Cut Property

Let an undirected graph $G = (V, E)$ with edge weights be given.

A *tree* in G is a subgraph $T = (V', E')$ which is connected and contains no cycles.

A *spanning tree* is one reaching all the vertices: $V' = V$.

In the rest of this discussion we will equate tree T with it's set of edges E'. Note that E' determines T since it is connected, i.e. $V' = \{u \in V : (u, v) \in E' \text{ for some } v \in V\}$.

The *weight* of a tree (or of any set of edges) is the sum of its edge weights.

A *minimal spanning tree* (MST) is a spanning tree whose weight is not greater than the weight of any other spanning tree of G.

The *cut* defined by a set of vertices S is the set of all edges that cross from S to V-S:

$$\text{cut}(S) = \{(u, v) \in E : u \in S, v \in V - S\}.$$

A *light* (or lightest) edge in a set of edges is one whose weight is no greater than that of any other edge of the set.

X = set of edges already in S

If X is a set of edges, a set of vertices S is said to *respect* X if $\text{cut}(S) \cap X = \phi$. In other words, no edge of X crosses from S to $V - S$.

Cut Property. Let X be a set edges that is a subset of some MST T. Let S be a set of vertices whose cut respects X and let (u, v) be a light edge of $\text{cut}(S)$. Then there is a MST containing $X \cup \{(u, v)\}$.

In other words, a light edge of $\text{cut}(S)$ can be added to X and it will still be a subset of some MST.

Proof. If T contains (u, v) we are done. If not, adjoin (u, v) to T forming a cycle within $T \cup \{(u, v)\}$. This cycle must contain at least one other edge (w, z) of $\text{cut}(S)$. Then $T' = T \cup \{u, v\} \cap \{(w, z)\}$ is a spanning tree of weight no greater than that of T, so T' is a MST. qed.

add (u,v) & remove (w,z)