1.

Suppose what is given is a field, † is the multiplicative identity since it is the only element in the set satisfy the condition:

$$\forall x \in \{\spadesuit, \dagger, \bigcirc, \mathrm{A}\} : x \cdot \dagger = x$$

But, since

$$\forall x \in \{\spadesuit, \dagger, \bigcirc, \mathrm{A}\} : x \cdot \bigcirc \neq x$$

which means that ◯ does not have all the inverse, hence what given is not a field.

2.

For any number $x = x_1 + ix_2, y = y_1 + iy_2, z = z_1 + iz_2 \in \mathbb{Q}[i]$ with $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{Q}$
F1: $\mathbb{Q}$ is closed under addition and multiplication, therefore
$x + y = (x_1 + y_1) + i(x_2 + y_2) \in \mathbb{Q}[i]$
F2: $\mathbb{Q}$ is commutative, therefore

$$x_1 + y_1 = y_1 + x_1$$

$$x_2 + y_2 = y_2 + x_2$$

$$x_1 \cdot y_1 = y_1 \cdot x_1$$

$$x_2 \cdot y_2 = y_2 \cdot x_2$$

Therefore, $x + y = y + x, x \cdot y = y \cdot x$
F3: $\mathbb{Q}$ is associative, therefore

$$x_1 + (y_1 + z_1) = (x_1 + y_1) + z_1$$

$$x_2 + (y_2 + z_2) = (x_2 + y_2) + z_2$$

Therefore, $x + (y + z) = (x + y) + z$
F4:

$$\begin{aligned}
x \cdot (y + z) &= (x_1 + ix_2) \cdot ((y_1 + z_1) + i(y_2 + z_2)) \\
&= x_1 \cdot (y_1 + z_1) - x_2 \cdot (y_2 + z_2) + i(x_2 \cdot (y_1 + z_1) + x_1 \cdot (y_2 + z_2)) \\
&= (x_1 \cdot y_1 - x_2 \cdot y_2 + i(x_2 \cdot y_1 + x_1 \cdot y_2)) + (x_1 \cdot z_1 - x_2 \cdot z_2 + i(x_2 \cdot z_1 + x_1 \cdot z_2)) \\
&= (x_1 + ix_2) \cdot (y_1 + iy_2) + (x_1 + ix_2) \cdot (y_1 + iy_2) \\
&= x \cdot y + x \cdot z
\end{aligned}$$

F5: If $x = 0 + i0$ and $y = 1 + i0$ then $\forall z \in \mathbb{Q}[i]$:

$$x + z = (x_1 + z_1) + i(x_2 + z_2) = (0 + z_1) + i(0 + z_2) = z$$

and

$$\begin{aligned}
y \cdot z &= (y_1 \cdot z_1 - y_2 \cdot z_2) + i(y_1 \cdot z_2 + y_2 \cdot z_1) \\
&= (z_1 - 0) + i(z_2 + 0) \\
&= z
\end{aligned}$$

Therefore, $x = 0 + i0$ and $y = 1 + i0$ are the neutral elements.
F6: $\forall x \in \mathbb{Q}[i] : \exists y = -x1 + i(-x2)$ such that

$$x + y = (x_1 + (-x_1)) + i(x_2 + (-x_2)) = 0 + 0i$$

and $\forall x \in \mathbb{Q}[i] \backslash \{0\} : \exists z = \dfrac{x_1 - ix_2}{x_1^2 + x_2^2}$ (since $x_1^2 + x_2^2 = 0$ if and only if $x_1 = x_2 = 0$ which means that $x = 0 + 0i$) such that

$$x \cdot z = \left(x_1 \cdot \frac{x_1}{x_1^2 + x_2^2} - x_2 \cdot \frac{-x_2}{x_1^2 + x_2^2} \right) + i \left(x_1 \cdot \frac{-x_2}{x_1^2 + x_2^2} + x_2 \cdot \frac{x_1}{x_1^2 + x_2^2} \right) = 1 + 0i$$

It is not possible to turn $\mathbb{Q}[i]$ into an ordered field because:
Supppose $\mathbb{Q}[i]$ is a field then

$$i \in \mathbb{Q}[i] \cap i \neq 0 + 0i \implies i \neq 0 \implies i^2 = -1 > 0$$

which is a contradiction and hence $\mathbb{Q}[i]$ is not a field.

3.

a. Since S is bounded below, there exists a number M such that

$$\forall s \in S : s \geq M \iff \forall s \in S : -s \leq -M$$

Therefore, the set $-S$ is bounded above by $-M$
b. Also, $-S$ is not empty and is a subset of $\mathbb{R}$ which is complete, hence -S has a supremum.
Let X, Y respectively be the set contains all the upper bound of $-S$ and the set contains all the lower bound of S.
We have

$$\forall x \in X : sup(-S) \geq x \iff \forall x \in X : -sup(-S) \leq -x \iff \forall y \in Y : -sup(-S) \leq y$$

because from part a, we know that M is a lower bound of S or $M \in X$ if and only if $-M$ is an upper bound of $-S$ $(-M \in Y)$.
Therefore, -sup(-S) is the infimum of S and inf(S) = -sup(-S)

4.

$\forall n \in \mathbb{N} : 1 > 1 - \dfrac{1}{n} \geq 0$
$\forall k \in \mathbb{N} : (-1)^{2k} = 1, (-1)^{2k-1} = -1$
$\forall n \in \mathbb{N}, \exists k \in \mathbb{N} : n = 2k \vee n = 2k - 1.$
If $n = 2k$ then $-1 < 0 \leq (-1)^n \left(1 - \dfrac{1}{n}\right) < 1$
Else if $n = 2k - 1$ then $-1 < (-1)^n \left(1 - \dfrac{1}{n}\right) \leq 0 < 1$
Hence, 1 and -1 is respectively the upper bound and upperbound for S and therefore the supremum of S must be less or equal to 1. If $\sup(S)$ is less than 1, then
Let $\sup(S) = 1 - \epsilon$, then $\forall n \in \mathbb{N}$ such that $2n > \dfrac{1}{\epsilon} : (-1)^{2n} \left(1 - \dfrac{1}{2n}\right) > 1 - \epsilon$ which is a contradiction because supremum must be larger than all number in the set. Therefore, $\sup(S) = 1$.
Similarly, $\inf(S)$ is larger than equal to -1 then
Let $\inf(S) = -1 + \epsilon$, then $\forall n \in \mathbb{N}$ such that $2n+1 > \dfrac{1}{\epsilon} : (-1)^{2n+1} \left(1 - \dfrac{1}{2n+1}\right) < -1 + \epsilon$ which is a contradiction because infimum must be smaller than all number in the set. Therefore, $\inf(S) = -1$

5.

We know that $\forall s \in S : \text{sup}S \geq s$ and $\forall t \in T : \text{sup}T \geq t$.
Therefore, $\forall s \in S$ and an arbitary $t \in T : \text{sup}S + t \geq s + t$ and hence $\forall s \in S\, \forall t \in T : \text{sup}S + \text{sup}(T) \geq s + t$.
Therefore $\text{sup}S + \text{sup}T$ is an upperbound of $S + T$, which means that $\text{sup}(S + T) \leq \text{sup}S + \text{sup}T$.
Suppose $\text{sup}(S + T) < \text{sup}S + \text{sup}T$, then there exists an $\epsilon > 0$ such that $\text{sup}(S + T) = \text{sup}S + \text{sup}T - \epsilon$. However,

$$\exists s' \in S \text{ satisfies } s \in \left(\text{sup}S - \frac{\epsilon}{2}, \text{sup}S\right]$$

bceause else, $\sup S - \dfrac{\epsilon}{4}$ is an upper bound smaller than $\sup S$
Similarly,

$$\exists t' \in T \text{ satisfies } s \in \left(\text{sup}T - \frac{\epsilon}{2}, \text{sup}T\right]$$

Hence, $s' + t' > \text{sup}S - \dfrac{\epsilon}{2} + \sup T - \dfrac{\epsilon}{2} = \text{sup}S + \text{sup}T$ - ϵ which is a contradiction because a supremum must be larger than all the elements in the set and therefore, $\text{sup}S + \text{sup}T = \text{sup}(S + T)$

6.

Let S be a non-empty set having at least one upper bound. Then the set U containing all upper bound of S is not empty.
First, choose a random element from S and a random element from U which we denote respectively s_1 and u_1. Then define $I_i := [s_i, u_i]$ as follows

$$I_i := \begin{cases} [s_1, u_1], \text{for } i = 1 \\ \\ \left[s_{i-1}, \dfrac{s_{i-1}+u_{i-1}}{2}\right], \text{for } \dfrac{s_{i-1}+u_{i-1}}{2} \text{ is an upper bound} \\ \\ \left[\dfrac{s_{i-1}+u_{i-1}}{2}, u_{i-1}\right], \text{for } \dfrac{s_{i-1}+u_{i-1}}{2} \text{ is not an upper bound} \end{cases}$$

Since $\forall i \in \mathbb{N} : \dfrac{s_i + u_i}{2} \in (s_1, u_1)$, therefore $I_{i+1} \subset I_i$ for all natural number i. Hence,

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing$$

and

$$\lim_{n\to\infty} \mu\left(\bigcap_{i=1}^{n} I_n\right) = \lim_{n\to\infty} \mu(I_n) = \lim_{n\to\infty} \frac{\mu(I_1)}{2^n} = 0$$

which means that $\bigcap\limits_{n=1}^{\infty} I_n$ contains a single point which we denote x because it is not empty, it cannot be an interval larger than 0 and from how we define the interval I_i, $\bigcap\limits_{n=1}^{\infty} I_n$ cannot contain two seperate points .
If x is not the supremum of S then there exists an upper bound y that is smaller than x, which means that $\nexists i \in \mathbb{N} : s_i \in [y, x]$ because s_i cannot be an upper bound and therefore s_i is always smaller than y and hence $\mu\left(\bigcap\limits_{i=1}^{n} I_n\right) \geq x - y > 0$, which is a contradiction. As a result, that single point x is the supremum of S.