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Without loss of generality, we can assume $\dim(\text{Image}(g)) = 1$. Assume that g is only defined in the set $[-N, N] \times [-M, M]$ where N, M are arbitary. We have that

$$\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} g(x^*, y^*)\Delta F_{i,j}$$

where with $x_i = \frac{-(n-i)N+iN}{n} = \frac{2iN-nN}{n}$, $y_i = \frac{-(m-i)M+iM}{m} = \frac{2iM-mM}{m}$, we have

$$\begin{aligned}\Delta F_{i,j} &= F(x_i, y_j) - F(x_{i-1}, y_j) - F(x_i, y_{j-1}) + F(x_{i-1}, y_{j-1}) \\ &= \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(x,y)dxdy\end{aligned}$$

Hence,

$$\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} g(x^*, y^*)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(x,y)dxdy$$

$$\begin{aligned}E[g(X,Y)] &= \int_{[-N,N]\times[-M,M]} g(x,y)f(x,y)dxdy \\ &= \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} g(x,y)f(x,y)dxdy\end{aligned}$$

We know that g is uniformly continuous as it is continuous in a compact set, we have that with large enough n,m

$$|g(x,y) - g(x^*, y^*)| < \epsilon$$

Hence,

$$\begin{aligned}&\left|\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) - E[g(X,Y)]\right| \\ =&\left|\lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} (g(x,y) - g(x^*, y^*))f(x,y)dxdy\right| \\ <&\left|\lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} \epsilon f(x,y)dxdy\right| \\ =&\epsilon\left|\lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(x,y)dxdy\right| \\ =&\epsilon\end{aligned}$$

Since, ϵ is arbitary, $\int_{[-N,N]\times[-M,M]} g(x,y)dF(x,y) = E[g(X,Y)]$, and since N, M are arbitary,

$$\int_{\mathbb{R}^2} g(x,y)dF(x,y) = E[g(X,Y)]$$

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Because $s_i > a_i$ for all i, $X(0)$ does not affect stationary distributions.

$$
\begin{aligned}
\pi(0) &= \left(1 + \frac{a_0}{s_1} + \frac{a_0 a_1}{s_1 s_2} + \dots\right)^{-1} \\
&= \left(1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 4 \cdot 5} + \dots\right)^{-1} \\
&= \left(1 + \sum_{i=1}^{\infty} \frac{2 \cdot i!}{(i+2)!}\right)^{-1} \\
&= \left(1 + \sum_{i=1}^{\infty} \frac{2}{(i+1)(i+2)}\right)^{-1} \\
&= \left(1 + 2\sum_{i=1}^{\infty} \left(\frac{1}{i+1} - \frac{1}{i+2}\right)\right)^{-1} \\
&= \left(1 + 2\left(\frac{1}{2}\right)\right)^{-1} \\
&= \frac{1}{2}
\end{aligned}
$$

Hence, we can calculate

$$
\pi(i) = \pi(0)\frac{a_0 a_1 \dots a_{i-1}}{s_1 s_2 \dots s_i} = \frac{1}{2} \cdot \frac{2 \cdot i!}{(i+2)!} = \frac{1}{(i+1)(i+2)}
$$