

1.

a.

$$f_Y(y) = \int_0^2 \frac{1}{6} dx = \frac{1}{3}$$

Since $0 < y < 3$, $2 < y^2 + 2 < 11$, and hence if $u \leq 2$, $f_U(u) = 0$ and if $u \geq 11$, $f_U(u) = 0$. For $2 < u < 11$, we have that

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(Y^2 + 2 \leq u) \\ &= P(Y^2 \leq u - 2) = P(Y \leq \sqrt{U - 2}) \\ &= \int_0^{\sqrt{U-2}} \frac{1}{3} dy = \frac{\sqrt{U-2}}{3} \end{aligned}$$

b.

Since $0 < x < 2$, $0 < y < 3$, $-3 < x - y < 2$. Therefore, if $v \leq -3$, $f_V(v) = 0$ and if $v \geq 2$, $f_V(v) = 0$. For $-3 < v < 2$, we have that

$$\begin{aligned} F_V(v) &= P(V \leq v) = P(X - Y \leq v) \\ &= P(Y \geq -v + X) \end{aligned}$$

For $-3 < v < -1$,

$$\begin{aligned} F_V(v) &= \int_{-v}^3 \int_0^{y+v} \frac{1}{6} dx dy \\ &= \int_{-v}^3 \frac{y+v}{6} dy = \frac{9}{12} + \frac{3v}{6} - \frac{v^2}{12} + \frac{v^2}{6} \\ &= \frac{(3+v)^2}{12} \\ \implies f_V(v) &= \frac{2v+6}{12} \end{aligned}$$

For $0 < v < 2$,

$$\begin{aligned} F_V(v) &= 1 - \int_v^2 \int_0^{-v+x} \frac{1}{6} dy dx \\ &= 1 - \int_v^2 \frac{1}{6} (-v+x) dx \\ &= 1 + \frac{2v}{6} - \frac{1}{3} - \frac{v^2}{6} + \frac{v^2}{12} \\ &= 1 - \frac{(v-2)^2}{12} \\ \implies f_V(v) &= \frac{-2v+4}{12} \end{aligned}$$

For $-1 < v < 0$

$$\begin{aligned} F_V(v) &= \int_0^2 \int_{-v+x}^3 \frac{1}{6} dy dx \\ &= \frac{4v+8}{12} \\ \implies f_V(v) &= \frac{1}{3} \end{aligned}$$

2.

Since $16 \geq U = Y^4 \geq 0$. We have that if $0 \leq u$, $f_U(u) = 0$ and if $u \geq 16$, $f_U(u) = 0$. For $0 < u < 16$, we have that

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(Y^4 \leq u) = P(-\sqrt[4]{u} \leq Y \leq \sqrt[4]{u}) \\ &= \int_0^{\sqrt[4]{u}} \frac{y}{6} dy + \int_{-\sqrt[4]{u}}^0 \frac{y^2}{4} dy \\ &= \frac{\sqrt{u}}{12} + \frac{\sqrt[4]{u^3}}{12} \end{aligned}$$

and hence

$$f_U(u) = \frac{1}{24u^{1/2}} + \frac{1}{16u^{1/4}}$$

3.

a.

$U = 1 - Y^2 \leq 1$ hence if $u > 1$ then $f_U(u) = 0$, otherwise if $u \leq 1$ then

$$f_Y(y) = \int_y^\infty \frac{1}{9} e^{-x/3} dx = \frac{1}{3} e^{-y/3}$$

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(1 - Y^2 \leq u) = P(Y^2 \geq 1 - u) = P(Y \geq \sqrt{1 - u}) \\ &= \int_{\sqrt{1-u}}^\infty \frac{1}{3} e^{-y/3} dy = e^{-\sqrt{1-u}/3} \end{aligned}$$

and hence

$$f_U(u) = e^{-\sqrt{1-u}/3} \cdot \frac{1}{6\sqrt{1-u}}$$

b.

Since $x > y > 0$, $v = x - y > 0$. Therefore, if $v \leq 0$ then $f_V(v) = 0$, else if $v > 0$ then

$$\begin{aligned} F_V(v) &= P(V \leq v) = P(X - Y \leq v) = P(X \leq v + Y) \\ &= \int_0^\infty \int_y^{y+v} \frac{1}{9} e^{-x/3} dx dy \\ &= \int_0^\infty \frac{\left(e^{\frac{v}{3}} - 1\right) e^{-\frac{y}{3} - \frac{v}{3}}}{3} dy \\ &= 1 - e^{-v/3} \end{aligned}$$

and hence

$$f_U(u) = \frac{1}{3} e^{-v/3}$$

4.

a.

$$x = h^{-1}(u) = \frac{u+3}{2} \implies \frac{d}{du}h^{-1}(u) = \frac{1}{2}$$

$$\text{If } 1 > \frac{u+3}{2} > 0 \iff -1 > u > -3$$

$$f_U(u) = f_X\left(\frac{u+3}{2}\right) \cdot \frac{1}{2} = \frac{-u-1}{2}$$

$$\text{If } u \leq -3 \text{ or } u \geq -1 \text{ then } f_U(u) = 0$$

b.

$$x = h^{-1}(v) = \sqrt[3]{v} \implies \frac{d}{dv}h^{-1}(v) = \frac{1}{3}v^{-2/3}$$

$$\text{If } 1 < \sqrt[3]{v} < 0 \iff 1 < v < 0$$

$$f_V(v) = f_X(\sqrt[3]{v}) \cdot \frac{1}{3}v^{-2/3} = \frac{2}{3}v^{-2/3}(1 - \sqrt[3]{v})$$

$$\text{If } v \geq 1 \text{ or } v \leq 0 \text{ then } f_V(v) = 0.$$

5.

$$y = h^{-1}(u) = \frac{2-u}{4} \implies \frac{d}{du}h^{-1}(u) = -\frac{1}{4}$$

$$\text{If } -1 < \frac{2-u}{4} < 1 \iff -2 < u < 6.$$

For $-2 < u < 6$,

$$f_U(u) = f_X\left(\frac{2-u}{4}\right) \cdot \left|-\frac{1}{4}\right| = \frac{6-u}{32}$$

Otherwise, $f_U(u) = 0$.

6.

Consider $V = Y$

$$x = h_1^{-1}(u, v) = \frac{v}{u}, \quad y = h_2^{-1}(u, v) = v$$

$$\det \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial h_2^{-1}} & \frac{\partial h_1^{-1}}{\partial h_2^{-1}} \\ \frac{\partial u}{\partial h_2^{-1}} & \frac{\partial v}{\partial h_2^{-1}} \end{vmatrix} = \det \begin{vmatrix} \frac{v}{u^2} & \frac{1}{u} \\ 0 & 1 \end{vmatrix} = -\frac{v}{u^2}$$

For $v > 0, u > 0$, we have

$$f_{U,V}(u, v) = f_{X,Y} \left(\frac{v}{u}, v \right) \cdot \left| -\frac{v}{u^2} \right| = \frac{v}{u} e^{-(v/u+v)} \cdot \frac{v}{u^2} = \frac{v^2}{u^3} e^{-(v+uv)/u}$$

Hence, for $u > 0$

$$f_U(u) = \int_0^\infty \frac{v^2}{u^3} e^{-(v+uv)/u} dv = \frac{2}{(u+1)^3}$$

and $f_U(u) = 0$ for $u \leq 0$

7.

a.

For $0 < \frac{u-v}{2} < y, x < \frac{u+v}{2} < 1 \iff 2 > u > v > 0, u < 2-v, v < 1$.

$$x = h_1^{-1}(u, v) = \frac{u-v}{2}, \quad y = h_2^{-1}(u, v) = \frac{u+v}{2}$$

$$\det \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial u} & \frac{\partial h_1^{-1}}{\partial v} \\ \frac{\partial h_2^{-1}}{\partial u} & \frac{\partial h_2^{-1}}{\partial v} \end{vmatrix} = \det \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

For $0 \leq v \leq 1$ and $v \leq u \leq 2-v$, we have that

$$f_{U,V}(u, v) = f_{X,Y} \left(\frac{u-v}{2}, \frac{u+v}{2} \right) \cdot \frac{1}{2} = 6 \cdot \frac{u-v}{2} \cdot \frac{1}{2} = \frac{3(u-v)}{2}$$

$f_{U,V}(u, v) = 0$ otherwise.

b.

For $0 \leq v \leq 1$, we have that

$$f_V(v) = \int_v^{2-v} \frac{3(u-v)}{2} du = 3v^2 - 6v + 3$$

and $f_V(v) = 0$ otherwise.

8.

$$x = h_1^{-1}(u, v) = \frac{uv}{u-1}, \quad y = h_2^{-1}(u, v) = \frac{v}{u-1}$$

For $0 < \frac{uv}{u-1} < 2, 0 < \frac{v}{u-1} < 3$

$$\det \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial u} & \frac{\partial h_1^{-1}}{\partial v} \\ \frac{\partial h_2^{-1}}{\partial u} & \frac{\partial h_2^{-1}}{\partial v} \end{vmatrix} = \det \begin{vmatrix} -v & u \\ (u-1)^2 & u-1 \end{vmatrix} = \frac{v}{(u-1)^2}$$

If $u > 1$ then for $v < \frac{2u-2}{u}, v < 3u-3, v > 0$

$$f_{U,V}(u, v) = f_{X,Y} \left(\frac{uv}{u-1}, \frac{v}{u-1} \right) \cdot \frac{v}{(u-1)^2} = \frac{v}{6(u-1)^2}$$

If $0 < u < 1$ then for $v > \frac{2u-2}{u}, v > 3u-3, v < 0$

$$f_{U,V}(u, v) = f_{X,Y} \left(\frac{uv}{u-1}, \frac{v}{u-1} \right) \cdot \frac{v}{(u-1)^2} = \frac{v}{6(u-1)^2}$$

For $0 < v < 2$,

$$f_V(v) = \int_{-2/(v-2)}^{\infty} \frac{v}{6(u-1)^2} du = -\frac{v-2}{6}$$

For $-1 < v < 0$,

$$f_V(v) = \int_0^{-2/(v-2)} -\frac{v}{6(u-1)^2} du = \frac{1}{3}$$

For $-3 < v < -1$,

$$f_V(v) = \int_0^{(v+3)/3} -\frac{v}{6(u-1)^2} du = \frac{v+3}{6}$$

and $f_V(v) = 0$ otherwise, which is the same as the answer in question 1b

9.

We have that

$$m_U(t) = E[e^{tu}] = E[e^{tcY}] = m_Y(tc) = (1 - \beta tc)^{-n/2}$$

If $U \sim \chi_n^2$ then

$$m_U(t) = m_\chi(t) \implies (1 - \beta tc)^{-n/2} = (1 - 2t)^{-n/2} \implies c = \frac{2}{\beta}$$

10.

$$\begin{aligned}m_U(t) &= E[e^{t(3y^2-2)}] = e^{-2t} E[e^{3ty^2}] \\&= e^{-2t} \int_0^3 e^{3ty^2} \cdot \frac{2y}{9} dy \\&= \frac{e^{-2t}(e^{27t} - 1)}{27t} \\&= \frac{e^{25t} - e^{-2t}}{27t}\end{aligned}$$

which is the moment generating function of Uniform(-2, 25).

11.

$$\begin{aligned}
m_U(t) &= E[e^{-at \ln(y/x)}] = E\left[\left(\frac{y}{x}\right)^{-at}\right] \\
&= \int_0^\infty \int_0^x \left(\frac{y}{x}\right)^{-at} \cdot \frac{e^{-x/2}}{4} dy dx \\
&= \int_0^\infty \frac{x^{-at+1}}{-at+1} \cdot \frac{1}{x^{-at}} \cdot \frac{e^{-x/2}}{4} dx \\
&= \frac{1}{4(-at+1)} \int_0^\infty x \cdot e^{-x/2} \\
&= \frac{1}{4(-at+1)} \cdot 4 \\
&= \frac{1}{-at+1}
\end{aligned}$$

which is the moment generating function of $\text{Exponential}(a)$

12.

We have that

$$F_Y(y) = y^5$$

and hence

$$f_{Y_7}(y) = 7(F_Y(y))^6 f_Y(y) = 35y^{34}$$

Therefore, the expected value of the biggest bonus is

$$55000 \cdot \int_0^1 y \cdot 35y^{34} dy = \frac{35}{36} y^{36} \Big|_0^1 = \frac{35}{36} \cdot 55000 = \frac{481250}{9}$$

13.

$$f_X(x) = \int_0^\infty \frac{1}{\beta\theta} e^{-(x/\beta+y/\theta)} dy = \frac{e^{-x/\beta}}{\beta}$$

$$f_Y(y) = \int_0^\infty \frac{1}{\beta\theta} e^{-(x/\beta+y/\theta)} dx = \frac{e^{-y/\theta}}{\theta}$$

and hence X and Y is independent. Let $U = \min(X, Y)$, then

$$\begin{aligned} F_U(u) &= P(U \leq u) = 1 - P(X \geq u, Y \geq u) \\ &= 1 - P(X \geq u)P(Y \geq u) \\ &= 1 - \int_u^\infty \frac{e^{-x/\beta}}{\beta} dx \cdot \int_u^\infty \frac{e^{-y/\theta}}{\theta} dy \\ &= 1 - e^{-u/\beta} \cdot e^{-u/\theta} \end{aligned}$$

Therefore,

$$f_U(u) = -e^{-u/\beta-u/\theta} \left(-\frac{1}{\beta} - \frac{1}{\theta} \right)$$

and hence

$$\begin{aligned} E[U] &= \int_0^\infty u \cdot -e^{-u/\beta-u/\theta} \left(-\frac{1}{\beta} - \frac{1}{\theta} \right) du \\ &= \left(-\frac{1}{\beta} - \frac{1}{\theta} \right) \cdot \left(-\frac{1}{\left(\frac{1}{\beta} + \frac{1}{\theta} \right)^2} \right) \\ &= \frac{1}{\frac{1}{\beta} + \frac{1}{\theta}} = \frac{\beta\theta}{\beta + \theta} \end{aligned}$$

14.

Let $U = \min(X, Y)$, then

$$\begin{aligned}F_U(u) &= P(U \leq u) = 1 - P(X \geq u, Y \geq u) \\&= 1 - \int_u^1 \int_u^1 \frac{4}{3}x + \frac{2}{3}y dx dy \\&= -u^3 + u^2 + u\end{aligned}$$

and hence

$$f_U(u) = -3u^2 + 2u + 1$$

Therefore,

$$E[U] = \int_0^1 u(-3u^2 + 2u + 1) du = \frac{5}{12}$$

15.

For $Y_1 < Y_2$, we have that

$$\begin{aligned} f_{Y_1, Y_2}(y_1, y_2) &= \frac{2!}{0!} [F_{Y_2}(y_2) - F_{Y_1}(y_1)]^0 f_Y(Y_2) \cdot f_Y(y_1) \\ &= 2 \cdot \frac{1}{\beta} e^{-y_1/\beta} \cdot \frac{1}{\beta} e^{-y_2/\beta} \\ &= 2 \cdot \frac{1}{\beta^2} e^{-(y_1+y_2)/\beta} \end{aligned}$$

Therefore,

$$\begin{aligned} P(3Y_1 < Y_2) &= \int_0^\infty \int_{3y_1}^\infty 2 \cdot \frac{1}{\beta^2} e^{-(y_1+y_2)/\beta} dy_2 dy_1 \\ &= \int_0^\infty \frac{2e^{-\frac{4y_1}{\beta}}}{\beta} dy_1 \\ &= \left(-\frac{e^{-\frac{4x}{\beta}}}{2} \right)_0^\infty \\ &= \frac{1}{2} \end{aligned}$$

Therefore,

$$P(3Y_1 > Y_2) = 1 - P(3Y_1 < Y_2) = \frac{1}{2}$$