1.

a.

Consider $a_k = \left|\frac{1}{k} - \frac{(-1)^k}{\sqrt{k}}\right|$. Then

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left|\frac{1}{k} - \frac{(-1)^k}{\sqrt{k}}\right| = 0$$

as

$$\lim_{k\to\infty} \frac{1}{k} = 0 \text{ and } \lim_{k\to\infty} -\frac{(-1)^k}{\sqrt{k}} = 0$$

For odd k, $(-1)^k a_k = -\frac{1}{k} - \frac{1}{\sqrt{k}}$.
For even k, $(-1)^k a_k = -\frac{1}{k} + \frac{1}{\sqrt{k}}$
Hence,

$$\sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} -\frac{1}{k} + \frac{(-1)^k}{\sqrt{k}} < \sum_{k=1}^{\infty} -\frac{1}{k} = -\infty$$

as for every natural number n_0, if n_0 is even

$$\sum_{k=1}^{n_0} \frac{(-1)^k}{\sqrt{k}} = \sum_{k=1}^{n_0/2} \underbrace{-\frac{1}{\sqrt{2k-1}} + \frac{1}{\sqrt{2k}}}_{<0} < 0$$

if n_0 is odd then

$$\sum_{k=1}^{n_0} \frac{(-1)^k}{\sqrt{k}} = -\frac{1}{\sqrt{n_0}} + \sum_{k=1}^{(n_0-1)/2} \underbrace{-\frac{1}{\sqrt{2k-1}} + \frac{1}{\sqrt{2k}}}_{<0} < 0$$

2.

If $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges then for any t and N such that $2^N > t$, we have that

$$\sum_{k=1}^{t} a_k \le \sum_{k=2}^{2^N-1} a_k = \sum_{k=1}^{N} \sum_{j=2^k}^{2^{k+1}-1} a_j \le \sum_{k=1}^{N} 2^k a_{2^k} \le \sum_{k=1}^{\infty} 2^k a_{2^k}$$

Hence, as $s_t = \sum_{k=1}^{t} a_k$ is non-decreasing and is bounded, $\sum_{k=1}^{\infty} a_k$ converges. On the other hand, if $\sum_{k=1}^{\infty} a_k$ converges then

$$\frac{\sum_{k=1}^{t} 2^k a_{2^k}}{2} \le \sum_{k=1}^{t} \sum_{j=2^{k-1}+1}^{2^k} a_j = \sum_{k=1}^{t} a_k \le \sum_{k=1}^{\infty} a_k$$

which means that $p_t = \sum_{k=1}^{t} 2^k a_{2^k}$ is non-decreasing and is bounded, hence $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges. As $\sum_{k=1}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=1}^{\infty} \left(\left(\frac{1}{2}\right)^{p-1}\right)^k$. $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $\left|\frac{1}{2^{p-1}}\right| < 1 \implies p > 1$

3.

Consider $b_k = \frac{1}{\sqrt{k}}$, then $a_k = (-1)^k b_k$ is convergent as the sequence $(b_k)_{k=1}^{\infty}$ clearly decreases monotonically to 0. We have that for all naturnal number N

$$\begin{aligned} c_n &= \sum_{k=0}^{n} a_{n-k} a_k = \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}} \\ &= (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \end{aligned}$$

Hence, applying Cauchy

$$|c_n| \geq \frac{2(n+1)}{n+2} = 2 - \frac{2}{n+2} > 1$$

Thus $\sum_{n=0}^{\infty} c_n$ diverges.

4.

Since f is Riemann integrable over $[a, b]$, f is bounded over $[a, b]$.
Let $M = \max\{|f(x)| : x \in [a, b]\}$, then for all $\epsilon > 0$, there exists $\delta = \frac{\epsilon}{M}$ such that

$$\left|\int_a^b f(x)dx - \int_a^{b-\delta} f(x)dx\right| \leq \left|\int_{b-\delta}^b f(x)dx\right| \leq M \cdot \frac{\epsilon}{M} = \epsilon$$

5.

For all $x \in [0, 1]$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{-nx^2} = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{x}{x^2 e^{nx^2}} = 0$$

For all $n \in N$, $\exists x_0 = \frac{1}{n} \leq 1$ such that

$$\lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty} n \cdot \frac{1}{n} e^{-n \cdot \frac{1}{n^2}} = \lim_{n\to\infty} e^{-1/n} = 1$$

6.

a.

$$\lim_{t\to\infty} e^{-t}t^{x+1} = \lim_{t\to\infty} \frac{t^{x+1}}{e^t} = \lim_{t\to\infty} \frac{(x+1)!}{e^t} = 0$$

We also have this simlarly for $e^{-t}t^x$. Hence, $\exists t_0$ such that for all $t > t_0$: $e^{-t}t^{x+1} < 1 \implies t^{x-1}e^{-t} < \frac{1}{t^2}$ Therefore,

$$\int_0^\infty t^{x-1}e^{-t}dt = \int_0^{t_0} t^{x-1}e^{-t}dt + \int_{t_0}^\infty t^{x-1}e^{-t}dt < \underbrace{\int_0^{t_0} t^{x-1}e^{-t}dt}_{\text{bounded}} + \underbrace{\int_{t_0}^\infty \frac{1}{t^2}}_{\text{bounded}}$$

exists.

b.

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t}dt \\ &= t^x \cdot (-e^{-t})\big|_0^\infty + x\int_0^\infty t^{x-1}e^{-t}dt \\ &= x\Gamma(x)\end{aligned}$$

c.

Using induction, we have the base case

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1 = 0!$$

For the inductive steps and from part b, if $\Gamma(n+1) = n!$ then $\Gamma(n+2) = (n+1)\cdot\Gamma(n+1) = (n+1)!$.