

1.

1.

For any $\varepsilon = (\varepsilon_n) \neq \tilde{\varepsilon} = (\tilde{\varepsilon}_n)$, there is a $N \in \mathbb{N}$ such that $\varepsilon_n = \tilde{\varepsilon}_n$ for all $n < N$ and $\varepsilon_N \neq \tilde{\varepsilon}_N$. WLOG, let $\varepsilon_N = 2$ and $\tilde{\varepsilon}_N = 0$. Thus we have

$$g(\varepsilon) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3^n} \geq \sum_{n=1}^{N-1} \frac{\varepsilon_n}{3^n} + \frac{2}{3^N}$$

and

$$g(\tilde{\varepsilon}) = \sum_{n=1}^{\infty} \frac{\tilde{\varepsilon}_n}{3^n} \leq \sum_{n=1}^{N-1} \frac{\tilde{\varepsilon}_n}{3^n} + \sum_{n=N+1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{N-1} \frac{\varepsilon_n}{3^n} + \underbrace{\frac{2}{3^{N+1}} \cdot \frac{1}{1-1/3}}_{\frac{1}{3^N}}$$

Thus $g(\varepsilon) > g(\tilde{\varepsilon})$ and therefore g is injective.

2.

Consider the function

$$p : \{0, 1\}^{\mathbb{N}} \rightarrow [0, 1], \quad (\varepsilon_n)_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}$$

Since there is a natural bijection $h : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 2\}^{\mathbb{N}}$, $p = p \circ h^{-1} \circ h = g \circ h$ is injective. We claim that p is also surjective. For every $x \in [0, 1]$, there exists a sequence $\varepsilon = (\varepsilon_n)_{n=1}^{\infty}$ such that $g(\varepsilon) = x$

$$\left| \sum_{n=1}^N \frac{\varepsilon_n}{2^n} - x \right| < \epsilon$$

To construct the sequence ε , start from $n = 0$,

- if $\sum_{i=1}^n \frac{1}{2^{n+1}} < x$, then let $\varepsilon_{n+1} = 1$
- if $\sum_{i=1}^n \frac{1}{2^{n+1}} > x$, then let $\varepsilon_{n+1} = 0$
- if $\sum_{i=1}^n \frac{1}{2^{n+1}} = x$, then let $\varepsilon_{n+1} = 1$ and $\varepsilon_i = 0$ for all $i > n + 1$ then stop the process

Increase n by 1 and start the process again.

Since $\frac{1}{2^n} \rightarrow 0$ as $n \rightarrow \infty$ and $\sum_{i=1}^n \frac{1}{2^i} \leq x$ for all $n \in \mathbb{N}$. Thus for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$, $0 \leq x - \sum_{i=1}^n \frac{1}{2^i} < \epsilon$, and therefore $p(\varepsilon) = \sum_{n=1}^{\infty} \frac{1}{2^n} = x$. Thus, p is bijective. Now consider the function

$$k : (0, 1) \rightarrow \mathbb{R}, \quad x \mapsto \tan(2x\pi - \pi)$$

We have that k is bijective thus $(0, 1) \sim \mathbb{R}$. $(0, 1) \sim [0, 1]$ as the map

$$\phi(x) = \begin{cases} \frac{1}{n+1} & , \text{ if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 1 & , \text{ if } x = 0 \\ x & , \text{ otherwise} \end{cases}$$

is bijective and thus $\{0, 1\}^{\mathbb{N}} \sim \mathbb{R}$.

3.

$\{0, 2\}^{\mathbb{N}} \sim \{0, 1\}^{\mathbb{N}}$, thus $\{0, 2\}^{\mathbb{N}} \sim \mathbb{R}$ and $\mathbb{R} \sim C$.

4.

Also, if N_x is finite, we set $a_{n,x} = 0$ for all $n > N_x$ so that $\sum_{n=1}^{\infty} \frac{a_{n,x}}{3^n} = x$ regardless of N_x .

For any $x \neq y$, that is $\sum_{n=1}^{N_x} \frac{a_{n,x}}{3^n} \neq \sum_{n=1}^{N_y} \frac{a_{n,y}}{3^n}$, then as we know the function from subquestion 1 is injective, we have that $(a_{n,x}) \neq (a_{n,y})$, that is there exists $N \in \mathbb{N}$ such that for all $n < N$, $a_{n,x} = a_{n,y}$ and $a_{N,x} \neq a_{N,y}$.

- In case $N_x > N, N_y > N, a_{N,x} > a_{N,y} \implies x > y$ as

$$\sum_{n=1}^{\infty} \frac{a_{n,x}}{3^n} - \frac{a_{n,y}}{3^n} = \frac{2}{3^N} + \sum_{n=N+1}^{\infty} \frac{a_{n,x} - a_{n,y}}{3^n} \geq \frac{2}{3^N} - \underbrace{\sum_{n=N+1}^{\infty} \frac{2}{3^n}}_{1/3^N} > 0$$

and thus because of $a_{N,x} \neq a_{N,y}, x \neq y$ by assumption and WLOG, we have $a_{N,x} < a_{N,y} \iff x < y$.

- In case $N_y > N, N_x \leq N$ which is $N_x = N$ then $a_{N,x} = 1$. If $a_{N,y} = 2$ then obviously $x < y$, if $a_{N,y} = 0$ then since $x \neq y$, there is n_0 such that $a_{n_0,y} - a_{n_0,x} < 2$ and hence

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_{n,y} - a_{n,x}}{3^n} &= -\frac{1}{3^N} + \sum_{n=N+1}^{\infty} \frac{a_{n,y} - a_{n,x}}{3^n} \\ &= -\frac{1}{3^N} + \sum_{\substack{n=N+1 \\ n \neq n_0}}^{\infty} \frac{a_{n,y} - a_{n,x}}{3^n} + \frac{a_{n_0,y} - a_{n_0,x}}{3^{n_0}} \\ &< -\frac{1}{3^N} + \sum_{n=N+1}^{\infty} \frac{2}{3^n} \\ &= 0 \end{aligned}$$

Finally, we can conclude that if $x < y$, then there exists $N \in \mathbb{N}$ such that for all $n < N$, $a_{n,x} = a_{n,y}$ and there is three cases

- $a_{N,x} = 0, a_{N,y} = 2$, thus $b_{N,x} = 0, b_{N,y} = 1$

$$\begin{aligned}
f(y) - f(x) &= \sum_{n=1}^{\infty} \frac{b_{n,y} - b_{n,x}}{2^n} \\
&= \frac{1}{2^N} + \sum_{n=N+1}^{\infty} \frac{b_{n,y} - b_{n,x}}{2^n} \\
&\geq \frac{1}{2^N} + \sum_{n=N+1}^{\infty} \frac{-1}{2^n} \\
&= \frac{1}{2^N} - \frac{1}{2^N} = 0
\end{aligned}$$

- $a_{N,x} = 1, a_{N,y} = 2$, thus $b_{N,x} = b_{N,y} = 1$

$$\begin{aligned}
f(y) - f(x) &= \sum_{n=1}^{\infty} \frac{b_{n,y} - b_{n,x}}{2^n} \\
&= \sum_{n=N+1}^{\infty} \frac{b_{n,y} - b_{n,x}}{2^n} \\
&= \sum_{n=N+1}^{\infty} \frac{b_{n,y}}{2^n} \geq 0
\end{aligned}$$

- $a_{N,x} = 0, a_{N,y} = 1$, thus $b_{N,x} = 0, b_{N,y} = 1$.

$$\begin{aligned}
f(y) - f(x) &= \sum_{n=1}^{\infty} \frac{b_{n,y} - b_{n,x}}{2^n} \\
&= \frac{1}{2^N} + \sum_{n=N+1}^{\infty} \frac{-b_{n,x}}{2^n} \\
&\geq \frac{1}{2^N} - \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\
&= 0
\end{aligned}$$

5.

We have that f is continuous and $f(0) = 0, f(1) = 1$. Thus by the intermediate value theorem, we have that for all $y \in [0, 1]$, there exists $x \in \mathbb{C}$ such that $f(x) = y$. Thus $f(C) = [0, 1]$.

2.

For any pairwise disjoint $E_i \in \mathcal{M}$, let $A_n = \sqcup_{i=1}^n E_i$ then we have that

$$\mu(\sqcup_{i=1}^{\infty} E_i) = \mu(\cup_{i=1}^{\infty} A_n) = \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \mu(\sqcup_{i=1}^n E_i) = \sum_{i=1}^{\infty} \mu(E_i)$$

as $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$.

3.

a.

If $E_j \in \mathcal{M}$ for all $j \in \mathbb{N}$ then $E_1 \setminus E_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ and thus $\bigcup_{n=2}^{\infty} (E_1 \setminus E_n) \in \mathcal{M}$, then $E_1 \setminus (\bigcup_{n=2}^{\infty} E_1 \setminus E_n) = E_1 \setminus (E_1 \setminus \bigcap_{n=2}^{\infty} E_n) = E_1 \cap (\bigcap_{n=2}^{\infty} E_n) = \bigcap_{n=1}^{\infty} E_n \in \mathcal{M}$

b.

We know that \mathcal{M} being a σ -algebra implies that $X \in \mathcal{M}$.

If $X \in \mathcal{M}$, then for every $A \in \mathcal{M}$, $X \setminus A = A^c \in \mathcal{M}$.

c.

If $A \in \mathcal{S}$, then $A \in \mathcal{M}$ or $A^c \in \mathcal{M}$. In both case $A^c \in \mathcal{S}$ as $(A^c)^c \in \mathcal{M}$ or $A \in \mathcal{M}$.

If $A_j \in \mathcal{S}$ for all $j \in \mathbb{N}$ then $A_j \in \mathcal{M}$ or $A_j^c \in \mathcal{M}$. Let A_{j_1}, A_{j_2} be subsequence of A such that $A_{j_1} \in \mathcal{M}$ and $A_{j_2}^c \in \mathcal{M}$. Then we know that $P := \bigcap_{j_1=1}^{\infty} A_{j_1} \in \mathcal{M}$ and since $Q := \bigcup_{j_2=1}^{\infty} A_{j_2}^c = (\bigcap_{j_2=1}^{\infty} A_{j_2})^c \in \mathcal{M}$ and thus $P \setminus Q = \bigcap_{j_1=1}^{\infty} A_{j_1} \cap \bigcap_{j_2=1}^{\infty} A_{j_2} = \bigcap_{j=1}^{\infty} A_j \in \mathcal{M} \subseteq \mathcal{S}$.

4.

a.

If $E \in \mathcal{M}$, then $E \cap X_\lambda \in \mathcal{M}_\lambda$ for all $\lambda \in \Lambda$,
then $X_\lambda \setminus (E \cap X_\lambda) = X \setminus (E \cap X_\lambda) = (X \setminus E) \cap X_\lambda \in \mathcal{M}_\lambda$ and thus $E^c \in \mathcal{M}$.
If $E_i \in \mathcal{M}$ for $i \in \mathbb{N}$, then $E_i \cap X_\lambda \in \mathcal{M}_\lambda$, and thus $(\cup_{i=1}^\infty E_i) \cap X_\lambda \in \mathcal{M}_\lambda$ for
all $\lambda \in \Lambda$ and thus $\cup_{i=1}^\infty E_i \in \mathcal{M}$.

b.

$$\mu(\emptyset) = \sum_{\lambda \in \Lambda} \mu_\lambda(\emptyset \cap X_\lambda) = \sum_{\lambda \in \Lambda} \underbrace{\mu_\lambda(\emptyset)}_0 = 0$$

For any $E_j \in \mathcal{M}$ for all $j \in \mathbb{N}$ such that E_j are pairwise disjoint, we have
that

$$\begin{aligned} \mu(\sqcup_{j=1}^\infty E_j) &= \sum_{\lambda \in \Lambda} u_\lambda(\sqcup_{j=1}^\infty E_j \cap X_\lambda) \\ &= \sum_{\lambda \in \Lambda} \sum_{j=1}^\infty u_\lambda(E_j \cap X_\lambda) \\ &= \sum_{j=1}^\infty \sum_{\lambda \in \Lambda} \mu_\lambda(E_j \cap X_\lambda) \\ &= \sum_{j=1}^\infty \mu(E_j) \end{aligned}$$

c.

If μ is σ -finite, then there exists $X_n \subseteq X_{n+1} \in \mathcal{M}$ such that $\cup_{n=1}^\infty X_n = X$
and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. Thus if we let $X_{n,\lambda} = X_n \cap X_\lambda$, we have that
 $X_{n,\lambda} \subseteq X_{n+1,\lambda}$, $X_{n,\lambda} \in \mathcal{M}_\lambda$,

$$\mu(X_n \cap X_\lambda) = \mu(X_{n,\lambda}) < \infty$$

and

$$X_\lambda = X \cap X_\lambda = (\cup_{n=1}^\infty X_n) \cap X_\lambda = \cup_{n=1}^\infty (X_n \cap X_\lambda) = \cup_{n=1}^\infty X_{n,\lambda}$$

for all $n \in \mathbb{N}$, which means that all but a countable measure μ_λ have
 $\mu_\lambda(X_\lambda) = 0$ and the rest are σ -finite.

Now suppose all but a countable measure μ_λ have $\mu_\lambda(X_\lambda) = 0$ and the
rest are σ -finite, then for every $\lambda \in \Lambda$, there exists $X_{n,\lambda} \in \mathcal{M}_\lambda$ such that
 $X_{n,\lambda} \subseteq X_{n+1,\lambda}$, $\cup_{n=1}^\infty X_{n,\lambda} = X_\lambda$ and $\mu_\lambda(X_{n,\lambda}) < \infty$ for every $n \in \mathbb{N}$. Since
 Λ is a collection of measure, there is a bijection $\mathbb{N} \sim \Lambda$

- $X = \cup_{\lambda \in \Lambda} X_\lambda = \cup_{\lambda \in \Lambda} \cup_{n=1}^\infty X_{n,\lambda} = \cup_{n=1}^\infty$

5.

We have that the definition of the outer measure for both parts a and b

$$\mu^*(A) := \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) : A \subseteq \cup_{j=1}^{\infty} E_j, E_j \in \mathcal{S} \right\}$$

a.

Then for any nonempty set $A \subseteq \mathcal{S}$, we have that if $\mu^*(A) = 0$ then $\sum_{j=1}^{\infty} \rho(E_j) = 0$ and thus $\rho(E_j) = 0$ for every $j \in \mathbb{N}$ and thus $E_j = \emptyset$ and $\cup_{j=1}^{\infty} E_j = \emptyset$. Therefore, $A = \emptyset$ and thus a contradiction. Therefore, $\mu^*(\emptyset) = 0$ and $\mu^*(A) > 0$. But since $\sum_{j=1}^{\infty} \rho(E_j)$ is either an integer or infinity, $A \subseteq X \cup (\cup_{j=2}^{\infty} \emptyset)$ and

$$\rho(X) + \sum_{j=2}^{\infty} \rho(\emptyset) = 1$$

Then from Caratheodory, we know that \mathcal{S} is the σ -algebra. we have that $\mu^*(A) = 1$ and $\mu^*(\emptyset) = 0$.

b.

From definition, we have that for any set A such that $\mu^*(A) \geq \rho(A)$. If $\rho(A) = N$ for some $N \in \mathbb{N}$ or $N = \infty$, then we can let

$$K = \{k \in A : k \text{ is an integer}\}$$

Then $A \subseteq \cup_{j=0}^{\infty} E_j$ where

$$E_j = \begin{cases} \left(\frac{j}{2}, \frac{j}{2} + 1 \right), & \text{if } 2|j \text{ and } \frac{j}{2} \notin K \\ \left(-\frac{j-1}{2} - 1, -\frac{j-1}{2}/2 \right), & \text{if } 2 \nmid j \text{ and } -\frac{j-1}{2} \notin K \\ \left[\frac{j}{2}, \frac{j}{2} + 1 \right), & \text{if } 2|j \text{ and } \frac{j}{2} \in K \\ \left(-\frac{j-1}{2} - 1, -\frac{j-1}{2}/2 \right], & \text{if } 2 \nmid j \text{ and } -\frac{j-1}{2} \in K \end{cases}$$

so that

$$\rho(E_j) = \begin{cases} 1, & \text{if } \frac{j}{2} \in K \text{ or } -\frac{j-1}{2} \in K \\ 0, & \text{otherwise} \end{cases}$$

and thus there is N interval E_j such that $\rho(E_j) = 1$. Therefore,

$$\begin{aligned} \mu^*(A) &\leq \sum_{j=0}^{\infty} \rho(E_j) \\ &= \sum_{\substack{j=0 \\ j/2 \in K \text{ or} \\ -(j-1)/2 \in K}}^{\infty} \rho(E_j) + \sum_{\substack{j=0 \\ j/2 \notin K \text{ and} \\ -(j-1)/2 \notin K}}^{\infty} \rho(E_j) \\ &= N + 0 = N = \rho(A) \end{aligned}$$

which concludes that $\mu^*(A) = \rho(A)$. Then from Caratheodory, we know that $\mathcal{P}(X)$ is the σ -algebra.