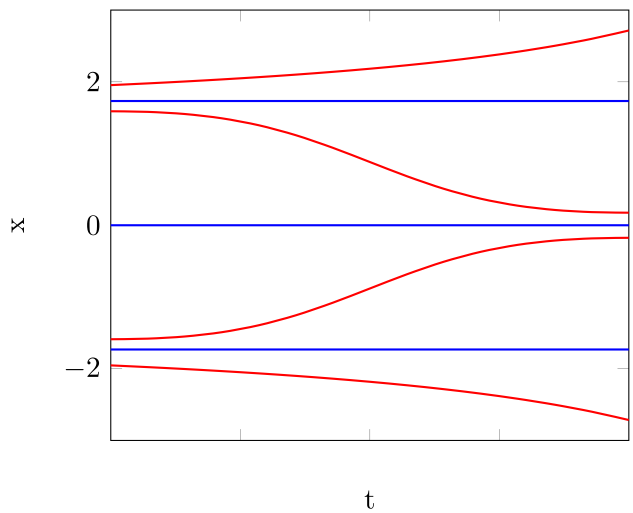
1.2

a.

$$\begin{aligned} g(x) &= x^3 - 3x = 0 \\ \implies & x(x^2-3) = 0 \\ \implies & x = 0 \text{ or } x = \pm\sqrt{3} \end{aligned}$$

$$g'(x) = 3x^2 - 3$$

Thus $g'(0) = -3, g'(\sqrt{3}) = g'(-\sqrt{3}) = 6$ and therefore, 0 is a sink, $\pm\sqrt{3}$ are sources.

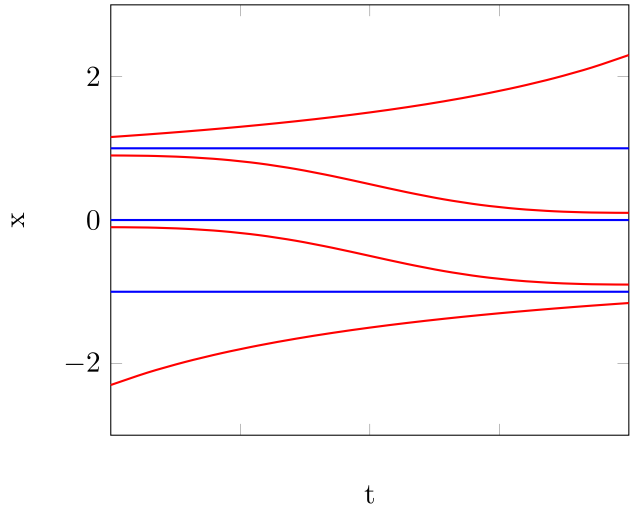


b.

$$\begin{aligned} g(x) &= x^4 - x^2 = 0 \\ \implies & x^2(x^2-1) = 0 \\ \implies & x = 0 \text{ or } x = \pm 1 \end{aligned}$$

$$g'(x) = 4x^3 - 2x$$

Thus $g'(0) = 0, g'(1) = 2, g'(-1) = -2$ and therefore, -1 is a sink, 1 is a source and 0 is neither.

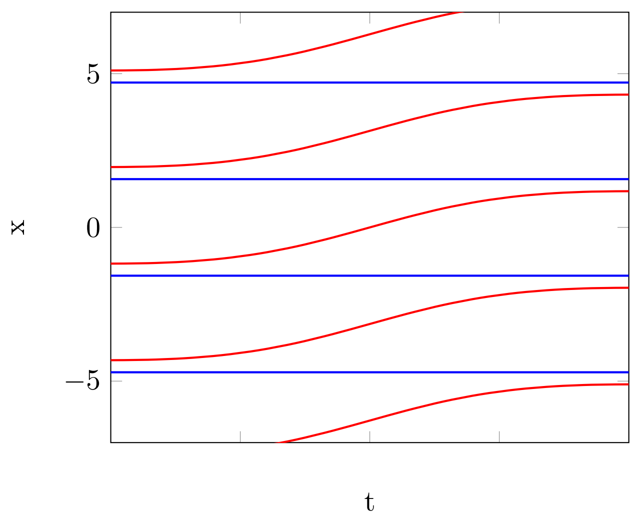


c.

$$g(x) = \cos(x)$$
$$\implies x = k\pi + \pi/2 \text{ for } k \in \mathbb{Z}$$

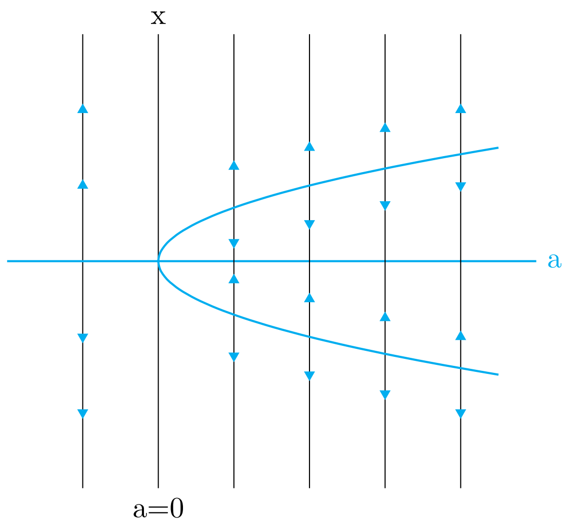
$$g'(x) = -\sin(x)$$

Thus $g'(k\pi+\pi/2) = -1$ for odd k, $g'(k\pi+\pi/2) = 1$ for even k and therefore, k, $k\pi + \pi/2$ is a sink when k is odd and a source when k is even.

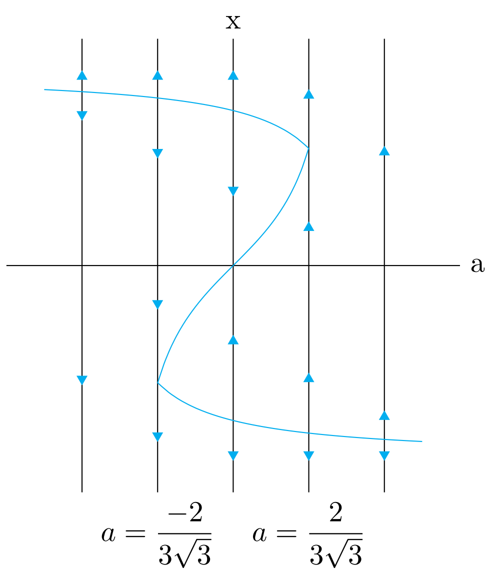


3.

b.

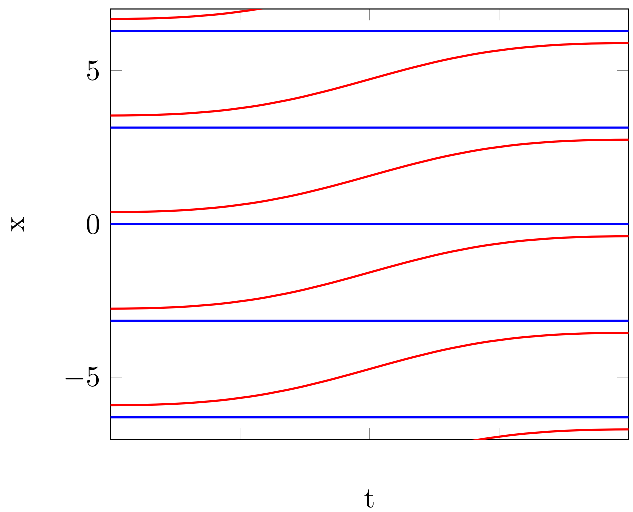


c.



5.

a.

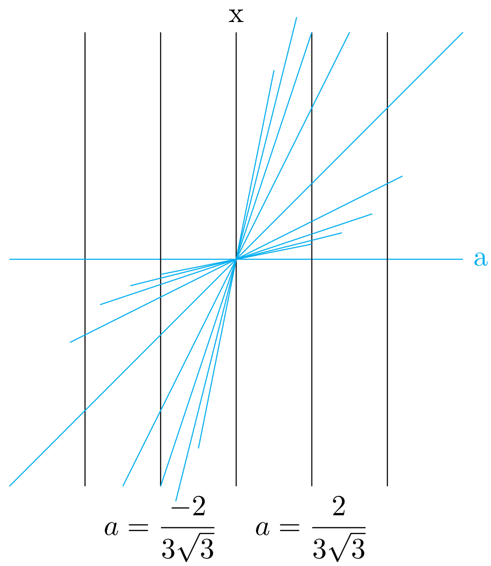


b.

We want to look at the zeros of $ax - \sin(x)$, or the intersection between $y = -ax$ and $y = \sin(x)$. we notice that

- $x = 0$ is always an intersection
- At $a = 1, -1$ there is no other intersection.
- As $|a|$ approaches 0, the slope of the line decrease thus there is more intersection.
- At $a = 0$, there is infinitely many intersections.

c.



The blue line have more intersection with the line $a = b$ as $b \to 0$ and less intersection until there is only 1 as $|b| \to \infty$. On each line $a = b$, we can write a partition of where the points of the partition are the intersections of $a = b$ with the blue line. The arrow will alternate on such interval. At $x = 0$, the derivative of $\sin(x) + ax$ have the same sign as a which will indicate the direction of the arrow at the intervals furthest from 0.
Note: the blue "line" should not be exactly line.

11.

a.

$$
\begin{aligned}
& x' = x^2 \\
\implies & \frac{dx}{dt} = x^2 \\
\implies & \int \frac{dx}{x^2} = \int dt \\
\implies & -\frac{1}{3x^3} = t + C \\
\implies & x(t) = \sqrt[3]{\frac{-1}{3(t+C)}}
\end{aligned}
$$

b.

For each $C \in \mathbb{R}$, the domain of t is $\mathbb{R} \backslash \{-C\}$ and the domain of x is $\mathbb{R} \backslash \{0\}$.

c.

Consider $x(t) = \tan(\pi t/2)$, then x is defined on $(-1, 1)$ but not on $[-1, 1)$ or $(-1, 1]$. We also have that $x(0) = 0$. The differential equation is then

$$x' = \frac{\pi}{2}(1 + \tan^2(\pi t/2)) = \frac{\pi}{2}(1 + x^2)$$

12.

a.

$$
\begin{aligned}
& x' = x^{1/3} \\
\implies & \int \frac{dx}{x^{1/3}} = \int dt \\
\implies & \frac{3}{2} x^{2/3} = t + C \\
\implies & x = \begin{cases} \frac{2}{3}(t+C)^{3/2}, & \text{if } t > -C \\ 0, & \text{if } t < -C \end{cases}
\end{aligned}
$$

since $x'(-C) = 0$. Hence, for any $-C > 0$ or $C < 0$, $x(t) = 0$.

b.

$$
\begin{aligned}
& x' = x/t \\
\implies & \int \frac{dx}{x} = \int \frac{dt}{t} \\
\implies & \ln(x) = \ln(t) + C \\
\implies & x = e^{\ln(t)+C} = te^C = tC'
\end{aligned}
$$

$x(0) = 0$ regardless of C' thus every solution in the family satisfy the initial condition.

c.

$$
\begin{aligned}
& x' = x/t^2 \\
\implies & \int \frac{dx}{x} = \int \frac{dt}{t^2} \\
\implies & \ln(x) = \frac{-1}{t} + C \\
\implies & x = e^{\frac{-1}{t}+C} = e^{-1/t}e^C = C'e^{-1/t}
\end{aligned}
$$

Since $t \neq 0$ from $x' = \dfrac{x}{t^2}$, and x is discontinuous at 0 regardless of $x(0)$:

$$\lim_{t \to 0^+} x(t) = 0 \neq \infty \lim_{t \to 0^-} x(t)$$

There is no continuous solution.

2.2

a.

We first find the eigenvalue

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0 \implies \lambda \in \{1, 3\}$$

For $\lambda = 1$,

$$\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} v = 0 \implies v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $\lambda = 3$,

$$\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} v = 0 \implies v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus,

$$X = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b.

We first find the eigenvalue

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 6-\lambda \end{vmatrix} = 0 \implies \lambda \in \{0, 7\}$$

For $\lambda = 0$,

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} v = 0 \implies v = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For $\lambda = 7$,

$$\begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix} v = 0 \implies v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Thus,

$$X = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{7t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

c.

We first find the eigenvalue

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & 0-\lambda \end{vmatrix} = 0 \implies \lambda \in \{-1, 2\}$$

For $\lambda = -1$,

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} v = 0 \implies v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $\lambda = 2$,

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} v = 0 \implies v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus,

$$X = C_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

d.

We first find the eigenvalue

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & -3-\lambda \end{vmatrix} = 0 \implies \lambda = -1 \pm \sqrt{10}$$

For $\lambda = -1 + \sqrt{10}$,

$$\begin{pmatrix} 2-\sqrt{10} & 2 \\ 3 & -2-\sqrt{10} \end{pmatrix} v = 0 \implies v = \begin{pmatrix} 2+\sqrt{10} \\ 3 \end{pmatrix}$$

For $\lambda = -1 - \sqrt{10}$,

$$\begin{pmatrix} 2+\sqrt{10} & 2 \\ 3 & -2+\sqrt{10} \end{pmatrix} v = 0 \implies v = \begin{pmatrix} 2-\sqrt{10} \\ 3 \end{pmatrix}$$

Thus,

$$X = C_1 e^{-1+\sqrt{10}} \begin{pmatrix} 2+\sqrt{10} \\ 3 \end{pmatrix} + C_2 e^{-1-\sqrt{10}} \begin{pmatrix} 2-\sqrt{10} \\ 3 \end{pmatrix}$$

2.3

- figure 1: c
- figure 2: b
- figure 3: d
- figure 4: a

2.6

The characteristic equation is $r^2 + br + k = 0$ thus for the system to have real and distinct eigenvalue, we need $b^2 - 4k > 0$ thus $b > 2k$ (assuming constant $b \geq 0, k > 0$) and the general solution should be

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where r_1, r_2 are the respective solution of the characteristic equation, that is

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4k}}{2}$$

We have $x(0) = 1$ thus $C_1 + C_2 = 1$. Also, $b, k > 0$, $r_{1,2} < 0$ and thus it is a damped harmonic oscillator.

2.8

Let the matrix be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If $b = 0$ or $c = 0$ then $(a, d) \in \{(1, 0), (0, 1)\}$. Otherwise, the eigenvalue is the solution of

$$(a - \lambda)(d - \lambda) - bc = \lambda^2 - \lambda(a + d) + (ad - bc) = 0$$

Thus we have the system of equation

$$\begin{cases} ad - bc = 0 \\ 1 - (a + d) + ad - bc = 0 \end{cases}$$

Thus,

$$ad = bc \text{ and } a + d = 1$$

and the matrix can be rewritten as

$$\begin{pmatrix} a & b \\ a(1-a)/b & 1 - a \end{pmatrix}$$

2.14

Suppose λ_1, λ_2 are eigenvalues of a 2 by 2 matrix with non-linearly dependent eigenvectors. That is $\lambda_1 \neq \lambda_2$ and $v_1 = xv_2$ for some $x \in \mathbb{R}$. Then we have

$$x\lambda_2 v_2 = xAv_2 = Axv_2 = Av_1 = \lambda_1 v_1 = \lambda_1 x v_2$$

which means that $\lambda_2 = \lambda_1$ are not distinct.
Therefore, the eigenvectors of a 2×2 matrix corresponding to distinct real eigenvalues are always linearly independent.