1

a.

$$u_0(t) = x_0 = 2$$

$$u_1(t) = x_0 + \int_0^t F(u_0(s))ds = 2 + \int_0^t F(2)ds = 2 + 4t$$

Induction:

$$u_{n+1}(t) = x_0 + \int_0^t F(u_n(s))ds = 2 + \int_0^t \sum_{i=0}^{n} \frac{4t^i}{i!}dt = 2 + \sum_{i=1}^{n+1} \frac{4t^i}{i!} = \sum_{i=0}^{n} \frac{4t^i}{i!} - 2$$

Thus, we see that as $n \to \infty$

$$u_n(t) = \sum_{i=0}^{n} \frac{4t^i}{i!} - 2 \to 4e^t - 2$$

Domain is $\mathbb{R}$.

b.

$$u_0(t) = x_0 = 0$$

$$u_1(t) = x_0 + \int_0^t F(u_0(s))ds = \int_0^t F(0)ds = 0$$

$$u_{n+1}(t) = x_0 + \int_0^t F(u_1(s))ds = \int_0^t F(0)ds = 0$$

Thus, we see that

$$u_n(t) = 0$$

Domain is $\mathbb{R}$.

e.

$$u_0(t) = x_0 = 1$$

$$u_1(t) = x_0 + \int_0^t F(u_0(s))ds = 1 + \int_0^t F(1)ds = 1 + t/2$$

$$u_2(t) = x_0 + \int_0^t F(u_1(s))ds = 1 + \int_0^t \frac{1}{2+s}ds = 1 + \ln|t+2| - \ln(2)$$

2.

We have that

$$u_0(t) = u_0 = X_0$$

$$u_1(t) = X_0 + \int_0^t f(X_0)ds = X_0 + AX_0 \int_0^t ds = X_0 + AX_0 t$$

Induction:

$$u_{n+1}(t) = X_0 + \int_0^t f(u_n(t))ds = X_0 + \int_0^t A \sum_{i=0}^{n} \frac{s^i A^i}{i!} ds = \sum_{i=0}^{n+1} \frac{(tA)^i}{i!}$$

Thus as $n \to \infty$

$$u_n(t) = X_0 \sum_{i=0}^{n} \frac{(tA)^i}{i!} \to \exp(tA)X_0$$

4.

If $Y(t)$ and $Z(t)$ are solutions of

$$X' = AX$$

Then

$$(\alpha Y(t)+\beta Z(t))' = \alpha Y'(t)+\beta Z'(t) = \alpha AY(t)+\beta AZ(t) = A(\alpha Y(t)+\beta Z(t))$$

Thus also satisfied

$$X' = AX$$

6.

In the cases of $a \notin \mathbb{Q}$ or irreducible $a = p/q \in \mathbb{Q}$ with even q, the domain of $f(x) = x^a$ is $\mathbb{R}^+$. Thus $f(x)$ is continuously differentiable everywhere in the domain. Hence, the solution would then be unique in the domain and not unique in $\mathbb{R}$.
In case off odd q, the domain for $f(x)$ would be $\mathbb{R}$. If $a \geq 1$, then $f(x)$ is continuously differentiable everywhere thus the solution is unique. If $a < 1$, then $f(x)$ is not continuously differentiable at 0 thus not unique solution.

7.

We have the solution

$$P(t) = P_0 \exp\left(\int_0^t A(s)ds\right)$$

Thus

$$\det(P(t)) = \det(P_0)\det\left(\exp\left(\int_0^t A(s)ds\right)\right)$$

Now let $T(t)$ be the matrix that transform $\int_0^t A(s)ds$ to its normal jordan form $J(t)$, we have that

$$\det\left(\exp\left(\int_0^t A(s)ds\right)\right) = \det(\exp(T(t)J(t)T^{-1}(t))) = \det(\exp(J(t)))$$

Since $J(t)$ is an upper-triangular matrix we have that

$$\det(\exp(J(t))) = \exp(\mathrm{Tr}(J(t))) = \exp\left(\mathrm{Tr}\left(\int_0^t A(s)ds\right)\right) = \exp\left(\int_0^t \mathrm{Tr}(A(s))ds\right)$$

Thus

$$\det(P(t)) = \det(P_0)\exp\left(\int_0^t \mathrm{Tr}(A(s)ds)\right)$$