1.

$$(I - P_X)_{i,i} SS_{\text{res}} = [e_i^T (I - P_X) e_i][y^T (I - P_X) y] \geq [e_i^T (I - P_X) y]^2 = r_i^2$$

from the hints and

$$SS_{\text{res}} = r^T r = (y^T - \hat{y}^T) r = y^T r - \underbrace{0}_{\text{since } x^T r = 0} = y^T (I - P_X) y$$

Thus we have

$$|s_i| \leq \frac{|r_i|}{\sqrt{r_i^2/(n-p-1)}} = \sqrt{n-p-1}$$

2.

a.

$$V[Y] = \mu^2 \implies s(\mu) = \mu$$
$$T(\mu) = \int \frac{1}{\mu} d\mu = \ln(\mu)$$

b.

$$V[Y] = \mu^k \implies s(\mu) = \mu^{k/2}$$
$$T(\mu) = \int \frac{1}{\mu^{k/2}} d\mu = \frac{\mu^{-k/2+1}}{-k/2+1}$$

c.

$$V[Y] = e^\mu \implies s(\mu) = e^{\mu/2}$$
$$T(\mu) = \int \frac{1}{e^{\mu/2}} d\mu = -\frac{2}{e^{\mu/2}}$$

d.

$$V[Y] = \mu^{-2} \implies s(\mu) = \mu^{-1}$$
$$T(\mu) = \int \mu d\mu = \mu^2$$

e.

$$V[Y] = \frac{\mu^2}{e^{2\mu}} \implies s(\mu) = \frac{\mu}{e^\mu}$$
$$T(\mu) = \int \frac{e^\mu}{\mu} d\mu$$

which is not integrable, though we can express e^μ as a Maclaurin series and take the antiderivative terms by terms to get

$$T(\mu) = \int \frac{e^\mu}{\mu} d\mu = \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

3.

a.

We have that

$$\frac{1}{n}\sum_{i=1}^{n} V(\varepsilon_i) = \sigma_1^2$$

and

$$\frac{1}{m}\sum_{i=m+1}^{m+n} V(\varepsilon_i) = \sigma_2^2$$

Thus

$$\frac{1}{n+m}\sum_{i=1}^{m+n} V(\varepsilon_i) = \frac{n}{m+n}\sigma_1^2 + \frac{m}{m+n}\sigma_2^2$$

and we can find

$$\delta = \frac{n}{m+n} < 1$$

b.

We have that

$$\begin{aligned}
E[SS_{\text{res}}] &= E[Y^T(I-P_X)Y] \\
&= \mathrm{Tr}(E[Y^T(I-P_X)Y]) \\
&= \mathrm{Tr}(I-P_X)E[YY^T] \\
&= \mathrm{Tr}((I-P_X)E[Y]E[Y^T]) + \mathrm{Tr}((I-P_X)\Sigma) \\
&= E[Y^T](I-P_X)E[Y] + \mathrm{Tr}((I-P_X)\Sigma) \\
&= 0 + \sigma_1^2 n + \sigma_2^2 m - \sigma_1^2 \sum_{i=1}^{n} P_{X_{i,i}} - \sigma_2^2 \sum_{i=n+1}^{m+n} P_{X_{i,i}}
\end{aligned}$$

since $(I-P_X)E[Y] = 0$. Thus

$$E\left[\frac{SS_{\text{res}}}{(n+m-p-1)}\right] = \frac{n - \sum_{i=1}^{n} P_{X_{i,i}}}{n+m-p-1}\sigma_1^2 + \frac{m - \sum_{i=n+1}^{m+n} P_{X_{i,i}}}{n+m-p-1}\sigma_2^2$$

Thus since $\sum_{i=1}^{m+n} P_{X_{i,i}} = p+1$, we have

$$E\left[\frac{SS_{\text{res}}}{(n+m-p-1)}\right] - \sigma_1^2 = \frac{m - \sum_{i=n+1}^{m+n} P_{X_{i,i}}}{n+m-p-1}(\sigma_2^2 - \sigma_1^2) < 0$$

and similarly,

$$E\left[\frac{SS_{\text{res}}}{(n+m-p-1)}\right] - \sigma_2^2 = \frac{m - \sum_{i=1}^{n} P_{X_{i,i}}}{n+m-p-1}(\sigma_1^2 - \sigma_2^2) < 0$$

c.

We first have additionally,

$$\frac{1}{k}\sum_{m+n+1}^{m+n+k} V(\varepsilon_i) = \sigma_3^2$$

and thus

$$\frac{1}{m+n+k}\sum_{i=1}^{m+n+k} V(\varepsilon_i) = \frac{n}{m+n+k}\sigma_1^2 + \frac{m}{m+n+k}\sigma_2^2 + \frac{k}{m+n+k}\sigma_3^2$$