Game Theory

Many practical problems require decision making in a competitive situation where there are two or more opposing parties with conflicting interests and where the action of one depends upon the action taken by the opponent. Such a situation is termed as a competitive situation.

A great variety of competitive situations are seen in every day life. Competitive situations occur frequently on Economic and Business activities. Management and Labour relations, Political battles and elections, war etc., are some of the examples of competitive situations.

Game theory is a theory of conflict and it is a mathematical theory which deals with competitive situations.

It is a type of decision theory which is concerned with the decision making in situations where two or more rational opponents are involved under conditions of competition and conflicting interests.

Theory of games became popular when Newman along with Morgenstern published the book titled "Theory of Games and Economic behaviour" in 1944.

Game:

Game is defined as an activity between two or more persons according to a set of rules at the end of which each person receives some benefit or satisfaction or suffers loss.

Therefore in a game there are two or more opposite parties with conflicting interests. They know the objectives and the rules of the game. An experienced player usually predicts with accuracy how his opponent will react if a particular strategy is adopted. When one player wins, his opponent loses.

Characteristics (features) of a competitive game:

A competitive situation is called a game if it has the following properties or characteristics.

1. There are finite number of competitors called players.
2. Each player has a list of finite number of possible courses of action.
3. A play is said to be played when each of the players chooses a single course of action from the list of courses of action available to him.
4. Evey play is associated with an outcome known as pay off. It determines a set of payments.
5. The possible gain or loss of each player depends not only on the choice made by him but also the choice made by his opponent.

Assumption of a game

1. The players act rationally and intelligently.
2. Each player has a finite set of possible courses of action.
3. The players attempt to maximise gains or minimise losses.
4. All relevant information are known to each player.
5. The players make individual decisions.
6. The players simultaneously select their respective courses of action.
7. The pay off is fixed and determined in advance.

Strategy:

The strategy of a player is the predetermined rule by which a player decides his course of action during the game. That is, a strategy for a given player is a set of rules or programmes that specify which, of the available courses of action, he should select at each play.

There are two types of strategies, Pure strategy and Mixed strategy.

Pure Strategy:

A pure strategy is a decision (in advance of all plays) always to choose a particular course of action. It is a pre-determined course of action. The player knows it in advance.

Mixed Strategy:

A player is said to adopt mixed strategy when he does not adopt a single strategy all the time but would play different strategies each at a certain time. A mixed strategy is a decision to choose a course of action for each play in accordance with some particular probability distribution. In a mixed strategy we can not definitely say which course of action the player will choose. We can only guess on the basis of probability.

Player:

Each participant of the game is called a player.

Pay offs

The outcome of a game in the form of gains or loses to the competing players for choosing different courses of action is known as pay offs.

Pay off matrix

In a game, the gains and the losses, resulting from different moves and counter moves, when represented in the form of a matrix, is known as a pay off matrix. Each element of the pay off matrix is the gain of the maximizing player when a particular course of action is chosen by him as against the course of action chosen by the opponent.

Given below is a pay off matrix

$$\begin{array}{cc} & B \\ & \begin{array}{cc} B_1 & B_2 \end{array} \\ A & \begin{array}{c} A_1 \\ A_2 \end{array} \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \end{array}$$

Here A is the maximizing player and B is the minimizing player. Each element in the matrix is the gain for A when he chooses a course of action against which B chooses another course of action. For example, when A chooses A_2 and B chooses B_1, the gain for 'A' is shown in the second row, first column. Hence it is 0.

Value of the game:

The value of the game is the maximum guaranteed gain to the maximising player (A) if both the players use their best strategies. It is the expected pay - off of a play when all the players of the game follow their optimal strategies.

Maximising and Minimising players

If there are two players A and B, generally the pay offs given in a pay off matrix indicate gains to A for each possible outcomes of the game. That is, each outcome of a game results into a gain for A. All such gains are shown in the pay off matrix. Usually each row of the pay off matrix indicates gains to A for his particular strategy. A is called the maximizing player and B is called minimizing player. The pay off values given in each column of pay off matrix indicates the losses for B for his particular course of action. Therefore if the element in the position (A_1, B_3) is a, then A's gain is 'a' and B's gain is $-a$ (or B's loss is a), when A chooses the strategy A_1 and B chooses the strategy B_3.

Maximin and Minimax:

Each row in a pay off matrix represents pay offs in respect of every strategy of the maximising player A. Similarly each column represents pay offs in respect of every strategy of minimising player B. Maximin is the maximum of minimum pay offs in each row. Minimax is the minimum of maximum pay offs in each column.

Eg:

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 5 & 3 & 2 \\ 1 & -2 & 0 \\ 8 & -1 & 1 \end{bmatrix} \end{array}$$

Minimum in row $A_1 = 2$

" " $A_2 = -2$

" " $A_3 = -1$

Maximum of these minima = Maximin = 2

Maximum for column $B_1 = 8$

" " $B_2 = 3$

" " $B_3 = 2$

Minimum of these maxima = Minimax = 2

Maximin Principle

Here, the maximizing player (A) lists worst possible pay offs of all his potential strategies and chooses that strategy which corresponds to the best. This is maximin principle.

Minimax principle

Here, the minimizing player (B) lists his maximum losses from each stra.egy and selects that strategy which corresponds to the least. This is minimax principle.

Saddle point:

A Saddle point of a pay off matrix is that position in the pay off matrix where the maximin coincides with the minimax. Pay off at the saddle point is the value of the game. In a game having a saddle point optimum strategy of maximizing player is always to choose the row containining saddle point and for minimisng player to choose the column contaiaing saddle point. If there are more than one saddle point there will b· more than one solution.

A game for which maximin for A = minimax for B, is called a game with saddle point. The element at the saddle point position is the value of the game denoted by v.

Eg:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ \text{Column max:} \end{array}
\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \left[\begin{array}{ccc} 3 & 2 & 4 \\ -2 & 1 & -3 \\ 0 & -2 & 3 \end{array}\right] \\ \begin{array}{ccc} 3 & 2 & 4 \end{array} \\ \begin{array}{ccc} & \uparrow & \end{array} \end{array}
\begin{array}{c} \text{Row min} \\ 2 \leftarrow \\ -3 \\ -2 \\ \\ \end{array}$$

Maximin = Max of Rowmin = 2

Minimax = Min of Column max = 2

Maximin = Minimax = 2 which refers to (A_1, B_2)

$\therefore$ Saddle point is (A_1, B_2)

value of the game (v) = 2

Different kinds of games

Games are categorized on the basis of

(1) Number of players (2) Number of moves

(3) Nature of the pay off (4) Nature of rules

Zero Sum game

In a game, if the algebraic sum of the outcomes (or gain) of all the players together is zero, the game is called zero sum game, otherwise it is called Non-zero sum game. That is, in a zero sum game, the amounts won by all winners together is equal to the sum of the amounts lost by all together.

Note: A game involving n players is called n – person game and a game with two players is called two person game.

Two-person zero sum game (Rectangular games): Two-person Zero sum game is the simplest of game models. There will be two persons in the conflict and the sum of the pay offs of both together is zero. That is, the gain of one is at the expense of the other.

The two person zero sum game may be (1) Pure strategy game or (2) Mixed strategy game.

Basic assumptions in two-person zero sum game

1. There are two players
2. They have opposite interests.
3. The number of strategies available to each player is finite.
4. For each specific strategy, selected by a player, there results a pay off.
5. The amount won by one player is exactly equal to the amount lost by the other.

Limitations of Game theory

1. In fact, a player may have infinite number of strategies. But we assume that there are only finite number of strategies.
2. It is assumed that each player has the knowledge of opponent's strategies. But it is not necessary in all cases.
3. The assumption that gain of one person is the loss of his opponent need not be true in all situations.
4. Game theory usually ignores the presence of risk and uncertainty.
5. It is assumed that pay off is always known in advance. But sometimes it is impossible to know the pay off accurately.
6. It is assumed that the two persons involved in the game have equal intelligence. But it need not be.

Fair game:

A game is said to be fair if the value of the game = 0

Solution of Pure Strategy Games

The maximising player arrives at his optimal strategy on the basis of maximin criterion, while minimising player's strategy is based on the minimax criterion. The game is solved by equating maximin value with minimax value. In this type of problems saddle point exists.

Ex. 1: For the following pay off matrix of firm A, determine the optimal strategies for both the firms and the value of the game (using maximin-minimax principle)

$$\begin{array}{cc} & \textbf{Firm B} \\ \textbf{Firm A} & \begin{bmatrix} 3 & -1 & 4 & 6 & 7 \\ -1 & 8 & 2 & 4 & 12 \\ 16 & 8 & 6 & 14 & 12 \\ 1 & 11 & -4 & 2 & 1 \end{bmatrix} \end{array}$$

Ans:

$$\begin{array}{c|ccccc|c} & B_1 & B_2 & B_3 & B_4 & B_5 & \text{Row min.} \\ \hline A_1 & 3 & -1 & 4 & 6 & 7 & -1 \\ A_2 & -1 & 8 & 2 & 4 & 12 & -1 \\ A_3 & 16 & 8 & 6 & 14 & 12 & 6 \leftarrow \\ A_4 & 1 & 11 & -4 & 2 & 1 & -4 \\ A_5 & & & & & & \\ \hline \text{Column max:} & 16 & 11 & 6 & 14 & 12 & \\ & & & \uparrow & & & \end{array}$$

Maximin = Maximum of row minimum = 6

Minimax = Minimum of column max = 6

∴ Maximin = Minimax = 6 at third row, third column.

∴ Saddle point exists (A_3, B_3)

∴ Optimal strategy for B is B_3 and optimal strategy for A is A_3.

The value of the game (v) = pay off at (A_3, B_3) = 6

Ex. 2: From the following game matrix, find the saddle point and state the game value.

$$\begin{array}{cc} & \begin{array}{cc} \textbf{M} & \textbf{N} \end{array} \\ \begin{array}{c} \textbf{P} \\ \textbf{Q} \end{array} & \begin{bmatrix} 6 & 2 \\ -1 & -4 \end{bmatrix} \end{array}$$

Ans:

$$\begin{array}{c|cc|c} & M & N & \text{Row min.} \\ \hline P & 6 & 2 & 2 \\ Q & -1 & -4 & -4 \\ \hline \text{Column max:} & 6 & 2 & \end{array}$$

Maximin = Max. (2, -4) = 2

Minimax = Min (6, 2) = 2

Maximin = Minimax at (P, N)

So saddle point is PN and game value = 2

Ex. 3: State whether the following game matrix has a saddle point.

$$\begin{bmatrix} 1 & 0 \\ -4 & 3 \end{bmatrix}$$

Ans:

$$\begin{array}{c|cc|c} & & & \text{Row minima} \\ & 1 & 0 & 0 \\ & -4 & 3 & -4 \\ \hline \text{Column max:} & 1 & 3 & \end{array}$$

Maximin = max (0, -4) = 0

Minimax = Min (1, 3) = 1

Maximin ≠ Minimax.

∴ There is no saddle point.

Ex. 4: The following is a pay off matrix

$$\begin{array}{cc} & \textbf{Y} \\ \textbf{X} & \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \end{array}$$

What is the value of game ? Who will be the winner of the game ? Why ?

Ans:

$$\begin{array}{c|cc|c} & Y & & \text{Row min} \\ \hline X & 1 & -2 & -2 \\ & 2 & -1 & -1 \\ \hline \text{Columan max:} & 2 & -1 & \end{array}$$

Maximin = Max: (-2, -1) = –1

Minimax = Min (2, -1) = –1

∴ Maximin = Minimax ∴ Saddle point exists

Value of game = -1

Gain of X = -1

Since the value of the game is negative, Y wins the game.

Ex. 5: Solve the game whose pay off matrix is given by

$$\begin{array}{cc} & \textbf{Player B} \\ \textbf{Player A} & \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix} \end{array}$$

Ans:

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{Row minima} \\ \hline A_1 & 15 & 2 & 3 & 2 \\ A_2 & 6 & 5 & 7 & 5 \leftarrow \\ A_3 & -7 & 4 & 0 & 7 \\ \hline \text{Column Maxima:} & 15 & 5 & 7 & \\ & & \uparrow & & \end{array}$$

∴ maximin = minimax = 5 at (A_2, B_2)

The matrix has a saddle point at position (A_2, B_2). Hence the solution to the game is

(i) the optimum strategy for player A is A_2,

(ii) the optimum strategy for player B is B_2.

(iii) the value of the game is 5

Ex. 6: Solve the game whose pay - off matrix is given below.

$$\begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix}$$

Ans:

$$\begin{array}{r c c}
 & & \text{Row Minima} \\
 & \begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix} & \begin{matrix} 0 \\ 4 \leftarrow \\ 2 \\ 1 \end{matrix} \\
\text{Column Maxima:} & \begin{matrix} 9 & 6 & 4 & 8 & 8 \\ & & \uparrow & & \end{matrix} &
\end{array}$$

$\therefore$ Maximin = Max (0, 4, 2, 1) = 4

Minimax = Min (9, 6, 4, 8, 8) = 4

$\therefore$ Maximin = Minimax = 4 at (A_2, B_3)

The matrix has a saddle point at the position (A_2, B_3)

(i) the optimum strategy for player A is A_2 and for B is B_3.

(ii) the value of the game is 4

Ex. 7: A company's management and labour union are negotiating a settlement. Each has the strategies (1) aggressive approach (2) logical approach (3) legalistic approach (4) conciliatory approach

The cost to the company for every pair of strategic choices are given below

$$\begin{array}{c c}
 & \textbf{Company strategies} \\
\textbf{Union strategies} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 40 & 30 & 24 & 70 \\ 50 & 28 & 16 & 20 \\ 80 & 4 & 20 & 10 \\ -10 & 8 & 22 & 0 \end{bmatrix}
\end{array}$$

Arrive at the decision what is the value of the game ?

Ans:

$$\begin{array}{r c c}
 & \text{B} & \\
 & \begin{matrix} B_1 & B_2 & B_3 & B_4 \end{matrix} & \text{Row min} \\
\begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{bmatrix} 40 & 30 & 24 & 70 \\ 50 & 28 & 16 & 20 \\ 80 & 4 & 20 & 10 \\ -10 & 8 & 22 & 0 \end{bmatrix} & \begin{matrix} 24 \leftarrow \\ 16 \\ 4 \\ -10 \end{matrix} \\
\text{Column max:} & \begin{matrix} 80 & 30 & 24 & 70 \\ & & \uparrow & \end{matrix} &
\end{array}$$

Maximin = Minimax = 24 at (A_1, B_3)

Saddle point exists which is (A_1, B_3).

ie Union's strategy : Aggressive approach

Company's strategy : Legalistic approach

Value of the game = 24

Ex. 8: For what value of λ, the game with the following matrix is determinable

$$\begin{array}{c c}
 & \textbf{B} \\
 & \begin{matrix} B_1 & B_2 & B_3 \end{matrix} \\
\begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix}
\end{array}$$

Treating λ is neither minimax nor maximin

Ans:

$$\begin{array}{r c c}
 & \text{B} & \text{Row min} \\
\text{A} & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} & \begin{matrix} 2 \leftarrow \\ -7 \\ -2 \end{matrix} \\
\text{Column max:} & \begin{matrix} -1 & 6 & \lambda \\ \uparrow & & \end{matrix} &
\end{array}$$

Maximin = 2 and Minimax = -1

$\therefore$ λ Cannot be less than −1 and greater than 2

λ lies between −1 and 2

Solution of Mixed Strategy Problems

When there is no saddle point for a game problem, the minimax – maximin principle cannot be applied to solve that problem. In those cases the concept of chance move is introduced. Here the choice among a number of strategies is not the decision of the player but by some chance mechanism. That is, pre-determined probabilities are used for deciding the course of action. The strategies thus used are called mixed strategies.

Solution to a Mixed strategy problem can be arrived at by any of the following methods.

(a) Probability method (Equal gain method)

(b) Graphic method

(c) Linear Programming method.

(1) Probability Method

This method is applied when there is no saddle point and the pay off matrix has two rows and two columns only. The players may adopt mixed strategies with certain probabilities. Here the problem is to determine the probabilities of different strategies of both players and the expected value of the game.

Consider the following pay off matrix

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & B_1 \; B_2 \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

Let 'p' be the probability for A using strategy A_1 and 1 - p be the probability for A using A_2. Then we have the equation. Expected gain of A if B chooses $B_1 = ap + c(1 - p)$.

Expected gain of A if B chooses $B_2 = bp + d(1 - p)$.

$\therefore \quad ap + c(1-p) = bp + d(1-p)$

Solving the equation we get p. $\qquad p = \dfrac{(d-c)}{(a+d)-(b+c)}$

Similarly let q and 1 - q be respectively probabilities for B choosing strategies B_1 and B_2, then $aq + b(1 - q) = cq + d(1 - q)$

Solving this equation, $q = \dfrac{(d-b)}{(a+d)-(b+c)}$

$$\begin{array}{c|cc|} & q & 1-q \\ \hline p & a & b \\ 1-p & c & d \\ \hline \end{array}$$

Expected value of the game (v)

$= \Sigma$ x. prob $= a\,pq + bp(1 - q)$

$+ c(1 - p)q + d(1 - p)(1 - q)$

x	prob	x × prob.
a	pq	a pq
b	p (1 − q)	bp (1 − q)
c	(1 − p) q	c (1 − p) q
d	(1 − p) (1 − q)	d (1 − p) (q − q)

Substituting the values of p and q and simplifying $v = \dfrac{(ad-bc)}{(a+d)-(b+c)}$

Therefore Solution is

Strategies of A are $(p, 1-p)$

Strategises of B are $(q, 1-q)$

Value of the game = v where

$$p = \frac{(d-c)}{(a+d)-(b+c)} \text{ and } q = \frac{(d-b)}{(a+d)-(b+c)} \text{ and } v = \frac{(ad-bc)}{(a+d)-(b+c)}$$

Ex. 9: Find p, q and v from the following problem

$$\begin{array}{cc} & \textbf{B} \\ \textbf{A} & \begin{bmatrix} -2 & -1 \\ 2 & -3 \end{bmatrix} \end{array}$$

Ans: Here a = -2, b = -1, c = 2 and d = -3

$$\therefore \quad p = \frac{(d-c)}{(a+d)-(b+c)} = \frac{-3-2}{(-2-3)-(-1+2)} = \frac{-5}{-5-1} = \frac{-5}{-6} = \frac{5}{6}$$

$$q = \frac{(d-b)}{(a+d)-(b+c)} = \frac{-3--1}{-5-1} = \frac{-3+1}{-6} = \frac{-2}{-6} = \frac{1}{3}$$

$$v = \frac{(ad-bc)}{(a+d)-(b+c)} = \frac{(-2\times-3)-(-1\times2)}{-6} = \frac{6+2}{-6} = -\frac{4}{3}$$

Ex. 10: Solve the following game

$$\begin{array}{cc} & \textbf{Player B} \\ \textbf{Player A} & (\mathbf{B_1}) \; (\mathbf{B_2}) \\ \begin{array}{c} \mathbf{A_1} \\ \mathbf{A_2} \end{array} & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

Ans: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

[There is no saddle point for this problem, since maximin ≠ minimax.]

Let Probability for player A using the strategy $A_1 = p$

Probability for player A using strategy $A_2 = 1 - p$.

Probability for player B using the strategy $B_1 = q$

Probability for player B using strategy $B_2 = 1 - q$.

$$p = \frac{(d-c)}{(a+d)-(b+c)} = \frac{1-4}{(3+1)-(4+5)} = \frac{-3}{4-9} = \frac{-3}{-5} = \frac{3}{5}$$

$$1 - p = 1 - \frac{3}{5} = \frac{2}{5}$$

$$q = \frac{(d-b)}{(a+d)-(b+c)} = \frac{1-5}{(3+1)-(4+5)} = \frac{-4}{-5} = \frac{4}{5}$$

$$1 - q = 1 - \frac{4}{5} = \frac{1}{5}$$

Expected value of the game $= \dfrac{(ad-bc)}{(a+d)-(b+c)} = \dfrac{3\times1-5\times4}{-5} = \dfrac{-17}{-5} = \dfrac{17}{5}$

Solution:

A's strategy $= (p, 1-p) = (\frac{3}{5}, \frac{2}{5})$

B's strategy $= (q, 1-q) = (\frac{4}{5}, \frac{1}{5})$

$v = \frac{17}{5}$

Ex. 11: Consider a modified form of "Matching based coins" game problem. The matching player A is paid Rs. 8 if two coins turn both heads and Re. 1.00 if two coins trun both tails. B is paid Rs. 3 when the two coins do not match. Given the choice of being A or B, and what would be your strategy ?

Ans: A's gain is loss of B. Therefore they are shown as negative figures. The pay off matrix completely represents the gain of A.

Pay off matrix

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cc} B_1 & B_2 \\ (H) & (T) \end{array} \\ \text{Player A} \begin{array}{c} A_1\ (H) \\ A_2\ (T) \end{array} & \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

Since maximin is not equal to minimax, pay - off matrix does not possess any saddle point. The players, will therefore use mixed strategies.

P (A using the strategy A_1) = p

P (A using the strategy A_2) = 1 - p

P (B using the strategy B_1) = q

P (B using the strategy B_2) = 1 - q

$$p = \frac{(d-c)}{(a+d)-(b+c)} = \frac{1--3}{(8+1)-(-3-3)} = \frac{4}{9+6} = \frac{4}{15}$$

and $1 - p = \frac{11}{15}$

$$q = \frac{(d-b)}{(a+d)-(b+c)} = \frac{1+3}{9+6} = \frac{5}{15} \text{ and } 1 - p = \frac{11}{15}$$

A's strategy = $(p, 1 - p) = (\frac{4}{15}, \frac{11}{15})$

B's strategy = $(q, 1 - q) = (\frac{4}{15}, \frac{11}{15})$

$$\text{value of the game} = \frac{(ad-bc)}{(a+d)-(b+c)} = \frac{8 \times 1 - (-3 \times -3)}{15} = -\frac{1}{15}$$

Since A's expected gain is $-\frac{1}{15}$, B's expected gain = $\frac{1}{15}$

Principle of dominance

The principle of dominance states that if the strategy of a player dominates over another strategy in all conditions, then the latter strategy can be ignored because it will not affect the solution in any way. A strategy dominates over the other only if it is preferable in all conditions.

1) If all the elements in a row of a pay off matrix are less than or equal to the corresponding elements of another row, then the latter dominates and so former is ignored.

Example: Consider $\begin{bmatrix} 1 & 3 & 4 & 5 \\ -2 & 1 & 4 & 0 \end{bmatrix}$

Here every element of second row is less than or equal to the corresponding elements of I row. Therefore first row dominates and so second row can be ignored.

2) If all the elements in a column of a pay off matrix are greater than or equal to the corresponding elements of a column then the latter dominates and so former is ignored.

Example: Consider $\begin{bmatrix} 2 & 2 \\ -3 & 1 \\ -1 & 1 \\ 4 & 5 \end{bmatrix}$

Here the elements II column are greater than or equal to the corresponding elements of first column. So first column dominates and second column can be ignored.

3) If the linear combination of two or more rows (or columns) dominates a row (or column), then the latter is ignored.

Note (1): If all the elements of a row are less than or equal to average of the corresponding elements of two other rows, then the former is ignored.

Example: Consider $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 1 & 4 \\ 1 & 1 & 0 & -2 \end{bmatrix}$

Here each element of first row is less than or equal to the average of the corresponding elements of second and third rows. Therefore first row can be ignored.

Note (2): If all the elements of a column are greater than or equal to the average of the corresponding elements of two other columns, then the former is ignored.

Example: consider $\begin{bmatrix} 2 & 0 & 2 \\ 3 & -1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$

Here the elements of third column are greater than or equal to the average of the corresponding elements first and second columns. Therefore third column can be ignored.

Principle of dominance is applicable to pure strategy and mixed strategy problems.

Ex. 12: Following is the pay off matrix for players A and B.

$$\begin{array}{cc} & \textbf{Player B} \\ & \begin{array}{ccccc} \textbf{I} & \textbf{II} & \textbf{III} & \textbf{IV} & \textbf{V} \end{array} \\ \textbf{Player A} \quad \begin{array}{c} \textbf{1} \\ \textbf{2} \\ \textbf{3} \\ \textbf{4} \end{array} & \begin{bmatrix} 2 & 4 & 3 & 3 & 4 \\ 5 & 6 & 3 & 7 & 8 \\ 6 & 7 & 9 & 8 & 7 \\ 4 & 2 & 8 & 4 & 3 \end{bmatrix} \end{array}$$

Using dominance property, obtain the optimum strategies for both the players and determine the value of game.

Ans: All elements of column IV are greater than or equal to the corresponding element of column I we eliminate column IV. The resulting matrix is

$$\text{Player B}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{V} \end{array} \\ \begin{bmatrix} 2 & 4 & 3 & 4 \\ 5 & 6 & 3 & 8 \\ 6 & 7 & 9 & 7 \\ 4 & 2 & 8 & 3 \end{bmatrix} \end{array}$$

All elements in row 4 are less than the corresponding elements of Row 3. So we delete the fourth row. So we have

$$\text{Player A}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \text{Player B} \\ \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{V} \end{array} \\ \begin{bmatrix} 2 & 4 & 3 & 4 \\ 5 & 6 & 3 & 8 \\ 6 & 7 & 9 & 7 \end{bmatrix} \end{array}$$

Now all elements of column V are greater than the corresponding elements of column I, we delete column V. The reduced matrix is

$$\text{Player A}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \text{Player B} \\ \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} 2 & 4 & 3 \\ 5 & 6 & 3 \\ 6 & 7 & 9 \end{bmatrix} \end{array}$$

As the elements of Row 1 and Row 2 are less than the corresponding elements of Row 3 we delete both row 1 and row 2 one by one. The resulting matrix is

$$\text{Player A}\quad 3\quad \begin{array}{c} \text{Player B} \\ \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ [6 \quad 7 \quad 9] \end{array}$$

Column I dominates columns II and III. So deleting them we have

$$3\quad \begin{array}{c} \text{I} \\ [6] \end{array}$$

$\therefore$ Optimum Strategies: A will choose 3 and B will choose I.

Value of the game $= 6$

Ex. 13: Solve the following game by the principle of dominance

$$\begin{bmatrix} 8 & 10 & 9 & 14 \\ 10 & 11 & 8 & 12 \\ 13 & 12 & 14 & 13 \end{bmatrix}$$

Ans:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 8 & 10 & 9 & 14 \\ 10 & 11 & 8 & 12 \\ 13 & 12 & 14 & 13 \end{bmatrix} \end{array}$$

B_4 is dominated by B_1. $\therefore$ Eliminate B_4. $\therefore$ The resulting matrix is

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 8 & 10 & 9 \\ 10 & 11 & 8 \\ 13 & 12 & 14 \end{bmatrix} \end{array}$$

Now A_1 is dominated by A_3. So eliminate A_1. The reduced matrix is

$$\begin{array}{c} \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 10 & 11 & 8 \\ 13 & 12 & 14 \end{bmatrix} \end{array}$$

Average of B_2 and B_3 dominates B_1. The reduced matrix is

$$\begin{array}{c} \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{bmatrix} 11 & 8 \\ 12 & 14 \end{bmatrix} \end{array}$$

Row A_3 dominates A_2. So eliminate A_2

$$\text{then,}\quad A_3 \begin{array}{c} \begin{array}{cc} B_2 & B_3 \end{array} \\ [12 \quad 14] \end{array}$$

Column B_2 dominates B_3. So eliminate B_3.

The solution is (A_3, B_2)

$\therefore$ Optimum Strategy for A is A_3 and for B is B_2.

The value of the game is 12

Ex. 14: Solve the game whose pay off matrix is given by.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{bmatrix} \end{array}$$

Ans: Applying principle of dominance B_3 is dominated by B_1: So ignore B_3. The reduced matrix is

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 1 & 7 \\ 6 & 2 \\ 5 & 1 \end{bmatrix} \end{array}$$

Now A_3 is dominated by A_2. So ignore A_3. The resulting matrix is

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix} \end{array}$$

Let p and $1-p$ be the probabilities for A choosing A_1 and A_2. Let q and $1-q$ be the probabilities for B choosing B_1 and B_2. Then

$$p = \frac{d-c}{(a+d)-(b+c)} = \frac{2-6}{(1+2)-(6+7)} = \frac{-4}{3-13} = \frac{-4}{-10} = \frac{2}{5} \text{ and } 1-p = \frac{3}{5}$$

$\therefore$ P (A choosing A_1) = 2/5

P (A choosing A_2) = 3/5

P (A choosing A_3) = 0

$$q = \frac{d-b}{(a+d)-(b+c)} = \frac{2-7}{3-13} = \frac{-5}{-10} = \frac{1}{2} \text{ and } 1-q = \frac{1}{2}$$

$\therefore$ P (B choosing B_1) = $\frac{1}{2}$

P (B choosing B_2) = $\frac{1}{2}$

P (B choosing B_3) = 0

Expected value of the game = $\frac{ad-bc}{(a+d)-(b+c)} = \frac{2-42}{-10} = .4$

$\therefore$ A's mixed strategy = $(\frac{2}{5}, \frac{3}{5}, 0)$

B's mixed strategy = $(\frac{1}{2}, \frac{1}{2}, 0)$

V=4

Ex. 15: Given the pay off matrix for player A, obtain the optimum strategies for both the players and determine the value of the game,

$$\begin{array}{cc} & \textbf{Player B} \\ \textbf{Player A} & \begin{bmatrix} 6 & -3 & 7 \\ -3 & 0 & 4 \end{bmatrix} \end{array}$$

Ans: By applying the principle of dominance we find that strategy B_1 dominates B_3. So B_3 can be eliminated. The reduced matrix is

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} \end{array}$$

$$p = \frac{0--3}{(6+0)-(-3-3)} = \frac{3}{6+6} = \frac{3}{12} = \frac{1}{4} \text{ and } 1-p = \frac{3}{4}$$

$\therefore$ $P(A_1) = \frac{1}{4}, P(A_2) = \frac{3}{4}$

$$q = \frac{0--3.}{(6+0)-(-3-3)} = \frac{3}{6+6} = \frac{3}{12} = \frac{1}{4} \text{ and } 1-q = \frac{3}{4}$$

$\therefore$ $P(B_1) = \frac{1}{4}, P(B_2) = \frac{3}{4}, P(B_3) = 0$

$\therefore$ A's mixed strategy = $(\frac{1}{4}, \frac{3}{4})$

B's mixed strategy = $(\frac{1}{4}, \frac{3}{4}, 0)$

Value of the game = $\frac{ad-bc}{(a+d)-(b+c)} = \frac{0-9}{12} = -\frac{3}{4}$

(2) Graphic Method

If the pay off matrix is of order $2 \times n$ or $m \times 2$, graphic method can be applied.

Solution for $2 \times n$ Games

Here A has only two strategies and B has n strategies. Let the strategies of A be A_1 and A_2. Let the strategies of B be $B_1, B_2, \ldots\ldots B_n$. Let p and $1-p$ be the probabilities with which player A uses his pure strategies. Player A will select the value of 'p' which to maximises the minimum expected pay off.

For this, we draw lines representing the strategies of B, say $B_1, B_2, \ldots\ldots B_n$. The lower boundary of these lines will give the minimum expected pay off and the highest point on this lower boundary will give the maximum expected pay off of player A and hence the optimum value of p. The lines which pass through this maximin point represent the optimum strategies of B. Thus we get a 2×2 matrix. From this matrix, we can find p and $1-p$ and expected value of the game.

Ex. 16: Solve the following game problem.

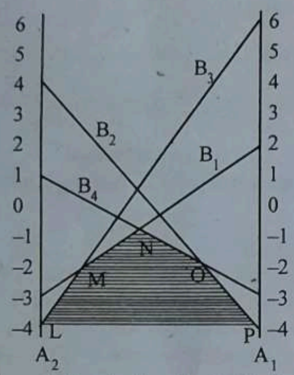
$$\begin{array}{cc} & \textbf{Player B} \\ \begin{array}{c} \textbf{Player} \\ \textbf{A}_1 \\ \textbf{A}_2 \end{array} & \begin{array}{c} \begin{array}{ccccc} \textbf{B}_1 & \textbf{B}_2 & \textbf{B}_3 & \textbf{B}_4 & \textbf{B}_5 \end{array} \\ \begin{bmatrix} 2 & -4 & 6 & -3 & 5 \\ -3 & 4 & -4 & 1 & 0 \end{bmatrix} \end{array} \end{array}$$

Ans: By dominance property, B_5 dominates B_1. B_5 can be eliminated. So the pay off matrix is

$$\begin{array}{cc} \begin{array}{c} \text{Player} \\ A_1 \\ A_2 \end{array} & \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 2 & -4 & 6 & -3 \\ -3 & 4 & -4 & 1 \end{bmatrix} \end{array} \end{array}$$

This is $2 \times n$ matrix, where $n = 4$.

Steps: First draw two vertical lines A_2 and A_1. On both A_2 and A_1, mark points. Then Draw lines representing B_1, B_2, B_3, B_4. To draw the line B_1: Take -3 on A_2 and 2 on A_1 and join. Similarly draw B_2, B_3, B_4.



Lower Area bounded by the lines B_1, B_2, B_3, B_4, is LMNOP.

$\therefore$ Of this, 'N' is the required point as it is the maximum point. N is the intersection of B_1 and B_4. So ignoring B_2 and B_3. Thus the reduced matrix is obtained by avoiding B_2 and B_3.

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

A's optimal strategy for choosing A_1 and A_2 is given by:

$p = \frac{1--3}{(1+2)-(-3-3)} = \frac{4}{9}$ and $1-p = \frac{5}{9}$

B's stratgy for choosing B_1 and B_4 is given by:

$q = \frac{1--3}{(1+2)-(-3-3)} = \frac{4}{9}$ and $1-q = \frac{5}{9}$

$\therefore$ $V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{2-9}{3-(-6)} = \frac{-7}{9}$

$\therefore$ Solution:

A's strategy $= \left(\frac{4}{9}, \frac{5}{9}\right)$

B's strategies are $\left(\frac{4}{9}, 0, 0, \frac{5}{9}\right)$

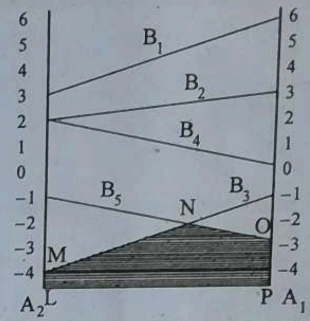
Value of the game $= \frac{-7}{9}$

Ex. 17: Use graphic method to solve the following game

$$\begin{array}{c c c} & & \text{Player Y} \\ & & \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \text{Player X} & \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 6 & 3 & -1 & 0 & -3 \\ 3 & 2 & -4 & 2 & -1 \end{bmatrix} \end{array}$$

Ans: Draw A_1 and A_2 (vertical lines)

Mark points from −3 to 6 on A_1 and A_2. Join the points on A_1 and A_2 to get the lines B_1, B_2, B_3, B_4, B_5.



Lower area bounded by the lines B_1, B_2, B_3, B_4, B_5 is LMNOP. The maximin point of the area is N. N is the point of intersection of B_3 and B_5.

So ignoring B_1, B_2 and B_4 reduced matrix is

$$\begin{array}{c c} & \begin{array}{cc} B_3 & B_5 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -1 & -3 \\ -4 & -1 \end{bmatrix} \end{array}$$

X's optimal strategies are given by

$p = \frac{d-c}{(a+d)-(b+c)} = \frac{-1-(-4)}{(-1-1)-(-4-3)} = \frac{3}{-2+7} = \frac{3}{5}$ and $1-p = 2/5$

Y's strategies are given by

$q = \frac{d-b}{(a+d)-(b+c)} = \frac{-1-(-3)}{(-1-1)-(-4-3)} = \frac{2}{-2+7} = \frac{2}{5}$ and $1-q = 3/5$

value of the game $= \frac{ad-bc}{(a+d)-(b+c)} = \frac{1-12}{5} = \frac{-11}{5}$

Solution to the game is

X's strategy (3/5, 2/5)

Y's strategy (0, 0, 2/5, 0, 3/5)

Value of the game $= \frac{-11}{5}$

Solution for $m \times 2$ matrix

Here B has only two strategies say B_1 and B_2. A has m strategies say $A_1, A_2, \ldots\ldots\ldots, A_m$. We draw two vertical lines B_1 and B_2. Draw the lines $A_1, A_2, \ldots\ldots\ldots, A_m$ by joining the respective points on B_1 and B_2. The upper boundary of these lines gives the maximum of the loss and lowest point of this upper boundary is minimax point. The lines passing through the minimax point represent the choices of A. Thus we get a 2×2 matrix. Solving this matrix, we get the strategies A and B.

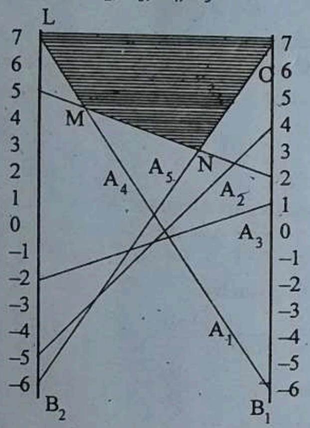
Ex. 18: Solve graphically the game whose pay off matrix is given below:

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \text{Player B} \\ \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -6 & 7 \\ 4 & -5 \\ 1 & -2 \\ 2 & 5 \\ 7 & -6 \end{bmatrix} \end{array}$$

Ans: Draw two vertical lines B_2 and B_1. Mark the values on B_1 and B_2. Then draw the lines A_1, A_2, A_3, A_4, A_5.

To draw A_1, take 7 on B_2 and -6 on B_1 and join them

Similarly draw the lines A_2, A_3, A_4, A_5.



The upper boundary is the polygon LMNO. Lowest point on the upper boundary is N. N is the point of intersection of the lines A_4 and A_5.

Thus the game reduces to

$$\begin{array}{c} \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -2 & 5 \\ 7 & -6 \end{bmatrix} \end{array}$$

$$p = \frac{d-c}{(a+d)-(b+c)} = \frac{-6-7}{-8-12} = \frac{-13}{-20} = \frac{13}{20} \text{ and } 1-p = \frac{7}{20}$$

$$q = \frac{d-b}{(a+d)-(b+c)} = \frac{-6-5}{-8-12} = \frac{-11}{-20} = \frac{11}{20} \text{ and } 1-q = \frac{9}{20}$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{12-35}{-8-12} = \frac{-23}{-20} = \frac{23}{20}$$

The solution is

Strategies of player A are $(0, 0, 0, \frac{13}{20}, \frac{7}{20})$

Strategies of B are $(\frac{11}{20}, \frac{9}{20})$

Value of the game = $\frac{23}{20}$

Ex. 19: Solve the following game problem

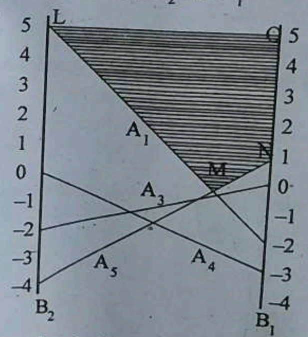
$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} B \\ \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -2 & 5 \\ -5 & 3 \\ 0 & -2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \end{array}$$

Ans: Elements of A_2 are smaller than elements of A_1.

$\therefore$ A_2 is ignored. Reduced matrix is

$$\begin{array}{c} \\ A_1 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -2 & 5 \\ 0 & -2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \end{array}$$

Draw two vertical lines B_2 and B_1. Draw the lines A_1, A_3, A_4 and A_5 by joining the respective values on B_2 and B_1.



The upper boundary in the polygon is LMNO. The lowest point on the upper boundary is M. M is the point of intersection of the lines A_1 and A_5. Therefore other strategies of A are ignored.

Reduced matrix is $\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & -2 & 5 \\ A_5 & 1 & -4 \end{array}$

$$p = \frac{d-c}{(a+d)-(b+c)} = \frac{-4-1}{-6-6} = \frac{-5}{-12} = \frac{5}{12} \text{ and } 1-p == \frac{7}{12}$$

$$q = \frac{d-b}{(a+d)-(b+c)} = \frac{-4-5}{-6-6} = \frac{-9}{-12} = \frac{3}{4} \text{ and } 1-q = \frac{1}{4}$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{8-5}{-6-6} = \frac{3}{-12} = -\frac{1}{4}$$

Solution:

A's strategy $= [\frac{5}{12}, 0, 0, 0, \frac{7}{12}]$

B's strategy [3/4, 1/4]

Value of the game $= -1/4$

Reducing the game to a Linear Programming Problem

Every finite two person zero sum game can be expressed as a Linear Programming Problem. For a '$m \times n$' rectangular game where m or n is greater than 2, Linear Programming technique can be applied.

Consider a game problem of '$m \times n$' pay off matrix given by

$$\begin{bmatrix} a_{11} & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} \cdots\cdots & a_{2n} \\ a_{m_1} & a_{m_2} \cdots\cdots & a_{m_n} \end{bmatrix}$$

Let the course of action of A be $A_1, A_2, \ldots\ldots A_m$ and those of B be $B_1, B_2, \ldots\ldots B_n$. Let $p_1, p_2, \ldots\ldots p_m$ be the probabilities of A choosing the strategies $A_1, A_2, \ldots\ldots A_m$. Similarly let $q_1, q_2, \ldots\ldots q_n$ be the probabilities for B choosing the strategies $B_1, B_2, \ldots\ldots B_n$.

Then $p_1 + p_2 + = \ldots\ldots p_m = 1$ and $q_1 + q_2 + \ldots\ldots q_n = 1$.

Let V be the value of the game.

Then the problem is to find the values of $p_1, p_2, \ldots\ldots p_m$ which maximise V or minimise $\frac{1}{V}$ subject to

$$a_{11} p_1 + a_{21} p_2 + \ldots\ldots + a_{m1} p_m \geq V$$
$$a_{12} p_1 + a_{22} p_2 + \ldots\ldots + a_{m2} p_m \geq V$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{1n} p_1 + a_{2n} p_2 + \ldots\ldots a_{mn}\ p_m \geq V$$
$$p_1,\ p_2,\ \ldots\ldots\ p_m \geq 0$$

Dividing the inequalities by V, and putting $\frac{p_1}{V} = x_1, \frac{p_2}{V} = x_2, \ldots \frac{p_m}{V} = x_m$.

The inequalities become,

$$a_{11} x_1 + a_{21} x_2 + \ldots\ldots\ldots + a_{m1} x_m \geq 1$$
$$a_{12} x_1 + a_{22} x_2 + \ldots\ldots\ldots + a_{m2} x_m \geq 1$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{1n} x_1 + a_{2n} x_2 + \ldots\ldots\ldots + a_{mn} x_m \geq 1$$

Further,

$x_1 + x_2 + \ldots + x_m$ becomes $\frac{P_1 + P_2 + \ldots\ldots P_m}{V} = \frac{1}{V}$

[as $p_1 + p_2 + \ldots\ldots + p_m = 1$]

$\therefore$ Minimising $\frac{1}{V}$, means minimisng $x_1 + x_2 + \ldots\ldots x_m$ (say z)

Thus, We have the L. P. P.

Minimise: $Z = x_1 + x_2 + \ldots\ldots x_m$.

Subject to

$$a_{11} x_1 + a_{21} x_2 + \ldots\ldots + a_{m1} x_m \geq 1$$
$$a_{12} x_1 + a_{22} x_2 + \ldots\ldots + a_{m2} x_m \geq 1$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{1n} x_1 + a_{2n} x_2 + \ldots\ldots + a_{mn} x_m \geq 1$$
$$x_1, x_2, \ldots\ldots x_m \geq 0$$

Similarly if we consider the player B, the L. P. P. is

Maximise $Z' = y_1 + y_2 + \ldots\ldots y_n$.

Subject to

$$a_{11} y_1 + a_{12} y_2 + \ldots\ldots + a_{1n} y_n \leq 1$$
$$a_{21} y_1 + a_{22} y_2 + \ldots\ldots + a_{2n} y_n \leq 1$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1} y_1 + a_{m2} y_2 + \ldots\ldots + a_{mn} y_n \leq 1$$
$$y_1, y_2, \ldots\ldots y_m \geq 0$$

(It can be seen that the second LPP is the dual of the first.)

Value of the game $= V = \frac{1}{y_1 + y_2 + \ldots\ldots y_n}$ or $\frac{1}{x_1 + x_2 + \ldots\ldots x_n}$

Ex. 20: Express the following game problem into an LPP.

$$\begin{array}{cc|ccc} & & & \textbf{Player B} & \\ & & B_1 & B_2 & B_3 \\ \hline \textbf{Player A} & A_1 & 6 & 2 & 7 \\ & A_2 & 1 & 9 & 3 \end{array}$$

Ans: In respect of A, the L. P. P is

Minimise $Z = x_1 + x_2$

Subject to
$$6x_1 + x_2 \geq 1$$
$$2x_1 + 9x_2 \geq 1$$
$$7x_1 + 3x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

On the part of B the L. P. P. is

Maximise $Z' = y_1 + y_2 + y_3$

Subject to $6y_1 + 2y_2 + 7y_3 \le 1$

$y_1 + 9y_2 + 3y_3 \le 1$

$y_1, y_2, y_3 \ge 0$

Value of the game is $V = \frac{1}{y_1+y_2+y_3}$ or $\frac{1}{x_1+x_2+x_3}$

Strategies of B are q_1, q_2 and q_3 where $q_1 = y_1 v$, $q_2 = y_2 v$ and $q_3 = y_3 v$.

Strategies of A are p_1 and p_2 where $p_1 = x_1 v$ and $p_2 = x_2 v$.

Ex. 16: For the following pay-off table, transform the zero sum game into an equivalent linear programming problem and solve it by simplex method.

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \text{Player B} \\ \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 9 & 1 & 4 \\ 0 & 6 & 3 \\ 5 & 2 & 8 \end{bmatrix} \end{array}$$

Ans: The problem of player A is to determine x_1, x_2, x_3 which

Minimise $Z = x_1 + x_2 + x_3$

Subject to the constraints:

$$9x_1 + 5x_2 \ge 1$$
$$x_1 + 6x_2 + 2x_3 \ge 1$$
$$4x_1 + 3x_2 + 8x_3 \ge 1$$
$$x_1, x_2, x_3 \ge 0$$

The problem of player B is to determine y_1, y_2, y_3 which maximise

$Z' = y_1 + y_2 + y_3$

Subject to the constraints:

$$9y_1 + y_2 + 4y_3 \le 1$$
$$6y_2 + 3y_3 \le 1$$
$$5y_1 + 2y_2 + 8y_3 \le 1$$
$$y_1, y_2, y_3 \ge 0$$

Let us now solve the problem of player B,

By introducing the slack variables s_1, s_2, s_3 respectively and then applying simplex method.

I Simplex Table

B	C	y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Mini ratio
s_1	0	1	$\boxed{9}$	1	4	1	0	0	$\frac{1}{9}$ ←
s_2	0	1	0	6	3	0	1	0	–
s_3	0	1	5	2	8	0	0	1	$\frac{1}{5}$
		Δ_j	–1	–1	–1	0	0	0	

II Simplex table

B	C	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Mini ratio
y_1	1	$\frac{1}{9}$	1	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	0	0	1
s_2	0	1	0	$\boxed{6}$	3	0	1	0	$\frac{1}{6}$ ←
s_3	0	$\frac{4}{9}$	0	$\frac{13}{9}$	$\frac{52}{9}$	$-\frac{5}{9}$	0	1	$\frac{4}{13}$
		Δ_j	0	$-\frac{8}{9}$	$-\frac{5}{9}$	$-\frac{1}{9}$	0	0	

III Simplex table

B	C	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Mini ratio
y_1	1	$\frac{5}{54}$	1	0	$\frac{7}{18}$	$\frac{1}{9}$	$-\frac{1}{54}$	0	$\frac{5}{21}$
y_2	1	$\frac{1}{9}$	0	1	$\frac{1}{2}$	0	$\frac{1}{6}$	0	$\frac{2}{9}$
s_3	0	$\frac{11}{54}$	0	0	$\boxed{\frac{91}{18}}$	$-\frac{5}{9}$	$-\frac{13}{54}$	1	$\frac{11}{273}$ ←
		Δ_j	0	0	$-\frac{1}{9}$	$\frac{1}{9}$	$-\frac{4}{27}$	0	

IV Simplex table

B	C	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3
y_1	1	$\frac{21}{273}$	1	0	0	$\frac{4}{91}$	0	$-\frac{7}{91}$
y_2	1	$\frac{40}{273}$	0	1	0	$-\frac{5}{91}$	$\frac{52}{273}$	$-\frac{9}{91}$
y_3	1	$\frac{11}{273}$	0	0	1	$-\frac{10}{91}$	$-\frac{13}{273}$	$\frac{18}{91}$
		Δ_j	0	0	0	$-\frac{9}{91}$	$-\frac{13}{91}$	$-\frac{2}{91}$

The simplex table gives the values of $y_1 = \frac{21}{273}$, $y_2 = \frac{40}{273}$, $y_3 = \frac{11}{273}$ and $x_1 = \frac{9}{91}$, $x_2 = \frac{13}{91}$, $x_3 = \frac{2}{91}$

Expected value of the game $= \frac{1}{y_1+y_2+y_3} = \frac{1}{\left(\frac{21+40+11}{273}\right)} = \frac{273}{72} = \frac{91}{24}$

The optimum strategies of player B are given by

$q_1 = y_1 v = \frac{21}{273} \times \frac{91}{24} = \frac{7}{24}$

$q_2 = y_2 v = \frac{40}{273} \times \frac{91}{24} = \frac{5}{9}$

$q_1 = y_3 v = \frac{11}{273} \times \frac{91}{24} = \frac{11}{72}$

The optimum strategy for A

$p_1 = x_1 v = \frac{9}{91} \times \frac{91}{24} = \frac{3}{8}$

$p_2 = x_2 v = \frac{13}{91} \times \frac{91}{24} = \frac{13}{24}$

$p_1 = x_3 v = \frac{2}{91} \times \frac{91}{24} = \frac{1}{12}$

Hence the optimum solution of the given game is
Strategies of A = [3/8, 13/24, 1/12], and
Strategies of B = [7/24, 5/9, 11/72]

Value of the game = $\frac{91}{24}$

Ex. 21: Solve the following game problem by L. P. technique

$$\begin{array}{c c} & B \\ A & \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \end{array}$$

Ans: The L. P. P. is

Min: $Z = x_1 + x_2 + x_3$
S.t $x_1 + 3x_2 + 6x_3 \geq 1$
$-x_1 + 5x_2 + 2x_3 \geq 1$
$3x_1 - 3x_2 - 2x_3 \geq 1$
$x_1, x_2, x_3 \geq 0$

or

Max: $Z' = y_1 + y_2 + y_3$
S.t $y_1 - y_2 + 3y_3 \leq 1$
$3y_1 + 5y_2 - 3y_3 \leq 1$
$6y_1 + 2y_2 - 2y_3 \leq 1$
$y_1, y_2, y_3 \geq 0$

Let us solve the second problem

constraints are $y_1 - y_2 + 3y_3 + S_1 = 1$
$3y_1 + 5y_2 - 3y_3 + S_2 = 1$
$6y_1 + 2y_2 - 2y_3 + S_3 = 1$

Objective function is $Z' = y_1 + y_2 + y_3 + 0S_1 + 0S_2 + 0S_3$

Starting Simplex Table

B	C_B	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	θ
s_1	0	1	1	-1	3	1	0	0	-1
s_2	0	1	3	$\boxed{5}$	-3	0	1	0	1/5 ←
s_3	0	1	6	2	-2	0	0	1	½
		Δ_j	-1	-1	-1	0	0	0	
				↑					

II Simplex Table

B	C_B	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	θ
s_1	0	6/5	8/5	0	$\boxed{12/5}$	1	1/5	0	1/2 ←
y_2	1	4/5	3/5	1	-3/5	0	1/5	0	–1/3
s_3	0	3/5	24/5	0	4/5	0	-2/5	1	3/4
		Δ_j	-2/5	0	-8/5	0	1/5	0	
					↑				

III Simplex Table

B	C_B	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3
y_3	1	1/2	2/3	0	1	5/12	1/12	0
y_2	1	1/2	1	1	0	1/4	1/4	0
s_3	0	1/5	16/15	0	0	-1/3	-7/15	1
		Δ_j	2/3	0	0	2/3	1/3	0

Solution is optimum.

Solution is

$y_1 = 1/2$, $y_2 = 1/2$ $y_3 = 0$ and

$Z' = (1 \times 1/2) + (1 \times 1/2) + (1 \times 0) = 1/2 + 1/2 = 1$

Solution of the primal is $x_1 = 2/3$, $x_2 = 1/3$, $x_3 = 0$ and

$Z = (1 \times \frac{2}{3}) + (1 \times \frac{1}{3}) + (1 \times 0) = \frac{2}{3} + \frac{1}{3} = 1$

Value of the game = $\frac{1}{Z}$ or $\frac{1}{Z'} = 1$

Optimum strategies of player B

$q_1 = y_1 v = 1/2 \times 1 = ½$
$q_2 = y_2 v = 1/2 \times 1 = ½$
$q_3 = y_3 v = 0 \times 1 = 0$

Optimum strategies of player A

$p_1 = x_1 v = \frac{2}{3} \times 1 = \frac{2}{3}$
$p_2 = x_2 v = \frac{1}{3} \times 1 = \frac{1}{3}$
$p_3 = x_3 v = 0 \times 1 = 0$

Solution to the game:

A' strategy (2/3, 1/3, 0)
B's strategy (1/2, 1/2, 0)
v = 1

Summary of the methods applied.

1. Try for the saddle point If saddle point exists, the strategies of A and B are those corresponding to the saddle point.
2. When saddle point does not exist, apply dominance rule (if possible) and find the reduced matrix.
3. When the reduced matrix (or given matrix) is
   (a) 2×2 matrix – apply probability method
   (b) $2 \times m$ or $n \times 2$, matrix – apply Graphic method
   (c) $m \times n$ matrix – apply L. P. Technique.

Non - zero sum Game: Games with less than complete conflict of interest are called non-zero sum games. In the business activities such situations are common. There, sum of the gains or losses is not equal to zero. That is, the gain of one competitor may not be completely at the expense of the other competitors. For example, when firm X advertises, some customers of Y switch over to X, but there may be new customers buying the product of X and Y. Therefore both the firms may gain, though their share in the total gain may not be equal.