1.

We know that $\frac{1}{4}\left(Z_1+Z_2+Z_3+Z_4\right) \sim Normal(0, \frac{1}{4})$
and hence $\frac{1}{2}\left(Z_1+Z_2+Z_3+Z_4\right) \sim Normal(0, 1)$
We also have that $W = Z_5^2+Z_6^2+Z_7^2+Z_8^2+Z_9^2+Z_{10}^2 \sim \chi_6^2$
Therefore, with $c = \sqrt{6}/2$

$$\frac{\sqrt{6}}{2} \cdot \frac{Z_1+Z_2+Z_3+Z_4}{\sqrt{Z_5^2+Z_6^2+Z_7^2+Z_8^2+Z_9^2+Z_{10}^2}} \sim t_6$$

2.

We have that

$$Z_1^2 + Z_2^2 + \ldots + Z_n^2 \sim \chi_n^2$$

and

$$Z_{n+1}^2 + Z_{n+2}^2 + \ldots + Z_{3n}^2 \sim \chi_{2n}^2$$

Therefore, with $c = 2$

$$2 \cdot \frac{Z_1^2 + Z_2^2 + \ldots + Z_n^2}{Z_{n+1}^2 + Z_{n+2}^2 + \ldots + Z_{3n}^2 \sim \chi_{2n}^2} \sim F_{2n}^n$$

3.

$$\overline{X} = 2\overline{Y} + 35 > 60 \iff \overline{Y} > 12.5$$

We have that

$$\overline{Y} \sim N(\mu_{\overline{Y}} \mu_Y = 15, \sigma^2_{\overline{Y}} = \sigma^2_Y/52 = 75/52)$$

Then since $\frac{15-12.5}{\sqrt{75/52}} = \frac{\sqrt{39}}{3}$

$$P(X > 60) = P(Y > 12.5) = 1 - 0.0188 = \frac{2453}{2500}$$

4.

Consider $Y = \sum_{i=1}^{100} Y_i \sim Normal(\mu_Y = 100 \cdot 2540 = 254000, \sigma_Y^2 = 2100^2 \cdot 100)$ Then $Z = \dfrac{300000 - 254000}{21000} = \dfrac{46}{21}$ and hence the proability that the total of 100 claims will be over 300000 dollars is 0.0143

5.

For each bulb, the probability that it is not a dud is

$$1 - \int_0^{2.5} \frac{1}{11} \cdot e^{-x/11} dx = e^{-5/22} = 0.2033$$

Then the probability that there is less than 45 duds follows a normal distribution with $\mu = 200 \cdot 0.2033 = 40.66$ and $\sigma = \sqrt{200 \cdot 0.2033 \cdot 0.7967} = 5.69156$, which hence is

$$1 - 0.2236 = 0.7486$$

because

$$\frac{45 - 40.66}{5.69156} = 0.7625$$

6.

We have that

$$\begin{aligned}
E[\overline{Y}]^2 &= E[\overline{Y}^2] - V[\overline{Y}] \\
&= E[\overline{Y}^2] - \frac{\beta^2}{m} \\
&= E[\overline{Y}^2] - \frac{E[\overline{Y}]^2}{m}
\end{aligned}$$

Therefore,

$$E[\overline{Y}]^2 = E[\overline{Y}^2] \cdot \frac{m}{m+1}$$

Hence,

$$\begin{aligned}
E[C] &= E[2Y^2 - 4Y] \\
&= 2E[Y^2] - 4E[Y] \\
&= 2(V[Y] + E[Y]^2) - 4E[Y] \\
&= 2(\beta^2 + \beta^2) - 4\beta \\
&= 4\beta^2 - 4\beta \\
&= 4E[\overline{Y}]^2 - 4E[\overline{Y}] \\
&= 4\frac{m}{m+1}E[\overline{Y}^2] - 4E[\overline{Y}]
\end{aligned}$$

Therefore, an unbiased estimator is $\dfrac{4m\overline{Y}^2}{m+1} - 4\overline{Y}$

7.

$X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$. Therefore,

$$\begin{aligned} F_{X_{(n)}}(x) &= P(X_{(n)} \le x) = P(X_1, X_2, \ldots, X_n < x) \\ &= \left(\frac{x}{\theta}\right)^n \\ f_{X_{(n)}}(x) &= n \cdot \frac{1}{\theta} \cdot \left(\frac{x}{\theta}\right)^{n-1} \end{aligned}$$

Therefore,

$$E[X_{(n)}] = \int_0^\theta x \frac{n}{\theta} \left(\frac{x}{\theta}\right)^{n-1} dx = \frac{n}{\theta^n} \int_0^\theta x^n dx = \frac{\theta n}{n+1}$$

and similarly

$$E[X_{(n)}^2] = \int_0^\theta x^2 \frac{n}{\theta} \left(\frac{x}{\theta}\right)^{n-1} dx = \frac{n}{\theta^n} \int_0^\theta x^{n+1} dx = \frac{\theta^2 n}{n+2}$$

We have that

$$\begin{aligned} V[Y] &= V[E[Y|X]] + E[V[Y|X]] \\ &= V\left[\frac{X}{3}\right] + E\left[\frac{X^2}{9}\right] \\ &= \frac{\theta^2}{108} + \frac{1}{9}(E[X]^2 + V[X]) \\ &= \frac{\theta^2}{108} + \frac{1}{9}\left(\frac{\theta^2}{4} + \frac{\theta^2}{12}\right) \\ &= \frac{5\theta^2}{108} \\ &= \frac{5(n+2)}{108n} E[X_{(n)}^2] \end{aligned}$$

8.

a.

$$F_{Y_{(n)}}(y) = \left(\int_0^y \frac{5y^4}{(\beta+1)^5}dy\right)^n = \left(\frac{y}{\beta+1}\right)^{5n}$$

$$f_{Y_{(n)}}(y) = 5n\left(\frac{y}{\beta+1}\right)^{5n-1} \cdot \frac{1}{\beta+1}$$

$$\begin{aligned}E[Y_{(n)}] &= \int_0^{\beta+1} y \cdot 5n\left(\frac{y}{\beta+1}\right)^{5n-1} \cdot \frac{1}{\beta+1}dy \\ &= \frac{5n}{(\beta+1)^{5n}}\int_0^{\beta+1} y^{5n}dy \\ &= \frac{5n\cdot(\beta+1)^{5n+1}}{(5n+1)(\beta+1)^{5n}} \\ &= \frac{5n(\beta+1)}{5n+1}\end{aligned}$$

$$\begin{aligned}\beta = E[\beta] = E[\hat{\beta}_1] &= a\cdot E[Y_{(n)}] + b \\ &= a\cdot\frac{5n(\beta+1)}{5n+1} + b\end{aligned}$$

Therefore, $a = \frac{5n+1}{5n}, b = -1$ and $aY_{(n)} + b$ is an estimator for β

b.

$$E[Y] = \int_0^{\beta+1} y\cdot\frac{5y^4}{(\beta+1)^5}dy = \frac{5(\beta+1)}{6}$$

$$\begin{aligned}\beta = E[\beta] = E[\hat{\beta}_2] &= a\cdot E[\overline{Y}] + b \\ &= a\cdot\frac{5(\beta+1)}{6} + b\end{aligned}$$

Therefore, $a = \frac{6}{5}, b = -1$ and $aY_{(n)} + b$ is an estimator for β

c.

$$\begin{aligned}V[Y_{(n)}] &= E[Y_{(n)}^2] - E[Y_{(n)}]^2 \\ &= \int_0^{\beta+1} y^2\cdot 5n\left(\frac{y}{\beta+1}\right)^{5n-1}\cdot\frac{1}{\beta+1}dy - \left(\frac{5n(\beta+1)}{5n+1}\right)^2 \\ &= \frac{5\left(\beta^2+2\beta+1\right)n}{5n+2} - \left(\frac{5n(\beta+1)}{5n+1}\right)^2 \\ &= \frac{5n(\beta+1)^2}{(5n+1)^2(5n+2)}\end{aligned}$$

Hence,

$$V[\hat{\beta}_1] = a^2\cdot V[Y_{(n)}] = \frac{(\beta+1)^2}{5n(5n+2)}$$

$$
\begin{aligned}
V[\overline{Y}] &= \frac{V[Y]}{n} \\
&= \frac{1}{n}\left(\int_0^{\beta+1} y^2 \cdot \frac{5y^4}{(\beta+1)^5} dy - \left(\frac{5(\beta+1)}{6}\right)^2\right) \\
&= \frac{1}{n}\left(\frac{5(\beta+1)^2}{7} - \left(\frac{5(\beta+1)}{6}\right)^2\right) \\
&= \frac{5(\beta+1)^2}{252n}
\end{aligned}
$$

Hence,

$$
V[\hat{\beta}_2] = a^2 \cdot V[\overline{Y}] = \frac{(\beta+1)^2}{35n}
$$

Therefore, if $n = 1$ then $V[\hat{\beta}_1] = V[\hat{\beta}_2]$, if $n > 1$ then $V[\hat{\beta}_1] < V[\hat{\beta}_2]$, which means that $\hat{\beta}_1$ is more efficient than $\hat{\beta}_2$ if $n > 1$ else, they have the same efficiency.

9.

a.

We have that

$$F_Y(y) = \int_0^y \frac{cy^{c-1}}{\theta} e^{-y^c/\theta} dy = 1 - e^{-y^c/\theta}$$

Therefore,

$$F_{Y^C}(y^c) = P(Y^C < y^c) = P(Y < y) = F_Y(y) = 1 - e^{-y^c/\theta}$$

Let $V = Y^C$, then

$$F_V(v) = 1 - e^{-v/\theta}$$

Hence, Y^C or V follows an exponential distribution with mean θ, which means that U follows the gamma distribution with $\alpha = n, \beta = \theta$.

$$L(\theta) = \prod_{i=1}^{n} f_Y(y_i|\theta) = \prod_{i=1}^{n} \frac{cy_i^{c-1}}{\theta} e^{-y_i^c/\theta}$$

Hence,

$$\begin{aligned} L(\theta|U) &= \frac{L(\theta)}{f_U(u)} \\ &= \frac{\prod_{i=1}^{n} \frac{cy_i^{c-1}}{\theta} e^{-y_i^c/\theta}}{\frac{1}{\Gamma(n)\theta^n} u^{n-1} e^{-u/\theta}} \\ &= \frac{\prod_{i=1}^{n} cy_i^{c-1}}{\frac{1}{\Gamma(n)} u^{n-1}} \end{aligned}$$

which means that U is sufficient for θ

b.

We know that

$$L(\theta) = \prod_{i=1}^{n} \frac{cy_i^{c-1}}{\theta} e^{-y_i^c/\theta} = \underbrace{e^{-u/\theta}}_{g(u,\theta)} \underbrace{\prod_{i=1}^{n} \frac{cy_i^{c-1}}{\theta}}_{h(y_1,y_2,\ldots,h_n)}$$

Therefore, U is sufficient for θ.

10.

a.

$$\begin{aligned} P(|Y_{(1)} - \beta| \leq c) = P(Y_{(1)} - \beta \leq c) = P(Y_{(1)} \leq c + \beta) \\ &= 1 - (Y \geq c + \beta)^n \\ &= 1 - \left(\int_{c+\beta}^{\infty} \frac{\alpha\beta^\alpha}{y^{\alpha+1}} dy \right)^n \\ &= 1 - \left(\frac{\beta}{c+\beta} \right)^{\alpha n} \end{aligned}$$

Then

$$P(|Y_{(1)} - \beta| \leq c) = \lim_{n\to\infty} P(|Y_{(1)} - \beta| \leq c) = \lim_{n\to\infty} 1 - \left(\frac{\beta}{c+\beta} \right)^{\alpha n} = 1$$

Hence, $Y_{(1)}$ is a consistent estimator for β.

b.

$$\begin{aligned} L(\alpha, \beta) &= \prod_{i=1}^{n} \frac{\alpha\beta^\alpha}{y^{\alpha+1}} I_{(\beta,\infty)}(y_i) \\ &= \underbrace{\frac{(\alpha\beta^\alpha)^n}{\left(\prod_{i=1}^{n} y_i\right)^{\alpha+1}} I_{(\beta,\infty)}(y_{(1)})}_{g(\alpha,\beta,u_1,u_2)} \cdot \underbrace{1}_{h(y_1,y_2,\dots,y_n)} \end{aligned}$$

Then $U_1 = \prod_{i=1}^{n} Y_i$ and $U_2 = Y_{(1)}$ are jointly sufficient for α and β

c.

If $y_{(1)} \geq \beta$ then

$$\begin{aligned} \ln(L(\alpha, \beta)) &= \ln \left(\frac{(\alpha\beta^\alpha)^n}{\left(\prod_{i=1}^{n} y_i\right)^{\alpha+1}} \right) \\ &= n\ln(\alpha) + n\alpha\ln(\beta) - (\alpha+1)\ln\left(\prod_{i=1}^{n} y_i \right) \end{aligned}$$

$$\frac{d}{d\beta} l(\alpha, \beta) = \frac{n\alpha}{\beta} \geq 0$$

Therefore, β has to be the largest possible, which is $y_{(1)}$

$$\frac{d}{d\alpha} l(\alpha, \beta) = \frac{n}{\alpha} + n\ln(\beta) - \ln\left(\prod_{i=1}^{n} y_i \right) = 0 \iff \alpha = \frac{-n}{n\ln(y_{(1)}) - \ln\left(\prod_{i=1}^{n} y_i\right)}$$

d.

$$F_{Y_{(1)}}(y) = P(Y_{(1)} < y) = 1 - \left(\int_y^\infty \frac{3\beta^3}{y^4} dy \right)^n = 1 - \left(\frac{\beta}{y} \right)^{3n}$$

Then

$$f_{Y_{(1)}}(y) = -3n \left(\frac{\beta}{y} \right)^{3n-1} \cdot \left(-\frac{\beta}{y^2} \right) = \frac{3n}{y} \left(\frac{\beta}{y} \right)^{3n}$$

Then

$$E[Y_{(1)}] = \int_\beta^\infty y \cdot \frac{3n}{y} \left(\frac{\beta}{y} \right)^{3n} dy = \frac{3\beta n}{3n-1}$$

Therefore,

$$\beta = \frac{3n-1}{3n} E[Y_{(1)}]$$

11.

$$E[Y] = \int_0^\theta y \cdot \frac{\beta(\theta-y)^{\beta-1}}{\theta^\beta} dy = \frac{\theta}{\beta+1}$$

$$E[Y^2] = \int_0^\theta y^2 \cdot \frac{\beta(\theta-y)^{\beta-1}}{\theta^\beta} dy = \frac{2\theta^2}{(\beta+1)\,(\beta+2)}$$

We know that

$$\frac{\theta}{\beta+1} = E[Y] = \mu_1', m_1' = \overline{Y}$$

$$\frac{2\theta^2}{(\beta+1)(\beta+2)} = E[Y^2] = \mu_2', m_2' = \frac{1}{n}\sum_{i=1}^n Y_i^2$$

and hence

$$V[Y] = \frac{2\theta^2}{(\beta+1)(\beta+2)} - \left(\frac{\theta}{\beta+1}\right)^2 = \frac{\beta\theta^2}{(\beta+1)^2(\beta+2)}$$

Therefore,

$$\frac{\theta_{MM}}{\beta_{MM}+1} = \overline{Y}$$

and

$$\frac{\beta_{MM}\theta_{MM}^2}{(\beta_{MM}+1)^2(\beta_{MM}+2)} = \frac{n-1}{n}S^2$$

Therefore,

$$\beta_{MM} = \frac{2}{1 - \frac{n-1}{n} \cdot \frac{S^2}{\overline{Y}^2}} - 2$$

and

$$\theta_{MM} = \overline{Y} \cdot \left(\frac{2}{1 - \frac{n-1}{n} \cdot \frac{S^2}{\overline{Y}^2}} - 1 \right)$$

12.

$$E[Y] = \int_{-\beta}^{0} \frac{24}{17\beta^3} y^3 dy + \int_{0}^{\beta} \frac{18}{17\beta^2} y^2 dy = -\frac{6\beta}{17} + \frac{6\beta}{17} = 0$$

$$V[Y] = E[Y^2] = \int_{-\beta}^{0} \frac{24}{17\beta^3} y^4 dy + \int_{0}^{\beta} \frac{18}{17\beta^2} y^3 dy = \frac{24\beta^2}{85} + \frac{9\beta^2}{34} = \frac{93\beta^2}{170}$$

Then,

$$\beta_{MM} = \sqrt{\frac{170(n-1)}{93n} S^2}$$

13.

$$E[\hat{\theta}] = 1 \cdot P(Y_i = k) + 0 \cdot P(Y_1 \neq k) = P(Y_1 = k) = p(1-p)^{k-1}$$

Since Y_i is a geometric distribution, $\sum_{i=1}^{n}$ is a negative binomial. Therefore,

$$\begin{aligned}
\hat{p}^* &= E[\hat{\theta}|U = u] \\
&= E[X|\sum_{i=1}^{n} Y_i = u] \\
&= P(Y_i = k|\sum_{i=1}^{s} Y_i = u) \\
&= \frac{P(Y_1 = k) \cdot P\left(\sum_{i=2}^{n} Y_i = u - k\right)}{P\left(\sum_{i=1}^{n} Y_i = u\right)} \\
&= \frac{p(1-p)^{k-1} \cdot \binom{u-k-1}{n-2} \cdot p^{n-1}(1-p)^{u-k-n+1}}{p^n(1-p)^{u-n} \cdot \binom{u-1}{n-1}} \\
&= \frac{\binom{u-k-1}{n-2}}{\binom{u-1}{n-1}}
\end{aligned}$$