1.

Apply speration of variables, $u(x,t) = X(x)T(t)$, we have

$$\begin{aligned}
&\rho_0 X(x)T''(t) = T_0 X''(x)T(t) - \beta X(x)T'(t) \\
\implies &X(x)(\rho_u T''(t) + \beta T'(t)) = T_0 X''(x)T(t) \\
\implies &\frac{\rho_u T''(t) + \beta T'(t)}{T_0 T(t)} = \frac{X''(x)}{X(x)} = -\lambda
\end{aligned}$$

and

$$\begin{cases} u(0,t) = 0 \implies X(0) = 0 \\ u(L,t) = 0 \implies X(L) = 0 \end{cases}$$

Thus for every integer $n \geq 1$,

$$X_n(x) = \sin \frac{n\pi x}{L}$$

and

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

Now we need to solve for

$$\rho_u T''(t) + \beta T'(t) + T_0 \frac{n^2\pi^2}{L^2} T(t) = 0$$

which has no solution as

$$\beta^2 - 4\rho_u \frac{T_0 n^2 \pi^2}{L^2} < \beta^2 - \frac{4\rho_u \pi^2 T_0}{L^2} < 0$$

2.

Apply speration of variables, $u(x,t) = X(x)T(t)$, we have

$$\begin{cases} u(0,t) = 0 \implies X(0) = 0 \\ u(L,t) = 0 \implies X(L) = 0 \\ u_t(x,0) = 0 \implies T'(0) = 0 \end{cases}$$

and

$$X(x)T''(t) = c^2 X''(x)T(t)$$
$$\implies \frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = -\lambda$$

and Thus for every integer $n \geq 1$,

$$X_n(x) = \sin \frac{n\pi x}{L}$$

and

$$\lambda_n = \frac{c^2 n^2 \pi^2}{L^2}$$

and thus

$$T''(t) + \lambda T(t) = 0$$

and

$$T(t) = c_1 \cos \frac{cn\pi t}{L} + c_2 \sin \frac{cn\pi t}{L}$$
$$T'(t) = \frac{cn\pi}{L} \left(-c_1 \sin \frac{cn\pi t}{L} + c_2 \cos \frac{cn\pi t}{L} \right)$$

Therefore,

$$T'(0) = \frac{cn\pi}{L} c_2 \cos \frac{cn\pi 0}{L} = \frac{cn\pi c_2}{L} = 0 \implies c_2 = 0$$

Thus $c_1 \neq 0$ to avoid trivial solution

$$T_n(t) = \cos \frac{cn\pi t}{L}$$

and

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L} \\ &= \sum_{n=1}^{\infty} \frac{A_n}{2} \left(\sin \frac{n\pi(x+ct)}{L} + \sin \frac{n\pi(x-ct)}{L} \right) \\ &= \frac{F(x+ct) - F(x-ct)}{2} \end{aligned}$$

As

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

from

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = f(x)$$

3.

Apply speration of variables, $u(x,t) = X(x)T(t)$, we have

$$\begin{cases} u(0,t) = 0 \implies X(0) = 0 \\ u(L,t) = 0 \implies X(L) = 0 \\ u(x,0) = 0 \implies T(0) = 0 \end{cases}$$

and

$$\begin{aligned} & X(x)T''(t) = c^2 X''(x)T(t) \\ \implies & \frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = -\lambda \end{aligned}$$

and Thus for every integer $n \geq 1$,

$$X_n(x) = \sin \frac{n\pi x}{L}$$

and

$$\lambda_n = \frac{c^2 n^2 \pi^2}{L^2}$$

and thus

$$T''(t) + \lambda T(t) = 0$$

and

$$T(t) = c_1 \cos \frac{cn\pi t}{L} + c_2 \sin \frac{cn\pi t}{L}$$

Therefore,

$$T(0) = c_1 \cos \frac{cn\pi 0}{L} = 0 \implies c_1 = 0$$

Thus $c_2 \neq 0$ to avoid trivial solution

$$T_n(t) = \sin \frac{cn\pi t}{L}$$

and

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sin \frac{cn\pi t}{L} \\ &= \sum_{n=1}^{\infty} \frac{A_n}{2} \left(\cos \frac{n\pi(x-ct)}{L} - \cos \frac{n\pi(x+ct)}{L} \right) \\ &= \sum_{n=1}^{\infty} \frac{A_n}{2} \cos \frac{n\pi \overline{x}}{L} \Bigg|_{x+ct}^{x-ct} \\ &= \sum_{n=1}^{\infty} \frac{A_n}{2} \int_{x-ct}^{x+ct} \frac{n\pi}{L} \sin \frac{n\pi \overline{x}}{L} d\overline{x} \\ &= \frac{1}{2c} \int_{x-ct}^{x+ct} \underbrace{\sum_{n=1}^{\infty} A_n \frac{n\pi c}{L} \sin \frac{n\pi \overline{x}}{L}}_{G(\overline{x})} d\overline{x} \end{aligned}$$

as we can get A_n to match

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} A_n \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L} d\overline{x}$$

Thus as $u_t(x,0) = f(x)$, we have

$$\frac{n\pi c}{L} A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

4.

1.

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^L u_t^2(x,t)dx + \frac{c^2}{2}\int_0^L u_x^2(x,t)dx \\ &= \frac{1}{2}\int_0^L u_t^2(x,t) + c^2 u_x^2(x,t)dx \end{aligned}$$

Thus

$$\begin{aligned} E'(t) &= \frac{1}{2}\int_0^L 2u_t u_{tt} + 2c^2 u_x u_{xt} dx \\ &= c^2 \int_0^L u_t u_{xx} + u_x u_{xt} dx \\ &= c^2 \left. (u_x u_t) \right|_0^L \end{aligned}$$

2.

a.

$$E'(t) = 0$$

Thus energy is conserved

b.

$$E'(t) = 0$$

Thus energy is conserved

c.

d.