Fall 2021, Math 328, Homework 2

Due: End of day on 2022-10-01

1 (10 points)

1. Compute the order of each of the elements in the following groups: $D_6,\ D_8,\ D_{10}$.
2. Suppose that σ is an element of S_n which has the form

$$\sigma = (a_1, a_2, \ldots, a_m).$$

   Let i be any integer. Prove that $\sigma^i(a_k) = a_r$ where $r \equiv k + i \bmod m$. Determine the order of σ as an element of S_n.
3. For $n = 3, 4$, write out all elements of S_n in cycle notation, and compute the order of each element.

2 (10 points)

Given a group G, an element $g \in G$ is called *central* provided that for all $h \in G$, one has $g \cdot h = h \cdot g$. The identity element is clearly central.

1. Prove that the product of two central elements is central.
2. Prove that the inverse of a central element is central.
3. Prove that the collection of all central elements of G forms a subgroup of G. This group is called *the centre* of G, and is denoted by $Z(G)$.
4. Find the centre of the following groups: S_4, Q_8, D_{2n} (arbitrary $n \geq 3$).

3 (10 points)

Let $\varphi : G \to H$ be a homomorphism of groups and let $g \in G$ be given. Prove that $\varphi(g^a) = \varphi(g)^a$ for all $a \in \mathbb{Z}$.

4 (10 points)

1. Prove that S_3 and D_6 are isomorphic.
2. Prove that S_4 and D_{24} are not isomorphic.
3. For a pair of groups G and H, prove that $G \times H$ and $H \times G$ are isomorphic.
4. Let G be a group, and let $\mathrm{Aut}(G)$ be the set of automorphisms of G. Prove that $\mathrm{Aut}(G)$ is a group under composition of automorphisms.
5. Suppose that G and H are isomorphic. Prove that $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{Aut}(H)$.

5 (20 points)

For each of the following, a subset of a group is specified. Determine (with proof) whether the given subset is a subgroup.

1. The set of complex numbers of the form $\{a + i \cdot a \mid a \in \mathbb{R}\}$, a subset of $\mathbb{C}$.
2. The set of complex numbers of absolute value 1, a subset of $(\mathbb{C}, +)$.
3. The set of complex numbers of absolute value 1, a subset of $(\mathbb{C}^\times, \times)$.
4. The set of transpositions in S_n, with $n \geq 3$.
5. The set of reflections in D_{2n}, with $n \geq 3$.
6. The set of odd integers in $\mathbb{Z}$.
7. The set of integers which are divisible by n in $\mathbb{Z}$, for $n \in \mathbb{N}$.
8. The set $\{1, r^2, s, sr^2\}$ in D_8.
9. The set $\{1, r^2, sr, sr^3\}$ in D_8.
10. The set $\{1, r^2, s, sr^2\}$ in D_{10}.
11. The set $\{1, r^2, sr, sr^3\}$ in D_{10}.
12. The set $\{1, i, j, k\}$ in Q_8.
13. The set $\{1, i, -1, -i\}$ in Q_8.
14. The subset of S_4 containing the following elements:
    (a) The identity.
    (b) All cycles of length 3.
    (c) The elements $(1,2)(3,4)$, $(1,3)(2,4)$, $(1,4)(2,3)$.