Paul's Online Notes

Section 1-8 : Improper Integrals

In this section we need to take a look at a couple of different kinds of integrals. Both of these are examples of integrals that are called Improper Integrals.

Let's start with the first kind of improper integrals that we're going to take a look at.

Infinite Interval
In this kind of integral one or both of the limits of integration are infinity. In these cases, the interval of integration is said to be over an infinite interval.

Let's take a look at an example that will also show us how we are going to deal with these integrals.

Example 1 Evaluate the following integral.

$$\int_1^\infty \frac{1}{x^2}\,dx$$

Show Solution

So, this is how we will deal with these kinds of integrals in general. We will replace the infinity with a variable (usually t), do the integral and then take the limit of the result as t goes to infinity.

On a side note, notice that the area under a curve on an infinite interval was not infinity as we might have suspected it to be. In fact, it was a surprisingly small number. Of course, this won't always be the case, but it is important enough to point out that not all areas on an infinite interval will yield infinite areas.

Let's now get some definitions out of the way. We will call these integrals **convergent** if the associated limit exists and is a finite number (*i.e.* it's not plus or minus infinity) and **divergent** if the associated limit either doesn't exist or is (plus or minus) infinity.

Let's now formalize up the method for dealing with infinite intervals. There are essentially three cases that we'll need to look at.

1. If $\displaystyle\int_a^t f(x)\,dx$ exists for every $t > a$ then,

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided the limit exists and is finite.

2. If $\int_{t}^{b} f(x)\,dx$ exists for every $t < b$ then,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t \to -\infty} \int_{t}^{b} f(x)\,dx$$

provided the limits exists and is finite.

3. If $\int_{-\infty}^{c} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$ are both convergent then,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

Where c is any number. Note as well that this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

Let's take a look at a couple more examples.

Example 2 Determine if the following integral is convergent or divergent and if it's convergent find its value.

$$\int_{1}^{\infty} \frac{1}{x}\,dx$$

Hide Solution

So, the first thing we do is convert the integral to a limit.

$$\int_{1}^{\infty} \frac{1}{x}\,dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x}\,dx$$

Now, do the integral and the limit.

$$\begin{aligned}\int_{1}^{\infty} \frac{1}{x}\,dx &= \lim_{t \to \infty} \ln(x)\big|_{1}^{t} \\ &= \lim_{t \to \infty} (\ln(t) - \ln 1) \\ &= \infty\end{aligned}$$

So, the limit is infinite and so the integral is divergent.

If we go back to thinking in terms of area notice that the area under $g(x) = \frac{1}{x}$ on the interval $[1, \infty)$ is infinite. This is in contrast to the area under $f(x) = \frac{1}{x^2}$ which was quite small. There really isn't all that much difference between these two functions and yet there is a large difference in the area under them. We can actually extend this out to the following fact.

Fact

If $a > 0$ then

$$\int_a^\infty \frac{1}{x^p}\, dx$$

is convergent if $p > 1$ and divergent if $p \le 1$.

One thing to note about this fact is that it's in essence saying that if an integrand goes to zero fast enough then the integral will converge. How fast is fast enough? If we use this fact as a guide it looks like integrands that go to zero faster than $\frac{1}{x}$ goes to zero will probably converge.

Let's take a look at a couple more examples.

Example 3 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^0 \frac{1}{\sqrt{3-x}}\, dx$$

Hide Solution ▾

There really isn't much to do with these problems once you know how to do them. We'll convert the integral to a limit/integral pair, evaluate the integral and then the limit.

$$\begin{aligned}\int_{-\infty}^0 \frac{1}{\sqrt{3-x}}\, dx &= \lim_{t\to-\infty}\int_t^0 \frac{1}{\sqrt{3-x}}\, dx\\ &= \lim_{t\to-\infty} -2\sqrt{3-x}\Big|_t^0\\ &= \lim_{t\to-\infty}\left(-2\sqrt{3}+2\sqrt{3-t}\right)\\ &= -2\sqrt{3}+\infty\\ &= \infty\end{aligned}$$

So, the limit is infinite and so this integral is divergent.

Example 4 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx$$

Hide Solution ▾

In this case we've got infinities in both limits. The process we are using to deal with the infinite limits requires only one infinite limit in the integral and so we'll need to split the integral up into two separate integrals. We can split the integral up at any point, so let's choose $x = 0$ since this will be a convenient point for the evaluation process. The integral is then,

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx = \int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx + \int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx$$

We've now got to look at each of the individual limits.

$$\begin{aligned}\int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx &= \lim_{t\to-\infty}\int_{t}^{0} x\mathbf{e}^{-x^2}\,dx \\ &= \lim_{t\to-\infty}\left.\left(-\frac{1}{2}\mathbf{e}^{-x^2}\right)\right|_{t}^{0} \\ &= \lim_{t\to-\infty}\left(-\frac{1}{2}+\frac{1}{2}\mathbf{e}^{-t^2}\right) \\ &= -\frac{1}{2}\end{aligned}$$

So, the first integral is convergent. Note that this does NOT mean that the second integral will also be convergent. So, let's take a look at that one.

$$\begin{aligned}\int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx &= \lim_{t\to\infty}\int_{0}^{t} x\mathbf{e}^{-x^2}\,dx \\ &= \lim_{t\to\infty}\left.\left(-\frac{1}{2}\mathbf{e}^{-x^2}\right)\right|_{0}^{t} \\ &= \lim_{t\to\infty}\left(-\frac{1}{2}\mathbf{e}^{-t^2}+\frac{1}{2}\right) \\ &= \frac{1}{2}\end{aligned}$$

This integral is convergent and so since they are both convergent the integral we were actually asked to deal with is also convergent and its value is,

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx = \int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx + \int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx = -\frac{1}{2}+\frac{1}{2} = 0$$

Example 5 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^{\infty} \sin x \, dx$$

Hide Solution

First convert to a limit.

$$\begin{aligned}\int_{-2}^{\infty} \sin x \, dx &= \lim_{t \to \infty} \int_{-2}^{t} \sin x \, dx \\ &= \lim_{t \to \infty} \left. \left(-\cos x \right) \right|_{-2}^{t} \\ &= \lim_{t \to \infty} \left(\cos 2 - \cos t \right)\end{aligned}$$

This limit doesn't exist and so the integral is divergent.

In most examples in a Calculus II class that are worked over infinite intervals the limit either exists or is infinite. However, there are limits that don't exist, as the previous example showed, so don't forget about those.

Discontinuous Integrand

We now need to look at the second type of improper integrals that we'll be looking at in this section. These are integrals that have discontinuous integrands. The process here is basically the same with one subtle difference. Here are the general cases that we'll look at for these integrals.

1. If $f(x)$ is continuous on the interval $[a, b)$ and not continuous at $x = b$ then,

   $$\int_{a}^{b} f(x) \, dx = \lim_{t \to b^-} \int_{a}^{t} f(x) \, dx$$

   provided the limit exists and is finite. Note as well that we do need to use a left-hand limit here since the interval of integration is entirely on the left side of the upper limit.

2. If $f(x)$ is continuous on the interval $(a, b]$ and not continuous at $x = a$ then,

   $$\int_{a}^{b} f(x) \, dx = \lim_{t \to a^+} \int_{t}^{b} f(x) \, dx$$

   provided the limit exists and is finite. In this case we need to use a right-hand limit here since the interval of integration is entirely on the right side of the lower limit.

3. If $f(x)$ is not continuous at $x = c$ where $a < c < b$ and $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent then,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

As with the infinite interval case this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

4. If $f(x)$ is not continuous at $x = a$and $x = b$and if $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent then,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Where c is any number. Again, this requires BOTH of the integrals to be convergent in order for this integral to also be convergent.

Note that the limits in these cases really do need to be right or left-handed limits. Since we will be working inside the interval of integration we will need to make sure that we stay inside that interval. This means that we'll use one-sided limits to make sure we stay inside the interval.

Let's do a couple of examples of these kinds of integrals.

Example 6 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$$

Hide Solution ▾

The problem point is the upper limit so we are in the first case above.

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to 3^-}\int_0^t \frac{1}{\sqrt{3-x}}\,dx\\ &= \lim_{t\to 3^-}\left(-2\sqrt{3-x}\right)\Big|_0^t\\ &= \lim_{t\to 3^-}\left(2\sqrt{3}-2\sqrt{3-t}\right)\\ &= 2\sqrt{3}\end{aligned}$$

The limit exists and is finite and so the integral converges and the integral's value is $2\sqrt{3}$.

Example 7 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^{3} \frac{1}{x^3}\,dx$$

Hide Solution

This integrand is not continuous at $x = 0$ and so we'll need to split the integral up at that point.

$$\int_{-2}^{3} \frac{1}{x^3}\,dx = \int_{-2}^{0} \frac{1}{x^3}\,dx + \int_{0}^{3} \frac{1}{x^3}\,dx$$

Now we need to look at each of these integrals and see if they are convergent.

$$\begin{aligned}\int_{-2}^{0} \frac{1}{x^3}\,dx &= \lim_{t\to 0^-} \int_{-2}^{t} \frac{1}{x^3}\,dx \\ &= \lim_{t\to 0^-} \left(-\frac{1}{2x^2}\right)\Bigg|_{-2}^{t} \\ &= \lim_{t\to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{8}\right) \\ &= -\infty\end{aligned}$$

At this point we're done. One of the integrals is divergent that means the integral that we were asked to look at is divergent. We don't even need to bother with the second integral.

Before leaving this section let's note that we can also have integrals that involve both of these cases. Consider the following integral.

Example 8 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{0}^{\infty} \frac{1}{x^2}\,dx$$

Hide Solution

This is an integral over an infinite interval that also contains a discontinuous integrand. To do this integral we'll need to split it up into two integrals so each integral contains only one point of discontinuity. It is important to remember that all of the processes we are working with in this section so that each integral only contains one problem point.

We can split it up anywhere but pick a value that will be convenient for evaluation purposes.

$$\int_0^{\infty} \frac{1}{x^2}\,dx = \int_0^{1} \frac{1}{x^2}\,dx + \int_1^{\infty} \frac{1}{x^2}\,dx$$

In order for the integral in the example to be convergent we will need BOTH of these to be convergent. If one or both are divergent then the whole integral will also be divergent.

We know that the second integral is convergent by the fact given in the infinite interval portion above. So, all we need to do is check the first integral.

$$\begin{aligned}\int_0^{1} \frac{1}{x^2}\,dx &= \lim_{t\to 0^+} \int_t^{1} \frac{1}{x^2}\,dx \\ &= \lim_{t\to 0^+} \left(-\frac{1}{x}\right)\Big|_t^1 \\ &= \lim_{t\to 0^+} \left(-1 + \frac{1}{t}\right) \\ &= \infty\end{aligned}$$

So, the first integral is divergent and so the whole integral is divergent.

 Page Last Modified : 1/21/2020