1 Preliminary

$$\mathrm{card}(X) < \mathrm{card}(\mathcal{P}(X))$$

$$\mathrm{card}(\mathcal{P}(\mathbb{N})) = \mathrm{card}(\mathbb{R})$$

Every nonempty open set $\mathcal{O}$ in $\mathbb{R}$ can be written as at most countable union of pairwise disjoint open intervals. that is, $\mathcal{O} = \sqcup_{j=1}^{\infty}(a_j, b_j)$ such that $(a_i, b_i) \cap (a_j, b_j) = \varnothing$ for all $i \neq j$. Some intervals may be empty, if empty intervals are ignored, the reperesentation is unique.

2 Measure Theory

2.1 Motivation

Theorem 2.1 (Axiom of Choice) *Let $\mathcal{F}$ be a nonempty family of nonempty sets. Then there exists a choice function f on $\mathcal{F}$ such that for each set $F \in \mathcal{F}, f(F) \subset F$.*

Theorem 2.2 (Fact) *There does not exist a set function $\mu : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$ such that*

- $\mu(\sqcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$ *for all pairwise disjoint $E_j \subseteq \mathbb{R}^d$ for $j \in \mathbb{N}$.*
- *If E is congruent to F (E can be transformed into F by translations, rotations, and reflections), then $\mu(E) = \mu(F)$.*
- $\mu([0, 1)^d) = 1$

There is a known non-measurable subset $\mathcal{N} \subseteq [0, 1)$.
Known $f_j(x) \uparrow f(x)$ for all $x \in [0, 1]$. f_j are Riemann integrable with $\int_{[0,1]} f_j = 0$ but $\lim_{j\to\infty} \int_{[0,1]} f_j = 0$ but f is not Riemann integrable.

2.2 σ-algebras and Borel sets

Definition 2.1. *Let X be a set. A nonempty subset $\mathcal{S}$ of $\mathcal{P}(X)$ is called a σ-algebra if*

- $A \in \mathcal{S} \implies A^c \in \mathcal{S}$
- $A_j \in \mathcal{S}, j \in \mathbb{N} \implies \cup_{j=1}^{\infty} A_j \in \mathcal{S}$

Definition 2.2 (Borel σ-algebra)**.** *For a metric space X, the Borel σ-algebra is the smallest algebra that contains all the open subsets of X.*

Definition 2.3. *(X, M) is a measurable space if $M \subseteq \mathcal{P}(X)$ is a σ-algebra. A measure μ on (X, M) is a set $\mu : M \to [0, \infty]$ satisfying*

1. $\mu(\varnothing) = 0$
2. $\mu(\sqcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$

Definition

1. μ *is called finite if* $\mu(X) < \infty$
2. μ *is* σ*-finite if* $X = \cup_{n=1}^{\infty} X_n$, $X_n \subseteq X_{n+1}$ *and* $\mu(X_n) < \infty$.
3. μ *is semifinite if for each* $E \in M$, *there is* $F \subseteq E$ *such that* $0 < \mu(F) < \infty$.

Theorem 2.3 (Properties of measure) *If either*

1. $E_1 \supseteq E_2 \ldots$, *and* $\mu(E_{j_0}) < \infty$ *for some* j.
2. $E_1 \subseteq E_2 \ldots$.

then the limit is the same as the union or intersection.

Definition 2.4 (Outer measure)**.** *set function* $\mu^* : \mathcal{P}(X) \to [0, \infty]$ *such that*

1. $\mu^*(\varnothing) = 0$
2. *if* $A \subseteq B$, *then* $\mu^*(A) \leq \mu^*(B)$
3. $\mu^*(\cup) \leq \sum \mu^*$

A set E *is* μ^* *measurable if*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for all $A \subseteq X$

Theorem 2.4 *(**Carathéodory's Theorem**) Let* μ^* *be an outer measure on* X *and let* $\mathcal{M}(X, \mu^*)$ *be the collection of all* μ^**-measurable subsets of* X. *Then* $\mathcal{M}(X, \mu^*)$ *is a* σ*-algebra and* $(X, \mathcal{M}(X, \mu^*), \mu)$ *is a complete measure space, where* $\mu := \mu^*|_{\mathcal{M}(X,\mu^*)}$.

Figure 1:

Lemma 2.5 *If* $(X, \mathcal{M}(X, \mu^*), \mu)$ *is the complete measure space derived from an outer measure* μ^* *on* X, *then for any pairwise disjoint* μ^**-measurable sets* E_j, $j \in \mathbb{N}$,

$$\mu^*(A \cap (\sqcup_{j=1}^{\infty} E_j)) = \sum_{j=1}^{\infty} \mu^*(A \cap E_j), \qquad \forall\, A \subseteq X.$$

Figure 2:

Definition: For any subset $A \subseteq \mathbb{R}^d$, define

$$m^*(A) = \inf\Big\{\sum_{j=1}^{\infty}|R_j| \;\Big|\; A \subseteq \cup_{j=1}^{\infty} R_j, \text{ each } R_j \text{ is a half-open rectangle in } \mathbb{R}^d\Big\}.$$

Note that the above infimum is taken over all possible countable coverings $A \subseteq \cup_{j=1}^{\infty} R_j$ by all possible half-open rectangles R_j. By Theorem 2.6, the above m^* is the (Lebesgue) **outer measure**.

Figure 3:

Theorem 2.11 *Let $\mathcal{E}$ denote the collection of all half-open rectangles in $\mathbb{R}^d$. Let $\mathcal{A}$ be the collection of all finite disjoint unions of members in $\mathcal{E}$. Define a set function $\mu_0 : \mathcal{A} \to [0, \infty]$ by*

$$\mu_0(A) := \sum_{j=1}^{n} |R_j|, \qquad A = \sqcup_{j=1}^{n} R_j \in \mathcal{A} \quad \text{with} \quad R_1, \ldots, R_n \in \mathcal{E}.$$

Then $\mathcal{E}$ is an elementary family, $\mathcal{A}$ is an algebra, μ_0 is a well-defined extension of $|\cdot|$ from $\mathcal{E}$ to $\mathcal{A}$ and is a premeasure on $\mathcal{A}$. Moreover, the Lebesgue outer measure m^* is the same outer measure induced by $\mu_0 : \mathcal{A} \to [0, \infty]$.

Figure 4:

Theorem 2.13 *Let $E \subseteq \mathbb{R}^d$. The following are equivalent:*

(1) E is measurable;

(2) For any given $\varepsilon > 0$, there exists an open set $\mathscr{O} \supseteq E$ such that $m^*(\mathscr{O} \backslash E) < \varepsilon$;

(3) For any given $\varepsilon > 0$, there exists a closed set $F \subseteq E$ such that $m^*(E \backslash F) < \varepsilon$;

(4) There exists a G_δ set $G = \cap_{n=1}^{\infty} \mathscr{O}_n$, with all $\mathscr{O}_n$ open, such that $G \supseteq E$ and $m^*(G \backslash E) = 0$;

(5) There exists an F_σ set $F = \cup_{n=1}^{\infty} F_n$, with all F_n closed, such that $F \subseteq E$ and $m^*(E \backslash F) = 0$.

(6) There exists a Borel set B such that $E \subseteq B$ and $m^*(B \backslash E) = 0$. [or $B \subseteq E$ and $m^*(E \backslash B) = 0$.]

If $m^*(E) < \infty$, then any of the above statements is also equivalent to

(7) For any $\varepsilon > 0$, there exists a finite union $U = \sqcup_{j=1}^{N} R_j$ of pairwise disjoint bounded half-open cubes (or almost disjoint bounded closed/open rectangles) such that $m^*(E \Delta U) < \varepsilon$.

(8) For any $\varepsilon > 0$, there exists a compact set K such that $K \subseteq E$ and $m^*(E \backslash K) < \varepsilon$.

Figure 5:

Theorem 3.5 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from X to $\overline{\mathbb{R}}$. Then*

(1) $g_1(x) := \sup_{n\in\mathbb{N}} f_n(x)$ and $g_2(x) := \inf_{n\in\mathbb{N}} f_n(x)$ are measurable;

(2) $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ are measurable;

(3) If $f(x) = \lim_{n\to\infty} f_n(x)$ for every $x \in X$, then f is measurable.

(4) If $f(x) = \lim_{n\to\infty} f_n(x)$ for μ-a.e. $x \in X$ and if $(X, \mathcal{M}, \mu)$ is complete, then f is measurable.

Figure 6:

Theorem 3.7 *Let $(X, \mathcal{M})$ be a measurable space. Let $f : X \to \overline{\mathbb{R}}$ be a measurable function. Then*

(1) *If f is nonnegative (i.e., $f(x) \geqslant 0\ \forall x \in X$), then there exists $(\phi_n)_{n\in\mathbb{N}}$ of nonnegative ($\mathcal{M}$-measurable) simple functions such that $0 \leqslant \phi_n(x) \uparrow f(x)$ for all $x \in X$. Moreover, $\phi_n \to f$ uniformly on any set on which f is bounded;*

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(2) *There exists $(\phi_n)_{n\in\mathbb{N}}$ of real-valued ($\mathcal{M}$-measurable) simple functions such that $|\phi_n(x)| \leqslant |\phi_{n+1}(x)| \leqslant |f(x)|$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \phi_n(x) = f(x)$ for all $x \in X$. Moreover, $\phi_n \to f$ uniformly on any set on which f is bounded.*

Figure 7:

Theorem 3.8 *(**Egorov**) Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of (complex-valued) measurable functions defined on a measurable set E with $\mu(E) < \infty$, and assume that $\lim_{n\to\infty} f_n(x) = f(x)$, μ-a.e. $x \in E$. Then for any given $\varepsilon > 0$, there is a subset $F \subseteq E$ such that $F \in \mathcal{M}$, $\mu(E \backslash F) < \varepsilon$ and $f_n \to f$ uniformly on F as $n \to \infty$.*

Figure 8:

Definition: Let $\varphi : X \to \mathbb{R}$ be a nonnegative simple function with the canonical representation $\varphi = \sum_{j=0}^{n} c_j \chi_{E_j}$. Define

$$\int \varphi = \int \varphi d\mu = \int_X \varphi(x) d\mu(x) := \sum_{j=0}^{n} c_j \mu(E_j).$$

If $E \in \mathcal{M}$ is a measurable set, then define

$$\int_E \varphi = \int_E \varphi d\mu = \int \varphi \chi_E d\mu = \sum_{j=0}^{n} c_j \mu(E_j \cap E).$$

Figure 9:

The Monotone Convergence Theorem (MCT): *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of **nonnegative** measurable functions in $\mathcal{M}^+(X, \mu)$ such that $f_n(x) \uparrow f(x)$ as $n \to \infty$ for all $x \in X$ (or μ-a.e. $x \in X$ and f is measurable). Then*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int \lim_{n\to\infty} f_n d\mu = \int_X f d\mu.$$

Figure 10:

Fatou's Lemma: If $(f_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative measurable functions in $\mathcal{M}^+(X, \mu)$, then

$$\int_X \liminf_{n\to\infty} f_n d\mu \leqslant \liminf_{n\to\infty} \int_X f_n d\mu.$$

Figure 11:

Definition: Let $f, f_n : X \to \overline{\mathbb{R}}$ (or $\mathbb{C}$) be measurable functions for $n \in \mathbb{N}$.

(1) $f_n \to f$ in $\mathcal{L}_1(X, \mathcal{M}, \mu)$ (denoted by $f_n \xrightarrow{\mathcal{L}_1} f$) if $\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$;

(2) $f_n \to f$ in measure ($f_n \xrightarrow{\mu} f$) if for every $\varepsilon > 0$, $\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geqslant \varepsilon\}) = 0$;

(3) $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure if for each $\varepsilon > 0$ such that $\lim_{m,n\to\infty} \mu(\{x \in X : |f_m(x) - f_n(x)| \geqslant \varepsilon\}) = 0$;

Figure 12:

Theorem 4.11 *Let $f, f_n : X \to \overline{\mathbb{R}}$ (or $\mathbb{C}$) be measurable functions.*

*(1) **Tchebychev's Inequality:** For $f \in \mathcal{L}_1(X, \mathcal{M}, \mu)$,*

$$\mu(\{x \in X \ : \ |f(x)| \geqslant \lambda\}) \leqslant \frac{1}{\lambda} \int_X |f| d\mu, \qquad \text{for all } \lambda > 0.$$

(2) If $f_n \xrightarrow{\mathcal{L}_1(X,\mathcal{M},\mu)} f$, then $f_n \xrightarrow{\mu} f$;

(3) If $f_n \to f$ μ-a.e. and $\mu(X) < \infty$, then $f_n \xrightarrow{\mu} f$;

(4) If $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, then there is a measurable function f such that $f_n \xrightarrow{\mu} f$, and there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ μ-a.e..

Figure 13:

Generalized (Lebesgue) Dominated Convergence Theorem: Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be measurable functions in $\mathcal{M}(X, \mu)$ such that $\lim_{n\to\infty} g_n(x) = g(x)$ for μ-a.e. $x \in X$ with $g, g_n \in \mathcal{L}_1(X, \mathcal{M}, \mu)$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n = f$ μ-a.e. (or $f_n \xrightarrow{\mu} f$) and f is measurable. If $|f_n| \leqslant g_n$ μ-a.e. for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \int_X g_n d\mu = \int_X g d\mu$, then

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu = \int_X f d\mu.$$

Figure 14:

(Lebesgue) Dominated Convergence Theorem: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on X and $\lim_{n\to\infty} f_n = f$ μ-a.e. (or $f_n \xrightarrow{\mu} f$) and f is measurable. If there is a nonnegative function $g \in \mathcal{L}_1(X, \mathcal{M}, \mu)$ such that $|f_n| \leqslant g$ μ-a.e. for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Figure 15:

Theorem 4.13 (**Absolutely Continuity of Integrals**) *Let $f \in \mathcal{L}_1(X, \mathcal{M}, \mu)$.*

(i) For any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_A |f| < \varepsilon, \qquad \text{whenever } A \in \mathcal{M} \text{ with } \mu(A) < \delta.$$

(ii) If $X = \cup_{n=1}^{\infty} X_n$ such that $X_n \in \mathcal{M}$ and $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \int_{X\setminus X_n} |f| = 0$.

Figure 16:

Theorem 4.14 *Let $f \in \mathcal{L}_1(\mathbb{R}^d)$ and $\varepsilon > 0$. Then*

(1) There is a compactly supported integrable simple function φ such that $\int_{\mathbb{R}^d} |f - \varphi| < \varepsilon$;

(2) There is an integrable step function ψ such that $\int_{\mathbb{R}^d} |f - \psi| < \varepsilon$;

(3) There is a compactly supported continuous function h such that $\int_{\mathbb{R}^d} |f - h| < \varepsilon$.

Figure 17:

Definition: The x-section A_x and y-section A^y of a subset $A \subseteq X \times Y$ are

$$A_x := \{y \in Y \ : \ (x, y) \in A\}, \qquad A^y := \{x \in X \ : \ (x, y) \in A\}.$$

For a function $f : X \times Y \to \mathbb{C}$, the x-section f_x and y-section f^y are

$$f_x(y) = f^y(x) = f(x, y).$$

Figure 18:

Definition: $\mathcal{C} \subseteq \mathcal{P}(X)$ is a **monotone class** on X if it is closed under countable increasing unions and countable decreasing intersections, that is, for $E_j \in \mathcal{C}, j \in \mathbb{N}$, if $E_j \uparrow$, then $\cup_{j=1}^{\infty} E_j \in \mathcal{C}$; if $E_j \downarrow$, then $\cap_{j=1}^{\infty} E_j \in \mathcal{C}$.

Figure 19:

The Monotone Class Lemma: If $\mathcal{A}$ is an algebra of subsets of X, then the monotone class $\mathcal{C}$ generated by $\mathcal{A}$ coincides with the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$.

Figure 20:

Theorem 5.3 *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \times \mathcal{N}$, then the function $x \mapsto \nu(E_x)$ is $\mathcal{M}$-measurable and the function $y \mapsto \mu(E^y)$ is $\mathcal{N}$-measurable, and*

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y).$$

Figure 21:

Theorem 5.4 *(**The Fubini-Tonelli Theorem**) Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. Define*

$$g(x) := \int f_x d\nu \quad \text{and} \quad h(y) = \int f^y d\mu. \tag{5.1}$$

Tonelli: *If $f : X \times Y \to [0, \infty]$ is a nonnegative $\mathcal{M} \times \mathcal{N}$-measurable function, then g and h are nonnegative measurable functions, and*

$$\int_{X \times Y} f(x, y) d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) d\mu(x) \right) d\nu(y). \tag{5.2}$$

Fubini: *If $f \in L_1(X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)$, then $f_x \in L_1(Y, \mathcal{N}, \nu)$ for μ-a.e. $x \in X$, $f^y \in L_1(X, \mathcal{M}, \mu)$ for ν-a.e. $y \in Y$, the a.e.-defined functions $g \in L_1(X, \mathcal{M}, \mu), h \in L_1(Y, \mathcal{N}, \nu)$, and (5.2) holds.*

Figure 22:

Theorem 5.5 *(**The Fubini-Tonelli Theorem for Complete Measure**) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be complete, σ-finite measure spaces, and let $(X \times Y, \overline{\mathcal{M} \times \mathcal{N}}, \mu \times \nu)$ be the completion of $(X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)$. If f is $\overline{\mathcal{M} \times \mathcal{N}}$-measurable and either (i) $f \geqslant 0$ or (ii) $f \in L_1(X \times Y, \overline{\mathcal{M} \times \mathcal{N}}, \mu \times \nu)$, then f_x is $\mathcal{N}$-measurable for a.e. $x \in X$ and f^y is $\mathcal{M}$-measurable for a.e. $y \in Y$. For (ii), f_x and f^y are also integrable for a.e. x and y. Moreover, the a.e.-defined functions g and h in (5.1) are measurable and for (ii), $g \in L_1(X, \mathcal{M}, \mu), h \in L_1(Y, \mathcal{N}, \nu)$, and*

$$\int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f(x, y) d\mu(x) d\nu(y) = \int_X \int_Y f(x, y) d\nu(y) d\mu(x).$$

Figure 23:

Definition: For a partition $\Gamma = \{x_0, \ldots, x_n\}$ of $[a, b]$ and a real-valued function f, define

$$P(f, \Gamma) := \sum_{j=1}^{n} [f(x_j) - f(x_{j-1})]^+, \qquad N(f, \Gamma) := \sum_{j=1}^{n} [f(x_j) - f(x_{j-1})]^-,$$

where $x^+ = \max(x, 0)$ and $x^- := \max(-x, 0)$.
Definition: The positive and negative variations of f on $[a, b]$ are defined by

$$P_a^b(f) := \sup\{P(f, \Gamma) \ : \ \text{all partitions } \Gamma \text{ of } [a, b]\}, \quad N_a^b(f) := \sup\{N(f, \Gamma) \ : \ \text{all partitions } \Gamma \text{ of } [a, b]\}.$$

Lemma 6.2 *Let $f : [a, b] \to \mathbb{R}$. Then*

$$P_a^x(f) = \frac{1}{2}(V_a^x(f) + f(x) - f(a)), \qquad N_a^x(f) = \frac{1}{2}(V_a^x(f) - f(x) + f(a)), \qquad \forall\, x \in [a, b].$$

In particular, $V_a^x(f) = P_a^x(f) + N_a^x(f)$ and $f(x) - f(a) = P_a^x(f) - N_a^x(f)$ for all $x \in [a, b]$.

Figure 24:

Definition: Let $f : [a,b] \to \mathbb{R}$. For a partition $\Gamma = \{x_0, x_1, \ldots, x_n\}$ (that is, $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$) of $[a,b]$, define

$$V(f,\Gamma) := \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|.$$

The **total variation** of f over $[a,b]$ is defined to be

$$V_a^b(f) := \sup\{V(f,\Gamma) \ : \ \text{all partitions } \Gamma \text{ of } [a,b]\}.$$

If $V_a^b(f) < \infty$, then f is said to be of **bounded variation** on $[a,b]$, written as $f \in \mathrm{BV}[a,b]$.

Figure 25:

Theorem 6.3 *A real-valued function $f \in BV[a,b] \iff f$ can be written as the difference of two increasing functions on $[a,b]$.*

Figure 26:

Theorem 6.4 (**Vitali Covering Theorem**) *Let $E \subseteq \mathbb{R}^d$ such that $m^*(E) < \infty$. Let $\mathcal{C}$ be a collection of closed balls (or cubes) that covers E in the sense of Vitali. Then for any given $\varepsilon > 0$, there is a finite collection $\{B_1, \ldots, B_N\}$ of pairwise disjoint balls/cubes in $\mathcal{C}$ such that*

$$m^*(E \backslash (\sqcup_{j=1}^{N} B_j)) < \varepsilon.$$

Moreover, there is an at most countable collection $(B_j)_{j\in\mathbb{N}}$ of pairwise disjoint balls in $\mathcal{C}$ such that $m^*(E \backslash \sqcup_{j=1}^{\infty} B_j) = 0$.

Figure 27:

Definition:

$$D^+ f(x) := \limsup_{t\to 0^+} \frac{f(x+t)-f(x)}{t} = \lim_{\to 0^+} \sup_{0<s\leqslant t} \frac{f(x+s)-f(x)}{s},$$

$$D_+ f(x) := \liminf_{t\to 0^+} \frac{f(x+t)-f(x)}{t} = \lim_{t\to 0^+} \inf_{0<s\leqslant t} \frac{f(x+s)-f(x)}{s},$$

$$D^- f(x) := \limsup_{t\to 0^+} \frac{f(x)-f(x-t)}{t}, \qquad D_- f(x) := \liminf_{t\to 0^+} \frac{f(x)-f(x-t)}{t}.$$

Note that there exists a sequence $(t_n)_{n=1}^{\infty}$ such that $t_n \to 0^+$ and $\lim_{n\to\infty} \frac{f(x+t_n)-f(x_n)}{t_n} = D^+ f(x)$. This also holds for other one-sided derivatives.

Figure 28:

Example: Let $f : [0,1] \to [0,1]$ be the Cantor function. Then f is increasing and continuous. But $f' = 0$ a.e.

Figure 29:

Lemma 6.6 *For $f \in \mathcal{L}_1([a,b])$ such that $\int_a^x f = 0$ for all $x \in [a,b]$, then $f = 0$ a.e. in $[a,b]$.*

Figure 30:

Lemma 6.9 *If f is absolutely continuous on $[a,b]$, then f is uniformly continuous and f is of bounded variation on $[a,b]$.*

Figure 31:

Theorem 6.10 *If f is absolutely continuous on $[a,b]$ and $f' = 0$ a.e. on $[a,b]$, then f must be a constant.*

Figure 32:

Theorem 6.11 *(i) If* $F(x) = \int_a^x f + C, x \in [a,b]$ *for some* $f \in L_1([a,b])$*, then* F *is absolutely continuous on* $[a,b]$*,* $F' = f$ *a.e., and* $C = F(a)$*.*

(ii) If f *is absolutely continuous on* $[a,b]$*, then*

$$f(x) = f(a) + \int_a^x f', \qquad \forall\, x \in [a,b].$$

*(iii) (**Integration by parts for absolutely continuous functions**) If both* f *and* g *are absolutely continuous on* $[a,b]$*, then* fg *is absolutely continuous on* $[a,b]$ *and*

$$\int_{[a,b]} f(x)g'(x)dx = [f(b)g(b) - f(a)g(a)] - \int_{[a,b]} f'(x)g(x)dx.$$

Figure 33: