# Proof of Kruskal's Algorithm

(Proof adapted from Goodaire & Parmenter's *Discrete Mathematics with Graph Theory.*)

**Theorem.** After running Kruskal's algorithm on a connected weighted graph $G$, its output $T$ is a minimum weight spanning tree.

**Proof.** First, $T$ is a spanning tree. This is because:

No cycles, connected ⇒ spanning

- **$T$ is a forest.** No cycles are ever created.
- **$T$ is spanning.** Suppose that there is a vertex $v$ that is not incident with the edges of $T$. Then the incident edges of $v$ must have been considered in the algorithm at some step. The first edge (in edge order) would have been included because it could not have created a cycle, which contradicts the definition of $T$.
- **$T$ is connected.** Suppose that $T$ is not connected. Then $T$ has two or more connected components. Since $G$ is connected, then these components must be connected by some edges in $G$, not in $T$. The first of these edges (in edge order) would have been included in $T$ because it could not have created a cycle, which contradicts the definition of $T$.

Second, $T$ is a spanning tree of minimum weight. We will prove this using induction. Let $T^*$ be a minimum-weight spanning tree. If $T = T^*$, then $T$ is a minimum weight spanning tree. If $T \neq T^*$, then there exists an edge $e \in T^*$ **of minimum weight** that is not in $T$. Further, $T \cup e$ contains a cycle $C$ such that:

a. Every edge in $C$ has weight less than $\mathrm{wt}\,(e)$. (This follows from how the algorithm constructed $T$.)

b. There is some edge $f$ in $C$ that is not in $T^*$. (Because $T^*$ does not contain the cycle $C$.)

Consider the tree $T_2 = T \setminus \{e\} \cup \{f\}$: switch out edge in T

a. $T_2$ is a spanning tree.

b. $T_2$ has more edges in common with $T^*$ than $T$ did.

c. And $\mathrm{wt}\,(T_2) \geq \mathrm{wt}\,(T)$. (We exchanged an edge for one that is no more expensive.)

We can redo the same process with $T_2$ to find a spanning tree $T_3$ with more edges in common with $T^*$. By induction, we can continue this process until we reach $T^*$, from which we see

$$\mathrm{wt}\,(T) \leq \mathrm{wt}\,(T_2) \leq \mathrm{wt}\,(T_3) \leq \cdots \leq \mathrm{wt}\,(T^*).$$

Since $T^*$ is a minimum weight spanning tree, then these inequalities must be equalities and we conclude that $T$ is a minimum weight spanning tree.