1.

a.

$$\begin{aligned}\frac{z}{z^3-z^2-z+1} &= -\frac{1}{4(z+1)}+\frac{1}{4(z-1)}+\frac{1}{2(z-1)^2}\\ &= -\frac{1}{4}\frac{1}{1-(-z)}-\frac{1}{4}\frac{1}{1-z}+\frac{1}{2}\left(\frac{1}{1-z}\right)'\\ &= \sum_{n=0}^{\infty}-\frac{1}{4}(-z)^n-\frac{1}{4}z^n+\frac{1}{2}\left(\sum_{n=0}^{\infty}z^n\right)'\\ &= \sum_{n=0}^{\infty}\left(-\frac{1}{4}(-1)^n-\frac{1}{4}\right)z^n+\frac{1}{2}\left(\sum_{n=0}^{\infty}(n+1)z^n\right)\end{aligned}$$

b.

$$\begin{aligned}\frac{z}{z^3-z^2-z+1} &= -\frac{1}{4(z+1)}+\frac{1}{4(z-1)}+\frac{1}{2(z-1)^2}\\ &= -\frac{1}{4z(1+1/z)}+\frac{1}{4z(1-1/z)}-\frac{1}{2}\left(\frac{1}{z(1-1/z)}\right)'\\ &= \sum_{n=0}^{\infty}-\frac{1}{4z}\left(-\frac{1}{z}\right)^n+\frac{1}{4z}\left(\frac{1}{z}\right)^n-\frac{1}{2}\left(\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right)'\\ &= \sum_{n=0}^{\infty}-\frac{1}{4}(-1)^n z^{-n-1}+\frac{1}{4}z^{-n-1}-\frac{1}{2}\left(\sum_{n=0}^{\infty}z^{n-1}\right)'\\ &= \sum_{n=0}^{\infty}-\frac{1}{4}(-1)^n z^{-n-1}+\frac{1}{4}z^{-n-1}-\frac{1}{2}\left(-\frac{1}{z^2}+\sum_{n=0}^{\infty}(n+1)z^n\right)\end{aligned}$$

c.

$$\begin{aligned}\frac{z}{z^3-z^2-z+1} &= -\frac{1}{4(1+z)}+\frac{1}{4(z-1)}+\frac{1}{2(z-1)^2}\\ &= -\frac{1}{4(z-2)}\frac{1}{1+\frac{3}{z-2}}+\frac{1}{4(1+(z-2))}+\frac{1}{2}\left(\frac{1}{z-1}\right)'\\ &= -\frac{1}{4(z-2)}\sum_{n=0}^{\infty}\left(-\frac{3}{z-2}\right)^n+\frac{1}{4}\sum_{n=0}^{\infty}(-(z-2))^n+\frac{1}{2}\left(\sum_{n=0}^{\infty}(-(z-2))^n\right)'\\ &= \sum_{n=-1}^{\infty}-\frac{(-3)^{n+1}}{4}\left(\frac{1}{z-2}\right)^n+\sum_{n=0}^{\infty}\frac{(-1)^n}{4}(z-2)^n+\frac{1}{2}\sum_{n=0}^{\infty}(n+1)(-1)^n(z-2)^n\end{aligned}$$

d.

$$\begin{aligned}
\frac{z}{z^3-z^2-z+1} &= -\frac{1}{4(1+z)} + \frac{1}{4(z-1)} + \frac{1}{2(z-1)^2} \\
&= -\frac{1}{4}(z+1)^{-1} + \frac{1}{4(z+1)}\frac{1}{1-\frac{2}{z+1}} + \frac{1}{2}\left(\frac{1}{z-1}\right)' \\
&= -\frac{1}{4}(z+1)^{-1} + \frac{1}{4(z+1)}\sum_{n=0}^{\infty}\frac{2^n}{(z+1)^n} + \frac{1}{2}\left(\frac{1}{z+1}\sum_{n=0}^{\infty}\frac{2^n}{(z+1)^n}\right)' \\
&= -\frac{1}{4}(z+1)^{-1} + \frac{1}{4}\sum_{n=0}^{\infty}\frac{2^n}{(z+1)^{n+1}} + \frac{1}{2}\left(\sum_{n=2}^{\infty}\frac{(-n+1)2^n}{(z+1)^n}\right)
\end{aligned}$$

2.

a.

We have that $\lim_{z\to 0} |\cos(z)| = \infty$ and $\lim_{z\to 0} |z^2 - z| = 0$. Hence, 0 is a pole.

b.

We have that $\lim_{\substack{x\to 0\\ y=0}} \cos\left(\frac{z+1}{z}\right)$ does not exists. Hence, it is essential.

3.

Consider

$$g(z) = \frac{z+a}{1+\overline{a}z} \text{ and } h(w) = \frac{w-b}{1-\overline{b}w}$$

Then f, g, h are biholomorphic maps and so $h \circ f \circ g$ is a biholomorphic map such that $(h \circ f \circ g)(0) = 0$. Thus

$$|(h \circ f \circ g)(0)| = 1$$

and hence schwarz's lemma implies that

$$|f'(a)| = \frac{1}{|h'(b)| \circ |g'(0)|} = \frac{1-|b|^2}{1-|a|^2}$$

4.

Every biholomorphic maps from the upper half plane to the unit disc has the form

$$e^{i\theta}\frac{z-\beta}{z-\overline{\beta}}$$

for some $\theta \in \mathbb{R}$ and β in the upper half plane. Then,

$$e^{(i\theta - i\pi/2)/2}|z|$$

maps the upper half plane to D. Hence, biholomorphic maps mapping D to the unit disc also has the form

$$e^{i\theta}\frac{z-w}{z-\overline{w}}$$

for some $w \in D$. We also know that biholomorphic mapping between the unit disc is of that form. Hence, all biholomorphic maps $D \to D$ is of this form as well.

5.

a.

If both f, g has pole at 0. Then $\lim |f(z)g(z)| = \infty$, which means that fg also has pole at 0.

b.

fg can be removable at 0, but f can be not removable at 0. Let $g = 0$. Then $fg = g = 0$ but f can be anything at 0 including essential or pole.

c.

Consider

$$f = \begin{cases} e^{iz}, & \text{Re(z)} \geq 0 \\ 1, & \text{otherwise} \end{cases}$$

and

$$g = \begin{cases} e^{-iz}, & \text{Re(z)} < 0 \\ 1, & \text{otherwise} \end{cases}$$

Then fg has a pole at 0 but both f, g has essential singularity at 0.

d.

If neither f and g has an essential singularity at 0, that is $a_n, b_n \neq 0$ for finitely many $n < 0$, then $a_n b_k \neq 0$ for finitely many $n + k < 0$, and hence fg does not have an essential singularity at 0.