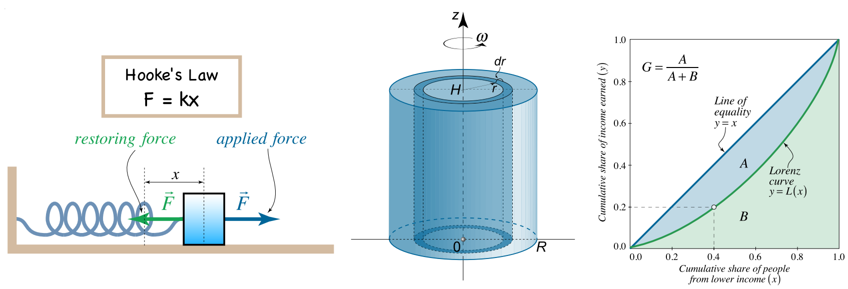
Math24

Menu

Calculus

Applications of Integrals



Volume of a Solid of Revolution: Disks and Washers

If a region in the plane is revolved about a line in the same plane, the resulting object is known as a solid of revolution.

For example, a solid right circular cylinder can be generated by revolving a rectangle.

The line about which we rotate the shape is called the axis of revolution.

The Disk Method

The disk method is used when we rotate a single curve $y = f(x)$ around the $x-$ (or $y-$) axis.

Suppose that $y = f(x)$ is a continuous non-negative function on the interval $[a, b]$.

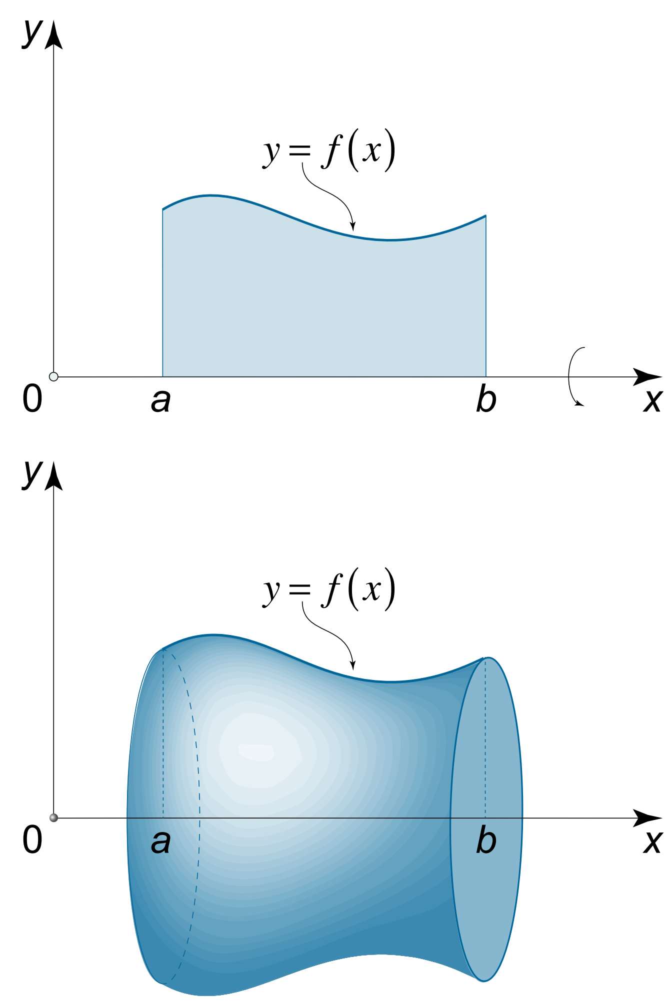


Figure 1.

The volume of the solid formed by revolving the region bounded by the curve $y = f(x)$ and the $x-$axis between $x = a$ and $x = b$ about the $x-$axis is given by

The cross section perpendicular to the axis of revolution has the form of a disk of radius $R = f(x)$.

Similarly, we can find the volume of the solid when the region is bounded by the curve $x = f(y)$ and the $y-$axis between $y = c$ and $y = d$, and is rotated about the $y-$axis.

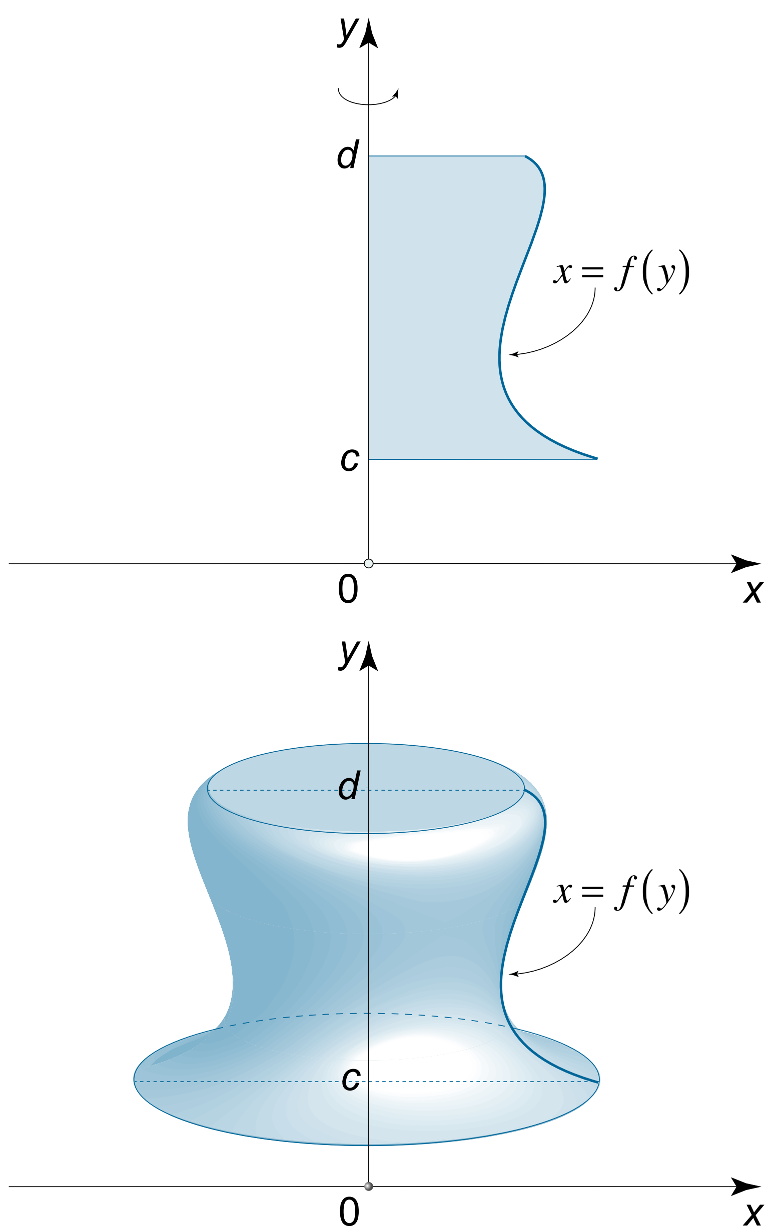


Figure 2.

$$V = \pi \int\limits_c^d {{{\left[{f\left(y \right)} \right]}^2}dy} .$$

The Washer Method

We can extend the disk method to find the volume of a hollow solid of revolution.

Assuming that the functions $f\left(x \right)$ and $g\left(x \right)$ are continuous and non-negative on the interval $\left[{a,b} \right]$ and $g\left(x \right) \le f\left(x \right)$, consider a region that is bounded by two curves $y = f\left(x \right)$ and $y = g\left(x \right)$, between $x = a$ and $x = b$.

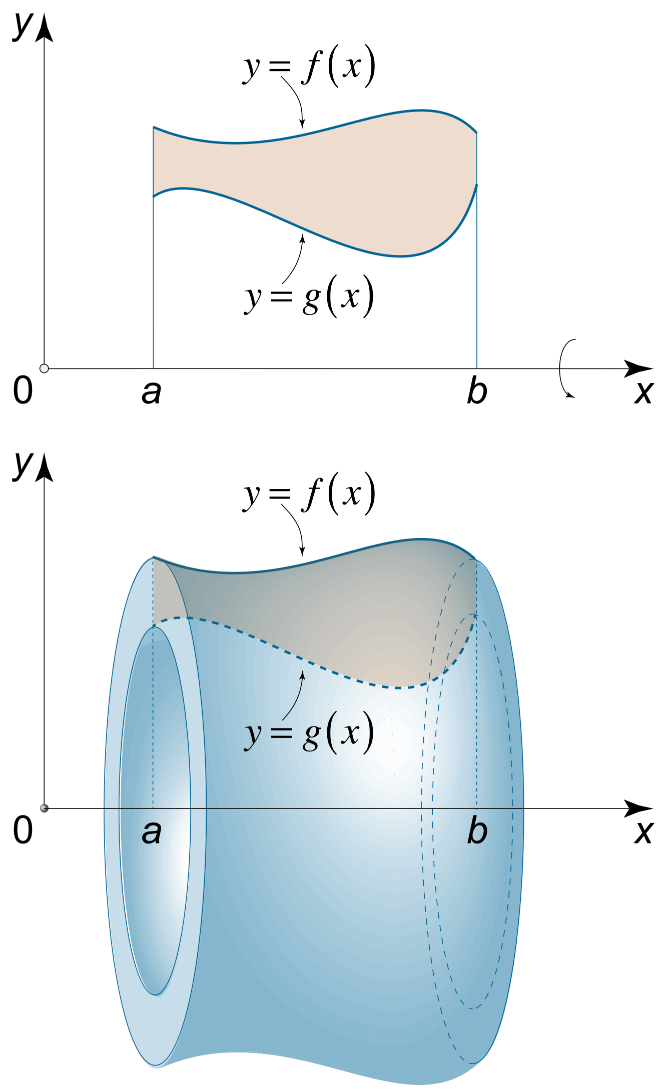


Figure 3

The volume of the solid formed by revolving the region about the $x-$axis is

$$V = \pi \int\limits_a^b \left(\left[f\left(x \right) \right]^2 - \left[g\left(x \right) \right]^2 \right) dx.$$

At a point x on the $x-$axis, a perpendicular cross section of the solid is washer-shape with the inner radius $r = g\left(x \right)$ and the outer radius $R = f\left(x \right).$

The volume of the solid generated by revolving about the $y-$axis a region between the curves $x = f\left(y \right)$ and $x = g\left(y \right),$ where $g\left(y \right) \le f\left(y \right)$ and $c \le y \le d$ is given by the formula

$$V = \pi \int\limits_c^d \left(\left[f\left(y \right) \right]^2 - \left[g\left(y \right) \right]^2 \right) dy.$$

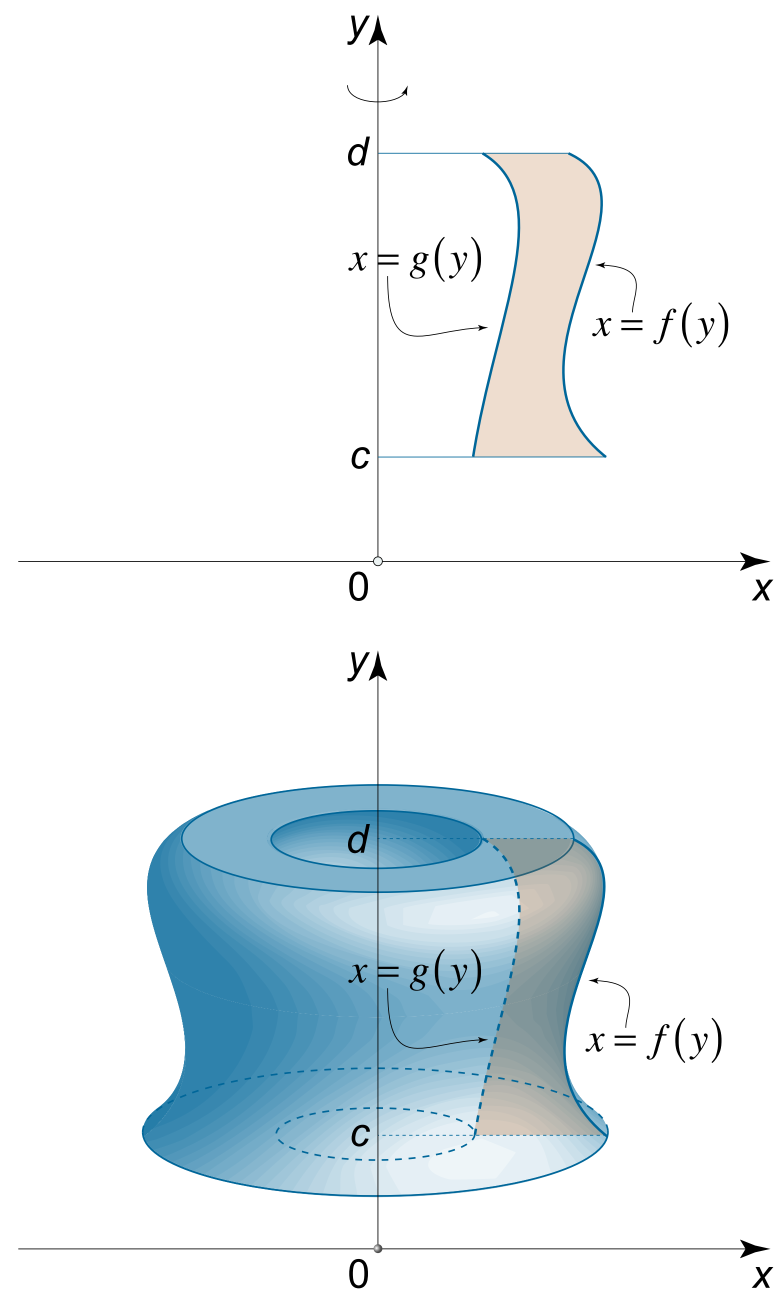


Figure 4.

Volume of a Solid of Revolution for a Parametric Curve

If a bounding curve is defined in parametric form by the equations $x = x(t), y = y(t),$ where the parameter t varies from α to β, then the volume of the solid generated by revolving the curve about the $x-$axis is given by

$$V_x = \pi \int\limits_{\alpha}^{\beta} y^2(t)\,\frac{dx}{dt}dt.$$

Respectively, when the curve is rotated about the $y-$axis, the volume of the solid of revolution is equal

$$V_y = \pi \int\limits_{\alpha}^{\beta} x^2(t)\,\frac{dy}{dt}dt.$$

Volume of a Solid of Revolution for a Polar Curve

There are many curves that are given by a polar equation $r = r(\theta)$. To convert from polar coordinates (r, θ) to Cartesian coordinates (x, y) , we use the known formulas

$$x = r(\theta)\cos\theta,\;\; y = r(\theta)\sin\theta.$$

So we come to the parametric form of the curve considered in the previous section.

It is important to keep in mind that the radius vector r also depends on the parameter θ. Therefore, the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are written as

$$\frac{dx}{dt} = \frac{d(r(\theta)\cos\theta)}{dt} = \frac{d(r(\theta))}{dt}\cos\theta - r(\theta)\sin\theta,$$

$$\frac{dy}{dt} = \frac{d(r(\theta)\sin\theta)}{dt} = \frac{d(r(\theta))}{dt}\sin\theta + r(\theta)\cos\theta.$$

Solved Problems

Click or tap a problem to see the solution.

Example 1

Using the disk method, calculate the volume of the right circular cone of height H and base radius R.

Example 2

Find the volume of the solid obtained by rotating the sine function between $x = 0$ and $x = \pi$ about the $x-$axis.

Example 3

Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = x^2$ and the square root function $y = \sqrt{x}$ around the $x-$axis.

Example 4

Find the volume of the solid obtained by rotating the region bounded by two parabolas $y = x^2 + 1$ and $y = 3 - x^2$ about the $x-$axis.

Example 5

A symmetrical parabolic segment with base a and height h rotates around the base. Calculate the volume of the resulting solid of revolution (Cavalieri's "lemon").

Example 6

The catenary line $y = \cosh x$ rotates around the $x-$axis and produces a surface called a catenoid. Find the volume of the solid bounded by the catenoid and two planes $x = -1$ and $x = 1$.

Example 7

Example 8

Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = 2x - x^2$ and the $x-$axis about the $y-$axis.

Example 9

Find the volume of the solid obtained by rotating an equilateral triangle with side a around one of its sides.

Example 10

One arch of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ revolves around its base. Calculate the volume of the body bounded by the given surface.

Example 1.

Using the disk method, calculate the volume of the right circular cone of height H and base radius R.

Solution.

The slant height of the cone is defined by the equation:

$$x = R - \frac{R}{H}y.$$

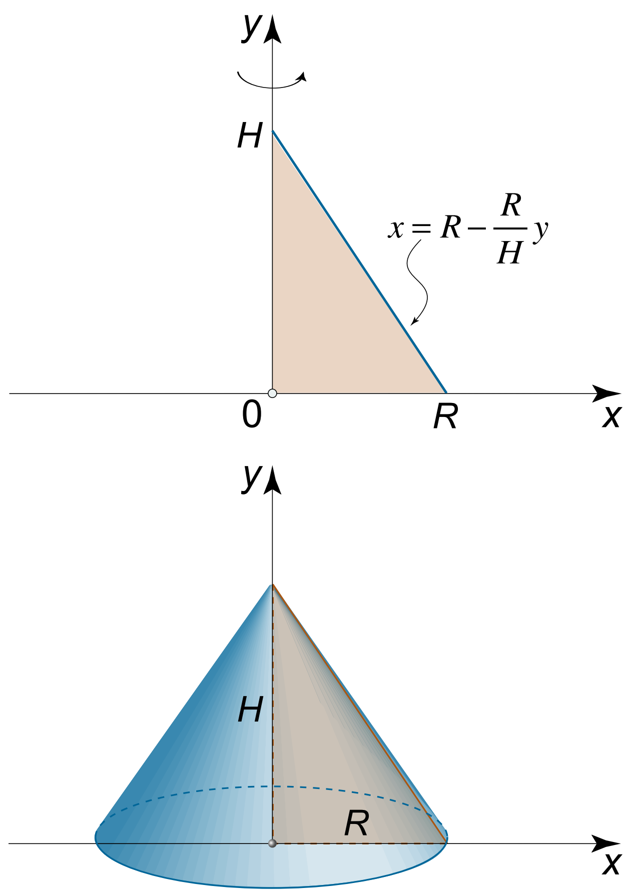


Figure 5.

Hence, the volume of the cone is given by

$$V = \pi \int\limits_0^H \left[x\left(y\right)\right]^2 dy = \pi \int\limits_0^H \left[R - \frac{R}{H}y\right]^2 dy = \pi R^2 \int\limits_0^H \left(1 - \frac{2y}{H} + \frac{y^2}{H^2}\right) dy$$

$$= \pi R^2 \left(H - \frac{y^2}{H} + \frac{y^3}{3H^2}\right)\Big|_0^H = \pi R^2 \left(\cancel{H} - \cancel{H} + \frac{H}{3}\right) = \frac{\pi R^2 H}{3}.$$

Example 2.

Find the volume of the solid obtained by rotating the sine function between $x = 0$ and $x = \pi$ about the $x-$axis.

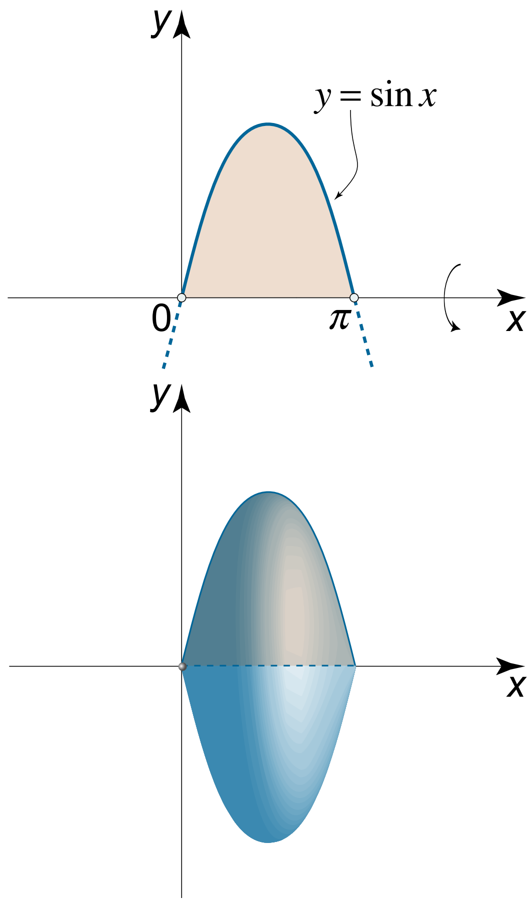


Figure 6.

By the disk method,

$$V = \pi \int\limits_0^\pi {{{\left[{\sin x} \right]}^2}dx} = \frac{\pi }{2}\int\limits_0^\pi {\left({1 - \cos 2x} \right)dx} = \frac{\pi }{2}\left({x - \frac{{\sin 2x}}{2}} \right)\Big|_0^\pi$$

$$= \frac{\pi }{2}\left[{\left({\pi - 0} \right) - \left({0 - 0} \right)} \right] = \frac{{{\pi ^2}}}{2}.$$

Example 3.

Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = x^2$ and the square root function $y = \sqrt{x}$ around the $x-$axis.

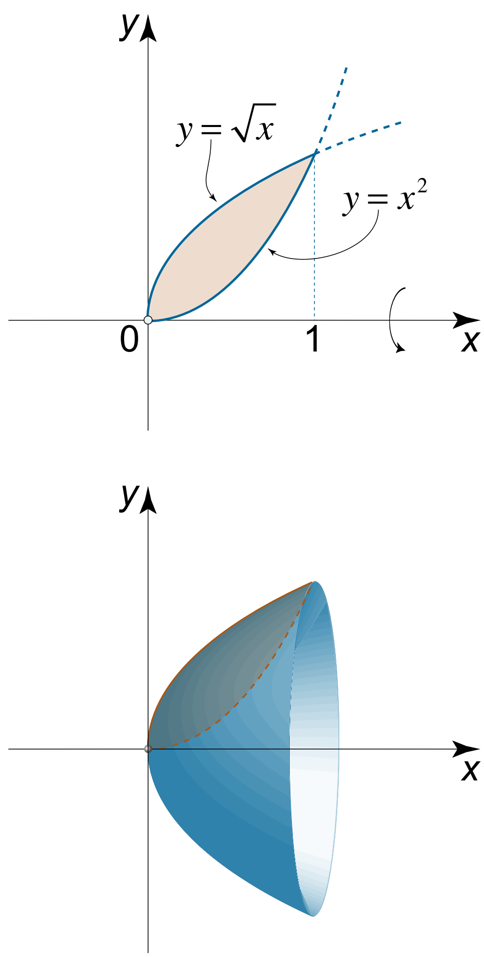


Figure 7.

Both curves intersect at the points $x = 0$ and $x = 1$. Using the washer method, we have

$$V = \pi \int\limits_0^1 \left(\left[\sqrt{x}\right]^2 - \left[x^2\right]^2 \right) dx = \pi \int\limits_0^1 \left(x - x^4\right) dx = \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right) \Big|_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right)$$
$$= \frac{3\pi}{10}.$$

Example 4.

Find the volume of the solid obtained by rotating the region bounded by two parabolas $y = x^2 + 1$ and $y = 3 - x^2$ about the $x-$axis.

First we determine the boundaries a and b :

$$x^2 + 1 = 3 - x^2, \quad \Rightarrow 2x^2 = 2, \quad \Rightarrow x^2 = 1, \quad \Rightarrow x_{1,2} = \pm 1.$$

Hence the limits of integration are $a = -1$, $b = 1$. We sketch the bounding region and the solid of revolution:

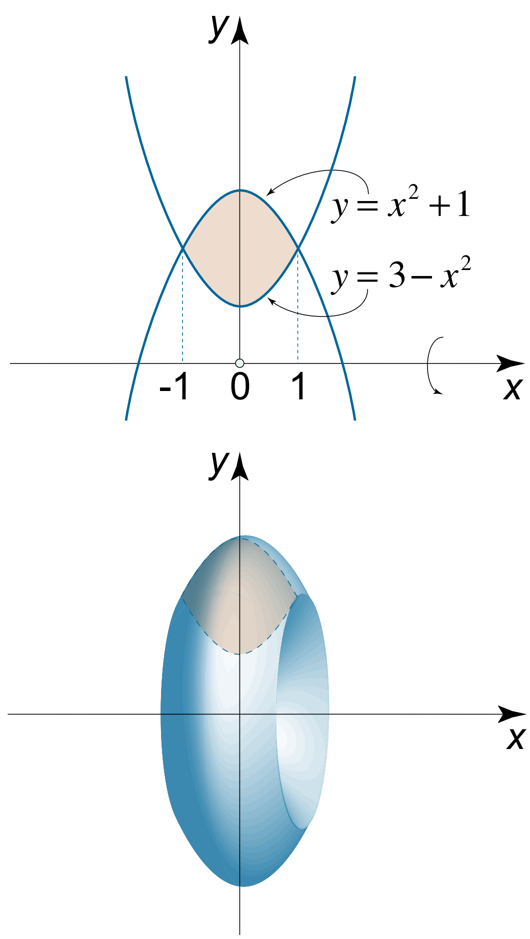


Figure 8.

Using the washer method, we find the volume of the solid:

$$V = \pi \int\limits_a^b \left(\left[f(x) \right]^2 - \left[g(x) \right]^2 \right) dx = \pi \int\limits_{-1}^{1} \left(\left(3 - x^2 \right)^2 - \left(x^2 + 1 \right)^2 \right) dx$$

$$= \pi \int\limits_{-1}^{1} \left(\left[3 - x^2 \right]^2 - \left[x^2 + 1 \right]^2 \right) dx = \pi \int\limits_{-1}^{1} \left(8 - 8x^2 \right) dx = 8\pi \int\limits_{-1}^{1} \left(1 - x^2 \right) dx$$

$$= 8\pi \left(x - \frac{x^3}{3} \right) \Big|_{-1}^{1} = 8\pi \left[\left(1 - \frac{1}{3} \right) - \left(-1 + \frac{1}{3} \right) \right] = 8\pi \cdot \frac{4}{3} = \frac{32\pi}{3}$$

Example 5.

A symmetrical parabolic segment with base a and height h rotates around the base. Calculate the volume of the resulting solid of revolution (Cavalieri's "lemon").

Solution.

The quadratic function is defined by the equation $y = kx(a - x)$, where the coefficient k can be found from the condition $y\left(\frac{a}{2}\right) = h$. Hence

$$y\left(\frac{a}{2}\right) = h, \quad \Rightarrow \frac{ka}{2}\left(a - \frac{a}{2}\right) = h, \quad \Rightarrow \frac{ka^2}{4} = h, \quad \Rightarrow k = \frac{4h}{a^2}.$$

So the parabolic segment is given by the expression $y = \frac{4h}{a^2} x (a - x)$.

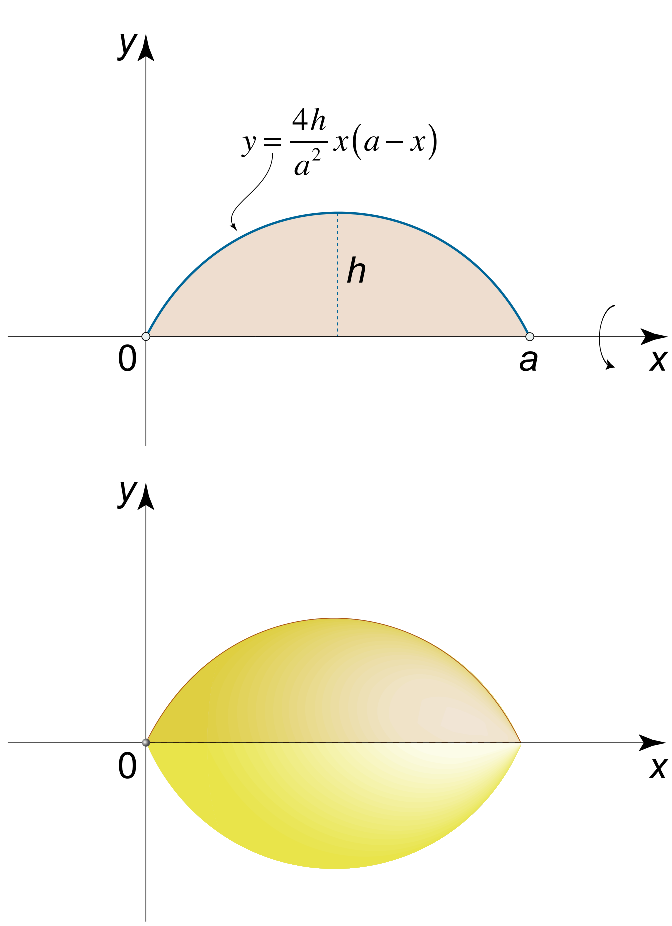


Figure 9.

Using the disk method, we obtain

$$V = \pi \int\limits_0^a [y(x)]^2 dx = \pi \int\limits_0^a \left[\frac{4h}{a^2} x(a-x)\right]^2 dx = \frac{16\pi h^2}{a^4} \int\limits_0^a \left(ax - x^2\right)^2 dx$$

$$= \frac{16\pi h^2}{a^4} \int\limits_0^a \left(a^2x^2 - 2ax^3 + x^4\right) dx = \frac{16\pi h^2}{a^4} \left(\frac{a^2x^3}{3} - \frac{2ax^4}{4} + \frac{x^5}{5}\right)\Bigg|_0^a$$

$$= \frac{16\pi h^2}{a^4} \left(\frac{a^5}{3} - \frac{a^5}{2} + \frac{a^5}{5}\right) = 16\pi h^2 a \cdot \frac{1}{30} = \frac{8\pi h^2 a}{15}$$

Example 6.

Solution.

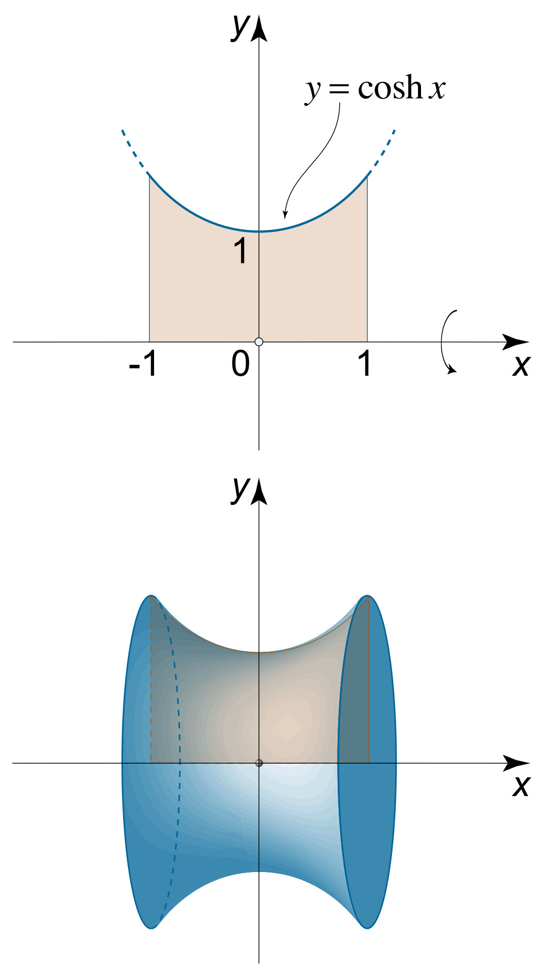


Figure 10.

Using the disk method and the hyperbolic identity

$$\cosh^2 x = \frac{1+\cosh 2x}{2},$$

we have

$$V = \pi \int_{-1}^{1} [\cosh x]^2 dx = \frac{\pi}{2} \int_{-1}^{1} [1+\cosh 2x]\, dx = \frac{\pi}{2} \left[x + \frac{\sinh 2x}{2} \right] \Bigg|_{-1}^{1}$$

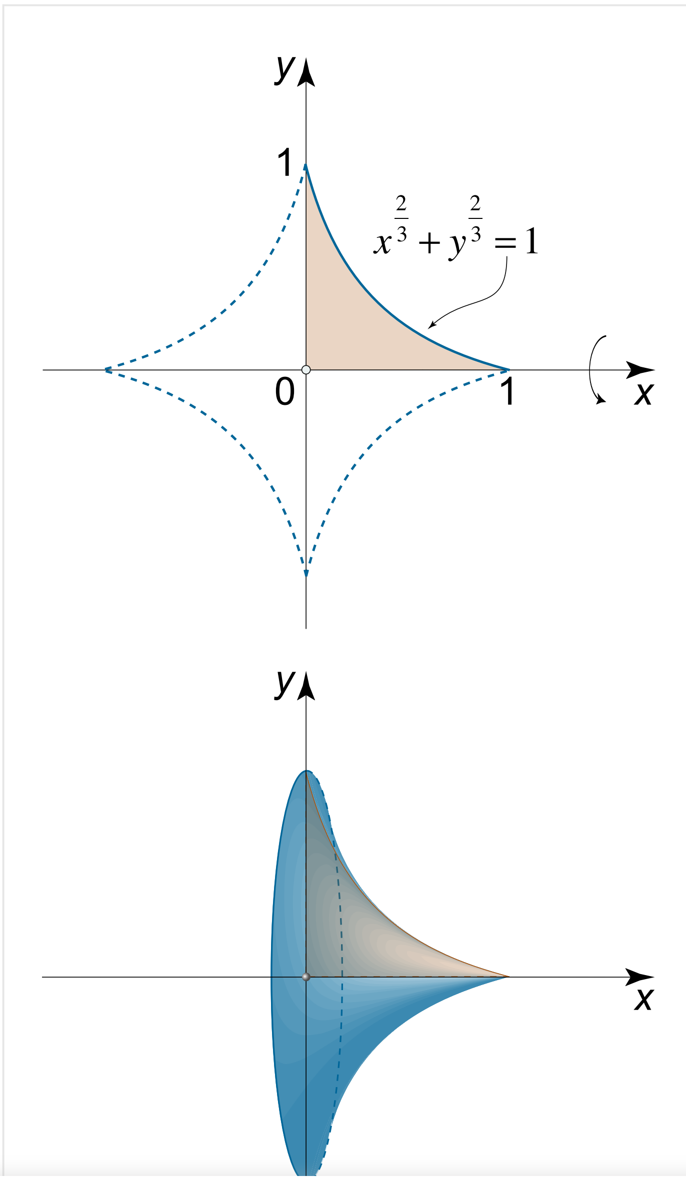
We took into account here that the hyperbolic sine function is odd, so $\sinh(-2) = -\sinh 2$.

Example 7.

Find the volume of the solid obtained by rotating the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ around its axis of symmetry.

Solution.

Due to symmetry, we can consider the region lying in the first quadrant and then multiply the volume of the region by two.



Let's solve the astroid equation for y^2 :

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1, \;\; \Rightarrow y^{\frac{2}{3}} = 1 - x^{\frac{2}{3}}, \;\; \Rightarrow \left(y^{\frac{2}{3}}\right)^3 = \left(1 - x^{\frac{2}{3}}\right)^3, \;\; \Rightarrow y^2 = 1 - 3x^{\frac{2}{3}} + 3x^{\frac{4}{3}} - x^2.$$

Hence, the total volume of the solid bounded by the astroid is given by

$$V = 2\pi \int\limits_0^1 y^2(x)\,dx = 2\pi \int\limits_0^1 \left(1 - 3x^{\frac{2}{3}} + 3x^{\frac{4}{3}} - x^2\right) dx$$

$$= 2\pi \left(x - \frac{9x^{\frac{5}{3}}}{5} + \frac{9x^{\frac{7}{3}}}{7} - \frac{x^3}{3}\right)\Bigg|_0^1 = 2\pi \left(1 - \frac{9}{5} + \frac{9}{7} - \frac{1}{3}\right) = 2\pi \cdot \frac{16}{105} = \frac{32\pi}{105}$$

Example 8.

Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = 2x - x^2$ and the $x-$axis about the $y-$axis.

Solution.

Find the points of intersection of the parabola with the $x-$axis:

$$2x - x^2 = 0, \;\; \Rightarrow x(2 - x) = 0, \;\; \Rightarrow x_1 = 0, x_2 = 2.$$

As the region is revolved about the $y-$axis, we express the equation of the bounding curve in terms of y :

$$y = 2x - x^2, \;\; \Rightarrow x^2 - 2x + 1 = 1 - y, \;\; \Rightarrow (x-1)^2 = 1 - y,$$
$$\Rightarrow x - 1 = \pm\sqrt{1-y}, \;\; \Rightarrow x = 1 \pm \sqrt{1-y}.$$

The signs "plus" and "minus" correspond to the two branches of the curve:

$$x = g(y) = 1 - \sqrt{1-y},$$

$$x = f(y) = 1 + \sqrt{1-y}$$

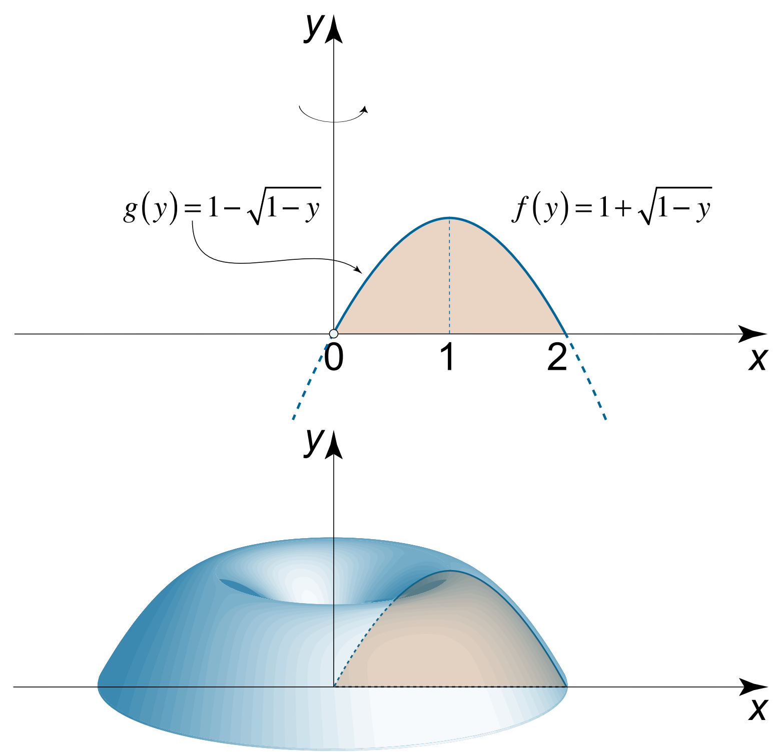


Figure 12.

Given that the variable y varies from 0 to 1 and using the washer method, we find the volume of the solid:

$$V = \pi \int\limits_0^1 \left([f(y)]^2 - [g(y)]^2 \right) dy = \pi \int\limits_0^1 \left(\left[1 + \sqrt{1-y}\right]^2 - \left[1 - \sqrt{1-y}\right]^2 \right) dy$$

$$= \pi \int\limits_0^1 \left(4\sqrt{1-y}\right) dy = 4\pi \int\limits_0^1 \sqrt{1-y} dy = \left[4\pi \cdot \frac{2(1-y)^{\frac{3}{2}}}{3} \cdot (-1) \right] \Bigg|_0^1$$

$$= \left[-\frac{8\pi\sqrt{(1-y)^3}}{3} \right] \Bigg|_0^1 = \frac{8\pi}{3}.$$

Example 9.

Find the volume of the solid obtained by rotating an equilateral triangle with side a about

Solution.

Let ABC be an equilateral triangle with side a.

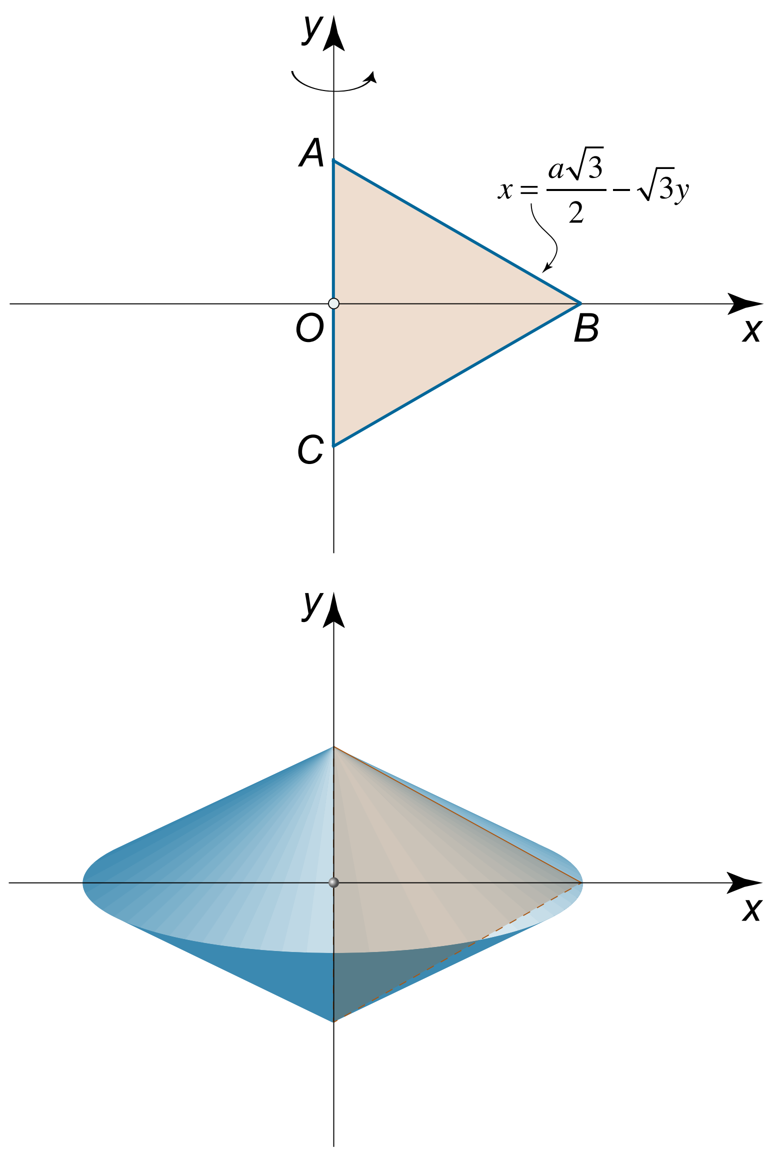


Figure 13.

Determine the height of the triangle OB :

$$OB = \sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \sqrt{\frac{3a^2}{4}} = \frac{\sqrt{3}a}{2}.$$

$$A\left(0,\frac{a}{2}\right),\ B\left(\frac{\sqrt{3}a}{2},0\right),\ C\left(0,-\frac{a}{2}\right).$$

Find the equation of the straight line AB using the two-point form:

$$\frac{x-x_A}{x_B-x_A}=\frac{y-y_A}{y_B-y_A},\quad \Rightarrow \frac{x-0}{\frac{\sqrt{3}a}{2}-0}=\frac{y-\frac{a}{2}}{0-\frac{a}{2}},\quad \Rightarrow \frac{x}{\sqrt{3}}=\frac{y-\frac{a}{2}}{-1},$$
$$\Rightarrow x=\frac{a\sqrt{3}}{2}-\sqrt{3}y.$$

By symmetry of the solid, we can integrate from 0 to $\frac{a}{2}$ and then multiply the result by two. Hence, the volume is given by the formula:

$$V=2\pi\int\limits_0^{\frac{a}{2}}\left[x\left(y\right)\right]^2dy=2\pi\int\limits_0^{\frac{a}{2}}\left[\frac{a\sqrt{3}}{2}-\sqrt{3}y\right]^2dy=6\pi\int\limits_0^{\frac{a}{2}}\left[\frac{a}{2}-y\right]^2dy$$
$$=6\pi\int\limits_0^{\frac{a}{2}}\left[\frac{a^2}{4}-ay+y^2\right]dy=6\pi\left[\frac{a^2y}{4}-\frac{ay^2}{2}+\frac{y^3}{3}\right]\Bigg|_0^{\frac{a}{2}}=6\pi\left[\cancel{\frac{a^3}{8}}-\cancel{\frac{a^3}{8}}+\frac{a^3}{24}\right]$$
$$=\frac{6\pi a^3}{24}=\frac{\pi a^3}{4}.$$

Example 10.

One arch of the cycloid $x=\theta-\sin\theta,\ y=1-\cos\theta$ revolves around its base. Calculate the volume of the body bounded by the given surface.

Solution.

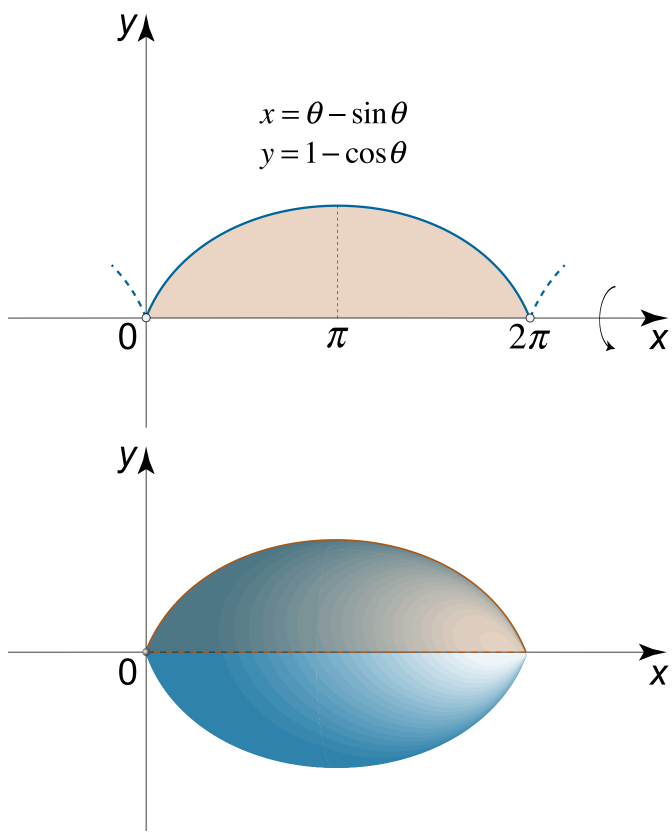


Figure 14.

The cycloid is given in parametric form. Therefore we express the integral $V = \pi \int\limits_0^{2\pi} y^2 dx$ in terms of the parameter θ :

$$y^2 = (1 - \cos\theta)^2,$$

$$dx = d\,(\theta - \sin\theta) = (1 - \cos\theta)\,d\theta.$$

Note that the variable x and the parameter θ change in the same range from 0 to 2π. Hence, the volume of the solid is given by the integral

$$V = \pi \int\limits_0^{2\pi} y^2 dx = \pi \int\limits_0^{2\pi} (1 - \cos\theta)^3 d\theta.$$

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta,$$

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta.$$

Hence, the volume of the solid is

$$V = \pi \int\limits_0^{2\pi} (1-\cos\theta)^3 d\theta = \pi \int\limits_0^{2\pi} \left(1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta\right) d\theta = \pi \int\limits_0^{2\pi} \left(1 - 3\cos\theta \right.$$

$$\left. + \frac{3}{2} + \frac{3}{2}\cos 2\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta\right) d\theta = \pi \int\limits_0^{2\pi} \left(\frac{5}{2} - \frac{15}{4}\cos\theta \right.$$

$$\left. + \frac{3}{2}\cos 2\theta - \frac{1}{4}\cos 3\theta\right) d\theta = \pi \left[\frac{5\theta}{2} - \frac{15}{4}\sin\theta + \frac{3}{4}\sin 2\theta - \frac{1}{12}\sin 3\theta\right]\Bigg|_0^{2\pi} = 5\pi^2.$$

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