1.

1.

Since a function has an inverse if and only if that function is bijective.
We have that a G-equivariant function $f : X \to Y$ is bijective if and only if it is an isomorphism.

2.

$1 \in G$, therefore $\forall g \in G$ and $\forall g_O \cdot x \in \mathrm{Orb}_G(x)$:

$$1 \cdot (g \cdot (g_O \cdot x)) = g \cdot (1 \cdot (g_O \cdot x))$$

Hence, there exists an action of G on $\mathrm{Orb}_G(x)$ such that the inclusion $\mathrm{Orb}_G(x) \hookrightarrow X$ is G-equivariant.
Since the inclusion maps $g_O \cdot x \to g_O \cdot x$ the action is unique.

3.

Let the map be f

$$g \cdot f(g \cdot \mathrm{Stab}_G(x)) = g \cdot g \cdot x = (g \cdot g) \cdot x = f(g \cdot g \cdot \mathrm{Stab}_G(x))$$

4.

$1 \in G, g_1 \cdot H = H \in G/H$ and $\mathrm{Orb}_G(g_1 \cdot H) = \{g_2 \cdot H | g_2 \in G\} = G/H$.

5.

Since the action of G on X is transitive, $\exists x_0 \in X : X = \mathrm{Orb}_G(x_0)$.
We can define the isomorphisms as follow:

$$f : X \to G/H, \qquad x \to g \cdot H \text{ where } g \cdot x_0 = x$$
$$h : G/H \to X, \qquad g \cdot H \to x \text{ where } g \cdot x_0 = x$$

We have that $\forall g \in G, \forall x \in X$:

$$f(g \cdot x) = g_1 \cdot H \text{ where } g_1 \cdot x_0 = g \cdot x$$
$$g \cdot f(x) = g \cdot g_2 \cdot H \text{ where } g_2 \cdot x_0 = x$$

Hence $g \cdot g_2 \cdot x_0 = g \cdot x = g_1 \cdot x_0$ which means that $g \cdot g_2 = g_1$, therefore

$$g \cdot f(x) = g_1 \cdot H = f(g \cdot x)$$

We have that $\forall g \in G, \forall g^* \cdot H \in G/H$:

$$h(g \cdot g^* \cdot H) = x_1 \text{ where } g \cdot g^* \cdot x_0 = x_1$$
$$g \cdot h(g^* \cdot H) = g \cdot x_2 \text{ where } g^* \cdot x_0 = x_2$$

Hence $g \cdot x_2 = g \cdot g^* \cdot x_0 = x_1$ which means that

$$h(g \cdot g^* \cdot H) = g \cdot h(g^* \cdot H)$$

It is also obvious that $f \circ h = 1_{G/H}$ and $h \circ f = 1_X$

6.

Let the inclusion be f $1 \in G$, let 1 act on the disjoint union, therefore $\forall i$, let $x_i \in X_i$, then

$$f(g \cdot x_i) = g \cdot x_i = g \cdot f(x_i)$$

Hence, there is an action of G on the disjoint union such that the inclusion are G-equivariant for all i Since the inclusion maps $x_i \to x_i$, the action is unique.

7.

We know that the action of G on G/H_i is transitive and therefore, there is a G-equivariant isomorphism between X and G/H_i. And since the action is unique, we know that the function maps X to the disjoint union is G-equivariant and isomorphism.

2.

1.

A_i is a distinct orbits of H acting on X. Hence, $\exists h_i \in H : h_i \cdot X = A_i$. Hence, as H is a normal subgroup, $\exists j : \forall g \in G : g \cdot h_i \cdot X = h_j \cdot g \cdot X = A_j$ $1 \in G$, therefore $\forall i : 1 \cdot A_i = A_i$.
$\forall g, h \in G, \forall i : (g \cdot h) \cdot A_i = \{g \cdot h \cdot a_i | a_i \in A_i\} = g \cdot (h \cdot A_i)$ Since $1 \in H, H \in \{A_1, A_2, \ldots, A_r\}$. Therefore, $Orb_G(H) = \{A_1, A_2, \ldots, A_r\}$ which means that the action is transitive. Since $A_1, A_2, \ldots, A_r$ are distinct orbits and each $A_i = h_i \cdot H$ for some h_i, they have the same size.

2.

Since H and $\mathrm{Stab}_G(x)$ are subgroups of G.

$$\#(H \cdot \mathrm{Stab}_G(x)) = \frac{\#H \cdot \#\mathrm{Stab}_G(x)}{\#(H \cap \mathrm{Stab}_G(x))}$$

Therefore,

$$\#A_1 = \frac{\#(A_1 \cdot \mathrm{Stab}_G(a))}{\#\mathrm{Stab}_G(a)} = \frac{\#H}{\#(H \cap \mathrm{Stab}_G(x))} = [H : H \cap \mathrm{Stab}_G(a)]$$

We also have

$$\#(\mathrm{Stab}_G(a) \cdot H) = \frac{\#\mathrm{Stab}_G(a) \cdot \#H}{\#(\mathrm{Stab}_G(a) \cap H)}$$

Hence,

$$\frac{\#G}{\#\mathrm{Stab}_G(a) \cdot H} = \frac{\#G \cdot \#(\mathrm{Stab}_G(a) \cap H)}{\#\mathrm{Stab}_G(a) \cdot \#H} = \frac{\#G \cdot \#\mathrm{Stab}_G(a)}{\#\mathrm{Stab}_G(a) \cdot \#H} = \frac{\#G}{\#H} = r$$

3.

1.

Since N is a normal subgroup of order 2, it includes the identity element and a non-identity element n which inverse is itself. Then $\forall g \in G$ we have that $g \cdot n \cdot g^{-1} \in \{a, 1\}$.
If $g \cdot n \cdot g^{-1} = n$, then $g \cdot n \cdot g^{-1} \cdot g = g \cdot n = n \cdot g$.
If $g \cdot n \cdot g^{-1} = 1$, then $gn = g$ which means that $n = 1$ which is a contradiction.

2.

Let the group be G, then as G has order 6, one of its element must have order 2.
Therefore, that element and 1 forms a subgroup H in G. Same argument, there is an element with order 3 and hence create a subgroup K with order 3. We know that there is a map $\rho : G \to S_2$ with $\rho(K) = 1$, also $[G : K] = 2$ hence K is normal.
Hence, if H is normal then $HK = G$ is abelian which is a contradiction.

3.

Since we know that there exists $x, y \in G$ has order 2,3 respectively.
If G is abelian then $xy = yx$ G is cyclic.
If G is not abelian then
If $yx = y^m$ for some m then $x = y^{m-1}$ which is a contradiction as then x, y commutes.
If $yx = y$ then $b = 1$ which is a contradiction. If $ba = ab^2$ then this gropu is isoromorphic to S_3

4.

1.

Let the group be G. Since $\{1\}$ is a conjugacy class, let S_a be the other. Since the conjugacy class of G partition G. $\#G = 1 + \#S_a$ also $\#S_a$ divides $\#S_a + 1$. Therefore $\#S_a = 1$ and $\#G = 2$ which means that G has the identity element and another element whose inverse is itself.

2.

Let $x \leq y$ be the sizes of the two conjugacy classes that is not $\{1\}$. Then $\#G = 1 + x + y$. Then since both x and y divide $\#G$, x divides $1 + y$ and $y divides 1 + x, x = y = 1 or y = 1 + x$ If $y = 1 + x$ then since x divides $1 + y = 2 + x, x \in \{1, 2\}, (x, y) \in \{(1, 1), (1, 2), or (2, 3)\}$. If $(x, y) = (1, 1)$ then $\#G = 3$ and hence $G = Z_3$. If $(x, y) = (1, 2)$ then $\#G = 4$ and since there are only up-to-isomorphism two abelian groups with order four, G has four conjugacy classes which leads to a contradiction. If $(x, y) = (2, 3)$ then $\#G = 6$, there is up-to-isomorphism only one nonabelian group of order 6, which is S3.

5.

1.

We have that $Z(G) \leq C(g_i)$, therefore $[G : C(g_i)] \leq [G : Z(G)] = n$

2.

Since $\#G = \sum_{i=1}^{r} [G : C_G(g_i)]$ we have that $[G : C_G(g_i)] = [G : Z(G)] = n$ and therefore each conjugacy class has 1 element, which means that the group is abelian.

6.

1.

$$\forall g^* \in G : \varphi_{g^*} : G \to G, \qquad g \to g^* \cdot g \cdot {g^*}^{-1}$$

is an automorphism and hence if $\varphi_{g^*}(H) = H$ then H is normal.

2.

Consider the additive group $\mathbb{Q}$. Then $\mathbb{Z}$ is a normal subgroup. However, the automorphism

$$\varphi : \mathbb{Q} \to \mathbb{Q}, \qquad x \to x/2$$

does not map $\mathbb{Z}$ to $\mathbb{Z}$.

3.

Since for all automorphism $\varphi : \#(\varphi(H)) = \#H$ which means that $\varphi(H) = H$ as H is the only subgroup with order n, which means that H is characteristic then normal.

4.

Since H is the unique subgroup of G of index n, we have that H is the unique subgroup of G of order n which means that H is characteristic and normal as proven above.