1.

Since f is integrable on $\mathbb{R}$, there is a compactly supported continuous function h on H such that $\int_{\mathbb{R}} |f - h| < \varepsilon/2$. Since $\int_{\mathbb{R}} |f| < \infty$, we have that h is bounded on H thus h is uniformly continuous on H and hence on $\mathbb{R}$. Thus for any $\varepsilon > 0$, there is $\delta > 0$ such that for all $|x-y| < \delta$, $|h(x)-h(y)| < \varepsilon/2m(H)$

$$\begin{aligned}
|F(x) - F(y)| &= \left| \int_{-\infty}^{x} f(t)dt - \int_{-\infty}^{y} f(t)dt \right| \\
&= \left| \int_{y}^{x} f(t)dt \right| \\
&\leq \int_{y}^{x} |f(t) - h(t)|dt + \int_{y}^{x} |h(t)|dt \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2m(H)} m(H) \\
&= \varepsilon
\end{aligned}$$

2.

a.

For any $\{E_n\} \in \mathcal{M}$, $\{F_n\} \in \mathcal{N}$, if $D \subseteq \cup_{n=1}^{\infty} E_n \times F_n$, then for any $(x, x) \in D$ there is $n \in \mathbb{N}$ such that $(x, x) \in E_n \times F_n$ which implies $x \in E_n \cap F_n$. Therefore, $[0, 1] \subseteq \cup_{n=1}^{\infty}(E_n \cap F_n)$. Thus $\mu(E_n \cap F_n) > 0$ for some $n \in \mathbb{N}$, which means that $\mu(E_n) > 0$ and $\nu(F_n) = \infty$. Therefore, $(\mu \times \nu)(D) = \infty$.

b.

$$A_x = \{x\} \text{ and } A^y = \{y\}$$

Thus

$$\nu(A_x) = 1 \text{ and } \mu(A^y) = 0$$

and hence

$$\int_Y \int_X \chi_D d\mu d\nu = \int_{[0,1]} \mu(A^y) d\nu = 0$$

$$\int_X \int_Y \chi_D d\nu d\mu = \int_0^1 \nu(A_x) d\mu = 1$$

and

$$\iint_{X \times Y} \chi_D d(\mu \times \nu) = \infty$$

3.

Apply Theorem 5.5

$$\begin{aligned}\int_0^a |g(x)|dx &= \int_0^a \int_x^a |t^{-1} f(t)|dtdx \\ &= \int_0^a \int_0^t \frac{|f(t)|}{t} dxdt \\ &= \int_0^a |f(t)|dt \\ &< \infty\end{aligned}$$

Thus g is integrable. Therefore,

$$\begin{aligned}\int_0^a g(x)dx &= \int_0^a \int_x^a t^{-1} f(t)dtdx \\ &= \int_0^a \int_0^t \frac{f(t)}{t} dxdt \\ &= \int_0^a f(t)dt\end{aligned}$$

4.

First, note that if $\lambda_f(\alpha) = \infty$ for some $\alpha > 0$ then

$$\int_X |f(x)|^p d\mu(x) = \infty = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha$$

Now suppose $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$, then for any x, we have that

$$\int_0^{|f(x)|} p\alpha^{p-1} d\alpha = \alpha^p \Big|_{\alpha=0}^{|f(x)|} = |f(x)|^p$$

Thus

$$\begin{aligned}
&\int_X |f(x)|^p d\mu(x) \\
&= \int_X \int_0^{|f(x)|} p\alpha^{p-1} d\alpha d\mu(x) \\
&= \int_X \int_0^\infty p\alpha^{p-1} 1_{|f(x)|>\alpha} d\alpha d\mu(x) \\
&= \int_0^\infty p\alpha^{p-1} \int_X 1_{|f(x)|>\alpha} d\mu(x) d\alpha \\
&= \int_0^\infty p\alpha^{p-1} \lambda_f(\alpha) d\alpha
\end{aligned}$$

5.

a.

Let $M = \int_{\mathbb{R}^d} |f(x)|dx$ and $N = \int_{\mathbb{R}^d} |g(y)|dy$, then from theorem 5.5,

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} |H(x,y)|d(x \times y) \\
=&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |H(x,y)|dydx \qquad \left(= \int_{\mathbb{R}^d} [f * g](x)dx\right) \\
=&\int_{\mathbb{R}^d} |g(y)| \int_{\mathbb{R}^d} |f(x-y)|dxdy \\
=&M\int_{\mathbb{R}^d} |g(y)|dy \\
=&MN < \infty
\end{aligned}$$

We also get from the above equations that

$$\int_{\mathbb{R}^d} |f(x-y)g(y)|dy < \infty$$

for a.e. $x \in \mathbb{R}^d$. Thus $[f * g]$ is well-defined a.e. $x \in \mathbb{R}^d$.

b.

Let $\xi_n \to \xi$, then for every $\varepsilon > 0$, we can find a uniformly continuous compact supported function h on X such that $\int_{\mathbb{R}^d} |f - h| < \varepsilon/4$. Now, for every $x \in X$ we have

$$\begin{aligned}
&\lim_{n\to\infty} |\exp(-ix \cdot (\xi - \xi_n)) - 1| \\
=&\lim_{n\to\infty} \sqrt{(\cos(-x\cdot(\xi-\xi_n)) - 1)^2 + (\sin(-x\cdot(\xi-\xi_n)))^2} \\
=&\lim_{n\to\infty} \sqrt{2 - 2\cos(-x\cdot(\xi-\xi_n))} \\
=&0
\end{aligned}$$

Thus, there is n_0 such that for all $n > n_0$,

$$|e^{-ix\cdot(\xi-\xi_n)} - 1| < \frac{\varepsilon}{2Mm(X)}$$

where

$$M = \sup_{x\in X} |x| < \infty$$

Therefore, we have

$$
\begin{aligned}
&|\widehat{f}(\xi)-\widehat{f}(\xi_n)| \\
\leq &\int_{\mathbb{R}^d} |f(x)||e^{-ix\cdot\xi}-e^{-ix\cdot\xi_n}|dx \\
\leq &\int_{\mathbb{R}^d} |f(x)-h(x)||e^{-ix\cdot\xi}-e^{-ix\cdot\xi_n}|dx \\
&+\int_{\mathbb{R}^d} |h(x)||e^{-ix\cdot\xi}-e^{-ix\cdot\xi_n}|dx \\
\leq &2\int_{\mathbb{R}^d} |f(x)-h(x)|dx \\
&+\int_{X} |h(x)||e^{-ix\cdot(\xi-\xi_n)}-1|dx \\
< &\frac{\varepsilon}{2}+m(X)M\frac{\varepsilon}{2Mm(X)} \\
= &\varepsilon
\end{aligned}
$$

and $\widehat{f}$ is continuous. Now since we know that H is integrable, $H(x)e^{-ix\cdot\xi}$ is also integrable, therefore

$$
\begin{aligned}
&\widehat{f*g}(\xi) \\
= &\int_{\mathbb{R}^d} [f*g](x)e^{-ix\cdot\xi}dx \\
= &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(y)e^{-ix\cdot\xi}dydx \\
= &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(y)e^{-ix\cdot\xi}dxdy \\
= &\widehat{f}(\xi)\int_{\mathbb{R}^d} g(y)e^{-i(y-x)\cdot\xi}e^{-ix\cdot\xi}dy \\
= &\widehat{f}(\xi)\widehat{g}(\xi)
\end{aligned}
$$