A1.1

We have

$$-1 = \phi(-1) = \phi(i \cdot i) = \phi(i) \cdot \phi(i) \implies \phi(i) = i \text{ or } \phi(i) = -i$$

First, consider the case $\phi(i) = i$, for any $z = x + iy$ where $x, y \in \mathbb{R}$ we have that

$$\phi(z) = \phi(x + iy) = \phi(x) + \phi(iy) = x + \phi(i) \cdot \phi(y) = x + iy = z$$

In the other case $\phi(i) = -i$, for any $z = x + iy$ where $x, y \in \mathbb{R}$

$$\phi(z) = \phi(x + iy) = \phi(x) + \phi(i) \cdot \phi(y) = x - iy = \overline{z}$$

Hence, ϕ is either the identity or the conjugate map.

A1.2

a

According to the Fundamental Theorem of Algebra, we know that every polynomials equation of degree n with complex number coefficents has n roots in $\mathbb{C}$, hence for every polynomials $f(z) \in \mathbb{R}[x]$, we can rewrite it in $\mathbb{C}$ as

$$f(z) = \sum_{i=1}^{n} a_i z^i = a_n \prod_{i=1}^{n} (z - z_i)$$

for some $z_i \in \mathbb{C}$. Suppose z is a root of the polynomial, then

$$\begin{aligned} f(\overline{z}) &= \sum_{i=1}^{n} a_i \overline{x}^i \\ &= \sum_{i=1}^{n} \overline{a_i}\overline{x^i} \\ &= \sum_{i=1}^{n} \overline{a_i x^i} \\ &= \overline{\sum_{i=1}^{n} a_i x^i} \\ &= 0 \end{aligned}$$

Hence, $\overline{z}$ is also a root.
Now suppose there is an irreducible polynomials in $\mathbb{R}[x]$ of degree greater than 2. If the degree of that polynomial f(x) is odd, then applying the intermediate value theorem when we consider $\lim_{x \to \infty} g(x)$ and $\lim_{x \to -\infty} g(x)$, yields that there is a real root. Therefore, the irreducible polynomial must have an even degree. Since it has an even degree, we can rewrite $f(x)$ as

$$f(x) = a_n \prod_{i=1}^{n/2} (x - z_i)(x - \overline{z_i}) = a_n \prod_{i=1}^{n/2} (x^2 - 2\mathrm{Re}(z_i)x + |z_i|^2)$$

b

$$\begin{aligned} & x^8 + 8 = 0 \\ \Longrightarrow\ & x^8 = -8 \\ \Longrightarrow\ & x^8 = 8(e^{i\pi}) \\ \Longrightarrow\ & x = x_k = \sqrt[8]{8} \left(\cos\left(\frac{\pi}{8} + \frac{2\pi k}{8}\right) + i \sin\left(\frac{\pi}{8} + \frac{2\pi k}{8}\right)\right) \end{aligned}$$

for natural $0 \leq k \leq 7$.
We have for any $0 \leq k \leq 3$,

$$\begin{aligned}
\overline{x_k} &= \sqrt[8]{8}\left(\cos\left(\frac{\pi}{8}+\frac{2\pi k}{8}\right) - i\sin\left(\frac{\pi}{8}+\frac{2\pi k}{8}\right)\right) \\
&= \sqrt[8]{8}\left(\cos\left(\frac{15\pi}{8}-\frac{2\pi k}{8}\right) + i\sin\left(\frac{15\pi}{8}-\frac{2\pi k}{8}\right)\right) \\
&= \sqrt[8]{8}\left(\cos\left(\frac{\pi}{8}+\frac{2\pi(7-k)}{8}\right) + i\sin\left(\frac{\pi}{8}+\frac{2\pi(7-k)}{8}\right)\right) \\
&= x_{7-k}
\end{aligned}$$

Therefore,

$$(x-x_k)(x-x_{7-k}) = x^2 - 2x\left(\sqrt{2}\cos\left(\frac{\pi}{8}+\frac{2\pi k}{8}\right)\right) + \sqrt[4]{8}$$

Hence,

$$x^8+8 = \prod_{k=0}^{3}\left(x^2 - 2x\left(\sqrt{2}\cos\left(\frac{\pi}{8}+\frac{2\pi k}{8}\right)\right) + \sqrt[4]{8}\right)$$

A1.3

First consider the isomorphic map:

$$\phi : (x, y) \to x + iy, \qquad \mathbb{R}^2 \to \mathbb{C}$$

We have that for all $n \in \mathbb{N}^*$

$$\phi(P_n) - \phi(P_{n-1}) = n \cdot e^{\frac{2\pi(n-1)}{3}i}$$

$e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$ Hence, we have that

$$\begin{aligned}
\phi(P_n) &= \sum_{j=1}^{n} (\phi(P_j) - \phi(P_{j-1})) + \phi(P_0) \\
&= \sum_{j=1}^{n} j e^{\frac{2\pi(j-1)}{3}i} \\
&= e^{-2i\pi/3} \sum_{j=1}^{n} j \left(e^{2\pi i/3}\right)^j \\
&= \frac{(ne^{2\pi i/3} - n - 1)e^{2\pi i(n+1)/3} + e^{2\pi i/3}}{(1 - e^{2\pi i/3})^2 e^{2\pi i/3}} \\
&= \frac{(ne^{2\pi i/3} - n - 1)e^{2\pi i n/3} + 1}{(1 - e^{2\pi i/3})^2} \\
&= \frac{(n\left(\frac{-1+i\sqrt{3}}{2}\right) - n - 1)e^{2\pi i n/3} + 1}{(1 - \left(\frac{-1+i\sqrt{3}}{2}\right))^2} \\
&= \frac{e^{2\pi i n/3}\left(\frac{-3n+ni\sqrt{3}}{2} - 1\right) + 1}{\left(\frac{3-i\sqrt{3}}{2}\right)^2} \\
&= \frac{2}{3} \cdot \frac{e^{2\pi i n/3}\left(\frac{-3n+ni\sqrt{3}}{2} - 1\right) + 1}{1 - i\sqrt{3}} \\
&= \frac{2}{3} \cdot \frac{\left(e^{2\pi i n/3}\left(\frac{-3n+ni\sqrt{3}}{2} - 1\right) + 1\right)(1 + i\sqrt{3})}{(1 - i\sqrt{3})(1 + i\sqrt{3})} \\
&= \frac{1}{6} \cdot \left(e^{2\pi i n/3}\left(\frac{-3n + ni\sqrt{3}}{2} - 1\right) + 1\right)(1 + i\sqrt{3}) \\
&= \frac{1}{6}\left(e^{2\pi i n/3}\left(\frac{-3n + ni\sqrt{3}}{2} - 1\right)(1 + i\sqrt{3}) + 1 + i\sqrt{3}\right) \\
&= \frac{1}{6} \cdot (e^{2\pi i n/3}(-1 - i\sqrt{3} - 3n - i\sqrt{3}n) + 1 + i\sqrt{3})
\end{aligned}$$

When $3|n$, $e^{2\pi i n/3} = 1$ and hence,

$$\phi(P_n) = \frac{1}{6} \cdot (-1 - i\sqrt{3} - 3n - i\sqrt{3}n + 1 + i\sqrt{3}) = -\frac{n}{2} - \frac{n\sqrt{3}}{6}i$$

which means

$$P_n = \left(-\frac{n}{2}, -\frac{n\sqrt{3}}{6}\right)$$

If remainder of n divides 3 is 1, then $e^{2\pi in/3} = -1/2 + \sqrt{3}i/2$ and hence

$$\begin{aligned}
\phi(P_n) &= \frac{1}{6} \cdot \left(\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)(-1 - i\sqrt{3} - 3n - i\sqrt{3}n) + 1 + i\sqrt{3}\right) \\
&= -i\frac{\sqrt{3}(-1+n)}{6} + \frac{1+n}{2}
\end{aligned}$$

which means

$$P_n = \left(\frac{1+n}{2}, -\frac{\sqrt{3}(-1+n)}{6}\right)$$

If remainder of n divides 3 is 2, then $e^{2\pi in/3} = -1/2 - \sqrt{3}i/2$ and hence

$$\begin{aligned}
\phi(P_n) &= \frac{1}{6} \cdot \left(\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)(-1 - i\sqrt{3} - 3n - i\sqrt{3}n) + 1 + i\sqrt{3}\right) \\
&= \frac{\sqrt{3}}{3}(n+1)i
\end{aligned}$$

which means

$$P_n = \left(0, \frac{\sqrt{3}}{3}(n+1)\right)$$

A1.4

Let $z = x + iy$, then define

$a = \dfrac{if(1) + f(i)}{2i}$

$b = \dfrac{if(1) - f(i)}{2i}$

We have that

$$\begin{aligned} az + b\overline{z} &= a(x+iy) + b(x-iy) \\ &= x(a+b) + iy(a-b) \\ &= x \cdot f(1) + iy \cdot \frac{f(i)}{i} \\ &= f(x) + f(iy) \\ &= f(x+iy) \\ &= f(z) \end{aligned}$$

A1.5

Define a linear transformation

$$g : \mathbb{C} \to \mathbb{C}, \qquad z \to \frac{i\overline{f(1)} - \overline{f(i)}}{\overline{f(1)}f(i) - f(1)\overline{f(i)}} z + \frac{if(1) - f(i)}{\overline{f(i)}f(1) - f(i)\overline{f(1)}} \overline{z}$$

so that

$$g(f(1)) = 1$$

$$g(f(i)) = i$$

Notice that the denominator $\overline{f(1)}f(i) - f(1)\overline{f(i)} \neq 0$ as it means that $\mathrm{Im}(f(1)\overline{f(i)}) = 0$, which means both $\mathrm{Im}(f(1))$ and $\mathrm{Im}(f(i)) = 0$. Hence, $|f(1) - f(i)| = 2 \neq \sqrt{2} = |1 - i|$.

$$g \circ f(z) = \frac{i\overline{f(1)} - \overline{f(i)}}{\overline{f(1)}f(i) - f(1)\overline{f(i)}} f(z) + \frac{if(1) - f(i)}{\overline{f(i)}f(1) - f(i)\overline{f(1)}} \overline{f(z)}$$

$$|g(f(z_1 - z_2))| = |g(f(z_1) - f(z_2))| = |g(f(z_1)) - g(f(z_2))|$$

We know that $|f(z_1) - f(z_2)| = |z_1 - z_2| = |f(z_1 - z_2) - f(0)| = |f(z_1 - z_2)|$ and g is a linear transformation, therefore

$$|g(f(z_1 - z_2))| = |g(f(z_1) - f(z_2))| = |g(f(z_1)) - g(f(z_2))|$$

Which means that $h := g \circ f$ is also an isometry. Since $g \circ f$ fixes $0, 1, i$, we have the following

$$|h(z)| = |z|, \qquad |h(z) - 1| = |z - 1|, \qquad |h(z) - i| = |z - i|$$

Square the three equations, we have

$$h(z)\overline{h(z)} = z\overline{z}, \qquad (h(z)-1)(\overline{h(z)}-1) = (z-1)(\overline{z}-1), \qquad (h(z)-i)(\overline{h(z)}+i) = (z-i)(\overline{z}+i)$$

Expanding the second and third equations yield,

$$h(z)\overline{h(z)} - h(z) - \overline{h(z)} = z\overline{z} - z - \overline{z}, \qquad h(z)\overline{h(z)} + ih(z) - i\overline{h(z)} = z\overline{z} + iz - i\overline{z}$$

Substitute in the 2 equations in the first equation $h(z)\overline{h(z)} = z\overline{z}$, we have

$$h(z) + \overline{h(z)} = z + \overline{z}, \qquad h(z) - \overline{h(z)} = z - \overline{z}$$

Hence, $h(z) = z$, which means $g \circ f$ is the identity map.
f is injective as $|f(z_1) - f(z_2)| = 0 \implies |z_1 - z_2| = 0$.
Consider B_r where r is the radius. Suppose f is not bijective. Then we can find some $z \in B_r \backslash f(B_r)$. Then the sequence $z_0 = z, z_1 = f(z_0), z_2 = f(z_1), \ldots$ has a convergent subsequence. However, we can choose $0 < \epsilon < \inf\{|z - z^*| : z^* \in \mathrm{Img}(f)\}$. so that $|f(z_n) - f(z_m)| = |f(z_0) - f(z_{m-n})| \geq \epsilon$ for $m > n$, which means that there is no convergent subsequence and therefore a contradiction. Hence, f is bijective, and thus f, the inverse of g is also linear.