigma(s+1) \end{cases}$$

and with the initial conditions (1), we have

$$\begin{cases} X(\sigma, s) = 2\sigma s^2 + 4\sigma s + \sigma \\ Y(\sigma, s) = s \\ U(\sigma, s) = \dfrac{8\sigma^2}{3}(s+1)^3 + \dfrac{\sqrt{5}\sigma}{2}(s+1)^2 - \dfrac{5\sigma^2}{3} - \dfrac{\sqrt{5}\sigma}{2} \end{cases}$$

Hence, we can obtain the inverse function,

$$\begin{cases} \sigma = \dfrac{x}{4y+1} \\ s = y \end{cases}$$

and the solution

$$u(x, y) = U(\sigma, s) = \left(\frac{x}{4y+1}\right)^2 \left(\frac{8}{3}(y+1)^3 - \frac{5}{3}\right) + \frac{\sqrt{5}x}{8y+2}((y+1)^2 - 1)$$

5.

Solve $u_x u_y = xy$ with $u(x,0) = x$.
First, let $F(x,y,u,p,q) = pq - xy$, we further have that

$$X_0(\sigma) = \sigma, Y_0(\sigma) = 0, U_0(\sigma) = \sigma \tag{2}$$

We have

$$\begin{cases} P_0Q_0 - X_0Y_0 = P_0Q_0 = 0 \\ 1 = U_0'(\sigma) = X_0'(\sigma)P_0 + Y_0'(\sigma)Q_0 = P_0 \end{cases} \implies \begin{cases} P_0 = 1 \\ Q_0 = 0 \end{cases}$$

Then we have the ODE system

$$\begin{cases} \dfrac{dX}{ds} = F_p = Q \\ \dfrac{dY}{ds} = F_q = P \\ \dfrac{dU}{ds} = PF_p + QF_q = 2PQ \\ \dfrac{dP}{ds} = -F_x - F_zP = Y \\ \dfrac{dQ}{ds} = -F_y - F_zQ = X \end{cases}$$

From which, we have

$$\frac{d^2Q}{ds^2} = \frac{dX}{ds} = Q$$

thus

$$Q(\sigma,s) = C_1(\sigma)e^s + C_2(\sigma)e^{-s} \text{ with } Q(\sigma,0) = 0$$

and

$$Q(\sigma,s) = C_5(\sigma)(e^s - e^{-s}) = C_5(\sigma)\sinh(s)$$

and similarly,

$$P(\sigma,s) = C_3(\sigma)e^s + C_4(\sigma)e^{-s} \text{ with } P(\sigma,0) = 1$$

and hence $C_3 + C_4 = 1$.
Finally, we can obtain

$$X = C_5\cosh(s)$$

and

$$Y = C_3(\sigma)e^s - C_4(\sigma)e^{-s} = C_3(\sigma)e^s + (C_3(\sigma) - 1)e^{-s} = C_3(\sigma)\cosh(s) - e^{-s}$$

Now, using the initial conditions, we can get

$$X = \sigma\cosh(s) \text{ and } Y = \cosh(s) - e^{-s} = \sinh(s)$$

and the inverse functions

$$\begin{cases} \sigma = \dfrac{x}{\cosh(\ln(y + \sqrt{1+y^2}))} = \dfrac{x}{\sqrt{y^2+1}} \\ s = \sinh^{-1}(y) = \ln(y + \sqrt{1+y^2}) \end{cases}$$

We also have

$$\frac{dQ}{ds} = X = \sigma \cosh(s) \implies Q = \sigma \sinh(s)$$

and

$$\frac{dP}{ds} = Y = \sinh(s) \implies P = \cosh(s)$$

Thus,

$$\frac{dU}{ds} = 2\sigma \sinh(s)\cosh(s)$$

Thus

$$U(\sigma, s) = \sigma \sinh^2(s) + C$$

and using the initial condition $U_0(\sigma) = \sigma$, we have that

$$U(\sigma, s) = \sigma \sinh^2(s) + \sigma$$

Finally, we have the solution

$$\begin{aligned} u(x,y) &= U\left(\frac{x}{\sqrt{y^2+1}}, \ln(y+\sqrt{1+y^2})\right) \\ &= \frac{x}{\sqrt{y^2+1}}\left(\sinh^2(\ln(y+\sqrt{1+y^2}))+1\right) \\ &= \frac{x(y^2+1)}{\sqrt{y^2+1}} = x\sqrt{y^2+1} \end{aligned}$$