1 p.144

Consider the characteristic equation $r^2 - 2r - 3 = (r-3)(r+1) = 0$. The characteristic roots are $3, 1$. Hence, a fundamental sets of solution is $y_1 = e^{-t}, y_2 = e^{3t}$.

$$\begin{aligned} & y'' - 2y' - 3y = 3e^{2t} \\ \iff & 4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 3e^{2t} \\ \iff & A = -1 \end{aligned}$$

Hence, the solution is

$$c_1 e^{-t} + c_2 e^{3t} - e^{2t}$$

2 p.144

Consider the characteristic equation $r^2 - r - 2 = (r-2)(r+1) = 0$. The characteristic roots are $3, 1$. Hence, a fundamental sets of solution is $y_1 = e^{-t}, y_2 = e^{2t}$.

$$
\begin{aligned}
& y'' - y' - 2y = -2t + 4t^2 \\
\Longleftrightarrow\ & (A_1t^2 + A_2t + A_3)'' - (A_1t^2 + A_2t + A_3)' - 2(A_1t^2 + A_2t + A_3) = -2t + 4t^2 \\
\Longleftrightarrow\ & -2A_1t^2 + (-2A_1 - 2A_2)t + 2A_1 - A_2 - 2A_3 = -2t + 4t^2 \\
\Longleftrightarrow\ & (A_1, A_2, A_3) = (-2, 3, -7/2)
\end{aligned}
$$

Hence, the solution is

$$c_1e^{-t} + c_2e^{2t} - 2t^2 + 3t - \frac{7}{2}$$

7 p.144

Consider the characteristic equation $r^2 + 1 = 0$. The characteristic roots are $i, -i$. Hence, a fundamental sets of solution is $y_1 = \sin(t), y_2 = \cos(t)$.

$$
\begin{aligned}
& y'' + y = 3\sin(2t) + t\cos(2t) \\
\iff & (A_1 t\cos(2t) + A_2\sin(2t))'' + A_1 t\cos(2t) + A_2\sin(2t) = 3\sin(2t) + t\cos(2t) \\
\iff & (-2A_1 t\sin(2t) + A_1\cos(2t) + 2A_2\cos(2t))' + A_1 t\cos(2t) + A_2\sin(2t) = 3\sin(2t) + t\cos(2t) \\
\iff & -2A_1\sin(2t) - 4A_1 t\cos(2t) - 2A_1\sin(2t) - 4A_2\sin(2t) + A_1 t\cos(2t) + A_2\sin(2t) \\
& = 3\sin(2t) + t\cos(2t) \\
\iff & -4A_1\sin(2t) - 3A_2\sin(2t) - 3A_1 t\cos(2t) = 3\sin(2t) + t\cos(2t) \\
\iff & (A_1, A_2) = (-1/3, -5/9)
\end{aligned}
$$

Hence, the solution is

$$c_1\sin(t) + c_2\cos(t) - \frac{1}{3}t\cos(2t) - \frac{5}{9}\sin(2t)$$

11 p.144

Consider the characteristic equation $r^2 + r - 2 = (r+2)(r-1) = 0$. The characteristic roots are $-2, 1$. Hence, a fundamental sets of solution is $y_1 = e^t, y_2 = e^{-2t}$.

$$\begin{aligned}
& y'' + y' - 2y = 2t \\
\iff & (A_1 t + A_2)'' + (A_1 t + A_2)' + 2(A_1 t + A_2) = 2t \\
\iff & A_1 + 2A_1 t + A_2 = 2t \\
\iff & (A_1, A_2) = (1, -1)
\end{aligned}$$

Hence,

$$y(t) = c_1 e^t + c_2 e^{-2t} + t - 1$$
$$y(0) = c_1 + c_2 - 1 = 0$$
$$y'(0) = c_1 - 2c_2 + 1 = 1$$

which means the solution is then

$$\frac{2}{3} e^t + \frac{1}{3} e^{-2t} + t - 1$$

1 p.149

Consider the characteristic equation $r^2 - 5r + 6 = (r-3)(r-2) = 0$. The characteristic roots are $3, 2$. Hence, a fundamental sets of solution is $y_1 = e^{3t}, y_2 = e^{2t}$. Consider

$$y_c = c_1 e^{3t} + c_2 e^{2t}$$

Let

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1(t)e^{3t} + u_2(t)e^{2t}$$

Then as

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix} = -e^{5t}$$

Hence,

$$u_1 = \int \frac{-e^{2t}2e^t}{-e^{5t}} dt = \int 2e^{-2t} dt = -e^{-2t}$$

$$u_2 = \int \frac{e^{3t}2e^t}{-e^{5t}} dt = \int -2e^{-t} = 2e^{-t}$$

Hence,

$$y_p(t) = -e^{-2t}e^{3t} + 2e^{-t}e^{2t} = e^t$$

$$\begin{aligned} & y'' - y' - 2y = -2t + 4t^2 \\ \iff & Ae^t - 5Ae^t + 6Ae^t = 2e^t \\ \iff & A = 1 \end{aligned}$$

which agrees with the other method. Hence, the solution is

$$y(t) = c_1 e^{3t} + c_2 e^{2t} + e^t$$

4 p.149

Consider the characteristic equation $r^2 + 1 = 0$. The characteristic root are $\pm i$. Hence, a fundamental sets of solution is $y_1 = \sin(t), y_2 = \cos(t)$. Consider

$$y_c = c_1 \sin(t) + c_2 \cos(t)$$

Let

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

Then as

$$W(y_1, y_2)(t) = \begin{vmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix} = -1$$

Hence,

$$u_1 = \int \frac{-\cos(t)\tan(t)}{-1} dt = \int \sin(t) dt = -\cos(t)$$

$$u_2 = \int \frac{\sin(t)\tan(t)}{-1} dt = \frac{\ln(\sin(x) + 1)}{2} - \frac{\ln(1 - \sin(x))}{2} - \sin(x)$$

Hence,

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} - \cos(t)\sin(t) + \cos(t)\left(\frac{\ln(\sin(t) + 1)}{2} - \frac{\ln(1 - \sin(t))}{2} - \sin(t)\right)$$

6 p.149

Consider the characteristic equation $r^2+4r+4 = (r+2)^2 = 0$. The characteristic root is 2. Hence, a fundamental sets of solution is $y_1 = e^{-2t}, y_2 = te^{-2t}$. Consider

$$y_c = c_1e^{-2t} + c_2te^{-2t}$$

Let

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1(t)e^{-2t} + u_2(t)te^{-2t}$$

Then as

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{2t} & e^{-2t} - 2te^{2t} \end{vmatrix} = e^{-4t}$$

Hence,

$$u_1 = \int \frac{-te^{-2t}t^{-2}e^{-2t}}{e^{-4t}}dt = \int -t^{-1}dt = -\ln(t)$$

$$u_2 = \int \frac{e^{-2t}t^{-2}e^{-2t}}{e^{-4t}}dt = \int t^{-2}dt = -t^{-1}$$

Hence,

$$y(t) = c_1e^{-2t} + c_2te^{-2t} - \ln(t)e^{-2t} - e^{-2t}$$

12 p.149

We have

$$y_c(t) = c_1(1+t) + c_2e^t$$

Plugging that in

$$\begin{aligned} ty'' - (1+t)y' + y &= t(c_2e^t) - (1+t)(c_1 + c_2e^t) + c_1(1+t) + c_2e^t \\ &= 0 \end{aligned}$$

Since $t > 0$, we need to solve the equation

$$y'' - \frac{1+t}{t}y' + \frac{1}{t}y = te^{2t}$$

Consider

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

Then as

$$W(y_1, y_2)(t) = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = te^t > 0$$

Hence,

$$u_1 = \int \frac{-e^t te^{2t}}{te^t} dt = \int dt = t$$

$$u_2 = \int \frac{(1+t)te^{2t}}{te^t} dt = \int (1+t)e^t dt = te^t$$

Therefore, the solution is

$$y(t) = c_1(1+t) + c_2e^t + t(1+t) + te^{2t}$$

13 p.149

We have

$$y_c(x) = c_1x^2 + c_2x^2\ln(x)$$

Plugging that in

$$\begin{aligned}
&x^2y'' - 3xy' + 4y \\
=&x^2(2c_2\ln(x) + 3c_2 + 2c_1) - 3x(x(2c_2\ln(x) + c_2 + 2c_1)) + 4(c_1x^2 + c_2x^2\ln(x)) \\
=&0
\end{aligned}$$

Since $t > 0$, we need to solve the equation

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln(x)$$

Consider

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

Then as

$$W(y_1, y_2)(t) = \begin{vmatrix} x^2 & x^2\ln(x) \\ 2x & 2x\ln(x) + x \end{vmatrix} = x^3 + 2x^3\ln(x) - 2x^3\ln(x) = x^3 > 0$$

Hence,

$$u_1 = \int \frac{-x^2\ln(x)\ln(x)}{x^3}dx = \int -\frac{\ln^2(x)}{x} = -\frac{\ln^3(x)}{3}$$

$$u_2 = \int \frac{x^2\ln(x)}{x^3}dx = \int \frac{\ln(x)}{x}dx = \frac{\ln^2(x)}{2}$$

Therefore, the solution is

$$y(t) = c_1x^2 + c_2x^2\ln(x) + \frac{x^2\ln^3(x)}{6}$$