MATH 217 (Fall 2022)

Honors Advanced Calculus, I

Assignment #5

1. Let $\varnothing \neq U \subset \mathbb{R}^N$ be open, and let $f, g\colon U \to \mathbb{R}$ be twice partially differentiable. Show that
$$\Delta(fg) = f\Delta g + 2(\nabla f)\cdot(\nabla g) + (\Delta f)g.$$

2. Let $\varnothing \neq U \subset \mathbb{R}^3$ be open, and let $f, g : U \to \mathbb{R}$ be twice continuously partially differentiable. Show that $\operatorname{div}(\nabla f \times \nabla g) = 0$ on U, where $\times$ denotes the cross product in $\mathbb{R}^3$.

3. Compute Δf for
$$f\colon \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}, \quad (x,y,z) \mapsto \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

4. Show that the function
$$f\colon \mathbb{R}^N \times (\mathbb{R}\setminus\{0\}) \to \mathbb{R}, \quad (x,t) \mapsto \frac{1}{t^{\frac{N}{2}}}\exp\left(-\frac{\|x\|^2}{4t}\right)$$
solves the *heat equation*
$$\Delta f - \frac{\partial f}{\partial t} = 0,$$
where Δ denotes the *spatial* Laplace operator, i.e.,
$$\Delta f = \sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable, let $c > 0$ and $v \in \mathbb{R}^N$ be arbitrary, and let $\omega := c\|v\|$. Show that
$$F\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, \quad (x,t) \mapsto f(x\cdot v - \omega t)$$
solves the *wave equation*
$$\Delta F - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = 0,$$
delta Δ again denoting the spatial Laplace operator.

6* Let
$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{otherwise.} \end{cases}$$
Show that f is twice partially differentiable everywhere, but that
$$\frac{\partial^2 f}{\partial y \partial x}(0,0) \neq \frac{\partial^2 f}{\partial x \partial y}(0,0).$$
Is f continuous at $(0,0)$?

Due Monday, October 24, 2020, at 10:00 a.m.; no late assignments.