1 p.160

Let

$$\phi = \tan^{-1}\left(\frac{c_2}{c_1}\right)$$

$$\begin{aligned} u &= 3\cos(2t) + 4\sin(2t) \\ &= \sqrt{3^2+4^2}\cos\left(2t - \tan^{-1}\left(\frac{4}{3}\right)\right) \\ &= 5\cos\left(2t - \tan^{-1}\left(\frac{4}{3}\right)\right) \end{aligned}$$

12 p.160

Substitute $u = v + w$ in the DE,

$$\begin{aligned} &m(v+w)'' + \gamma(v+w)' + k(v+w) = 0 \\ &(mv'' + \gamma v' + kv) + (mw'' + \gamma w' + kw) = 0 \end{aligned}$$

If we set $mv'' + \gamma v' + kv = 0$ which means v is a solution of the original DE, then w is also a solution from the equation above. Setting

- $v(t_0) = u_0$
- $v'(t_0) = 0$
- $w(t_0) = 0$
- $w'(t_0) = u'_0$

We can find a solution to $vandw$ and hence

$$u(t_0) = v(t_0) + w(t_0) = u_0 \text{ and } u(t_0) = v'(t_0) + w'(t_0)$$

13 p.160

If the system is critically damped, the general solution is

$$c_1 e^{-ct/2m} + c_2 t e^{-ct/2m}$$

At the equilibrium point, $c_1 e^{-ct/2m} + c_2 t e^{-ct/2m} = 0$ which means that $c_1 + c_2 t = 0$. Therefore, $t = \frac{-c_1}{c_2}$ is the time if $c_2 \neq 0$. If $c_2 = 0$ and $c_1 \neq 0$ then it never pass the equilibrium point and if $c_2 = c_1 = 0$, it always at equilibrium point. If the system is overdamped the general solution is

$$c_1 \exp\left(\frac{-c + \sqrt{c^2 - 4km}}{2m} t\right) + c_2 \exp\left(\frac{-c - \sqrt{c^2 - 4km}}{2m} t\right)$$

At the equilibrium point

$$\begin{aligned}
& c_1 \exp\left(\frac{-c + \sqrt{c^2 - 4km}}{2m} t\right) + c_2 \exp\left(\frac{-c - \sqrt{c^2 - 4km}}{2m} t\right) = 0 \\
\implies & c_1 + c_2 \exp\left(\frac{-c - \sqrt{c^2 - 4km}}{2m} t - \frac{-c + \sqrt{c^2 - 4km}}{2m} t\right) = 0 \\
\implies & c_1 + c_2 \exp\left(-\frac{\sqrt{c^2 - 4km}}{m} t\right) = 0
\end{aligned}$$

Hence, if $c_1 = 0, c_2 = 0$, it is always at equilibrium point. If $c_1 c_2 > 0$ or either c_1, c_2 are 0, it never reaches equilibrium point and if $c_1 c_2 < 0, t = \frac{m}{\sqrt{c^2 - 4km}} \ln\left(\frac{-c_2}{c_1}\right)$

11 p.170

First, let establish that

$$u_c = c_1\cos(t) + c_2\sin(t)$$

Then we have that

$$u(0) = u_c(0) + u_p(0) = c_1 + u_p(0) = 0 \implies c_1 = -u_p(0)$$

and

$$u'(0) = u_c'(0) + u_p'(0) = c_2 + u_p'(0) = 0 \implies c_2 = -u_p'(0)$$

Now consider

$$u'' + u = F(t)$$

Let $u_p(t) = u_1(t)v_1(t) + u_2(t)v_2(t)$, then

$$W = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} = 1 \neq 0$$

$$v_1 = -\int \frac{-\sin(t)F(t)}{1}dt = \int \sin(t)F(t)dt$$

and

$$v_2 = -\int \frac{\cos(t)F(t)}{1}dt = -\int \cos(t)F(t)dt$$

When $0 \leq t \leq \pi$,

$$v_1 = \int \sin(t)F_0 t dt = F_0(\sin(t) - t\cos(t))$$

$$v_2 = -\int \cos(t)F_0 t dt = -F_0(\cos(t) + t\sin(t))$$

and thus

$$\begin{aligned} u_p(t) &= \cos(t)F_0(\sin(t) - t\cos(t)) - \sin(t)F_0(\cos(t) + t\sin(t)) \\ &= -F_0 t \end{aligned}$$

Hence,

$$c_1 = 0 \text{ and } c_2 = -F_0$$

and therefore

$$u(t) = -F_0\sin(t) + F_0 t$$

When $\pi < t \leq 2\pi$,

$$v_1 = \int \sin(t)F_0(2\pi - t)dt = F_0((t-2\pi)\cos(t) - \sin(t))$$

$$v_2 = -\int \cos(t)F_0(2\pi - t)dt = F_0((t-2\pi)\sin(t) + \cos(t))$$

and thus

$$\begin{aligned}
u_p(t) &= \cos(t)F_0((t-2\pi)\cos(t)-\sin(t))+\sin(t)F_0((t-2\pi)\sin(t)+\cos(t)) \\
&= F_0((t-2\pi)\cos^2(t)-\sin(t)\cos(t)+(t-2\pi)\sin^2(t)+\sin(t)\cos(t)) \\
&= F_0(t-2\pi)
\end{aligned}$$

Hence,

$$c_1 = -2\pi F_0 \text{ and } c_2 = -F_0$$

and therefore,

$$u(t) = -2\pi F_0\cos(t) - F_0\sin(t) + F_0(t-2\pi)$$

When $t > 2\pi$,

$$u = 0$$

Hence,

$$u(t) = \begin{cases} -F_0\sin(t) + F_0 t, & \text{if } 0 \le t \le \pi \\ -2\pi F_0\cos(t) - F_0\sin(t) + F_0(t-2\pi), & \text{if } \pi \le t \le 2\pi \\ 0, & \text{if } t > 2\pi \end{cases}$$

and

$$u'(t) = \begin{cases} -F_0\cos(t) + F_0, & \text{if } 0 \le t \le \pi \\ 2\pi F_0\sin(t) - F_0\cos(t) + F_0, & \text{if } \pi \le t \le 2\pi \\ 0, & \text{if } t > 2\pi \end{cases}$$

and hence u, u' are continuous.

12 p.170

We have that $C = 0.25 \cdot 10^{-6}F$, $R = 5000\Omega$, $L = 1H$, $E(t) = 12V$ Hence,

$$Q'' + 5000Q' + 4 \cdot 10^6 Q = 12$$

Hence,

$$Q_c(t) = c_1 e^{-1000t} + c_2 e^{-4000t}$$

Obviously, $Q_p(t) = 12$. Therefore,

$$Q(t) = c_1 e^{-1000t} + c_2 e^{-4000t} + 12$$

$$Q(0) = c_1 + c_2 + 12 = 0$$

$$Q'(0) = -1000c_1 - 4000c_2 = \frac{12}{5000}$$

Hence,

$$c_1 = -16 \text{ and } c_2 = 4$$

Thus

$$Q(t) = -16e^{-1000t} + 4e^{-4000t} + 12$$

$$Q(0.001) = -16e^{-1} + 4e^{-4} + 12$$

and

$$Q(0.01) = -16e^{-10} + 4e^{-40} + 12$$