1.

1.

If $x \in \liminf_{n\to\infty} E_n$ then, there exists $j \in \mathbb{N}$ such that $\forall k \geq j, x \in E_k$ which means that $x \in \cup_{k=j}^{\infty} E_k \subseteq \cup_{k=j-1}^{\infty} E_k \subseteq \ldots \subseteq \cup_{k=1}^{\infty} E_k$i thus $x \in \cap_{j=1}^{\infty} \cup_{k=j}^{\infty} E_k =: \limsup_{n\to\infty} E_n$.

2.

$x \in \liminf_{n\to\infty}(E_n \cap F_n) \iff$ there exists $j \in \mathbb{N}$ such that $\forall k \geq j, x \in E_k \cap F_k$ which is equivalent to $x \in E_k$ and $x \in F_k \iff x \in \liminf_{n\to\infty} E_n$ and $x \in \liminf_{n\to\infty} F_n \iff x \in \liminf_{n\to\infty} E_n \cap \liminf_{n\to\infty} F_n$

3.

Base case: n=1

$$\cup_{j=1}^{1} E_j := E_1 = E_1 \backslash \underbrace{\cup_{k=1}^{0} E_k}_{\varnothing} := F_1$$

Inductive steps: if the equation holds for n then it also holds for $n+1$.

$$\cup_{j=1}^{n+1} E_j = \cup_{j=1}^{n} E_j \cup E_{n+1} = \cup_{j=1}^{n} F_j \cup E_{n+1} \overset{(1)}{=} \cup_{j=1}^{n} F_j \cup \underbrace{E_{n+1} \backslash (\cup_{k=1}^{n} E_k)}_{F_{n+1}} = \cup_{j=1}^{n+1} F_j$$

as $\cup_{k=1}^{n} E_k = \cup_{k=1}^{n} F_k \implies E_{n+1} \backslash (\cup_{k=1}^{n} E_k) = F_{n+1}$.

2.

Assume X is non-empty, then there exists $z_0 \in X$, thus fixing $z_0 = (x_0, y_0)$, we can define an equivalence class, ξ_r as follows: p is in the class ξ_r if

$$|p - x_0| = r$$

where r is some rational number so that the set containing all classes ξ_r is countable. Next, define the function

$$f_r : [0, 2\pi] \to \mathbb{R}^2, \qquad t \to (x_0 + r\cos(t), x_0 + r\sin(t))$$

such that $\xi_r \subseteq \mathrm{Img}(f_r)$.

WLOG, assume that $f_r(0) \in X$, then as the function $g(\theta) = |f_r(\theta) - f_r(0)|$ is a continuous function that strictly increasing on $[0, \pi]$ from 0 to $2r$ and then strictly decreasing on $[\pi, 2\pi]$ back to 0, there is a bijective function that maps $[0, \pi] \to [0, 2r]$ and $[\pi, 2\pi] \to [0, 2r]$. Thus, the set of all points in each ξ_r is countable because $\mathbb{Q}_{[0,2r]}$ is countable. And since the set containing all classes ξ_r is also countable, X is countable.

3.

1.

Since $A \subseteq A \cup B$, $\text{card}(A) \leq \text{card}(A \cup B)$. Now suppose A is countable, then we can write A and B as $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$. Therefore, it is possible to construct a bijection function between A and $A \cup B$

$$\phi(a_i) = \begin{cases} b_{i/2}, & \text{if } i \text{ is even} \\ a_{(i+1)/2}, & \text{if } i \text{ is odd} \end{cases}$$

and thus having the same cardinality. We can extend that to the case where A is not countableas there is an infinite countable subset $\{a_1, a_2, \ldots\} = \tilde{A} \subset A$ and therefore we can construct a bijection between A and $A \cup B$ based on the previous bijection.

$$\psi(a) = \begin{cases} \phi(a), & \text{if } a \in \tilde{A} \\ a, & \text{if } a \in A \backslash \tilde{A} \end{cases}$$

where in the case $a \in \tilde{A}$, there is a_i such that $a = a_i$

$$\phi(a) = \phi(a_i) = \begin{cases} b_{i/2}, & \text{if } i \text{ is even} \\ a_{(i+1)/2}, & \text{if } i \text{ is odd} \end{cases}$$

2.

For every $x \in E$, there exists $\delta_x > 0$ such that $(x - \delta_x, x) \cap B = \varnothing$, and since $\mathbb{Q}$ is dense, we can create a function f that maps x to a rational number in $(x - \delta_x, x)$. We claim that f is injective.
For any $x_1 \neq x_2 \in E$, if $f(x_1) = f(x_2)$, then $f(x_1) \in (x_1 - \delta_{x_1}, x_1)$ and $f(x_2) \in (x_2 - \delta_{x_2}, x_2)$. But we have $(x_1 - \delta_{x_1}, x_1) \cap (x_2 - \delta_{x_2}, x_2) = \varnothing$ else $B \ni x_1 \in (x_2 - \delta_{x_2}, x_2)$ or $B \ni x_2 \in (x_1 - \delta_{x_1}, x_1)$. Therefore, $f(x_1) \neq f(x_2)$ and hence a contradiction. Thus f is injective.

4.

a.

For any set $E_k = \{x \in E : x \geq \frac{1}{k}\}$, we can see that $\sum_{x \in E} x \geq \sum_{x \in E_k} x \geq \frac{j}{k}$ where j is the number of elements in E_k. Thus as $\sum_{x \in E} x < \infty$, there is finite countable elements in E_k. We also have that

$$E = \bigcup_{k \in N} E_k$$

Thus E is at most countable.

b.

Let $E_k = \{x_i : i \leq k\}$ and since every element in E is positive, the series strictly increasing and $\lim_{n\to\infty} \sum_{i=1}^{n} x_i = \infty$ or $\lim_{n\to\infty} \sum_{i=1}^{n} x_i = L$ for some $L > 0 \in \mathbb{R}$.

If $\lim_{n\to\infty} \sum_{i=1}^{n} x_i = \infty$, then for every $M > 0$, there exists k_0 such that for every $k > k_0$, $\sum_{x \in E_k} x = \sum_{i=1}^{k} x_i > M$ thus $\sup_{F \in \mathcal{F}} s_F = \sum_{x \in E} x = \infty = \lim_{n\to\infty} \sum_{i=1}^{n} x_i$.

If $\lim_{n\to\infty} \sum_{i=1}^{n} x_i = L$, then we consider two cases

- If $\sup_{F \in \mathcal{F}} s_F > L$, then there is a subset F such that $s_F \geq L$ and since E is infinitely countable, we can let $F' = F \cup \{x_0\}$ so that $s_{F'} > L$. Since F' is finite, and the mapping which we called f is bijective. We can find the largest index $k = \max\{f^{-1}(x) : x \in F'\}$, and thus $\lim_{n\to\infty} \sum_{i=1}^{n} x_i > \sum_{i=1}^{k} x_i > s_{F'} > L$ which is a contradiction.

- If $\sup_{F \in \mathcal{F}} s_F < L$, then there exists an $\epsilon > 0$ so that for every $F \in \mathcal{F}$, $L - s_F > \epsilon$ but this contradicts with $\lim_{n\to\infty} \sum_{i=1}^{n} x_i = L$ as for every $\epsilon > 0$ there exists k_0 such that for every $k > k_0$, $L - \underbrace{\sum_{i=1}^{k} x_i}_{s_{E_k}} < \epsilon$.

  Therefore, $\sup_{F \in \mathcal{F}} s_F = L = \lim_{n\to\infty} \sum_{i=1}^{n} x_i$

5.

1. $\varnothing$ is countable.
2. If $E \subseteq S$, then either E or E^c is countable, therefore $E^c \subseteq S$.
3. If $E_k \subseteq S$ are all countable then $\bigcup_{n=1}^{\infty} E_n \in S$. If one or more of E_k are not countable then $\bigcap_{n=1}^{\infty} E_n$ is countable thus $\bigcup_{n=1}^{\infty} E_n \in S$ Therefore, S is a σ-algebra. Every singleton is contained in S.