1.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. As $\mathcal{G}_u \subseteq \mathcal{G}_t$, we have that

$$E[E[X_t|\mathcal{G}_t]|\mathcal{G}_u] = \int_{G_u} E[X_t|\mathcal{G}_t]dP(\mathcal{G}_u)$$

$$= \int_{G_u} \int_{G_t} XdP(\mathcal{G}_t)dP(\mathcal{G}_u)$$

$$= \int_{G_t} \int_{G_u} XdP(\mathcal{G}_u)dP(\mathcal{G}_t)$$

$$= E[X_t|\mathcal{G}_u]$$

Similarly,

$$E\left[\int_0^t E\left[Y_s|\mathcal{G}_s\right]ds\middle|\mathcal{G}_u\right] = E\left[\int_0^t \int_{G_s} YdP(G_s)ds\middle|\mathcal{G}_u\right]$$

$$= \int_{G_u} \int_0^t \int_{G_s} YdP(G_s)dsdP(G_u)$$

$$= \int_0^t \int_{G_u} \int_{G_s} YdP(G_s)dP(G_u)ds$$

$$= \int_0^t E[Y_s|\mathcal{G}_u]ds$$

We also know that

$$\max\left(\int_{G_t} X_+dP(G_t), \int_{G_t} X_-dP(G_t)\right) \leq \max\left(\int_\Omega X_+dP, \int_\Omega X_-dP\right)$$

$$= \max(E[X_+], E[X_-]) < \infty$$

Therefore, $E[|X_t||G_t] < \infty$. Similarly, using Fubini and the steps above, we can also show that

$$\int_0^t E[|Y_s||\mathcal{G}_s]ds < \infty$$

Therefore,

$$E[E[X_t|\mathcal{G}_t]|\mathcal{G}_u] - E\left[\int_0^t E\left[Y_s|\mathcal{G}_s\right]ds\middle|\mathcal{G}_u\right]$$

$$= E[X_t|\mathcal{G}_u] - \int_0^t E[Y_s|\mathcal{G}_u]ds$$

which confirms it is indeed a martingale.

2.

To match the state equations, we have that

$$a^1_{i,j} = iK^1, \qquad s^1_{i,j} = i^2$$

and

$$a^2_{i,j} = rj, \qquad s^2_{i,j} = \frac{r}{K^2}(j^2 + \alpha_{21}ij)$$

and

$$Lf(i,j) = a^1_{i,j}[f(i+1,j) - f(i,j)] + a^2_{i,j}[f(i,j+1) - f(i,j)]$$
$$+ s^1_{i,j}[f(i-1,j) - f(i,j)] + s^2_{i,j}[f(i,j-1) - f(i,j)]$$

Then the 2 state equations are consistent with the martingale problem:

$$f(X^1_t, X^2_t) - f(X^1_0, X^2_0) - \int_0^t Lf(X^1_u, X^2_u)du$$

which is a $\sigma(X^1_s, X^2_s, s \leq t)$-martingale.

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3.

We know that
$$F_{X_1 \vee X_2}(x) = (F_{X_i}(x))^2$$

Hence,
$$f_{X_1 \vee X_2}(x) = 2F_{X_i}(x)f_{X_i}(x)$$
$$= 2\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$
$$= \frac{e^{-\frac{x^2}{2}}}{\pi} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

Therefore, we can calculate

$$f_{X_1 \vee X_2}(0) = \frac{1}{\pi} \underbrace{\int_{-\infty}^{0} e^{-\frac{u^2}{2}} du}_{\sqrt{\pi/2}} = \frac{1}{\sqrt{2\pi}}$$

$$\int_{0}^{1} f_{X_1 \vee X_2}(x)dx$$
$$= F_{X_1 \vee X_2}(1) - F_{X_1 \vee X_2}(0)$$
$$= F_{X_i}(1)^2 - F_{X_i}(0)^2$$
$$= 0.84134475^2 - 0.5^2$$
$$= 0.45786098835$$

4.

Let $g(B_t, t) = f\left(\exp\left(\frac{\sigma^2}{2}t\right)\right)$, hence we have that

$$\frac{\partial}{\partial x}g(B, x) = \frac{\sigma^2}{2}e^{\frac{\sigma^2 x}{2}}\left(f' \circ e^{\frac{\sigma^2 x}{2}}\right)$$

$$\frac{\Delta}{2}g(B, s) = 0$$

and that

$$g(B_t, t) - g(0, 0) - \int_0^t \frac{\Delta}{2}g(B_s, s) + \frac{\partial}{\partial s}g(B_s, s)ds$$

is a $\sigma(B_u, u \leq t)$-martingale problem. However, we have that

$$\frac{\Delta}{2}g(B_s, s) + \frac{\partial}{\partial s}g(B_s, s) = \frac{\sigma^2}{2}\exp\left(\frac{\sigma^2 s}{2}\right)f'\left(\exp\left(\frac{\sigma^2 s}{2}\right)\right)$$

Hence, if we let $X_t = \exp\left(\frac{\sigma^2}{2}t\right)$, then we get

$$f(X_t) - f(X_0) - \int_0^t \frac{\sigma^2 X_u}{2}f'(X_u)du$$

is a martingale. If there is another solution Y_t that has $Y_0 = X_0$. Let $f(x) = \ln(x)$ so that $f'(x) = \frac{1}{x}$ and therefore, $X_u f'(X_u) = 1$. Thus,

$$\ln(X_t) - \ln(X_0) - \int_0^t \frac{\sigma^2}{2}du = \ln(Y_t) - \ln(Y_0) - \int_0^t \frac{\sigma^2}{2}du$$

and hence $\ln(X_t) = \ln(Y_t) \implies X_t = Y_t$ for all t which proves its uniqueness.

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