

MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Assignment #9

1. Let $I \subset \mathbb{R}^N$ be a compact interval. Show that

$$\mathcal{A} := \{A \subset I : A \text{ has content}\}$$

is an *algebra* over I , i.e.,

- (a) $\emptyset, I \in \mathcal{A}$,
 - (b) if $A \in \mathcal{A}$, then $I \setminus A \in \mathcal{A}$, and
 - (c) if $A_1, \dots, A_n \in \mathcal{A}$, then $A_1 \cup \dots \cup A_n \in \mathcal{A}$.
2. The function

$$f: [0, 1] \times [0, 1] \rightarrow \mathbb{R}, \quad (x, y) \mapsto xy$$

is continuous and thus Riemann integrable. Evaluate $\int_{[0,1] \times [0,1]} f$ using only the definition of the Riemann integral, i.e., in particular, without using Fubini's Theorem.

3. Define

$$f: [0, 1]^3 \rightarrow \mathbb{R}, \quad (x, y, z) \mapsto \begin{cases} xy, & z \leq xy, \\ z, & z \geq xy. \end{cases}$$

Evaluate $\int_{[0,1]^3} f$.

4. Let

$$D := \{(x, y) \in \mathbb{R} : x, y \geq 0, x^2 + y^2 \leq 1\},$$

and let

$$f: D \rightarrow \mathbb{R}, \quad (x, y) \mapsto \frac{4y^3}{(x+1)^2}$$

Evaluate $\int_D f$.

5. Let $a < b$, let $f: [a, b] \rightarrow [0, \infty)$ be continuous, and let

$$D := \{(x, y) : x \in [a, b], y \in [0, f(x)]\}.$$

Show that D has content and that

$$\mu(D) = \int_a^b f(x) dx.$$

6*. Define $f: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ by letting

$$f(x, y) = \begin{cases} 2^{2n}, & \text{if } (x, y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n}, 2^{-n+1}) \text{ for some } n \in \mathbb{N}, \\ -2^{2n+1}, & \text{if } (x, y) \in [2^{-n-1}, 2^{-n}) \times [2^{-n}, 2^{-n+1}) \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the iterated integrals

$$\int_0^1 \left(\int_0^1 f(x, y) dy \right) dx \quad \text{and} \quad \int_0^1 \left(\int_0^1 f(x, y) dx \right) dy$$

both exist, but that

$$\int_0^1 \left(\int_0^1 f(x, y) dy \right) dx \neq \int_0^1 \left(\int_0^1 f(x, y) dx \right) dy.$$

Why doesn't this contradict Fubini's Theorem?

Due Monday, November 28, 2020, at 10:00 a.m.; no late assignments.