1.

1.

For any $\varepsilon = (\varepsilon_n) \neq \tilde{\varepsilon} = (\tilde{\varepsilon}_n)$, there is a $N \in \mathbb{N}$ such that $\varepsilon_n = \tilde{\varepsilon}_n$ for all $n < N$ and $\varepsilon_N \neq \tilde{\varepsilon}_N$. WLOG, let $\varepsilon_N = 2$ and $\tilde{\varepsilon}_N = 0$. Thus we have

$$g(\varepsilon) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3^n} \geq \sum_{n=1}^{N-1} \frac{\varepsilon_n}{3^n} + \frac{2}{3^N}$$

and

$$g(\tilde{\varepsilon}) = \sum_{n=1}^{\infty} \frac{\tilde{\varepsilon}_n}{3^n} \leq \sum_{n=1}^{N-1} \frac{\tilde{\varepsilon}_n}{3^n} + \sum_{n=N+1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{N-1} \frac{\varepsilon_n}{3^n} + \underbrace{\frac{2}{3^{N+1}} \cdot \frac{1}{1-1/3}}_{\frac{1}{3^N}}$$

Thus $g(\varepsilon) > g(\tilde{\varepsilon})$ and therefore g is injective.

2.

Consider the function

$$p : \{0,1\}^{\mathbb{N}} \to [0,1], \qquad (\epsilon_n)_{n=1}^{\infty} \to \sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n}$$

Since there is a natural bijection $h : \{0,1\}^{\mathbb{N}} \to \{0,2\}^{\mathbb{N}}$, $p = p \circ h^{-1} \circ h = g \circ h$ is injective. We claim that p is also surjective. For every $x \in [0,1]$, there exists a sequence $\varepsilon = (\varepsilon_n)_{n=1}^{\infty}$ such that $g(\varepsilon) = x$

$$\left| \sum_{n=1}^{N} \frac{\varepsilon_n}{2^n} - x \right| < \epsilon$$

To construct the sequence ε, start from $n = 0$,

- if $\sum_{i=1}^{n} + \dfrac{1}{2^{n+1}} < x$, then let $\varepsilon_{n+1} = 1$
- if $\sum_{i=1}^{n} + \dfrac{1}{2^{n+1}} > x$, then let $\varepsilon_{n+1} = 0$
- if $\sum_{i=1}^{n} + \dfrac{1}{2^{n+1}} = x$, then let $\varepsilon_{n+1} = 1$ and $\varepsilon_i = 0$ for all $i > n+1$ then stop the process

Increase n by 1 and start the process again.
Since $\dfrac{1}{2^n} \to 0$ as $n \to \infty$ and $\sum_{i=1}^{n} \frac{1}{2^i} \leq x$ for all $n \in \mathbb{N}$. Thus for every $\varepsilon > 0$, there exists $n_0 \in N$ such that for every $n > n_0$, $0 \leq x - \sum_{i=1}^{n} \frac{1}{2^i} < \epsilon$, and therefore $p(\varepsilon) = \sum_{n=1}^{\infty} \frac{1}{2^n} = x$. Thus, p is bijective. Now consider the function

$$k : (0,1) \to \mathbb{R}, \qquad x \to \tan(2x\pi - \pi)$$

We have that k is bijective thus $(0,1) \sim \mathbb{R}$. $(0,1) \sim [0,1]$ as the map

$$\phi(x) = \begin{cases} \frac{1}{n+1} & \text{, if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 1 & \text{, if } x = 0 \\ x & \text{, otherwise} \end{cases}$$

is bijective and thus $\{0,1\}^{\mathbb{N}} \sim \mathbb{R}$.

3.

$\{0,2\}^{\mathbb{N}} \sim \{0,1\}^{\mathbb{N}}$, thus $\{0,2\}^{\mathbb{N}} \sim \mathbb{R}$ and $\mathbb{R} \sim C$.

4.

Also, if N_x is finite, we set $a_{n,x} = 0$ for all $n > N_x$ so that $\sum_{n=1}^{\infty} \frac{a_{n,x}}{3^n} = x$ regardless of N_x.
For any $x \neq y$, that is $\sum_{n=1}^{N_x} \frac{a_{n,x}}{3^n} \neq \sum_{n=1}^{N_y} \frac{a_{n,y}}{3^n}$, then as we know the function from subquestion 1 is injective, we have that $(a_{n,x}) \neq (a_{n,y})$, that is there exists $N \in \mathbb{N}$ such that for all $n < N, a_{n,x} = a_{n,y}$ and $a_{N,x} \neq a_{N,y}$.

- In case $N_x > N, N_y > N$, $a_{N,x} > a_{N,y} \implies x > y$ as

$$\sum_{n=1}^{\infty} \frac{a_{n,x}}{3^n} - \frac{a_{n,y}}{3^n} = \frac{2}{3^N} + \sum_{n=N+1}^{\infty} \frac{a_{n,x} - a_{n,y}}{3^n} \geq \frac{2}{3^N} - \underbrace{\sum_{n=N+1}^{\infty} \frac{2}{3^n}}_{1/3^N} > 0$$

  and thus because of $a_{N,x} \neq a_{N,y}, x \neq y$ by assumption and WLOG, we have $a_{N,x} < a_{N,y} \iff x < y$.

- In case $N_y > N, N_x \leq N$ which is $N_x = N$ then $a_{N,x} = 1$. If $a_{N,y} = 2$ then obviously $x < y$, if $a_{N,y} = 0$ then since $x \neq y$, there is n_0 such that $a_{n_0,y} - a_{n_0,x} < 2$ and hence

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a_{n,y} - a_{n,x}}{3^n} &= -\frac{1}{3^N} + \sum_{n=N+1}^{\infty} \frac{a_{n,y} - a_{n,x}}{3^n} \\ &= -\frac{1}{3^N} + \sum_{\substack{n=N+1 \\ n \neq n_0}}^{\infty} \frac{a_{n,y} - a_{n,x}}{3^n} + \frac{a_{n_0,y} - a_{n_0,x}}{3^{n_0}} \\ &< -\frac{1}{3^N} + \sum_{n=N+1}^{\infty} \frac{2}{3^n} \\ &= 0\end{aligned}$$

Finally, we can conclude that if $x < y$, then there exists $N \in \mathbb{N}$ such that for all $n < N$, $a_{n,x} = a_{n,y}$ and there is three cases

- $a_{N,x} = 0, a_{N,y} = 2$, thus $b_{N,x} = 0, b_{N,y} = 1$

$$\begin{aligned} f(y) - f(x) &= \sum_{n=1}^{\infty} \frac{b_{n,y} - b_{n,x}}{2^n} \\ &= \frac{1}{2^N} + \sum_{n=N+1}^{\infty} \frac{b_{n,y} - b_{n,x}}{2^n} \\ &\geq \frac{1}{2^N} + \sum_{n=N+1}^{\infty} \frac{-1}{2^n} \\ &= \frac{1}{2^N} - \frac{1}{2^N} = 0 \end{aligned}$$

- $a_{N,x} = 1, a_{N,y} = 2$, thus $b_{N,x} = b_{N,y} = 1$

$$\begin{aligned} f(y) - f(x) &= \sum_{n=1}^{\infty} \frac{b_{n,y} - b_{n,x}}{2^n} \\ &= \sum_{n=N+1}^{\infty} \frac{b_{n,y} - b_{n,x}}{2^n} \\ &= \sum_{n=N+1}^{\infty} \frac{b_{n,y}}{2^n} \geq 0 \end{aligned}$$

- $a_{N,x} = 0, a_{N,y} = 1$, thus $b_{N,x} = 0, b_{N,y} = 1$.

$$\begin{aligned} f(y) - f(x) &= \sum_{n=1}^{\infty} \frac{b_{n,y} - b_{n,x}}{2^n} \\ &= \frac{1}{2^N} + \sum_{n=N+1}^{\infty} \frac{-b_{n,x}}{2^n} \\ &\geq \frac{1}{2^N} - \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\ &= 0 \end{aligned}$$

5.

We know that $C \sim \{0,2\}^{\mathbb{N}} \sim \{0,1\}^{\mathbb{N}} \sim \mathbb{R}$ therefore $C \sim \mathbb{R}$.

2.

For any pairwise disjoint $E_i \in \mathcal{M}$, let $A_n = \sqcup_{i=1}^n E_i$ then we have that

$$\mu(\sqcup_{i=1}^\infty E_i) = \mu(\cup_{i=1}^\infty A_n) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(\sqcup_{i=1}^n E_i) = \sum_{i=1}^\infty \mu(E_i)$$

as $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$.

3.

a.

If $E_j \in \mathcal{M}$ for all $j \in \mathbb{N}$ then $E_1 \backslash E_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ and thus $\cup_{n=2}^{\infty}(E_1 \backslash E_n) \in \mathcal{M}$, then $E_1 \backslash (\cup_{n=2}^{\infty} E_1 \backslash E_n) = E_1 \backslash (E_1 \backslash \cap_{n=2}^{\infty} E_n) =$
$E_1 \cap (\cap_{n=2}^{\infty} A_n) = \cap_{n=1}^{\infty} A_n \in \mathcal{M}$

b.

We know that $\mathcal{M}$ being a σ-algebra implies that $X \in \mathcal{M}$.
If $X \in \mathcal{M}$, then for every $A \in \mathcal{M}$, $X \backslash A = A^c \in \mathcal{M}$.

c.

If $A \in \mathcal{S}$, then $A \in \mathcal{M}$ or $A^c \in \mathcal{M}$. In both case $A^c \in \mathcal{S}$ as $(A^c)^c \in \mathcal{M}$ or $A \in \mathcal{M}$.
If $A_j \in \mathcal{S}$ for all $j \in \mathbb{N}$ then $A_j \in \mathcal{M}$ or $A_j^c \in \mathcal{M}$. Let A_{j_1}, A_{j_2} be subsequence of A such that $A_{j_1} \in \mathcal{M}$ and $A_{j_2}^c \in \mathcal{M}$. Then we know that $P := \cap_{j_1=1}^{\infty} A_{j_1} \in \mathcal{M}$ and since $Q := \cup_{j_2=1}^{\infty} A_{j_2}^c = (\cap_{j_2=1}^{\infty} A_{j_2})^c \in \mathcal{M}$ and thus $P \backslash Q = \cap_{j_1=1}^{\infty} A_{j_1} \cap \cap_{j_2=1}^{\infty} A_{j_2} = \cap_{j=1}^{\infty} A_j \in \mathcal{M} \subseteq \mathcal{S}$.

4.

a.

If $E \in \mathcal{M}$, then $E \cap X_\lambda \in \mathcal{M}_\lambda$ for all $\lambda \in \Lambda$,
then $X_\lambda \backslash (E \cap X_\lambda) = X \backslash (E \cap X_\lambda) = (X \backslash E) \cap X_\lambda \in \mathcal{M}_\lambda$ and thus $E^c \in \mathcal{M}$.
If $E_i \in \mathcal{M}$ for $i \in \mathbb{N}$, then $E_i \cap X_\lambda \in \mathcal{M}_\lambda$, and thus $(\cup_{i=1}^\infty E_i) \cap X_\lambda \in \mathcal{M}_\lambda$ for all $\lambda \in \Lambda$ and thus $\cup_{i=1}^\infty E_i \in \mathcal{M}$.

b.

$$\mu(\varnothing) = \sum_{\lambda \in \Lambda} \mu_\lambda(\varnothing \cap X_\lambda) = \sum_{\lambda \in \Lambda} \underbrace{\mu_\lambda(\varnothing)}_{0} = 0$$

For any $E_j \in \mathcal{M}$ for all $j \in \mathbb{N}$ such that E_j are pairwise disjoint, we have that

$$\begin{aligned}
\mu(\sqcup_{j=1}^\infty E_j) &= \sum_{\lambda \in \Lambda} u_\lambda(\sqcup_{j=1}^\infty E_j \cap X_\lambda) \\
&= \sum_{\lambda \in \Lambda} \sum_{j=1}^\infty u_\lambda(E_j \cap X_\lambda) \\
&= \sum_{j=1}^\infty \sum_{\lambda \in \Lambda} \mu_\lambda(E_j \cap X_\lambda) \\
&= \sum_{j=1}^\infty \mu(E_j)
\end{aligned}$$

c.

If μ is σ-finite, then there exists $X_n \subseteq X_{n+1} \in \mathcal{M}$ such that $\cup_{n=1}^\infty X_n = X$ and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. Thus if we let $X_{n,\lambda} = X_n \cap X_\lambda$, we have that $X_{n,\lambda} \subseteq X_{n+1,\lambda}$, $X_{n,\lambda} \in \mathcal{M}_\lambda$,

$$\mu(X_n \cap X_\lambda) = \mu(X_{n,\lambda}) < \infty$$

and

$$X_\lambda = X \cap X_\lambda = (\cup_{n=1}^\infty X_n) \cap X_\lambda = \cup_{n=1}^\infty (X_n \cap X_\lambda) = \cup_{n=1}^\infty X_{n,\lambda}$$

for all $n \in \mathbb{N}$, which means that all but a countable measure μ_λ have $\mu_\lambda(X_\lambda) = 0$ and the rest are σ-finite.
Now suppose all but a countable measure μ_λ have $\mu_\lambda(X_\lambda) = 0$ and the rest are σ-finite, then for every $\lambda \in \Lambda$, there exists $X_{n,\lambda} \in \mathcal{M}_\lambda$ such that $X_{n,\lambda} \subseteq X_{n+1,\lambda}$, $\cup_{n=1}^\infty X_{n,\lambda} = X_\lambda$ and $\mu_\lambda(X_{n,\lambda}) < \infty$ for every $n \in \mathbb{N}$. Since Λ is a collection of measure, there is a bijection $\mathbb{N} \sim \Lambda$

- $X = \cup_{\lambda \in \Lambda} X_\lambda = \cup_{\lambda \in \Lambda} \cup_{n=1}^\infty X_{n,\lambda} = \cup_{n=1}$

5.

We have that the definition of the outer measure for both parts a and b

$$\mu^*(A) := \inf\left\{\sum_{j=1}^{\infty}\rho(E_j) : A \subseteq \cup_{j=1}^{\infty}E_j, E_j \in \mathcal{S}\right\}$$

a.

Then for any nonempty set $A \subseteq \mathcal{S}$, we have that if $\mu^*(A) = 0$ then $\sum_{j=1}^{\infty}\rho(E_j) = 0$ and thus $\rho(E_j) = 0$ for every $j \in \mathbb{N}$ and thus $E_j = \varnothing$ and $\cup_{j=1}^{\infty}E_j = \varnothing$. Therefore, $A = \varnothing$ and thus a contradiction. Therefore, $\mu^*(\varnothing) = 0$ and $\mu^*(A) > 0$. But since $\sum_{j=1}^{\infty}\rho(E_j)$ is either an integer or infinity, $A \subseteq X \cup (\cup_{j=2}^{\infty}\varnothing)$ and

$$\rho(X) + \sum_{j=2}^{\infty}\rho(\varnothing) = 1$$

Then from Caratheodory, we know that S is the σ-algebra. we have that $\mu^*(A) = 1$ and $\mu^*(\varnothing) = 0$.

b.

From definition, we have that for any set A such that $\mu^*(A) \geq \rho(A)$.
If $\rho(A) = N$ for some $N \in \mathbb{N}$ or $N = \infty$, then we can let

$$K = \{k \in A : \text{k is an integer}\}$$

Then $A \subseteq \cup_{j=0}^{\infty}E_j$ where

$$E_j = \begin{cases} \left(\frac{j}{2}, \frac{j}{2}+1\right), & \text{if } 2|j \text{ and } \frac{j}{2} \notin K \\ \left(-\frac{j-1}{2}-1, -\frac{j-1}{2}/2\right), & \text{if } 2 \nmid j \text{ and } -\frac{j-1}{2} \notin K \\ \left[\frac{j}{2}, \frac{j}{2}+1\right), & \text{if } 2|j \text{ and } \frac{j}{2} \in K \\ \left(-\frac{j-1}{2}-1, -\frac{j-1}{2}/2\right], & \text{if } 2 \nmid j \text{ and } -\frac{j-1}{2} \in K \end{cases}$$

so that

$$\rho(E_j) = \begin{cases} 1, & \text{if } \frac{j}{2} \in K \text{ or } -\frac{j-1}{2} \in K \\ 0, & \text{otherwise} \end{cases}$$

and thus there is N interval E_j such that $\rho(E_j) = 1$. Therefore,

$$\begin{aligned} \mu^*(A) &\leq \sum_{j=0}^{\infty}\rho(E_j) \\ &= \sum_{\substack{j=0 \\ j/2 \in K \text{ or} \\ -(j-1)/2 \in K}}^{\infty}\rho(E_j) + \sum_{\substack{j=0 \\ j/2 \notin K \text{ and} \\ -(j-1)/2 \notin K}}^{\infty}\rho(E_j) \\ &= N + 0 = N = \rho(A) \end{aligned}$$

which concludes that $\mu^*(A) = \rho(A)$. Then from Caratheodory, we know that $\mathcal{P}(X)$ is the σ-algebra.