1.

We can first rewrite,

$$e^z = e^{a+ib} = e^a(\cos(b) + i\sin(b))$$

Thus, for any $w = r(\cos\theta + i\sin(\theta)) \neq 0$, if $a = \ln(r), b = \theta$, then

$$e^z = r(\cos(\theta) + i\sin(\theta)) = w$$

2.

We will first calculate

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2}$$

whose real part is

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2} dx$$

The roots of z^2+1 is $\pm i$, thus consider $0 < \delta < R < \infty$

$$\Omega_{R,\delta} = \{z \in \mathbb{C}, |z| < R, |z-i| > \delta\}$$

so that $\dfrac{e^{iz}}{z^2+1}$ is differentiable in $\Omega_{R,\delta}$, and thus

$$\int_{-R}^{R} \frac{e^{iz}}{1+z^2} dz + \int_{\Gamma_R} \frac{e^{iz}}{1+z^2} dz = \oint_{|z-i|=\delta} \frac{e^{iz}}{1+z^2} dz$$

We then calculate

$$\begin{aligned}
\left| \int_{\Gamma_R} \frac{e^{iz}}{1+z^2} \right| dz &= \left| \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{1+(Re^{i\theta})^2} Rie^{i\theta} d\theta \right| \\
&\leq |Ri| \int_0^{\pi} \left| \frac{e^{iRe^{i\theta}}}{1+(Re^{i\theta})^2} \right| |e^{i\theta}| d\theta \\
&= R \int_0^{\pi} \left| \frac{e^{iRe^{i\theta}}}{1+(Re^{i\theta})^2} \right| d\theta
\end{aligned}$$

Notice that

$$|1+(Re^{i\theta})^2| \geq ||Re^{i\theta}|^2 - |-1|| \geq R^2 - 1$$

Thus we have

$$\begin{aligned}
\left| \int_{\Gamma_R} \frac{e^{iz}}{1+z^2} \right| dz &\leq R \int_0^{\pi} \left| \frac{e^{iRe^{i\theta}}}{R^2-1} \right| d\theta \\
&= \frac{R}{|R^2-1|} \int_0^{\pi} |e^{iR\cos(\theta)}||e^{-R\sin(\theta)}| d\theta \\
&= \frac{R}{|R^2-1|} \int_0^{\pi} |e^{-R\sin(\theta)}| d\theta \\
&\leq \frac{2R}{|R^2-1|} \int_0^{\pi/2} |e^{-2R\theta/\pi}| d\theta \\
&= 0 \text{ as } R \to \infty
\end{aligned}$$

Thus

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{iz}}{1+z^2} = 0$$

Finally,

$$\begin{aligned}\oint_{|z-i|=\delta} \frac{e^{iz}dz}{z^2+1} &= \oint_{|z-z_0|=\delta} \frac{1}{2i}\left(\frac{e^{iz}}{z-i} - \frac{e^{iz}}{z+i}\right)dz \\ &= \oint_{|z-z_0|=\delta} \frac{1}{2i}\frac{e^{iz}}{z-i}dz \\ &= \frac{2\pi i}{2i}e^{-1} = \frac{\pi}{e}\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1} = \frac{\pi}{e}$$

3.

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{-ie^{-\pi^2\xi^2}}{\pi\xi} e^{i2\pi x\xi} d\xi \\
=&\frac{-i}{\pi}\mathcal{F}\left(\frac{e^{-\pi^2\xi^2}}{\xi}\right)(x) \\
=&\frac{-i}{\pi}\left(\mathcal{F}\left(\frac{1}{\xi}\right) * \mathcal{F}\left(e^{-\pi^2\xi^2}\right)\right)(x) \\
=&\frac{-i}{\pi}\int_{-\infty}^{\infty} \mathcal{F}\left(\frac{1}{t}\right)(-x-s)\mathcal{F}(e^{-\pi^2 s^2})(s)ds \\
=&\frac{-i}{\pi}\int_{-\infty}^{\infty} -i\pi\operatorname{sgn}(x+s)\frac{1}{\sqrt{\pi}}e^{-s^2}ds \\
=&\int_{-\infty}^{\infty} \operatorname{sgn}(x-s)\frac{1}{\sqrt{\pi}}e^{-s^2}ds \\
=&\underbrace{\int_{-\infty}^{-x} \frac{1}{\sqrt{\pi}}e^{-s^2}ds - \int_{x}^{\infty} \frac{1}{\sqrt{\pi}}e^{-s^2}ds}_{0} + \int_{-x}^{x} \frac{1}{\sqrt{\pi}}e^{-s^2}ds \\
=&2\int_{0}^{x} \frac{1}{\sqrt{\pi}}e^{-s^2}ds
\end{aligned}
$$

4.

Consider

$$\Gamma_1 : t \to -R + (b/2a)(1-t), \qquad [0,1] \to \mathbb{C}$$
$$\Gamma_2 : t \to 2Rt - R, \qquad [0,1] \to \mathbb{C}$$
$$\Gamma_3 : t \to (b/2a)t + R, \qquad [0,1] \to \mathbb{C}$$
$$\Gamma_4 : b/2a + R - 2Rt, \qquad [0,1] \to \mathbb{C}$$

so that $\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4$ is the rectangle with vertices $-R, R, R+b/2a, -R+b/2a$

$$\begin{aligned}
&\int_{-R}^{R} e^{-ix\xi} f(x)dx \\
&= \int_{-R}^{R} e^{-ix\xi - a(x+b/2a)^2 + b^2/4a - c} dx \\
&= -\int_{\Gamma_4} e^{-i(x-b/2a)\xi - ax^2 + b^2/4a - c} dx \\
&= -e^{ib\xi/2a + b^2/4a - c} \int_{\Gamma_4} e^{-ix\xi - ax^2} dx
\end{aligned}$$

Now we have

$$\lim_{R\to\infty} \int_{\Gamma_1} |e^{-ix\xi - ax^2}| dx = \lim_{R\to\infty} \int_{\Gamma_1} |e^{-ax^2}| dx = 0$$

as the length of the line is fixed at $|b/2a|$ while $|e^{-ax^2}| \to 0$ on Γ_1 as $R \to \infty$. Similarly, for Γ_3. Thus

$$\lim_{R\to\infty} \int_{\Gamma_1} e^{-ix\xi - ax^2} dx = \lim_{R\to\infty} \int_{\Gamma_3} e^{-ix\xi - ax^2} dx = 0$$

Thus as $R \to \infty$

$$-\frac{1}{\sqrt{2\pi}} \int_{\Gamma_4} e^{-ix\xi - ax^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\Gamma_2} e^{-ix\xi - ax^2} dx \to \frac{1}{\sqrt{2a}} e^{-\xi^2/4a}$$

Hence, the final answer is

$$\frac{1}{\sqrt{2a}} e^{ib\xi/2a + b^2/4a - c - \xi^2/4a}$$

5.

$$U_{tt} + (i\xi)^4 U = U_{tt} + \xi^4 U = 0, \qquad U(\xi,0) = F(\xi), U_t(\xi,0) = 0$$

Thus,

$$U(\xi,t) = c_1 \sin(\xi^2 t) + c_2 \cos(\xi^2 t)$$

and

$$U_t(\xi,t) = \xi^2 (c_1 \cos(\xi^2 t) - c_2 \sin(\xi^2 t))$$

Using the initial condition, we have $c_1 = 0$, thus

$$U(\xi,t) = C\cos(\xi^2 t)$$

Apply the other initial condition, we have

$$U(\xi,0) = C = F(\xi)$$

Thus

$$U(\xi,t) = F(\xi)\cos(\xi^2 t)$$

Finally, we can get

$$u(x,t) = \mathcal{F}^{-1}(F(\xi)\cos(\xi^2 t)) = \frac{1}{\sqrt{2\pi}} f * \mathcal{F}^{-1}(\cos(\xi^2 t))$$

Thus, we only need to calculate

$$\mathcal{F}^{-1}(\cos(\xi^2 t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi + i\xi^2 t} + e^{ix\xi - i\xi^2 t} d\xi$$

We also have

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi + i\xi^2 t} d\xi \\
=&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\left[(\xi + x/2t)^2 - x^2/4t^2\right]} d\xi \\
=&\frac{e^{-ix^2/4t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\xi^2} d\xi \\
=&\frac{e^{-ix^2/4t}}{\sqrt{2\pi}} e^{i\pi/4}\sqrt{\pi/t} \\
=&\frac{e^{-i(x^2/4t - \pi/4)}}{\sqrt{2t}}
\end{aligned}$$

while for

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\xi - i\xi^2 t} d\xi \\
=&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it\left[(\xi - x/2t)^2 - x^2/4t^2\right]} d\xi \\
=&\frac{e^{ix^2/4t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it\xi^2} d\xi \\
=&\frac{e^{ix^2/4t}}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-i\xi^2} d\xi
\end{aligned}
$$

We solve $\int_{-\infty}^{\infty} e^{-ix^2}dx = 2\int_0^{\infty} e^{-ix^2}dx$. Let consider the function e^{-z^2} and

$$\Gamma_1 : t \to Rt, \qquad [0,1] \to \mathbb{C}$$

$$\Gamma_2 : t \to Re^{it\pi/4}, \qquad [0,1] \to \mathbb{C}$$

$$\Gamma_3 : t \to (1-t)Re^{i\pi/4}, \qquad [0,1] \to \mathbb{C}$$

Hence, as

$$\int_{\Gamma_3} e^{-z^2}dz = -e^{i\pi/4}\int_0^R e^{-iz^2}dz$$

$$\int_0^R e^{-x^2}dx + \int_{\Gamma_2} e^{-z^2}dz - e^{i\pi/4}\int_0^R e^{-iz^2}dz = 0$$

We have

$$\int_0^R e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and

$$
\begin{aligned}
\int_{\Gamma_2} |e^{-z^2}|dz &= \int_0^{\pi/4} \left|Rie^{i\theta}\right| |e^{-\left(Re^{i\theta}\right)^2}|d\theta \\
&= |R|\int_0^{\pi/4} \left|e^{-R^2\cos(2\theta)}\right| d\theta \\
&\leq |R|\int_0^{\pi/4} \left|e^{-R^2(1-4\theta/\pi)}\right| d\theta \\
&= |R|/e^{-R^2}\int_0^{\pi/4} \underbrace{\left|e^{4R^2\theta/\pi}\right| d\theta}_{\text{const}} \to 0 \text{ as } R \to \infty
\end{aligned}
$$

since

$$\cos(2\theta) = \sin(\pi/2 - 2\theta) \geq \frac{2}{\pi}(\pi/2 - 2\theta) = 1 - 4\theta/\pi$$

for $\theta \in [0, \pi/4]$. Thus

$$\int_{-\infty}^{\infty} e^{-ix^2}dx = 2e^{-i\pi/4}\frac{\sqrt{\pi}}{2} = \sqrt{\pi}e^{-i\pi/4}$$

Therefore, we have

$$\mathcal{F}^{-1}(\cos(\xi^2 t)) = \frac{e^{-i(x^2/4t-\pi/4)}}{\sqrt{2t}} + \frac{e^{ix^2/4t}}{\sqrt{2t}}e^{-i\pi/4}$$