MATH 217 (Fall 2022)
Honors Advanced Calculus, I

Assignment #6

1. Determine the Jacobians of

$$\mathbb{R}^3 \to \mathbb{R}^3, \quad (r,\theta,\phi) \mapsto (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

   and

$$\mathbb{R}^3 \to \mathbb{R}^3, \quad (r,\theta,z) \mapsto (r\cos\theta, r\sin\theta, z).$$

2. An $N \times N$ matrix X is *invertible* if there is $X^{-1} \in M_N(\mathbb{R})$ such that $XX^{-1} = X^{-1}X = I_N$ where I_N denotes the unit matrix.

   (a) Show that $U := \{X \in M_N(\mathbb{R}) : X \text{ is invertible}\}$ is open. (*Hint*: $X \in M_N(\mathbb{R})$ is invertible if and only if $\det X \neq 0$.)

   (b) Show that the map

$$f\colon U \to M_N(\mathbb{R}), \quad X \mapsto X^{-1}$$

   is totally differentiable on U, and calculate $Df(X_0)$ for each $X_0 \in U$. (*Hint*: You may use that, by Cramer's Rule, f is continuous.)

3. Let

$$p\colon (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \to \mathbb{R}^2, \quad (r,\theta) \mapsto (r\cos\theta, r\sin\theta),$$

   let $\varnothing \neq U \subset \mathbb{R}^2$ be open, and let $f : U \to \mathbb{R}$ be twice continuously partially differentiable. Show that

$$(\Delta f) \circ p = \frac{\partial^2 (f\circ p)}{\partial r^2} + \frac{1}{r}\frac{\partial (f\circ p)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 (f\circ p)}{\partial \theta^2}$$

   on $p^{-1}(U)$. (*Hint*: Apply the Chain Rule twice.)

4. Let

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \begin{cases} \frac{xy^3}{x^2+y^4}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

   Show that:

   (a) f is continuous at $(0,0)$;

   (b) for each $v \in \mathbb{R}^2$ with $\|v\| = 1$, the directional derivative $D_v f(0,0)$ exists and equals 0;

   (c) f is not totally differentiable at $(0,0)$.

(*Hint for* (c): Assume towards a contradiction that f is totally differentiable at $(0,0)$, and compute the first derivative of $\mathbb{R} \ni t \mapsto f(t^2, t)$ at 0 first directly and then using the chain rule. What do you observe?)

5. Let $U \subset \mathbb{R}^N$ be open and convex and contain 0, let $n \in \mathbb{N}_0$, and let $f : \mathbb{R}^N \to \mathbb{R}$ be $n+1$ times continuously partially differentiable such that $\frac{\partial^\alpha f}{\partial x^\alpha} \equiv 0$ on U for all multiindices $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = n+1$. Show that, there are $(c_\alpha)_{\alpha \in \mathbb{N}_0^N, |\alpha| \leq n}$ in $\mathbb{R}$ such that

$$f(x) = \sum_{|\alpha| \leq n} c_\alpha x^\alpha$$

for all $x \in U$.

6*. Let $\varnothing \neq C \subset \mathbb{R}^N$ be open and connected, and let $f : C \to \mathbb{R}$ be differentiable such that $\nabla f \equiv 0$. Show that f is constant. (*Hint*: First, treat the case where C is convex using the chain rule; then, for general C, assume that f is not constant, let $x, y \in C$ such that $f(x) \neq f(y)$, and show that $\{U, V\}$ with $U := \{z \in C : f(z) = f(x)\}$ and $V := \{z \in C : f(z) \neq f(x)\}$ is a disconnection for C.)

Due Monday, October 31, 2020, at 10:00 a.m.; no late assignments.