## 1.

$$\int_0^b a(b-x)dx = abx - \frac{ax^2}{2}\Big|_0^b = \frac{ab^2}{2} = 1 \implies a = \frac{2}{b^2}$$

$$E[X] = \int_0^b xa(b-x)dx = \frac{abx^2}{2} - \frac{ax^3}{3}\Big|_0^b = \frac{ab^3}{6}$$

$$E[X^2] = \int_0^b x^2 a(b-x)dx = \frac{abx^3}{3} - \frac{ax^4}{4}\Big|_0^b = \frac{ab^4}{12}$$

Therefore,

$$V[X] = E[X^2] - (E[X])^2 = \frac{ab^4}{12} - \frac{a^2b^6}{36} = \frac{b^2}{6} - \frac{b^2}{9} = \frac{b^2}{18}$$

and hence

$$\sigma_X = \frac{b\sqrt{2}}{6}$$

$$F(x) = \int ab - axdx = abx - \frac{ax^2}{2} = \frac{2x}{b} - \frac{x^2}{b^2} + C$$

$C = 0$ so that $F(0) = 0$ and $F(b) = 1$. Therefore,

$$P(X > E[X] + \sigma_X) = 1 - F\left(\frac{2b + b\sqrt{2}}{6}\right) =$$

**2.**

$$
\begin{aligned}
M_X(t) &= \frac{3}{2}\int_0^2 e^{tx}(x-1)^2 dx \\
&= \frac{3}{2}\int_{-1}^1 e^{tx+t}x^2 dx \\
&= \frac{3}{2}\left( x^2 \cdot \frac{1}{t}e^{tx+t}\bigg|_{-1}^{1} - \int_{-1}^1 \frac{2x}{t}e^{tx+t}dx \right) \\
&= \frac{3}{2}\left( \frac{1}{t}e^{2t} - \frac{1}{t} - \frac{2x}{t^2}e^{tx+t}\bigg|_{-1}^{1} + \int_{-1}^1 \frac{2}{t^2}e^{tx+t}dx \right) \\
&= \frac{3}{2}\left( \frac{1}{t}e^{2t} - \frac{1}{t} - \frac{2}{t^2}e^{2t} - \frac{2}{t^2} + \frac{2}{t^3}e^{tx+t}\bigg|_{-1}^{1} \right) \\
&= \frac{3}{2}\left( \frac{1}{t}e^{2t} - \frac{1}{t} - \frac{2}{t^2}e^{2t} - \frac{2}{t^2} + \frac{2e^{2t}}{t^3} - \frac{2}{t^3} \right)
\end{aligned}
$$

**3.**

$$\begin{aligned}
f(x) &= \frac{1}{3-x} \\
&= \frac{1}{5} \cdot \frac{1}{1-\frac{1}{5}(2+x)} \\
&= \frac{1}{5} \sum_{k=0}^{\infty} \left( \frac{1}{5}(2+x) \right)^k
\end{aligned}$$

**4.**

$$
\begin{aligned}
E[N] &= \sum_{n=0}^{\infty} n \cdot 2^{-n-1} \\
&= \sum_{n=0}^{\infty} (n-1) \cdot 2^{-n} + 1 \\
&= \sum_{n=0}^{\infty} n \cdot 2^{-n} + \sum_{n=0}^{\infty} 2^{-n} + 1 \\
&= \frac{\frac{1}{2}}{\left(\frac{1}{2}-1\right)^2} - \frac{1}{1-\frac{1}{2}} + 1 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
E[N^2] &= \sum_{n=0}^{\infty} n^2 \cdot 2^{-n-1} \\
&= \sum_{n=0}^{\infty} (n-1)^2 \cdot 2^{-n} - 1 \\
&= \sum_{n=0}^{\infty} n^2 \cdot 2^{-n} - 2\sum_{n=0}^{\infty} n \cdot 2^{-n} + \sum_{n=0}^{\infty} 2^{-n} - 1 \\
&= \frac{\frac{1}{2^2}+\frac{1}{2}}{\left(1-\frac{1}{2}\right)^3} - 2 \cdot \frac{\frac{1}{2}}{\left(\frac{1}{2}-1\right)^2} + \frac{1}{1-\frac{1}{2}} - 1 \\
&= 6 - 2 \cdot 2 + 2 - 1 \\
&= 3
\end{aligned}
$$

Hence, we have that

$$
V[N] = E[N^2] - (E[N])^2 = 2
$$

We also have that

$$
\begin{aligned}
V[Y] &= E[V[X|N]] + V[E[X|N]] \\
&= E\left[\frac{1}{3} \cdot \frac{2}{3} \cdot n\right] + V\left[\frac{1}{3} n\right] \\
&= \frac{2}{9} E[N] + \frac{1}{9} V[N] \\
&= \frac{4}{9}
\end{aligned}
$$