1.

a.

Suppose $m(F) = 0$, then for any $x \notin F$, $x \notin E_n$ for finitely many $n \in \mathbb{N}$, thus

$$\lim_{n\to\infty} \chi_{E_n}(x) = 0$$

a.e. $x \in \mathbb{R}^d$.
Now suppose $m(F) > 0$, then let $x \in F$, thus $x \in E_n$ for infinitely many $n \in \mathbb{N}$, which means $\limsup_{n\to\infty} \chi_{E_n}(x) = 1$ for all $x \in F$. Thus

$$\lim_{n\to\infty} \chi_{E_n}(x) = 0$$

for some set $X \subseteq F^c$ thus contradiction as $m(F) > 0$.

b.

Apply fatou's lemma, we have that

$$\int_{\mathbb{R}^d} \liminf_{n\to\infty} f\chi_{E_n} dm = 0$$

which means that

$$m(f \liminf \chi_{E_n} \neq 0) = 0$$

hence

$$m(\liminf \chi_{E_n} \neq 0) = 0$$

Therefore, $\liminf \chi_{E_n}(x) = 1$ on a set X where $m(X) = 0$. But for every $x \in G$, $\liminf \chi_{E_n}(x) = 1$ thus $m(G) = 0$.

2.

a.

We have that $\frac{x}{n} \geq \sin\left(\frac{x}{n}\right)$ thus $x > n\sin\left(\frac{x}{n}\right)$, and let $t = x^2$, we have

$$\int_0^\infty \frac{x}{x^4+1} = \int_0^\infty \frac{1}{2(t^2+1)}dt = \frac{\pi}{4}$$

Thus the solution is

$$\int_0^\infty \frac{\lim_{n\to\infty} x \cdot n/x\sin(x/n)}{x^4+1} = \int_0^\infty \frac{x}{x^4+1} = \frac{\pi}{4}$$

b.

Since $n^2(1-\cos(x/n)) \leq n^2(1-\cos^2(x/n)) = n^2\sin^2(x/n) \leq x^2$ and let $t = x^3$,

$$\int_{-\infty}^\infty \frac{x^2}{x^6+1}dx = \int_{-\infty}^\infty \frac{1}{3(t^2+1)}dt = \frac{\pi}{3}$$

We have the solution through L'Hopital and let $y = x^2$

$$\int_{-\infty}^\infty \lim_{n\to\infty} \frac{n^2(1-\cos(x/n))}{1+x^6} = \int_{-\infty}^\infty \frac{x^2/2}{x^6+1}dx = \frac{\pi}{6}$$

as

$$\lim_{n\to\infty} n^2(1-\cos(x/n)) = \lim_{n\to\infty} \frac{-x\sin(x/n)/n^2}{-2/n^3} = \frac{x}{2}x\lim_{n\to\infty} \frac{sin(x/n)}{x/n} = \frac{x^2}{2}$$

3.

If $\lim_{n\to\infty} \int_E |f_n - f| = 0$ then for every $\varepsilon > 0$ there is n_0 such that for all $n > n_0$,

$$\left|\int_E |f_n| - \int_E |f|\right| \leq \left|\int_E (|f_n| - |f|)\right| \leq \int_E |f_n - f| \leq \varepsilon$$

Thus $\int_E |f_n| \to \int_E |f|$.
Now suppose $\int_E |f_n| \to \int_E |f|$, then we know that

$$|f_n| + |f| \to 2|f|$$

a.e. $x \in E$

$$\int_E |f_n - f| \leq \int_E |f_n| + |f|$$

$$\int_E |f_n - f| = 0$$

as $f_n \to f$ a.e. $x \in E$ and

$$\lim_{n\to\infty} \int_E |f_n| + |f| = \int_E 2|f|$$

Thus applying the Generalized Dominance Convergence Theorem on $|f_n - f|$ and $|f_n| + |f|$, we have that

$$\lim_{n\to\infty} \int_E |f_n - f| = \int_E 0 = 0$$

4.

a.

For all $\varepsilon > 0$ we can find a uniformly continuous function g such that $\int_{\mathbb{R}^d} |f-g| < \varepsilon/3$ and small enough $t > 0$ such that $|g(x-t)-g(x)| < \varepsilon/3m(E)$ for all $x \in \mathbb{R}^d$. Then

$$\begin{aligned}&\int_{\mathbb{R}^d} |f_t(x) - f(x)|\\ \leq&\int_{\mathbb{R}^d} |f(x-t) - g(x-t)| + |g(x-t) - g(x)| + |g(x) - f(x)|\\ <&\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\\ =&\varepsilon\end{aligned}$$

Thus

$$\int_{\mathbb{R}^d} |f_t(x) - f(x)| = 0$$

b.

Since $\chi_E \in \mathcal{L}_1(\mathbb{R}^d)$, for all $\varepsilon > 0$, there is a uniformly continuous function h such that $\int_{\mathbb{R}^d} |\chi_E - h| < \varepsilon/3$, then let the sequence $x_n \to x$ and thus there is an n_0 such that for all $n > n_0$, $|h(x_n) - h(x)| < \varepsilon/3m(E)$. Then

$$\begin{aligned}&|\phi(x) - \phi(x_n)|\\ =&\left|\int_{\mathbb{R}^d} \chi_E(x+t)\chi_E(t) - \chi_E(x_n+t)\chi_E(t)dt\right|\\ =&\left|\int_{\mathbb{R}^d} \chi_E(t)\left(\chi_E(x+t) - \chi_E(x_n+t)\right)dt\right|\\ \leq&\int_{\mathbb{R}^d} |\chi_E(x+t) - \chi_E(x_n+t)|dt\\ \leq&\int_{\mathbb{R}^d} |\chi_E(x+t) - h(x+t)| + |h(x+t) - h(x_n+t)| + |h(x_n+t) - \chi_E(x_n+t)|\\ <&\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\\ =&\varepsilon\end{aligned}$$

c.

We first have that for $x \in E$,

$$\phi(x) = \int_{\mathbb{R}^d} \chi_E(x+t)\chi_E(t)dt = m(E \cap (E - x)) = m(E_x)$$

where $E_x = \{y : y \in E, x + y \in E\}$.
Notice that if $y \in E_x$ then $y \in E$ and $x + y \in E$ thus $x \in E - E$. Thus if $m(E_x) > 0$ then $x \in E - E$.

Now since $m(E) > 0$, we have that there is $B_\varepsilon(x_0) \subseteq E$ and thus for any $\delta < \varepsilon/2$, we have that $\phi(x) = m(E_x) > 0$ for all $x \in B_{\delta/2}(0)$. Thus $B_\delta(0) \subseteq E - E$

5.

For every $\varepsilon > 0$, we can find a respective integrable step function ϕ such that $\int_{\mathbb{R}} |f - \phi| < \varepsilon/2$, where

$$\phi = \sum_{k=1}^{N} a_k \chi_{R_k}$$

where $a_k \in \mathbb{R}$ and R_k are bounded intervals. Thus, there is an open interval R such that $\cup_{k=1}^{N} R_k \subseteq R$, now we have that

$$\lim_{n\to\infty} \int_R |\sin(nx)| dx \leq \lim_{n\to\infty} \int_{R_{n,1}} 1 dx + \int_{R_{n,2}} 1 dx + \int_{R_n} \sin(nx) dx = 0$$

where $R_n = (a, b)$ is the largest interval such that $\int_{R_n} \sin(nx)dx = 0$ and $\sin(na) = \sin(nb) = 0$. $R_{n,1}$ and $R_{n,2}$ are the intervals on the left and right of R_n respectively so that $m(R_{n,1}) \to 0$ and $m(R_{n,2}) \to 0$. Thus

$$\lim_{n\to\infty} \int_R |\sin(nx)| dx = 0$$

and hence

$$\lim_{n\to\infty} \int_R |\phi(x) \sin(nx)| dx = 0$$

Therefore, for every ε, there exists λ_0 such that for all $\lambda > \lambda_0$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}} f(x) \sin(\lambda x) dx \right| \\
&\leq \int_{\mathbb{R}} |f(x) - \phi(x)| |\sin(\lambda x)| dx + \int_{\mathbb{R}} |\phi(x) \sin(\lambda x)| dx \\
&\leq \int_{\mathbb{R}} |f(x) - \phi(x)| dx + \int_R |\phi(x)| |\sin(\lambda x)| dx \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$