1.

Since the set is compact with content zero, we have that there exists $I_1, I_2, \ldots, I_n$ such that

$$I_1 \cup I_2 \cup \ldots \cup I_n \subset U \text{ and } \sum_{j=1}^{n} \mu(I_j) < \frac{\epsilon}{2}$$

Consider the interval $I_i = (a_{i,1}, b_{i,1}) \times \ldots \times (a_{i,N}, b_{i,N})$. Since the set of rational number is dense, we can find b'_1 such that

$$\mathbb{Q} \ni b'_{i,1} - b_{i,1} < \frac{\epsilon(b_{i,1} - a_{i,1})}{2nN\mu(A_i)}$$

Therefore, $O_{i,1} = [a_{i,1}, b'_{i,1}] \times \ldots \times [a_{i,N}, b_{i,N}]$ satisfies

$$\mu(O_{i,1}) - \mu(I_i) = \mu(I_i)\frac{b'_{i_1} - a_{i,1}}{b_{i,1} - a_{i,1}} - \mu(I_i) = \mu(I_i)\frac{b'_{i_1} - b_{i,1}}{b_{i,1} - a_{i,1}} < \frac{\mu(I_i)}{b_{i,1} - a_{i,1}} \cdot \frac{\epsilon(b_{i,1} - a_{i,1})}{2nN\mu(A_i)} = \frac{\epsilon}{2nN}$$

Then using $O_{i,1}$, we can find $O_{i,2}$ such that $\mu(O_{i,2}) - \mu(O_{i,1}) < \dfrac{\epsilon}{2nN}$ using the same process.
Do this process for the rest $n-1$ subintervals, we have that $O_i = O_{i,n} = [a_{i,1}, b'_{i,1}] \times \ldots \times [a_{i,N}, b'_{i,N}]$ satisfies $I_i \subset O_i$ and

$$\mu(O_i) - \mu(I_i) < \frac{\epsilon}{2nN} \cdot N = \frac{\epsilon}{2n}$$

and hence

$$\sum_{j=1}^{n} \mu(O_i) - \sum_{j=1}^{n} \mu(I_i) < \frac{\epsilon}{2n} \cdot n = \frac{\epsilon}{2}$$

Therefore,

$$\sum_{j=1}^{n} \mu(O_i) < \sum_{j=1}^{n} \mu(I_i) + \frac{\epsilon}{2} < \epsilon$$

Since each intervals in O_i has a rational length, we can split it into cubes. Suppose we have a interval $I = [a_1, b_1] \times \ldots \times [a_N, b_N]$ such that $b_i - a_i = \dfrac{x_i}{y_i}$ where x_i and y_i are integers. Then since for every subintervals $[a_i, b_i]$, we have that $\dfrac{\frac{x_i}{y_i}}{\frac{1}{\prod_{j=1}^{N} y_j}} = x_i \cdot \prod_{\substack{j=1 \\ i \neq j}}^{N} y_j$ is also an integer and hence we can split the interval I into cubes with sides of length $\frac{1}{\prod_{j=1}^{N} y_j}$. Therefore, we get the results.

2.

Let I be a compact interval such that $D \subset I_1 \cup I_2 \cup \ldots \cup I_n \subset I$, we have that

$$\mu(D) = \int_I \chi_D \leq \int_I \chi_{\bigcup_{j=1}^n I_j} \leq \int_I \sum_{j=1}^n \chi_{I_j} = \sum_{j=1}^n \int_I \chi_{I_j} = \sum_{j=1}^n \mu(I_j)$$

and hence $\mu(D) \leq \inf \sum_{j=1}^n \mu(I_j)$.
For all $\epsilon > 0$ then there is a partition P of I such that

$$\left| \mu(D) - \sum_v \chi_D(x_v)\mu(I_v) \right| < \epsilon$$

Next, we choose $I_1, I_2, \ldots, I_n$ satisfy the condition $D \cap I_v \neq \varnothing$ and choose $x_v \in D \cap I_v$. Therefore,

$$\begin{aligned} \sum_{j=1}^n \mu(I_j) &= \sum_v \chi_D(x_v)\mu(I_v) \\ &\leq \mu(D) + \left| \mu(D) - \sum_v \chi_D(x_v)\mu(I_v) \right| \\ &< \mu(D) + \epsilon \end{aligned}$$

Therefore,

$$\mu(D) = \inf \sum_{j=1}^n \mu(I_j)$$

3.

Since f and g are continuous and I is compact. f and g are Riemann integrable on I and hence fg is also Riemann integrable, which means that there exists a partition P such that for all refinement P_ϵ of P

$$\left|\int_I fg - \sum_v f(x_v)g(x_v)\mu(I_v)\right| < \frac{\epsilon}{2}$$

for arbitary $x_v \in I_v$. We know that $f(I)$ are compact since f is continuous and I is compact. Therefore, there exists $M = \sup_{x\in I} f(x)$. Then since g is uniformly continuous on I, we can find a refinement Q of P such that for all subdivision, $|g(x_v) - g(y_v)| < \frac{\epsilon}{2M\mu(I)}$. Hence, for all refinement Q_ϵ of Q, we have

$$\begin{aligned}
\left|\int_I fg - \sum_v f(x_v)g(y_v)\mu(I_v)\right| \leq& \left|\int_I fg - \sum_v f(x_v)g(x_v)\mu(I_v)\right| \\
&+ \left|\sum_v f(x_v)g(x_v)\mu(I_v) - \sum_v f(x_v)g(y_v)\mu(I_v)\right| \\
<& \frac{\epsilon}{2} + \left|\sum_v f(x_v)\mu(I_v)(g(x_v) - g(y_v))\right| \\
<& \frac{\epsilon}{2} + \sum_v M\mu(I_v)\frac{\epsilon}{2M\mu(I)} \\
=& \frac{\epsilon}{2} + \frac{\epsilon}{2\mu(I)}\sum_v \mu(I_v) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

4.

a.

Since D has content, ∂D has content zero, and hence

$$\mu(\overline{D}) = \mu(D \cup \partial D) \leq \mu(D) + \mu(\partial D) = \mu(D)$$

Since $D \subseteq \overline{D}, \mu(D) \leq \mu(\overline{D})$ and hence $\mu(D) = \mu(\overline{D})$

b.

Z is a set of content zero, therefore Z is bounded and hence $\overline{Z}$ is compact and has content zero. Therefore, $\phi(\overline{Z})$ has content zero, which means that $\phi(Z) \subset \phi(Z) \cup \phi(\partial Z) = \phi(\overline{Z})$ also has content zero.

5.

a.

Consider the function

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \to \left(\frac{zx}{h}, \frac{zy}{h}\right)$$

Then

$$\det \phi(x,y) = \det \begin{pmatrix} \frac{z}{h} & 0 \\ 0 & \frac{z}{h} \end{pmatrix} = \frac{z^2}{h^2}$$

$$\begin{aligned}
\mu_3 &= \int_{0+}^{h} \int_{\mathbb{R}^2} \chi_D \left(\frac{hx}{z}, \frac{hy}{z}\right) dxdydz \\
&= \int_{0+}^{h} \int_{\mathbb{R}^2} \chi_D(x,y) \cdot \frac{z^2}{h^2} dxdydz \\
&= \mu_2(D) \int_{0+}^{h} \frac{z^2}{h^2} dz \\
&= \mu_2(D) \left.\frac{z^3}{3h^2}\right|_{0+}^{h} \\
&= \mu_2(D) \cdot \frac{h}{3}
\end{aligned}$$

which is the volume of a “cone” formed by projecting D to the origin of $\mathbb{R}^3$

b.

$$\begin{aligned}
\mu_n &= \int_{0+}^{h} \int_{\mathbb{R}^{n-1}} \chi_D \left(x, y\right) dxdydz \\
&= \mu_{n-1}(D) \int_{0+}^{h} 1 dz \\
&= \mu_{n-1}(D) \left. z\right|_{0+}^{h} \\
&= \mu_{n-1}(D) \cdot h
\end{aligned}$$

c.

When $n = 1$, the object generated is a line with $\mu_1 = h$ When $n = 2$, the object generated is a rectangle with $\mu_2 = h \cdot \mu_1$

6.

Suppose $x_0 \in K$ such that $\phi(x_0) \in \partial\phi(K)$, then $x_0 \in K \backslash Z$ or $x_0 \in Z$.
In the case where $x_0 \in K \backslash Z$. Suppose that $x_0 \notin \partial K$. Since Z has content zero, there exists an open neighborhood around x_0: $B(x_0)$ such that $B(x_0) \subset K$ and $\det J_\phi(x) \neq 0$ for all $x \in B(x_0)$. Hence, $\phi(B(x_0)) \subset \phi(K)$ is open. However, we know that $\phi(x_0) \in \partial\phi(K)$, which means that for every open neighborhood around $\phi(x_0)$, there exists a point not in K. Therefore, this is a contradiction and $x_0 \in \partial K$.
Finally, since if $x_0 \in K$ satisifes $\phi(x_0) \in \partial\phi(K)$ then $x_0 \in \partial K$ or $x_0 \in Z$, we have that $\phi(\partial K) \subset \phi(\partial K) \cup \phi(Z)$.