Math 266: Autonomous equation and population dynamics

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Such equations are always separable:

$$\frac{dy}{f(y)} = dt \Rightarrow \int \frac{dy}{f(y)} = t + C.$$

However, the solutions are often implicit.

Basic properties of solutions

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- Algebraically, in the formula for solutions $\int \frac{dy}{f(y)} = t + C$, this means changing the arbitrary constants.
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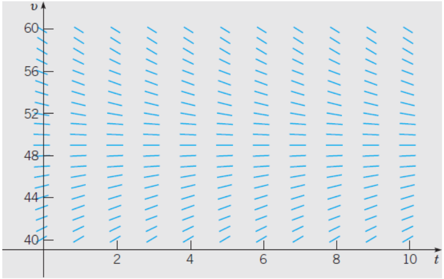
Equilibrium solutions

A constant function $y(t) \equiv y_0$ is a solution if and only if $f(y_0) = 0$. Such numbers y_0 are often called **critical points**.

An old example through dfield

We recall the first example of falling objects

$$\frac{dv}{dt} = 9.8 - \frac{1}{5}v$$



The only equilibrium solution is $v(t) \equiv 49$.

Non-equilibrium solutions

From now on, in the autonomous equation

$$y' = f(y)$$

we assume that f and f' are both continuous, so we have local existence and uniqueness of solutions. (Recall the bad example: $f(y) = y^{1/3}$.)

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- A non-equilibrium solution is confined in a region where either f is always positive or f is always negative.
- Monotonicity: All non-equilibrium solutions are either always increasing ($f(y) > 0$) or decreasing ($f(y) < 0$).
- Limits: As $t \to \pm\infty$, each non-equilibrium solution either converges to an equilibrium solution or infinity (maybe in finite time).

Population dynamics

As examples, we study some basic models in population dynamics of a single species. Without interaction with other species, the change of the population $y = y(t)$ can be assumed to only depend on the population itself:

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- Thomas Robert Malthus 1798: growth rate
- Pierre Franois Verhulst 1838: logistic model
- Critical threshold, etc.

Malthus: exponential growth

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- With initial value $y(0) = y_0 > 0$, the solution is an exponential function $y(t) = y_0 e^{rt}$.
- In a limited time period and ideal situation (abundance of food, no change of environment, etc.), the model is observed to be quite accurate.
- But obviously, it cannot go on forever. Limitation of food, space, etc.

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The simplest example for such a function is a linear function

$$h(y) = r - ay = r(1 - \frac{y}{K}).$$

Analyzing logistic growth

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The number K is often called the **saturation level** or the **environment carrying capacity**.

Second derivative and inflection points

By chain rule and the equation itself, we have

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(f(y)) = f'(y)\frac{dy}{dt} = f'(y)f(y).$$

Here $f'(y) = r(1 - \frac{2y}{K})$, so

- If $0 < y < K/2$, then $y'' > 0$ and y is convex
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To sum up, for the initial value problem

$$\frac{dy}{dt} = ry(1 - \frac{y}{K}), \quad y(0) = y_0 > 0.$$

- If $0 < y_0 < K$, then the solution is increasing with a single inflection point when $y = K/2$ where it changes from convex to concave.
- If $y_0 > K$, then the solution is decreasing and convex.

Graphs and the phase line

It is easier to summarize the analysis with the following graphs.

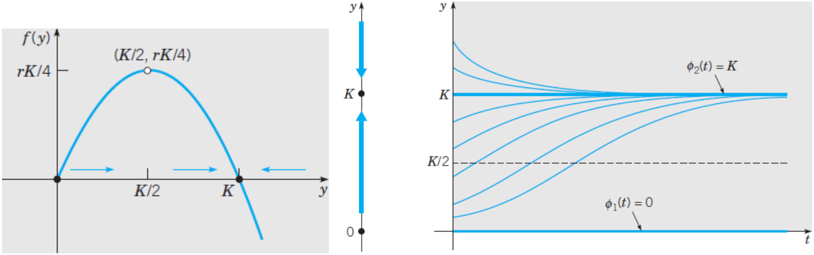

Figure: Left: Graph of $f = f(y)$; Middle: phase line; Right: the graphs of the solutions $y = y(t)$

On the **phase line**, a dot represents an equilibrium solution while on other parts we use arrows to describe the monotonicity of solutions.

Explicit solutions

For the logistic model, we can find the solution in explicit form:

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Therefore

$$\frac{y}{1-y/K} = Ce^{rt} \quad \Rightarrow \quad y = \frac{Ce^{rt}}{1+Ce^{rt}/K}.$$

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It is easy to check all the properties that we derived without solving the equation. In particular, as long as $y_0 > 0$,

$$y(t) \to K \quad \text{as} \quad t \to \infty.$$

Logistic functions

The special case $K = 1$ and $y_0 = \frac{1}{2}$ gives the so called standard logistic function

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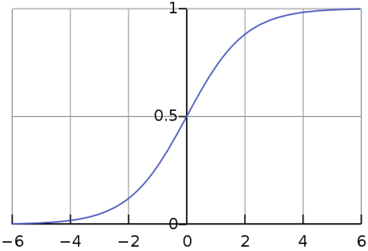
whose graph is a "sigmoid" curve (looking like the letter "S").



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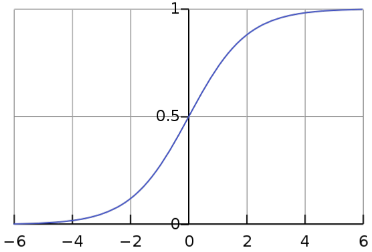


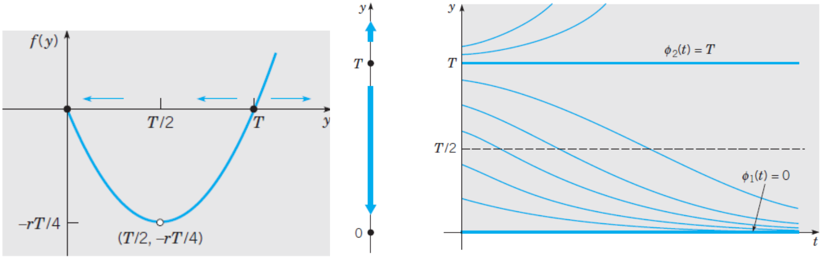
Applications in a range of fields, including artificial neural networks, biology (especially ecology), biomathematics, chemistry, demography, economics, geoscience, mathematical psychology, probability, sociology, political science, linguistics, and statistics.

A critical threshold

For certain species, when population is too small, there is a danger that it could go extinct as there are not enough partners to breed new generations. In such cases, we introduce the following model

$$\frac{dy}{dt} = -ry(1 - \frac{y}{T})$$

where $T > 0$ is the **threshold level**.

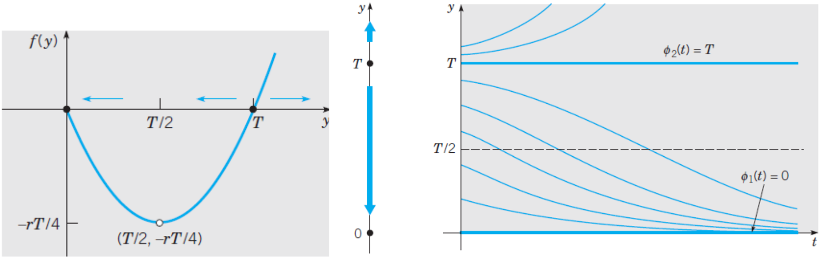


Analyzing the critical threshold

The analysis of

$$\frac{dy}{dt} = f(y) := -ry(1 - \frac{y}{T})$$

is very similar to the logistic model except all the signs are reversed. In this case, the equilibrium solution $y(t) \equiv T$ is said to be **unstable** as both solutions in $(0, T)$ and (T, ∞) diverge from T.



Logistic growth with a threshold

However, from the explicit form of the solution to

$$\frac{dy}{dt} = -ry(1 - \frac{y}{T}), \quad y(0) = y_0$$

given by

$$y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$$

we see that if $y_0 > T$, then $y(t) \to +\infty$ in finite time

$$t \to T_* = \frac{1}{r} \ln \frac{y_0}{y_0 - T}.$$

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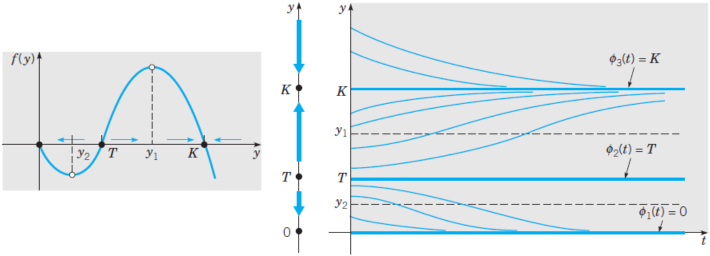
To avoid such behaviors, we combine the logistic model with the critical threshold:

$$\frac{dy}{dt} = -ry(1 - \frac{y}{T})(1 - \frac{y}{K}).$$

Logistic growth with a threshold

$$\frac{dy}{dt} = f(y) := -ry(1-\frac{y}{T})(1-\frac{y}{K}).$$

Here $r > 0$ is the natural growth rate; $T > 0$ is the threshold; $K > 0$ is the environment carrying capacity. We assume that $K > T$.



Equilibrium solutions:

- $y(t) \equiv T$ is unstable;
- $y(t) \equiv K$ is asymptotically stable.

Analyzing logistic growth with a threshold

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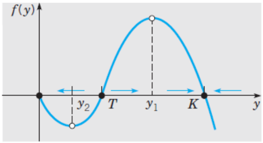
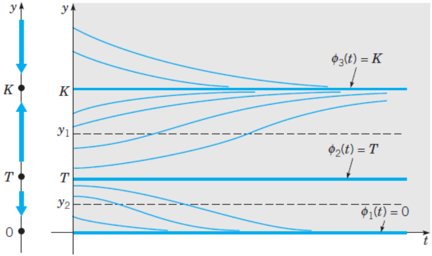
Other solutions:

- If initial population is below the threshold, then the species goes extinct: $y(t) \to 0$.
- If initial population is above the threshold, then the population tends to the saturation level: $y(t) \to K$.





Summary

In general, for an autonomous equation

$$y' = f(y)$$

The equilibrium solutions are given by $y(t) \equiv y_0$ where y_0 are solutions to $f(y_0) = 0$.

Classification of equilibrium solution

There are three kinds of equilibrium solutions $y(t) \equiv y_0$:

- Asymptotically stable: Nearby solutions converge to y_0.
- Unstable: Nearby solutions diverge from y_0.
- Semistable: On one side, nearby solutions converge to y_0, while on the other side, they diverge from y_0.

It is very easy to use phase line to picture these situations.