1.

Since f is integrable on $\mathbb{R}$, there is a compactly supported continuous function h on H such that $\int_{\mathbb{R}} |f-h| < \varepsilon/2$. Since $\int_{\mathbb{R}} |f| < \infty$, we have that h is bounded on H thus h is uniformly continuous on H and hence on $\mathbb{R}$. Thus for any $\varepsilon > 0$, there is $\delta > 0$ such that for all $|x-y| < \delta$, $|h(x)-h(y)| < \varepsilon/2m(H)$

$$\begin{aligned}
|F(x) - F(y)| &= \left| \int_{-\infty}^{x} f(t)dt - \int_{-\infty}^{y} f(t)dt \right| \\
&= \left| \int_{y}^{x} f(t)dt \right| \\
&\leq \int_{y}^{x} |f(t) - h(t)|dt + \int_{y}^{x} |h(t)|dt \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2m(H)} m(H) \\
&= \varepsilon
\end{aligned}$$

2.

For any $\{E_n\} \in \mathcal{M}$, $\{F_n\} \in \mathcal{N}$, if $D \subseteq \cup_{n=1}^{\infty} E_n \times F_n$, then for any $(x, x) \in D$ there is $n \in \mathbb{N}$ such that $(x, x) \in E_n \times F_n$ which implies $x \in E_n \cap F_n$. Therefore, $[0, 1] \subseteq \cup_{n=1}^{\infty}(E_n \cap F_n)$. Thus $\mu(E_n \cap F_n) > 0$ for some $n \in \mathbb{N}$, which means that $\mu(E_n) > 0$ and $\nu(F_n) = \infty$. Therefore, $(\mu \times \nu)(D) = \infty$.

3.

Apply Theorem 5.5

$$\begin{aligned}
\int_0^a |g(x)|dx &= \int_0^a \int_x^a |t^{-1} f(t)| dt dx \\
&= \int_0^a \int_0^t \frac{|f(t)|}{t} dx dt \\
&= \int_0^a |f(t)| dt \\
&< \infty
\end{aligned}$$

Thus g is integrable. Therefore,

$$\begin{aligned}
\int_0^a g(x)dx &= \int_0^a \int_x^a t^{-1} f(t) dt dx \\
&= \int_0^a \int_0^t \frac{f(t)}{t} dx dt \\
&= \int_0^a f(t) dt
\end{aligned}$$

4.

First, note that if $\lambda_f(\alpha) = \infty$ for some $\alpha > 0$ then

$$\int_X |f(x)|^p d\mu(x) = \infty = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha$$

Now suppose $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$, then for any x, we have that

$$\int_0^{|f(x)|} p\alpha^{p-1} d\alpha = \alpha^p \big|_{\alpha=0}^{|f(x)|} = |f(x)|^p$$

Thus

$$\begin{aligned}
&\int_X |f(x)|^p d\mu(x) \\
&= \int_X \int_0^{|f(x)|} p\alpha^{p-1} d\alpha d\mu(x) \\
&= \int_X \int_0^\infty p\alpha^{p-1} 1_{|f(x)|>\alpha} d\alpha d\mu(x) \\
&= \int_0^\infty p\alpha^{p-1} \int_X 1_{|f(x)|>\alpha} d\mu(x) d\alpha \\
&= \int_0^\infty p\alpha^{p-1} \lambda_f(\alpha) d\alpha
\end{aligned}$$

5.

Let $M = \int_{\mathbb{R}^d} f(x)dx$ and $N = \int_{\mathbb{R}^d} g(y)dy$, then from theorem 5.5,

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}} |H(x,y)| d(x \times y) \\
=&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} H(x,y)dxdy \\
=&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y)g(y)dydx \\
=&M \int_{\mathbb{R}^d} g(y)dy \\
=&MN < \infty
\end{aligned}
$$