Heap Algorithms

PARENT(A, i)

// *Input*: A: an array representing a heap, i: an array index
// *Output*: The index in A of the parent of i
// *Running Time*: $O(1)$
if $i == 1$ **return** NULL
return $\lfloor i/2 \rfloor$

LEFT(A, i)

// *Input*: A: an array representing a heap, i: an array index
// *Output*: The index in A of the left child of i
// *Running Time*: $O(1)$
if $2 * i \leq$ *heap-size*$[A]$
  return $2 * i$
else return NULL

RIGHT(A, i)

// *Input*: A: an array representing a heap, i: an array index
// *Output*: The index in A of the right child of i
// *Running Time*: $O(1)$
if $2 * i + 1 \leq$ *heap-size*$[A]$
  return $2 * i + 1$
else return NULL

MAX-HEAPIFY(A, i)

// *Input*: A: an array where the left and right children of i root heaps (but i may not), i: an array index
// *Output*: A modified so that i roots a heap
// *Running Time*: $O(\log n)$ where $n =$ *heap-size*$[A] - i$
$l \leftarrow$ LEFT(i)
$r \leftarrow$ RIGHT(i)
if $l \leq$ *heap-size*$[A]$ and $A[l] > A[i]$
  largest $\leftarrow l$
else *largest* $\leftarrow i$
if $r \leq$ *heap-size*$[A]$ and $A[r] < A[largest]$
  largest $\leftarrow r$
if *largest* $\neq i$
  exchange $A[i]$ and $A[largest]$
  MAX-HEAPIFY(A, LARGEST)

BUILD-MAX-HEAP(A)

// *Input*: A: an (unsorted) array
// *Output*: A modified to represent a heap.
// *Running Time*: $O(n)$ where $n =$ *length*$[A]$
heap-size$[A] \leftarrow$ *length*$[A]$
for $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
  MAX-HEAPIFY(A, i)

Heap-Increase-Key(A, i, key)

```
// Input: A: an array representing a heap, i: an array index, key: a new key greater than A[i]
// Output: A still representing a heap where the key of A[i] was increased to key
// Running Time: O(log n) where n = heap-size[A]
if key < A[i]
    error("New key must be larger than current key")
A[i] ← key
while i > 1 and A[Parent(i)] < A[i]
    exchange A[i] and A[Parent(i)]
    i ← Parent(i)
```

Heap-Sort(A)

```
// Input: A: an (unsorted) array
// Output: A modified to be sorted from smallest to largest
// Running Time: O(n log n) where n = length[A]
Build-Max-Heap(A)
for i = length[A] downto 2
    exchange A[1] and A[i]
    heap-size[A] ← heap-size[A] − 1
    Max-Heapify(A, 1)
```

Heap-Extract-Max(A)

```
// Input: A: an array representing a heap
// Output: The maximum element of A and A as a heap with this element removed
// Running Time: O(log n) where n = heap-size[A]
max ← A[1]
A[1] ← A[heap-size[A]]
heap-size[A] ← heap-size[A] − 1
Max-Heapify(A, 1)
return max
```

Max-Heap-Insert(A, key)

```
// Input: A: an array representing a heap, key: a key to insert
// Output: A modified to include key
// Running Time: O(log n) where n = heap-size[A]
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
Heap-Increase-Key(A[heap-size[A]], key)
```

1 Overview

- Overview of Heaps
- Heap Algorithms (Group Exercise)
- More Heap Algorithms!
- Master Theorem Review

2 Heap Overview

Things we can do with heaps are:

- insert $\log N$
- max $O(1)$
- extract_max $\log N$
- increase_key $\log N$
- build them $N \log N$
- sort with them $N \log N$

(Max-)Heap Property For any node, the keys of its children are less than or equal to its key.

3 Heap Algorithms (Group Exercise)

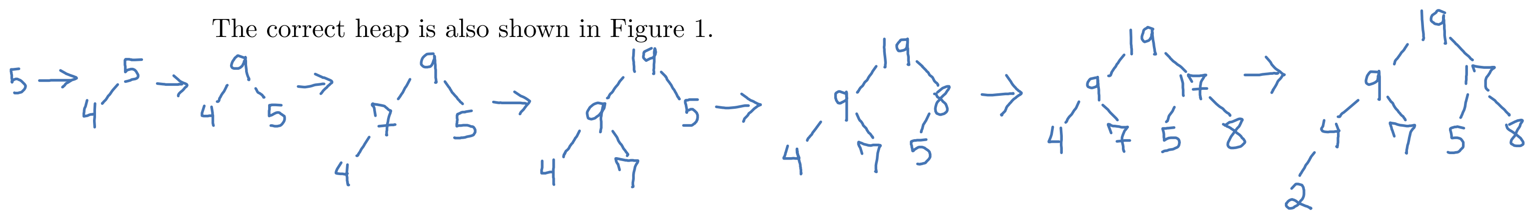
We split into three groups and took 5 or 10 minutes to talk. Then each group had to work their example algorithm on the board.

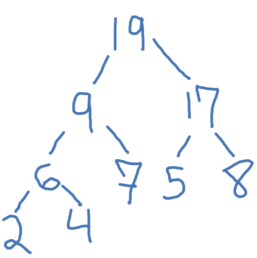
Group 1: Max-Heapify and Build-Max-Heap

Given the array in Figure 1, demonstrate how Build-Max-Heap turns it into a heap. As you do so, make sure you explain:

- How you visualize the array as a tree (look at the Parent and Child routines).
- The Max-Heapify procedure and why it is $O(\log(n))$ time.
- That early calls to Max-Heapify take less time than later calls.

The correct heap is also shown in Figure 1.





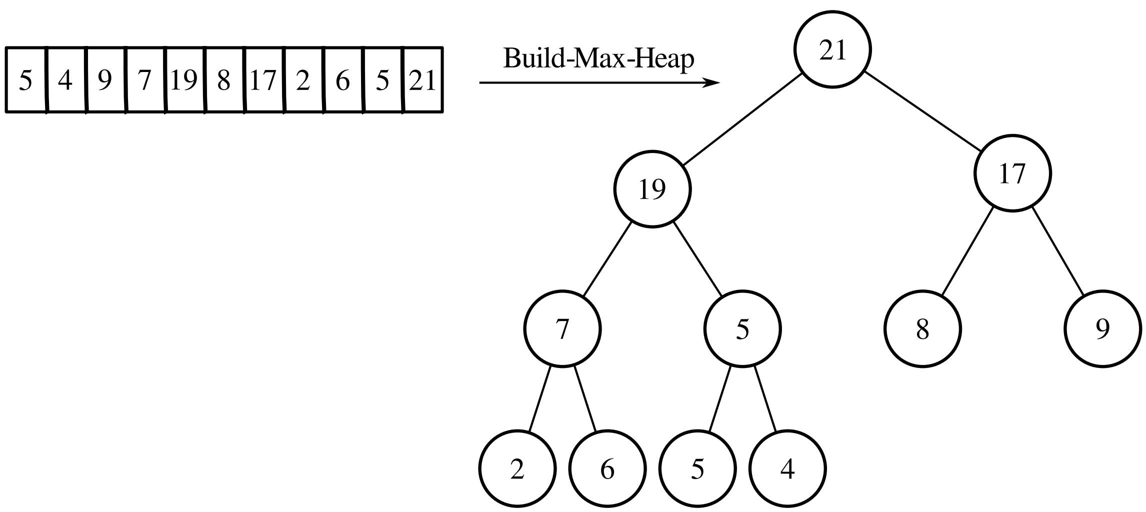


Figure 1: The array to sort and the heap you should find.

Group 2: Heap-Increase-Key

For the heap shown in Figure 2 (which Group 1 will build), show what happens when you use Heap-Increase-Key to increase key 2 to 22. Make sure you argue why what you're doing is $O(\log n)$. (Hint: Argue about how much work you do at each level)

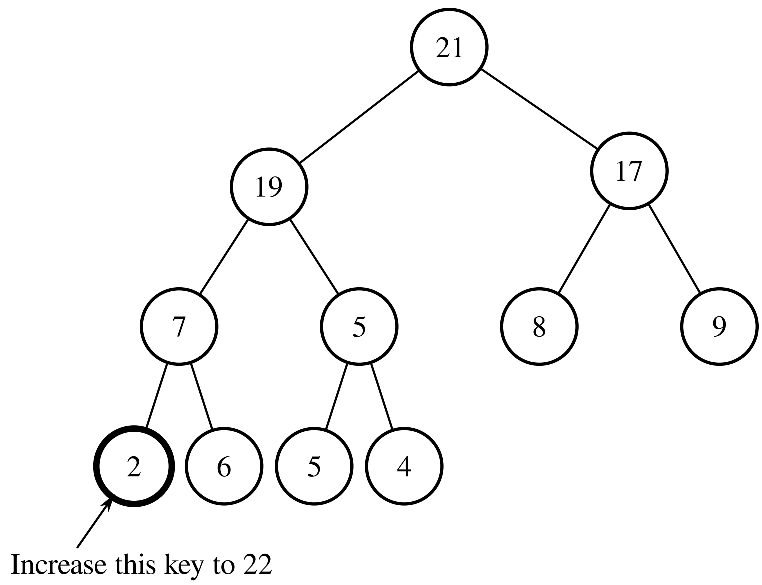


Figure 2: The heap on which to increase a key. You should increase the key of the bottom left node (2) to be 22.

Group 3: Heap-Sort

Given the heap shown in Figure 3 (which Groups 1 and 2 will build for you), show how you use it to sort. You do not need to explain the Max-Heapify or the Build-Max-Heap routine, but you should make sure you explain why the runtime of this algorithm is $O(n \log n)$. Remember the running time of Max-Heapify is $O(\log n)$.

Keep removing the max from the heap until its empty $O(N \log N)$

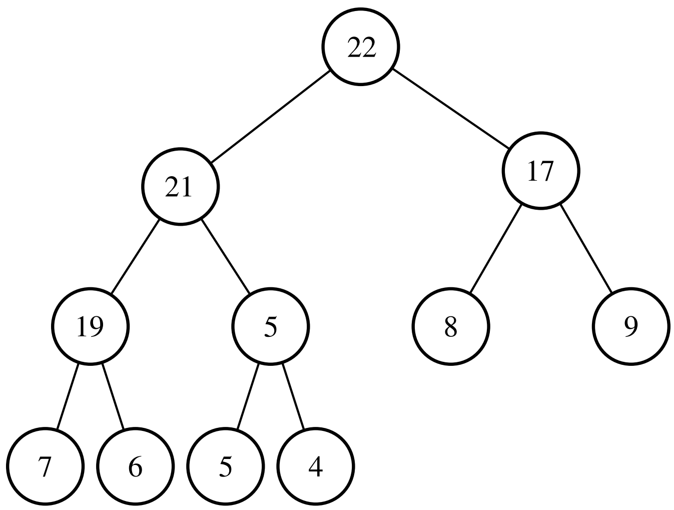


Figure 3: Sort this heap.

4 More Heap Algorithms

Note HEAP-EXTRACT-MAX and MAX-HEAP-INSERT procedures since we didn't discuss them in class:

HEAP-EXTRACT-MAX(A)
$max \leftarrow A[1]$
$A[1] \leftarrow A[heap\text{-}size[A]]$
$heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
MAX-HEAPIFY($A, 1$)
return max

MAX-HEAP-INSERT(A, key)
$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow -\infty$
HEAP-INCREASE-KEY($A[heap\text{-}size[A]], key$)

5 Running Time of BUILD-MAX-HEAP

Trivial Analysis: Each call to MAX-HEAPIFY requires $\log(n)$ time, we make n such calls $\Rightarrow O(n \log n)$.

Tighter Bound: Each call to MAX-HEAPIFY requires time $O(h)$ where h is the height of node i. Therefore running time is

$$\begin{aligned} \sum_{h=0}^{\log n} \underbrace{\frac{n}{2^h + 1}}_{\text{Number of nodes at height } h} \times \underbrace{O(h)}_{\text{Running time for each node}} &= O\left(n \sum_{h=0}^{\log n} \frac{h}{2^h}\right) \\ &= O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h}\right) \\ &= O(n) \qquad (1) \end{aligned}$$

Note $\sum_{h=0}^{\infty} h/2^h = 2$.

6 Proving BUILD-MAX-HEAP Using Loop Invariants

(We didn't get to this in this week's recitation, maybe next time).

Loop Invariant: Each time through the **for** loop, each node greater than i is the root of a max-heap.

Initialization: At the first iteration, each node larger than i is at the root of a heap of size 1, which is trivially a heap.

Maintainance: Since the children of i are larger than i, by our loop invariant, the children of i are roots of max-heaps. Therefore, the requirement for MAX-HEAPIFY is satisfied and, at the end of the loop, index i also roots a heap. Since we decrement i by 1 each time, the invariant holds.

Termination: At termination, $i = 0$ so $i = 1$ is the root of a max-heap and therefore we have created a max-heap.

Discussion: What is the loop invariant for HEAP-SORT? (All keys greater than i are sorted).

Initialization: Trivial.

Maintainance: We always remove the largest value from the heap. We can call MAX-HEAPIFY because we have shrunk the size of the heap so that the root's children are root's of good heaps (although the root is not the root of a good heap).

Termination: $i = 0$

7 Master Theorem Review: More Examples

TRAVERSE-TREE(T)

if *left-child*(*root*[T]) == NULL and *right-child*(*root*[T]) == NULL **return**
output *left-child*(*root*[T]), *right-child*(*root*[T])
TRAVERSE-TREE(*right-child*(*root*[T]))
TRAVERSE-TREE(*left-child*(*root*[T]))

Recurrence is $T = 2T(n/2) + O(1)$. $a = 2, b = 2, n^{\log_b(a)} = n, f(n) = 1$. Master Theorem Case 1, Running Time $O(1)$.

MULTIPLY(x, y)

$n \leftarrow \max(|x|, |y|)$ // $|x|$ *is size of* x *in bits*
if $n = 1$ **return** xy
$x_L \leftarrow x[1 : n/2]$, $x_R \leftarrow x[n/2 + 1 : n]$, $y_L \leftarrow y[1 : n/2]$, $y_R \leftarrow y[n/2 + 1 : n]$
$P_1 = \text{MULTIPLY}(x_L, y_L)$
$P_2 = \text{MULTIPLY}(x_R, y_R)$
$P_3 = \text{MULTIPLY}(x_L + x_R, y_L + y_R)$
return $2^n P_1 + 2^{n/2}(P_3 - P_1 - P_2) + P_2$

Recurrence Relation: $T(n) = 3T(n/2) + O(n)$ (Note: Addition takes linear time in number of bits). $a = 3, b = 2, n^{\log_b(a)} = n^{\log_3(2)}$, $f(n) = O(n)$, Case 1 of Master Theorem, $O(n^{\log_3(2)})$

MatrixMultiply(X, Y)

$n \leftarrow sizeof(X)$ // *Assume X and Y are the same size and square*
if $n = 1$, return XY
// *Split X and Y into four quadrants:*
$A \leftarrow UpperLeft(X),\ B \leftarrow UpperRight(X),\ C \leftarrow LowerLeft(X),\ D \leftarrow LowerRight(X)$
$E \leftarrow UpperLeft(Y),\ F \leftarrow UpperRight(Y),\ G \leftarrow LowerLeft(Y),\ H \leftarrow LowerRight(Y)$
$UL \leftarrow$ MatrixMultiply(A, E) + MatrixMultiply(B, G)
$UR \leftarrow$ MatrixMultiply(A, F) + MatrixMultiply(B, H)
$LL \leftarrow$ MatrixMultiply(C, E) + MatrixMultiply(D, G)
$LR \leftarrow$ MatrixMultiply(C, F) + MatrixMultiply(D, H)
return matrix with UL as upper left quadrant, UR as upper right, LL as lower left, LR as lower right.

Recurrence Relation: $T(n) = 8T(n/2) + O(n^2)$. $a = 8, b = 2, n^{\log_b(a)} = n^3, f(n) = n^2$. Case 1 of the Master Theorem, $O(n^3)$.